Γ-CONVERGENCE OF FREE-DISCONTINUITY PROBLEMS

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ABSTRACT. We study the Γ-convergence of sequences of free-discontinuity functionals depending on vector-valued functions u which can be discontinuous across hypersurfaces whose shape and location are not known a priori. The main novelty of our result is that we work under very general assumptions on the integrands which, in particular, are not required to be periodic in the space variable. Further, we consider the case of surface integrands which are not bounded from below by the amplitude of the jump of u.

We obtain three main results: compactness with respect to Γ-convergence, representation of the Γ-limit in an integral form and identification of its integrands, and homogenisation formulas without periodicity assumptions. In particular, the classical case of periodic homogenisation follows as a by-product of our analysis. Moreover, our result covers also the case of stochastic homogenisation, as we will show in a forthcoming paper.

Keywords: Free-discontinuity problems, Γ-convergence, homogenisation.

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1. INTRODUCTION

In this paper we study the Γ-convergence, as $k \to +\infty$, of sequences of free-discontinuity functionals of the form

$$E_k(u,A) = \int_A f_k(x,\nabla u(x))\,dx + \int_{S_u\cap A} g_k(x,[u](x),\nu_u(x))d\mathcal{H}^{n-1}(x), \tag{1.1}$$

where $A \subset \mathbb{R}^n$ is a bounded open set, $u\colon A \to \mathbb{R}^m$ is a generalised special function of bounded variation, ∇u is its approximate gradient, S_u is the jump set of u and $[u]$ is its jump on S_u, while ν_u is the approximate normal to S_u and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

Functionals of the form (1.1) appear naturally in the study of quasistatic crack growth in nonlinear elasticity (see [21, 14, 11, 20, 13] and the monograph [5]), and represent the energy associated to a deformation u of an elastic body with cracks. The parameter k may have different meanings: it may represent the scale of a regularisation of the energy, the size of a microstructure, the ratio of the contrasting values of the mechanical response of the material in different parts of the body. For example, for a high-contrast medium f_k and g_k represent the strength and the toughness of the material, respectively, and may have a very different behaviour in each component. In the classical case of periodic homogenisation, where $f_k(x,\xi) = f(x/\varepsilon_k,\xi)$, $g_k(x,\zeta,\nu) = g(x/\varepsilon_k,\zeta,\nu)$, and f and g are periodic in the first variable, ε_k describes the scale of the microstructure, with $\varepsilon_k \to 0+$ as $k \to +\infty$. In this case taking the limit of E_k, in the sense of Γ-convergence, corresponds to computing the *effective* energy of the material.

The first paper on this subject (and still one of the most general results to date) is [8], where the authors consider the classical case of *periodic* homogenisation, assuming

$$c_4(1+|\zeta|) \le g_k(x,\zeta,\nu) \le c_5(1+|\zeta|) \quad \text{for } x \in \mathbb{R}^n,\ \zeta \in \mathbb{R}^m, \text{ and } \nu \in \mathbb{R}^n \text{ with } |\nu| = 1, \tag{1.2}$$

for suitable constants $0 < c_4 \le c_5 < +\infty$ independent of k. Conditions (1.2), together with standard p-growth assumptions on f_k, guarantee that sequences (u_k) with bounded energy $E_k(u_k,A)$ are bounded in the space $BV(A,\mathbb{R}^m)$ of functions with bounded variation on A with values in $\mathbb{R}^m$. When ∂A is regular, $BV(A,\mathbb{R}^m)$ is compactly embedded into $L^1(A,\mathbb{R}^m)$, hence in this case it is natural to study the Γ-convergence of $E_k(\cdot,A)$ in $L^1(A,\mathbb{R}^m)$.

In [19] the Γ-convergence of (1.1) is studied under the assumption that u is scalar ($m = 1$) and that g_k does not depend on ζ and satisfies

$$c_4 \le g_k(x,\nu) \le c_5 \quad \text{for } x \in \mathbb{R}^n \text{ and } \nu \in \mathbb{R}^n \text{ with } |\nu| = 1. \tag{1.3}$$

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These assumptions ensure that sequences (u_k) with bounded $\|u_k\|_{L^\infty(A)}$ and bounded energy $E_k(u_k, A)$ are also bounded in $BV(A)$, hence relatively compact in $L^1(A)$. Since for scalar functions (u_k) a bound on $\|u_k\|_{L^\infty(A)}$ can be easily obtained by truncation, also in this case it is natural to study the Γ-convergence of $E_k(\cdot, A)$ with respect to the strong $L^1(A)$-convergence.

The aim of the present paper is to study the Γ-convergence of (1.1) in the vector-valued case ($m \geq 1$) *without any periodicity assumptions*, and *under the more general growth conditions*

$$c_4 \leq g_k(x, \zeta, \nu) \leq c_5(1 + |\zeta|) \quad \text{for } x \in \mathbb{R}^n,\ \zeta \in \mathbb{R}^m,\ \text{and } \nu \in \mathbb{R}^n \text{ with } |\nu| = 1, \tag{1.4}$$

which include both (1.2) and (1.3). Unlike (1.2), assumption (1.4) does not guarantee a bound in $BV(A, \mathbb{R}^m)$ for sequences (u_k) with bounded energy $E_k(u_k, A)$. Moreover, unlike in the scalar case considered in [19], in the vector-valued case an estimate for $\|u_k\|_{L^\infty(A,\mathbb{R}^m)}$ cannot be easily obtained by a standard truncation procedure. For these reasons, in our setting sequences (u_k) with bounded energy $E_k(u_k, A)$ are, in general, not relatively compact in $L^1(A, \mathbb{R}^m)$.

Therefore, we study the Γ-convergence in the larger space $L^0(A, \mathbb{R}^m)$ of all $\mathcal{L}^n$-measurable functions $u\colon A \to \mathbb{R}^m$, endowed with the metrisable topology of convergence in measure. This is the natural choice of convergence in our case: using compactness theorems for free-discontinuity functionals, it is indeed possible to prove that sequences (u_k) with equi-bounded energy $E_k(u_k, A)$ are relatively compact in $L^0(A, \mathbb{R}^m)$, under a very weak integral bound on (u_k). Therefore, Γ-convergence of $(E_k(\cdot, A))$ in $L^0(A, \mathbb{R}^m)$ implies convergence of the solutions of some associated minimisation problems obtained, for instance, by adding a lower order term to E_k (see Corollary 6.1).

The present paper contains three main results. The first one, Theorem 3.5, is a compactness result with respect to Γ-convergence for sequences as in (1.1). Namely, we show that for every sequence (E_k) with $(f_k) \subset \mathcal{F}$ (see assumptions $(f1)$-$(f4)$ in Definition 3.1) and $(g_k) \subset \mathcal{G}$ (see assumptions $(g1)$-$(g7)$ in Definition 3.1) there exists a subsequence, not relabelled, such that, for every bounded open set $A \subset \mathbb{R}^n$, $E_k(\cdot, A)$ Γ-converges to a functional $E_\infty(\cdot, A)$, which can be written in the form (1.1) for suitable functions $f_\infty \in \mathcal{F}$ and $g_\infty \in \mathcal{G}$.

The second result, Theorem 3.7, identifies the Γ-limit $E_\infty(\cdot, A)$. That is, it provides a connection between the functions f_k and g_k, used to define E_k, and the functions f_∞ and g_∞, which appear in the integral representation of E_∞. More precisely, we show that the volume density f_∞ is obtained as

$$f_\infty(x, \xi) = \limsup_{\rho \to 0+} \lim_{k \to +\infty} \frac{1}{\rho^n} \inf \int_{Q_\rho(x)} f_k(y, \nabla u(y)) dy, \tag{1.5}$$

where the infimum is taken among all the functions $u \in W^{1,p}(Q_\rho(x), \mathbb{R}^m)$ with $u(y) = \xi \cdot y$ near $\partial Q_\rho(x)$, and $Q_\rho(x) := x + (-\rho/2, \rho/2)^n$. For the surface density g_∞, we prove instead that

$$g_\infty(x, \zeta, \nu) = \limsup_{\rho \to 0+} \lim_{k \to +\infty} \frac{1}{\rho^{n-1}} \inf \int_{S_u \cap Q^\nu_\rho(x)} g_k(y, [u](y), \nu_u(y)) d\mathcal{H}^{n-1}(y). \tag{1.6}$$

Here, the cube $Q^\nu_\rho(x)$ is a suitable rotation of $Q_\rho(x)$ (see item (l) of Section 2), and the infimum is taken among all the functions $u \in SBV(Q^\nu_\rho(x), \mathbb{R}^m)$ with $\nabla u = 0$ $\mathcal{L}^n$-a.e. in $Q^\nu_\rho(x)$ and that near $\partial Q^\nu_\rho(x)$ agree with the pure-jump function $u_{x,\zeta,\nu}$ (see item (n) of Section 2). This result shows, in particular, that the problems for the volume and surface integrals are decoupled; *i.e.,* f_∞ depends only on the sequence (f_k), while g_∞ depends only on the sequence (g_k).

The third result (Theorem 3.8) deals with the case of homogenisation, that is $f_k(x, \xi) = f(x/\varepsilon_k, \xi)$ and $g_k(x, \zeta, \nu) = g(x/\varepsilon_k, \zeta, \nu)$ for a sequence $\varepsilon_k \to 0+$ as $k \to +\infty$. In this case, for given x, ξ, ζ, and ν, a natural change of variables in (1.5) and (1.6) leads to consider, for every $r > 0$, the two rescaled minimisation problems

$$\frac{1}{r^n} \inf \left\{ \int_{Q_r(rx)} f(y, \nabla u(y)) dy \colon u \in W^{1,p}(Q_r(rx), \mathbb{R}^m),\ u(y) = \xi \cdot y \text{ near } \partial Q_r(rx) \right\}, \tag{1.7}$$

and

$$\frac{1}{r^{n-1}} \inf \int_{S_u \cap Q^\nu_r(rx)} g(y, [u](y), \nu_u(y)) d\mathcal{H}^{n-1}(y). \tag{1.8}$$

In the last formula, the infimum is taken among all the functions $u \in SBV(Q^\nu_r(rx), \mathbb{R}^m)$ with $\nabla u = 0$ $\mathcal{L}^n$-a.e. in $Q^\nu_r(rx)$ and that near $\partial Q^\nu_r(rx)$ agree with the pure jump function $u = u_{rx,\zeta,\nu}$ (see item (n) of Section 2).

Assume that the limits as $r \to 0+$ of the expressions in (1.7) and (1.8) exist and are independent of x, and denote them by $f_{\rm hom}(\xi)$ and $g_{\rm hom}(\zeta, \nu)$, respectively (see (3.14) and (3.15)). Then, we prove that for every bounded open set $A \subset \mathbb{R}^n$ the sequence $E_k(\cdot, A)$ with integrands $f(x/\varepsilon_k, \xi)$ and $g(x/\varepsilon_k, \zeta, \nu)$ Γ-converges to the functional $E_{\rm hom}(\cdot, A)$ with integrands $f_{\rm hom}(\xi)$ and $g_{\rm hom}(\zeta, \nu)$. In particular, we recover

the case where $f(x,\xi)$ and $g(x,\zeta,\nu)$ are periodic with respect to x, which was previously studied in [8] assuming (1.2) for g.

