

# SOME REMARKS ON HUISKEN'S MONOTONICITY FORMULA FOR MEAN CURVATURE FLOW

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ABSTRACT. We discuss a monotone quantity related to Huisken's monotonicity formula and some technical consequences for mean curvature flow.

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## 1. MAXIMIZING HUISKEN'S MONOTONICITY FORMULA

For an immersed hypersurface  $M \subset \mathbb{R}^{n+1}$ , we call  $A$  and  $H$  respectively its *second fundamental form* and its *mean curvature*.

Let  $M_t = \varphi(M, t)$  be the *mean curvature flow* (MCF) of an  $n$ -dimensional compact hypersurface in  $\mathbb{R}^{n+1}$ , defined by the smooth family of immersions  $\varphi : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$  which satisfies  $\partial_t \varphi = H\nu$  where  $\nu$  is the "inner" unit normal vector field to the hypersurface.

Huisken in [7] found his fundamental monotonicity formula

$$(1.1) \quad \frac{d}{dt} \int_M \frac{e^{-\frac{|x-p|^2}{4(C-t)}}}{[4\pi(C-t)]^{n/2}} d\mu_t(x) = - \int_M \frac{e^{-\frac{|x-p|^2}{4(C-t)}}}{[4\pi(C-t)]^{n/2}} \left| H + \frac{\langle x-p | \nu \rangle}{2(C-t)} \right|^2 d\mu_t(x) \leq 0,$$

for every  $p \in \mathbb{R}^{n+1}$ , in the time interval  $[0, \min\{C, T\})$ . Here  $d\mu_t$  is the canonical measure on  $M$  associated to the metric induced by the immersion at time  $t$ .

We call the quantity  $\int_M \frac{e^{-\frac{|x-p|^2}{4(C-t)}}}{[4\pi(C-t)]^{n/2}} d\mu_t(x)$ , the *Huisken's functional*.

Such formula was generalized by Hamilton in [5, 6] as follows, suppose that we have a positive smooth solution of  $u_t = -\Delta u$  in  $\mathbb{R}^{n+1} \times [0, C)$  then, in the time interval  $[0, \min\{C, T\})$ , there holds

$$(1.2) \quad \begin{aligned} \frac{d}{dt} \left[ \sqrt{2(C-t)} \int_M u d\mu_t \right] &= - \sqrt{2(C-t)} \int_M u |H - \langle \nabla \log u | \nu \rangle|^2 d\mu_t \\ &\quad - \sqrt{2(C-t)} \int_M \left( \nabla^\perp \nabla^\perp u - \frac{|\nabla^\perp u|^2}{u} + \frac{u}{2(C-t)} \right) d\mu_t \end{aligned}$$

where  $\nabla^\perp$  denotes the covariant derivative along the normal direction.

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**Definition 1.1.** Let  $\varphi : M \rightarrow \mathbb{R}^{n+1}$  be a smooth, compact, immersed hypersurface. Given  $\tau > 0$ , we consider the family  $\mathcal{F}_\tau$  of smooth positive functions  $u : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  such that  $\int_{\mathbb{R}^{n+1}} u \, dx = 1$  and there exists a smooth positive solution of the problem

$$\begin{cases} v_t = -\Delta v \text{ in } \mathbb{R}^{n+1} \times [0, \tau), \\ v(x, 0) = u(x) \text{ for every } p \in \mathbb{R}^{n+1}. \end{cases}$$

Then, we define the following quantity

$$\sigma(\varphi, \tau) = \sup_{u \in \mathcal{F}_\tau} \sqrt{4\pi\tau} \int_M u \, d\mu.$$

*Remark 1.2.* The heat kernel  $K_{\mathbb{R}^{n+1}}(x, p, \tau) = \frac{e^{-\frac{|x-p|^2}{4\tau}}}{(4\pi\tau)^{(n+1)/2}}$  of  $\mathbb{R}^{n+1}$  at time  $\tau > 0$  and point  $p \in \mathbb{R}^{n+1}$  clearly belongs to the family  $\mathcal{F}_\tau$ .

