

Exact periodic stripes for minimizers of a local/non-local interaction functional in general dimension

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Abstract

We study the functional considered in [13, 14, 16] and a continuous version of it, analogous to the one considered in [18]. The functionals consist of a perimeter term and a non-local term which are in competition. For both the continuous and discrete problem, we show that the global minimizers are exact periodic stripes. One striking feature of the functionals is that the minimizers are invariant under a smaller group of symmetry than the functional itself. To our knowledge this is the first example of a model with local/nonlocal terms in competition such that the functional is invariant under permutation of coordinates and the minimizers display a pattern formation which is one dimensional.

1 Introduction

In this paper we study a discrete local/nonlocal functional considered in a series of papers by Giuliani, Lebowitz, Lieb and Seiringer (cf. [13, 14, 16]) and a continuous version of it.

The discrete functional is the following: given $E \subset \mathbb{Z}^d$

$$\tilde{\mathcal{F}}_{J,L}^{\text{disc}}(E) := J \sum_{x \in [0,L)^d} \sum_{y \sim x} |\chi_E(x) - \chi_E(y)| - \sum_{\substack{x \in [0,L)^d, y \in \mathbb{Z}^d \\ x \neq y}} \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^p}, \quad (1.1)$$

where $p > 2d$, $d \geq 1$, $y \sim x$ if x and y are neighbour points in the lattice, and

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

In the continuous setting we consider the following functional: for $E \subset \mathbb{R}^d$, $L > 0$,

$$\tilde{\mathcal{F}}_{J,L}(E) = \frac{1}{L^d} \left(J \text{Per}_1(E) - \int_{[0,L)^d \times \mathbb{R}^d} \frac{|\chi_E(x) - \chi_E(y)|}{(|x - y| + 1)^p} dx dy \right), \quad (1.2)$$

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where

$$\text{Per}_1(E) := \int_{\partial E \cap [0,L]^d} |\nu^E(x)|_1 d\mathcal{H}^{d-1}(x), \quad |z|_1 = \sum_{i=1}^d |z_i|$$

and $\nu^E(x)$ exterior normal to E in x , with $d \geq 1$, $p > 2d$. The Per_1 is the so-called 1-perimeter of E . This functional is analogous to the one considered in [18], differing only in the choice of the kernel $K(\zeta) = \frac{1}{(|\zeta|+1)^p}$ instead of $\frac{1}{|\zeta|^{p+1}}$. This choice, as we will explain in Remark 3.1, is done only for simplicity of calculations but does not change the physics underlying the problem. Since all the results in [18] that we will use remain true for this kernel, in this introduction we will assume them to hold also for the slightly modified functional.

The objective is to study the structure of the minimizers of the two functionals.

Both of the models describe systems of particles in which there is a strong short-range attracting force (the perimeter) and a weak repulsive long-range force (non-local term). The competition between short-range and long-range forces is, in several areas of physics and biology, at the base of patterns formation (see e.g. [11, 19, 22, 7, 26]). However, to prove patterns formation is a difficult problem which is rigorously proved in very few models: in the discrete case, to our knowledge, only in [16, 28, 6], and, in the continuous setting, in [15]. On the general issue of crystallization see [5]. In the continuous setting, the closest and most famous model is the sharp interface version of Ohta-Kawasaki [25]. This functional is well-studied (e.g. [8, 10, 20, 2, 26, 24, 9, 17, 21, 23]). Even though periodic pattern formation is expected due to physical and numerical simulations (e.g. [27, 8]), the problem is still open.

In order to make our problem well-posed we impose periodic boundary conditions, namely we restrict the functional to $[0, L]^d$ -periodic sets.

For both the discrete and continuous problem, there exists a critical constant J_c^{dsc} (J_c respectively) such that if $J > J_c^{\text{dsc}}$ (respectively $J > J_c$), then the global minimizers are trivial, namely either empty or the whole domain. The critical constants J_c^{dsc} and J_c are

$$J_c^{\text{dsc}} := \sum_{y_1 > 0, y^\perp \in \mathbb{Z}^{d-1}} \frac{y_1}{(y_1^2 + |y^\perp|^2)^{p/2}} \quad \text{and} \quad J_c := \int_{\mathbb{R}^d} \frac{|\zeta_1|}{(|\zeta| + 1)^p} d\zeta.$$

When $J = J_c^{\text{dsc}} - \tau$ (resp. $J = J_c - \tau$) with $0 < \tau \leq \bar{\tau}$ for some $\bar{\tau} > 0$ small enough, it has been conjectured that the minimizers should be periodic stripes of optimal period. By stripes we mean in the continuous setting a $[0, L]^d$ -periodic set which is, up to Lebesgue null sets, of the form $e_i^\perp + \hat{E}e_i$ for some $i \in \{1, \dots, d\}$ and $\hat{E} = \cup_{k=1}^N (t_i, s_i) \subset \mathbb{R}$, where $\{e_i\}_{i=1}^d$ is the canonical basis and e_i^\perp the $(d-1)$ -dimensional plane orthogonal to e_i . A stripe is periodic if $\exists h > 0$, $\nu \in \mathbb{R}$ s.t. $\hat{E} = \cup_{k=0}^N (2kh + \nu, (2k+1)h + \nu)$. In this case we write $E_h = e_1^\perp + \hat{E}e_1$, with $\nu = 0$ in \hat{E} . In the discrete setting the concept of stripe is the same, up to intersecting with the discrete lattice.

We will show that this conjecture holds both for the discrete and continuous setting. Indeed, our method applies also requiring, instead of periodic boundary conditions, optimal stripes–Dirichlet boundary conditions, namely asking that the sets E are, outside $[0, L]^d$, periodic stripes of optimal period. We prefer periodic boundary conditions since the problem is invariant under coordinate exchange, and then we are not pre-selecting a certain direction.

In order to formulate the precise results, we notice, as in [18], that stripes with optimal energy for $\tilde{\mathcal{F}}_{J,L}^{\text{dsc}}$ (respectively $\tilde{\mathcal{F}}_{J,L}$) have width of order $\tau^{-1/p-d-1}$ and energy of order $\tau^{(p-d)/(p-d-1)}$. Then it

is natural to rescale the functional in such a way the the width and the energy of the stripes are of order 1 for τ small. The new functionals are respectively called $\mathcal{F}_{\tau,L}^{\text{dsc}}$ and $\mathcal{F}_{\tau,L}$ (see § 2 and § 4). For the continuous setting, our main theorems are the following:

Theorem 1.1. *For $d \geq 2$, $p > 2d$, $L > 0$, there exists $\bar{\tau} > 0$ s.t. $\forall 0 < \tau \leq \bar{\tau}$ the minimizers of $\mathcal{F}_{\tau,L}$ are periodic stripes.*

Theorem 1.2. *For every $0 < \tau \leq \bar{\tau}$, there exist $h_\tau^* > 0$ such that the width h_τ of the minimizers of $\mathcal{F}_{\tau,L}$ satisfies*

$$|h_\tau^* - h_\tau| \lesssim \frac{1}{L}. \quad (1.3)$$

Moreover, for $L \in 2h_\tau^\mathbb{N}$, $E_{h_\tau^*}$ is the unique minimizer, up to translations and exchange of coordinates.*

For the discrete setting, we prove equivalently:

Theorem 1.3. *For $d \geq 2$, $p > 2d$, $L > 0$, there exists $\bar{\tau} > 0$ s.t. $\forall 0 < \tau \leq \bar{\tau}$ the minimizers of $\mathcal{F}_{\tau,L}^{\text{dsc}}$ are periodic stripes.*

Theorem 1.4. *For every $0 < \tau \leq \bar{\tau}$, there exist $h_\tau^* > 0$ such that the width h_τ of the minimizers of $\mathcal{F}_{\tau,L}^{\text{dsc}}$ satisfies*

$$|h_\tau^* - h_\tau| \lesssim \frac{1}{L}. \quad (1.4)$$

Moreover, for $L \in 2h_\tau^\mathbb{N}$, $E_{h_\tau^*}$ is the unique minimizer, up to translations and exchange of coordinates.*

For the discrete problem, the strongest result known prior to this paper is contained in [16]. Their setting is different from ours, thus a direct comparison is difficult to make. Indeed, they do not restrict the sums w.r.t. x in the functional to $[0, L)^d$, but they allow x to run over \mathbb{Z}^d . In this way the energy is in general not well defined if E is unbounded, but one can make sense of the problem, introducing the following concept: a set $E \subset \mathbb{Z}^d$ is a ground state whenever any local perturbation “increases” its energy. In order to define “increase” in the energy, one has to disregard contributions in the functional coming from the interactions at infinity. Then, they show that periodic stripes of a certain width h^* are ground states. From the definition, in principle, two ground states which differ on an infinite set are not comparable.

In order to partially overcome this problem we impose periodic boundary conditions, thus giving a meaning to the functional, and we consider global minimizers of the energy. In this setting, we prove not only that optimal periodic stripes are minimizers, but also that all minimizers must have that form.

One key element that is used in [13, 14, 16], is the concept of “angles” (for the exact definition see [13, 14, 16]). Intuitively “angles” are what are what you would expect for a discrete set. They can be thought as the obstructions from being union of stripes. Due to a minimal size of the “angle”, they carry a strictly positive energy which is bounded from below (depending on the step of the lattice). Thus from energetic arguments one can show an upper bound on “the number angles”.