In the forthcoming paper [9] we shall prove that, under our more general assumptions (1.3), the existence of these limits and their independence of x can be proved even in the more general context of stochastic homogenisation. Therefore Theorem 3.8 of the present paper will be a key ingredient in the proof of the results on stochastic homogenisation for free-discontinuity problems.

The paper is organised as follows. In Section 2 we fix the notation and give the references for the background material used in the paper. In Section 3 we list the general hypotheses on the integrands f_k and g_k and state our main results. We also prove that the result on homogenisation follows, through a change of variables, from the result on the identification of the Γ-limit.

In Section 4 we prove a compactness theorem for the perturbed functionals obtained by adding to $E_k(u,A)$ the regularising term $\varepsilon\int_{S_u\cap A}|[u]|d\mathcal{H}^{n-1}$, which allows us to use the results of [8]. This section contains also some technical lemmas on smooth truncations that are used throughout the paper.

In Section 5 we begin the proof of Theorem 3.5, which gives the compactness of sequences of functionals of the form (1.1) with respect to Γ-convergence. The main tool is the analysis of the limit as $\varepsilon\to 0+$ of the Γ-limits of the perturbed functionals of Section 4. The conclusion of the proof is based on Theorem 5.2, where the integrands of the functional obtained in this way are compared with (1.5) and (1.6). The proof of this theorem is very technical and is given in Sections 7 and 8.

In Section 6 we prove the identification result for the Γ-limit (Theorem 3.7) using Theorem 5.2. Moreover we show that, for some minimisation problems involving an $L^p(A,\mathbb{R}^m)$-perturbation of the functionals (1.1), Γ-convergence in $L^0(A,\mathbb{R}^m)$ implies convergence of the minimum values and, for a subsequence, convergence in $L^p(A,\mathbb{R}^m)$ of the minimum points.

In Sections 7 and 8 we prove the statements of Theorem 5.2 concerning the volume and the surface integrals, respectively.

The final section is an appendix which collects some technical results used in the paper.

2. Preliminaries and notation

In this section we give a brief account of the mathematical tools that will be needed in the paper.

For the general notions on BV, SBV, and $GSBV$ functions and their fine properties we refer to [3] (see also [16, 22]). For $u\in BV$, Du and D^su denote the distributional derivative of u and its singular part with respect to the Lebesgue measure, respectively, while ∇u stands for the density of the absolutely continuous part of Du with respect to the Lebesgue measure. ∇u coincides with the approximate gradient of u, which makes sense also for $u\in GSBV$. Moreover, S_u denotes the set of approximate discontinuity points of u, and ν_u the measure theoretic normal to S_u. The symbols $u^\pm$ denote the one-sided approximate limits of u at a point of S_u, from the side of $\pm\nu_u$.

For the general theory of Γ-convergence we refer to the monograph [12]. Other results on this subject can be found in [6] and [7].

We introduce now some notation that will be used throughout the paper.

(a) m and n are fixed positive integers, $\mathbb{R}$ is the set of real numbers, and $\mathbb{R}^m_0:=\mathbb{R}^m\setminus\{0\}$.

(b) $\mathbb{S}^{n-1}:=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n : x_1^2+\dots+x_n^2=1\}$ and $\widehat{\mathbb{S}}^{n-1}_\pm:=\{x\in\mathbb{S}^{n-1}:\pm x_{i(x)}>0\}$, where $i(x)$ is the largest $i\in\{1,\dots,n\}$ such that $x_i\neq 0$.

(c) $\mathcal{L}^n$ denotes the Lebesgue measure on $\mathbb{R}^n$ and $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$.

(d) $\mathscr{A}$ denotes the collection of all bounded open subsets of $\mathbb{R}^n$; if A, $B\in\mathscr{A}$, by $A\subset\subset B$ we mean that A is relatively compact in B.

(e) For $u\in GSBV(A,\mathbb{R}^m)$, with $A\in\mathscr{A}$, the jump of u across S_u is defined by $[u]:=u^+-u^-$.

(f) For $A\in\mathscr{A}$ we define

$$SBV_{\rm pc}(A,\mathbb{R}^m):=\{u\in SBV(A,\mathbb{R}^m):\nabla u=0\ \mathcal{L}^n\text{-a.e., }\mathcal{H}^{n-1}(S_u)<+\infty\};$$

it is known (see [3, Theorem 4.23]) that every u in $SBV_{\rm pc}(A,\mathbb{R}^m)\cap L^\infty(A,\mathbb{R}^m)$ is piecewise constant in the sense of [3, Definition 4.21], namely there exists a Caccioppoli partition (E_i) of A such that u is constant $\mathcal{L}^n$-a.e. in each set E_i. We note that same result holds for $u\in SBV_{\rm pc}(A,\mathbb{R}^m)$, however this property will never be used in the paper.

(g) For $A\in\mathscr{A}$ and $p>1$ we define

$$SBV^p(A,\mathbb{R}^m):=\{u\in SBV(A,\mathbb{R}^m):\nabla u\in L^p(A,\mathbb{R}^{m\times n}),\ \mathcal{H}^{n-1}(S_u)<+\infty\}.$$

(h) For $A \in \mathscr{A}$ and $p > 1$ we define

$$GSBV^p(A,\mathbb{R}^m) := \{u \in GSBV(A,\mathbb{R}^m) : \nabla u \in L^p(A,\mathbb{R}^{m\times n}),\ \mathcal{H}^{n-1}(S_u) < +\infty\};$$

it is known that $GSBV^p(A,\mathbb{R}^m)$ is a vector space and that $\psi(u) \in SBV^p(A,\mathbb{R}^m) \cap L^\infty(A,\mathbb{R}^m)$ for every $u \in GSBV^p(A,\mathbb{R}^m)$ and for every $\psi \in C^1_c(\mathbb{R}^m,\mathbb{R}^m)$ (see, e.g., [13, page 172]).

(i) For every $\mathcal{L}^n$-measurable set $A \subset \mathbb{R}^n$ let $L^0(A,\mathbb{R}^m)$ be the space of all $\mathcal{L}^n$-measurable functions $u\colon A \to \mathbb{R}^m$, endowed with the topology of convergence in measure on bounded subsets of A; we observe that this topology is metrisable and separable.

(j) For $x \in \mathbb{R}^n$ and $\rho > 0$ we define

$$B_\rho(x) := \{y \in \mathbb{R}^n :\ |y-x| < \rho\},$$
$$Q_\rho(x) := \{y \in \mathbb{R}^n :\ |(y-x)\cdot e_i| < \rho/2 \quad \text{for } i = 1,\dots,n\},$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$, $e_1,\dots,e_n$ is the canonical basis of $\mathbb{R}^n$, and $\cdot$ denotes the Euclidean scalar product; we omit the subscript ρ when $\rho = 1$ ($|\cdot|$ denotes the absolute value in $\mathbb{R}$ or the Euclidean norm in $\mathbb{R}^n$, $\mathbb{R}^m$, or $\mathbb{R}^{m\times n}$, depending on the context).

(k) For every $\nu \in \mathbb{S}^{n-1}$ let R_ν be an orthogonal $n\times n$ matrix such that $R_\nu e_n = \nu$; we assume that the restrictions of the function $\nu \mapsto R_\nu$ to the sets $\widehat{\mathbb{S}}^{n-1}_\pm$ defined in (b) are continuous and that $R_{-\nu}Q(0) = R_\nu Q(0)$ for every $\nu \in \mathbb{S}^{n-1}$; a map $\nu \mapsto R_\nu$ satisfying these properties is provided in Example A.1 in the Appendix.

(l) For $x \in \mathbb{R}^n$, $\rho > 0$, and $\nu \in \mathbb{S}^{n-1}$ we set

$$Q^\nu_\rho(x) := R_\nu Q_\rho(0) + x;$$

we omit the subscript ρ when $\rho = 1$.

(m) For $\xi \in \mathbb{R}^{m\times n}$, the linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$ with gradient ξ is denoted by ℓ_ξ; *i.e.*, $\ell_\xi(x) := \xi x$, where x is considered as an $n\times 1$ matrix.

(n) For $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^m_0$, and $\nu \in \mathbb{S}^{n-1}$ we define the function $u_{x,\zeta,\nu}$ as

$$u_{x,\zeta,\nu}(y) := \begin{cases} \zeta & \text{if } (y-x)\cdot\nu \ge 0,\\ 0 & \text{if } (y-x)\cdot\nu < 0.\end{cases}$$

(o) For $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$, we set

$$\Pi^\nu_0 := \{y \in \mathbb{R}^n : y\cdot\nu = 0\} \quad \text{and} \quad \Pi^\nu_x := \{y \in \mathbb{R}^n : (y-x)\cdot\nu = 0\}.$$

3. Statement of the main results

Throughout the paper we fix six constants $p, c_1,\dots,c_5$, with $1 < p < +\infty$, $0 < c_1 \le c_2 < +\infty$, $1 \le c_3 < +\infty$, and $0 < c_4 \le c_5 < +\infty$, and two nondecreasing continuous functions σ_1, $\sigma_2\colon [0,+\infty) \to [0,+\infty)$ such that $\sigma_1(0) = \sigma_2(0) = 0$.

Definition 3.1 (Volume and surface integrands)**.** Let $\mathcal{F} = \mathcal{F}(p, c_1, c_2, \sigma_1)$ be the collection of all functions $f\colon \mathbb{R}^n\times\mathbb{R}^{m\times n} \to [0,+\infty)$ satisfying the following conditions:

($f1$) (measurability) f is Borel measurable on $\mathbb{R}^n\times\mathbb{R}^{m\times n}$;

($f2$) (continuity in ξ) for every $x \in \mathbb{R}^n$ we have

$$|f(x,\xi_1) - f(x,\xi_2)| \le \sigma_1(|\xi_1-\xi_2|)\big(1 + f(x,\xi_1) + f(x,\xi_2)\big)$$

for every ξ_1, $\xi_2 \in \mathbb{R}^{m\times n}$;

($f3$) (lower bound) for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^{m\times n}$

$$c_1|\xi|^p \le f(x,\xi);$$

($f4$) (upper bound) for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^{m\times n}$

$$f(x,\xi) \le c_2(1 + |\xi|^p).$$

Let $\mathcal{G} = \mathcal{G}(c_3, c_4, c_5, \sigma_2)$ be the collection of all functions $g\colon \mathbb{R}^n\times\mathbb{R}^m_0\times\mathbb{S}^{n-1} \to [0,+\infty)$ satisfying the following conditions:

($g1$) (measurability) g is Borel measurable on $\mathbb{R}^n\times\mathbb{R}^m_0\times\mathbb{S}^{n-1}$;

($g2$) (continuity in ζ) for every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$ we have

$$|g(x,\zeta_2,\nu) - g(x,\zeta_1,\nu)| \le \sigma_2(|\zeta_1-\zeta_2|)\big(g(x,\zeta_1,\nu) + g(x,\zeta_2,\nu)\big)$$

for every ζ_1, $\zeta_2 \in \mathbb{R}^m_0$;