It is immediate to see by this remark that the quantity  $\sigma(\varphi, \tau)$  is positive and precisely, for every  $p \in \mathbb{R}^{n+1}$  and  $\tau > 0$ ,

$$\sigma(\varphi, \tau) \geq \sqrt{4\pi\tau} \int_M \frac{e^{-\frac{|x-p|^2}{4\tau}}}{(4\pi\tau)^{(n+1)/2}} d\mu(x) = \int_M \frac{e^{-\frac{|x-p|^2}{4\tau}}}{(4\pi\tau)^{n/2}} d\mu(x) > 0,$$

which is the quantity of the ‘‘classical’’ Huisken’s monotonicity formula. Hence,

$$(1.3) \quad \sigma(\varphi, \tau) \geq \sup_{p \in \mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-p|^2}{4\tau}}}{(4\pi\tau)^{n/2}} d\mu(x) > 0.$$

We want to see that actually this inequality is an equality, that is, we can take the sup only on heat kernels. Moreover, the sup is a maximum.

We work out some properties of the functions  $u \in \mathcal{F}_\tau$ .

We recall the integrated version of Li–Yau Harnack inequality (see [11]).

**Proposition 1.3** (Li–Yau integral Harnack inequality). *Let  $u : \mathbb{R}^{n+1} \times (0, T) \rightarrow \mathbb{R}$  be a smooth positive solution of heat equation, then for every  $0 < t \leq s < T$  we have*

$$u(x, t) \leq u(y, s) \left(\frac{s}{t}\right)^{(n+1)/2} e^{\frac{|x-y|^2}{4(s-t)}}.$$

Since the functions  $v : \mathbb{R}^{n+1} \times [0, \tau) \rightarrow \mathbb{R}$  associated to any  $u \in \mathcal{F}_\tau$  are positive solutions of the backward heat equation, such inequality reads, for  $0 \leq s \leq t < \tau$ ,

$$v(x, t) \leq v(y, s) \left(\frac{\tau-s}{\tau-t}\right)^{(n+1)/2} e^{\frac{|x-y|^2}{4(t-s)}}.$$

This estimate, together with the uniqueness theorem for positive solution of the heat equation (see again [11]), implies that the function  $u = v(\cdot, 0)$  is obtained by convolution of the function  $v(\cdot, t)$  with the *forward* heat kernel at time  $t > 0$ . This fact implies that the condition  $\int_{\mathbb{R}^{n+1}} v(x, t) \, dx = 1$  holds for every  $t \in [0, \tau)$ , and that every derivative of every function  $v$  is bounded in the strip  $[0, \tau - \varepsilon]$ , for every  $\varepsilon > 0$ .

The functions  $v(\cdot, t)$  weakly\* converge as probability measures, as  $t \rightarrow \tau$ , to some positive unit measure  $\lambda$  on  $\mathbb{R}^{n+1}$  such that

$$(1.4) \quad v(x, t) = \int_{\mathbb{R}^{n+1}} \frac{e^{-\frac{|x-y|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{(n+1)/2}} d\lambda(y).$$

Conversely, every probability measure  $\lambda$ , by convolution with the heat kernel, gives rise to a function  $v$  such that  $v(\cdot, \tau) \in \mathcal{F}_\tau$ , the most interesting case being

$\lambda = \delta_p$  for  $p \in \mathbb{R}^{n+1}$ .

Indeed, we know that for every  $t \in [0, \tau)$  and  $s \in (t, \tau)$ ,

$$v(x, t) = \int_{\mathbb{R}^{n+1}} v(y, s) \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{[4\pi(s-t)]^{(n+1)/2}} dx$$

hence, choosing a sequence of times  $s_i \nearrow \tau$  such that the measures  $v(\cdot, s_i) \mathcal{L}^{n+1}$  weakly\* converge to some measure  $\lambda$ , we get equality (1.4), since  $\frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{[4\pi(s-t)]^{(n+1)/2}}$

converges uniformly to  $\frac{e^{-\frac{|x-y|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{(n+1)/2}}$  on  $\mathbb{R}^{n+1}$ , as  $s \rightarrow \tau$ .

This representation formula also implies that the limit measure  $\lambda$  is unique and that actually  $\lim_{s \rightarrow \tau} v(\cdot, s) \mathcal{L}^{n+1} = \lambda$  in the weak\* convergence of measures on  $\mathbb{R}^{n+1}$ .