In the continuous setting, using different ideas, it was proved (see [18] or Section 2) that, as τ tends to 0, the minimizers of $\mathcal{F}_{\tau,L}$ almost-Hausdorff converge (see Definition 2.1) to periodic stripes of

period h . Moreover, h is close to some h^* not depending on L (in the spirit of Theorems 1.2 and 1.4).

In this paper we first show that the Γ -convergence result of [18] can be extended to the discrete setting. Thus, in both cases, when τ is sufficiently small, minimizers are almost-Hausdorff close to periodic stripes. Our strategy is then to show that, once in this configuration, deviations from being exactly a stripe increase the value of the functionals.

This paper is organized as follows: in Section 2, we explain the setting and some preliminary result that will be used in the following; in Section 3 we show that for $\tau > 0$ if one minimizes $\mathcal{F}_{\tau,L}^{\text{dsc}}$ or $\mathcal{F}_{\tau,L}$ among sets composed of stripes then the minimizers are periodic stripes (this is done by the so-called reflection positivity); in Section 4, we show analogous results to [18] for the discrete setting; in Section 5, we finally prove Theorems 1.1-1.4.

2 Setting and Preliminary results

In this section, we set the notation and recall some preliminary results on the functional (1.2) proven in [18] which will be used in the proof of our main theorems.

In the following, we let $Q_L = [0, L]^d$, $|z|_1 = \sum_{i=1}^d |z_i|$ the 1-norm of $z \in \mathbb{R}^d$ and $|z|$ its Euclidean norm. We let (e_1, \dots, e_d) be the canonical basis in \mathbb{R}^d and, for $x \in \mathbb{R}^d$, $\zeta = (\zeta_1, \dots, \zeta_d) \in \mathbb{R}^d$ and $i \in \{1, \dots, d\}$, we let $x + \zeta_i := x + \zeta_i e_i$, $\zeta_i^\perp := \zeta - \zeta_i e_i$. \mathcal{H}^{d-1} is the $(d-1)$ -dimensional Hausdorff measure. Given a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, Df denotes its distributional derivative. For a measure μ on \mathbb{R}^d we denote by $|\mu|$ its total variation. For $x \in \mathbb{R}^d$, $A \subset \mathbb{R}^d$, $d(x, A)$ denotes the usual Euclidean distance of x from the set A .

Definition 2.1. Let $A, B \subset [0, L]^d$. $\mathcal{N}_\varepsilon(A)$ (resp. $\mathcal{N}_\varepsilon(B)$) are the Euclidean ε -neighborhoods of A (resp. B) and $d_H(A, B) = \sup\{\inf\{\varepsilon \geq 0 : A \subset \mathcal{N}_\varepsilon(B)\}, \inf\{\varepsilon \geq 0 : B \subset \mathcal{N}_\varepsilon(A)\}\}$ is the Hausdorff distance between A and B .

Moreover, let $A, B \subset \mathbb{R}^d$ such that $\mathcal{H}^{d-1}(A), \mathcal{H}^{d-1}(B) < +\infty$. We say that A, B are ε -almost close in the Hausdorff distance if there exists $A' \subset A$ and $B' \subset B$ such that $\mathcal{H}^{d-1}(A \setminus A') \leq \varepsilon$ and $\mathcal{H}^{d-1}(B \setminus B') \leq \varepsilon$ such that $d_H(A', B') \leq \varepsilon$.

Given a sequence $(A_n) \subset \mathbb{R}^d$ such that $\sup_n \mathcal{H}^{d-1}(A_n) < +\infty$. We say that A_n almost-Hausdorff converges to A if for every $\varepsilon > 0$, there exists N such that for every $n > N$ one has that A_n and A are ε -almost close in Hausdorff distance.

For an L -periodic set $E \subset \mathbb{R}$, E is of finite perimeter if $E \cap [0, L) = \cup_{i=1}^N I_i$, where $N \in \mathbb{N}$ and I_i are disjoint intervals. We let $\text{Per}(E)$ be the relative perimeter of E in $[0, L)$, namely

$$\text{Per}(E) := \text{Per}(E, [0, L)) = 2N$$

For a Q_L -periodic set $E \subset \mathbb{R}^d$, the definition of set of finite perimeter is the following

Definition 2.2. A Q_L -periodic set E is of finite perimeter if the distributional derivative of χ_E is a locally finite measure. We let ∂E be the reduced boundary of E , namely the set of points $x \in \text{spt}(D\chi_E)$ such that the limit

$$\nu_E(x) := -\lim_{r \downarrow 0} \frac{D\chi_E(B(x, r))}{|D\chi_E|(B(x, r))}$$

exists and satisfies $|\nu^E(x)| = 1$. We call ν^E the exterior normal to E . In particular, $D\chi_E = -\nu^E \mathcal{H}^{d-1} \llcorner \partial E$.

We define

$$\text{Per}_1(E) := \int_{\partial E \cap Q_L} |\nu^E(x)|_1 d\mathcal{H}^{d-1}(x)$$

and, for $i \in \{1, \dots, d\}$

$$\text{Per}_{1i}(E) = \int_{\partial E \cap Q_L} |\nu_i^E(x)| d\mathcal{H}^{d-1}(x),$$

thus $\text{Per}_1(E) = \sum_{i=1}^d \text{Per}_{1i}(E)$.

Because of periodicity, we always assume that $|D\chi_E|(\partial Q_L) = 0$.

For $i \in \{1, \dots, d\}$, $x_i^\perp \in [0, L)^{d-1}$ we let \tilde{x}_i^\perp the point in e_i^\perp with coordinates x_i^\perp and for $E \subset Q_L$ we define the 1-dimensional slices

$$E_{x_i^\perp} := \{t \in [0, L) : te_i + \tilde{x}_i^\perp \in E\}.$$

The following slicing formula holds for every $i \in \{1, \dots, d\}$

$$\text{Per}_{1i}(E) = \int_{\partial E \cap Q_L} |\nu_i^E(x)| d\mathcal{H}^{d-1}(x) = \int_{[0, L)^{d-1}} \text{Per}(E_{x_i^\perp}) dx_i^\perp.$$

Let $\tau := J_c - J$ for $J < J_c$. We let

$$K_1(\zeta) = \frac{1}{(|\zeta| + 1)^p}, \quad K_\tau(\zeta) = \frac{1}{(|\zeta| + \tau^{1/(p-d-1)})^p}.$$

Then,

$$\tilde{\mathcal{F}}_{J,L}(E) = \frac{1}{L^d} \left(-\tau \text{Per}(E, Q_L) + \int_{\mathbb{R}^d} K_1(\zeta) \left[\int_{\partial E \cap Q_L} \sum_{i=1}^d |\nu_i^E(x)| |\zeta_i| d\mathcal{H}^{d-1}(x) - \int_{Q_L} |\chi_E(x) - \chi_E(x + \zeta)| dx \right] d\zeta \right).$$

We recall now the crucial observation made in the proof of [18, Lemma 3.2]

$$\begin{aligned} |\chi_E(x) - \chi_E(x + \zeta)| &= |\chi_E(x) - \chi_E(x + \zeta_i)| + |\chi_E(x + \zeta_i) - \chi_E(x + \zeta)| \\ &\quad - 2|\chi_E(x) - \chi_E(x + \zeta_i)| |\chi_E(x + \zeta_i) - \chi_E(x + \zeta)|. \end{aligned} \quad (2.1)$$

Inserting it in the expression for $\tilde{\mathcal{F}}_{J,L}$, and assuming $E = E_h$ is made of periodic stripes of width h in the direction e_i , up to relabeling coordinates we can assume that $i = 1$, thus $E_h = \hat{E}_h \times [0, L)^{d-1}$, and

$$\tilde{\mathcal{F}}_{J,L}(E_h) = -\frac{\tau}{h} + \int_{\mathbb{R}^d} K_1(\zeta) \left(\frac{|\zeta_1|}{h} - \frac{1}{L^d} \int_{[0, L)^d} |\chi_{E_h}(x) - \chi_{E_h}(x + \zeta_1)| dx \right) d\zeta.$$

As in [18], it is possible to compute the energy $\tilde{\mathcal{F}}_{J,L}(E_h)$ to get

$$\tilde{\mathcal{F}}_{J,L}(E_h) \simeq -\frac{\tau}{h} + h^{-(p-d)}.$$

Optimizing in h , one finds that the optimal stripes have a width of order $\tau^{-1/(p-d-1)}$ and energy of order $-\tau^{(p-d)/(p-d-1)}$. Letting $\beta := p - d - 1$, this motivates the rescaling

$$x := \tau^{-1/\beta} \tilde{x}, \quad L := \tau^{-1/\beta} \tilde{L} \quad \text{and} \quad \tilde{\mathcal{F}}_{J,L}(E) := \tau^{(p-d)/\beta} \mathcal{F}_{\tau, \tilde{L}}(\tilde{E}). \quad (2.2)$$

In these variables, the optimal stripes have width of order one.

Making the substitutions in (4.4) letting also $\zeta = \tau^{-1/\beta} \tilde{\zeta}$ and in the end dropping the tildes, one has

$$\mathcal{F}_{\tau,L}(E) = \frac{1}{L^d} \left(-\text{Per}_1(E) + \int_{\mathbb{R}^d} K_\tau(\zeta) \left[\int_{\partial E \cap Q_L} \sum_{i=1}^d |\nu_i^E(x)| |\zeta_i| d\mathcal{H}^{d-1}(x) - \int_{Q_L} |\chi_E(x) - \chi_E(x+\zeta)| dx \right] d\zeta \right). \quad (2.3)$$

Let us now recall an important estimate from below for $\mathcal{F}_{\tau,L}$ (see [18, Lemma 3.6]).