($g3$) (estimate for $|\zeta_1| \le |\zeta_2|$) for every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$ we have

$$g(x,\zeta_1,\nu) \le c_3\, g(x,\zeta_2,\nu)$$

for every ζ_1, $\zeta_2 \in \mathbb{R}^m_0$ with $|\zeta_1| \le |\zeta_2|$;

($g4$) (estimate for $c_3|\zeta_1| \le |\zeta_2|$) for every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$ we have

$$g(x,\zeta_1,\nu) \le\ g(x,\zeta_2,\nu)$$

for every ζ_1, $\zeta_2 \in \mathbb{R}^m_0$ with $c_3|\zeta_1| \le |\zeta_2|$;

($g5$) (lower bound) for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^m_0$, and $\nu \in \mathbb{S}^{n-1}$

$$c_4 \le g(x,\zeta,\nu);$$

($g6$) (upper bound) for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^m_0$, and $\nu \in \mathbb{S}^{n-1}$

$$g(x,\zeta,\nu) \le c_5(1+|\zeta|);$$

($g7$) (symmetry) for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^m_0$, and $\nu \in \mathbb{S}^{n-1}$

$$g(x,\zeta,\nu) = g(x,-\zeta,-\nu).$$

Remark 3.2 (Assumptions ($g3$) and ($g4$)). Let $g\colon \mathbb{R}^n\times\mathbb{R}^m_0\times\mathbb{S}^{n-1} \to [0,+\infty)$ be a function satisfying the following "monotonicity" condition: for every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$

$$g(x,\zeta_1,\nu) \le g(x,\zeta_2,\nu)$$

for every ζ_1, $\zeta_2 \in \mathbb{R}^m_0$ with $|\zeta_1| \le |\zeta_2|$; then it is immediate to verify that g satisfies ($g3$) and ($g4$).

On the other hand ($g3$) and ($g4$) are weaker than monotonicity in $|\zeta|$. For instance, the function $g(x,\zeta,\nu) := \hat g(|\zeta|)$, with $\hat g : [0,+\infty) \to [0,+\infty)$ given by

$$\hat g(t) = \begin{cases} t & \text{if } t \in [0,1],\\ \in \left[\frac{t}{c_3},1\right] & \text{if } t \in [1,c_3],\\ \frac{t}{c_3} & \text{if } t \ge c_3, \end{cases}$$

satisfies ($g3$) and ($g4$), but its behaviour in $[1,c_3]$ can be chosen quite freely, in particular it can be nonmonotone.

Remark 3.3. We remark that assumptions ($g3$) and ($g4$) on the surface integrand g will be crucial to prove that the functional E defined in (3.4) decreases by smooth truncations up to an error term (see (4.13) and the proof of Lemma 4.1). We also notice that ($g3$) and ($g4$) could be omitted if assumption ($g5$) were replaced by the stronger lower bound

$$c(1+|\zeta|) \le g(x,\zeta,\nu) \quad \text{for every } (x,\zeta,\nu) \in \mathbb{R}^n \times \mathbb{R}^m_0 \times \mathbb{S}^{n-1} \tag{3.1}$$

for some $c > 0$ (see, e.g., the proof of [8, Lemma 3.5]). However, a lower bound as in (3.1) would rule out, for instance, functionals of Mumford-Shah type, which we would like to cover in our analysis. For this reason we prefer to work under the weaker growth condition ($g5$) on g and under the additional "monotonicity" assumptions ($g3$) and ($g4$).

Given $f \in \mathcal{F}$ and $g \in \mathcal{G}$, we consider the integral functionals F, G, $E\colon L^0(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A} \longrightarrow [0,+\infty]$ defined as

$$F(u,A) := \begin{cases} \displaystyle\int_A f(x,\nabla u)\,dx & \text{if } u|_A \in GSBV^p(A,\mathbb{R}^m),\\ +\infty & \text{otherwise in } L^0(\mathbb{R}^n,\mathbb{R}^m). \end{cases} \tag{3.2}$$

$$G(u,A) := \begin{cases} \displaystyle\int_{S_u\cap A} g(x,[u],\nu_u)d\mathcal{H}^{n-1} & \text{if } u|_A \in GSBV^p(A,\mathbb{R}^m),\\ +\infty & \text{otherwise in } L^0(\mathbb{R}^n,\mathbb{R}^m), \end{cases} \tag{3.3}$$

$$E(u,A) := F(u,A) + G(u,A). \tag{3.4}$$

We also consider the integral functional $E^p\colon L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A} \longrightarrow [0,+\infty]$, defined as the restriction of E to $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}$.

Remark 3.4. Since $[u]$ is reversed when the orientation of ν_u is reversed, the functional G is well defined thanks to ($g7$).

The following compactness theorem, with respect to Γ-convergence, is one of the main results of this paper.

Theorem 3.5 (Compactness for Γ-convergence). *Let (f_k) be a sequence in $\mathcal{F}$, let (g_k) be a sequence in $\mathcal{G}$, let $E_k\colon L^0(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}\to[0,+\infty]$ be the integral functionals defined by (3.4) corresponding to f_k and g_k, and let $E_k^p\colon L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}\to[0,+\infty]$ be their restrictions to $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}$. Then there exist a subsequence, not relabelled, and two functions $f\in\mathcal{F}$ and $g\in\mathcal{G}$ such that for every $A\in\mathscr{A}$*

$$E_k(\cdot,A)\ \Gamma\text{-converges to } E(\cdot,A) \text{ in } L^0(\mathbb{R}^n,\mathbb{R}^m),$$
$$E_k^p(\cdot,A)\ \Gamma\text{-converges to } E^p(\cdot,A) \text{ in } L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m),$$

where the integral functional $E\colon L^0(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}\to[0,+\infty]$ is given by (3.4) and E^p is its restriction to $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}$.

Similar results, under different hypotheses on the surface densities g_k, have been obtained in [8] and [19].

Let X be a subspace of $L^0(\mathbb{R}^n,\mathbb{R}^m)$. For every $H\colon X\times\mathscr{A}\longrightarrow[0,+\infty]$, $A\in\mathscr{A}$, and $w\in L^0(\mathbb{R}^n,\mathbb{R}^m)$, we set

$$m_H^{1,p}(w,A):=\inf\left\{H(u,A):u\in X,\ u|_A\in W^{1,p}(A,\mathbb{R}^m),\ u=w \text{ near } \partial A\right\},\tag{3.5}$$
$$m_H^{\rm pc}(w,A):=\inf\left\{H(u,A):u\in X,\ u|_A\in SBV_{\rm pc}(A,\mathbb{R}^m),\ u=w \text{ near } \partial A\right\},\tag{3.6}$$
$$m_H(w,A):=\inf\left\{H(u,A):u\in X,\ u|_A\in SBV^p(A,\mathbb{R}^m),\ u=w \text{ near } \partial A\right\},\tag{3.7}$$

with the standard convention $\inf\emptyset=+\infty$. In all the formulas above, by "$u=w$ near ∂A" we mean that there exists a neighbourhood U of ∂A in $\mathbb{R}^n$ such that $u=w$ $\mathcal{L}^n$-a.e. in $U\cap A$.

Let (f_k) be a sequence in $\mathcal{F}$ and let (g_k) be a sequence in $\mathcal{G}$. For every k, we consider the integral functionals $F_k,G_k,E_k\colon L^0(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}\to[0,+\infty]$ defined by (3.2), (3.3), and (3.4) corresponding to f_k and g_k. For every $x\in\mathbb{R}^n$, $\xi\in\mathbb{R}^{m\times n}$, $\zeta\in\mathbb{R}^m_0$, and $\nu\in\mathbb{S}^{n-1}$ we define

$$f'(x,\xi):=\limsup_{\rho\to0+}\liminf_{k\to+\infty}\frac{m_{F_k}^{1,p}(\ell_\xi,Q_\rho(x))}{\rho^n},\tag{3.8}$$

$$f''(x,\xi):=\limsup_{\rho\to0+}\limsup_{k\to+\infty}\frac{m_{F_k}^{1,p}(\ell_\xi,Q_\rho(x))}{\rho^n},\tag{3.9}$$

$$g'(x,\zeta,\nu):=\limsup_{\rho\to0+}\liminf_{k\to+\infty}\frac{m_{G_k}^{\rm pc}(u_{x,\zeta,\nu},Q_\rho^\nu(x))}{\rho^{n-1}},\tag{3.10}$$

$$g''(x,\zeta,\nu):=\limsup_{\rho\to0+}\limsup_{k\to+\infty}\frac{m_{G_k}^{\rm pc}(u_{x,\zeta,\nu},Q_\rho^\nu(x))}{\rho^{n-1}}.\tag{3.11}$$

Remark 3.6. It turns out that f', $f''\in\mathcal{F}$ (see Lemma A.6), and g', $g''\in\mathcal{G}$ (see Lemma A.7).

The second main result of this paper is the identification of the Γ-limit.

Theorem 3.7 (Identification of the Γ-limit). *Let (f_k), (g_k), (E_k), and (E_k^p) be as in Theorem 3.5, let $f_\infty\in\mathcal{F}$ and $g_\infty\in\mathcal{G}$, let E_∞ be defined as in (3.4) with f_∞ and g_∞, and let E_∞^p be its restriction to $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}$. Assume that the following equalities are satisfied:*

(a1) *for $\mathcal{L}^n$-a.e. $x\in\mathbb{R}^n$ we have*

$$f_\infty(x,\xi)=f'(x,\xi)=f''(x,\xi)\quad \text{for every } \xi\in\mathbb{R}^{m\times n};$$

(a2) *for every $A\in\mathscr{A}$, for every $u\in GSBV^p(A,\mathbb{R}^m)$, and for $\mathcal{H}^{n-1}$-a.e. $x\in S_u$ we have*

$$g_\infty(x,[u](x),\nu_u(x))=g'(x,[u](x),\nu_u(x))=g''(x,[u](x),\nu_u(x)).$$

Then

$$E_k(\cdot,A)\ \Gamma\text{-converges to } E_\infty(\cdot,A) \text{ in } L^0(\mathbb{R}^n,\mathbb{R}^m),\tag{3.12}$$
$$E_k^p(\cdot,A)\ \Gamma\text{-converges to } E_\infty^p(\cdot,A) \text{ in } L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m),\tag{3.13}$$

for every $A\in\mathscr{A}$.

The third main result of the paper concerns the case of homogenisation, where $f_k(x,\xi):=f(x/\varepsilon_k,\xi)$ and $g_k(x,\zeta,\nu):=g(x/\varepsilon_k,\zeta,\nu)$ for a sequence $\varepsilon_k\to0+$.

