A STUDY OF THE DUAL PROBLEM OF THE ONE-DIMENSIONAL L^∞-OPTIMAL TRANSPORT PROBLEM WITH APPLICATIONS

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Abstract. The Monge-Kantorovich problem for the $\mathcal{W}_\infty$ distance presents several peculiarities. Among them the lack of convexity and then of a direct duality. We study in dimension 1 the dual problem introduced by Barron-Bocea-Jensen. We construct a couple of Kantorovich potentials which is "as less trivial as possible". More precisely, we build a potential which is non constant around any point that the plan which is locally optimal moves at maximal distance. As an application, we show that the set of points which are displaced to maximal distance by a locally optimal transport plan is minimal.

1. Introduction

Given two probability measures μ, ν on $\mathbb{R}^d$, the infinite Wasserstein distance between μ and ν is defined as

$$\mathcal{W}_\infty(\mu,\nu) := \min_{\gamma\in\Pi(\mu,\nu)} \gamma - \underset{\mathbb{R}^d\times\mathbb{R}^d}{\text{ess.sup}}\, |x-y|, \tag{1.1}$$

where $\Pi(\mu,\nu)$ denotes the set of positive measures on $\mathbb{R}^d$ whose first and second marginals are μ, ν respectively. This distance is the natural limit as $p \to \infty$ of the more common Kantorovich-Wasserstein distances $\mathcal{W}_p$.

Problem (1.1) above was first studied in [7] where it was observed that, in spite of its proximity with the $\mathcal{W}_p$ distances, it presents several peculiarities. The two most striking phenomena are the lack of linearity or even convexity of (1.1) with respect to γ (in fact the functional $\gamma \mapsto \|y-x\|_{L^\infty_\gamma(\mathbb{R}^d)^2}$ is only level-convex), and the non-uniqueness of the minimizer: indeed, one may guess that, from any optimal plan γ, any small perturbation of γ around a point (x,y) which is not moved at maximal distance will provide a new and different optimal transport plan. It is therefore necessary to introduce a suitable solution of *local solution* of (1.1). The following definition turned out to be the more suitable.

Definition 1.1. A transport plan $\gamma \in \Pi(\mu,\nu)$ is a *restrictable solution* of (1.1) if any positive and nonzero Borel measure γ' on $\mathbb{R}^{2d}$ that is majorized by γ is a solution to the problem

$$\inf_{\lambda\in\Pi(\mu',\nu')} \lambda - \underset{\mathbb{R}^d\times\mathbb{R}^d}{\text{ess.sup}}\, |y-x|,$$

where $\mu' = \pi^1_\sharp\gamma'$ and $\nu' = \pi^2_\sharp\gamma'$.

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It is likely that a small "local modification" of a restrictable solution will not be another restrictable solution. From the analyis of the problem (1.1) as a limit of the classical optimal transport problem with cost $c(x,y) = |y-x|^p$ as $p \to +\infty$, it turns out that the notion of restrictable solution is equivalent to the one of *infinite cyclical monotonicity* of a transport plan, whose definition is the following:

Definition 1.2. A transport plan $\gamma \in \Pi(\mu,\nu)$ is ∞-cyclically-monotone (∞-**cm** in the paper) if, for all $n \in \mathbb{N}$ and $(x_0,y_0),\dots,(x_n,y_n) \in \operatorname{spt}(\gamma)$, it holds

$$\max_i\{|x_i - y_i|\} \leq \max_i\{|x_i - y_{i+1}|\},$$

where, as usual, we use the notation $y_{n+1} = y_0$.

The infinite cyclical monotonicity is the L^∞ version of the c-cyclical monotonicity, which is a fundamental notion in the classical theory of optimal transportation. The equivalence between the two definitions above, as well as the existence of a transport plan which satisfies them, is proven in [7], see Theorems 3.2 and 4.4 therein (we also mention [15] where these results are generalized to the case where $|y-x|$ is replaced by more general cost functions).

The lack of convexity with respect to γ of the energy

$$\gamma \mapsto \|y-x\|_{L^\infty_\gamma}$$

does not allow to extend the widespread use of convex duality to this transport problem. In fact, this obstacle brought to use the cyclical monotonicity together with a regularity property of transport plans and geometric measure theory arguments to produce a far reaching alternative method for the study of optimal transport problems (see, for example[8, 9, 14, 3]).

In the classical theory of optimal transportation, the duality method plays a crucial role not only in the proof of existence of optimal transport maps but also in the characterizations of this map. Heuristically, the reasoning which leads to the characterizations is the following: from the duality formula

$$\inf\left\{\int c\,\mathrm{d}\gamma \,:\, \gamma \in \Pi(\mu,\nu)\right\} = \sup\left\{\int \varphi\,\mathrm{d}\mu + \int \psi\,\mathrm{d}\nu \,:\, \varphi(x)+\psi(y) \leq c(x,y)\right\},$$

one can deduce that, if γ and (φ,ψ) are respectively a minimizer and a maximizing pair for these problems, the condition $\varphi \oplus \psi \leq c$ is saturated (in sense that equality holds) on the support of γ. Therefore, if c and φ are regular enough, one can deduce

$$\nabla_1 c(x,y) = \nabla\varphi(x) \quad \text{for } \gamma\text{-a.e. } (x,y) \in (\mathbb{R}^d)^2.$$

If the cost function c is such that this relationship may be inverted, one concludes that an optimal transport maps exists and is characterized by φ (namely, it is exactly the map $x \mapsto (\nabla_1 c(x,\cdot))^{-1}(\nabla\varphi(x))$). Following these arguments, Brenier [4, 5] proved the existence of an optimal map for the quadratic cost and that this map is induced by a convex potential; it must then be a solution of the *Monge-Ampère equation*, which serves as basis for a whole regularity theory of optimal transport maps, *cf.* [16, Chapter 4]. Similar results have been generalized to a much larger class of cost functions, first for strictly convex costs with respect to the difference [12, 13, 6] and later on for those satisfying the so-called *twist condition*, see [11, 10].