First, notice that (2.1) implies

$$\begin{aligned} \int_{Q_L \times \mathbb{R}^d} K_\tau(\zeta) |\chi_E(x) - \chi_E(x+\zeta)| dx d\zeta &\leq \int_{Q_L \times \mathbb{R}^d} K_\tau(\zeta) \sum_{i=1}^d |\chi_E(x) - \chi_E(x+\zeta_i)| dx d\zeta \\ &\quad - \frac{2}{d} \int_{Q_L \times \mathbb{R}^d} K_\tau(\zeta) \sum_{i=1}^d |\chi_E(x) - \chi_E(x+\zeta_i)| |\chi_E(x) - \chi_E(x+\zeta_i^\perp)| dx d\zeta. \end{aligned} \quad (2.4)$$

Then define, for $i \in \{1, \dots, d\}$,

$$\mathcal{G}_{\tau,L}^i(E) := \int_{\mathbb{R}^d} \hat{K}_\tau(\zeta_i) \left[\int_{\partial E \cap Q_L} |\nu_i^E(x)| |\zeta_i| d\mathcal{H}^{d-1}(x) - \int_{Q_L} |\chi_E(x) - \chi_E(x+\zeta_i)| dx \right] d\zeta,$$

where $\hat{K}_\tau(z) = \int_{\mathbb{R}^{d-1}} \frac{1}{(|(z,\zeta)| + \tau^{1/(p-d-1)})^p} d\zeta$ and

$$I_{\tau,L}^i(E) := \frac{2}{d} \int_{Q_L \times \mathbb{R}^d} K_\tau(\zeta) |\chi_E(x) - \chi_E(x+\zeta_i)| |\chi_E(x) - \chi_E(x+\zeta_i^\perp)| dx d\zeta.$$

Estimate (2.4) implies

$$\mathcal{F}_{\tau,L}(E) \geq \frac{1}{L^d} \left(-\text{Per}_1(E) + \sum_{i=1}^d \mathcal{G}_{\tau,L}^i(E) + \sum_{i=1}^d I_{\tau,L}^i(E) \right). \quad (2.5)$$

Before introducing the second key estimate, let us set some notations.

Let $E = \bigcup_{i \in \mathbb{Z}} (s_i, t_i)$ be a set of finite perimeter. We will denote by

$$h(t_i) := h(s_i) := t_i - s_i \quad g(t_i) := s_{i+1} - t_i \quad g(s_i) := s_i - t_{i-1}.$$

For $z \in \mathbb{R}$, and $i \in \mathbb{Z}$, we define

$$\eta(t_i, z) := \min(z_+, h(t_i)) + \min(z_-, g(t_i))$$

and

$$\eta(s_i, z) := \min(z_+, g(s_i)) + \min(z_-, h(s_i)).$$

Moreover, given a Q_L -periodic set E of finite perimeter, then the functions $g_{x_i^\perp}$, $h_{x_i^\perp}$ and $\eta_{x_i^\perp}$ are defined as above for the slices $E_{x_i^\perp}$.

The estimate we need is the following (see Lemma 3.4 of [18]): for every L -periodic set E of finite perimeter and every $z \in \mathbb{R}$,

$$\int_0^L |\chi_E(x) - \chi_E(x+z)| dx \leq \sum_{x \in \partial E \cap [0, L]} \eta(x, z). \quad (2.6)$$

For every $\tau \geq 0$, $\zeta \in \mathbb{R}^d$, Q_L -periodic set E of finite perimeter and $i \in \{1, \dots, d\}$, by integrating one has that

$$\int_{Q_L \times \mathbb{R}^d} K_\tau(\zeta) |\chi_E(x) - \chi_E(x+\zeta_i)| dx d\zeta \leq \int_{\partial E \cap Q_L} |\nu_i^E(x)| \int_{\mathbb{R}^d} K_\tau(\zeta) \eta_{x_i^\perp}(x_i, \zeta_i) d\zeta d\mathcal{H}^{d-1}(x). \quad (2.7)$$

Notice now that

$$\mathcal{G}_{\tau, L}^i(E) = \int_{[0, L^{d-1})} \mathcal{G}_{\tau, L}^{1d}(E_{x_i^\perp}) dx_i^\perp,$$

where

$$\mathcal{G}_{\tau, L}^{1d}(E) := \int_{\mathbb{R}} \widehat{K}_\tau(z) \left(\text{Per}(E) |z| - \int_0^L |\chi_E(x) - \chi_E(x+z)| dx \right) dz$$

and that, as a consequence of (2.7), $\mathcal{G}_{\tau, L}^i(E) \geq 0$.

More precisely, one has the following estimate from below for $\mathcal{G}_{\tau, L}^{1d}$ (see [18, Lemma 3.7]): for every L -periodic set E of finite perimeter (recall that $\beta = p - d - 1$),

$$\mathcal{G}_{\tau, L}^{1d}(E) \gtrsim \sum_{x \in \partial E \cap [0, L]} \min(h(x)^{-\beta}, \tau^{-1}) + \min(g(x)^{-\beta}, \tau^{-1}). \quad (2.8)$$

Moreover, for every $\delta \geq \tau^{1/\beta}$,

$$\text{Per}(E) - 1 \lesssim L\delta^{-1} + \delta^\beta \mathcal{G}_{\tau, L}^{1d}(E). \quad (2.9)$$

These estimates hold also for the new kernel due to the simple observation

$$\frac{1}{C} \frac{1}{|\zeta|^p + \tau^{\beta p}} \leq K_\tau(\zeta) \leq C \frac{1}{|\zeta|^p + \tau^{\beta p}}. \quad (2.10)$$

In particular,

$$\mathcal{F}_{\tau, L}(E) \gtrsim -1 + \frac{1}{L^d} \left(\sum_{i=1}^d \mathcal{G}_{\tau, L}^i(E) + \sum_{i=1}^d I_{\tau, L}^i(E) \right) \quad (2.11)$$

and

$$\text{Per}_1(E) \lesssim L^d \max(1, \mathcal{F}_{\tau, L}(E)). \quad (2.12)$$

Combining these estimates with measure-theoretic density arguments, it is proven in [18] that the sets of $\mathcal{F}_{0, L}$ -finite energy are 1-dimensional sets of finite perimeter and the following Γ -convergence result

Theorem 2.1 ([18, Theorem 1.1]). *Let $p > 2d$ and $L > 0$. Then one has that $\mathcal{F}_{\tau, L}$ Γ -converge in the L^1 -topology as $\tau \rightarrow 0$ to the functional $\mathcal{F}_{0, L}$ which is finite on sets of the form $E = F \times \mathbb{R}^{d-1}$ where $F \subset \mathbb{R}$ is L -periodic with $\#\{\partial F \cap [0, L]\} < \infty$, defined by*

$$\mathcal{F}_{0, L}(E) = \frac{1}{L} \left(-\#\{\partial F \cap [0, L]\} + \int_{\mathbb{R}^d} \frac{1}{|\zeta|^q} \left[\sum_{x \in \partial F \cap [0, L]} |\zeta_1| - \int_0^L |\chi_F(x) - \chi_F(x+\zeta_1)| dx \right] d\zeta \right). \quad (2.13)$$

Moreover, if there exists M such that for every τ one has that $\mathcal{F}_{\tau,L}(E^\tau) < M$, then up to a relabeling of the coordinate axes, one has that there is a subsequence which converges in L^1 and almost-Hausdorff converges to some set $E = F \times \mathbb{R}^{d-1}$ with $\#\{\partial F \cap [0, L]\} < \infty$.

In the second part of the statement we underline the almost-Hausdorff convergence of sets of uniformly bounded energy to the sets of finite action for $\mathcal{F}_{0,L}$. This fact is contained in the proof even though not explicitly stated in [18].

3 The 1-dimensional problem

In this section we consider the 1-dimensional functional

$$\mathcal{F}_{\tau,L}^1(E) = \frac{1}{L} \left(-\text{Per}(E) + \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \frac{1}{(\tau^\beta + |(z, \zeta)|)^p} d\zeta \left[\text{Per}(E)|z| - \int_0^L |\chi_E(x) - \chi_E(x+z)| dx \right] dz \right)$$

on an L -periodic set $E \subset \mathbb{R}$ of finite perimeter.

Computing the $(d-1)$ -dimensional integral, one obtains

$$\mathcal{F}_{\tau,L}^1(E) = \frac{1}{L} \left(-\text{Per}(E) + C_q \int_{\mathbb{R}} \frac{1}{(\tau^\beta + |z|)^q} \left[\text{Per}(E)|z| - \int_0^L |\chi_E(x) - \chi_E(x+z)| dx \right] dz \right)$$

where $q := p - d + 1$.

Analogously to Theorem 6.4 in [18], our aim is to show the following

Theorem 3.1. *For every $\tau > 0$, $\exists h_\tau^* > 0$ s.t., for every $L > 0$, the minimizers of $\mathcal{F}_{\tau,L}^1$ are periodic stripes of period h_τ for some $h_\tau > 0$ satisfying*

$$|h_\tau - h_\tau^*| \lesssim \frac{1}{L}.$$

Moreover, for $L \in 2h_\tau^* \mathbb{N}$, $E_{h_\tau^*}$ is, up to translations, the unique minimizer.