Theorem 3.8 (Homogenisation). *Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$, and let F and G be the functionals defined as in* (3.2) *and* (3.3)*, respectively. Assume that for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m\times n}$, $\zeta \in \mathbb{R}^m_0$, and $\nu \in \mathbb{S}^{n-1}$ the limits*

$$\lim_{r\to+\infty} \frac{m^{1,p}_F(\ell_\xi, Q_r(rx))}{r^n} =: f_{\rm hom}(\xi), \tag{3.14}$$

$$\lim_{r\to+\infty} \frac{m^{\rm pc}_G(u_{rx,\zeta,\nu}, Q^\nu_r(rx))}{r^{n-1}} =: g_{\rm hom}(\zeta,\nu) \tag{3.15}$$

exist and are independent of x. Then $f_{\rm hom} \in \mathcal{F}$ and $g_{\rm hom} \in \mathcal{G}$.

Let (ε_k) be a sequence of positive real numbers converging to 0, let f_k and g_k be defined by

$$f_k(x,\xi) := f(x/\varepsilon_k, \xi) \quad \text{and} \quad g_k(x,\zeta,\nu) := g(x/\varepsilon_k, \zeta, \nu),$$

let E_k be defined as in (3.4) *with f_k and g_k, let $E_{\rm hom}$ be defined as in* (3.4) *with $f_{\rm hom}$ and $g_{\rm hom}$, and let E^p_k and $E^p_{\rm hom}$ be their restrictions to $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}$. Then*

$$E_k(\cdot,A) \ \Gamma\text{-converges to } E_{\rm hom}(\cdot,A) \text{ in } L^0(\mathbb{R}^n,\mathbb{R}^m),$$
$$E^p_k(\cdot,A) \ \Gamma\text{-converges to } E^p_{\rm hom}(\cdot,A) \text{ in } L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m),$$

for every $A \in \mathscr{A}$.

Arguing as in [8] (see also [7] for the volume part) one can prove that (3.14) and (3.15) are always satisfied when f and g are periodic of period 1 with respect to the space coordinates $x_1,\dots,x_n$. We omit here the proof of this property, since in [9] we shall prove that (3.14) and (3.15) are satisfied almost surely under the natural assumptions of stochastic homogenisation, which include, in particular, the case of deterministic periodic homogenisation.

The complete proofs of Theorems 3.5 and 3.7 require several intermediate results which will be established in the next sections. Theorem 3.8 instead follows easily from Remark 3.6 and from Theorem 3.7 by means of a natural change of variables, as we show below.

Proof of Theorem 3.8. By Theorem 3.7 it is enough to show that

$$f'(x,\xi) = f''(x,\xi) = f_{\rm hom}(\xi) \quad \text{and} \quad g'(x,\zeta,\nu) = g''(x,\zeta,\nu) = g_{\rm hom}(\zeta,\nu) \tag{3.16}$$

for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m\times n}$, $\zeta \in \mathbb{R}^m_0$, and $\nu \in \mathbb{S}^{n-1}$. Indeed, if these equalities are satisfied, then $f_{\rm hom} \in \mathcal{F}$ and $g_{\rm hom} \in \mathcal{G}$ by Remark 3.6, and the Γ-convergence follows from Theorem 3.7 applied with $f_\infty = f_{\rm hom}$ and $g_\infty = g_{\rm hom}$.

To prove the first equality in (3.16) we fix $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m\times n}$, $\rho > 0$, and $k \in \mathbb{N}$. Given $u \in W^{1,p}(Q_\rho(x),\mathbb{R}^m)$, let $u_k \in W^{1,p}(Q_{\rho/\varepsilon_k}(x/\varepsilon_k),\mathbb{R}^m)$ be defined by $u_k(z) = u(\varepsilon_k z)/\varepsilon_k$ for every $z \in Q_{\rho/\varepsilon_k}(x/\varepsilon_k)$. By the change of variables $z = y/\varepsilon_k$ we obtain $F_k(u, Q_\rho(x)) = \varepsilon_k^n F(u_k, Q_{\rho/\varepsilon_k}(x/\varepsilon_k))$. Since $u = \ell_\xi$ near $\partial Q_\rho(x)$ if and only if $u_k = \ell_\xi$ near $\partial Q_{\rho/\varepsilon_k}(x/\varepsilon_k)$, we deduce that $m^{1,p}_{F_k}(\ell_\xi, Q_\rho(x)) = \varepsilon_k^n\, m^{1,p}_F(\ell_\xi, Q_{\rho/\varepsilon_k}(x/\varepsilon_k)) = (\rho^n/r_k^n)\, m^{1,p}_F(\ell_\xi, Q_{r_k}(r_k x/\rho))$, where $r_k := \rho/\varepsilon_k$. By applying (3.14) with x replaced by x/ρ we obtain

$$\lim_{k\to+\infty} \frac{1}{\rho^n} m^{1,p}_{F_k}(\ell_\xi, Q_\rho(x)) = f_{\rm hom}(\xi).$$

By (3.8) and (3.9) this implies that $f'(x,\xi) = f''(x,\xi) = f_{\rm hom}(\xi)$.

To prove the second equality in (3.16) we fix $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^m_0$, $\nu \in \mathbb{S}^{n-1}$, $\rho > 0$, and $k \in \mathbb{N}$. Given $v \in SBV_{\rm pc}(Q^\nu_\rho(x),\mathbb{R}^m)$, let $v_k \in SBV_{\rm pc}(Q^\nu_{\rho/\varepsilon_k}(x/\varepsilon_k),\mathbb{R}^m)$ be defined by $v_k(z) = v(\varepsilon_k z)$ for every $z \in Q^\nu_{\rho/\varepsilon_k}(x/\varepsilon_k)$. Then $S_{v_k} = (1/\varepsilon_k)S_v$ and, thanks to $(g7)$, we may assume that $[v_k](z) = [v](\varepsilon_k z)$ for $\mathcal{H}^{n-1}$-a.e. $z \in S_{v_k}$. By the change of variables $z = y/\varepsilon_k$ we obtain $G_k(v, Q^\nu_\rho(x)) = \varepsilon_k^{n-1} G(v_k, Q^\nu_{\rho/\varepsilon_k}(x/\varepsilon_k))$. From the fact that $v = u_{x,\zeta,\nu}$ near $\partial Q^\nu_\rho(x)$ if and only if $v_k = u_{x/\varepsilon_k,\zeta,\nu}$ near $\partial Q^\nu_{\rho/\varepsilon_k}(x/\varepsilon_k)$, we deduce that $m^{\rm pc}_{G_k}(u_{x,\zeta,\nu}, Q^\nu_\rho(x)) = \varepsilon_k^{n-1}\, m^{\rm pc}_G(u_{x/\varepsilon_k,\zeta,\nu}, Q^\nu_{\rho/\varepsilon_k}(x/\varepsilon_k)) = (\rho^{n-1}/r_k^{n-1})\, m^{\rm pc}_G(u_{r_k x/\rho,\zeta,\nu}, Q^\nu_{r_k}(r_k x/\rho))$, where $r_k := \rho/\varepsilon_k$. By applying (3.15) with x replaced by x/ρ we obtain

$$\lim_{k\to+\infty} \frac{1}{\rho^{n-1}} m^{\rm pc}_{G_k}(u_{x,\zeta,\nu}, Q^\nu_\rho(x)) = g_{\rm hom}(\zeta,\nu).$$

By (3.10) and (3.11) this implies that $g'(x,\zeta,\nu) = g''(x,\zeta,\nu) = g_{\rm hom}(\zeta,\nu)$. □

4. Compactness result for perturbed functionals

In this section we prove a compactness result, Theorem 4.3, for the perturbed functionals obtained by adding to $E^p_k(u,A)$ the regularising term $\varepsilon \int_{S_u\cap A} |[u]|\, d\mathcal{H}^{n-1}$, with $\varepsilon > 0$. Theorem 4.3 will then be pivotal to prove our main compactness result, Theorem 3.5.

In order to prove Theorem 4.3 we need some technical tools.

We start with a result (Lemma 4.1) establishing the existence of smooth truncations of u by which the functionals F and E "almost decrease" (see (4.12) and (4.13) below). Similar truncation results can be found in [10, proof of Proposition 2.6] and [8, Lemma 3.5].

In what follows we use the shorthand $\{|u|>\lambda\} := \{x\in\mathbb{R}^n : |u(x)|>\lambda\}$, where $u\in L^0(\mathbb{R}^n,\mathbb{R}^m)$ and $\lambda>0$.

Smooth truncations. Let $\varphi\in C^\infty(\mathbb{R})$ be fixed and such that $\varphi(t)=t$ for every $t\le 1$, $\varphi(t)=0$ for every $t\ge 3$, while $\varphi(t)\ge 0$ and $|\varphi'(t)|\le 1$ for every $t\ge 0$. We define $\psi\in C^\infty_c(\mathbb{R}^m,\mathbb{R}^m)$ by

$$\psi(\zeta) := \begin{cases} \varphi(|\zeta|)\zeta/|\zeta| & \text{if } \zeta\neq 0,\\ 0 & \text{if } \zeta=0.\end{cases}$$

Then $\psi(\zeta)=\zeta$ for every $|\zeta|\le 1$, $\psi(\zeta)=0$ for every $|\zeta|\ge 3$, and $|\psi(\zeta)|\le 2$ for every $\zeta\in\mathbb{R}^m$. Moreover for every η, $\tilde\eta\in\mathbb{R}^m$ we have

$$\partial_\eta\psi(\zeta)\cdot\tilde\eta = (\zeta\cdot\eta)\,(\zeta\cdot\tilde\eta)\,\varphi'(|\zeta|)/|\zeta|^2 + (\eta\cdot\tilde\eta)\,\varphi(|\zeta|)/|\zeta| - (\zeta\cdot\eta)\,(\zeta\cdot\tilde\eta)\,\varphi(|\zeta|)/|\zeta|^3.$$

Let $\eta^{||}$ and $\tilde\eta^{||}$ be the orthogonal projections of η and $\tilde\eta$ onto the one-dimensional space generated by ζ, and let $\eta^\perp$ and $\tilde\eta^\perp$ be the orthogonal projections of η and $\tilde\eta$ onto the space orthogonal to ζ. Then

$$\begin{aligned}\partial_\eta\psi(\zeta)\cdot\tilde\eta &= (\eta^{||}\cdot\tilde\eta^{||})\,\varphi'(|\zeta|) + (\eta\cdot\tilde\eta)\,\varphi(|\zeta|)/|\zeta| - (\eta^{||}\cdot\tilde\eta^{||})\,\varphi(|\zeta|)/|\zeta|\\ &= (\eta^{||}\cdot\tilde\eta^{||})\,\varphi'(|\zeta|) + (\eta^\perp\cdot\tilde\eta^\perp)\,\varphi(|\zeta|)/|\zeta|.\end{aligned}$$

Since $|\varphi'(t)|\le 1$ and $0\le\varphi(t)/t\le 1$ for every $t\in\mathbb{R}$, we obtain that

$$\partial_\eta\psi(\zeta)\cdot\tilde\eta \le |\eta^{||}\cdot\tilde\eta^{||}| + |\eta^\perp\cdot\tilde\eta^\perp| \le |\eta|\,|\tilde\eta|.$$

Since $\tilde\eta$ is arbitrary, this implies that $|\partial_\eta\psi(\zeta)|\le|\eta|$ for every $\eta\in\mathbb{R}^m$. By the mean value theorem this inequality gives $|\psi(\zeta_2)-\psi(\zeta_1)|\le|\zeta_2-\zeta_1|$ for every ζ_1, $\zeta_2\in\mathbb{R}^m$.

