

# Self-similar minimizers of a branched transport functional

Michael Goldman\*

## Abstract

We solve here completely an irrigation problem from a Dirac mass to the Lebesgue measure. The functional we consider is a two dimensional analog of a functional previously derived in the study of branched patterns in type-I superconductors. The minimizer we obtain is a self-similar tree.

## 1 Introduction

In this paper we consider the functional (see Section 2 for a more precise definition)

$$\mathcal{E}(\mu) := \int_a^b \#\{\varphi_i \neq 0\} + \sum_i \varphi_i |\dot{X}_i|^2 dt, \quad (1.1)$$

where  $\mu = \mu_t \otimes dt$  with  $\mu_t = \sum_i \varphi_i \delta_{X_i}$  for a.e.  $t \in (a, b)$  for some (pairwise distinct)  $X_i \in \mathbb{R}$  and where  $\dot{X}_i$  denotes the time derivative of  $X_i(t)$ . For two given measures  $\mu_{\pm}$  of equal mass, we study the following Dirichlet problem

$$\min_{\mu} \{\mathcal{E}(\mu) : \mu_a = \mu_-, \mu_b = \mu_+\}. \quad (1.2)$$

Our main result is a full characterization of the minimizers of (1.2) in the case  $\mu_-$  is a Dirac mass,  $\mu_+$  is the Lebesgue measure restricted to an interval of length  $\mu_-(\mathbb{R})$  and  $b - a$  is large enough. In order to fix notation, since the problem is invariant by translations, we may assume that  $a = 0$ ,  $b = T$ ,  $\mu_- = \varphi \delta_X$  and  $\mu_+ = dx \llcorner [-\varphi/2, \varphi/2]$  for some  $T, \varphi > 0$  and  $X \in \mathbb{R}$ . As will be apparent below, up to rescalings and shears, we may further normalize to  $X = 0$  and  $\varphi = 1$ , so that

$$\mu_0 = \delta_0 \quad \text{and} \quad \mu_T = dx \llcorner [-1/2, 1/2].$$

In this case, as will become clearer in the proof, the threshold value  $T = 1/4$  naturally appears. In order to state our main theorem, let us define for  $t \in [0, 1/4]$ , the measure

---

\*LJLL, Université Paris Diderot, CNRS, UMR 7598, France email: goldman@math.univ-paris-diderot.fr

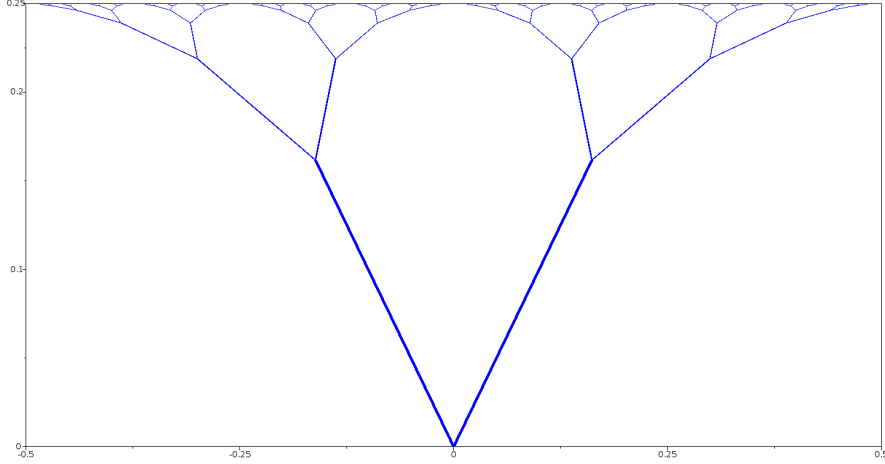


Figure 1: The optimal configuration  $\mu^*$

$\mu_t^*$  (see Figure 1). For  $k \geq 0$ , let  $t_k := \frac{1}{4} \left(1 - \left(\frac{1}{2}\right)^{3k/2}\right)$ . We define recursively  $\mu_t^*$  in the intervals  $[t_{k-1}, t_k]$ . Let  $X_1^0 = 0$  and  $\mu_0^* = \delta_0$ . Assume that  $\mu_t^*$  is defined in  $[0, t_{k-1}]$  and that  $\mu_{t_{k-1}}^* = 2^{-(k-1)} \sum_{i=1}^{2^{k-1}} \delta_{X_i^{k-1}}$ . For  $t \in [t_{k-1}, t_k]$  and  $1 \leq i \leq 2^k$ , we now define  $X_i^k(t)$ . For this, let us divide  $[-1/2, 1/2]$  in  $2^k$  intervals of equal size and let  $\bar{X}_i^k$  be the barycenter of the  $i$ -th such interval i.e.  $\bar{X}_i^k := \frac{-1}{2} + \frac{i-1}{2^k} + \frac{1}{2^{k+1}}$ . We then let

$$X_i^k(t) := \frac{t - t_{k-1}}{\frac{1}{4} - t_{k-1}} \left( \bar{X}_i^k - X_{\lfloor i/2 \rfloor}^{k-1} \right) + X_{\lfloor i/2 \rfloor}^{k-1},$$

and  $\mu_t^* := 2^{-k} \sum_{i=1}^{2^k} \delta_{X_i^k(t)}$ . Notice that with this definition, for every  $t \in [0, 1/4)$ ,  $k \in \mathbb{N}$  and  $1 \leq i \leq 2^k$ , the mass at  $X_i^k(t)$  is irrigating the interval  $(\bar{X}_i^k - 2^{-(k+1)}, \bar{X}_i^k + 2^{-(k+1)})$  and  $X_i^k$  is moving at constant speed towards  $\bar{X}_i^k$  (and would reach it at time  $T = 1/4$  if there were no further branching points). Our main theorem is the following

**Theorem 1.1.** *For  $T = 1/4$ ,  $\mu_0 = \delta_0$  and  $\mu_T = dx \llcorner [-1/2, 1/2]$ ,  $\mu^*$  is the unique minimizer of (1.2). Moreover, if  $T \geq 1/4$ , the unique minimizer of (1.2) is given by  $\mu_t = \delta_0$  for  $t \in [0, T - 1/4]$  and  $\mu_t = \mu_{t-(T-1/4)}^*$  for  $t \in (T - 1/4, T)$ , with*

$$\mathcal{E}(\mu) = \frac{1}{2 - \sqrt{2}} + T.$$

As a consequence, we obtain the following corollary (see Lemma 3.1 for the exact definitions of the rescaling and shear)

**Corollary 1.2.** *For  $X \in \mathbb{R}$ ,  $T, \varphi > 0$  with  $T\varphi^{-3/2} \geq 1/4$ , the unique minimizer of (1.2) with  $\mu_0 = \varphi \delta_X$  and  $\mu_T = dx \llcorner [-\varphi/2, \varphi/2]$  is given by a suitably sheared and rescaled*

version of the optimal measure for  $X = 0$ ,  $\varphi = 1$  and  $T' = T\varphi^{-3/2}$ . Moreover

$$\mathcal{E}(\mu) = \varphi^{3/2} \frac{1}{2 - \sqrt{2}} + T + \frac{\varphi}{T} |X|^2.$$

As an application of Corollary 1.2, we will further derive a full characterization of symmetric (with respect to  $t = 0$ ) minimizers in the case  $a = -b = -T$ ,  $\mu_{\pm} = dx \llcorner [-1/2, 1/2]$  and  $T \geq 1/4$  (see Theorem 4.1).

The proof of Theorem 1.1 is based on the tree structure of the minimizers of (1.2) (see Proposition 2.7) which together with invariance by scaling and shearing (Lemma 3.1) leads to a recursive characterization of the minimizers (see (3.4)). This in turn, allows for the first branching time  $T = 1/4$ , to rewrite the geometrical problem (1.2) in term of a purely analytical minimization problem (see (3.12)). Using a computer assisted proof, this permits to exclude the possibility of branching of three or more branches (see Section 3.1). Once it is known that only pairs of branches appear at each branching point, it is possible to prove that mass splits always in half and to compute the distance between branching points (see Section 3.2).

The variational problem (1.2) may be seen as a two dimensional (one for time and one for space) analog of the three dimensional (one for time and two for space) problem derived in [8] as a reduced model for the description of branching in type-I superconductors in the regime of very small applied external field. We refer the reader to [8] for more precise physical motivations and references. In this regime, the natural Dirichlet conditions appearing are  $\mu_{\pm} = dx \llcorner [-1/2, 1/2]$ . Let us point out that in the three dimensional model, the term  $\#\{\varphi_i \neq 0\}$  is replaced by  $\sum_i \varphi_i^{1/2}$ . This is in line with the interpretation of the first term in (1.1) as an interfacial term penalizing the creation of many flux tubes. That is, if we are in  $(1 + d)$ -dimensions, it is proportional to the perimeter of a union of  $d$ -dimensional balls of volume  $\varphi_i$  (which is  $2\sqrt{\pi} \sum_i \varphi_i^{1/2}$  if  $d = 2$  and  $2\#\{\varphi_i \neq 0\}$  if  $d = 1$ ). The second term in (1.1) may be interpreted as the Wasserstein transportation cost of moving such balls. In many models describing pattern formation in material sciences, branching patterns similar to the one observed here are expected. However, it is usually very hard to go beyond scaling laws [10, 16, 5, 9]. In some cases, reduced models have been derived [12, 8, 7] but so far the best results concerning the minimizers are local energy bounds leading to the proof of asymptotic self-similarity [6, 14]. Our result is thus the first complete characterization of a minimizer in this context. Of course, this was possible thanks to the simplicity of our model (one dimensional trees in a two dimensional ambient space). We should however point out that our result is not fully satisfactory since we are essentially able to study only the situation of an isolated microstructure (due to the constraint  $T \geq 1/4$ ) whereas one is typically interested in the case  $T \ll 1$  where many microstructures are present and where the lateral boundary conditions have limited effect i.e. one tries to capture an extensive behavior of the system. As detailed in the final section 4, we believe that even in the regime  $T \ll 1$ , every microstructure is of the type described in Corollary 1.2.

As pointed out in [8], the functional (1.1) bears many similarities with so-called branched transport (or irrigation) models [2]. Also in this class of problems, there has been a strong interest for the possible fractal behavior of minimizers. Besides results on scaling laws [4] and fractal regularity [3], to the best of our knowledge, the only explicit minimizers exhibiting infinitely many branching points have been obtained in [13] for a Dirac irrigating a Cantor set and in [11] in an infinite dimensional context. In particular, the optimal irrigation pattern from a Dirac mass to the Lebesgue measure is currently not known for the classical branched transportation model. One important difference between our model and branched transportation is that in our case, minimality does not imply triple junctions nor conditions on the angles between the branches.

The organization of the paper is the following. In Section 2, we recall the definition and the basic properties of the functional  $\mathcal{E}$ . Then, in Section 3 we prove Theorem 1.1. In the final Section 4, we give an application of Theorem 1.1 to the irrigation of the Lebesgue measure by itself and state an open problem.

**Notation** In the paper we will use the following notation. The symbols  $\simeq$ ,  $\gtrsim$ ,  $\lesssim$ ,  $\ll$  indicate estimates that hold up to a global constant. For instance,  $f \lesssim g$  denotes the existence of a constant  $C > 0$  such that  $f \leq Cg$ . We denote by  $\mathcal{H}^1$  the 1-dimensional Hausdorff measure. For a Borel measure  $\mu$ , we will denote by  $\text{supp } \mu$  its support.