From the Γ -convergence result of Theorem 2.1, due to the L -periodicity of the sets E_{h_τ} , one also has the following

Corollary 3.1. *For every $L > 0$, there exists $\tau_L > 0$ such that, for all $0 \leq \tau \leq \tau_L$, $h_\tau = h$, where h is the width of stripes minimizing $\mathcal{F}_{0,L}^1$.*

In order to prove Theorem 3.1, we proceed using the method of reflection positivity as in [18]. We go along the same path, stating lemmas analogous to those used in [18], adapted to the setting $\tau > 0$. Instead of providing complete proofs, we point out only the differences with respect to the case $\tau = 0$.

For $h > 0$, recall that $E_h := \cup_{k \in \mathbb{Z}} [(2k)h, (2k+1)h]$. Then, define

$$e_{\infty,\tau}(h) := \mathcal{F}_{\tau,2h}^i(E_h) = \lim_{L \rightarrow +\infty} \mathcal{F}_{\tau,L}^i(E_h).$$

Analogously to [18, Lemma 6.1], one can see that

$$e_{\infty,\tau}(h) = -\frac{1}{h} + \bar{C}_{q,\tau} h^{-(q-1)}, \quad (3.1)$$

which has a positive minimizer h_τ^* .

The main ingredient to prove Theorem 3.1 is the following estimate, which recalls [18, Lemma 6.3].

Lemma 3.1. *For every L -periodic set E of finite perimeter, it holds*

$$\mathcal{F}_{\tau,L}(E) \geq \frac{1}{2L} \sum_{x \in \partial E \cap [0,L)} h(x)e_{\infty,\tau}(h(x)) + g(x)e_{\infty,\tau}(g(x)). \quad (3.2)$$

Theorem 3.1 follows from (3.2) in a few lines as in [18].

Lemma 3.1 is a direct consequence of the following two Lemmas.

The first, analogous to [18, Lemma 6.8], rewrites the second term of the functional $\mathcal{F}_{\tau,L}^1$ as a Laplace transform.

Lemma 3.2. *For every $E \in [0, L)$, $\tau > 0$,*

$$\begin{aligned} \mathcal{F}_{\tau,L}^1(E) &= \frac{1}{L} \text{Per}(E) \left(-1 + C_q \int_{\mathbb{R}} \frac{|z|}{(\tau^\beta + |z|)^q} dz \right) \\ &\quad - \int_0^{+\infty} \frac{C_q \alpha^{q-1} e^{-\alpha\tau^\beta}}{\Gamma(q)} \left(\int_{[0,L) \times \mathbb{R}} |\chi_E(x) - \chi_E(y)| e^{-\alpha|x-y|} dx dy \right) d\alpha, \end{aligned} \quad (3.3)$$

where Γ is the Euler's Gamma function.

Proof. Notice that, in comparison with the functional $\mathcal{F}_{0,L}^1$ studied in [18], the part of $\mathcal{F}_{\tau,L}$ which multiplies the perimeter of E has finite energy (due to the presence of τ^β in the kernel) and then it does not need to be transformed through Laplace transform together with the last term.

For the last term, it is sufficient to recall that, for $s > 0$,

$$\frac{1}{s^q} = \frac{1}{\Gamma(q)} \int_0^{+\infty} \alpha^{q-1} e^{-\alpha s} d\alpha$$

and the rule that expresses the translate of a Laplace transform

$$\frac{1}{(s + \tau^\beta)^q} = \frac{1}{\Gamma(q)} \int_0^{+\infty} \alpha^{q-1} e^{-\alpha\tau^\beta} e^{-\alpha s} d\alpha.$$

□

Remark 3.1. *The reason why we have chosen as a kernel for the original functional $\frac{1}{(|\xi|+1)^p}$ instead of $\frac{1}{|\xi|^{p+1}}$ is that the translate of a Laplace transform is easier to calculate and its inverse Laplace transform stays positive whenever the inverse of the original function is positive (in this case α^{q-1} becomes $\alpha^{q-1} e^{-\alpha\tau^\beta}$). This is used to deduce Lemma 3.1 from (3.3) and the next Lemma.*

Lemma 3.3 ([18, Lemma 6.10]). *For every $\alpha > 0$ and every L -periodic set E*

$$- \int_{[0,L) \times \mathbb{R}} |\chi_E(x) - \chi_E(y)| e^{-\alpha|x-y|} dx dy \geq \frac{1}{2} \sum_{x \in \partial E \cap [0,L)} h(x)e_{\alpha,\infty}(h(x)) + g(x)e_{\alpha,\infty}(g(x)). \quad (3.4)$$

where, for $\alpha, h > 0$,

$$e_{\alpha,\infty}(h) := -\frac{1}{2h} \int_0^{2h} \int_{\mathbb{R}} |\chi_{E_h}(x) - \chi_{E_h}(y)| e^{-\alpha|x-y|} = \lim_{L \rightarrow +\infty} -\frac{1}{L} \int_{[0,L) \times \mathbb{R}} |\chi_{E_h}(x) - \chi_{E_h}(y)| e^{-\alpha|x-y|}.$$

For the proof of this result, based on the so-called reflection positivity technique, see [18, Lemma 6.10].

Remark 3.2. *Using the same ideas and the reflection positivity technique in the same way, one can prove a result like Theorem 3.1 for the following discrete 1-dimensional functional. For $\tau > 0$ and $E \subset \tau^{1/\beta}\mathbb{Z} \cap \mathbb{R}$ L -periodic, let*

$$\begin{aligned} \mathcal{F}_{\tau,L}^{1,dsc}(E) = & \frac{1}{L} \left(- \sum_{x \in [0,L) \cap \tau^{1/\beta}\mathbb{Z}} \sum_{y \sim x} |\chi_E(x) - \chi_E(y)| \right. \\ & \left. + \sum_{(z,\zeta) \in \tau^{1/\beta}\mathbb{Z}} \frac{1}{(\tau^\beta + |(z,\zeta)|)^p} \left[\sum_{x \in [0,L) \cap \tau^{1/\beta}\mathbb{Z}} \sum_{y \sim x} |\chi_E(x) - \chi_E(y)| |z| - \sum_{x \in [0,L) \cap \tau^{1/\beta}\mathbb{Z}} |\chi_E(x) - \chi_E(x+z)| \right] \right). \end{aligned}$$

Then, there exist h_τ^* and h_τ positive numbers with the same properties as in Theorem 3.1 for $\mathcal{F}_{\tau,L}^{1,dsc}$.

4 Discrete Γ -convergence Result

In this section, we will show how to adapt the proof of [18, Theorem 1.1] in order to obtain the same result for the discrete setting.

Let $E \subset \varepsilon\mathbb{Z}^d$. We define

$$\tilde{E}^\varepsilon := \bigcup_{i \in E} \left(i + [\varepsilon/2, \varepsilon/2)^d \right). \quad \text{Per}_{1,\varepsilon}(E, Q_L) := \sum_{x \in [0,L)^d \cap \varepsilon\mathbb{Z}^d} \sum_{y \sim x} |\chi_E(x) - \chi_E(y)| \varepsilon^{d-1}. \quad (4.1)$$

We call $\text{Per}_{1,\varepsilon}$ the $(1, \varepsilon)$ -perimeter. From the above definitions, it is clear that

$$\text{Per}_{1,\varepsilon}(E, Q_L) = \text{Per}_1(\tilde{E}^\varepsilon, Q_L).$$

The discrete functional can be rewritten as

$$\tilde{\mathcal{F}}_{J,L}^{dsc}(E) := \frac{1}{L^d} \left(J\text{Per}_{1,1}(E, Q_L) - \int_{Q_L \times \mathbb{R}^d} \sum_{\zeta \in \mathbb{Z}^d} K_1(\zeta) |\chi_{\tilde{E}^1}(x + \zeta) - \chi_{\tilde{E}^1}(x)| dx \right),$$

where $K_1(\zeta) := \frac{1}{|\zeta|^p}$ for some $p > 2d$. We will also denote by $K_\varepsilon(\zeta) := \frac{\varepsilon^{d/\beta}}{|\zeta|^p}$, $\beta = p - d - 1$.

Lemma 4.1 ([18, Lemma 3.2]). *For every Q_L -periodic set $E \subset \varepsilon\mathbb{Z}^d$, it holds*

$$\begin{aligned} \int_{Q_L} \sum_{\zeta \in \varepsilon\mathbb{Z}^d} K_\varepsilon(\zeta) |\chi_{\tilde{E}^\varepsilon}(x) - \chi_{\tilde{E}^\varepsilon}(x + \zeta)| dx & \leq \sum_{i=1}^d \int_{Q_L} \sum_{\zeta \in \varepsilon\mathbb{Z}^d} K_\varepsilon(\zeta) |\chi_{\tilde{E}^\varepsilon}(x) - \chi_{\tilde{E}^\varepsilon}(x + \zeta_i)| dx \\ & - \frac{2}{d} \sum_{i=1}^d \int_{Q_L} \sum_{\zeta \in \varepsilon\mathbb{Z}^d} K_\varepsilon(\zeta) |\chi_{\tilde{E}^\varepsilon}(x) - \chi_{\tilde{E}^\varepsilon}(x + \zeta_i)| |\chi_{\tilde{E}^\varepsilon}(x) - \chi_{\tilde{E}^\varepsilon}(x + \zeta_i^\perp)| dx. \end{aligned} \quad (4.2)$$

Proof. The proof is analogous to the one of (2.4) in [18]. The idea is to apply (2.1) and use periodicity in order to obtain the claim. \square

By considering the set \tilde{E}^ε , we define h, g and η as in the continuous setting. Recalling (2.6), we also have