For every $\lambda>0$ we set

$$\psi^\lambda(\zeta) := \lambda\psi(\zeta/\lambda). \tag{4.1}$$

Then $\psi^\lambda\in C^\infty_c(\mathbb{R}^m,\mathbb{R}^m)$ and

$$\psi^\lambda(\zeta)=\zeta \quad \text{for every } \zeta\in\mathbb{R}^m\colon |\zeta|\le\lambda, \tag{4.2}$$

$$|\psi^\lambda(\zeta)|\le 2\lambda \quad \text{for every } \zeta\in\mathbb{R}^m, \tag{4.3}$$

$$\psi^\lambda(\zeta)=0 \quad \text{for every } \zeta\in\mathbb{R}^m\colon |\zeta|\ge 3\lambda, \tag{4.4}$$

$$|\psi^\lambda(\zeta_2)-\psi^\lambda(\zeta_1)|\le|\zeta_2-\zeta_1| \quad \text{for every } \zeta_1,\zeta_2\in\mathbb{R}^m. \tag{4.5}$$

From (4.2) and (4.5) it follows that

$$|\psi^\lambda(\zeta)|\le|\zeta| \quad \text{for every } \zeta\in\mathbb{R}^m. \tag{4.6}$$

Lemma 4.1. *Let $\eta>0$ and let $h\in\mathbb{N}$, $h\ge 1$, be such that*

$$c_2/(c_1h)<\eta \quad \textit{and} \quad 2c_3/h<\eta, \tag{4.7}$$

where c_1, c_2, and c_3 are as in Definition 3.1. Let moreover $\alpha\ge 3$ be such that $\alpha-1\ge c_3$. Given $\lambda>0$, let $\lambda_1,\dots,\lambda_{h+1}\in\mathbb{R}$ be such that

$$\lambda_1\ge\lambda \tag{4.8}$$

$$\lambda_{i+1}\ge\alpha\lambda_i \quad \textit{for } i=1,\dots,h. \tag{4.9}$$

We set $\mu:=\lambda_{h+1}$ and, for $i=1,\dots,h$, we define $\psi_i:=\psi^{\lambda_i}$, where ψ^{λ_i} is given by (4.1). Then for every $i=1,\dots,h$ we have $\psi_i\in C^\infty_c(\mathbb{R}^m,\mathbb{R}^m)$,

$$|\psi_i(\zeta)|\le\mu \quad \textit{for every } \zeta\in\mathbb{R}^m, \tag{4.10}$$

$$\psi_i(\zeta)=\zeta \quad \textit{for every } \zeta\in\mathbb{R}^m \textit{ with } |\zeta|\le\lambda. \tag{4.11}$$

Moreover, the following property holds: if the function $f\colon\mathbb{R}^n\times\mathbb{R}^{m\times n}\to[0,+\infty)$ satisfies $(f1)$, $(f3)$, $(f4)$, and the function $g\colon\mathbb{R}^n\times\mathbb{R}^m_0\times\mathbb{S}^{n-1}\to[0,+\infty)$ satisfies $(g1)$, $(g3)$, $(g4)$, $(g7)$, then for every $u\in L^0(\mathbb{R}^n,\mathbb{R}^m)$ and every $A\in\mathscr{A}$ there exist $\hat\imath$, $\hat\jmath\in\{1,\dots,h\}$ (depending also on f, g, u, and A) such that

$$F(\psi_{\hat\imath}(u),A)\le(1+\eta)F(u,A)+c_2\mathcal{L}^n(A\cap\{|u|\ge\lambda\}), \tag{4.12}$$

$$E(\psi_{\hat\jmath}(u),A)\le(1+\eta)E(u,A)+c_2\mathcal{L}^n(A\cap\{|u|\ge\lambda\}), \tag{4.13}$$

where F and E are as in (3.2) and (3.4), respectively.

Proof. Since $\alpha \geq 3$, inequalities (4.10) and (4.11) follow from (4.2), (4.3), (4.8), and (4.9).

Let f, g, u, A, be as in the statement. To prove (4.12) and (4.13) it is enough to consider the case $u|_A \in GSBV^p(A,\mathbb{R}^m)$. For every $i = 1,\dots,h$ let $v_i := \psi_i(u)$. Then $v_i = u$ $\mathcal{L}^n$-a.e. in $\{|u| \leq \lambda_i\}$ by (4.2) and $v_i = 0$ $\mathcal{L}^n$-a.e. in $\{|u| \geq \lambda_{i+1}\}$ by (4.4) and (4.9). Moreover (4.5) gives $|\nabla v_i| \leq |\nabla u|$ $\mathcal{L}^n$-a.e. in A. Therefore $(f3)$, $(f4)$, (4.8), and (4.9) yield

$$\begin{aligned} F(v_i,A) &\leq \int_{A\cap\{|u|\leq\lambda_i\}} f(x,\nabla u)\,dx + c_2\mathcal{L}^n(A\cap\{|u|\geq\lambda_{i+1}\}) + c_2\int_{A\cap\{\lambda_i<|u|<\lambda_{i+1}\}} |\nabla u|^p\,dx \\ &\leq \int_A f(x,\nabla u)\,dx + c_2\mathcal{L}^n(A\cap\{|u|\geq\lambda\}) + \frac{c_2}{c_1}\int_{A\cap\{\lambda_i<|u|<\lambda_{i+1}\}} f(x,\nabla u)\,dx. \end{aligned} \tag{4.14}$$

Since

$$\sum_{i=1}^{h}\int_{A\cap\{\lambda_i<|u|<\lambda_{i+1}\}} f(x,\nabla u)\,dx \leq \int_A f(x,\nabla u)\,dx,$$

there exists $\hat\imath \in \{1,\dots,h\}$ such that

$$\int_{A\cap\{\lambda_i<|u|<\lambda_{i+1}\}} f(x,\nabla u)\,dx \leq \frac{1}{h}\int_A f(x,\nabla u)\,dx.$$

By (4.14) this implies

$$F(v_i,A) \leq \Big(1+\frac{c_2}{c_1 h}\Big)F(u,A) + c_2\mathcal{L}^n(A\cap\{|u|\geq\lambda\}),$$

which gives (4.12) thanks to (4.7).

To estimate $G(v_i,A)$ we use the inclusion $S_{v_i} \subset S_u \cap \big(\{|u^+|<\lambda_{i+1}\}\cup\{|u^-|<\lambda_{i+1}\}\big)$. Moreover, thanks to $(g7)$, we can choose the orientation of ν_{v_i} so that $\nu_{v_i} = \nu_u$ $\mathcal{H}^{n-1}$-a.e. in S_{v_i}. This leads to $v_i^\pm = \psi_i(u^\pm)$ $\mathcal{H}^{n-1}$-a.e. in S_{v_i}. By (4.5) this implies that

$$|[v_i]| \leq |[u]| \quad \mathcal{H}^{n-1}\text{-a.e. on } S_{v_i}. \tag{4.15}$$

Therefore we have

$$\begin{aligned} G(v_i,A) \leq{}& \int_{S_u\cap A\cap\{|u^+|\leq\lambda_i\}\cap\{|u^-|\leq\lambda_i\}} g(x,[u],\nu_u)\,d\mathcal{H}^{n-1} + \int_{S_u\cap A\cap\{\lambda_i<|u^+|<\lambda_{i+1}\}} g(x,[v_i],\nu_u)\,d\mathcal{H}^{n-1} + \int_{S_u\cap A\cap\{\lambda_i<|u^-|<\lambda_{i+1}\}} g(x,[v_i],\nu_u)\,d\mathcal{H}^{n-1} \\ &+ \int_{S_u\cap A\cap\{|u^+|\geq\lambda_{i+1}\}\cap\{|u^-|\leq\lambda_i\}} g(x,[v_i],\nu_u)\,d\mathcal{H}^{n-1} + \int_{S_u\cap A\cap\{|u^+|\leq\lambda_i\}\cap\{|u^-|\geq\lambda_{i+1}\}} g(x,[v_i],\nu_u)\,d\mathcal{H}^{n-1}. \end{aligned} \tag{4.16}$$

For $\mathcal{H}^{n-1}$-a.e. point of $\{|u^+|\geq\lambda_{i+1}\}\cap\{|u^-|\leq\lambda_i\}$ we have $[v_i] = -u^-$, hence $|[v_i]|\leq\lambda_i$, while (4.9) implies that

$$|[u]| = |u^+ - u^-| \geq |u^+| - |u^-| \geq \lambda_{i+1} - \lambda_i \geq (\alpha-1)\lambda_i \geq c_3\lambda_i,$$

hence $c_3|[v_i]| \leq |[u]|$. By $(g4)$ this implies

$$g(x,[v_i],\nu_u) \leq g(x,[u],\nu_u) \quad \mathcal{H}^{n-1}\text{-a.e. on } \quad \{|u^+|\geq\lambda_{i+1}\}\cap\{|u^-|\leq\lambda_i\}.$$

The same inequality holds $\mathcal{H}^{n-1}$-a.e. on $\{|u^+|\leq\lambda_i\}\cap\{|u^-|\geq\lambda_{i+1}\}$. Therefore, from (4.15), (4.16), and $(g3)$ we obtain

$$G(v_i,A) \leq \int_{S_u\cap A} g(x,[u],\nu_u)\,d\mathcal{H}^{n-1} + c_3\int_{S_u\cap A\cap\{\lambda_i<|u^+|<\lambda_{i+1}\}} g(x,[u],\nu_u)\,d\mathcal{H}^{n-1} + c_3\int_{S_u\cap A\cap\{\lambda_i<|u^-|<\lambda_{i+1}\}} g(x,[u],\nu_u)\,d\mathcal{H}^{n-1}. \tag{4.17}$$

Since

$$\begin{aligned} &\sum_{i=1}^{h}\Big(\frac{c_2}{c_1}\int_{A\cap\{\lambda_i<|u|<\lambda_{i+1}\}} f(x,\nabla u)\,dx + c_3\int_{S_u\cap A\cap\{\lambda_i<|u^+|<\lambda_{i+1}\}} g(x,[u],\nu_u)\,d\mathcal{H}^{n-1} + c_3\int_{S_u\cap A\cap\{\lambda_i<|u^-|<\lambda_{i+1}\}} g(x,[u],\nu_u)\,d\mathcal{H}^{n-1}\Big) \\ &\qquad\leq \frac{c_2}{c_1}\int_A f(x,\nabla u)\,dx + 2c_3\int_{S_u\cap A} g(x,[u],\nu_u)\,d\mathcal{H}^{n-1}, \end{aligned}$$

there exists $\hat\jmath \in \{1,\dots,h\}$ such that

$$\begin{aligned} &\frac{c_2}{c_1}\int_{A\cap\{\lambda_j<|u|<\lambda_{\hat\jmath+1}\}} f(x,\nabla u)\,dx + c_3\int_{S_u\cap A\cap\{\lambda_j<|u^+|<\lambda_{\hat\jmath+1}\}} g(x,[u],\nu_u)\,d\mathcal{H}^{n-1} + c_3\int_{S_u\cap A\cap\{\lambda_j<|u^-|<\lambda_{\hat\jmath+1}\}} g(x,[u],\nu_u)\,d\mathcal{H}^{n-1} \\ &\qquad\leq \frac{c_2}{c_1 h}\int_A f(x,\nabla u)\,dx + \frac{2c_3}{h}\int_{S_u\cap A} g(x,[u],\nu_u)\,d\mathcal{H}^{n-1}. \end{aligned} \tag{4.18}$$

Inequality (4.13) follows then from (4.7), (4.14), (4.17), and (4.18). □

The estimate in the previous lemma can be extended to the Γ-liminf, as the following result shows.