Finally, we show that  $|\lambda| = 1$ . This follows by Fubini–Tonelli’s theorem for positive product measures, as  $\int_{\mathbb{R}^{n+1}} u(x) dx = 1$ ,

$$\begin{aligned} 1 &= \int_{\mathbb{R}^{n+1}} u(x) dx = \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \frac{e^{-\frac{|x-y|^2}{4\tau}}}{[4\pi\tau]^{(n+1)/2}} d\lambda(y) dx \\ &= \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \frac{e^{-\frac{|x-y|^2}{4\tau}}}{[4\pi\tau]^{(n+1)/2}} dx d\lambda(y) \\ &= \int_{\mathbb{R}^{n+1}} d\lambda(y) = |\lambda|. \end{aligned}$$

By this discussion it follows that the family  $\mathcal{F}_\tau$  consists of

$$u(x, t) = \int_{\mathbb{R}^{n+1}} \frac{e^{-\frac{|x-y|^2}{4\tau}}}{[4\pi\tau]^{(n+1)/2}} d\lambda(y)$$

where  $\lambda$  varies among the convex set of Borel probability measures on  $\mathbb{R}^{n+1}$  (which is weak\*-compact).

A consequence of this fact is that since the integral  $\sqrt{4\pi\tau} \int_M u d\mu$  is a linear functional in the function  $u$ , the sup in defining  $\sigma(\varphi, \tau)$  can be taken considering only the extremal points of the above convex, which are the *delta* measures in  $\mathbb{R}^{n+1}$ . Consequently, the functions  $u$  to be considered can be restricted to be heat kernels at time  $\tau > 0$ .

It is then easy to conclude that as the hypersurface  $M$  is compact in  $\mathbb{R}^{n+1}$ , the sup is actually a maximum.

**Proposition 1.4.** *The quantity  $\sigma(\varphi, \tau)$  is given by*

$$\sigma(\varphi, \tau) = \max_{p \in \mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-p|^2}{4\tau}}}{(4\pi\tau)^{n/2}} d\mu(x).$$

We have also easily that

$$\sigma(\varphi, \tau) = \sup_{p \in \mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-p|^2}{4\tau}}}{(4\pi\tau)^{n/2}} d\mu(x) \leq \int_M \frac{1}{(4\pi\tau)^{n/2}} d\mu(x) \leq \frac{\text{Area}(M)}{(4\pi\tau)^{n/2}}.$$

**Proposition 1.5** (Rescaling Invariance). *For every  $\lambda > 0$  we have*

$$\sigma(\lambda\varphi, \lambda^2\tau) = \sigma(\varphi, \tau).$$

*Proof.* Let  $u \in \mathcal{F}_\tau$  with associate solution of backward heat equation  $v : \mathbb{R}^{n+1} \times [0, \tau) \rightarrow \mathbb{R}$  and consider the rescaled function  $\tilde{u}(y) = u(y/\lambda)\lambda^{-(n+1)}$ .

It is easy to see that

$$\int_{\mathbb{R}^{n+1}} \tilde{u}(y) dy = \lambda^{-(n+1)} \int_{\mathbb{R}^{n+1}} u(y/\lambda) dy = \int_{\mathbb{R}^{n+1}} u(x) dx = 1$$

with the change of variable  $x = \lambda^{-(n+1)}y$ , moreover the function  $\tilde{v}(y, s) = v(y/\lambda, s/\lambda^2)\lambda^{-(n+1)}$  is a positive solution of the backward heat equation on the time interval  $\lambda^2\tau$ , hence  $\tilde{u} \in \mathcal{F}_{\lambda^2\tau}$ .

It is now a straightforward computation to see that

$$\sqrt{4\pi\lambda^2\tau} \int_M \tilde{u} d\mu_{\lambda\varphi} = \sqrt{4\pi\tau} \int_M u d\mu_\varphi,$$

for every smooth immersion of a compact hypersurface  $\varphi : M \rightarrow \mathbb{R}^{n+1}$ . The statement clearly follows.  $\square$

By formula (1.2), as the second term vanishes when  $v$  is a backward heat kernel, it follows that if  $\varphi : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$  is the MCF of a compact hypersurface  $M$ , we have

$$\begin{aligned} & \frac{d}{dt} \left[ \sqrt{2(C-t)} \int_M K_{\mathbb{R}^{n+1}}(x, p, C-t) d\mu_t(x) \right] \\ &= -\sqrt{2(C-t)} \int_M K_{\mathbb{R}^{n+1}}(x, p, C-t) |\mathbf{H} + \langle x-p | \nu \rangle / 2(C-t)|^2 d\mu_t(x) \end{aligned}$$

which is clearly negative in the time interval  $[0, \min\{C, T\})$ .