## 2 The variational problem and main properties of the functional

**Definition 2.1.** For  $a < b$  we denote by  $\mathcal{A}_{a,b}$  the set of pairs of measures  $\mu \geq 0$ ,  $m$  with  $m \ll \mu$ , satisfying the continuity equation

$$\partial_t \mu + \partial_x m = 0 \quad \text{in } \mathbb{R} \times (a, b) \quad (2.1)$$

and such that  $\mu = \mu_t \otimes dt$  where, for a.e.  $t \in (a, b)$ ,  $\mu_t = \sum_i \varphi_i \delta_{X_i}$  for some  $\varphi_i \geq 0$  and  $X_i \in \mathbb{R}$ . We denote by  $\mathcal{A}_{a,b}^* := \{\mu : \exists m, (\mu, m) \in \mathcal{A}_{a,b}\}$  the set of admissible  $\mu$ .

Further, we define  $\mathcal{E} : \mathcal{A}_{a,b} \rightarrow [0, \infty]$  by

$$\mathcal{E}(\mu, m) := \int_a^b \#\{\text{supp } \mu_t\} dt + \int_{\mathbb{R} \times (a,b)} \left( \frac{dm}{d\mu} \right)^2 d\mu \quad (2.2)$$

and (with abuse of notation)  $\mathcal{E} : \mathcal{A}_{a,b}^* \rightarrow [0, \infty]$  by

$$\mathcal{E}(\mu) := \min\{\mathcal{E}(\mu, m) : m \ll \mu, \partial_t \mu + \partial_x m = 0\}. \quad (2.3)$$

Equation (2.1) is understood in the sense of distributions (testing with test functions in  $C_c^\infty(\mathbb{R} \times (0, T))$ ). Contrary to [8], we use free boundary conditions instead of periodic ones but this makes only minor differences. In the sequel we will only deal with measures

$\mu$  of bounded support. In this case, because of (2.1),  $\mu_t(\mathbb{R})$  does not depend on  $t$ . Let us point out that for such measures, the minimum in (2.3) is attained thanks to [1, Th. 8.3.1]. Moreover, the minimizer is unique by strict convexity of  $m \rightarrow \int_{\mathbb{R} \times (a,b)} \left(\frac{dm}{d\mu}\right)^2 d\mu$ . Let us also notice that by the Benamou-Brenier formula [1], we have for every measure  $\mu$ , and every  $t, t' \in (a, b)$ ,

$$W_2^2(\mu_t, \mu_{t'}) \leq \mathcal{E}(\mu)|t - t'|, \quad (2.4)$$

where the 2-Wasserstein distance between two measures  $\mu$  and  $\nu$  of bounded second moment with  $\mu(\mathbb{R}) = \nu(\mathbb{R})$  is defined by

$$W_2^2(\mu, \nu) := \min \left\{ \int_{\mathbb{R} \times \mathbb{R}} |x - y|^2 d\Pi(x, y) : \Pi_1 = \mu, \Pi_2 = \nu \right\},$$

where the minimum is taken over measures on  $\mathbb{R} \times \mathbb{R}$  and  $\Pi_1$  and  $\Pi_2$  are respectively the first and second marginal of  $\Pi$ . In particular for every measure  $\mu$  with  $\mathcal{E}(\mu) < \infty$ , the curve  $t \mapsto \mu_t$  is Hölder continuous with exponent one half in the space of measures (endowed with the metric  $W_2$ ) and the traces  $\mu_a$  and  $\mu_b$  are well defined.

Given two measures  $\mu_{\pm}$  on  $\mathbb{R}$  with  $\mu_+(\mathbb{R}) = \mu_-(\mathbb{R})$  and bounded support, we are interested in the variational problem

$$\inf \{ \mathcal{E}(\mu) : \mu_a = \mu_-, \mu_b = \mu_+ \}. \quad (2.5)$$

Let us first notice that if  $L > 0$  is such that  $\text{supp } \mu_- \cup \text{supp } \mu_+ \subseteq [-L/2, L/2]$ , then we may restrict the infimum in (2.5) to measures satisfying  $\text{supp } \mu_t \subseteq [-L/2, L/2]$  for a.e.  $t \in (a, b)$ . Indeed, if  $\mu$  is admissible with  $\mu_t = \sum_i \varphi_i \delta_{X_i}$  then letting  $\tilde{X}_i := \min(L/2, |X_i|) \text{sign} X_i$  and then  $\tilde{\mu}_t := \sum_i \varphi_i \delta_{\tilde{X}_i}$ , we get that  $\tilde{\mu}$  is admissible and has lower energy than  $\mu$  (i.e. the energy decreases by projection on  $[-L/2, L/2]$ ). From now on we will only consider such measures.

As in [8, Prop 5.2] (to which we refer for the proof), a simple branching construction shows that any pair of measures with equal flux may be connected with finite cost.

**Proposition 2.2.** *For every pair of measures  $\mu_{\pm}$  with  $\text{supp } \mu_{\pm} \subseteq [-L/2, L/2]$  and  $\mu_+(\mathbb{R}) = \mu_-(\mathbb{R}) = \varphi$ , there is  $\mu \in \mathcal{A}_{a,b}^*$  such that letting  $b - a = 2T$ ,  $\mu_a = \mu_-$ ,  $\mu_b = \mu_+$  and*

$$\mathcal{E}(\mu) \lesssim T + \frac{\varphi L^2}{T}.$$

*If  $\mu_+ = \mu_-$ , then there is a construction with*

$$\mathcal{E}(\mu) \lesssim T + T^{1/3} \varphi^{1/3} L^{2/3}.$$

From this, arguing as in [8, Prop. 5.5], we obtain that

**Proposition 2.3.** *For every pair of measures  $\mu_{\pm}$  with bounded support and  $\mu_+(\mathbb{R}) = \mu_-(\mathbb{R})$ , the infimum in (2.5) is finite and attained.*

We now give some regularity results for minimizers of (2.5). These can be mostly proven as in [8] so we state them without proof. Let us first recall the notion of subsystem.

**Proposition 2.4** (Definition of a subsystem). *Given a point  $(X, t) \in [-L/2, L/2] \times (a, b)$  and  $\mu \in \mathcal{A}_{a,b}^*$  with  $\mathcal{E}(\mu) < \infty$ , there exists a subsystem  $\mu'$  of  $\mu$  emanating from  $(X, t)$  in the sense that there exists  $\mu'$  such that*

- (i).  $\mu' \leq \mu$  in the sense that  $\mu - \mu'$  is a positive measure,
- (ii).  $\mu'_t = a\delta_X$ , where  $a = \mu_t(X)$ ,
- (iii). if  $m$  is such that  $\mathcal{E}(\mu) = \mathcal{E}(\mu, m)$ , then

$$\partial_t \mu' + \partial_x \left( \frac{dm}{d\mu} \mu' \right) = 0.$$

In particular, (ii) implies that  $(\mu_t - \mu'_t) \perp \delta_X$  in the sense of the Radon-Nikodym decomposition. We call  $\mu^+ := \mu' \llcorner \mathbb{R} \times (t, b)$  the forward subsystem emanating from  $(X, t)$  and  $\mu^- := \mu' \llcorner \mathbb{R} \times (a, t)$  the backward subsystem emanating from  $X$ .

**Lemma 2.5** (No loops). *Let  $\mu$  be a minimizer for the Dirichlet problem (2.5),  $\bar{t} \in (a, b)$ . Let  $X_1, X_2$  be two points in the line  $\{(x, t) : t = \bar{t}\}$ . Let  $\mu_1$  and  $\mu_2$  be subsystems of  $\mu$  emanating from  $(X_1, \bar{t})$ , resp.  $(X_2, \bar{t})$ . Let  $(X_+, t_+)$  be a point with  $t_+ > \bar{t}$  and  $(X_-, t_-)$  a point with  $t_- < \bar{t}$ , and such that  $\mu_1$  and  $\mu_2$  both have Diracs at both  $X_+$  and  $X_-$  with nonzero mass. Then  $X_1 = X_2$ .*

As in [8], a consequence of this lemma is that we have a representation of the form

$$\mu = \sum_i \frac{\varphi_i}{\sqrt{1 + |\dot{X}_i|^2}} \mathcal{H}^1 \llcorner \Gamma_i \quad (2.6)$$

where the sum is countable and  $\Gamma_i = \{(X_i(t), t) : t \in [a_i, b_i]\}$  where  $X_i$  are absolutely continuous and almost everywhere non overlapping.

Another consequence is that if there are two levels at which  $\mu$  is a finite sum of Diracs, then it is the case for all the levels in between. Since  $\mathcal{E}(\mu) < \infty$  implies in particular that  $\#\{\varphi_i \neq 0\} < \infty$  for a.e.  $t \in (a, b)$ , this means that  $\mu_t$  is in fact a locally finite (in time) sum of Dirac masses and thus, the sum in (2.6) is finite away from the initial and finite time.

For measures which are concentrated on finitely many curves, we have as in [8, Lem. 5.9], a representation formula for  $\mathcal{E}(\mu)$ .

**Lemma 2.6.** *Let  $\mu = \sum_{i=1}^N \frac{\varphi_i}{\sqrt{1 + |\dot{X}_i|^2}} \mathcal{H}^1 \llcorner \Gamma_i \in \mathcal{A}_{a,b}^*$  with  $\Gamma_i = \{(X_i(t), t) : t \in [a_i, b_i]\}$  for some absolutely continuous curves  $X_i$ , disjoint up to the endpoints. Every  $\varphi_i$  is then constant on  $[a_i, b_i]$  and we have conservation of mass. That is, for  $z := (x, t)$ , letting*

$$\begin{aligned} \mathcal{I}^-(z) &:= \{i \in [1, N] : t = b_i, X_i(b_i) = x\} \\ \mathcal{I}^+(z) &:= \{i \in [1, N] : t = a_i, X_i(a_i) = x\}, \end{aligned}$$

it holds

$$\sum_{i \in \mathcal{I}^-(z)} \varphi_i = \sum_{i \in \mathcal{I}^+(z)} \varphi_i.$$

Moreover,  $m = \sum_i \frac{\varphi_i}{\sqrt{1+|\dot{X}_i|^2}} \dot{X}_i \mathcal{H}^1 \llcorner \Gamma_i$  and

$$\mathcal{E}(\mu) = \sum_i \int_{a_i}^{b_i} 1 + \varphi_i |\dot{X}_i|^2 dt. \quad (2.7)$$

In particular, this proves that for minimizers, formula (2.7) holds (where the sum is at most countable). By a slight abuse of notation, for such measures we will denote

$$\mathcal{E}(\mu) = \int_a^b \#\{\varphi_i \neq 0\} + \sum_i \varphi_i |\dot{X}_i|^2 dt.$$

We gather below some properties of the minimizers

**Proposition 2.7.** *A minimizer of the Dirichlet problem (2.5) with boundary conditions  $\mu_{\pm}$  satisfies*

- (i). *Each  $X_i$  is affine.*
- (ii). *There is monotonicity of the traces in the sense that for every  $t \in (a, b)$ , if  $\mu_t = \sum_i \varphi_i \delta_{X_i}$  with  $X_i$  ordered (i.e.  $X_i \leq X_{i+1}$ ) and if  $\mu^{i,+}$  is the forward subsystem emanating from  $X_i$ , then the traces  $\mu_b^{i,+}$  satisfy  $\text{supp } \mu_b^{i,+} = [x_i^+, y_i^+]$  with  $y_i^+ \leq x_{i+1}^+$ . The analogous statement holds for the backward subsystems.*
- (iii). *If  $\mu_- = \varphi \delta_X$  then  $\mu$  has a tree structure.*
- (iv). *If  $\mu_- = \mu_+$ , then letting  $a = -T$  and  $b = T$ , there exists a minimizer which is symmetric with respect to the  $t = 0$  plane. For every such minimizer, the number of Dirac masses at time  $t$  is minimal for  $t = 0$ .*

*Proof.* Item (i) follows from fixing the branching points and minimizing in  $X_i$ . The other points are simple consequences of Lemma 2.5.  $\square$

The monotonicity property (ii), is analogous to the monotonicity of optimal transport maps in one space dimension [15]. As a consequence of (iv), in the case  $\mu_{\pm T} = \varphi/Ldx \llcorner [-L/2, L/2]$ , we may get an estimate on the number of Dirac masses on the mid-plane  $t = 0$ .