Lemma 4.2. *Let f_k and g_k be as in Theorem 3.5, let E_k be as in (3.4), with integrands f_k and g_k, and let E_k^p be the restriction of E_k to $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$. Finally, let $E'\colon L^0(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}\to[0,+\infty]$ and $E'^p\colon L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}\to[0,+\infty]$ be defined as*

$$E'(\cdot,A):=\Gamma\text{-}\liminf_{k\to+\infty}E_k(\cdot,A)\quad\text{and}\quad E'^p(\cdot,A):=\Gamma\text{-}\liminf_{k\to+\infty}E_k^p(\cdot,A),$$

where for E' we use the topology of $L^0(\mathbb{R}^n,\mathbb{R}^m)$, while for E'^p we use the topology of $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$. Under the assumptions of Lemma 4.1 the following property holds: for every $u\in L^0(\mathbb{R}^n,\mathbb{R}^m)$, $v\in L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ and $A\in\mathscr{A}$, there exist $\hat\imath$, $\hat\jmath\in\{1,\dots,h\}$ (depending also on u, v, and A) such that

$$E'(\psi_i(u),A)\le(1+\eta)E'(u,A)+c_2\mathcal{L}^n(A\cap\{|u|\ge\lambda\}),\tag{4.19}$$

$$E'^p(\psi_{\hat\jmath}(u),A)\le(1+\eta)E'^p(u,A)+c_2\mathcal{L}^n(A\cap\{|u|\ge\lambda\}).\tag{4.20}$$

Proof. Let $u\in L^0(\mathbb{R}^n,\mathbb{R}^m)$ and $A\in\mathscr{A}$ be fixed. Let (u_k) be a sequence in $L^0(\mathbb{R}^n,\mathbb{R}^m)$ converging to u in measure on bounded sets and such that

$$E'(u,A)=\liminf_{k\to+\infty}E_k(u_k,A).$$

There exists a subsequence (u_{k_j}) such that

$$E'(u,A)=\lim_{j\to+\infty}E_{k_j}(u_{k_j},A).\tag{4.21}$$

By Lemma 4.1 for every j there exists $i_j\in\{1,\dots,h\}$ such that

$$E_{k_j}(\psi_{i_j}(u_{k_j}),A)\le(1+\eta)E_{k_j}(u_{k_j},A)+c_2\mathcal{L}^n(A\cap\{|u_{k_j}|\ge\lambda\}).$$

Therefore there exist $\hat\imath\in\{1,\dots,h\}$ and a sequence $j_\ell\to+\infty$ such that $i_{j_\ell}=\hat\imath$ for every ℓ. This implies that

$$E_{k_{j_\ell}}(\psi_i(u_{k_{j_\ell}}),A)\le(1+\eta)E_{k_{j_\ell}}(u_{k_{j_\ell}},A)+c_2\mathcal{L}^n(A\cap\{|u_{k_{j_\ell}}|\ge\lambda\}).$$

Since $u_{k_{j_\ell}}\to u$ and $\psi_i(u_{k_{j_\ell}})\to\psi_i(u)$ in measure on bounded sets, taking the limit as $\ell\to+\infty$ and using (4.21) we obtain (4.19). The same argument, with obvious changes, also proves (4.20). □

We are now ready to prove the Γ-convergence of the perturbed functionals $E_k^{\varepsilon,p}$, which are defined on $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}\to[0,+\infty]$ by

$$E_k^{\varepsilon,p}(u,A):=\begin{cases}\displaystyle\int_A f_k(x,\nabla u)\,dx+\int_{S_u\cap A}g_k^\varepsilon(x,[u],\nu_u)d\mathcal{H}^{n-1} & \text{if } u|_A\in SBV^p(A,\mathbb{R}^m),\\ +\infty & \text{otherwise in } L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m),\end{cases}\tag{4.22}$$

where

$$g_k^\varepsilon(x,\zeta,\nu):=g_k(x,\zeta,\nu)+\varepsilon|\zeta|.\tag{4.23}$$

Theorem 4.3. *Under the assumptions of Theorem 3.5, for every $\varepsilon>0$ there exist a subsequence, not relabelled, and a functional $E^{\varepsilon,p}\colon L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}\to[0,+\infty]$ such that for every $A\in\mathscr{A}$ the sequence $E_k^{\varepsilon,p}(\cdot,A)$ defined in (4.22) Γ-converges to $E^{\varepsilon,p}(\cdot,A)$ in $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$. Let $f^{\varepsilon,p}\colon\mathbb{R}^n\times\mathbb{R}^{m\times n}\to[0,+\infty]$ and $g^{\varepsilon,p}\colon\mathbb{R}^n\times\mathbb{R}^m_0\times\mathbb{S}^{n-1}\to[0,+\infty]$ be the functions defined by*

$$f^{\varepsilon,p}(x,\xi)=\limsup_{\rho\to0+}\frac{m_{E^{\varepsilon,p}}(\ell_\xi,Q_\rho(x))}{\rho^n},\tag{4.24}$$

$$g^{\varepsilon,p}(x,\zeta,\nu)=\limsup_{\rho\to0+}\frac{m_{E^{\varepsilon,p}}(u_{x,\zeta,\nu},Q_\rho^\nu(x))}{\rho^{n-1}}.\tag{4.25}$$

Then $f^{\varepsilon,p}\in\mathcal{F}$, $g^{\varepsilon,p}$ satisfies (g1), (g3), (g4), and (g7), with c_3 replaced by $\hat c_3:=\max\{c_2/c_1,c_3\}$, and

$$E^{\varepsilon,p}(u,A)=\begin{cases}\displaystyle\int_A f^{\varepsilon,p}(x,\nabla u)\,dx+\int_{S_u\cap A}g^{\varepsilon,p}(x,[u],\nu_u)d\mathcal{H}^{n-1} & \text{if } u|_A\in SBV^p(A,\mathbb{R}^m),\\ +\infty & \text{otherwise in } L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m),\end{cases}\tag{4.26}$$

for every $A\in\mathscr{A}$.

Proof. For fixed $\varepsilon>0$ by (f3), (f4), (4.23), (g5), and (g6), for every $A\in\mathscr{A}$, we have

$$\begin{aligned}&c_1\int_A|\nabla u|^p\,dx+\int_{S_u\cap A}\big(c_4+\varepsilon|[u]|\big)d\mathcal{H}^{n-1}\le E_k^{\varepsilon,p}(u,A)\\&\le c_2\int_A\big(1+|\nabla u|^p\big)\,dx+(c_5+\varepsilon)\int_{S_u\cap A}\big(1+|[u]|\big)d\mathcal{H}^{n-1}\end{aligned}\tag{4.27}$$

if $u|_A\in SBV^p(A,\mathbb{R}^m)$, while $E_k^{\varepsilon,p}(u,A)=+\infty$ if $u|_A\notin SBV^p(A,\mathbb{R}^m)$.

Since the functionals $E_k^{\varepsilon,p}$ satisfy all assumptions of [8, Proposition 3.3], there exist a subsequence, not relabelled, and a functional $E^{\varepsilon,p}\colon L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}\to[0,+\infty]$ such that for every $A\in\mathscr{A}$ the sequence $E_k^{\varepsilon,p}(\cdot,A)$ Γ-converges to $E^{\varepsilon,p}(\cdot,A)$ in $L^p(A,\mathbb{R}^m)$.

Let $\Phi^\varepsilon\colon L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}\to[0,+\infty]$ be defined by

$$\Phi^\varepsilon(u,A):=\begin{cases} c_1\displaystyle\int_A|\nabla u|^p\,dx+\int_{S_u\cap A}\big(c_4+\varepsilon|[u]|\big)d\mathcal{H}^{n-1} & \text{if } u|_A\in SBV^p(A,\mathbb{R}^m),\\ +\infty & \text{otherwise.}\end{cases}\tag{4.28}$$

Since $\Phi^\varepsilon(\cdot,A)$ is lower semicontinuous in $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ (see [1, Theorems 2.2 and 3.7] or [2, Theorem 4.5 and Remark 4.6]), from (4.27) we deduce that for every $u\in L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ and every $A\in\mathscr{A}$ it holds

$$\begin{aligned} &c_1\int_A|\nabla u|^p\,dx+\int_{S_u\cap A}\big(c_4+\varepsilon|[u]|\big)d\mathcal{H}^{n-1}\le E^{\varepsilon,p}(u,A)\\ &\le c_2\int_A\big(1+|\nabla u|^p\big)\,dx+(c_5+\varepsilon)\int_{S_u\cap A}\big(1+|[u]|\big)d\mathcal{H}^{n-1}\end{aligned}\tag{4.29}$$

if $u|_A\in SBV^p(A,\mathbb{R}^m)$, while $E^{\varepsilon,p}(u,A)=+\infty$ if $u|_A\notin SBV^p(A,\mathbb{R}^m)$.

In order to apply the integral representation result [4, Theorem 1] we need a functional defined on $SBV^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}$. Since $E^{\varepsilon,p}(u,A)$ is not defined in $SBV^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\setminus L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$, we now introduce the functional $E^\varepsilon\colon SBV^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}\to[0,+\infty)$ defined by

$$E^\varepsilon(u,A):=\lim_{\lambda\to+\infty}E^{\varepsilon,p}(u^\lambda,A),\tag{4.30}$$

where $u^\lambda:=\psi^\lambda(u)$ and ψ^λ is as in (4.1).