Lemma 4.2 ([18, Lemma 3.4]). *For every Q_L -periodic set $E \subset \varepsilon\mathbb{Z}^d$ of finite $(1, \varepsilon)$ -perimeter,*

$$\int_{Q_L} \sum_{\zeta \in \varepsilon\mathbb{Z}^d} K(\zeta) |\chi_{\tilde{E}^\varepsilon}(x) - \chi_{\tilde{E}^\varepsilon}(x + \zeta_i)| dx \leq \int_{\partial\tilde{E}^\varepsilon \cap Q_L} |\nu_i^{\tilde{E}^\varepsilon}| \sum_{\zeta \in \varepsilon\mathbb{Z}^d} K(\zeta) \eta_{x_i^\perp}(x_i, \zeta_i) d\mathcal{H}^{d-1}(x). \quad (4.3)$$

Let $\tau := J_c^{\text{dsc}} - J$ for $J < J_c^{\text{dsc}}$. One can, as well as in the continuous case, estimate the energy of periodic stripes E_h of period h to get

$$\tilde{\mathcal{F}}_{J,L}^{\text{dsc}}(E_h) \simeq -\frac{\tau}{h} + h^{-(p-d)}.$$

After optimizing in h , we find that the optimal stripes have a width of order $\tau^{-1/(p-d-1)}$ and energy of order $-\tau^{(p-d)/(p-d-1)}$. Letting $\beta := p - d - 1$, this motivates the rescaling

$$x := \tau^{-1/\beta} \hat{x}, \quad L := \tau^{-1/\beta} \hat{L} \quad \text{and} \quad \tilde{\mathcal{F}}_{J,L}^{\text{dsc}}(E) := \tau^{(p-d)/\beta} \mathcal{F}_{\tau, \hat{L}}^{\text{dsc}}(\hat{E}), \quad (4.4)$$

where now $\hat{E} \subset \tau^{1/\beta} \mathbb{Z}^d$.

In these variables, the optimal stripes have width of order one. We would now like to proceed with the scaling of the functional. We denote by $\kappa = \tau^{1/\beta}$.

Lemma 4.3. *For every Q_L -periodic set $E \subset \kappa\mathbb{Z}^d$, we have*

$$\begin{aligned} \mathcal{F}_{\tau,L}^{\text{dsc}}(E) = & \frac{1}{L^d} \left(-\text{Per}_{1,\kappa}(E, Q_L) + \sum_{\zeta \in \kappa\mathbb{Z}^d} K_\tau(\zeta) \left[\int_{\partial\tilde{E}^\kappa \cap Q_L} \sum_{i=1}^d |\nu_i^{\tilde{E}^\kappa}(x)| |\zeta_i| d\mathcal{H}^{d-1}(x) \right. \right. \\ & \left. \left. - \int_{Q_L} |\chi_{\tilde{E}^\kappa}(x) - \chi_{\tilde{E}^\kappa}(x + \zeta)| dx \right] \right). \end{aligned}$$

Proof. We start by noticing that

$$J\text{Per}_{1,1}(E, Q_L) = -\tau\text{Per}_{1,1}(E, Q_L) + \sum_{\zeta \in \mathbb{Z}^d} K_1(\zeta) \int_{\partial\tilde{E}^1 \cap Q_L} \sum_{i=1}^d |\nu_i^{\tilde{E}^1}(x)| |\zeta_i| d\mathcal{H}^{d-1}(x),$$

thus we have that

$$\begin{aligned} \tilde{\mathcal{F}}_{J,L}^{\text{dsc}}(E) = & \frac{1}{L^d} \left(-\tau\text{Per}_{1,1}(E, Q_L) + \sum_{\zeta \in \mathbb{Z}^d} K_1(\zeta) \left[\int_{\partial\tilde{E}^1 \cap Q_L} \sum_{i=1}^d |\nu_i^{\tilde{E}^1}(x)| |\zeta_i| d\mathcal{H}^{d-1}(x) \right. \right. \\ & \left. \left. - \int_{Q_L} |\chi_{\tilde{E}^1}(x) - \chi_{\tilde{E}^1}(x + \zeta)| dx \right] \right). \end{aligned}$$

Let $\hat{E} := \tau^{1/\beta} E$, hence $\hat{E} \subset \kappa\mathbb{Z}^d$. Making the change of variables given in (4.4) and letting also $\hat{\zeta} := \tau^{-1/\beta} \zeta$, we obtain

$$\begin{aligned} \tilde{\mathcal{F}}_{J,L}(E) = & \frac{\tau^{(p-d)/\beta}}{\hat{L}^d} \left(-\text{Per}_{1,\kappa}(\hat{E}, Q_{\hat{L}}) + \sum_{i=1}^d \sum_{\hat{\zeta} \in \kappa\mathbb{Z}^d} K_\tau(\hat{\zeta}) \left[\int_{\partial\tilde{\hat{E}}^\kappa \cap Q_{\hat{L}}} |\nu_i^{\tilde{\hat{E}}^\kappa}(x)| |\hat{\zeta}_i| d\mathcal{H}^{d-1}(x) \right. \right. \\ & \left. \left. - \int_{Q_{\hat{L}}} |\chi_{\tilde{\hat{E}}^\kappa}(\hat{x}) - \chi_{\tilde{\hat{E}}^\kappa}(\hat{x} + \hat{\zeta})| dx \right] \right). \end{aligned}$$

□

Hence notice that, for $E \subset \kappa\mathbb{Z}^d$ Q_L -periodic, $\mathcal{F}_{\tau,L}^{\text{dsc}}(E)$ differs from $\mathcal{F}_{\tau,L}(\tilde{E}^\kappa)$ only from the kernel (a discrete sum over $\kappa\mathbb{Z}^d$ in the discrete case and an integral over \mathbb{R}^d in the continuous case).

Therefore, in what follows, the only difference w.r.t. the continuous case consists in showing that the new kernel has the same properties that are useful in the continuous case. The results on the terms not containing the kernel are then exactly the same for the sets \tilde{E}^κ .

Let us define

$$\hat{K}_\tau(\zeta) := \sum_{\xi \in \kappa\mathbb{Z}^{d-1}} \frac{\kappa^d}{(|\zeta|^2 + |\xi|^2)^{p/2}}.$$

Notice that as in the continuous case, for $z \in \kappa\mathbb{Z}^d$, we have that

$$\frac{1}{|z|^{p-d+1} + \tau^{(p-d+1)/\beta}} \lesssim \sum_{\xi \in \kappa\mathbb{Z}^{d-1}} \frac{\kappa^d}{(|z|^2 + |\xi|^2)^{p/2}} \lesssim \frac{1}{|z|^{p-d+1} + \tau^{(p-d+1)/\beta}}, \quad (4.5)$$

thus

$$\hat{K}_\tau(\zeta) \simeq \frac{1}{|\zeta|^q + \tau^{q/\beta}}, \quad (4.6)$$

where $q := p - d + 1$. Estimate (4.5) is the crucial condition for the kernel analogous to that needed in the continuous setting.

Fix $i \in \{1, \dots, d\}$. For any $E \subset \kappa\mathbb{Z}^d$, we denote by

$$\mathcal{G}_{\tau,L}^{i,\text{dsc}}(E) := \sum_{\zeta_i \in \kappa\mathbb{Z}} \hat{K}_\tau(\zeta_i) \left[\int_{\partial\tilde{E}^\kappa \cap Q_L} |\nu_i^{\tilde{E}^\kappa}(x)| |\zeta_i| d\mathcal{H}^{d-1}(x) - \int_{Q_L} |\chi_{\tilde{E}^\kappa}(x) - \chi_{\tilde{E}^\kappa}(x + \zeta_i)| dx \right],$$

and

$$I_{\tau,L}^{i,\text{dsc}}(E) := \frac{2}{d} \int_{Q_L} \sum_{\zeta \in \kappa\mathbb{Z}^d} K_\tau(\zeta) |\chi_{\tilde{E}^\kappa}(x) - \chi_{\tilde{E}^\kappa}(x + \zeta_i)| |\chi_{\tilde{E}^\kappa}(x) - \chi_{\tilde{E}^\kappa}(x + \zeta_i^\perp)| dx,$$

$$I_{\tau,L}^{\text{dsc}}(E) := \frac{2}{d} \sum_{i=1}^d \int_{Q_L} \sum_{\zeta \in \kappa\mathbb{Z}^d} K_\tau(\zeta) |\chi_{\tilde{E}^\kappa}(x) - \chi_{\tilde{E}^\kappa}(x + \zeta_i)| |\chi_{\tilde{E}^\kappa}(x) - \chi_{\tilde{E}^\kappa}(x + \zeta_i^\perp)| dx.$$

From Lemma 4.1, one has that for $E \in \kappa\mathbb{Z}^d$ it holds

$$\mathcal{F}_{\tau,L}^{\text{dsc}}(E) \geq \frac{1}{L^d} \left(-\text{Per}_{1,\kappa}(E, Q_L) + \sum_{i=1}^d \mathcal{G}_{\tau,L}^{i,\text{dsc}}(E) + I_{\tau,L}^{i,\text{dsc}}(E) \right).$$

Equation (4.3) implies that for every ζ and $i \in \{1, \dots, d\}$, one has that

$$\int_{\partial\tilde{E}^\kappa \cap Q_L} |\nu_i^{\tilde{E}^\kappa}(x)| |\zeta_i| d\mathcal{H}^{d-1}(x) - \int_{Q_L} |\chi_{\tilde{E}^\kappa}(x) - \chi_{\tilde{E}^\kappa}(x + \zeta_i)| dx \geq 0. \quad (4.7)$$

Let $E \subset \kappa\mathbb{Z}$, L -periodic and of finite $(1, \kappa)$ -perimeter. We define the one-dimensional functionals

$$\mathcal{G}_{\tau,L}^{1d,\text{dsc}}(E) := \sum_{z \in \kappa\mathbb{Z}} \hat{K}_\tau(z) \left(\text{Per}(\tilde{E}^\kappa, [0, L]) |z| - \int_0^L |\chi_{\tilde{E}^\kappa}(x) - \chi_{\tilde{E}^\kappa}(x + z)| dx \right).$$

By Fubini theorem, it holds

$$\mathcal{G}_{\tau,L}^{i,\text{dsc}}(E) = \int_{[0,L^{d-1})} \mathcal{G}_{\tau,L}^{1d,\text{dsc}}(\tilde{E}_{x_i^\perp}^\kappa) dx_i^\perp.$$

Let $I_t(r) = (t - r, t + r) \subset \mathbb{R}$.