Altought the functional $\gamma \mapsto \|y-x\|_{L^\infty_\gamma}$ is not convex, it is still level-convex (in the sense that the level sets of the functional are convex sets and this, sometimes, goes under the name of quasi-convex). Then it is still possible to consider a (sort of) duality theory. This theory has been recently introduced and investigated by Barron, Bocea and Jensen in [2] where it is proven that the minimal value of (1.1) is equal to

$$\inf_{\lambda\geq 0}\left(\sup\left\{\lambda+\int\varphi\,\mathrm{d}\mu+\int\psi\,\mathrm{d}\nu\ :\ \begin{array}{c}\varphi(x)+\psi(y)\leq 0\\ \text{whenever } |y-x|\leq\lambda\end{array}\right\}\right), \tag{1.2}$$

and that the infimum with respect to λ in (1.2) is attained for $\lambda=\mathcal{W}_\infty(\mu,\nu)$ (see the next section for more details). It follows that $\mathcal{W}_\infty(\mu,\nu)$ is the smallest λ such that

$$\sup\left\{\int\varphi\,\mathrm{d}\mu+\int\psi\,\mathrm{d}\nu\ :\ \varphi(x)+\psi(y)\leq 0 \text{ whenever}|y-x|\leq\lambda\right\}=0.$$

Unfortunatelly, the supremum with respect to (φ,ψ) always admits a trivial solution $\varphi=\psi=0$. It is therefore important to look for the "most interesting solutions" of this dual problem, and to study what information can be obtained from them about the optimal plans for (1.1).

The aim of this paper is to provide a solution of the dual problem (1.2) which is precisely "not constant on an as large as possible set" in the one-dimensional case. More precisely, let us notice that, if (φ,ψ) is a solution of (1.2) for $\lambda=\mathcal{W}_\infty$ and γ is a fixed optimal transport plan for (1.1), it is (at least heuristically) easy to check that φ is locally constant around any point x which is sended by γ at smaller distance that $\mathcal{W}_\infty$. In the present paper, we construct a pair (φ,ψ) of solutions of (1.2) (see Theorem 3.6) from a fixed infinite cyclically monotone transport plan $\overline{\gamma}$, and such that φ is BV function whose derivative is exactly supported on the set of points which are moved by $\overline{\gamma}$ at maximal distance: in this sense, our solution has a maximal non-constant set. As an application, we will prove that there exists a set of points of the support of the source measure which are sended at maximal distance by all the optimal transport plans for (1.2); in particular, this set is shared by all the inifinitely cyclically monotone transport plans. The proof of this last result is based on the properties of the non trivial solution (φ,ψ) of the dual problem that we construct.

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2. Notations and the dual problem

In this section, we quickly collect all the notations and known facts of measure theory and optimal transportation that we will use throughout the paper.

Let X and Y be two Polish spaces, and μ, ν be two positive measures on X, Y whose total masses are finite and equal. We denote $\Pi(\mu,\nu)$ the set of *transport plans* from μ to ν, that is, the set of positive measures on $X \times Y$ satisfying

$$\text{for any Borel sets } A \subset X \text{ and } B \subset Y,\ \gamma(A \times Y) = \mu(A) \text{ and } \gamma(X \times B) = \nu(B);$$

recall that this constraint can be reforumated as

$$\text{for any } (u,v) \in C_b(X) \times C_b(Y), \qquad \begin{array}{r} \displaystyle\iint u(x)\,\mathrm{d}\gamma(x,y) = \int u\,\mathrm{d}\mu \\ \text{and} \displaystyle\iint v(y)\,\mathrm{d}\gamma(x,y) = \int v\,\mathrm{d}\nu. \end{array}$$

In our settings, $X = Y = \mathbb{R}^d$ (and $d = 1$ along almost the whole paper) and the "primal" problem that we consider is the minimization of the supremal functional (1.1) above. We will denote by $\mathcal{O}_\infty(\mu,\nu)$ the set of its minimizers. The definition of an *infinitely cyclically monotone transport plan* has been recalled in the introduction, see Definition 1.2. We will use by simplicity the abreviation "∞-cm plan"; recall that, from [7, Theorems 3.2 and 4.4], we know that at least such a plan exists provided that μ,ν are both compactly supported in $\mathbb{R}^d$, and that these plans are exactly those which are *restrictable solutions* of (1.1), in sense given by Definition 1.1.

The following "dual problem" is introduced in Theorem 2.3 and Remark 2.4 of [2].