**Lemma 2.8.** *For  $T \ll \varphi^{1/2}L$ , if  $\mu$  is a symmetric minimizer of*

$$\min\{\mathcal{E}(\mu) : \mu_{\pm T} = \varphi/Ldx \llcorner [-L/2, L/2]\}$$

*and if  $\mu_0 = \sum_{i=1}^N \varphi_i \delta_{X_i}$ , then  $N \simeq \varphi^{1/3}L^{2/3}T^{-2/3}$  and the number  $N_{\text{good}}$  of  $\varphi_i$  satisfying  $\varphi_i \simeq (T\varphi L^{-1})^{2/3}$  is of order  $N$ .*

*Proof.* Since  $\mu$  is symmetric, by Lemma 2.7, letting for  $i = 1, \dots, N$ ,  $\mu^{i,+}$  be the forward subsystem emanating from  $X_i$ , we have by (2.4),

$$\mathcal{E}(\mu) \gtrsim NT + \frac{1}{T} \sum_{i=1}^N W_2^2(\varphi_i \delta_{X_i}, \mu_T^{i,+}) \gtrsim NT + \frac{1}{T} \sum_{i=1}^N \varphi_i^3.$$

By Proposition 2.2, we have  $\mathcal{E}(\mu) \lesssim T^{1/3} \varphi^{1/3} L^{2/3}$ , so that

$$T^{1/3} \varphi^{1/3} L^{2/3} \gtrsim NT + \frac{1}{T} \sum_{i=1}^N \varphi_i^3.$$

Since  $\sum_{i=1}^N \varphi_i^3 \geq \frac{\varphi L^2}{N^2}$ , we readily conclude the proof.  $\square$

### 3 Irrigation of the Lebesgue measure by a Dirac mass

In this section we consider (1.2) with  $a = 0$  and  $b = T$ ,  $\mu_- = \varphi \delta_X$  and  $\mu_+ = dx \llcorner [-\varphi/2, \varphi/2]$ . We will denote

$$E(T, \varphi, X) := \min\{\mathcal{E}(\mu) : \mu_0 = \varphi \delta_X, \mu_T = dx \llcorner [-\varphi/2, \varphi/2]\}.$$

For simplicity we let  $E(T, \varphi) := E(T, \varphi, 0)$  be the energy required to connect the Lebesgue measure to the centered Dirac mass and  $E(T) := E(T, 1)$ . The following lemma shows that understanding  $E(T)$  is enough for understanding  $E(T, \varphi, X)$ .

**Lemma 3.1.** *For every  $T, \varphi, X$ , it holds,*

$$E(T, \varphi, X) = E(T, \varphi) + \frac{1}{T} \varphi |X|^2. \quad (3.1)$$

Moreover, if  $\mu_t = \sum_i \varphi_i X_i(t)$  is optimal for  $E(T, \varphi)$ , then letting  $\widehat{X}_i(t) := (1 - \frac{t}{T})X + X_i(t)$ ,  $\widehat{\mu}_t = \sum_i \varphi_i \widehat{X}_i(t)$  is optimal for  $E(T, \varphi, X)$ .

Furthermore, we have

$$E(T, \varphi) = \varphi^{3/2} E(T \varphi^{-3/2}). \quad (3.2)$$

In addition, if  $\mu_t = \sum_i \varphi_i X_i(t)$  is optimal for  $E(T \varphi^{-3/2})$  then letting  $\hat{t} := \varphi^{3/2} t$ ,  $\widehat{\varphi}_i := \varphi \varphi_i$ , and  $\widehat{X}_i := \varphi X_i(\hat{t})$ , then  $\widehat{\mu}_{\hat{t}} = \sum_i \widehat{\varphi}_i \delta_{\widehat{X}_i}$  is optimal for  $E(T, \varphi)$ .

*Proof.* For  $\mu_t = \sum_i \varphi_i \delta_{X_i}$  admissible for  $E(T, X)$ , we define  $\widehat{\mu}_t := \sum_i \varphi_i \delta_{\widehat{X}_i}$ , where  $\widehat{X}_i(t) := (1 - \frac{t}{T})X + X_i(t)$ . Then,  $\widehat{\mu}_t$  is admissible for  $E(T, \varphi, X)$  and

$$\begin{aligned} \mathcal{E}(\widehat{\mu}) &= \int_0^T \#\{\varphi_i \neq 0\} + \sum_i \varphi_i |\dot{X}_i - \frac{1}{T} X|^2 dt \\ &= \int_0^T \#\{\varphi_i \neq 0\} + \sum_i \varphi_i |\dot{X}_i|^2 dt + \frac{|X|^2}{T^2} \int_0^T \sum_i \varphi_i dt - 2 \frac{X}{T} \int_0^T \sum_i \varphi_i \dot{X}_i dt. \end{aligned} \quad (3.3)$$



For  $\varepsilon > 0$ , thanks to Lemma 2.6 and the fact that  $X_i(0) = 0$ , we have

$$\int_0^{T-\varepsilon} \sum_i \varphi_i \dot{X}_i dt = \sum_i \varphi_i X_i(T - \varepsilon).$$

Furthermore, testing the weak convergence of  $\mu_t$  to  $dx$  as  $t \rightarrow T$ , with the function  $x$ , we get

$$\lim_{\varepsilon \rightarrow 0} \sum_i \varphi_i X_i(T - \varepsilon) = \int_{-\varphi/2}^{\varphi/2} x dx = 0.$$

Finally, since by Hölder's inequality applied twice and  $\sum_i \varphi_i = \varphi$ ,

$$\int_{T-\varepsilon}^T \sum_i \varphi_i |\dot{X}_i| \leq \varphi^{1/2} \int_{T-\varepsilon}^T (\sum_i \varphi_i |\dot{X}_i|^2)^{1/2} \leq \varphi^{1/2} \varepsilon^{1/2} \left( \int_{T-\varepsilon}^T \sum_i \varphi_i |\dot{X}_i|^2 \right)^{1/2},$$

we get

$$\int_0^T \sum_i \varphi_i \dot{X}_i dt = \lim_{\varepsilon \rightarrow 0} \left( \int_0^{T-\varepsilon} \sum_i \varphi_i \dot{X}_i dt + \int_{T-\varepsilon}^T \sum_i \varphi_i |\dot{X}_i| dt \right) = 0.$$

Combining this with  $\sum_i \varphi_i = \varphi$  and (3.3), we get

$$\mathcal{E}(\hat{\mu}) = \mathcal{E}(\mu) + \frac{1}{T} \varphi |X|^2,$$

from which the first part of the proposition follows.

The second part follows simply by using the rescaling  $\hat{t} := \varphi^{3/2} t$ ,  $\hat{\varphi}_i := \varphi \varphi_i$ , and  $\hat{X}_i := \varphi X_i(\hat{t})$ . □

**Remark 3.2.** *Using the same type of rescalings as the one leading to (3.2), it is not hard to prove that  $T \rightarrow E(T)$  is a continuous function.*

As a consequence of the monotonicity of the support and of the previous lemma, we can derive the following fundamental recursive characterization of  $E(T)$ .

**Lemma 3.3.** *For every  $T > 0$ ,*

$$E(T) = \min_{\sum_{i=1}^N \varphi_i = 1} \sum_{i=1}^N \varphi_i^{3/2} E(T \varphi_i^{-3/2}) + \frac{1}{T} \sum_{i=1}^N \varphi_i |\bar{X}_i|^2, \quad (3.4)$$

where  $\bar{X}_i$  are the centers of the intervals of length  $\varphi_i$  i.e.  $\bar{X}_i = -\frac{1}{2} + \sum_{j < i} \varphi_j + \frac{\varphi_i}{2}$ .

*Proof.* Let  $\varphi_1, \dots, \varphi_N$  be the fluxes of the branches leaving from the point  $(0, 0)$  (if it is not a branching point then  $N = 1$ ). Up to relabeling, we may assume that the  $\varphi_i$  are ordered i.e.  $\varphi_1$  corresponds to the first branch,  $\varphi_2$  to the second and so on. By the monotonicity of the traces (Proposition 2.7), the  $N$  branches are independent and the mass from the first

branch will go to  $[-1/2, -1/2 + \varphi_1]$ , the second branch will go to  $[-1/2 + \varphi_1, -1/2 + \varphi_1 + \varphi_2]$  and so on. Therefore, from (3.1) and (3.2),

$$E(T) = \min_{\sum_{i=1}^N \varphi_i = 1} \sum_{i=1}^N E(T, \varphi_i, \bar{X}_i) = \min_{\sum_{i=1}^N \varphi_i = 1} \sum_{i=1}^N \varphi_i^{3/2} E(T \varphi_i^{-3/2}) + \frac{1}{T} \sum_{i=1}^N \varphi_i |\bar{X}_i|^2.$$

□

Before going further, let us point out that for  $T, t > 0$  using as test configuration for  $E(T+t)$ ,  $\delta_0$  in  $[0, t]$  extended by the minimizer of  $E(T)$  in  $[t, T+t]$ , we obtain

$$E(T+t) \leq E(T) + t. \quad (3.5)$$

This together with (3.4), motivates the introduction of the largest branching time:

$$T_* = \inf\{T : E(T+t) = E(T) + t, \quad \forall t \geq 0\}. \quad (3.6)$$

By definition of  $T_*$  and (3.5), we see that for every  $\varepsilon > 0$ ,  $E(T_* - \varepsilon) + \varepsilon > E(T_*)$  which means that every minimizer of (3.4) for  $T = T_*$  must have  $N > 1$  branches (there must be branching at time zero).

We will also need the following simple lemma.

**Lemma 3.4.** *Let  $X \in \mathbb{R}$ ,  $T, \varphi > 0$  with  $T\varphi^{-3/2} > T_*$ . And let  $\mu_t$  be a minimizer for  $E(T, \varphi, X)$ . Letting  $X(t) := (1 - \frac{t}{T})X$ , for  $t \in [0, T - \varphi^{3/2}T_*]$  it holds,  $\mu_t = \varphi\delta_{X(t)}$ .*

*Proof.* Since  $T\varphi^{-3/2} > T_*$ , by definition of  $T_*$ , in  $[0, T\varphi^{-3/2} - T_*]$ , every minimizer of  $E(T\varphi^{-3/2})$  is of the form  $\delta_0$ . Therefore, by (3.1) and (3.2), if  $X(t) := (1 - \frac{t}{T})X$ , then for  $t \in [0, T - \varphi^{3/2}T_*]$ ,  $\mu_t = \varphi\delta_{X(t)}$ . □

We can now state the main result of this section.