Lemma 4.4 ([18, Lemma 3.7]). *Let $E \subset \kappa\mathbb{Z}$ be an L -periodic set of finite $(1, k)$ -perimeter and let $\beta := p - d - 1$. Then, for every $\tau \geq 0$,*

$$\mathcal{G}_{\tau,L}^{1d,\text{dsc}}(E) \gtrsim \sum_{x \in \partial \tilde{E}^\kappa \cap [0,L)} \min(h(x)^{-\beta}, \tau^{-1}) + \min(g(x)^{-\beta}, \tau^{-1}). \quad (4.8)$$

As a consequence, if $\mathcal{G}_{\tau,L}^{1d,\text{dsc}}(E) \lesssim \tau^{-1}$, then

$$\min_{x \in \partial \tilde{E}^\kappa \cap [0,L)} \min(h(x), g(x)) \gtrsim \mathcal{G}_{\tau,L}^{1d,\text{dsc}}(E)^{-1/\beta}. \quad (4.9)$$

For $L \geq r > 0$ and $t \in [0, L)$. Then, for every $\delta \geq \tau^{1/\beta}$,

$$\text{Per}(\tilde{E}^\kappa, I_t(r)) - 1 \lesssim r\delta^{-1} + \delta^\beta \mathcal{G}_{\tau,L}^{1d,\text{dsc}}(E). \quad (4.10)$$

Proof. The proof is contained in [18]. The basic ingredient to deal with the discrete kernel is to use (4.6). \square

Lemma 4.5 ([18, Lemma 3.8]). *Let $E \subset \kappa\mathbb{Z}^d$ be an Q_L -periodic set. Then we have that*

$$\mathcal{F}_{\tau,L}^{\text{dsc}}(E) \gtrsim -1 + \frac{1}{L^d} \left(\sum_{i=1}^d \mathcal{G}_{\tau,L}^{i,\text{dsc}}(E) + I_{\tau,L}^{\text{dsc}}(E) \right) \quad (4.11)$$

and

$$\text{Per}_{1,\kappa}(E, Q_L) \lesssim L^d \max(1, \mathcal{F}_{\tau,L}^{\text{dsc}}(E)). \quad (4.12)$$

Proof. The proof is obtained from (4.10) as in [18, Lemma 3.8]. \square

Lemma 4.6 ([18, Lemma 3.9]). *Let $E \subset \kappa\mathbb{Z}^d$ be a Q_L -periodic set. Then for every $m \in \mathbb{N}$, $m \geq 2$, $t \in [0, L)$, $L \geq r > 0$ and $i \in \{1, \dots, d\}$, one has that*

$$|\{x_i^\perp \in [0, L)^{d-1} : \text{Per}(\tilde{E}_{x_i^\perp}^\kappa, (t - r/2, t + r/2)) = m\}| \lesssim \mathcal{G}_{0,L}^i(\tilde{E}^\kappa) L^d r^\beta \left(\frac{1}{m-1} \right)^{p-d}. \quad (4.13)$$

Proof. The proof uses (4.10) again, as in [18, Lemma 3.9]. \square

We now recall the rigidity proposition contained in [18].

Proposition 4.1 ([18, Proposition 4.3]). *Let E be a Q_L -periodic set of finite perimeter such that $\sum_{i=1}^d \mathcal{G}_{0,L}^i(E) + I_{0,L}(E) < +\infty$. Then, E is one-dimensional, i.e. up to permutation of the coordinates, $E = \hat{E} \times \mathbb{R}^{d-1}$ for some L -periodic set \hat{E} .*

4.1 The Gamma-convergence result

In this section we obtain the discrete-to-continuous limit for the discrete problem. Recall the definition of $\mathcal{F}_{0,L}$

$$\mathcal{F}_{0,L}(E) := \begin{cases} \frac{1}{L}(-\text{Per}(\hat{E}, [0, L]) + \mathcal{G}_{0,L}^{1d}(\hat{E})) & \text{if } E = \hat{E} \times \mathbb{R}^{d-1} \text{ for some } L\text{-periodic} \\ & \text{set } \hat{E} \text{ of finite perimeter,} \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 4.1.

(i) Let $\{E_\tau\}_{\tau>0}$, $E_\tau \subset \kappa\mathbb{Z}^d$ be Q_L -periodic sets such that $\sup_\tau \mathcal{F}_{\tau,L}^{\text{dsc}}(E_\tau) < +\infty$. Consider $\{\tilde{E}_\tau^\kappa\}$ defined in (4.1). Then up to subsequences there exists an one-dimensional, Q_L -periodic set E_0 such that \tilde{E}_τ^κ converges in the almost-Hausdorff sense and in L^1 to E_0 . Moreover

$$\liminf_{\tau \downarrow 0} \mathcal{F}_{\tau,L}^{\text{dsc}}(E_\tau) \geq \mathcal{F}_{0,L}(E). \quad (4.14)$$

(ii) For any $E \subset \mathbb{R}^d$ which is Q_L -periodic with $\mathcal{F}_{0,L}(E) < +\infty$ there exists a sequence $E_\tau \subset \kappa\mathbb{Z}^d$ such that \tilde{E}_τ^κ converges to E in L^1 as $\tau \downarrow 0$ and such that

$$\limsup_{\tau \rightarrow 0} \mathcal{F}_{\tau,L}^{\text{dsc}}(E_\tau) \leq \mathcal{F}_{0,L}(E). \quad (4.15)$$

Proof. This proof will follow the continuous version contained in [18, Theorem 1.1]. Let $E_\tau \subset \kappa\mathbb{Z}^d$ be such that $\sup_\tau \mathcal{F}_{\tau,L}^{\text{dsc}}(E_\tau) < +\infty$.

Because of (4.12), one has that $\sup_\tau \text{Per}(\tilde{E}_\tau^\kappa, Q_L) < +\infty$. Hence one can extract a sequence converging to some Q_L -periodic set E of finite perimeter. Let us initially show that for any $i \in \{1, \dots, d\}$, one has that

$$\liminf_{\tau \rightarrow 0} \mathcal{G}_{\tau,L}^{i,\text{dsc}}(E_\tau) \geq \mathcal{G}_{0,L}^i(E) \quad \text{and} \quad \liminf_{\tau \rightarrow 0} I_{\tau,L}(E_\tau) \geq 0. \quad (4.16)$$

Because of (4.11) and Proposition 4.1, one has that E is one-dimensional. Let us initially show the inequality for $\mathcal{G}_{\tau,L}^{i,\text{dsc}}$. The proof for lower bound of $I_{\tau,L}$ is similar. Because $\hat{K}_\tau \leq \hat{K}_{\tau'}$ whenever $\tau > \tau'$, and because of (4.7), one has that

$$\mathcal{G}_{\tau,L}^{i,\text{dsc}}(E_{\tau'}) \leq \mathcal{G}_{\tau',L}^{i,\text{dsc}}(E_{\tau'}).$$

Without loss of generality let $i = 1$. Now, if τ is fixed, by Fatou,

$$\begin{aligned} \liminf_{\tau' \rightarrow 0} \mathcal{G}_{\tau',L}^{1,\text{dsc}}(E_{\tau'}) &\geq \sum_{\zeta_1 \in \mathbb{Z}} \hat{K}_\tau(\zeta_1) \liminf_{\tau' \rightarrow 0} \left[\int_{\partial \tilde{E}^{\kappa'} \cap Q_L} |\nu_1^{\tilde{E}^{\kappa'}}(x)| |\zeta_1| d\mathcal{H}^{d-1}(x) - \int_{Q_L} |\chi_{\tilde{E}^{\kappa'}}(x) - \chi_{\tilde{E}^{\kappa'}(x + \zeta_1)}| dx \right] \\ &\geq \sum_{\zeta_1 \in \mathbb{Z}} \hat{K}_\tau(\zeta_1) \left[\int_{\partial E \cap Q_L} |\nu_1^E(x)| |\zeta_1| d\mathcal{H}^{d-1}(x) - \int_{Q_L} |\chi_E(x) - \chi_E(x + \zeta_1)| dx \right] \\ &= \mathcal{G}_{\tau,L}^{1,\text{dsc}}(E), \end{aligned}$$

where we have used that for fixed ζ_1 , $\int_{\partial E \cap Q_L} |\nu_1^E(x)| |\zeta_1| d\mathcal{H}^{d-1}(x) - \int_{Q_L} |\chi_E(x) - \chi_E(x + \zeta_1)| dx$ is lower semicontinuous with respect to L^1 convergence. Finally, using again (4.7) and the monotone convergence theorem, we have

$$\liminf_{\tau' \rightarrow 0} \mathcal{G}_{\tau', L}^{1, \text{dsc}}(E_{\tau'}) \geq \lim_{\tau \rightarrow 0} \mathcal{G}_{\tau, L}^{1, \text{dsc}}(E) = \mathcal{G}_{0, L}^1(E),$$

which proves (4.16). From (4.16), in order to show the lower bound (4.14), we are left to check that

$$\liminf_{\tau \rightarrow 0} -\text{Per}_{1, \kappa}(E_\tau, Q_L) \geq -\text{Per}_1(E, Q_L).$$