**Proposition 1.6** (Monotonicity and Differentiability). *Along a MCF,  $\varphi : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$ , if  $\tau(t) = C - t$  for some constant  $C > 0$ , the quantity  $\sigma(\varphi_t, \tau)$  is monotone nonincreasing in the time interval  $[0, \min\{C, T\})$ , hence it is differentiable almost everywhere. Moreover, letting  $p_\tau$  a point in  $\mathbb{R}^{n+1}$  such that  $K_{\mathbb{R}^{n+1}}(x, p_\tau, \tau)$  is one of maximizer for  $\sigma(\varphi_t, \tau(t))$  of Proposition 1.4, we have for almost every  $t \in [0, \min\{C, T\})$ ,*

$$(1.5) \quad \frac{d}{dt} \sigma(\varphi_t, \tau) \leq - \int_M \frac{e^{-\frac{|x-p_\tau|^2}{4\tau}}}{(4\pi\tau)^{n/2}} \left| \mathbf{H} + \frac{\langle x-p_\tau | \nu \rangle}{2\tau} \right|^2 d\mu_t$$

or, since this inequality has to be intended in distributional sense, for every  $0 \leq r < t \leq \min\{C, T\}$ ,

$$(1.6) \quad \sigma(\varphi_r, \tau(r)) - \sigma(\varphi_t, \tau(t)) \geq \int_r^t \int_M \frac{e^{-\frac{|x-p_{\tau(s)}|^2}{4\tau(s)}}}{(4\pi\tau(s))^{n/2}} \left| \mathbf{H} + \frac{\langle x-p_{\tau(s)} | \nu \rangle}{2\tau(s)} \right|^2 d\mu_s ds$$

*Proof.* As the function  $\sigma(\varphi_t, \tau)$  is the maximum of monotone nonincreasing smooth functions, it also must be monotone nonincreasing, hence, differentiable at almost every time  $t \in [0, \min\{C, T\})$ .

The last assertion is standard, using *Hamilton's trick* (see [4]) to exchange the sup and derivative operations.  $\square$

*Remark 1.7.* Notice that the quantity  $\sigma$  can be defined also for any  $n$ -dimensional countably rectifiable subset  $S$  of  $\mathbb{R}^{n+1}$ , by substituting in the definition the term  $\int_M u d\mu$  with  $\int_S u d\mathcal{H}^n$ , where  $\mathcal{H}^n$  is the  $n$ -dimensional Hausdorff measure (possibly counting multiplicities). If then  $S$  is the support of a compact rectifiable *variety*, with finite Area, moving by mean curvature according to Brakke's definition (see [1]), Huisken's monotonicity formula (1.2) holds, hence, also this proposition.

**Definition 1.8.** We define, in the same hypothesis, for  $\tau = C - t$  with  $C \leq T$ ,

$$\Sigma(C) = \lim_{t \rightarrow C^-} \sigma(\varphi_t, \tau),$$

and  $\Sigma = \Sigma(T)$ .

By the previous discussion,  $\Sigma \geq \sup_{p \in \mathbb{R}^{n+1}} \Theta(p)$ , where this latter quantity is defined as

$$(1.7) \quad \Theta(p) = \lim_{t \rightarrow T^-} \int_M \frac{e^{-\frac{|x-p|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} d\mu_t(x),$$

the existence of this limit for every  $p \in \mathbb{R}^{n+1}$  is an obvious consequence of Huisken's monotonicity formula.

Moreover, it is easy to see also the existence of  $\max_{p \in \mathbb{R}^{n+1}} \Theta(p)$ .

**Definition 1.9.** Let  $\varphi : M \rightarrow \mathbb{R}^{n+1}$  be a smooth, compact, immersed hypersurface. Then we define

$$\nu(\varphi) = \sup_{\tau > 0} \sigma(\varphi, \tau).$$

**Proposition 1.10.** *The quantity  $\nu(\varphi)$  is finite and actually reached by some  $\tau_\varphi$ .*

*Proof.* Indeed, we have

$$\lim_{\tau \rightarrow 0^+} \sigma(\varphi, \tau) = \Theta(\varphi) > 0,$$

where  $\Theta(\varphi)$  is the maximum (which clearly exists as  $M$  is compact) of the  $n$ -dimensional density of  $\varphi(M)$  in  $\mathbb{R}^{n+1}$ . Then, if  $\varphi$  is an embedding,  $\Theta(\varphi) = 1$ , otherwise it will be the highest multiplicity of the points of  $\varphi(M)$ .

