Well posedness of ODE's and continuity equations with nonsmooth vector fields, and applications

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1 Preface

In this paper I will make an overview of the theory of flows associated to nonsmooth vector fields, describing also some applications to PDE's. My presentation reflects the content and the style of my Santalo lecture, aimed at the description of the origin and of the main developments of the theory, including the most recent ones. In the years more extended lecture notes (in chronological order, [10, 11, 25, 29]) on this subject appeared, which contain proofs and many more technical details, and the interested reader is referred to them.

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In the theory that I am going to illustrate, the basic idea is to exploit some randomness on the set of initial conditions of the ODE; this point of view has many connections with stochastic ODE's, where randomness acts in an additive or multiplicative way on the drift (see for instance [80, 81, 99] and the references therein), and with the theory of invariant measures for specific classes of PDE's, typically of Hamiltonian type (viewed as infinite-dimensional ODE's, see [37]). To keep the exposition simple, I will not describe these connections, that can be easily recognized by the experts. Even with these limitations, the subject is very wide and the references do not pretend to be exhaustive.

2 Introduction to the problem

Let $T > 0$, $d \geq 1$ and let $\boldsymbol{b}(t,x) = \boldsymbol{b}_t(x) : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ be a vector field. The flow map $\boldsymbol{X}(t,x)$ associated to $\boldsymbol{b}$ is defined by

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{X}(t,x) = \boldsymbol{b}_t\big(\boldsymbol{X}(t,x)\big), \\ \\ \boldsymbol{X}(0,x) = x, \end{cases}$$

where, as typical in the theory of ODE's, we assume that the function $\boldsymbol{X}(\cdot,x)$ is absolutely continuous in $[0,T]$, so that the derivative above makes sense for $\mathscr{L}^1$-a.e. $t \in (0,T)$ and also the initial condition makes sense. On the other hand, we are going to assume very little regularity on $\boldsymbol{b}$; already in the classical theory one can assume only a measurable dependence w.r.t. t, but weaking the regularity assumptions w.r.t. x (e.g. a uniform in time Lipschitz property) is much more demanding.

The goal is to provide a good notion of flow map $\boldsymbol{X}(t,x)$ even in situations when $\boldsymbol{b}$ is not so regular, and even defined only up to $\mathscr{L}^1 \times \mathscr{L}^d$-negligible sets. From the point of view of the applications, "good" means in particular stable w.r.t reasonable smooth approximations of $\boldsymbol{b}$, since this leads to a consistency between weak and relaxed models in many PDE's of Mathematical Physics (see for instance [90], [91]). In the classical theories, more refined existence/uniqueness results are available (one-sided Lipschitz condition [47], Osgood condition,....), but still these results arise as variants of the standard Lipschitz condition (another special case is given by 1d problems, see for instance [44]). More refined results arise if a special structure of the vector field is assumed: the most important case is maybe the case of gradient flows $\boldsymbol{b}(x) = -\nabla V(x)$: by looking at refined energy dissipation identities and monotonicity properties (even in the more general case of maximal monotone operators) the theory of gradient flows provides very general and elegant existence, uniqueness and stability results, see [14], [53], [58], [59].

The philosophy of the theory illustrated here, initiated by DiPerna-Lions in their seminal paper [76], is a bit different, since also a reference measure (typically $\mathscr{L}^d$ in Euclidean

spaces) plays a major role. As a matter of fact, the reference measure is used to quantify the concentration of trajectories and to specify exceptional sets.
The following axiomatization, slightly different from the one of the original paper [76], has been introduced in [8]. In order to introduce it, let us recall a notation typical of Optimal Transport: if U, V are metric spaces and $f : U \to V$ is a Borel map, any Borel measure μ in U induces a Borel measure $f_\sharp\mu$ in V, defined by

$$f_\sharp\mu(B) := \mu\big(f^{-1}(B)\big) \qquad B \in \mathscr{B}(V).$$

This construction works if μ is nonnegative, or with finite total variation, and in the first case preserves the total mass or the σ-finiteness property. It is also useful to remind the change of variables formula for the push-forward measure:

$$\int_V g\, df_\sharp\mu = \int_U g \circ f\, d\mu$$

whenever $g \in L^1(V, f_\sharp\mu)$, or g is Borel and nonnegative.

Definition 1 (Regular Lagrangian Flow) *We say that $\boldsymbol{X}(t,x)$ is a Regular Lagrangian Flow associated to $\boldsymbol{b}_t$ if:*

(1) for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$, $t \mapsto \boldsymbol{X}(t,x)$ is an absolutely continuous solution to the ODE $\frac{d}{dt}\boldsymbol{X}(t,x) = \boldsymbol{b}_t(\boldsymbol{X}(t,x))$ with $\boldsymbol{X}(0,x) = x$;

(2) for some constant C, $\boldsymbol{X}(t,\cdot)_\sharp\mathscr{L}^d \le C\mathscr{L}^d$ for all $t \in [0,T]$.

Here "regular" refers to (2), which encodes a non-concentration property of the trajectories at any given time t and fully involves the reference measure $\mathscr{L}^d$ (unlike (1), which involves only the σ-ideal of $\mathscr{L}^d$-negligible sets). Indeed, one of the open problems of the theory is to see if one can develop a good theory using only a σ-ideal of "exceptional sets".
Only for very special classes of vector fields, as the 2-dimensional, autonomous, curl-free vector fields $\boldsymbol{b} = \nabla^\perp f$, the sharp regularity assumption for existence/uniqueness of the RLF is known, see [4, 5, 6], and it involves a weak Sard property and the factorization of the dynamics into a family of 1-dimensional problems along the level sets of f. Without restriction on dimension, and for general classes of vector fields, one of the key results of [76] deals with Sobolev regularity w.r.t. the space variable. The statement below is a small refinement of the corresponding one in [76], since only one-sided bounds on divergence are assumed (see [8]).

Theorem 2 *If $\boldsymbol{b} \in L^1_t\big(W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)\big)$, under the global growth conditions*

$$\frac{|\boldsymbol{b}|}{1+|x|} \in L^1_t(L^1_x) + L^1_t(L^\infty_x), \qquad (\mathrm{div}\,\boldsymbol{b})^- \in L^1_t(L^\infty_x)$$

the regular Lagrangian flow exists, is unique and stable w.r.t. smooth approximations.

I will not specify exactly here what stability means, see [25] for more precise statements; it deals with a sequence of vector fields $\boldsymbol{b}_h$ for which the flows $\boldsymbol{X}_h$ are classically defined, and stability means convergence in measure of the path-valued maps $x \mapsto \boldsymbol{X}_h(\cdot, x)$ to $x \mapsto \boldsymbol{X}(\cdot, x)$.

Looking at the assumptions of Theorem 2, it is is useful to separate the growth assumptions from the regularity assumptions. Indeed, if we remove the latter, we can still provide existence (but not uniqueness, in general) of a generalized solution. It involves, in a natural probabilistic language, probability measures on paths concentrated on absolutely continuous solutions to the ODE. See also [48, 49, 50] for an application of these ideas to the construction of (very) weak solutions to the system of incompressible Euler equations.