**Proposition 3.5.** *We have  $T_* = 1/4$  and if  $\varphi_1, \dots, \varphi_N$  are optimal in (3.4) for  $T = T_*$ , then  $N = 2$  and  $\varphi_1 = \varphi_2 = 1/2$ . Moreover,*

$$E(1/4) - 1/4 = \frac{1}{2 - \sqrt{2}}.$$

The proof of this proposition will consist of the remaining part of this section. Before doing so, let us see how it implies Theorem 1.1. In the proof, we will use the following notation

**Definition 3.6.** *For  $\mu \in \mathcal{A}_{a,b}^*$  with  $\mu_t = \sum_i \varphi_i \delta_{X_i}$  and  $X \in \mathbb{R}$ , let  $S_X(\mu)$  be the measure defined by  $(S_X(\mu))_t := \sum_i \varphi_i \delta_{X_i + X}$ .*

*Proof of Theorem 1.1.* By definition of  $T_*$ , for  $T \geq 1/4$ , we have  $E(T) = E(1/4) + (T - 1/4)$  and if  $\mu$  is a minimizer for  $E(T)$ , then it coincides with  $\delta_0$  in  $[0, T - 1/4]$  and with (a translated version of) a minimizer for  $E(1/4)$  in  $[T - 1/4, T]$ . Therefore, it is enough proving that for  $T = 1/4$ , the only minimizer of  $E(1/4)$  is given by  $\mu^*$ .

Let  $\mu$  be such a minimizer and let us prove by induction that  $\mu = \mu^*$ . Recall first that we defined  $t_k = \frac{1}{4} \left(1 - \left(\frac{1}{2}\right)^{3k/2}\right)$ . Assume that  $\mu_t = \mu_t^*$  for  $t \in [0, t_{k-1}]$  and that  $\mu_{t_{k-1}} = 2^{-(k-1)} \sum_{i=1}^{2^{k-1}} \delta_{X_i^{k-1}}$  for some ordered  $X_i^{k-1} \in \mathbb{R}$ . By monotonicity of the support, in  $[t_{k-1}, 1/4]$ , each of the forward subsystems  $\mu^{+,i}$  emanating from  $X_i^{k-1}$  must be of the form  $\mu_t^{+,i} = S_{\bar{X}_i^{k-1}}(\mu_{t-t_{k-1}}^i)$  where  $\mu^i$  is a minimizer of  $E(\frac{1}{4} - t_{k-1}, 2^{-(k-1)}, X_i^{k-1} - \bar{X}_i^{k-1}) = E(\frac{1}{4}(2^{-(k-1)})^{3/2}, 2^{-(k-1)}, X_i^{k-1} - \bar{X}_i^{k-1})$ . By (3.2) and Proposition 3.5, every minimizer of  $E(\frac{1}{4}(2^{-(k-1)})^{3/2}, 2^{-(k-1)}, X_i^{k-1} - \bar{X}_i^{k-1})$  must branch into two pieces of equal mass. Thus, we can further decompose  $\mu^i = \mu^{i,1} + \mu^{i,2}$  where  $\mu^{i,1} = S_{-2^{-(k+1)}}(\nu^{i,1})$  with  $\nu^{i,1}$  a minimizer for  $E(\frac{1}{4}(2^{-(k-1)})^{3/2}, 2^{-k}, 2^{-(k+1)} + X_i^{k-1} - \bar{X}_i^{k-1})$  and similarly for  $\mu^{i,2}$ . Let

$$Y_{2i-1}^k(s) := -2^{-(k+1)} + \left(1 - \frac{s}{\frac{1}{4} - t_{k-1}}\right)(X_i^{k-1} - \bar{X}_{2i-1}^k) \quad \text{and}$$

$$Y_{2i}^k(s) := 2^{-(k+1)} + \left(1 - \frac{s}{\frac{1}{4} - t_{k-1}}\right)(X_i^{k-1} - \bar{X}_{2i}^k),$$

Since  $\bar{X}_i^{k-1} - 2^{-(k+1)} = \bar{X}_{2i-1}^k$  and  $\bar{X}_i^{k-1} + 2^{-(k+1)} = \bar{X}_{2i}^k$ , by Lemma 3.4, for  $s \in [0, t_k - t_{k-1}]$ ,  $\mu_s^i = 2^{-k}(\delta_{Y_{2i-1}^k(s)} + \delta_{Y_{2i}^k(s)})$  and thus letting

$$X_{2i-1}^k(t) := \frac{t - t_{k-1}}{\frac{1}{4} - t_{k-1}}(\bar{X}_{2i-1}^k - X_i^{k-1}) + X_i^{k-1} \quad \text{and}$$

$$X_{2i}^k(t) := \frac{t - t_{k-1}}{\frac{1}{4} - t_{k-1}}(\bar{X}_{2i}^k - X_i^{k-1}) + X_i^{k-1},$$

we finally obtain as claimed that for  $t \in [t_{k-1}, t_k]$ ,

$$\mu_t^{+,i} = 2^{-k}(\delta_{X_{2i-1}^k(t)} + \delta_{X_{2i}^k(t)}).$$

□

We may start investigating the properties of  $T_*$ .

**Lemma 3.7.** *It holds*

$$0 < T_* < \infty.$$

*As a consequence, the infimum in (3.6) is attained.*

*Proof.* We first observe that for every  $T > 0$ , by (2.4)

$$E(T) \geq T + \frac{W_2^2(\delta_0, dx \llcorner [-1/2, 1/2])}{T}. \quad (3.7)$$

Let us prove that  $T_* < \infty$ . Let  $T \geq 1$  and  $\mu_t$  be a minimizer for  $E(T)$ . By (3.5), for every  $T \geq 1$ ,

$$E(T) \leq E(1) + (T - 1).$$

On the other hand, by the no-loop condition, if  $\mu_t$  has its first branching at time  $t_0$  then in  $[t_0, T]$  it has at least two branches and thus

$$E(T) \geq 2(T - t_0).$$

Putting these two inequalities together we get  $t_0 \geq T - (E(1) - 1)$ . Letting  $T_1 := E(1) - 1$ , which is positive by (3.7), and assuming that  $T \geq T_1$ , this implies that before  $T - T_1$ , no branching may occur. Hence, for  $T \geq T_1$ ,

$$E(T) = E(T_1) + (T - T_1),$$

that is  $T_* \leq T_1$ .

We now prove that  $T_* > 0$ . By (3.7), if  $T_* = 0$ , for every  $T_1 \leq T$ ,

$$E(T) = E(T_1) + T - T_1 \geq T + \frac{W_2^2(\delta_0, dx \llcorner [-1/2, 1/2])}{T_1}$$

which letting  $T_1 \rightarrow 0$  would give a contradiction to  $E(T) < \infty$ . The fact that the infimum in (3.6) is attained follows by continuity of  $T \rightarrow E(T)$ .  $\square$

The next result is a form of equipartition of energy.

**Lemma 3.8.** *If  $\varphi_1, \dots, \varphi_N > 0$  are such that*

$$E(T_*) = \sum_{i=1}^N \varphi_i^{3/2} E(T_* \varphi_i^{-3/2}) + \frac{1}{T_*} \sum_{i=1}^N \varphi_i |\bar{X}_i|^2,$$

then

$$T_*(N - 1) = \frac{1}{T_*} \sum_{i=1}^N \varphi_i |\bar{X}_i|^2. \quad (3.8)$$

*Proof.* Let  $\mu_t = \sum_i \varphi_i \delta_{X_i}$  be optimal for  $T_*$ . By definition of  $T_*$ , the fact that  $T_* \varphi_i^{-3/2} > T_*$  and Lemma 3.4, there is  $\bar{\varepsilon} > 0$  such that for  $t \in [0, \bar{\varepsilon}]$ ,  $X_i(t) = \frac{t}{T_*} \bar{X}_i$ . For  $\bar{\varepsilon} > \varepsilon > 0$ , we are going to construct a competitor for  $E(T_* - \varepsilon)$ . In  $[\bar{\varepsilon} - \varepsilon, T_* - \varepsilon]$ , let  $Y_i(t) := X_i(t + \varepsilon)$  and in  $[0, \bar{\varepsilon} - \varepsilon]$ ,  $Y_i(t) := \frac{1}{T_*} \frac{\bar{\varepsilon}}{\bar{\varepsilon} - \varepsilon} t \bar{X}_i$  so that  $Y_i(\bar{\varepsilon} - \varepsilon) = X_i(\bar{\varepsilon})$ . Therefore,

$$\begin{aligned} E(T_* - \varepsilon) &\leq E(T_*) - N\varepsilon + \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 (\bar{\varepsilon} - \varepsilon) \frac{\bar{\varepsilon}^2}{(\bar{\varepsilon} - \varepsilon)^2} - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 \bar{\varepsilon} \\ &= E(T_*) - \left( N - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 \right) \varepsilon + o(\varepsilon^2) \end{aligned}$$

Hence,

$$\frac{E(T_* - \varepsilon) - E(T_*)}{-\varepsilon} \geq N - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 + o(\varepsilon).$$

Using (3.5) and letting  $\varepsilon \rightarrow 0$ , we get

$$N - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 \leq 1.$$

We similarly define a competitor for  $E(T_* + \varepsilon)$  by letting  $Y_i(t) := X_i(t - \bar{\varepsilon})$  in  $[\varepsilon + \bar{\varepsilon}, T_* + \varepsilon]$  and  $Y_i(t) := \frac{1}{T_*} \frac{\bar{\varepsilon}}{\varepsilon + \bar{\varepsilon}} t \bar{X}_i$  in  $[0, \varepsilon + \bar{\varepsilon}]$  and get

$$\begin{aligned} E(T_* + \varepsilon) &\leq E(T_*) + N\varepsilon + \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 (\bar{\varepsilon} + \varepsilon) \frac{\bar{\varepsilon}^2}{(\bar{\varepsilon} + \varepsilon)^2} - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 \bar{\varepsilon} \\ &= E(T_*) + \left( N - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 \right) \varepsilon + o(\varepsilon^2). \end{aligned}$$

From this we infer that

$$\frac{E(T_* + \varepsilon) - E(T_*)}{\varepsilon} \leq N - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 + o(\varepsilon).$$

By definition of  $T_*$  and by continuity of  $E$ ,

$$\frac{E(T_* + \varepsilon) - E(T_*)}{\varepsilon} = 1$$

and therefore

$$N - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 \geq 1,$$

which conclude the proof of (3.8).  $\square$

**Remark 3.9.** Notice that (3.8) is compatible with  $N = 2$ ,  $\varphi_1 = \varphi_2 = 1/2$  and  $T_* = \frac{1}{4}$ .

The following lemma gives an explicit formula for the second term in (3.4).

**Lemma 3.10.** For every  $(\varphi_i)_{i=1}^N$  with  $\sum_{i=1}^N \varphi_i = 1$ , it holds

$$\sum_{i=1}^N \varphi_i |\bar{X}_i|^2 = \frac{1}{12} \left( 1 - \sum_{i=1}^N \varphi_i^3 \right). \quad (3.9)$$

*Proof.* We prove this by induction on  $N$ . For  $N = 1$ , there is nothing to prove. For  $N = 2$ , since  $\varphi_2 = 1 - \varphi_1$ , the left-hand side of (3.9) is equal to

$$\varphi_1 \left( \frac{-1 + \varphi_1}{2} \right)^2 + (1 - \varphi_1) \left( \frac{\varphi_1}{2} \right)^2 = \frac{1}{4} \varphi_1 (1 - \varphi_1),$$

which is equal to the right-hand side of (3.9).