Theorem 2.1 (Duality formula for the L^∞-optimal transport problem)**.** *Let $\mu,\nu \in \mathcal{P}(\mathbb{R}^d)$ be compactly supported. For any $\lambda > 0$, denote by $\mathcal{U}_\lambda$ the set of couples $(\varphi,\psi) \in L^1_\mu \times L^1_\nu$ such that $\varphi(x) + \psi(y) \leq 0$ μ-a.e. x and ν-a.e. y such that $|y - x| \leq \lambda$. Then*

$$\mathcal{W}_\infty(\mu,\nu) = \inf_{\lambda \geq 0} \left(\sup \left\{ \lambda + \int \varphi\,\mathrm{d}\mu + \int \psi\,\mathrm{d}\nu \,:\, (\varphi,\psi) \in \mathcal{U}_\lambda \right\} \right).$$

For shorter notations we introduce $\lambda_C := \mathcal{W}_\infty(\mu,\nu)$ where C is for critical. As a consequence

$$\max \left\{ \int \varphi\,\mathrm{d}\mu + \int \psi\,\mathrm{d}\nu \,:\, (\varphi,\psi) \in \mathcal{U}_{\lambda_C} \right\} = 0 \tag{2.1}$$

and we observe, given an optimal transport plan γ for the primal problem (1.1), a pair (φ,ψ) is optimal for the dual problem (2.1) if and only if the equality $\varphi(x) + \psi(y) = 0$ holds for γ-a.e. (x,y). Such functions φ,ψ will be then called *Kantorovich potentials.* We notice that among the maximizers for problem (2.1) above there are always $\varphi \equiv 0$ and $\psi \equiv 0$.

Remark 2.2*.* More generally, for any maximizer (φ,ψ) and any optimal plan γ (which is therefore concentrated on the set $\{|y - x| \leq \lambda_C\}$), by using the fact that

$$\varphi(x) + \psi(y) \leq 0 \text{ whenever } |x - y| \leq \lambda_C$$
$$\text{and} \quad \varphi(x) + \psi(y) = 0 \quad \gamma\text{-a.e.},$$

one can prove that if $\overline{x} \in \operatorname{spt}(\mu)$ is such that

$$\max_{(\overline{x},y) \in \operatorname{spt}(\gamma)} |\overline{x} - y| = \lambda < \lambda_c,$$

then φ is μ-a.e. costant in $B(\overline{x}, \lambda_c - \lambda)$.

3. Construction of non-trivial potentials

In this section, we construct in the one-dimensional case a couple of Kantorovich potentials (φ, ψ) which is "as less trivial as possible", meaning that φ is not locally constant on the largest possible set. Our assumptions on the datas are simply that μ, ν are two probability measures on $\mathbb{R}$ with compact support and having no atom.

We will moreover the following notations:

- as before, we denote by λ_C the optimal value of the problem (1.1), and by $\mathcal{O}_\infty$ the set of optimizers;
- for any $\gamma \in \mathcal{O}_\infty$, we introduce the sets

$$\mathcal{M}_\gamma^+ := \{(x,y) \in \operatorname{spt}(\gamma) \,:\, y - x = \lambda_C\},$$
$$\mathcal{M}_\gamma^- := \{(x,y) \in \operatorname{spt}(\gamma) \,:\, y - x = -\lambda_C\},$$
$$\mathcal{M}_\gamma := \mathcal{M}_\gamma^+ \cup \mathcal{M}_\gamma^-$$
$$\text{and} \quad M_\gamma^+ = \pi_1(\mathcal{M}_\gamma^+),\ M_\gamma^- = \pi_1(\mathcal{M}_\gamma^-),\ M_\gamma = \pi_1(\mathcal{M}_\gamma),$$

  where π_1 is the projection with respect to the first variable, that is, $\pi_1(x,y) = x$ for any $(x,y) \in \mathbb{R}^2$. Therefore, $M_{\bar\gamma}^+, M_\gamma^-$ are the sets of points which are moved by γ at maximal distance.

We notice that $\mathcal{M}_\gamma^+$, $\mathcal{M}_\gamma^-$, $\mathcal{M}_\gamma$ and M_γ^+, M_γ^-, M_γ are compact subsets of $\mathbb{R}^2$ and $\mathbb{R}$ respectively; moreover, the sets M_γ^+ and M_γ^- are never simultaneously empty (and so is not M_γ).

Remark 3.1. When $A \subset \mathbb{R}$ is a compact set (in particular, when $A = \operatorname{spt}\mu, \operatorname{spt}\nu$ or $M_{\bar\gamma}^{\pm}$), a point $\bar x \in A$ will be called *right-extreme point of* A (resp. *left-extreme point* of A) if there exists $\delta > 0$ such that the interval $(\bar x, \bar x + \delta)$ (resp. $(\bar x - \delta, \bar x)$) does not meet A. We notice that since A is closed, the set $\mathbb{R} \setminus A$ may be written as

$$\mathbb{R} \setminus A = \bigcup_i (a_i, b_i)$$

where the union is taken on an at most countable set of indexes i. In particlar, the left-extreme and right-extreme points are all part of the a_i, b_i, so that such points are at most countably many: since μ and ν have no atom, the sets of left-extreme or right-extreme points of any closed set have always zero mass for μ, ν.

3.1. Properties of the maximal displacement set for ∞-cm plans.

In this paragraph, we fix $\bar\gamma$ an ∞-cm plan. We can then prove some additional properties of the maximal dispacement set.