We show then that

$$\lim_{\tau \rightarrow +\infty} \sigma(\varphi, \tau) = 0.$$

By the rescaling property of  $\sigma$ , we have  $\sigma(\varphi, \tau) = \sigma(\varphi/\sqrt{4\pi\tau}, 1/4\pi)$ , hence we need to show that

$$\limsup_{\tau \rightarrow +\infty} \sup_{u \in \mathcal{F}_1} \int_{\frac{M}{\sqrt{4\pi\tau}}} u d\mu = 0.$$

But we already saw that any function  $u \in \mathcal{F}_1$  satisfies  $0 \leq u(x) \leq \frac{1}{(4\pi)^{(n+1)/2}}$  hence,

$$\limsup_{\tau \rightarrow +\infty} \sup_{u \in \mathcal{F}_1} \int_{\frac{M}{\sqrt{4\pi\tau}}} u d\mu \leq \limsup_{\tau \rightarrow +\infty} \frac{\text{Vol}(M/\sqrt{4\pi\tau})}{(4\pi)^{(n+1)/2}} = \limsup_{\tau \rightarrow +\infty} \frac{\text{Vol}(M)}{(4\pi)^{(2n+1)/2}} \tau^{-n/2} = 0.$$

□

The following statement can be proved by the same argument of the proof of Proposition 1.6.

**Proposition 1.11 (Monotonicity and Differentiability – II).** *Along a MCF,  $\varphi : M \times [0, T] \rightarrow \mathbb{R}^{n+1}$ , the quantity above  $\nu(\varphi_t)$  is monotone non increasing in the time interval  $[0, T]$ , hence it is differentiable almost everywhere.*

*Moreover, letting  $p_\varphi \in \mathbb{R}^{n+1}$  and  $\tau_\varphi$  to be some of the maximizers whose existence is granted by Propositions 1.4 and 1.10, we have for almost every  $t \in [0, T]$ ,*

$$(1.8) \quad \frac{d}{dt} \nu(\varphi_t) \leq - \int_M \frac{e^{-\frac{|x-p_{\varphi_t}|^2}{4\tau_{\varphi_t}}}}{(4\pi\tau_{\varphi_t})^{n/2}} \left| \mathbf{H} + \frac{\langle x - p_{\varphi_t} | \nu \rangle}{2\tau_{\varphi_t}} \right|^2 d\mu_t(x)$$

*or, since this inequality has to be intended in distributional sense, for every  $0 \leq r < t < T$ ,*

$$(1.9) \quad \nu(\varphi_r) - \nu(\varphi_t) \geq \int_r^t \int_M \frac{e^{-\frac{|x-p_{\varphi_s}|^2}{4\tau_{\varphi_s}}}}{(4\pi\tau_{\varphi_s})^{n/2}} \left| \mathbf{H} + \frac{\langle x - p_{\varphi_s} | \nu \rangle}{2\tau_{\varphi_s}} \right|^2 d\mu_s(x) ds.$$

*Remark 1.12.* One can repeat all this analysis for a compact, immersed hypersurface of a flat Riemannian manifold  $T$ . Moreover, if the original hypersurface  $\varphi : M \rightarrow \mathbb{R}^{n+1}$  is immersed in  $\mathbb{R}^{n+1}$ , we can choose a Riemannian covering map  $I : \mathbb{R}^{n+1} \rightarrow T$  and consider the immersion  $\tilde{\varphi} = I \circ \varphi : M \rightarrow T$ . Then, we define as above, for every  $\tau > 0$ , the family  $\mathcal{F}_{T,\tau}$  of smooth positive functions  $u : T \rightarrow \mathbb{R}$  such that  $\int_T u dx = 1$  and there exists a smooth positive solution of the problem

$$\begin{cases} v_t = -\Delta v \text{ in } T \times [0, \tau) \\ v(p, 0) = u(x) \text{ for every } p \in T. \end{cases}$$

Then, we define the following quantity

$$\sigma_T(\varphi, \tau) = \sup_{u \in \mathcal{F}_{T,\tau}} \sqrt{4\pi\tau} \int_{\tilde{M}} u d\tilde{\mu}$$

where  $\tilde{M}$  refers to the fact that we are considering the immersion  $\tilde{\varphi} : M \rightarrow T$ .