Theorem 3 ([8]) *Under the only assumptions*

$$\frac{|\boldsymbol{b}|}{1+|x|} \in L^1_t(L^1_x) + L^1_t(L^\infty_x), \qquad (\mathrm{div}\,\boldsymbol{b})^- \in L^1_t(L^\infty_x)$$

there exists a regular generalized flow, namely a family of probability measures $\boldsymbol{\eta}_x$ in $C([0,T];\mathbb{R}^d)$ concentrated on absolutely continuous solutions to the Cauchy problem with $\gamma(0) = x$, with

$$\int_{\mathbb{R}^d} \left(\int \phi(\gamma(t))\, d\boldsymbol{\eta}_x(\gamma) \right) dx \le C \int_{\mathbb{R}^d} \phi(y)\, dy \qquad \forall t \in [0,T],\ \phi \ge 0,$$

with $C = \exp\left[\int_0^T (\|(\mathrm{div}\,\boldsymbol{b}_t)^-\|_\infty)\, dt\right]$.

The strategy of proof is natural: first one proves that the global bounds on $\boldsymbol{b}$ provide existence of weak solutions $w \in L^\infty_t(L^1 \cap L^\infty(\mathbb{R}^d))$ to the continuity equation

$$\text{(CE)} \qquad \frac{d}{dt} w_t + \mathrm{div}\,(\boldsymbol{b}_t w_t) = 0$$

for any $w_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$. This can be obtained by mollifying $\boldsymbol{b}_t$, to get approximate solutions w^ϵ to

$$\frac{d}{dt} w^\epsilon_t + \mathrm{div}\,(\boldsymbol{b}^\epsilon_t w^\epsilon_t) = 0.$$

Then, since the equation is linear w.r.t. w, any weak* limit point of w^ϵ is a solution. This construction provides also the informations

$$\int_{\mathbb{R}^d} w_t\, dx = \int_{\mathbb{R}^d} w_0\, dx, \qquad w_t \ge 0 \text{ if } w_0 \ge 0.$$

Then, one can apply the following superposition principle (going back to L.C.Young [106], see also [97] in the context of 1-dimensional currents) adapted to nonnegative solutions to continuity equations [14, Thm. 8.2.1]. See also [92] for the case of inhomogeneous transport equations and [32] for more on the theory of Young measures and its applications in the study of the limiting behaviour of oscillating functions.

Theorem 4 (Superposition principle) *Let $\mu_t = w_t \mathscr{L}^d$ be a weakly continuous solution to (CE), with $w_t \geq 0$, $\int w_t \, dx = 1$ and*

$$\int_0^T \int_{\mathbb{R}^d} |\boldsymbol{b}_t| w_t \, dx dt < \infty.$$

Then there exists a probability measure $\boldsymbol{\eta}$ in $C([0,T];\mathbb{R}^d)$ concentrated on absolutely continuous solutions to the ODE such that $(e_t)_\sharp \boldsymbol{\eta} = w_t \mathscr{L}^d$ for all $t \in [0,T]$ (where $e_t : C([0,T];\mathbb{R}^d) \to \mathbb{R}^d$ is the evaluation map at time $t \in [0,T]$). In particular, if $\|w_t\|_\infty \in L^\infty(0,T)$, $\boldsymbol{\eta}$ satisfies the non-concentration condition

$$\int_{C([0,T];\mathbb{R}^d)} \phi(\gamma(t)) \, d\boldsymbol{\eta}(\gamma) \leq C \int_{\mathbb{R}^d} \phi(y) \, dy \qquad \forall t \in [0,T], \ \phi \geq 0.$$

Finally the regular generalized flow $\boldsymbol{\eta}_x$ is simply given by the conditional probabilities of $\boldsymbol{\eta}$, $\boldsymbol{\eta}_x = \mathbf{E}(\boldsymbol{\eta}|\gamma(0) = x)$.
In analogy with optimal transportation (think to the dichotomy Kantorovich plans versus transport maps, [14, 103]) it is the local regularity of $\boldsymbol{b}$ which prevents branching and leads from $\boldsymbol{\eta}_x$ to a "deterministic" flow $\delta_{\boldsymbol{X}(\cdot,x)}$, and therefore to existence *and* uniqueness. How regularity of $x \mapsto \boldsymbol{b}_t(x)$ enters into play? This topic is discussed in the next section.

3 Uniqueness of RLF via PDE well-posedness

One of the main tools which provide uniqueness of the regular Lagrangian flow is provided by the following general principle: as soon as we are able to prove uniqueness, at the PDE level, in the class of bounded nonnegative solutions, uniqueness at the Lagrangian level follows.

Theorem 5 ([8]) *If (CE) is well posed in $L^\infty\big(L^1_+ \cap L^\infty_+(\mathbb{R}^d)\big)$, then any regular generalized flow $\boldsymbol{\eta}_x$ is deterministic, namely $\boldsymbol{\eta}_x$ is a Dirac mass for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$.*

Heuristically, the proof in [8] goes along this line: if there exists a non-deterministic regular generalized flow $\boldsymbol{\eta}_x$, then one can carefully select paths γ to build distinct solutions w_1, w_2 to (CE) in the class $L^\infty\big(L^1_+ \cap L^\infty_+(\mathbb{R}^d)\big)$ and with the same initial condition.
In view of Theorem 5 we need then to prove well-posedness of (CE). Even for bounded and divergence-free vector fields this is a nontrivial problem, since oscillations might lead to loss of uniqueness (see the nice counterexamples in [72], [1]) and, more generally, to the typical phenomenon of the deterioration of regularity [52], [71], [65]. A possible strategy, which goes back to DiPerna-Lions, is to write down the PDE satisfied by $w_t^\epsilon = w_t * \rho_\epsilon$, namely

$$w_t^\epsilon + \operatorname{div}(w_t^\epsilon \boldsymbol{b}_t) = \mathscr{C}_\epsilon[\boldsymbol{b}_t, w_t]$$

where $\mathscr{C}_\epsilon[\boldsymbol{b}_t, w_t]$ is the commutator

$$\mathscr{C}_\epsilon[\boldsymbol{b}_t, w_t] := \operatorname{div}\left((w_t * \rho_\epsilon)\boldsymbol{b}_t\right) - \big(\operatorname{div}\left(w_t\ \boldsymbol{b}_t\right)\big) * \rho_\epsilon.$$

Notice that the commutator acts on the single time slices and that, obviously, it converges to 0 weakly in the sense of distributions as $\epsilon \to 0$. However, in order to transfer so to speak uniqueness from the approximate problems to the limit one, the crucial technical point is to show *strong* $L^1_{\rm loc}$ convergence of $\mathscr{C}_\epsilon[\boldsymbol{b}_t, w_t]$ as $\epsilon \to 0$. This improved convergence is useful to estabilish the so-called renormalized property of w.