Lemma 3.2. *For any $\bar x \in M_{\bar\gamma}^+$, the intersection $(\bar x, \bar x + 2\lambda_C) \cap M_{\bar\gamma}^-$ is empty. Similarly, if $\bar x \in M_{\bar\gamma}^-$, then $(\bar x - 2\lambda_c, \bar x) \cap M_{\bar\gamma}^+ = \emptyset$*

Proof. We give a proof of the first case, being the second analogous. Assume by contradiction that there exists some $z \in M_{\bar\gamma}^-$ such that $\bar x < z < \bar x + 2\lambda_C$. Both $(z, z - \lambda_C)$ and $(\bar x, \bar x + \lambda_C)$ belong to the support of $\overline{\gamma}$, so that the ∞-cm property implies

$$\lambda_C = \max\Big(|(z - \lambda_C) - z|, |(\bar x + \lambda_C) - \bar x|\Big) \le \max\Big(|(z - \lambda_C) - \bar x|, |(\bar x + \lambda_C) - z|\Big). \quad (3.1)$$

On the other hand, the fact that $\bar{x} < z < \bar{x} + 2\lambda_C$ implies immediately that the right-handside in (3.1) is smaller than λ_C, a contradiction. □

Lemma 3.3. *The set $M^+_{\bar{\gamma}} \cap M^-_{\bar{\gamma}}$ is finite.*

Proof. Since $M^+_{\bar{\gamma}} \cap M^-_{\bar{\gamma}}$ is included in the support of μ, which is bounded, it is sufficient to prove that all of its points are isolated, and this is true since, by Lemma 3.2, above, if $\bar{x} \in M^+_\gamma \cap M^-_\gamma$, then $(M^+_\gamma \cap M^-_\gamma) \cap (\bar{x}, \bar{x}+2\lambda_C) = \emptyset$ and $(M^+_\gamma \cap M^-_\gamma) \cap (\bar{x}-2\lambda_c, \bar{x}) = \emptyset$. □

3.2. **Construction of a non-trivial potential.** In this paragraph, starting from an ∞-cm optimal transport plan $\bar{\gamma}$, we introduce a couple (φ, ψ) of non-trivial solutions of the dual problem (2.1). The couple enjoys the property that the points around which φ is not locally constant are exactly those that $\bar{\gamma}$ moves at maximal distance. We will need the following lemma of measure theory.

Lemma 3.4. *There exists two postive measures ρ^+, ρ^- on $\mathbb{R}$, having finite mass and such that, denoting by $\rho = \rho^+ - \rho^-$, the following properties are satisfied:*

(i) *the support of ρ^+ is $M^+_{\bar{\gamma}}$, the one of ρ^- is $M^-_{\bar{\gamma}}$ and the one ρ is $M_{\overline{\gamma}}$;*
(ii) *for any point of $M^+_{\bar{\gamma}}$ which is left-isolated or righ-isolated in $M^+_{\bar{\gamma}}$, we have $\rho(\{\bar{x}\}) > 0$.*

Proof. By lemma 3.3, the set $M^+_{\bar{\gamma}} \cap M^-_{\bar{\gamma}}$ is finite, we denote by $\{z_1, \dots, z_N\}$ the (possibly empty) set of its elements. We also select two at most countable (and also possibly empty) families $(x_i)_i$, $(y_j)_j$ of points of $M^+_{\bar{\gamma}} \setminus M^-_{\bar{\gamma}}$, and of $M^-_{\bar{\gamma}} \setminus M^+_{\bar{\gamma}}$ which are dense in $M^+_{\bar{\gamma}} \setminus M^-_{\bar{\gamma}}$ and $M^-_{\bar{\gamma}} \setminus M^+_{\bar{\gamma}}$ respectively. Moreover, by Remark 3.1, the left-isolated (resp. right-isolated) points of $M^+_{\bar{\gamma}}$ are at most countably many, so we can assume that all of them are part of the family $(x_i)_i$ or $(z_k)_k$. It is then straightforward to check that the measures

$$\rho^+ := \sum_i 2^{-i}\delta_{x_i} + \sum_{k=1}^N \delta z_k \quad \text{and} \quad \rho^- := \sum_j 2^{-j}\delta_{y_j} + \sum_{k=1}^N 2\delta_{z_k}$$

satisfy the required property. □

Define, now, a candidate pair of Kantorovich potentials.

Definition 3.5. For $x, y \in \mathbb{R}$, we define

$$\varphi_r(x) := -\rho((-\infty, x]) \text{ and } \psi_r(x) := \inf\big\{ -\varphi_r(x) : x \in [y-\lambda_C, y+\lambda_C]\big\}.$$

Theorem 3.6. *The functions φ_r and ψ_r defined above satisfy the following properties:*

(i) *φ_r has bounded variation (in particular, it has a left-limit and a right-limit at any point) and is right-continuous;*
(ii) *the support of $(\varphi_r')^-$ is exactly $M^+_{\bar{\gamma}}$, and the support of φ_r' is exactly $M_{\overline{\gamma}}$;*
(iii) *for any point $\bar{x} \in M^+_{\bar{\gamma}}$ which is left-isolated or right-isolated in $M^+_{\bar{\gamma}}$, we have*
$$\lim_{\substack{x \to \bar{x} \\ x < \bar{x}}} \varphi(x) > \varphi_r(\bar{x});$$
(iv) *the couple (φ_r, ψ_r) is a Kantorovich potential, i.e. a maximizer of* (2.1).