Assume now that (3.9) holds for  $N - 1$ . Let then  $\varphi = \varphi_1 + \varphi_2$  and  $\bar{X} = \bar{X}_1 + \frac{\varphi_2}{2}$ . By the induction hypothesis,

$$\sum_{i=3}^N \varphi_i |\bar{X}_i|^2 + \varphi |\bar{X}|^2 = \frac{1}{12} \left( 1 - \sum_{i=3}^N \varphi_i^3 - \varphi^3 \right)$$

so that

$$\sum_{i=1}^N \varphi_i |\bar{X}_i|^2 = \frac{1}{12} \left( 1 - \sum_{i=1}^N \varphi_i^3 \right) + \frac{1}{12} (\varphi_1^3 + \varphi_2^3 - \varphi^3) + \varphi_1 |\bar{X}_1|^2 + \varphi_2 |\bar{X}_2|^2 - \varphi |\bar{X}|^2.$$

We are thus left to prove that

$$\frac{1}{12} (\varphi_1^3 + \varphi_2^3 - \varphi^3) + \varphi_1 |\bar{X}_1|^2 + \varphi_2 |\bar{X}_2|^2 - \varphi |\bar{X}|^2 = 0. \quad (3.10)$$

By definition of  $\varphi$ ,  $\bar{X}$  and since  $\bar{X}_2 = \bar{X}_1 + \frac{\varphi_1 + \varphi_2}{2}$ , we have

$$\frac{1}{12} (\varphi_1^3 + \varphi_2^3 - \varphi^3) = -\frac{\varphi_1 \varphi_2}{4} (\varphi_1 + \varphi_2) = -\frac{\varphi_1 \varphi_2 \varphi}{4}.$$

Analogously we can compute

$$\begin{aligned} \varphi_1 |\bar{X}_1|^2 + \varphi_2 |\bar{X}_2|^2 - \varphi |\bar{X}|^2 &= \varphi_1 |\bar{X}_1|^2 + \varphi_2 \left| \bar{X}_1 + \frac{\varphi}{2} \right|^2 - \varphi \left| \bar{X}_1 + \frac{\varphi_2}{2} \right|^2 \\ &= \varphi_1 |\bar{X}_1|^2 + \varphi_2 |\bar{X}_1|^2 + \varphi_2 \varphi \bar{X}_1 + \frac{\varphi_2 \varphi^2}{4} - \varphi |\bar{X}_1|^2 - \varphi \varphi_2 \bar{X}_1 - \frac{\varphi \varphi_2^2}{4} \\ &= \frac{\varphi_2 \varphi}{4} (\varphi - \varphi_2) \\ &= \frac{\varphi_1 \varphi_2 \varphi}{4}. \end{aligned}$$

Adding these two equalities we get (3.10) which concludes the proof of (3.9).  $\square$

As a consequence, we obtain from (3.4) that for every  $T > 0$ ,

$$E(T) = \min_{\sum_{i=1}^N \varphi_i = 1} \sum_{i=1}^N \varphi_i^{3/2} E(\varphi_i^{-3/2} T) + \frac{1}{12T} \left( 1 - \sum_{i=1}^N \varphi_i^3 \right). \quad (3.11)$$

In this form, it is clear that the energy does not change if we reorder the  $\varphi_i$ .

Using the characterization (3.11), we show another characterization of  $E(T_*)$  which has the advantage of not being recursive anymore.

**Proposition 3.11.** *It holds*

$$\frac{E(T_*) - T_*}{T_*} = \min_{\sum_{i=1}^N \varphi_i = 1} \frac{(N-1) + \frac{1}{12T_*^2} \left(1 - \sum_{i=1}^N \varphi_i^3\right)}{1 - \sum_{i=1}^N \varphi_i^{3/2}}. \quad (3.12)$$

Moreover, if  $\varphi_1, \dots, \varphi_N$  are minimizers for  $E(T_*)$  then they also minimize the right-hand side of (3.12) and vice-versa.

*Proof.* Let  $\bar{\varphi}_i$  be the optimal fluxes for (3.11). By definition of  $T_*$ , we have for every  $\varphi_i$  with  $\sum_i \varphi_i = 1$  (since  $\varphi_i^{-3/2} \geq 1$ )

$$\begin{aligned} E(T_*) &\leq \sum_{i=1}^N \varphi_i^{3/2} E(T_* \varphi_i^{-3/2}) + \frac{1}{12T_*} \left(1 - \sum_{i=1}^N \varphi_i^3\right) \\ &= \sum_{i=1}^N \varphi_i^{3/2} (E(T_*) + (T_* - T_* \varphi_i^{-3/2})) + \frac{1}{12T_*} \left(1 - \sum_{i=1}^N \varphi_i^3\right) \\ &= E(T_*) \left(\sum_{i=1}^N \varphi_i^{3/2}\right) + T_* \sum_{i=1}^N (1 - \varphi_i^{3/2}) + \frac{1}{12T_*} \left(1 - \sum_{i=1}^N \varphi_i^3\right). \end{aligned}$$

Therefore

$$E(T_*) - T_* \leq (E(T_*) - T_*) \left(\sum_{i=1}^N \varphi_i^{3/2}\right) + (N-1)T_* + \frac{1}{12T_*} \left(1 - \sum_{i=1}^N \varphi_i^3\right),$$

and then

$$\frac{E(T_*) - T_*}{T_*} \leq \frac{(N-1) + \frac{1}{12T_*^2} \left(1 - \sum_{i=1}^N \varphi_i^3\right)}{1 - \sum_{i=1}^N \varphi_i^{3/2}}$$

with equality for  $\varphi_i = \bar{\varphi}_i$ . □

For  $N \geq 2$ , we introduce a quantity which will play a central role in our analysis. Let

$$\alpha_N := \inf_{\varphi_i \geq 0} \left\{ \frac{1 - \sum_{i=1}^N \varphi_i^3}{1 - \sum_{i=1}^N \varphi_i^{3/2}} : \sum_{i=1}^N \varphi_i = 1 \right\}.$$

**Proposition 3.12.** *It holds*

$$T_* \leq \frac{1}{4}.$$

Moreover, the number  $N > 1$  of branches of the minimizer for  $T_*$ , satisfies

$$\sqrt{N}(\sqrt{N} + 1)T_* + \frac{\alpha_N}{12T_*} \leq \frac{\sqrt{2}}{2(\sqrt{2} - 1)}. \quad (3.13)$$

As a consequence,

$$\sqrt{N} \leq \frac{1}{2} \left( -1 + \left( 1 + \frac{6}{\alpha_N(\sqrt{2}-1)^2} \right)^{1/2} \right). \quad (3.14)$$

In particular, since  $\alpha_N \geq 1$ , this gives  $N \leq 6$ .

*Proof.* We start with the upper bound  $T_* \leq \frac{1}{4}$ . Let  $\varphi_1, \dots, \varphi_N$  be optimal for  $E(T_*)$ . Then, by (3.12) and (3.8),

$$\frac{E(T_*) - T_*}{T_*} = \frac{2(N-1)}{\left(1 - \sum_i \varphi_i^{3/2}\right)}.$$

Using that for every  $\varphi_1, \dots, \varphi_N$  with  $\sum \varphi_i = 1$ ,

$$\frac{1}{1 - \sum_i \varphi_i^{3/2}} \geq \frac{1}{1 - \sqrt{N}^{-1}} \quad (3.15)$$

we get

$$\frac{E(T_*) - T_*}{T_*} \geq \frac{2(N-1)\sqrt{N}}{\sqrt{N}-1}.$$

Since the right-hand side is minimized for  $N = 2$  (among  $N \in \mathbb{N}$ ,  $N \geq 2$ ), we have

$$\frac{E(T_*) - T_*}{T_*} \geq \frac{2\sqrt{2}}{\sqrt{2}-1}. \quad (3.16)$$

We proceed further by proving an upper bound for the left-hand side of (3.16). For every  $T > 0$ , we can construct the self similar competitor for which every branch is divided into two branches of half the mass at every branching point (which is at  $T_k = T(1 - (\frac{1}{2})^{3k/2})$ ). Let  $\tilde{E}(T)$  be its energy. Then, arguing as in (3.4), we have

$$\tilde{E}(T) = 2 \left( \frac{1}{2} \right)^{3/2} \tilde{E}(T) + 2 \left( T - \left( \frac{1}{2} \right)^{3/2} T \right) + \frac{1}{16T}$$

that is

$$(\tilde{E}(T) - T) \left( 1 - \frac{1}{\sqrt{2}} \right) = T + \frac{1}{16T}$$

from which we get

$$\frac{\tilde{E}(T) - T}{T} = \frac{\sqrt{2}}{\sqrt{2}-1} \left( 1 + \frac{1}{16T^2} \right).$$

Since by definition  $E(T) \leq \tilde{E}(T)$ , we get

$$\frac{E(T) - T}{T} \leq \frac{\sqrt{2}}{\sqrt{2}-1} \left( 1 + \frac{1}{16T^2} \right). \quad (3.17)$$



For  $T > \frac{1}{4}$ , the right-hand side is strictly smaller than  $\frac{2\sqrt{2}}{\sqrt{2}-1}$  hence by (3.16), we cannot have  $T = T_*$ . This gives the upper bound.

We now turn to (3.13). For this, we notice that

$$\frac{E(T_*) - T_*}{T_*} = \frac{N-1}{1 - \sum_i \varphi_i^{3/2}} + \frac{(1 - \sum_i \varphi_i^3)}{12T_*^2(1 - \sum_i \varphi_i^{3/2})} \geq \frac{N-1}{1 - N^{-1/2}} + \frac{\alpha_N}{12T_*^2}. \quad (3.18)$$

Since  $T_* \leq 1/4$ , by definition of  $T_*$  (recall (3.6)),  $E(1/4) - 1/4 = E(T_*) - T_*$ , combining (3.18) with (3.17) for  $T = 1/4$  and

$$\frac{N-1}{1 - N^{-1/2}} = \sqrt{N}(\sqrt{N} + 1),$$

yields (3.13). We finally derive (3.14). For this, multiply (3.13) by  $T_*$  to obtain that

$$\sqrt{N}(\sqrt{N} + 1)T_*^2 - \frac{\sqrt{2}}{2(\sqrt{2}-1)}T_* + \frac{\alpha_N}{12} \leq 0.$$

This implies that the polynomial  $\sqrt{N}(\sqrt{N} + 1)X^2 - \frac{\sqrt{2}}{2(\sqrt{2}-1)}X + \frac{\alpha_N}{12}$  has real roots (and that  $T_*$  lies between these two roots) so that

$$\Delta = \frac{1}{2(\sqrt{2}-1)^2} - \frac{\alpha_N}{3}\sqrt{N}(\sqrt{N} + 1) \geq 0,$$

which is equivalent to

$$(\sqrt{N})^2 + \sqrt{N} - \frac{3}{2\alpha_N(\sqrt{2}-1)^2} \leq 0.$$

Since the largest root of this polynomial (in the variable  $\sqrt{N}$ ) is given by  $\frac{1}{2}(-1 + (1 + \frac{6}{\alpha_N(\sqrt{2}-1)^2})^{1/2})$ , we have obtained (3.14).  $\square$

**Remark 3.13.** *From the proof of the previous proposition, one could also get a lower bound for  $T_*$ . We will make use of this fact later on to study the case  $N = 2$ .*

### 3.1 Excluding the case $N > 2$

In this section, we show by a computer assisted proof that for  $T = T_*$ , there cannot be more than two branches.

**Lemma 3.14.** *For  $3 \leq N \leq 6$ ,*

$$\alpha_N > \frac{6}{(\sqrt{2}-1)^2(2\sqrt{N}+1)^2}. \quad (3.19)$$

*As a consequence, the number of branches at  $T = T_*$  equals two.*

$N$	3	4	5	6
$\alpha_N >$	1.76	1.4	1.17	1.01

Table 1: Critical values of  $\alpha_N$ .

*Proof.* By inverting the relation between  $\alpha_N$  and  $N$  in (3.14), it is readily seen that (3.19) and (3.14) are incompatible. Therefore, proving the lower bound (3.19) directly excludes the possibility of having  $N > 2$  branches.