Definition 6 (Renormalized solutions) *Let $\boldsymbol{b} \in L^1_{\rm loc}((0,T)\times\mathbb{R}^d)$ be such that $D\cdot\boldsymbol{b}_t \ll \mathscr{L}^d$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$, with density* $\operatorname{div}\boldsymbol{b}_t$ *satisfying*

$$\operatorname{div}\boldsymbol{b}_t \in L^1_{\rm loc}\left((0,T); L^1_{\rm loc}(\mathbb{R}^d)\right).$$

Let $w \in L^\infty_{\rm loc}((0,T)\times\mathbb{R}^d)$ and assume that, in the sense of distributions, one has

$$c := \frac{d}{dt}w + \boldsymbol{b}\cdot\nabla w \in L^1_{\rm loc}((0,T)\times\mathbb{R}^d). \tag{1}$$

Then, we say that w is a renormalized solution of (1) *if*

$$\frac{d}{dt}\beta(w) + \boldsymbol{b}\cdot\nabla\beta(w) = c\beta'(w) \qquad \forall \beta \in C^1(\mathbb{R}).$$

Equivalently, recalling the definition of the distribution $\boldsymbol{b}\cdot\nabla w = D\cdot(\boldsymbol{b}w) - wD\cdot b$, the definition could be given in a conservative form, writing

$$\frac{d}{dt}\beta(w) + D_x\cdot(\boldsymbol{b}\beta(w)) = c\beta'(w) + \beta(w)\operatorname{div}\boldsymbol{b}_t.$$

This concept is also reminiscent (see [70], [75]) of Kruzkhov's concept of *entropy* solution for a scalar conservation law

$$\frac{d}{dt}u + D_x\cdot(\boldsymbol{f}(u)) = 0, \qquad u : (0,+\infty)\times\mathbb{R}^d \to \mathbb{R}.$$

In this case only a distributional one-sided inequality is required:

$$\frac{d}{dt}\eta(u) + D_x\cdot(\boldsymbol{q}(u)) \le 0$$

for any convex entropy-entropy flux pair $(\eta, \boldsymbol{q})$ (i.e. η is convex and $\eta'\boldsymbol{f}' = \boldsymbol{q}'$).
If we assume strong convergence of the commutators, by writing down the PDE satisfied by $\beta(w^\epsilon)$ and passing to the limit as $\epsilon \to 0$ it is easy to obtain that w is a renormalized

solution. Now, if we are given two bounded solutions to (CE) with the same initial condition, we can call w the difference, which still solves (CE), and apply the renormalization property with a smooth approximation of $\beta(z) = |z|$ (for instance $\beta_\delta(z) = \sqrt{z^2 + \delta^2} - \delta$) to obtain that w vanishes identically. This proves the well posedness of (CE) in the class of bounded solutions. In this connection, it is worthwile to remark that it has been proved in [41] that the relation between the existence of strong approximations and renormalization is not incidental.

All in all, according to this strategy, everything depends on the possibility to prove strong convergence of commutators. For even convolution kernels ρ it is not hard to obtain the explicit expression

$$\mathscr{C}_\epsilon[\boldsymbol{c}, v](x) = \int_{\mathbb{R}^d} v(x - \epsilon y)\langle \frac{\boldsymbol{c}(x + \epsilon y) - \boldsymbol{c}(x)}{\epsilon}, \nabla \rho(y)\rangle \, dy. \tag{2}$$

Therefore, if $\boldsymbol{c}$ is of class $W^{1,1}$, the strong convergence of difference quotients

$$\frac{\boldsymbol{c}(x + \epsilon y) - \boldsymbol{c}(x)}{\epsilon} \to \nabla \boldsymbol{c}(x)(y) \qquad \text{in } L^1_{\rm loc}(\mathbb{R}^n), \text{ for all } y \in \mathbb{R}^n$$

and and integration w.r.t. x, together with the uniform bound in L^∞ on v, provide the strong convergence.

In the BV case, where the difference quotients converge only weakly as measures, this scheme has to be refined, using very anisotropic smoothing kernels. This idea, introduced by F.Bouchut, used in [89] for piecewise Sobolev vector fields, then in [46] for "split" vector fields $\boldsymbol{b}(x, v) = (v, \mathbf{F}(x))$ (see also [62], [61] and the more recent paper [39] that we mention later on), shows that one can make the contribution of the commutator very small by choosing a "local" convolution kernel adapted to the local structure of $\boldsymbol{c}$. For instance for the split vector fields above, one has to mollify faster in the x direction, compared to the v direction. In general one can use the celebrated Alberti's rank one theorem [2] to detect the "good" and "bad" directions.

Theorem 7 ([8]) *If $\boldsymbol{b} \in L^1_t\big(BV_{\rm loc}(\mathbb{R}^d; \mathbb{R}^d)\big)$, under the global growth conditions*

$$\frac{|\boldsymbol{b}|}{1 + |x|} \in L^1_t(L^1_x) + L^1_t(L^\infty_x), \qquad (\operatorname{div} \boldsymbol{b})^- \in L^1_t(L^\infty_x)$$

the regular Lagrangian flow exists, is unique and stable w.r.t. smooth approximations.

See also [13], [19], [63], [88], [83], [84], [85], [93], [95], [94] for more results, not only in the BV setting, more will be quoted later on.

In particular, I want to discuss in this section some more recent developments motivated by [9] (that I describe in more detail in the next section) and by Bressan's compactness conjecture [51], [52]. Bressan's conjecture predicts the strong compactness of the flows $\boldsymbol{X}_h(t,x)$ associated to sequences of smooth and uniformly bounded vector fields $\boldsymbol{b}_h$ with the properties

$$\sup_n \int_0^T \int_{\mathbb{R}^d} |\nabla_{t,x}\boldsymbol{b}_h(t,x)|\, dxdt < \infty,$$
$$C^{-1} \leq \det\bigl(\nabla_x \boldsymbol{X}_h(t,x)\bigr) \leq C \qquad \text{for some } C > 0$$