Proof. Since φ_r is the cumulative distribution function of the measure $-\rho$, it is a BV function whose distributional derivative is the measure $-\rho$; from this fact and from the definition of ρ, the properties (i) and (ii) are straightforward to prove. As for (iii), let $\bar{x}$ be an isolated point of $M^+_{\bar\gamma}$: from Property (ii) of Lemma 3.4, such a point is an atom of ρ^+ and not of ρ^-. We then write for any $x < \bar{x}$

$$\varphi_r(\bar{x}) - \varphi_r(x) = -\rho((x, \bar{x}]) = -\rho^+(\{\bar{x}\}) - \rho((x, \bar{x})),$$

the fist term being negative and the second one vanishing as $x \to \bar{x}$, which proves (iii).

It remains to show that (φ_r, ψ_r) is a pair of Kantorovich potentials. First of all, from the definition of ψ_r, it follows that

$$\text{for any } (x, y) \in \mathbb{R}^2 \text{ with } |y - x| \le \lambda_C, \quad \varphi_r(x) + \psi_r(y) \le 0.$$

This proves that (φ_r, ψ_r) is an admissible couple for the dual problem. We moreover claim that

$$\varphi(\bar{x}) + \psi(\bar{y}) = 0 \text{ for } \overline{\gamma}\text{-a.e. } (x, y); \tag{3.2}$$

more precisely, we will prove that, if $(\bar{x}, \bar{y}) \in \operatorname{spt}\mu \times \operatorname{spt}\nu$ is such that (3.2) does not hold, then either $\bar{x}$ belongs to an at most countable subset of $\operatorname{spt}\mu$ or $\bar{y}$ belongs to an at most countable subset of $\operatorname{spt}\nu$. This will be enough to prove (iv) since it implies that

$$\int \varphi_r \, \mathrm{d}\mu + \int \psi_r \, \mathrm{d}\nu = \int (\varphi_r(x) + \psi_r(y)) \, \mathrm{d}\overline{\gamma}(x, y) = 0,$$

so that (φ_r, ψ_r) is optimal.

Let then $(\bar{x}, \bar{y})$ be a point of $\operatorname{spt}(\overline{\gamma})$ such that $\varphi_r(\bar{x}) + \psi_r(\bar{y}) < 0$. Then:

$$-\varphi_r(\bar{x}) > \psi_r(\bar{y}) = \inf\big\{ -\varphi_r(x) \, : \, x \in [\bar{y} - \lambda_C, \bar{y} + \lambda_C]\big\}.$$

We deduce that there exists $z \in [\bar{y} - \lambda_C, \bar{y} + \lambda_C]$ such that $\varphi_r(z) > \varphi_r(\bar{x})$, and we distinguish the cases $z < \bar{x}$ and $z > \bar{x}$.

First case: $z < \bar{x}$. In this case, the definition of z reads

$$\varphi_r(\bar{x}) - \varphi_r(z) = -\rho((z, \bar{x}]) < 0.$$

In particular, there exists $z' \in (z, \bar{x}]$ which belongs to $M^+_{\bar\gamma}$, so that $(z', z' + \lambda_C) \in \operatorname{spt}(\overline{\gamma})$. Now we observe that

$$\max\big\{|\bar{y} - \bar{x}|, |(z' + \lambda_C) - z'|\big\} = \lambda_C \tag{3.3}$$

and consider

$$\max\big\{|\bar{y} - z'|, \ |(z' + \lambda_C) - \bar{x}|\big\}. \tag{3.4}$$

By the cyclical monotonicity of $\bar\gamma$, the max in (3.4) must be at least λ_C too, meaning that either $|\bar{y} - z'|$ or $|(z' + \lambda_C) - \bar{x}|$ is larger or equal to λ_C. Now we observe that:

- From the fact that $z \in (z, \bar{x}]$ and that $z, \bar{x} \in [\bar{y} - \lambda_C, \bar{y} + \lambda_C]$, we have

$$-\lambda_C < z' - \bar{y} \le \bar{x} - \bar{y} \le \lambda_C.$$

  Therefore, the inequality $|\bar{y} - z'| \ge \lambda_C$ is only possible if $z' = \bar{x} = \bar{y} + \lambda_C$. This implies that $\bar{x}$ belongs to $M^+_{\bar\gamma} \cap M^-_{\bar\gamma}$, which is finite by Lemma 3.3.