By Fatou and slicing, it is enough to prove that for $i \in \{1, \dots, d\}$ and a.e. $x_i^\perp \in [0, L)^{d-1}$,

$$\liminf_{\tau \rightarrow 0} -\text{Per}(\tilde{E}_{\tau, x_i^\perp}, [0, L)) \geq -\text{Per}(E_{x_i^\perp}, [0, L)). \quad (4.17)$$

Because of (4.11), $\mathcal{G}_{\tau, L}^{1d, \text{dsc}}(E_{\tau, x_i^\perp})$ is bounded by a constant (x_i^\perp dependent) for a.e. $x_i^\perp \in [0, L)^{d-1}$. Hence, because of (4.9), one has that

$$\min_{t \in \partial \tilde{E}_{\tau, x_i^\perp}} \min(h_{x_i^\perp}^\tau(t), g_{x_i^\perp}^\tau(t)) \geq c_{x_i^\perp}, \quad (4.18)$$

for some constant $c_{x_i^\perp} > 0$. Consider the section E_{τ, x_i^\perp} . From the L^1 -convergence of \tilde{E}_τ^κ , we have that $\tilde{E}_{\tau, x_i^\perp}^\kappa$ converges in L^1 to $E_{x_i^\perp}$, for a.e. x_i^\perp . Because of (4.18), we have that $\tilde{E}_{\tau, x_i^\perp}^\kappa$ converges to $E_{x_i^\perp}$ in Hausdorff distance and in particular, since $\tilde{E}_{\tau, x_i^\perp}^\kappa \subset \mathbb{R}$, that for τ small enough $\text{Per}(\tilde{E}_{\tau, x_i^\perp}^\kappa, [0, L)) = \text{Per}(E_{x_i^\perp}, [0, L))$, which yields (4.17).

Moreover this also implies that $\partial \tilde{E}_\tau^\kappa$ converges to ∂E almost-Hausdorff.

The proof of (4.15) is almost immediate. Indeed because of the definition of $\mathcal{G}_{0, L}^{1d}$, one has that E is union of stripes. Hence by taking the as E_τ such that \tilde{E}_τ^κ approximates in Hausdorff distance E one has the desired result. \square

5 Structure of minimizers

In this section we prove Theorems 1.1 and 1.3, asserting that minimizers of $\mathcal{F}_{\tau, L}$, resp. $\mathcal{F}_{\tau, L}^{\text{dsc}}$, are periodic stripes, provided τ is small enough.

Then, we provide a short proof of Theorems 1.2 and 1.4 dealing with the width of the optimal stripes.

Lemma 5.1. *Let $E \subset \mathbb{R}^d$ be a Q_L -periodic set of finite perimeter and S be a set composed of periodic stripes, i.e. (up to exchange of coordinates and translations) there exists $\hat{E} \subset \mathbb{R}$ such that $S = \hat{E} \times \mathbb{R}^{d-1}$ and*

$$\hat{E} := \bigcup_{i \in \mathbb{Z}} [2ih, (2i+1)h),$$

for a suitable h .

Denote by

$$H(E, S, \varepsilon) := \{x \in E \setminus S : d(x_1, h\mathbb{Z}) > \varepsilon\} \cup \{x \in S \setminus E : d(x_1, h\mathbb{Z}) > \varepsilon\}.$$

There exists $\bar{\varepsilon} > 0$, such that, for all $\varepsilon \leq \bar{\varepsilon}$, there exist $\delta(\varepsilon) > 0$ and $\tau(\varepsilon) > 0$ (independent on E) such that if $|H(E, S, \varepsilon)| < \delta(\varepsilon)$ and $\tau \leq \tau(\varepsilon)$, one has that

$$-\sum_{i=2}^d \text{Per}_{1i}(E, Q_L) + \sum_{i=2}^d \mathcal{G}_{\tau, L}^i(E) + I_{\tau, L}^i(E) \geq 0. \quad (5.1)$$

Moreover, the left hand side of (5.1) is equal to zero if and only if E is union of stripes.

Proof. As E, S and ε are fixed, we denote by $H = H(E, S, \varepsilon)$.

Let $i \in \{2, \dots, d\}$. Then,

$$-\text{Per}_{1i}(E) + \mathcal{G}_{\tau, L}^i(E) = \int_{[0, L]^{d-1}} \left[-\#\partial E_{x_i^\perp} + \mathcal{G}_{\tau, L}^{1d}(E_{x_i^\perp}) \right] d\mathcal{H}^{d-1}(dx_i^\perp).$$

For (2.8),

$$\begin{aligned} -\text{Per}_{1i}(E) + \mathcal{G}_{\tau, L}^i(E) &\geq \int_{[0, L]^{d-1}} \sum_{t \in \partial E_{x_i^\perp}} [-1 + C \min(h_{x_i^\perp}(t)^{-\beta}, \tau^{-1}) \\ &\quad + C \min(g_{x_i^\perp}(t)^{-\beta}, \tau^{-1})]. \end{aligned} \quad (5.2)$$

Consider the following decomposition

$$[0, L]^{d-1} = A_1^i(\eta) \cup A_2^i \cup A_3^i(\eta),$$

where

$$A_1^i(\eta) = \{x_i^\perp \in [0, L]^{d-1} : \min_{t \in \partial E_{x_i^\perp}} (h_{x_i^\perp}(t), g_{x_i^\perp}(t)) \geq \eta\}; \quad (5.3)$$

$$A_2^i = \{x_i^\perp \in [0, L]^{d-1} : \partial E_{x_i^\perp} = \emptyset\}; \quad (5.4)$$

$$A_3^i(\eta) = \{x_i^\perp \in [0, L]^{d-1} : \exists t \in \partial E_{x_i^\perp} \text{ s.t. } h_{x_i^\perp}(t) < \eta \text{ or } g_{x_i^\perp}(t) < \eta\} \quad (5.5)$$

and

$$0 < \eta = \sup\{s > 0 : -1 + C \min(s^{-\beta}, \tau^{-1}) > \frac{L}{s}\}. \quad (5.6)$$

To show that such an η exists, notice that $-1 + Cs^{-\beta} > L/s$ as soon as $s \leq \eta$ for a sufficiently small η (since $\beta > 1$). Then, by choosing $\tau \leq \eta^\beta$, which we can always assume since the result we want to show has to be valid for $\tau \leq \bar{\tau}$, the inequality in (5.6) holds.

Observe then that, if $x_i^\perp \in A_2^i$, then the integrand in the r.h.s. of (5.2) is zero, if $x_i^\perp \in A_1^i(\eta)$ then it is bigger or equal than $-\frac{L}{\eta}$ and if $x_i^\perp \in A_3^i(\eta)$ then thanks to (5.6) it is strictly bigger than 0.

Hence,

$$-\text{Per}_{1i}(E) + \mathcal{G}_{\tau, L}^i(E) \geq -\frac{L}{\eta} \mathcal{H}^{d-1}(A_1^i(\eta)) + c \mathcal{H}^{d-1}(A_3^i(\eta)) \quad (5.7)$$

for some $c > 0$.

We will proceed now to show that $I_{\tau, L}^i(E) > 2\frac{L}{\eta} \mathcal{H}^{d-1}(A_1^i(\eta))$. In order to do so, we will estimate via disintegration the contribution in $I_{\tau, L}^i(E)$ for fixed $x_i^\perp \in A_1^i$ and show that it is larger than $2\frac{L}{\eta}$ for a certain choice of parameters ε, δ and τ .

Let us now estimate $I_{\tau,L}^i(E)$. Recall that

$$I_{\tau,L}^i(E) = \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \int_{[0,L]^{d-1}} \int_{[0,L]} K_{\tau}(\zeta) f_E(x_i^{\perp}, x_i, \zeta_i^{\perp}, \zeta_i) dx_i dx_i^{\perp} d\zeta_i d\zeta_i^{\perp},$$

where

$$f_E(x_i^{\perp}, x_i, \zeta_i^{\perp}, \zeta_i) := |\chi_E(x_i^{\perp} + x_i + \zeta_i) - \chi_E(x_i + x_i^{\perp})| |\chi_E(x_i^{\perp} + x_i + \zeta_i^{\perp}) - \chi_E(x_i + x_i^{\perp})|.$$

Let us now fix $x_i^{\perp} \in A_1^i(\eta)$. Because of the definition of $A_1^i(\eta)$, there exists $t \in \partial E_{x_i^{\perp}}$ such that one of the followings hold

- (i) $(t - \eta, t) \subset E_{x_i^{\perp}}$ and $(t, t + \eta) \subset E_{x_i^{\perp}}^c$
- (ii) $(t - \eta, t) \subset E_{x_i^{\perp}}^c$ and $(t, t + \eta) \subset E_{x_i^{\perp}}$.