(so, in some sense, an integral version of the divergence along the flow is uniformly bounded). In connection with these papers and problems, one realizes that the most natural assumption at the level of the divergence of $\boldsymbol{b}(t,\cdot)$ is not really absolute continuity with bounded density, but rather the existence of a nonnegative density ρ, transported by $\boldsymbol{b}$, namely

$$\frac{d}{dt}\rho + D\cdot(\rho\boldsymbol{b}) = 0,$$

with the property that both ρ and ρ^{-1} are uniformly bounded. For instance, this is the case of the vector field arising in [9] and vector fields with this property have been called in [19] "nearly incompressible". It was also proved in [19] that the well posedness of (CE) in the class of nearly incompressible BV vector fields implies the validity of Bressan's conjecture, but well-posedness was proved only for the class of SBV vector fields (those for which the distributional derivative has no "Cantor" part, see [7]). In a very recent work [36], S.Bianchini and P.Bonicatto estabilished, among other things, the property for all nearly incompressible BV vector fields in all space dimensions (see [34], [35] for the 2d case), and then Bressan's conjecture. This important progress has been possible by a careful analysis of the decomposition of the space-time vectorfield $(\rho, \rho\boldsymbol{b})$ (viewed as a 1-dimensional current) as a superposition of elementary currents associated to curves, as in Smirnov's theorem [97]. A structural assumption leading essentially to uniqueness of the decomposition is formulated in terms of the flux of the vector field along suitable "tubular neighbourhoods" of the curves used in the decomposition. The structural assumption is then proved to be hold for nearly incompressible BV vector fields; here Alberti's rank 1 theorem plays again a role, as in the proof in [8].
Let me close this section with a remark in another direction, for instance understanding how one can study the problem in Riemannian manifolds via commutator estimates. In this connection, we realize that (2) is too "Euclidean-like", since it involves difference of vectors at different points. Of course one can localize the problem in many ways, for instance working in local coordinates (which is however the source of a new difficulty, since the estimates have eventually to be patched together) or using parallel transport. We will see in the next sections (see (4) in particular) how a coordinate-free approach to the commutator estimate can bypass these difficulties.

4 Some applications

Before moving to the more recent developments of the theory, I want to illustrate some applications of the theory of flows to PDE.

The Keyfitz-Kranzer system. This system, first studied in [86], can be written as

$$\frac{d}{dt}u + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\mathbf{f}_i(|u|)u\right) = 0, \qquad u : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}^k.$$

Formally, with the position $u = \theta\rho$, $\rho = |u|$, this system decouples into a scalar conservation law for ρ, namely

$$\frac{d}{dt}\rho + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\mathbf{f}_i(\rho)\rho\right) = 0,$$

and a transport equation driven by $\boldsymbol{b} = \mathbf{f}(\rho)$, namely

$$\frac{d}{dt}\theta + \boldsymbol{b} \cdot \nabla\theta = 0.$$

In [9], [12], this formal decoupling is made rigorous by using on one hand the notion of entropy solution for the scalar conservation law (which provides BV regularity of ρ and then of $\boldsymbol{b}$, see [70]), on the other hand the regular Lagrangian flow associated to $\boldsymbol{b}$ to build solutions to the transport equation. Stability plays a key role here, in order to prove that the function u built in this way is a distributional solution.

The semi-geostrophic system. This system is widely used in the modelling of atmospheric phenomena on a large scale, see [67] for more informations on this topic. In the case $d = 2$, and considering for the sake of simplicity the problem on the 2-dimensional torus $\mathbb{T}^2$, it reads

$$\begin{cases} \dfrac{d}{dt}J\nabla p_t + J\nabla^2 p_t u_t + \nabla p_t + J u_t = 0 \\ \nabla \cdot u_t = 0 \\ p_0 = p^0 \end{cases}$$

with u_t (the velocity) and p_t (the pressure) periodic,

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Setting $\nu = \mathscr{L}_{\mathbb{T}^2}$, physical considerations suggest the change of variables $\omega_t\nu = (\nabla P_t)_\sharp\nu$, with $P_t(x) := p_t(x) + |x|^2/2$, leading to the so-called dual problem

$$\frac{d}{dt}\omega_t + \operatorname{div}\left((T_\nu^{\omega_t} - Id)^\perp \omega_t\right) = 0.$$

Here $T_\nu^{\omega_t}$ is the optimal map from a reference measure ν to $\omega_t\nu$, so that the velocity field $\boldsymbol{b}_t = (T_\nu^{\omega_t} - Id)^\perp$ is divergence free. In connection with its regularity, since optimal maps are gradients of convex functions (see for instance [14], [103], [104]), BV regularity comes for free [3], [78], [7]. Using these informations, a “Lagrangian” solution to the semi-geostrophic system has been built in [69] starting from the weak solutions to the dual problem provided by [33], [68]. In more recent times, progress on the regularity of the optimal transport map (which turns out to be of class $W^{1,p}$ for some $p > 1$ [73], [74], [96] even in the borderline case of Caffarelli-Urbas theory [54], [55], [56]), [101], [102] has led in [23] to genuine distributional solutions to the semigeostrophic system, see also [24] for a 3-dimensional version of the problem.

Semiclassical limits in quantum mechanics. The problem of convergence of the linear Schrödinger equation in quantum mechanics

$$\begin{cases} i\epsilon\partial_t\psi_t^\epsilon = -\frac{\epsilon^2}{2}\Delta\psi_t^\epsilon + U\psi_t^\epsilon \\ \\ \psi_0^\epsilon = \psi_{0,\epsilon}. \end{cases}$$

to the Hamiltonian of classical mechanics (i.e. the continuity equation in phase space with velocity $\boldsymbol{b}(x,p) = (p, -\nabla U(x))$ has been studied in several papers, under regularity assumptions on U, via the so-called Wigner transform

$$W_\epsilon(x,p) := \frac{1}{(2\pi)^n}\int \psi\big(x + \frac{\epsilon}{2}y\big)\overline{\psi(x - \frac{\epsilon}{2}y\big)}e^{-ipy}\,dy.$$

However, in some physical models (the so-called Born-Oppenheimer theory), U arises from the minimization of an interaction energy and, because of the crossing of eigenvalues, no more than BV regularity of ∇U can be expected. In [15] the theory of regular Lagrangian flows has been used to prove the convergence result under this "critical" regularity assumption on ∇U. Furthermore, to prove a stability result also at the level of flows, in [16] it has been crucial to extend the notion of flow from the space $\mathbb{R}^d$ to the space $\mathscr{P}(\mathbb{R}^d)$ (viewing the continuity equation as an infinite-dimensional ODE in $\mathscr{P}(\mathbb{R}^d)$) and to provide a good, at least for this purpose, notion of flow in this larger state space (see in particular [22]).

5 Differentiability of the flow

In [87] LeBris-Lions proved that the general theory of flows I outlined before still works for $2d$-dimensional vector fields of the form

$$\boldsymbol{B}_t(x,v) = \big(\boldsymbol{b}_t(x), \nabla\boldsymbol{b}_t(x)v\big) \qquad (x,v) \in \mathbb{R}^d \times \mathbb{R}^d$$

with $\boldsymbol{b} \in W^{1,1}$. This is remarkable, since the last d components are only measurable w.r.t. the x variable. Again, the key is an anisotropic mollification scheme. The vector field $\boldsymbol{B}$ is natural in connection with the linearization of the ODE, since at least formally one has

$$\frac{d}{dt}\nabla \boldsymbol{X}(t,x)v = \nabla \boldsymbol{b}_t\big(\boldsymbol{X}(t,x)\big)\nabla \boldsymbol{X}(t,x)v \qquad \forall v \in \mathbb{R}^d.$$