- On the other hand, using again that $z \in (z, \bar{x}]$ and $z, \bar{x} \in [\bar{y} - \lambda_C, \bar{y} + \lambda_C]$, we observe
$$-\lambda_C \leq \bar{y} - \bar{x} < z' + \lambda_C - \bar{x} \leq \lambda_C.$$
The inequality $|(z' + \lambda_C) - \bar{x}| \geq \lambda_C$ can then only hold if $\bar{x} = z'$. In this case, both $(\bar{x}, \bar{y})$ and $(\bar{x}, \bar{x} + \lambda_C)$ belong to the support of $\overline{\gamma}$. As a consequence of ∞-cm condition, for every $(x, y') \in \operatorname{spt}\overline{\gamma}$ with $\bar{y} < y' < \bar{x} + \lambda_C$ it must hold $x = \bar{x}$ (in fact, if $x < \bar{x}$ the couples (x, y') and $(\bar{x}, \bar{y})$ would violate the condition while if $\bar{x} < x'$ the bad couples would be (x, y') and $(\bar{x}, \bar{x} + \lambda_C)$). It follows that
$$\begin{aligned} 0 = \mu(\{\bar{x}\}) &= \overline{\gamma}(\{\bar{x}\} \times \mathbb{R}) \\ &\geq \overline{\gamma}(\{\bar{x}\} \times (\bar{y}, \bar{x} + \lambda_C)) \\ &= \overline{\gamma}(\mathbb{R} \times (\bar{y}, \bar{x} + \lambda_C)) \\ &= \nu((\bar{y}, \bar{x} + \lambda_C)). \end{aligned}$$
Then $\bar{y}$ is a right-isolated point of $\operatorname{spt}\nu$ which, as discussed in Remark 3.1, are at most countably many.

Second case: $z > \bar{x}$. This case is treated in a very similar way as the previous one. First of all, we start by deducing from the inequality $\varphi_r(z) > \varphi_r(\bar{x})$ that there exists $z' \in (\bar{x}, z]$ such that $(z', z' - \lambda_C) \in \operatorname{spt}(\overline{\gamma})$. Then, by the ∞-cm condition,
$$\lambda_C = \max\left\{|\bar{y} - \bar{x}|, |(z' - \lambda_C) - z'|\right\} \leq \max\left\{|\bar{y} - z'|, |(z' - \lambda_C) - \bar{x}|\right\}, \tag{3.5}$$
so that either $|\bar{y} - z'|$ or $|z' - \lambda_C - \bar{x}|$ is at least equal to λ_C. On the other hand, the fact that $z' \in (\bar{x}, z]$ with $\bar{y} - \lambda_C \leq \bar{x} \leq z \leq \bar{y} + \lambda_C$ enforces
$$-\lambda_C < z' - \lambda_C - \bar{x} \leq \bar{y} - \bar{x} \leq \lambda_C. \tag{3.6}$$
$$\text{and } -\lambda_C < z' - \bar{y} \leq \lambda_C \tag{3.7}$$
Therefore, the condition (3.5) is only satisfied in the two following cases:

- there is equality in the two last inequalities of (3.6): in this case, we have simultaneously $z' = \bar{y} + \lambda_C$ and $\bar{x} = \bar{y} - \lambda_C$. Therefore, $(\bar{y} - \lambda_C, \bar{y})$ and $(\bar{y} + \lambda_C, \bar{y})$ both belong to the support of $\overline{\gamma}$: this means that $\bar{y} \in M^+_{\tilde{\gamma}} \cap M^-_{\tilde{\gamma}}$, where $\tilde{\gamma}$ is the "symmetric plan" of $\overline{\gamma}$, that is
$$\tilde{\gamma} = \tau_{\#}\overline{\gamma} \quad \text{with } \tau(x, y) = (y, x).$$
The transport plan $\tilde{\gamma}$ (which belongs to $\Pi(\nu, \mu)$) being itself ∞-cm, Lemma 3.3 also applies, yielding finiteness of $M^+_{\tilde{\gamma}} \cap M^-_{\tilde{\gamma}}$. Therefore, the equality case in (3.6) is only possible for a finite number of $\bar{y}$.
- If now there is equality in the last inequality of (3.7), we deduce that both $(\bar{x}, \bar{y})$ and $(\bar{y} + \lambda_C, \bar{y})$ belong to the support of $\overline{\gamma}$, with $\bar{x} < \bar{y} + \lambda_C$. As in the previous case, we conclude that $\bar{x}$ is a boundary point of one of the connected components of $\mathbb{R} \backslash (\operatorname{spt}\mu)$, which are at most coubtaly many thanks to Remark 3.1, concluding the proof of (iv). □

4. Application: existence of a minimal set of maximal displacements

In this section, we will use the pair of Kantorovich potentials furnished by Theorem 3.6 to get some informations on *any* optimal transport plan for (1.1) (no necessarily ∞-cm). Such a plan γ being concentrated on the set

$$\Big\{(x,y)\in\mathbb{R}^2\,:\,\varphi_r(x)+\psi_r(y)=0 \text{ and } |y-x|\leq\lambda_C\Big\},$$

it is natural to start by collecting properties of this set, and this is precisely the goal of the next proposition.

Proposition 4.1. *For any x, denote by*

$$B_r(x):=\Big\{y\in[x-\lambda_C,x+\lambda_C]\,:\,\varphi_r(x)+\psi_r(y)=0\Big\}.$$

Let $\bar{x}\in M^+_{\bar{\gamma}}$. Then, for any x such that $\bar{x}\leq x<\bar{x}+2\lambda_C$, we have $B_r(x)\subseteq[\bar{x}+\lambda_C,x+\lambda_C]$. In particular, the set $B_r(\bar{x})$ is reduced to $\{\bar{x}+\lambda_C\}$.