In Table 1, the values of the right-hand side of (3.19) are given for  $N \in [3, 6]$ . Since we are not able to prove these bounds analytically, we resort to a computer assisted proof. From Table 1, we see that we want to compute  $\alpha_N$  with a precision of  $10^{-2}$ .

*Step 1 (Computation of  $\alpha_2$ ):*

We start by computing  $\alpha_2$ . We claim that

$$\alpha_2 = 2. \tag{3.20}$$

Since

$$\alpha_2 = \min_{\varphi \in [0, 1/2]} \frac{3\varphi(1-\varphi)}{1-\varphi^{3/2}-(1-\varphi)^{3/2}},$$

in order to prove that the minimum is attained for  $\varphi = 0$ , it is enough to show that for  $\varphi \in [0, 1/2]$ ,

$$\varphi(1-\varphi) \geq \frac{2}{3}(1-\varphi^{3/2}-(1-\varphi)^{3/2}).$$

Since the two expressions agree for  $\varphi = 0$ , by differentiating, we are left with the proof of

$$1-2\varphi \geq (1-\varphi)^{1/2} - \varphi^{1/2}$$

or equivalently of

$$1-2\varphi + \varphi^{1/2} \geq (1-\varphi)^{1/2}.$$

Squaring both sides (notice that  $1-2\varphi + \varphi^{1/2} \geq 0$ ), this amounts to prove that for  $\varphi \in [0, 1/2]$ ,

$$2\varphi^2 - 2\varphi^{3/2} - \varphi + \varphi^{1/2} \geq 0. \tag{3.21}$$

Since the polynomial,

$$P(X) = 2X^4 - 2X^3 - X^2 + X$$

has roots  $\{-1/\sqrt{2}, 0, 1/\sqrt{2}, 1\}$ , it is positive in  $[0, 1/\sqrt{2}]$  and thus considering  $X = \varphi^{1/2}$ , we see that (3.21) holds and thus (3.20) is proven.

*Step 2 (Lower bound for  $\alpha_N$ ):* Consider now  $N \geq 3$ . Since  $\frac{1-\sum_i \varphi_i^3}{1-\sum_i \varphi_i^{3/2}}$  is continuous on the compact convex set  $K = \{0 \leq \varphi_i \leq 1 : \sum_i \varphi_i = 1\}$  (the only problem could arise when all the  $\varphi_i$ 's but one go to zero but then it is easy to see that  $\frac{1-\sum_i \varphi_i^3}{1-\sum_i \varphi_i^{3/2}} \rightarrow 2$ ), the

minimum is attained. If the minimum is attained at the boundary then  $\alpha_N = \alpha_{N-1}$ . Since the values in Table 1 are decreasing with  $N$  and since  $\alpha_2 = 2$ , in that case by a simple induction we would be over. Otherwise, we claim that for every  $N$ , the optimal  $\varphi_i$  may take only two values:  $\varphi$  repeated  $k \in [1, N]$  times and  $\frac{1-k\varphi}{N-k}$  repeated  $N-k$  times. Indeed, fix  $i \neq j \in [1, N]$  and for  $|\varepsilon|$  small enough, define  $\hat{\varphi}_i := \varphi_i + \varepsilon$ ,  $\hat{\varphi}_j := \varphi_j - \varepsilon$  and  $\hat{\varphi}_k := \varphi_k$  for  $k \neq i, j$ . By minimality, the derivative in zero of

$$\varepsilon \rightarrow \frac{1 - \sum_k \hat{\varphi}_k^3}{1 - \sum_k \hat{\varphi}_k^{3/2}}$$

is equal to zero. From this, we get the condition

$$\varphi_i^2 - \frac{\alpha_N}{2} \varphi_i^{1/2} = \varphi_j^2 - \frac{\alpha_N}{2} \varphi_j^{1/2} \quad \forall i \neq j.$$

Letting  $P_N(X) := X^4 - \frac{\alpha_N}{2} X$ , this means that for every  $i \neq j$ ,  $P_N(\varphi_i^2) = P_N(\varphi_j^2)$ . Since  $P'_N$  has only one positive root, this means that  $\varphi_i$  may take at most two values, proving the claim.

Therefore,

$$\alpha_N = \min_{k \in [1, N]} \min_{\varphi \in [0, k^{-1}]} f_{k, N}(\varphi), \quad (3.22)$$

where

$$f_{k, N}(\varphi) := \frac{1 - k\varphi^3 - (N - k)^{-2}(1 - k\varphi)^3}{1 - k\varphi^{3/2} - (N - k)^{-1/2}(1 - k\varphi)^{3/2}}.$$

We are thus left to estimate a finite number of one dimensional functions. Let us first point out that the case  $k = N$  corresponds to  $\varphi = 1/N$ . In this case, a simple numerical check shows that  $f_{k, N}(1/N)$  is larger than the critical values computed in Table 1.

For  $3 \leq N \leq 6$  and  $1 \leq k \leq N - 1$ , we want to estimate  $\min_{\varphi \in [0, k^{-1}]} f_{k, N}(\varphi)$  with a precision of  $10^{-2}$ . For this we will compute the values of  $f_{k, N}$  for a sufficiently fine discretization of  $[0, k^{-1}]$ . Let  $I \subseteq [0, 1/k]$  and let  $\Lambda := \sup_{\varphi \in I} |f'_{k, N}(\varphi)|$ . Since for  $\varphi, \psi \in I$ ,

$$|f_{k, N}(\varphi) - f_{k, N}(\psi)| \leq \Lambda |\varphi - \psi|,$$

in order to get a precision of  $10^{-2}$  on  $\inf_I f_{k, N}$ , it is enough to use a discretization step  $\delta \leq 10^{-2} \Lambda^{-1}$ . We are thus naturally led to estimate  $\sup |f'_{k, N}|$ . We can compute

$$f'_{k, N}(\varphi) = \frac{3k}{1 - k\varphi^{3/2} - (N - k)^{-1/2}(1 - k\varphi)^{3/2}} \left( -\varphi^2 + (N - k)^{-2}(1 - k\varphi)^2 - \frac{1}{2}(-\varphi^{1/2} + (N - k)^{-1/2}(1 - k\varphi)^{1/2})f_{k, N}(\varphi) \right). \quad (3.23)$$

*Step 2.1 (the case  $k \in [2, N-2]$ ):* We start with the easier case  $k \in [2, N-2]$  (which in particular implies  $N \geq 4$ ). In this case,  $\sup_{[0,1/k]} |f'_{k,N}| < \infty$  and we can take  $I = [0, 1/k]$ . Since it will appear in several places, we first study the function

$$g_{k,N}(\varphi) := 1 - k\varphi^{3/2} - (N-k)^{-1/2}(1 - k\varphi)^{3/2}.$$

In particular, we want to study  $\min g_{k,N}$ . Taking the derivative, we obtain

$$g'_{k,N}(\varphi) = \frac{3k}{2}(-\varphi^{1/2} + (N-k)^{-1/2}(1 - k\varphi)^{1/2}).$$

Therefore,  $g'_{k,N}$  is zero only if  $\varphi = 1/N$  so that  $g'_{k,N}$  is first positive and then negative and  $g_{k,N}$  attains its minimum on the boundary. Hence, for  $k \in [2, N-2]$ ,

$$\min_{\varphi \in [0, 1/k]} g_{k,N}(\varphi) = \min(1 - k^{-1/2}, 1 - (N-k)^{-1/2}) \geq 1 - 2^{-1/2}.$$

Injecting this into (3.23), we find

$$\begin{aligned} |f'_{k,N}(\varphi)| &\leq \frac{3k}{1 - 2^{-1/2}} \left( k^{-2} + (N-k)^{-2} + \frac{1}{2}(1 - 2^{-1/2})^{-1}(k^{-1/2} + (N-k)^{-1/2}) \right) \\ &\leq \frac{6}{1 - 2^{-1/2}} \left( 1 + \frac{2}{\sqrt{2} - 1} \right) \\ &\leq 120. \end{aligned}$$

In this case, we may take

$$\delta \leq 12^{-1} \times 10^{-3}. \quad (3.24)$$

*Step 2.2 (the case  $k \in \{1, N-1\}$ ):* We turn to the more delicate case of  $k = N-1$  (the case  $k = 1$  being analogous). In this case,  $\sup_{\varphi \in [0, 1/k]} |f'_{k,N}(\varphi)| = \lim_{\varphi \rightarrow 0} |f'_{k,N}(\varphi)| = \infty$ . Therefore, we need to be more careful. A Taylor expansion shows that

$$\lim_{\varphi \rightarrow 0} f_{k,N}(\varphi) = 2,$$

which is always strictly bigger than the critical values computed in Table 1. Hence, if we can find  $\eta > 0$ , such that in  $[0, \eta]$ ,  $f'_{k,N}$  is positive, we will have

$$\alpha_N = \min \left( \min_{\varphi \in [0, \eta]} f_{k,N}(\varphi), \min_{\varphi \in [\eta, 1/k]} f_{k,N}(\varphi) \right) = \min(2, \min_{\varphi \in [\eta, 1/k]} f_{k,N}(\varphi)).$$

From (3.23), we see that  $f'_{N-1,N}$  is of the same sign as

$$h_N(\varphi) := -\varphi^2 + (1 - (N-1)\varphi)^2 + \frac{1}{2}(\varphi^{1/2} - (1 - (N-1)\varphi)^{1/2})f_{N-1,N}(\varphi).$$

Since for every  $\varphi \geq 0$ ,

$$-\varphi^2 + (1 - (N-1)\varphi)^2 \geq 1 - 2(N-1)\varphi,$$

and

$$-(1 - (N - 1)\varphi)^{1/2} \geq -1,$$

we can first infer that

$$h_N(\varphi) \geq 1 - 2(N - 1)\varphi + \frac{1}{2}(\varphi^{1/2} - 1)f_{N-1,N}(\varphi). \quad (3.25)$$

We may now bound from above  $f_{N-1,N}(\varphi)$ . Notice that regarding the numerator,

$$1 - (N - 1)\varphi^3 - (1 - (N - 1)\varphi)^3 \leq 3(N - 1)\varphi + (N - 1)^3\varphi^3. \quad (3.26)$$

In order to bound from below the denominator, we first claim that for  $\varphi \leq \left(\frac{8}{1+16(N-1)}\right)^2$ ,

$$(1 - (N - 1)\varphi)^{3/2} \leq 1 - \frac{3}{2}(N - 1)\varphi + \frac{(N - 1)}{4}\varphi^{3/2}. \quad (3.27)$$

Indeed, letting

$$\Psi_N(\varphi) := (1 - (N - 1)\varphi)^{3/2} - 1 + \frac{3}{2}(N - 1)\varphi - \frac{(N - 1)}{4}\varphi^{3/2},$$

we have  $\Psi_N(0) = 0$  and

$$\Psi'_N(\varphi) = \frac{3}{2}(N - 1) \left( 1 - \frac{1}{4}\varphi^{1/2} - (1 - (N - 1)\varphi)^{1/2} \right).$$

Since for  $\varphi \leq \left(\frac{8}{1+16(N-1)}\right)^2$ ,  $1 - \frac{1}{4}\varphi^{1/2} - (1 - (N - 1)\varphi)^{1/2} \leq 0$ , we have  $\Psi'_N \leq 0$  and thus also  $\Psi_N \leq 0$ , proving (3.27). From this we get that for  $\varphi \leq \left(\frac{8}{1+16(N-1)}\right)^2$ ,

$$1 - (N - 1)\varphi^{3/2} - (1 - (N - 1)\varphi)^{3/2} \geq \frac{3}{2}(N - 1)\varphi - \frac{5}{4}(N - 1)\varphi^{3/2}. \quad (3.28)$$