It turns out that the linearization can be made rigorous *even* without adding one derivative to $\boldsymbol{b}$ (as done in the classical $C^{k,1}$ theory), provided the derivative is understood in a *very weak* sense, namely

$$\lim_{\epsilon\to 0^+} \frac{\boldsymbol{X}(t,x+\epsilon v)-\boldsymbol{X}(t,x)}{\epsilon} = \boldsymbol{Z}(t,x)v \quad \text{locally in measure in } \mathbb{R}^d_x \times \mathbb{R}^d_v,$$

where $(\boldsymbol{X}(t,x), \boldsymbol{Z}(t,x)v)$ is the regular Lagrangian flow relative to $\boldsymbol{B}_t$.
A natural question, then, arises. In Geometric Measure Theory there is a well estabilished weak notion of differentiability, Federer's approximate differentiability [79]: we say that f is approximately differentiable at x if, for some linear map L (the approximate differential) one has

$$\mathscr{L}^d\left(\left\{y \in B_r(x)\setminus\{x\} : \ \frac{|f(y)-f(x)-L(y-x)|}{|y-x|} > \delta\right\}\right) = o(r^d) \qquad \forall \delta > 0.$$

Functions in $W^{1,p}$ are approximately differentiable at $\mathscr{L}^d$-a.e. x, with approximate differential equal to the weak gradient of the Sobolev theory, and this weak differentiability property improves to classical differentiability as soon as $p > d$. More generally, approximate differentiability $\mathscr{L}^d$-a.e. on a Borel set A is equivalent to the property of being Lipschitz on "large" sets, i.e., Lipschitz on subsets of A whose complement has arbitrarily small Lebesgue measure. In [18], we proved that Federer's notion is strictly stronger.

Theorem 8 *For a Borel map $f : \mathbb{R}^d \to \mathbb{R}^m$, approximate differentiability $\mathscr{L}^d$-a.e. implies differentiability in measure, but the converse does not hold in general. The differential in measure is always linear w.r.t. v.*

Is the flow map $\boldsymbol{X}$ differentiable only in the very weak sense of differentiability in measure, or can we improve the differentiability to Federer's one? In a nutshell, we want to control "on large sets" the gradient or even the difference quotients of the flow $\boldsymbol{X}$.
Let us start from an elementary apriori estimate, in the smooth setting:

$$\frac{d}{dt}\ln(1+|\nabla \boldsymbol{X}|) = \frac{1}{1+|\nabla \boldsymbol{X}|}\frac{\nabla \boldsymbol{X}}{|\nabla \boldsymbol{X}|}\cdot \nabla \boldsymbol{b}_t(\boldsymbol{X})\cdot \nabla \boldsymbol{X} \le |\nabla \boldsymbol{b}_t(\boldsymbol{X})|.$$

An integration in time and the regularity of the flow (namely the quantitative non-concentration condition) yield

$$\int_{\mathbb{R}^d} \ln(1+|\nabla \boldsymbol{X}(T,x)|)\,dx \le C\int_0^T\int_{\mathbb{R}^d} |\nabla \boldsymbol{b}_s|(y)\,dyds.$$

This estimate is promising, since in the right hand side only the L^1 norm of the derivative of the vector field appears. However, there is no way to pass to the limit in this form, since the logarithm is a concave function, and therefore lower semicontinuity of the left hand side fails. In [17] replacing gradients by difference quotients, it was possible indeed to pass to the limit, showing the following result:

Theorem 9 *If $p > 1$ and $\boldsymbol{b} \in L^1_t\big(W^{1,p}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)\big)$ is bounded, then for any R, $\epsilon > 0$ there exists a compact set $K_\epsilon \subset B_R$ such that $\mathscr{L}^d(B_R \setminus K_\epsilon) < \epsilon$ and $\boldsymbol{X}$ restricted to $[0,T] \times K_\epsilon$ is Lipschitz. In particular $\boldsymbol{X}$ is approximately differentiable $\mathscr{L}^{1+d}$-a.e. in $(0,T) \times \mathbb{R}^d$.*

This has been the first indication that the DiPerna-Lions theory is much closer to the classical Cauchy-Lipschitz theory than expected, at least for not so wild velocity fields: as for measurable functions versus continuous functions, or Sobolev functions versus Lipschitz functions, the removal of a set of arbitrarily small (but positive) measure provides a gain in regularity.

This "logarithmic estimate of $|\nabla \boldsymbol{X}|$" has been very much improved by Crippa-De Lellis in [66]. First, using L^p maximal estimates [98] for the gradient of the vector field, they made quantitative the Lipschitz estimate on "large" sets (getting $\operatorname{Lip} \boldsymbol{X}|_{K_\epsilon \times [0,T]} \leq \exp(c/\epsilon^{1/p})$). Then, if $\boldsymbol{X}_i$ are RLF's relative to $\boldsymbol{b}_i$, $i = 1, 2$, by differentiation of

$$\int_{B_R} \ln\biggl(1 + \frac{|\boldsymbol{X}_1(t,x) - \boldsymbol{X}_2(t,x)|}{\delta}\biggr)\, dx$$

and playing with the choice of δ, they got a global quantitative stability estimate, of the form

$$\int_{B_R} \max_{[0,T]} |\boldsymbol{X}_1 - \boldsymbol{X}_2|\, dx \leq c(R, T, \|\boldsymbol{b}_i\|_\infty, \|\nabla \boldsymbol{b}_i\|_p) \ln^{-1}\bigl(\frac{1}{1 \wedge \|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^1}}\bigr).$$

This result provides a totally Lagrangian approach to the uniqueness of RLF's. In addition, refinements of this technique have led to new classes of vector fields for which existence and uniqueness of regular Lagrangian flows is available:

(a) Vector fields $\boldsymbol{b} = \sum_j g_j * K_j$, with $g_j \in L^1$, $K_j(x) \sim x/|x|^d$, see [43], [64] and [42]. The inclusion of this class of vector fields in the theory requires more advanced tools of harmonic analysis, compared to those of [66]. Notice that this result covers $W^{1,1}$ vector fields, thanks to the formula (with G Green's function for the Laplacian)

$$\boldsymbol{b}_i(x) = \int_{\mathbb{R}^d} \langle \nabla_y G(x,y), \nabla \boldsymbol{b}_i(y) \rangle\, dy$$

but not BV vector fields, which are representable still in this way, but with g_j Radon measures. On the other hand, not all vector fields representable in the form $\sum_j g_j * K_j$ with $g_j \in L^1$ are $W^{1,1}$.

(b) Vector fields with a split structure

$$\boldsymbol{b}(x,v) = (v, F(x)) \qquad F \in H^{3/4,2},$$

see [60]. This is the first positive result assuming only existence of a fractional derivative of order strictly less than 1! Notice that

$$BV \cap L^\infty \subset \bigcap_{s\in(0,1/2)} H^{s,2}, \qquad BV \cap L^\infty \not\subset H^{3/4,2}.$$

Hence, also this result is not comparable with the BV case.