Without loss of generality we may assume that the first item above holds. Then if

$$x_i \in (t - \eta, t), \quad x_i + \zeta_i \in (t, t + \eta) \quad \text{and} \quad \zeta_i^{\perp} \in \left\{ z \in \mathbb{R}^{d-1} : \varepsilon \leq \|z\| \leq 2\varepsilon, x + z \in S^c \setminus E \right\},$$

one has that

$$f_E(x_i^{\perp}, x_i, \zeta_i^{\perp}, \zeta_i) = 1 \quad \text{and} \quad K_{\tau}(\zeta) \gtrsim \frac{1}{\varepsilon^p + \tau^{\beta p}}.$$

Thus, letting $H_{x_i + \zeta_i} = H \cap \{z \in \mathbb{R}^d : z_i = x_i + \zeta_i\}$,

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \int_{[0,L]} f_E(x_i^{\perp}, x_i, \zeta_i^{\perp}, \zeta_i) K_{\tau}(\zeta) dx_i d\zeta_i d\zeta_i^{\perp} &\gtrsim \eta^2 \frac{1}{\varepsilon^p + \tau^{\beta p}} \varepsilon^{d-1} \\ &\quad - \frac{1}{\varepsilon^p + \tau^{\beta p}} \int_{t-\eta}^t \int_{t-x_i}^{t+\eta-x_i} \mathcal{H}^{d-1}(H_{x_i + \zeta_i}) d\zeta_i dx_i. \end{aligned} \tag{5.8}$$

On the other side, by integrating we have that

$$\frac{1}{\varepsilon^p + \tau^{\beta p}} \int_{t-\eta}^t \int_{t-x_i}^{t+\eta-x_i} \mathcal{H}^{d-1}(H_{x_i + \zeta_i}) d\zeta_i dx_i \leq \frac{1}{\varepsilon^p + \tau^{\beta p}} \int_{t-\eta}^t \int_0^L \mathcal{H}^{d-1}(H_{x_i + \zeta_i}) d\zeta_i dx_i \lesssim \frac{\eta \delta}{\varepsilon^p + \tau^{\beta p}},$$

where in the last inequality we have used that $|H| \leq \delta$. Integrating both sides of (5.8) with respect to dx_i^{\perp} , we obtain that

$$I_{\tau,L}^i(E) \gtrsim \left(\frac{\eta^2 \varepsilon^{d-1}}{\varepsilon^p + \tau^{\beta p}} - \frac{\eta \delta}{\varepsilon^p + \tau^{\beta p}} \right) \mathcal{H}^{d-1}(A_1^i(\eta)).$$

For any ε we can choose δ such $\eta^2 \varepsilon^{d-1} - \eta \delta \geq \eta^2 \varepsilon^{d-1} / 2$ and moreover choose ε, τ such that

$$\frac{1}{2} \frac{\eta^2 \varepsilon^{d-1}}{\varepsilon^p + \tau^{\beta p}} > L / \eta.$$

Hence we have just shown that for $\delta, \varepsilon, \tau$ small enough one has that, for $i \geq 2$,

$$-\text{Per}_{1i}(E) + \mathcal{G}_{\tau,L}^i(E) + I_{\tau,L}^i(E) \geq 0.$$

Summing over i , we obtain have the first part of claim. In order to obtain the last part of the claim it is enough to notice that the above is null if and only if $\mathcal{H}^{d-1}(A_1^i(\eta)) = 0$ and $\mathcal{H}^{d-1}(A_3^i(\eta)) = 0$ which implies that E is composed of stripes. \square

Proof of Theorem 1.1. Before proceeding to the proof, let us explain the strategy of the proof. In the first step of the proof we will show that the minimizers are almost-Hausdorff close to some set S that consists of optimal periodic stripes. Without loss of generality let us assume that $S = \hat{E} \times \mathbb{R}^{d-1}$, where

$$\hat{E} = \bigcup_{i \in \mathbb{Z}} [2i, (2i+1)h).$$

Once in this configuration, we will do a slicing argument. Namely, we will split the contributions of the functional in two parts

$$- \int_{[0,L]^{d-1}} \text{Per}(E_{x_1^\perp}) dx_1^\perp + \int_{[0,L]^{d-1}} \mathcal{G}_{\tau,L}^{1d}(E_{x_1^\perp}) dx_1^\perp + I_{\tau,L}^1 \quad (5.9)$$

and one which is larger than

$$- \sum_{i=2}^d \text{Per}_{1,i}(E) + \sum_{i=2}^d \mathcal{G}_{\tau,L}^i(E) + \sum_{i=2}^d I_{\tau,L}^i(E). \quad (5.10)$$

Afterwards, we will notice that optimal stripes minimize the first term (5.9) and both the second part of $\mathcal{F}_{\tau,L}$ and (5.10) are equal to zero. On the other side, for anything that does not consist of stripes and is almost-Hausdorff close to S , one has that the contribution given from (5.10) is strictly positive. Thus optimal stripes are optimizers for $\mathcal{F}_{\tau,L}$. To prove the positivity of (5.10) in case of non-optimal stripes, we will use Lemma 5.1.

Step 1: From the Γ -convergence result, we have that for every $\varepsilon, \delta > 0$, there exists a $\tau_0 = \tau_0(\varepsilon, \delta) > 0$ such that, for every $0 < \tau < \tau_0$ and for every minimizer E_τ of $\mathcal{F}_{\tau,L}$, one has that

1. E_τ is ε -almost close to S , where S is a periodic stripe of size $2h$. Without loss of generality, we may assume that $S = \hat{E} \times [0, L]^{d-1}$, where

$$\hat{E} = \bigcup_{i=0}^k [2ih, (2i+1)h).$$

2. $|H(E_\tau, S, \varepsilon)| < \delta$, where similarly to Lemma 5.1, we denote by

$$H(E_\tau, S, \varepsilon) := \{x \in E_\tau \setminus S : d(x_1, h\mathbb{Z}) > \varepsilon\} \cup \{x \in S \setminus E_\tau : d(x_1, h\mathbb{Z}) > \varepsilon\}.$$

We fix $\varepsilon \leq \bar{\varepsilon}$, $\delta \leq \delta(\varepsilon)$ and $\bar{\tau} < \min\{\tau(\varepsilon), \tau_0(\varepsilon, \delta)\}$ where $\bar{\varepsilon}$, $\delta(\varepsilon)$ and $\tau(\varepsilon)$ are given in Lemma 5.1.

Step 2: Let us consider the original functional $\mathcal{F}_{\tau,L}$, for $\tau \leq \bar{\tau}$ as in Step 1 and set $E = E_\tau$.

Remember, with the help of (2.1), that

$$\begin{aligned} \mathcal{F}_{\tau,L}(E) &= \frac{1}{L^d} \left(-\text{Per}_1(E) + \int_{\mathbb{R}^d} K_\tau(\zeta) \left[\int_{\partial E \cap Q_L} \sum_{i=1}^d |\nu_i^E| |\zeta_i| - \int_{Q_L} |\chi_E(x) - \chi_E(x + \zeta)| \right] \right) \\ &= \frac{1}{L^d} \left(-\text{Per}_{11}(E) + \int_{\mathbb{R}^d} K_\tau(\zeta) \left[\int_{\partial E \cap Q_L} |\nu_1^E| |\zeta_1| - \int_{Q_L} |\chi_E(x) - \chi_E(x + \zeta_1)| \right] \right) \end{aligned} \quad (5.11)$$

$$+ \frac{1}{L^d} \left(\sum_{i \geq 2} -\text{Per}_{1i}(E) + \int_{\mathbb{R}^d} K_\tau(\zeta) \left[\int_{\partial E \cap Q_L} \sum_{i \geq 2} |\nu_i^E| |\zeta_i| - \int_{Q_L} |\chi_E(x) - \chi_E(x + \zeta_1^\perp)| \right] \right) \quad (5.12)$$

$$+ 2 \int_{\mathbb{R}^d} K_\tau(\zeta) \int_{Q_L} |\chi_E(x) - \chi_E(x + \zeta_1)| |\chi_E(x) - \chi_E(\zeta_1^\perp)| \quad (5.13)$$

$$\begin{aligned} &\geq \frac{1}{L^d} \left(-\text{Per}_{11}(E) + \int_{\mathbb{R}^d} K_\tau(\zeta) \left[\int_{\partial E \cap Q_L} |\nu_1^E| |\zeta_1| - \int_{Q_L} |\chi_E(x) - \chi_E(x + \zeta_1)| \right] \right) \\ &- \sum_{i=2}^d \text{Per}_{1,i}(E) + \sum_{i=2}^d \mathcal{G}_{\tau,L}^i(E) + \sum_{i=2}^d I_{\tau,L}^i(E). \end{aligned} \quad (5.14)$$

One notices immediately that, thanks to Theorem 3.1, if $E = \mathring{E}_\tau \times [0, L]^{d-1}$ with \mathring{E}_τ 1-dimensional periodic set of period h_τ , then the first term (5.11) of $\mathcal{F}_{\tau,L}$ is minimized, while the second terms (5.12), (5.13) and (5.14) are equal to zero.

On the other hand, from Lemma 5.1, if a minimizer E does not have such a structure, or more in general E is not a stripe in direction e_1 , then the last term (5.14) is strictly positive. \square

Proof of Theorem 1.3. The proof of Theorem 1.3 is a minor modification of the proof of Theorem 1.1 by considering \tilde{E}^κ with $\kappa = \tau^\beta$. Indeed, Step 1 follows in the same way from the Γ -convergence result of Theorem 4.1. For Step 2 one uses the analogous results developed in Section 4. \square

Proof of Theorems 1.2 and 1.4. This result is already contained in the second step of the proof of Theorem 1.1, where the 1-dimensional result of Theorem 3.1 determines the width of the optimal stripes.

Analogously, thanks to Remark 3.2, one obtains analogous features for the optimal width of the optimal stripes in the discrete case. \square

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