Putting (3.26) and (3.28) together, we get that for  $\varphi \leq \left(\frac{8}{1+16(N-1)}\right)^2$ ,

$$f_{N-1,N}(\varphi) \leq 2 \frac{1 + \frac{(N-1)^2}{3}\varphi^2}{1 - \frac{5}{6}\varphi^{1/2}}. \quad (3.29)$$

Inserting this back into (3.25), we get that for  $\varphi \leq \left(\frac{8}{1+16(N-1)}\right)^2$ ,

$$h_N(\varphi) \geq 1 - 2(N - 1)\varphi + (\varphi^{1/2} - 1) \frac{1 + \frac{(N-1)^2}{3}\varphi^2}{1 - \frac{5}{6}\varphi^{1/2}}.$$

Since for  $\varphi \leq \left(\frac{8}{1+16(N-1)}\right)^2$ ,  $5/6\varphi^{1/2} \leq 2/3$  and thus  $\frac{\varphi^{2(N-1)^2}}{3(1-\frac{5}{6}\varphi^{1/2})} \leq \varphi^2(N-1)^2 \leq \varphi(N-1)$ , this may be further simplified into

$$h_N(\varphi) \geq 1 - 3(N-1)\varphi + (\varphi^{1/2} - 1)\frac{1}{1 - \frac{5}{6}\varphi^{1/2}}.$$

Since for  $x \leq 1/7$ ,

$$\frac{1}{1-x} \leq 1 + \frac{7}{6}x,$$

and since  $\varphi \leq \left(\frac{8}{1+16(N-1)}\right)^2$  implies  $5/6\varphi^{1/2} \leq 1/7$ , we have

$$\begin{aligned} h_N(\varphi) &\geq 1 - 3(N-1)\varphi + (\varphi^{1/2} - 1)\left(1 + \frac{35}{36}\varphi^{1/2}\right) \\ &\geq \frac{1}{36}\varphi^{1/2} - 3(N-1)\varphi. \end{aligned}$$

In particular, if  $\varphi \leq \frac{1}{108(N-1)}$ , then  $h_N \geq 0$ . To sum up, we have proven that

$$f'_{N-1,N} \geq 0 \quad \text{in} \quad \left[0, \min\left(\left(\frac{8}{1+16(N-1)}\right)^2, \frac{1}{108(N-1)}\right)\right].$$

Letting  $\eta := \frac{1}{540}$  (which corresponds to  $\frac{1}{108(N-1)}$  with  $N = 6$ ), then for  $N \in [3, 6]$ ,  $\eta \leq \min\left(\left(\frac{8}{1+16(N-1)}\right)^2, \frac{1}{108(N-1)}\right)$ , so that  $f'_{N-1,N} \geq 0$  in  $[0, \eta]$ .

We finally estimate  $|f'_{N-1,N}|$  in  $[\eta, (N-1)^{-1}]$ . Recall that by (3.23)

$$\begin{aligned} f'_{N-1,N}(\varphi) &= \frac{3(N-1)}{g_{N-1,N}(\varphi)} \left(-\varphi^2 + (1 - (N-1)\varphi)^2\right. \\ &\quad \left.- \frac{1}{2}(-\varphi^{1/2} + (1 - (N-1)\varphi)^{1/2})\frac{1 - (N-1)\varphi^3 - (1 - (N-1)\varphi)^3}{g_{N-1,N}(\varphi)}\right). \end{aligned}$$

As before,

$$\begin{aligned} \min_{[\eta, (N-1)^{-1}]} g_{N-1,N}(\varphi) &= \min(g_{N-1,N}(\eta), g_{N-1,N}(1/(N-1))) \\ &= \min\left(1 - (N-1)\eta^{3/2} - (1 - (N-1)\eta)^{3/2}, 1 - (N-1)^{-1/2}\right) \\ &\geq 1 - 5\eta^{3/2} - (1 - 5\eta)^{3/2} \geq 1, 3 \times 10^{-2}. \end{aligned}$$

We can thus estimate for  $\varphi \in [\eta, (N-1)^{-1}]$ ,

$$|f'_{N-1,N}(\varphi)| \leq \frac{18 \times 10^2}{1, 3} \left(2 + \frac{10^2}{1, 3}\right) \leq 2 \times 10^5.$$

$N$	3	4	5	6
$\alpha_N \simeq$	2	1.88	1.74	1.64

Table 2: Numerical values of  $\alpha_N$

Hence, we may finally take

$$\delta \leq 0.5 \times 10^{-7}. \quad (3.30)$$

Using a simple Scilab code, we find the values of  $\alpha_N$  given in Table 2. We see that the values we find are well above the critical values given in Table 1.  $\square$

**Remark 3.15.** *Although we are not able to prove it, the numerics show that for  $N \geq 4$ , the minimum in  $\alpha_N$  is attained for equidistributed masses i.e.  $\varphi = 1/N$ .*

**Remark 3.16.** *Arguing as in Step 2, it could have been proven that also for the original minimization problem  $E(T)$ , the optimal masses may take at most two distinct values.*

### 3.2 Conclusion in the case $N = 2$

In this section, we end the proof of Proposition 3.5 by studying the case  $N = 2$ .

Let first

$$T_- := \frac{1}{4} \left( 1 - \left( 1 - \frac{4}{3} (2 - \sqrt{2})^{1/2} \right) \right).$$

Then  $1/4 \geq T_* \geq T_-$ . Indeed, (3.13) for  $N = 2$  (recalling that  $\alpha_2 = 2$ ) may be seen to be equivalent to

$$T_*^2 - \frac{1}{2}T_* + \frac{1}{6\sqrt{2}(\sqrt{2}+1)} \leq 0. \quad (3.31)$$

In particular,  $T_*$  has to lie between the two roots of the right-hand side of (3.31) which yields  $T_* \geq T_-$  (recall that the bound  $1/4 \geq T_*$  was already derived in Proposition 3.12).

Now, if  $T_* \leq 1/4$  then  $E(1/4) = E(T_*) + 1/4 - T_*$  so that using (3.17) for  $T = 1/4$ , we get

$$E(T_*) - T_* = E(1/4) - 1/4 \leq \frac{\sqrt{2}}{2(\sqrt{2}-1)}.$$

Using (3.12), we obtain for  $\varphi$  optimal for  $T_*$ , the bound

$$T_* \frac{1 + \frac{1}{4T_*^2} \varphi(1-\varphi)}{1 - \varphi^{3/2} - (1 - \varphi^{3/2})} \leq \frac{\sqrt{2}}{2(\sqrt{2}-1)}. \quad (3.32)$$

The next proposition shows the reverse inequality which concludes the proof of Proposition 3.5.

**Proposition 3.17.** For  $T \in [T_-, 1/4]$  and  $\varphi \in [0, 1]$ ,

$$T \frac{1 + \frac{1}{4T^2}\varphi(1-\varphi)}{1 - \varphi^{3/2} - (1-\varphi)^{3/2}} \geq \frac{\sqrt{2}}{2(\sqrt{2}-1)}, \quad (3.33)$$

with equality if and only if  $T_* = \frac{1}{4}$  and  $\varphi = 1/2$ .

*Proof.* By symmetry we can assume that  $\varphi \in [0, 1/2]$ . Letting  $\lambda := 2T$ , (3.33) is equivalent to show that for  $\lambda \in [2T_-, 1/2]$  and  $\varphi \in [0, 1/2]$ ,

$$1 + \frac{1}{\lambda^2}\varphi(1-\varphi) \geq \frac{\sqrt{2}}{\lambda(\sqrt{2}-1)}(1 - \varphi^{3/2} - (1-\varphi)^{3/2}). \quad (3.34)$$

It will be more convenient to work with  $a := \frac{3\sqrt{2}\lambda}{2(\sqrt{2}-1)}$ . Letting

$$a_- := \frac{3\sqrt{2}}{\sqrt{2}-1}T_- = \frac{3}{2(2-\sqrt{2})}(1 - (1 - \frac{4}{3}(2-\sqrt{2}))^{1/2}) \simeq 1.36,$$

we are reduced to  $a \in [a_-, \frac{3\sqrt{2}}{4(\sqrt{2}-1)}]$ . Inequality (3.34) then reads

$$L(a, \varphi) := 1 + \frac{9}{2a^2(\sqrt{2}-1)^2}\varphi(1-\varphi) \geq \frac{3}{a(\sqrt{2}-1)^2}(1 - \varphi^{3/2} - (1-\varphi)^{3/2}) =: R(a, \varphi). \quad (3.35)$$

Let us first notice that  $L(a, 0) = 1 > 0 = R(a, 0)$  and that for  $\varphi = \frac{1}{2}$ , (3.35) reads,

$$1 + \frac{9}{8a^2(\sqrt{2}-1)^2} \geq \frac{3}{a\sqrt{2}(\sqrt{2}-1)},$$

which always holds true (in terms of  $\lambda$ , this amounts to  $1 + \frac{1}{4\lambda^2} \geq \frac{1}{\lambda}$ ). Moreover, the inequality above is strict if  $a < \frac{3\sqrt{2}}{4(\sqrt{2}-1)}$ . We are going to study the variations (for fixed  $a$ ) of  $L(a, \varphi) - R(a, \varphi)$ . By differentiating, this is equivalent to study the sign of

$$D(\varphi) := 1 - 2\varphi - a((1-\varphi)^{1/2} - \varphi^{1/2}). \quad (3.36)$$

Let  $X := \varphi^{1/2}$ . For  $a \in [a_-, \frac{3\sqrt{2}}{4(\sqrt{2}-1)}]$  and  $X \in [0, 1/\sqrt{2}]$ , since  $1 - 2X^2 + aX \geq 0$ , the sign of (3.36) is the same as the sign of

$$P(X) := (1 - 2X^2 + aX)^2 - a^2(1 - X^2) = 4X^4 - 4aX^3 + 2(a^2 - 2)X^2 + 2aX + (1 - a^2).$$

Since  $P$  has roots  $\{\pm 1/\sqrt{2}\}$ , we can factor it to obtain

$$P(X) = 2(X^2 - \frac{1}{2})(2X^2 - 2aX + (a^2 - 1)).$$



For  $a > \sqrt{2}$ ,  $2X^2 - 2aX + (a^2 - 1)$  has no real roots and therefore, for  $a \in [\sqrt{2}, \frac{3\sqrt{2}}{4(\sqrt{2}-1)}]$ ,  $P$  is negative inside  $[0, 1/\sqrt{2}]$  and thus  $\partial_\varphi L - \partial_\varphi R \leq 0$  implying that

$$\min_{\varphi \in [0, 1/2]} L(a, \varphi) - R(a, \varphi) = L(a, 1/2) - R(a, 1/2) \geq 0, \quad (3.37)$$

with strict inequality if  $a < \frac{3\sqrt{2}}{4(\sqrt{2}-1)}$  or  $\varphi \neq 1/2$ . This proves (3.34) for  $a \in [\sqrt{2}, \frac{3\sqrt{2}}{4(\sqrt{2}-1)}]$ . If now  $a \in [a_-, \sqrt{2}]$ , besides  $\pm 1/\sqrt{2}$ ,  $P$  has two more roots

$$X_\pm := \frac{a \pm \sqrt{2 - a^2}}{2}. \quad (3.38)$$

For  $a \in [a_-, \sqrt{2}]$ ,

$$0 \leq X_- \leq 1/\sqrt{2} \leq X_+,$$

and thus  $P$  is negative in  $[0, X_-]$  and positive in  $[X_-, 1/\sqrt{2}]$  from which,

$$\Psi(a) := \min_{\varphi \in [0, 1/2]} L(a, \varphi) - R(a, \varphi) = L(a, X_-^2) - R(a, X_-^2). \quad (3.39)$$