(c) Vector fields $\boldsymbol{b}(x,v)$ with a split structure, arising from a convolution with measures

$$D\boldsymbol{b} = D_{x,v}(\boldsymbol{b}_1, \boldsymbol{b}_2) = \begin{pmatrix} S_1 * L^1 & S_2 * L^1 \\ S_3 * \mathcal{M} & S_4 * L^1 \end{pmatrix}$$

with $S_i(x) \sim x/|x|^{d+1}$, see [39]. Its proof, in a fully Lagrangian scheme, arises from a deep combination of the anisotropic smoothing (here the split structure of the vector field comes into play) with the logarithmic estimates.

The last result is relevant in connection with the Vlasov-Poisson system, where $\boldsymbol{b}(x,v) = (v, \rho * x/|x|^d)$, as we will see in the next section.

6 Local theory and applications to the Vlasov-Poisson system

In the previous section we pointed out some analogies between the classical Cauchy-Lipschitz theory and the DiPerna-Lions theory. Recall that in the former theory one needs only *local* regularity and growth conditions to get existence and uniqueness of maximal solutions; in addition, the maximal existence time $T_{\boldsymbol{X}} : \mathbb{R}^d \to (0,\infty)$ for the ODE is lower semicontinuous and the condition

$$\lim_{t\uparrow T_{\boldsymbol{X}}} |\boldsymbol{X}(t,x)| = \infty$$

characterizes the blow-up time. Can we get something like this inside the weaker theory, dropping completely the growth bounds on $\boldsymbol{b}$ and on its spatial divergence? The definition of regular Lagrangian flow can be easily localized, by looking at arbitrary sets of initial points and arbitrary intervals $[0,\tau(x)]$; the non-concentration condition then reads

$$\boldsymbol{X}(t,\cdot)_\sharp(\mathscr{L}^d \llcorner \{\tau > t\}) \le C\mathscr{L}^d \qquad \forall t > 0.$$

This theme is investigated in [27], where among other things we proved the following result.

Theorem 10 ([27]) *Assume that* $\boldsymbol{b} \in L^1\big((0,T); W^{1,1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)\big)$*,* $\big(\mathrm{div}\, \boldsymbol{b}_t\big)^- \in L^1_t(L^\infty_{{\rm loc},x})$ *and no other growth condition on* $\boldsymbol{b}$*. Then there exists a unique maximal regular Lagrangian flow* $\boldsymbol{X}(t,x)$*, defined for* $0 \le t < T_{\boldsymbol{X}}(x)$*, with* $T_{\boldsymbol{X}} > 0$ $\mathscr{L}^d$*-a.e. and*

$$\text{(BU)} \qquad \limsup_{t\uparrow T_{\boldsymbol{X}}(x)} |\boldsymbol{X}(t,x)| = \infty \ \textit{ for } \mathscr{L}^d\textit{-a.e. } x \in \{T_{\boldsymbol{X}} < T\}.$$

In addition, if $\big(\mathrm{div}\, \boldsymbol{b}_t\big)^- \in L^1_t(L^\infty_x)$*, then the* lim sup *in* (BU) *is a limit.*

In connection with the proof of the above theorem, one should notice that at the Eulerian level the assumptions we made seem hard to localize (except maybe in the divergence-free case, when one can use vector potentials of the vector field, provided by Hodge theorem, and use cutoff functions for the potentials): if χ is a smooth cutoff function then the divergence of $\boldsymbol{b}\chi$ is not even $L^\infty_{\rm loc}$, since $\boldsymbol{b}$ is not locally bounded, and very little is known when the divergence does not belong to $L^\infty_{\rm loc}$. So, the actual proof uses localization, but really at the level of curves (with stopping times, as in probability theory) and the superposition principle is used to transfer results from the ODE to the PDE level.
In addition, also the regularity assumption can be made somehow more local, and independent of the Sobolev theory, having in mind Theorem 5: it suffices to assume that uniqueness holds for the continuity equation in the class of bounded and compacty supported solutions. This assumption covers the more general classes of vector fields considered in the previous sections and, in particular, those appearing in the Vlasov-Poisson system that we describe below.
In connection with the "pointwise" characterization of the blow-up time, we proved that $\lim_{t\uparrow T_{\boldsymbol{X}}(x)} |\boldsymbol{X}(t,x)| = \infty$, as in the classical Cauchy-Lipschitz theory, if the global assumption $\big(\mathrm{div}\, \boldsymbol{b}_t\big)^- \in L^1_t(L^\infty_x)$ is made at the level of the divergence. If the bounds on $(\mathrm{div}\, \boldsymbol{b}_t)^-$ are local in space, we have indeed an example of 3-dimensional vector field $\boldsymbol{b}$ whose maximal regular flow $\boldsymbol{X}$ satisfies $T_{\boldsymbol{X}} < \infty$ and

$$\limsup_{t\uparrow T_{\boldsymbol{X}}(x)} |\boldsymbol{X}(t,x)| = \infty \qquad \text{and} \qquad \liminf_{t\uparrow T_{\boldsymbol{X}}(x)} |\boldsymbol{X}(t,x)| < \infty$$

in a set with positive Lebesgue measure. Hence, local bounds on the divergence are compatible with an oscillatory behaviour of the solution, unlike the classical case.
We also remark that in [27] we provide a new characterization of renormalized solutions, independent of [41]: essentially our result gives that renormalized solutions are in 1-1 correspondence with densities transported by the maximal regular flow, see Theorem 4.10 of [27] for a precise statement. Finally, we have no analog of the lower semicontinuity property of $T_{\boldsymbol{X}}$ of the classical theory. One can only use the techniques of the previous section to obtain local bounds on the distribution function of $T_{\boldsymbol{X}}$.
Let us close this section with the illustration of the application, given in [28], of the maximal regular Lagrangian flow to the Vlasov-Poisson system. We consider in the phase

space $\mathbb{R}^d_x \times \mathbb{R}^d_\xi$ a time-dependent probability density $f(t,x,\xi) = f_t(x,\xi)$ satisfying

$$\text{(VP)} \qquad \begin{cases} \dfrac{d}{dt} f + \xi \cdot \nabla_x f + \mathbf{E}_t \cdot \nabla_\xi f = 0 \\ \\ f(0,x,\xi) = \bar{f}(x,\xi). \end{cases}$$

This PDE has the structure of continuity equation with the divergence-free vector field $\boldsymbol{b}_t(x,\xi) = (\xi, \mathbf{E}_t(x))$. Setting

$$\rho_f(x) := \int f(x,\xi)\, d\xi,$$

the vector field is coupled to f_t via $\mathbf{E}_t = -\epsilon \nabla \phi_t$ where $-\Delta \phi_t = \rho_{f_t}$, i.e. (for $d \geq 3$)

$$\mathbf{E}_t(x) = \epsilon\, c_d \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d} \rho_{f_t}(y)\, dy.$$