Proof. Let us fix $\bar{x}\in M^+_{\bar{\gamma}}$ and select $x\in[\bar{x},\bar{x}+2\lambda_C)$. By definition, all the elements of $B_r(x)$ are at most equal to $x+\lambda_C$, thus we only have to prove that $B_r(x)$ does not contain any y which is smaller than $\bar{x}+\lambda_C$. Consider then $y<\bar{x}+\lambda_C$, and let us prove that the equality $\varphi_r(x)+\psi_r(y)=0$ can not hold. We observe that, by Lemma 3.2, if $x>\bar{x}$ then $\rho^-((\bar{x},x])=0$, so that

$$\varphi_r(\bar{x})\geq\varphi_r(x) \tag{4.1}$$

which is of course also true when $x=\bar{x}$. On the other hand, by definition of ψ_r and since $[y-\lambda_C,\bar{x})\subset[y-\lambda_C,y+\lambda_C]$, we also have

$$\psi_r(y)\leq\inf\Big\{-\varphi_r(z)\,:\,y-\lambda_C\leq z<\bar{x}\Big\}. \tag{4.2}$$

From (4.1) and (4.2), it is clear that proving that

$$\exists z\in[y-\lambda_C,\bar{x}),\ \varphi_r(z)>\varphi_r(\bar{x}) \tag{4.3}$$

would be enough to conclude that $\psi_r(y)<-\varphi_r(x)$. We prove (4.3) by separating two cases.

- *First case:* $\bar{x}$ *is a left-isolated point of* $M^+_{\bar{\gamma}}$. In this case, Property (iii) of Theorem 3.6 ensures
$$\lim_{\substack{z\to\bar{x}\\ z<\bar{x}}}\varphi_r(z)>\varphi_r(\bar{x})$$
which is clearly enough to find the desired z.
- *Second case:* $\bar{x}$ *is not a left-isolated point of* $M^+_{\bar{\gamma}}$. Let us select some $\tilde{x}\in(\bar{x}-\lambda_C,\bar{x})\cap M^+_{\bar{\gamma}}$. Again by Lemma 3.2, $(\tilde{x},\bar{x}]\cap M^-_{\bar{\gamma}}=\emptyset$. Then we observe
$$\rho^+((\tilde{x},\bar{x}])>0\quad\text{and}\quad\rho^-((\tilde{x},\bar{x}])=0$$
which implies $\varphi_r(\tilde{x})>\varphi_r(\bar{x})$, as expected. □

We are now able to state the second main result on this paper, which asserts the existence of a maximal displacement set that is shared by all the optimal transport plans:

Theorem 4.2. *Let $\gamma, \overline{\gamma} \in \mathcal{O}_\infty$ be optimal transport plans and assume that $\overline{\gamma}$ satisfies the ∞-cm condition. Then*

$$M^+_{\overline{\gamma}} \subseteq M^+_\gamma \quad \text{and} \quad M^-_{\overline{\gamma}} \subseteq M^-_\gamma.$$

In particular, if $\overline{\gamma}_1$ and $\overline{\gamma}_2$ are two ∞-cm transport plans from μ to ν, then their maximal displacement sets are equal.

Proof. First we consider some consequences of Prop. 4.1 on γ.

Step I. *For any $\bar{x} \in M^+_{\overline{\gamma}}$, the quarter of plane $Q := [\bar{x}, +\infty) \times (-\infty, \bar{x} + \lambda_C]$ has zero mass for γ.* To prove this result, we first observe that, by the optimality of γ and of (φ_r, ψ_r) it follows that γ is concentrated on the set

$$B := \Big\{(x, y) \in \mathbb{R}^2 \,:\, |y - x| \le \lambda_C \text{ and } \varphi_r(x) + \psi_r(y) = 0\Big\}$$

which can be written as $\Big\{(x, y) \,:\, y \in B_r(x)\Big\}$. Now, for $x \ge \bar{x}$ and $y \in B_r(x)$ we have:

- if $x < \bar{x} + 2\lambda_C$, then Prop. 4.1 ensures that $y \ge \bar{x} + \lambda_C$;
- if $x \ge x + 2\lambda_C$, the same holds since $|y - x| \le \lambda_C$.

This exactly means that B has no intersection with the interior of the quarter of plane Q, which has therefore zero mass for γ; since moreover the lines $\{\bar{x}\} \times \mathbb{R}$ and $\mathbb{R} \times \{\bar{x} + \lambda_C\}$ have also zero mass for γ (because μ and ν are non atomic), the Step I is proved.

Step II: *if there exists $\bar{x} \in M^+_{\overline{\gamma}} \setminus M^+_\gamma$, then such an $\bar{x}$ satisfies $\mu([\bar{x}, \bar{x} + \delta]) = \nu([\bar{x} + \lambda_C - \delta, \bar{x} + \lambda_C]) = 0$ for $\delta > 0$ small enough.* Let $\bar{x}$ be such a point; since $\bar{x} \notin M^+_\gamma$, we can select $\delta > 0$ such that the square with side-length 2δ and centered in the point $(\bar{x}, \bar{x} + \lambda_C)$ has zero γ-measure. Without loss of generality we may also assume that $0 < \delta < \lambda_C$. Since μ is the first marginal of γ, we then have

$$\mu([\bar{x}, \bar{x} + \delta]) = \gamma([\bar{x}, \bar{x} + \delta] \times \mathbb{R}).$$

Since $\bar{x} \in M^+_{\overline{\gamma}}$ the result of Step I applies and gives $\gamma([\bar{x}, \bar{x} + \delta] \times (-\infty, \bar{x} + \lambda_C]) = 0$, so that

$$\gamma([\bar{x}, \bar{x} + \delta] \times \mathbb{R}) = \gamma([\bar{x}, \bar{x} + \delta] \times [\bar{x} + \lambda_C, +\infty)).$$