Step 1: E^ε is well defined and $E^\varepsilon=E^{\varepsilon,p}$ on $(SBV^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\cap L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m))\times\mathscr{A}$. We start by proving that E^ε is well defined; *i.e.,* that the limit in (4.30) exists. We prove it by contradiction. Namely, if the limit in (4.30) does not exist we can find $u\in SBV^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$, $A\in\mathscr{A}$, $a<b$, $\lambda_j\to+\infty$, and $\mu_j\to+\infty$ such that

$$E^{\varepsilon,p}(u^{\lambda_j},A)>b\quad\text{and}\quad E^{\varepsilon,p}(u^{\mu_j},A)<a.\tag{4.31}$$

Fix η, h, α as in Lemma 4.1, with $(1+\eta)a+\eta<b$. By possibly removing a finite number of terms in these sequences, it is not restrictive to assume that

$$c_2\mathcal{L}^n(A\cap\{|u|\ge\lambda_1\})<\eta,\tag{4.32}$$

and that $\lambda_{i+1}\ge\alpha\lambda_i$ for $i=1,\dots,h$. Then by Lemma 4.2 for every j there exists $i_j\in\{1,\dots,h\}$ such that

$$E^{\varepsilon,p}(\psi_{i_j}(u^{\mu_j}),A)\le(1+\eta)E^{\varepsilon,p}(u^{\mu_j},A)+c_2\mathcal{L}^n(A\cap\{|u^{\mu_j}|\ge\lambda_1\}),\tag{4.33}$$

where, here and below, we use the shorthand ψ_k for ψ^{λ_k}. Therefore there exist $\hat\imath\in\{1,\dots,h\}$ and a sequence $j_\ell\to+\infty$ such that $i_{j_\ell}=\hat\imath$ for every ℓ. Since $u^{\mu_{j_\ell}}\to u$ in measure on bounded sets we have that $\limsup_\ell\mathcal{L}^n(A\cap\{|u^{\mu_{j_\ell}}|\ge\lambda_1\})\le\mathcal{L}^n(A\cap\{|u|\ge\lambda_1\})$. Moreover $\psi_i(u^{\mu_{j_\ell}})\to\psi_i(u)$ in $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ as $\ell\to+\infty$. By the lower semicontinuity of the Γ-limits, from (4.33) we obtain

$$E^{\varepsilon,p}(\psi_i(u),A)\le(1+\eta)\limsup_{\ell\to+\infty}E^{\varepsilon,p}(u^{\mu_{j_\ell}},A)+c_2\mathcal{L}^n(A\cap\{|u|\ge\lambda_1\}).\tag{4.34}$$

By (4.31) and (4.32) this implies that

$$b<E^{\varepsilon,p}(\psi_i(u),A)\le(1+\eta)a+\eta,$$

which contradicts the inequality $(1+\eta)a+\eta<b$ and hence yields the existence of the limit in (4.30).

We note that (4.30) and (4.34) imply that, under the assumptions of Lemma 4.1, for every $u\in SBV^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ and every $A\in\mathscr{A}$, there exists $\hat\imath\in\{1,\dots,h\}$ such that

$$E^{\varepsilon,p}(\psi_i(u),A)\le(1+\eta)E^\varepsilon(u,A)+c_2\mathcal{L}^n(A\cap\{|u|\ge\lambda_1\}).\tag{4.35}$$

We now show that

$$E^\varepsilon(u,A)=E^{\varepsilon,p}(u,A)\quad\forall\,(u,A)\in\big(SBV^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\cap L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\big)\times\mathscr{A}.\tag{4.36}$$

Fix u and A; since $u^\lambda\to u$ in $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ as $\lambda\to+\infty$ by (4.2) and (4.6), by the lower semicontinuity of the Γ-limits we have

$$E^{\varepsilon,p}(u,A)\le\liminf_{\lambda\to+\infty}E^{\varepsilon,p}(u^\lambda,A)=E^\varepsilon(u,A).$$

To prove the opposite inequality we fix η, h, and α as in Lemma 4.1 and we consider a sequence (λ_i), $\lambda_i \to +\infty$ as $i \to +\infty$, such that $\lambda_{i+1} \geq \alpha\lambda_i$ for every i. We now apply Lemma 4.2 to $\lambda_{i+1}, \dots, \lambda_{i+h}$ and obtain that for every i there exists $j_i \in \{i+1, \dots, i+h\}$ such that

$$E^{\varepsilon,p}(u^{\lambda_{j_i}}, A) \leq (1+\eta)E^{\varepsilon,p}(u, A) + c_2\mathcal{L}^n(A \cap \{|u| \geq \lambda_i\}).$$

Taking the limit as $i \to +\infty$, by (4.30) we get

$$E^{\varepsilon}(u, A) \leq (1+\eta)E^{\varepsilon,p}(u, A),$$

and taking the limit as $\eta \to 0+$ we obtain

$$E^{\varepsilon}(u, A) \leq E^{\varepsilon,p}(u, A),$$

which concludes the proof of (4.36).

Step 2: Lower semicontinuity of E^ε with respect to the strong convergence in $L^1_{\rm loc}$. For fixed $A \in \mathscr{A}$ we now prove that $E^{\varepsilon}(\cdot, A)$ is lower semicontinuous on $SBV^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$ with respect to the strong convergence in $L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$. Let us fix $u \in SBV^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$ and a sequence (u_k) in $SBV^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$ converging to u in $L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$ and such that $\lim_k E^{\varepsilon}(u_k, A)$ exists. Let η, h, α, and (λ_i) be as in the previous step. We now apply (4.35) to $\lambda_{i+1}, \dots, \lambda_{i+h}$ and obtain that for every i and every k there exists $j_{i,k} \in \{i+1, \dots, i+h\}$ such that

$$E^{\varepsilon,p}(\psi_{j_{i,k}}(u_k), A) \leq (1+\eta)E^{\varepsilon}(u_k, A) + c_2\mathcal{L}^n(A \cap \{|u_k| \geq \lambda_i\}).$$

For every i there exist $N_i \in \{i+1, \dots, i+h\}$ and sequence $k^i_\ell \to +\infty$ as $\ell \to +\infty$ E such that $j_{i,k^i_\ell} = N_i$ for every ℓ. Since $\psi_{N_i}(u_{k^i_\ell})$ converges to $\psi_{N_i}(u)$ in $L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$ as $\ell \to +\infty$, by the lower semicontinuity of the Γ-limits we obtain

$$\begin{aligned} E^{\varepsilon,p}(\psi_{N_i}(u), A) &\leq \liminf_{\ell \to +\infty} E^{\varepsilon,p}(\psi_{N_i}(u_{k^i_\ell}), A) \\ &\leq (1+\eta) \lim_{\ell \to +\infty} E^{\varepsilon}(u_{k^i_\ell}, A) + c_2\mathcal{L}^n(A \cap \{|u| \geq \lambda_i\}) \\ &= (1+\eta) \lim_{k \to +\infty} E^{\varepsilon}(u_k, A) + c_2\mathcal{L}^n(A \cap \{|u| \geq \lambda_i\}). \end{aligned}$$

Taking the limit first as $i \to +\infty$ and then as $\eta \to 0+$, from (4.30) and from the previous inequalities we obtain

$$E^{\varepsilon}(u, A) \leq \lim_{k \to +\infty} E^{\varepsilon}(u_k, A),$$

which proves the lower semicontinuity of $E^{\varepsilon}(\cdot, A)$.

Step 3: Integral representation of $E^{\varepsilon,p}$. By [8, Proposition 3.3] for every $u \in SBV^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m) \cap L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m))$ the function $A \mapsto E^{\varepsilon,p}(u, A)$ is the restriction to $\mathscr{A}$ of a measure defined on the σ-algebra of all Borel subsets of $\mathbb{R}^n$. By (4.29) and (4.30), this implies that for every $u \in SBV^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$ the function $A \mapsto E^{\varepsilon}(u, A)$ is the restriction to $\mathscr{A}$ of a measure defined on the Borel σ-algebra of $\mathbb{R}^n$ (see, e.g., [15, Théorème 5.7]).

It follows from the definition that $E^{\varepsilon,p}$ is local; *i.e.,* if $u, v \in L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$, $A \in \mathscr{A}$, and $u = v$ $\mathcal{L}^n$-a.e. in A, then $E^{\varepsilon,p}(u, A) = E^{\varepsilon,p}(v, A)$. By (4.30), this property immediately extends to E^{ε}; *i.e.,* for every u, $v \in SBV^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$, $A \in \mathscr{A}$, with $u = v$ $\mathcal{L}^n$-a.e. in A, we have $E^{\varepsilon}(u, A) = E^{\varepsilon}(u, A)$. Moreover, by (4.5) we have $|\nabla u^\lambda| \leq |\nabla u|$ $\mathcal{L}^n$-a.e. in A and $|[u^\lambda]| \leq |[u]|$ $\mathcal{H}^{n-1}$-a.e. in $S_{u^\lambda} \cap A \subset S_u \cap A$. Taking into account the lower semicontinuity of Φ^ε defined in (4.28), these inequalities, together with (4.29) and (4.30), yield

$$\begin{aligned} &c_1 \int_A |\nabla u|^p \, dx + \int_{S_u \cap A} \big(c_4 + \varepsilon|[u]|\big) d\mathcal{H}^{n-1} \leq E^{\varepsilon}(u, A) \\ &\leq c_2 \int_A \big(1 + |\nabla u|^p\big) \, dx + (c_5 + \varepsilon) \int_{S_u \cap A} \big(1 + |[u]|\big) d\mathcal{H}^{n-1} \end{aligned}$$

for every $u \in SBV^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$ and every $A \in \mathscr{A}$.

Therefore $E^{\varepsilon,p}$ satisfies all the assumptions of the integral representation result [4, Theorem 1]. Consequently, using also (4.36), for every $u \in SBV^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m) \cap L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$ and every $A \in \mathscr{A}$ we have the integral representation (4.26) with $f^{\varepsilon,p}$ and $g^{\varepsilon,p}$ defined by (4.24) and (4.25). Indeed, it is easy to deduce from (3.7), (4.2), (4.30), and (4.36) that for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m \times n}$, $\zeta \in \mathbb{R}^m_0$, $\nu \in \mathbb{S}^{n-1}$, and $\rho > 0$ we have

$$\begin{aligned} m_{E^{\varepsilon,p}}(\ell_\xi, Q_\rho(x)) &= \inf\{E^{\varepsilon}(u, Q_\rho(x)) : u \in SBV^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m),\ u = \ell_\xi \text{ near } \partial Q_\rho(x)\}, \\ m_{E^{\varepsilon,p}}(u_{x,\zeta,\nu}, Q^\nu_\rho(x)) &= \inf\{E^{\varepsilon}(u, Q^\nu_\rho(x)) : u \in SBV^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m),\ u = u_{x,\zeta,\nu} \text{ near } \partial Q^\nu_\rho(x)\}, \end{aligned}$$

which coincide with the definitions used in [4]. By locality and inner regularity, formula (4.26) holds also for every $u \in L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$ and every $A \in \mathscr{A}$ such that $u|_A \in SBV^p(A, \mathbb{R}^m)$.

The Borel measurability of $f^{\varepsilon,p}$ and $g^{\varepsilon,p}$ are then proved in Lemma A.5.