Notice that $\epsilon = -1$ corresponds to the attractive (gravitational) case, considered in Cosmology, while $\epsilon = 1$ corresponds to the repulsive (electrostatic) case considered in Plasma Physics.
There exists a wide literature on classical solutions in smooth spaces or with assumptions on moments and integrability of the initial condition that, if sufficiently strong, are propagated and lead even to global solutions. In [28] we look for solutions in the natural energy space $L^1_+(\mathbb{R}^{2d})$. Notice that in this case the definition $|\boldsymbol{b}_t| f_t$ fails even to be locally integrable in space time; to overcome this difficulty, one adopts the weaker notion of renormalized solution, where only renormalization functions $\beta \in C^1_c(\mathbb{R})$ are allowed. Then, using as a tool the maximal regular flow, we prove:

Theorem 11 *In the repulsive case, if the initial density $\bar{f}$ satisfies*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |\xi|^2 \bar{f}(x,\xi)\, dx d\xi + \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\rho_{\bar{f}}(x) \rho_{\bar{f}}(y)}{|x-y|^{d-2}}\, dx dy < \infty,$$

then there exists a global renormalized solution f to (VP).
In addition, if $d = 3$ or $d = 4$, for this solution f the maximal regular flow is globally defined, hence

$$t \mapsto \int \beta(f_t)\, dx d\xi$$

is constant for any $\beta \geq 0$ Borel.

A closely related result in dimension $d = 3$ is proved in [40]; in addition, for the verification that the vector field $\boldsymbol{b}_t$ arising in the (VP) system satisfies the assumption of well posedness in the class of bounded and compactly supported solutions to (CE) (needed to

get existence and uniqueness of the maximal regular Lagrangian flow) we adapt to our case the proof the main result in [39] we already mentioned in Section 5, dealing with vector fields with a split structure.
Notice that since the flow curves in phase space can blow up and disappear in finite time (both forward and backwards), the total mass $\int f_t dx d\xi$ of the global renormalized solution need not to be constant, or monotonic. However, when $d = 3, 4$ the apriori bound on potential and kinetic energy is sufficient to provide global existence of the maximal regular Lagrangian flow and then constancy of all Casimir integrals $\int \beta(f_t)\, dx d\xi$.

7 The theory in metric measure spaces

Assume that we have a metric measure structure $(X, d, \boldsymbol{m})$, namely (X, d) is a metric space and $\boldsymbol{m}$ is a nonnegative Borel measure on X, that we assume for the sake of simplicity to be finite. We denote in the sequel by $\mathrm{Lip}_b(X, d)$ the space of bounded Lipschitz functions on X, by $|\nabla f|$ the local Lipschitz constant of f, namely

$$|\nabla f|(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y, x)}$$

and by $L^0(\boldsymbol{m})$ the space of $\boldsymbol{m}$-measurable functions on X. In this section I will illustrate how the DiPerna-Lions theory can be extended to this very abstract context, a task performed in [26] (see also [29]).
In order to extend the theory of flows to this quite abstract context we have to understand what we mean by "regular vector field" and "flow" relative to that vector field. The concept of vector field, or derivation, goes back to the seminal work of Weaver [105].

Definition 12 (Weaver's derivation) *A derivation $\boldsymbol{b}$ is a linear map from $\mathrm{Lip}_b(X, d)$ to $L^0(\boldsymbol{m})$ satisfying the Leibniz rule*

$$\boldsymbol{b}(fg) = f\boldsymbol{b}(g) + g\boldsymbol{b}(f) \qquad \boldsymbol{m}\text{-a.e. in } X.$$

We keep the more familiar notation $df(\boldsymbol{b})$ for the action of a derivation $\boldsymbol{b}$ on f.

The definition is clearly inspired by basic Differential Geometry, where derivations are thought as operators on C^∞ satisfying the Leibniz rule. In Weaver's theory, however, there is no attempt to define the fibers of the tangent bundle, for a given point: the philosophy is that the tangent bundle is implictly defined by the collection of its sections, and that only these sections matter, from the calculus point of view (see also [82] for much more on this subject).

Definition 13 (Modulus of a derivation) *By duality, we define the modulus $|\boldsymbol{b}|$ of a derivation as the smallest (in the $\boldsymbol{m}$-a.e. sense) function g satisfying*

$$|df(\boldsymbol{b})| \le |\nabla f| g \qquad \text{for } \boldsymbol{m}\text{-a.e. in } X, \text{ for all } f \in \mathrm{Lip}_b(X,d).$$

Definition 14 (Divergence of a derivation) *Still duality can be used to define the divergence of a derivation $\boldsymbol{b}$ with $|\boldsymbol{b}| \in L^1(X,\boldsymbol{m})$. We say that $\boldsymbol{b}$ has divergence in $L^1(X,\boldsymbol{m})$ if*

$$\int_X df(\boldsymbol{b})\, d\boldsymbol{m} = -\int_X fk\, d\boldsymbol{m} \qquad \forall f \in \mathrm{Lip}_b(X,d)$$

for some $k \in L^1(X,\boldsymbol{m})$. If k exists it is unique, and denoted $\operatorname{div}\boldsymbol{b}$.

The language of derivations is also very appropriate to deal with weak solutions to the continuity equation $\frac{d}{dt}u_t + \operatorname{div}(u_t\boldsymbol{b}_t) = 0$, that we denoted (CE): a family of probability densities u_t, with $t \mapsto u_t\boldsymbol{m}$ continuous, is said to be a weak solution of (CE) if

$$\frac{d}{dt}\int_X f u_t\, d\boldsymbol{m} = \int_X u_t df(\boldsymbol{b}_t)\, d\boldsymbol{m} \quad \text{for } \mathscr{L}^1\text{-a.e. } t>0, \text{ for all } f \in \mathrm{Lip}_b(X,d).$$

By Hilbert space techniques one obtains existence of solutions to (CE) under mild growth bounds on $\boldsymbol{b}$ and $\operatorname{div}\boldsymbol{b}$.

Theorem 15 ([26]) *Assume $|\boldsymbol{b}| \in L^1_t(L^2_x)$ and $(\operatorname{div}\boldsymbol{b})^- \in L^1_t(L^\infty_x)$. Then, for all $p \in [2,\infty]$ and $\bar u \in L^p$ there exists a solution $u \in L^\infty_t(L^p_x)$ to (CE) with initial condition $\bar u$.*

As in the Euclidean theory, some regularity of $\boldsymbol{b}_t$ in space is needed for uniqueness. The notion of $W^{1,2}$ vector field can be studied in a more systematic way, see [82], but we learned from [57] that bounds on the symmetric part of the derivative can be sufficient to provide strong convergence of commutators. It turns out that the Riemannian identity (emphasized by Bakry and collaborators)

$$\int_X \langle D^{\mathrm{sym}}\boldsymbol{b}\nabla f, \nabla g\rangle\, d\boldsymbol{m} := -\frac{1}{2}\int_X \langle \boldsymbol{b}, \nabla f\rangle \Delta g + \langle \boldsymbol{b}, \nabla g\rangle \Delta f - \langle \nabla f, \nabla g\rangle \operatorname{div}\boldsymbol{b}\, d\boldsymbol{m}$$

can be used, also in the abstract context, to define $D^{\mathrm{sym}}\boldsymbol{b}$ as a functional on (gradient) sections of the tangent bundle:

$$\int_X D^{\mathrm{sym}}\boldsymbol{b}(f,g)\, d\boldsymbol{m} := -\frac{1}{2}\int_X df(\boldsymbol{b})\Delta g + dg(\boldsymbol{b})\Delta f - \Gamma(f,g)\operatorname{div}\boldsymbol{b}\, d\boldsymbol{m}. \tag{3}$$