Moreover, since γ is an optimal transport plan, we have $|y - x| \le \lambda_C$ for γ-a.e. (x, y); in particular, for γ-a.e. (x, y) with $x \le \bar{x} + \delta$, we have $y \le \bar{x} + \delta + \lambda_C$. Consequently,

$$\gamma([\bar{x}, \bar{x} + \delta] \times [\bar{x} + \lambda_C, +\infty)) = \gamma([\bar{x}, \bar{x} + \delta] \times [\bar{x} + \lambda_C, \bar{x} + \lambda_C + \delta]),$$

which is zero since, by assumption on $\bar{x}$ and δ, the square $[\bar{x} \pm \delta, \bar{x} + \lambda_C \pm \delta]$ has zero mass for γ. The three last equalities imply that $\mu([\bar{x}, \bar{x} + \delta]) = 0$. The proof of the other equality is similar, and may be described as follows:

$$\begin{aligned}
\nu([\bar{x} + \lambda_C - \delta, \bar{x} + \lambda_C]) &= \gamma(\mathbb{R} \times [\bar{x} + \lambda_C - \delta, \bar{x} + \lambda_C]) \text{ since } \nu = (\pi_2)_\# \gamma \\
&= \gamma((-\infty, \bar{x}] \times [\bar{x} + \lambda_C - \delta, \bar{x} + \lambda_C]) \text{ by Step I} \\
&= \gamma([\bar{x} - \delta, \bar{x}] \times [\bar{x} + \lambda_C - \delta, \bar{x} + \lambda_C]) \text{ since } |y - x| \le \lambda_C \ \gamma\text{-a.e.} \\
&= 0 \text{ by assumption.}
\end{aligned}$$

Step III: $M^+_{\overline{\gamma}} \subset M^+_{\gamma}$. By contradiction, assume that there exists $\bar{x} \in M^+_{\overline{\gamma}}$ which does not belong to M^+_{γ}. Since $\overline{\gamma}$ is concentrated on the half-space $\Big\{(x,y) \in \mathbb{R}^2 \,:\, y \leq x+\lambda_C\Big\}$, we notice that

$$\overline{\gamma}([\bar{x}-\delta,\bar{x}] \times [\bar{x}+\lambda_C, \bar{x}+\lambda_C+\delta]) = 0$$

because, in this square, only the point $(\bar{x}, \bar{x}+\lambda_C)$ satisfies $|y-x| \leq \lambda_C$ and $\overline{\gamma}$ has no atom (because μ and ν don't).

From this property, and from the fact that $\overline{\gamma}$ gives no mass to any horizontal or vertical line, it follows

$$\begin{aligned} \overline{\gamma}([\bar{x} \pm \delta, \bar{x}+\lambda_C \pm \delta]) = \; & \overline{\gamma}([\bar{x},\bar{x}+\delta] \times [\bar{x}-\lambda_C \pm \delta]) \\ & + \overline{\gamma}([\bar{x}-\delta,\bar{x}] \times [\bar{x}-\lambda_C-\delta, \bar{x}-\lambda_C]). \end{aligned} \tag{4.4}$$

But, from the result of Step II, it follows that

$$\begin{aligned} \overline{\gamma}([\bar{x},\bar{x}+\delta] \times [\bar{x}-\lambda_C \pm \delta]) \quad & \leq \overline{\gamma}([\bar{x},\bar{x}+\delta] \times \mathbb{R}) \\ & = \mu([\bar{x},\bar{x}+\delta]) \\ & = 0 \end{aligned}$$

and that, similarly, $\overline{\gamma}([\bar{x}-\delta,\bar{x}] \times [\bar{x}-\lambda_C-\delta, \bar{x}-\lambda_C])$ is zero. The equality (4.4) therefore implies that the open square with side-length 2δ and with center $(\bar{x}, \bar{x}+\lambda_C)$ has zero mass for $\overline{\gamma}$, which gives the desired contradiction and concludes Step III.

Step IV: $M^-_{\overline{\gamma}} \subset M^-_{\gamma}$. Consider the "opposite transport plans"

$$\begin{cases} \overline{\gamma}_{\text{neg}} := \chi_{\#}\overline{\gamma} \\ \gamma_{\text{neg}} := \chi_{\#}\gamma \end{cases} \qquad \text{where } \chi(x,y) = (-x,-y).$$

It is clear that $\overline{\gamma}_{\text{neg}}, \gamma_{\text{neg}}$ have same marginals which are atomless and compactly supported (they are $\mathrm{d}\mu(-x)$ and $\mathrm{d}\nu(-y)$), have same maximal displacement which is still equal to λ_C and that $\overline{\gamma}_{\text{neg}}$ is still an ∞-cm plan, and that

$$x \in M^+_{\overline{\gamma}_{\text{neg}}} \Longleftrightarrow (-x) \in M^-_{\overline{\gamma}} \quad \text{and} \quad x \in M^+_{\gamma_{\text{neg}}} \Longleftrightarrow (-x) \in M^-_{\gamma}.$$

We can then apply the result if Steps I-III to the plans $\overline{\gamma}_{\text{neg}}$ and γ_{neg}, which concludes the proof. □

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