Let us now prove that for  $a \in [a_-, \sqrt{2}]$ ,  $\Psi'(a) \leq 0$ . We first compute

$$\Psi'(a) = \partial_a L(a, X_-^2) - \partial_a R(a, X_-^2) + 2X_- \partial_a X_- (\partial_\varphi L(a, X_-^2) - \partial_\varphi R(a, X_-^2)).$$

By minimality of  $X_-$ ,  $\partial_\varphi L(a, X_-^2) - \partial_\varphi R(a, X_-^2) = 0$  so that

$$\Psi'(a) = \partial_a L(a, X_-^2) - \partial_a R(a, X_-^2) = \frac{3}{a^2(\sqrt{2} - 1)^2} \left( 1 - X_-^3 - (1 - X_-^2)^{3/2} - \frac{3}{a} X_-^2 (1 - X_-^2) \right).$$

A simple computation shows that  $X_-^2 = \frac{1}{2}(1 - 2a\sqrt{2 - a^2})$  so that  $\Psi' \leq 0$  is equivalent to

$$\frac{1}{2\sqrt{2}}(1 - 2a\sqrt{2 - a^2})^{3/2} + \frac{1}{2\sqrt{2}}(1 + 2a\sqrt{2 - a^2})^{3/2} + \frac{3}{4a}(1 - a^2(2 - a^2)) \geq 1.$$

This indeed holds since for  $a \in [a_-, \sqrt{2}]$ ,

$$\begin{aligned} & \frac{1}{2\sqrt{2}}(1 - 2a\sqrt{2 - a^2})^{3/2} + \frac{1}{2\sqrt{2}}(1 + 2a\sqrt{2 - a^2})^{3/2} + \frac{3}{4a}(1 - a^2(2 - a^2)) \geq \\ & \frac{1}{2\sqrt{2}}((1 - 2a_-\sqrt{2 - a_-^2})^{3/2} + 1) + \frac{3}{4\sqrt{2}}(1 - a_-^2(2 - a_-^2)) \simeq 1.02 > 1. \end{aligned}$$

Therefore,  $\Psi' \leq 0$  and thus for  $a \in [a_-, \sqrt{2}]$ , by (3.37)

$$\Psi(a) \geq \Psi(\sqrt{2}) > 0,$$

which ends the proof of (3.33). □

**Remark 3.18.** From (3.31), one could infer the simpler bound  $T_* \geq \frac{1}{3\sqrt{2}(\sqrt{2}+1)}$  which leads to  $a \geq 1$ . For  $a \in [1, \sqrt{2}]$ , we still have (3.38) and (3.39). Numerically, it seems that  $\Psi$  is decreasing not only in  $[a_-, \sqrt{2}]$  but actually on the whole  $[1, \sqrt{2}]$ . We were unfortunately not able to prove this fact which would have yielded a more elegant proof of (3.33).

## 4 Applications and open problems

In this section we use Theorem 1.1 to characterize the symmetric minimizers of

$$\min\{\mathcal{E}(\mu) : \mu_{\pm T} = \varphi/L dx \llcorner [-L/2, L/2]\}, \quad (4.1)$$

at least for  $T$  large enough. By rescaling, it is enough to consider  $\varphi = L = 1$ .

**Theorem 4.1.** *For  $T \in [\frac{1}{4}, \frac{1}{4(2\sqrt{2}-2)})$ , the unique symmetric minimizer of (4.1) is equal in  $[0, T]$  to  $S_{-1/4}(\mu^1) + S_{1/4}(\mu^1)$ , where  $\mu^1$  is equal to the unique minimizer of  $E(T, 1/2)$  given by Corollary 1.2 in  $[0, T]$  and to its symmetric in  $[-T, 0]$ . For  $T > 4(2\sqrt{2} - 2)$ , it is given by the unique minimizer of  $E(T)$  given by Theorem 1.1 in  $[0, T]$  and to its symmetric in  $[-T, 0]$ .*

*Proof.* By Proposition 2.7, we know that a symmetric minimizer exists. Let  $\mu$  be such a minimizer. Thanks to the symmetry, we can restrict ourselves to study its structure in  $[0, T]$ . We let

$$\mathcal{E}^+(\mu) := \int_0^T \#\{\text{supp } \mu_t\} + \sum_i \varphi_i |\dot{X}_i|^2 dt.$$

Let  $\mu_0 = \sum_{i=1}^N \varphi_i \delta_{X_i}$ . We first claim that  $X_i = \bar{X}_i$ , where as before,  $\bar{X}_i = \frac{-1}{2} + \sum_{j<i} \varphi_j + \frac{\varphi_i}{2}$ . Indeed, applying the same shear as in (3.1), we obtain by minimality of  $\mu$ ,

$$\mathcal{E}^+(\mu) \geq \mathcal{E}^+(\hat{\mu}) + \frac{1}{T} \sum_{i=1}^N \varphi_i |X_i - \bar{X}_i|^2 \geq \mathcal{E}^+(\mu) + \frac{1}{T} \sum_{i=1}^N \varphi_i |X_i - \bar{X}_i|^2,$$

where the inequality could arise from a decreasing of the number of branches after the shear. This proves the claim. For  $i = 1, \dots, N$ , let  $\mu^{+,i}$  be the forward system emanating from  $(X_i, 0)$ . Then, by monotonicity of the traces (Proposition 2.7),  $\mu_T^{+,i} = dx \llcorner [\bar{X}_i - \varphi_i/2, \bar{X}_i + \varphi_i/2]$  and thus, by the no-loop property

$$\mathcal{E}^+(\mu) = \sum_{i=1}^N \mathcal{E}^+(\mu^{+,i}) = \sum_{i=1}^N E(T, \varphi_i). \quad (4.2)$$

Moreover,  $\mu^{+,i} = S_{\bar{X}_i}(\mu^i)$  where  $\mu^i$  is some minimizer of  $E(T, \varphi_i)$ . Let now  $T \geq 1/4$ . Since  $\varphi_i \leq 1$ , we have  $\varphi_i^{-3/2} T \geq 1/4$  and thus by Corollary 1.2,

$$\mathcal{E}^+(\mu) = \frac{1}{2 - \sqrt{2}} \sum_{i=1}^N \varphi_i^{3/2} + NT.$$

For fixed  $N$ , this is minimized by  $\varphi_i = 1/N$  so that

$$\mathcal{E}^+(\mu) = \frac{1}{2 - \sqrt{2}} N^{-1/2} + NT.$$

The function  $x \rightarrow \frac{1}{2 - \sqrt{2}} x^{-1/2} + xT$  is minimized by  $x_{opt} = (2T(2 - \sqrt{2}))^{-2/3}$ . Since  $x_{opt} < 2$  for  $T \geq \frac{1}{4(2\sqrt{2}-2)}$  and  $3 > x_{opt} > 2$  for  $\frac{1}{4} \leq T < \frac{1}{4(2\sqrt{2}-2)}$ , this concludes the proof.  $\square$

As already explained in the introduction, this theorem is not completely satisfactory. Indeed, physically, the most significant case is  $T \ll 1$  (where many microstructures should appear), which is not covered by Theorem 4.1. However, if we could prove the following conjecture,

### Conjecture

For  $T < T_*$  every minimizer of  $E(T)$  branches at time zero (or equivalently  $E(T - \varepsilon) < E(T) - \varepsilon$  for  $\varepsilon$  small enough),

then the picture would be almost complete. Indeed, in that case, arguing as in the proof of Theorem 4.1, we would have that every symmetric minimizer  $\mu$  with  $\mu_0 = \sum_{i=1}^N \varphi_i \delta_{X_i}$  would satisfy (4.2). Now for  $1 \leq i \leq N$ , let  $\varphi_{i,1}, \dots, \varphi_{i,N_i}$  be the  $N_i$  branches starting from  $(0, X_i)$ . As in the proof of (3.4), we would have

$$E(T, \varphi_i) = \sum_{k=1}^{N_i} E(T, \varphi_{i,k}, \bar{X}_{i,k}),$$

where  $\bar{X}_{i,k} := -\frac{1}{2} + \sum_{j < k} \varphi_{i,j} + \frac{\varphi_{i,k}}{2}$ . Since the minimizer corresponding to  $E(T, \varphi_{i,k}, \bar{X}_{i,k})$  cannot branch at time zero, we would have (if the conjecture holds) that  $\varphi_{i,k}^{-3/2} T \geq 1/4$  so that Corollary 1.2 applies and the structure of the minimizers would be fully determined (recall also Lemma 2.8). Let us point out that our conjecture would be for instance implied by the convexity of  $T \rightarrow E(T)$ .

## Acknowledgment

I warmly thank F. Otto for many useful discussions and constant support as well as A. Lemenant for a careful reading of a preliminary version of the paper. The hospitality of the Max Planck Institute for Mathematics in the Sciences of Leipzig, where part of this research was carried is gratefully acknowledged. Part of this research was funded by the program PGMO of the FMJH through the project COCA.

## References

- [1] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.
- [2] M. Bernot, V. Caselles, and J.-M. Morel. *Optimal transportation networks*, volume 1955 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Models and theory.

- [3] A. Brancolini and S. Solimini. Fractal regularity results on optimal irrigation patterns. *J. Math. Pures Appl.*, 102:854–890, 2014.
- [4] A. Brancolini and B. Wirth. Optimal energy scaling for micropatterns in transport networks. *SIAM J. Math. Anal.*, 49(1):311–359, 2017.
- [5] R. Choksi, R. V. Kohn, and F. Otto. Domain branching in uniaxial ferromagnets: a scaling law for the minimum energy. *Comm. Math. Phys.*, 201:61–79, 1999.
- [6] S. Conti. A lower bound for a variational model for pattern formation in shape-memory alloys. *Cont. Mech. Thermod.*, 17:469–476, 2006.
- [7] S. Conti, J. Diermeier, and B. Zwicknagl. Deformation concentration for martensitic microstructures in the limit of low volume fraction. *Calc. Var. Partial Differential Equations*, 56(1):Art. 16, 24, 2017.
- [8] S. Conti, M. Goldman, F. Otto, and S. Serfaty. A branched transport limit of the Ginzburg-Landau functional. *preprint*, 2017.
- [9] S. Conti, F. Otto, and S. Serfaty. Branched microstructures in the Ginzburg-Landau model of type-I superconductors. *SIAM J. Math. Anal.*, 48(4):2994–3034, 2016.
- [10] R. V. Kohn and S. Müller. Surface energy and microstructure in coherent phase transitions. *Comm. Pure Appl. Math.*, 47:405–435, 1994.
- [11] A. Marchese and A. Massaccesi. An optimal irrigation network with infinitely many branching points. *ESAIM: COCV*, 2014.
- [12] F. Otto and T. Viehmann. Domain branching in uniaxial ferromagnets: asymptotic behavior of the energy. *Calc. Var. Partial Differential Equations*, 38(1-2):135–181, 2010.
- [13] E. Paolini, E. Stepanov, and Y. Teplitskaya. An example of an infinite steiner tree connecting an uncountable set. *Adv. Calc. Var.*, 2015.
- [14] T. Viehmann. *Uniaxial Ferromagnets*. PhD thesis, Universität Bonn, 2009.
- [15] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.
- [16] B. Zwicknagl. Microstructures in low-hysteresis shape memory alloys: Scaling regimes and optimal needle shapes. *Arch. Ration. Mech. Anal.*, 213:355–421, 2014.