Notice that another possibility is simply to embed isometrically a convex set  $\Omega \subset \mathbb{R}^{n+1}$  containing  $\varphi(M)$  in a flat Riemannian manifold  $T$  (during the mean curvature flow a hypersurface  $\varphi$  initially contained in  $\Omega$  stays “inside” for all the evolution).

As before, these quantities are well defined, finite, positive and monotonically decreasing if  $\varphi_t$  moves by mean curvature.

## 2. APPLICATIONS

### 2.1. A No-Breathers Result.

**Definition 2.1.** A *breather* (following Perelman [12]) for mean curvature flow in  $\mathbb{R}^{n+1}$  is a smooth  $n$ -dimensional hypersurface evolving by mean curvature  $\varphi : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$ , such that there exists a time  $\bar{t} > 0$ , an isometry  $L$  of  $\mathbb{R}^{n+1}$  and a positive constant  $\lambda \in \mathbb{R}$  with  $\varphi(M, \bar{t}) = \lambda L(\varphi(M, 0))$ .

*Remark 2.2.* It is useless to consider nonshrinking (steady or expanding) compact breather of MCF, by the comparison with evolving spheres, they simply do not exist.

To authors’ knowledge, it is unknown if there exist nonhomothetic, noncompact “steady” or “expanding” breathers.

**Theorem 2.3.** *Every compact breather is a homothetic solution to MCF.*

*Proof.* By the rescaling property of  $\sigma$  in Proposition 1.5, fixing  $C > 0$  we have

$$\sigma(\varphi_0, C) \geq \sigma(\varphi_{\bar{t}}, C - \bar{t}) = \sigma(\lambda\varphi_0, C - \bar{t}) = \sigma(\varphi_0, (C - \bar{t})/\lambda^2)$$

hence, if we choose  $C = \frac{\bar{t}}{1-\lambda^2}$  we have  $C > \bar{t}$ , as  $\lambda < 1$  and  $(C - \bar{t})/\lambda^2 = C$ . It follows that

$$\sigma(\varphi_0, C) = \sigma(\varphi_{\bar{t}}, C - \bar{t})$$

and (for such special  $C$ ), by Proposition 1.6, if  $\tau(t) = C - t$

$$\int_0^{\bar{t}} \int_M \frac{e^{-\frac{|x-p_{\tau(t)}|^2}{4\tau(t)}}}{(4\pi\tau(t))^{n/2}} \left| \mathbf{H} + \frac{\langle x - p_{\tau(t)} | \nu \rangle}{2\tau(t)} \right|^2 d\mu_t dt = 0.$$

This implies that for every  $t \in (0, \bar{t})$  we have  $\mathbf{H}(x, t) = -\frac{\langle x - p_{\tau(t)} | \nu \rangle}{2(C-t)}$  for some  $p_{\tau(t)} \in \mathbb{R}^{n+1}$ , which is the well known equation characterizing a homothetically shrinking solution of MCF.  $\square$

*Remark 2.4.* This is the same argument to show that compact shrinking breathers of Ricci flow are actually Ricci gradient solitons.

Recalling the monotone nondecreasing quantity  $\mu$  of Perelman in [12], along a Ricci flow  $g(t)$  of a compact,  $n$ -dimensional Riemannian manifold  $M$ ,

$$\mu(g, \tau) = \inf_{\int_M u=1, u>0} \int_M \left( \tau \left[ R + \frac{|\nabla u|^2}{u} \right] - u \log u - \frac{un}{2} \log [4\pi\tau] - un \right) dV.$$

By the rescaling property  $\mu(\lambda g, \lambda\tau) = \mu(g, \tau)$ , if we have that  $g(\bar{t}) = \lambda dL^* g(0)$  for some diffeomorphism  $L : M \rightarrow M$  and  $0 < \lambda < 1$ , fixing  $C > 0$  we have

$$\mu(g(0), C) \leq \mu(g(\bar{t}), C - \bar{t}) = \mu(\lambda dL^* g(0), C - \bar{t}) = \mu(\lambda g(0), C - \bar{t}) = \mu(g(0), (C - \bar{t})/\lambda)$$

hence, if we choose  $C = \frac{\bar{t}}{1-\lambda}$  we have  $C > \bar{t}$ , as  $\lambda < 1$  and  $(C - \bar{t})/\lambda = C$ . It follows that

$$\mu(g(0), C) = \mu(g(\bar{t}), C - \bar{t})$$

and by the results of Perelman,  $g(t)$  is a shrinking soliton.