In the formula above, Δ should be thought as the infinitesimal generator of the heat semigroup P_t and Γ as the Carré du champ (so that $\Gamma(f) = \Gamma(f,f)$ plays the role of the square

of the gradient); both objects can be canonically derived either from a (sufficiently regular) Dirichlet form, or from an infinitesimally Hilbertian metric measure space $(X, d, \boldsymbol{m})$, see [31], [82] for a more precise discussion.
All in all, by saying that the right hand side in (3) is controlled by $\|\sqrt{\Gamma(f)}\|_4\|\sqrt{\Gamma(g)}\|_4$, we are saying that $D^{\rm sym}\boldsymbol{b}$ is in L^2. Besides this, for uniqueness we need also a mild regularity property of the heat semigroup P_t, always true under lower bounds on Ricci curvature: we say that L^4-Γ gradient inequalities hold if

$$\left(\int_X \Gamma^2(P_t f)\, d\boldsymbol{m}\right)^{1/4} \le \frac{c}{\sqrt{t}}\|f\|_4 \qquad \forall f \in L^4(X, \boldsymbol{m}),\ t \in (0,1).$$

Theorem 16 ([26]) *If $|\boldsymbol{b}| \in L^1_t(L^2_x)$, $(\mathrm{div}\,\boldsymbol{b})^- \in L^1_t(L^\infty_x)$, $\|D^{\rm sym}\boldsymbol{b}\|_2 \in L^1_t$ and if L^4-Γ gradient inequalities hold, then the continuity equation is well posed in $L^\infty_t(L^4_x)$ for any initial condition $\bar{u} \in L^4$.*

In the proof we follow a smoothing scheme based on the heat semigroup, $u^\epsilon_t = P_\epsilon u_t$:

$$u^\epsilon_t + \mathrm{div}\,(u^\epsilon_t \boldsymbol{b}_t) = \mathscr{C}_\epsilon[\boldsymbol{b}_t, u_t] \quad \text{with} \quad \mathscr{C}_\epsilon[\boldsymbol{b}_t, u_t] := \mathrm{div}\,((P_\epsilon u_t)\boldsymbol{b}_t) - P_\epsilon\big(\mathrm{div}\,(u_t \boldsymbol{b}_t)\big).$$

This strategy already proved to be succesful in the Wiener space, see [21] for the Sobolev case and [100] for the BV case (and also [38] for closely related results), but in that case the explicit knowledge of the transition probabilities of the Ornstein-Uhlenbeck semigroup P_t played a role:

$$P_t f(x) = \int_X f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\, d\gamma(y).$$

In this more general case we apply Bakry's interpolation to provide a new (even in the Euclidean case) expression of $\mathscr{C}_\epsilon[\boldsymbol{c}, v]$:

$$\begin{aligned} \mathscr{C}_\epsilon[\boldsymbol{c}, v] &= \int_0^\epsilon \frac{d}{ds} P_{\epsilon - s}(\mathrm{div}\,((P_s v)\boldsymbol{c}))\, ds \\ &= \int_0^\epsilon -P_{\epsilon-s}\Delta(\mathrm{div}\,((P_s v)\boldsymbol{c})) + P_{\epsilon-s}(\,\mathrm{div}\,((\Delta P_s v)\boldsymbol{c}))\, ds. \end{aligned}$$

Playing carefully with integration by parts and using the L^4-Γ gradient estimates as well as the regularity of $\boldsymbol{c}$ one can show strong convergence of the commutator. This can be understood better if we write the expression above in the easier case when $\boldsymbol{c}$ is divergence-free, testing the commutator in duality with a function w:

$$\int_X w \mathscr{C}_\epsilon[\boldsymbol{c}, v]\, d\boldsymbol{m} = \int_0^\epsilon \int_X D^{\rm sym}\boldsymbol{c}(P_s v, P_{\epsilon-s} w)\, d\boldsymbol{m}\, ds. \tag{4}$$

Now, let us come to the notion of flow associated to a family $\boldsymbol{b}_t$ of time-dependent derivations. For gradient flows, it is a well-known fact that one can characterize the solution to the ODE by looking at the maximal energy dissipation rate. Here, since we have no specific energy at our disposal, we use all the energies in $\mathrm{Lip}_b(X,d)$, having in mind that $(f\circ\gamma)'(t)=d_{\gamma(t)}f(\gamma'(t))$ in a smooth setup:

Definition 17 *We say that $\gamma\in AC_{\rm loc}([0,\infty);X)$ solves the ODE $\dot\gamma=\boldsymbol{b}_t(\gamma)$ if*

$$\frac{d}{dt}(f\circ\gamma)(t)=df(\boldsymbol{b}_t)(\gamma_t)\qquad \text{for } \mathscr{L}^1\text{-a.e. } t\in(0,\infty)$$

for all $f\in\mathrm{Lip}_b(X,d)$.

Given this, the definition of regular Lagrangian flow, relative to $(X,d,\boldsymbol{m})$, follows verbatim the Euclidean one. Notice also that since $df(\boldsymbol{b}_t)$ is only defined up to $\boldsymbol{m}$-negligible sets, Definition 17 makes little sense for single paths, unless there is the possibility to specify $df(\boldsymbol{b}_t)$ in a more precise way. On the other hand, for non-concentrating families of paths, as those appearing in the definition of regular lagrangian flow, the definition makes sense, i.e., it is independent of modifications on $df(\boldsymbol{b}_t)$ in $\boldsymbol{m}$-negligible sets.
One can then reproduce the superposition principle to get existence of a generalized flow $\boldsymbol{\eta}_x$ (assuming upper bounds on $(\mathrm{div}\,\boldsymbol{b}_t)^-$) and then use the well posedness of (CE) (assuming bounds on P_t and on $\|D^{\rm sym}\boldsymbol{b}\|_2$) to obtain the following existence and uniqueness result:

Theorem 18 ([26]) *If $|\boldsymbol{b}|\in L^1_t(L^2_x)$, $\mathrm{div}\,\boldsymbol{b}\in L^1_t(L^\infty_x)$, $\|D^{\rm sym}\boldsymbol{b}\|_2\in L^1_t$ and if L^4-Γ gradient inequalities hold, then the regular Lagrangian flow exists and is unique.*

Finally, let us mention that, at this level of generality, it makes sense even to consider stability across different metric measure structures. This theme is investigated in [30], in connection with measured Gromov-Hausdorff convergence and with the theory of metric measure spaces with Ricci curvature bounded from below.

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