**2.2. Singularities.** If  $\varphi : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$  is a MCF of a smooth, compact, *embedded* hypersurface, it is well known that during the flow it remains embedded and there exists a finite maximal time  $T > 0$  of smooth existence when the curvature  $A$  is unbounded as  $t \nearrow T$ .

Moreover for every  $t \in [0, T)$

$$\sup_{p \in M} |A(p, t)| \geq \frac{1}{\sqrt{2(T-t)}}.$$

If there exists a constant  $C > 0$  such that also

$$\sup_{p \in M} |A(p, t)| \leq \frac{C}{\sqrt{2(T-t)}}.$$

we say that at  $T$  we have a *type I* singularity, otherwise we say the singularity is of *type II*.

We want to show that if at time  $T$  we have a singularity, the associated quantity  $\Sigma = \lim_{t \rightarrow T^-} \sigma(\varphi_t, \tau)$  is larger than one.

Indeed, for every  $p \in \mathbb{R}^{n+1}$  such that there exists a sequence of points  $q_i \in M$  and times  $t_i \nearrow T$  with  $p = \lim_{i \rightarrow \infty} \varphi(q_i, t_i)$ , we consider the function  $\Theta(p)$  defined in equation (1.7). By a simple semicontinuity argument, we can see that  $\Theta(p) \geq 1$  for every  $p \in \mathbb{R}^{n+1}$  like above, see [2, Corollary 4.20], hence, as  $\Sigma \geq \sup_{p \in \mathbb{R}^{n+1}} \Theta(p)$  we get  $\Sigma \geq 1$ .

If then  $\Sigma = 1$ , it forces  $\Theta(p) = 1$  for all such points  $p$  which implies, by the local regularity result of White [15], that the flow cannot develop a singularity at time  $T$  (see also Ecker [2]).

Suppose now to have a type I singularity at time  $T$ .

By Proposition 1.6 we know that along this flow, for  $C = T$ , hence,  $\tau = T - t$ ,

$$\sigma(\varphi_r, T - r) - \sigma(\varphi_t, T - t) \geq \int_r^t \int_M \frac{e^{-\frac{|x - p_{T-s}|^2}{4(T-s)}}}{[4\pi(T-s)]^{n/2}} \left| \mathbf{H} + \frac{\langle x - p_{T-s} | \nu \rangle}{2(T-s)} \right|^2 d\mu_s(x) ds$$

for every  $0 \leq r \leq t \leq T$ , hence,

(2.1)

$$C(\varphi_0) \geq \sigma(\varphi_0, T) - \Sigma \geq \int_0^T \int_M \frac{e^{-\frac{|x - p_{T-s}|^2}{4(T-s)}}}{[4\pi(T-s)]^{n/2}} \left| \mathbf{H} + \frac{\langle x - p_{T-s} | \nu \rangle}{2(T-s)} \right|^2 d\mu_s(x) ds$$

Rescaling every hypersurface  $\varphi_t$  as in [7], around the point  $p_{T-t}$  as follows,

$$\tilde{\varphi}_s(q) = \frac{\varphi(q, t(s)) - p_{T-t(s)}}{\sqrt{2(T-t(s))}} \quad s = s(t) = -\frac{1}{2} \log(T-t)$$

and changing variables in formula (2.1), we get

$$(2.2) \quad C \geq \int_M e^{-\frac{|y|^2}{2}} d\tilde{\mu}_{-\frac{1}{2} \log T} \geq \int_{-\frac{1}{2} \log T}^{+\infty} \int_M e^{-\frac{|y|^2}{2}} |\tilde{H} + \langle y | \tilde{\nu} \rangle|^2 d\tilde{\mu}_s(y) ds.$$

It follows that reasoning like in [7] and [13] (or [14]), if the singularity is of type I, the curvature of the rescaled hypersurfaces  $\tilde{\varphi}_s : M \rightarrow \mathbb{R}^{n+1}$  is uniformly bounded and any sequence converges (up to a subsequence) to a limit embedded hypersurface  $\tilde{M}_\infty$  satisfying  $\tilde{H} = -\langle x | \tilde{\nu} \rangle$  which is the defining equation for a homothetic solution of MCF.

Moreover, By the estimates of Stone [13, Lemma 2.9], this limit hypersurface satisfies

$$\frac{1}{(2\pi)^{n/2}} \int_{\tilde{M}_\infty} e^{-\frac{|y|^2}{2}} d\mathcal{H}^n(y) = \lim_{t \rightarrow T^-} \sigma(\varphi_t, T - t) = \Sigma > 1.$$

Clearly, by this equation, this embedded limit hypersurface cannot be empty. Moreover, it cannot be flat also, as it would be an hyperplane for the origin of  $\mathbb{R}^{n+1}$  (the only hyperplanes satisfying  $H = -\langle x | \nu \rangle$  must pass through the origin) as the above integral would be one.

**Proposition 2.5.** *At a singular time  $T$  of the MCF of an embedded compact hypersurface the quantity  $\Sigma$  is larger than one.*

*If the singularity of the flow is of type I, any sequence of rescaled hypersurfaces (with the maximal curvature) around the maximizer points for the Huisken's functional at times  $t_i \nearrow T$  converges, up to a subsequence, to a nonempty and nonflat, smooth embedded limit hypersurface, satisfying  $H = -\langle x | \nu \rangle$ .*

Suppose now that we are dealing with the special case of an embedded closed curve  $\gamma_t$  evolving in the plane  $\mathbb{R}^2$ . Rescaling as above, *without assuming anything* about the “type” of a singularity at some time  $T$ , we can extract a subsequence  $\tilde{\gamma}_{s_i}$  of rescaled curves such that:

$$\int_{\gamma_{s_i}} e^{-\frac{|y|^2}{2}} |\tilde{H} + \langle y | \tilde{\nu} \rangle|^2 d\mathcal{H}^1(y) \rightarrow 0$$

with locally equibounded lengths. This implies that the curves  $\tilde{\gamma}_{s_i}$  have also locally equibounded  $L^2$  norms of the curvature. Possibly passing to another subsequence (not relabeled) we can assume that

- the curves  $\tilde{\gamma}_{s_i}$  converges in  $C_{loc}^1$  to a limit curve  $\tilde{\gamma}_\infty$  with equibounded curvatures  $\tilde{k}$  in  $L_{loc}^2$ ;
- the curve  $\tilde{\gamma}_\infty$  satisfies  $\tilde{k} = -\langle x | \nu \rangle$  distributionally;
- there holds  $\frac{1}{(2\pi)^{n/2}} \int_{\tilde{\gamma}_\infty} e^{-\frac{|y|^2}{2}} d\mathcal{H}^1(y) = \Sigma > 1$ ;
- finally, the curve  $\tilde{\gamma}_\infty$  is embedded, that is, without self-intersections, by the geometric argument of Huisken in [8].

By a bootstrap argument, using the condition  $\tilde{k} = -\langle x | \nu \rangle$ , it follows that  $\tilde{\gamma}_\infty$  is a smooth curve and since the only embedded curves in the plane satisfying such condition are the lines through the origin and the unit circle,  $\tilde{\gamma}_\infty$  has to be among them.

Then, the curve  $\tilde{\gamma}_\infty$  cannot be a line through the origin, because for all of them the value of the integral  $\frac{1}{(2\pi)^{n/2}} \int_{\tilde{\gamma}_\infty} e^{-\frac{|y|^2}{2}} d\mathcal{H}^1(y)$  is one. Hence,  $\tilde{\gamma}_\infty$  must be the unit circle.

This implies that at some time the curve  $\gamma_t$  has become  $C^1$ -close, hence a graph, over a round circle (in particular, it is starshaped). It is then straightforward to see by means of maximum principle that this last property is preserved during the evolution.



Then, by means of the interior estimates of Ecker and Huisken [3], one can find a close (in time) sequence of rescaled curves converging in  $C_{loc}^2$  to the unit circle. Then, as this implies that at some time the curve has become convex, the singularity can only be a type I *vanishing* singularity. As a consequence, type II singularities for embedded closed curves are not possible.

*Remark 2.6.* It would be very interesting if this argument can be extended in higher dimensions, that is, if rescaling the moving hypersurface around the points maximizing the Huisken's functional could help to produce homothetic blowups also in the case of a Type II singularity. Some results in this direction has been obtained by Ilmanen in [9, 10].

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