Step 4: $f^{\varepsilon,p}$ satisfies $(f2)$, $(f3)$ and $(f4)$. We now show that $f^{\varepsilon,p}$ satisfies $(f2)$. Since $(f2)$ holds for f_k, for every $A \in \mathscr{A}$ we have

$$E_k^{\varepsilon,p}(u+\ell_\xi, A) \le E_k^{\varepsilon,p}(u, A) + \sigma_1(|\xi|)\big(\mathcal{L}^n(A) + E_k^{\varepsilon,p}(u+\ell_\xi, A) + E_k^{\varepsilon,p}(u, A)\big)$$

for every $\xi \in \mathbb{R}^{m\times n}$ and for every $u \in L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$. We have

$$(1-\sigma_1(|\xi|))E_k^{\varepsilon,p}(u+\ell_\xi, A) \le (1+\sigma_1(|\xi|))E_k^{\varepsilon,p}(u, A) + \sigma_1(|\xi|)\mathcal{L}^n(A), \tag{4.37}$$

thus if $\sigma_1(|\xi|) < 1$ taking the Γ-limit gives

$$(1-\sigma_1(|\xi|))E^{\varepsilon,p}(u+\ell_\xi, A) \le (1+\sigma_1(|\xi|))E^{\varepsilon,p}(u, A) + \sigma_1(|\xi|)\mathcal{L}^n(A).$$

This implies that

$$(1-\sigma_1(|\xi_2-\xi_1|))\, m_{E^{\varepsilon,p}}(\ell_{\xi_2}, Q_\rho(x)) \le (1+\sigma_1(|\xi_2-\xi_1|))\, m_{E^{\varepsilon,p}}(\ell_{\xi_1}, Q_\rho(x)) + \sigma_1(|\xi_2-\xi_1|)\rho^n \tag{4.38}$$

for every $\rho > 0$, $x \in \mathbb{R}^n$, and ξ_1, $\xi_2 \in \mathbb{R}^{m\times n}$ with $\sigma_1(|\xi_2-\xi_1|) < 1$. Dividing by ρ^n and taking the limsup as $\rho \to 0+$ we obtain from (4.24) and (4.38)

$$(1-\sigma_1(|\xi_2-\xi_1|))f^{\varepsilon,p}(x,\xi_2) \le (1+\sigma_1(|\xi_2-\xi_1|))f^{\varepsilon,p}(x,\xi_1) + \sigma_1(|\xi_2-\xi_1|).$$

which implies

$$f^{\varepsilon,p}(x,\xi_2) \le f^{\varepsilon,p}(x,\xi_1) + \sigma_1(|\xi_2-\xi_1|)(1 + f^{\varepsilon,p}(x,\xi_1) + f^{\varepsilon,p}(x,\xi_2)).$$

This inequality is trivial if $\sigma_1(|\xi_2-\xi_1|) \ge 1$. Exchanging the roles of ξ_1 and ξ_2 we obtain $(f2)$ for $f^{\varepsilon,p}$.

Let us prove that $f^{\varepsilon,p}$ satisfies $(f3)$. By (4.27) for every $u \in L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$ and every $A \in \mathscr{A}$ we have that $E_k^{\varepsilon,p}(u, A) \ge \Phi^\varepsilon(u, A)$ for every k, where Φ^ε is defined by (4.28). Since $\Phi^\varepsilon(\cdot, A)$ is lower semicontinuous in $L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$, this inequality is preserved in the Γ-limit and hence we get

$$E^{\varepsilon,p}(u, A) \ge \Phi^\varepsilon(u, A) \tag{4.39}$$

for every $u \in L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$ and every $A \in \mathscr{A}$.

Let $\phi^\varepsilon : \mathbb{R}^n \times \mathbb{R}^{m\times n} \to [0, +\infty]$ be defined by

$$\phi^\varepsilon(x,\xi) := \limsup_{\rho\to 0+} \frac{m_{\Phi^\varepsilon}(\ell_\xi, Q_\rho(x))}{\rho^n}. \tag{4.40}$$

Note that, by translation invariance, $\phi^\varepsilon(x,\xi) = \phi^\varepsilon(0,\xi)$ for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^{m\times n}$. We can now apply the integral representation result [4, Theorem 1] to Φ^ε and, taking $u = \ell_\xi$ and $A = Q(0)$, we obtain

$$c_1|\xi|^p = \Phi^\varepsilon(\ell_\xi, Q(0)) = \int_{Q(0)} \phi^\varepsilon(y,\xi)\, dy = \phi^\varepsilon(0,\xi) = \phi^\varepsilon(x,\xi)$$

for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^{m\times n}$. Together with (4.24), (4.39), and (4.40), this gives the lower bound $(f3)$ for $f^{\varepsilon,p}$.

To prove the upper bound $(f4)$, we observe that $E_k^{\varepsilon,p}(\ell_\xi, Q_\rho(x)) \le c_2(1+|\xi|^p)\rho^n$ for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m\times n}$, $\rho > 0$ and k. This implies that $E^{\varepsilon,p}(\ell_\xi, Q_\rho(x)) \le c_2(1+|\xi|^p)\rho^n$, hence $m_{E^{\varepsilon,p}}(\ell_\xi, Q_\rho(x)) \le c_2(1+|\xi|^p)\rho^n$. The upper bound $(f4)$ for $f^{\varepsilon,p}$ follows from (4.24).

Step 5: $g^{\varepsilon,p}$ satisfies $(g3)$, $(g4)$ and $(g7)$. To prove $(g3)$ we fix ζ_1, $\zeta_2 \in \mathbb{R}^m_0$, with $|\zeta_1| \le |\zeta_2|$, and a rotation R on $\mathbb{R}^m$ such that $aR\zeta_2 = \zeta_1$, where $a := |\zeta_1|/|\zeta_2| \le 1$. Since f_k and g_k^ε (see (4.23)) satisfy $(f3)$, $(f4)$, and $(g3)$, for every $A \in \mathscr{A}$ and every $u \in L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$, with $u|_A \in SBV^p(A, \mathbb{R}^m)$, we have

$$\begin{aligned}
E_k^{\varepsilon,p}(aRu, A) &= \int_A f_k(x, aR\nabla u)\, dx + \int_{S_u\cap A} g_k^\varepsilon(x, aR[u], \nu_u) d\mathcal{H}^{n-1} \\
&\le c_2\mathcal{L}^n(A) + c_2\int_A |\nabla u|^p\, dx + c_3\int_{S_u\cap A} g_k^\varepsilon(x, [u], \nu_u) d\mathcal{H}^{n-1} \\
&\le c_2\mathcal{L}^n(A) + \frac{c_2}{c_1}\int_A f_k(x, \nabla u)\, dx + c_3\int_{S_u\cap A} g_k^\varepsilon(x, [u], \nu_u) d\mathcal{H}^{n-1}.
\end{aligned}$$

Passing to the Γ-limit, we obtain $E^{\varepsilon,p}(aRu, A) \le c_2\mathcal{L}^n(A) + \hat{c}_3 E^{\varepsilon,p}(u, A)$, with $\hat{c}_3 = \max\{c_2/c_1, c_3\}$. This implies that $m_{E^{\varepsilon,p}}(u_{x,aR\zeta_2,\nu}, Q^\nu_\rho(x)) \le c_2\rho^n + \hat{c}_3 m_{E^{\varepsilon,p}}(u_{x,\zeta_2,\nu}, Q^\nu_\rho(x))$ for every $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, and $\rho > 0$. Since $aR\zeta_2 = \zeta_1$, using (4.25) we obtain $g^{\varepsilon,p}(x,\zeta_1,\nu) \le \hat{c}_3\, g^{\varepsilon,p}(x,\zeta_2,\nu)$, which proves $(g3)$, with c_3 replaced by $\hat{c}_3$.

To prove $(g4)$ we fix $\zeta_1, \zeta_2 \in \mathbb{R}^m_0$, with $\hat{c}_3|\zeta_1| \le |\zeta_2|$, and a rotation R on $\mathbb{R}^m$ such that $aR\zeta_2 = \zeta_1$, where $a := |\zeta_1|/|\zeta_2| \le 1/\hat{c}_3 \le 1$. Since f_k and g_k^ε satisfy $(f3)$, $(f4)$, and $(g4)$, the inequalities $c_3 a \le \hat{c}_3 a \le 1$ imply that for every $A \in \mathscr{A}$ and every $u \in L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$, with $u|_A \in SBV^p(A, \mathbb{R}^m)$, we have

$$\begin{aligned} E_k^{\varepsilon,p}(aRu, A) &= \int_A f_k(x, aR\nabla u)\, dx + \int_{S_u \cap A} g_k^\varepsilon(x, aR[u], \nu_u) d\mathcal{H}^{n-1} \\ &\le c_2 \mathcal{L}^n(A) + c_2 a^p \int_A |\nabla u|^p\, dx + \int_{S_u \cap A} g_k^\varepsilon(x, [u], \nu_u) d\mathcal{H}^{n-1} \\ &\le c_2 \mathcal{L}^n(A) + \frac{c_2 a^p}{c_1} \int_A f_k(x, \nabla u)\, dx + \int_{S_u \cap A} g_k^\varepsilon(x, [u], \nu_u) d\mathcal{H}^{n-1}. \end{aligned}$$

Since $a \le 1$ and $\hat{c}_3 a \le 1$, we have $c_2 a^p/c_1 \le c_2 a/c_1 \le \hat{c}_3 a \le 1$. Therefore $E_k^{\varepsilon,p}(aRu, A) \le c_2 \mathcal{L}^n(A) + E_k^{\varepsilon,p}(u, A)$. Passing to the Γ-limit, we obtain $E^{\varepsilon,p}(aRu, A) \le c_2 \mathcal{L}^n(A) + E^{\varepsilon,p}(u, A)$. This implies that $m_{E^{\varepsilon,p}}(u_{x,aR\zeta_2,\nu}, Q^\nu_\rho(x)) \le c_2 \rho^n + m_{E^{\varepsilon,p}}(u_{x,\zeta_2,\nu}, Q^\nu_\rho(x))$ for every $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, and $\rho > 0$. Since $aR\zeta_2 = \zeta_1$, using (4.25) we obtain $g^{\varepsilon,p}(x, \zeta_1, \nu) \le g^{\varepsilon,p}(x, \zeta_2, \nu)$, which proves $(g4)$, with c_3 replaced by $\hat{c}_3$.

To prove the symmetry condition $(g7)$ for $g^{\varepsilon,p}$, we observe that $u_{x,-\zeta,-\nu} = u_{x,\zeta,\nu} - \zeta$ for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^m_0$, and $\nu \in \mathbb{S}^{n-1}$. Therefore $u \in SBV^p(Q^\nu_\rho(x), \mathbb{R}^m) \cap L^p(Q^\nu_\rho(x), \mathbb{R}^m)$ satisfies $u = u_{x,-\zeta,-\nu}$ in a neighbourhood of $\partial Q^\nu_\rho(x)$ if and only if $u = v - \zeta$ for some $v \in SBV^p(Q^\nu_\rho(x), \mathbb{R}^m) \cap L^p(Q^\nu_\rho(x), \mathbb{R}^m)$ satisfying $v = u_{x,\zeta,\nu}$ in a neighbourhood of $\partial Q^\nu_\rho(x)$. Since $Q^{-\nu}_\rho(x) = Q^\nu_\rho(x)$ by (k) in Section 2, it follows that $m_{E^{\varepsilon,p}}(u_{x,-\zeta,-\nu}, Q^{-\nu}_\rho(x)) = m_{E^{\varepsilon,p}}(u_{x,\zeta,\nu}, Q^\nu_\rho(x))$. By (4.25) this implies that $g^{\varepsilon,p}(x, \zeta, \nu) = g^{\varepsilon,p}(x, -\zeta, -\nu)$, which proves $(g7)$ for $g^{\varepsilon,p}$. □

5. Proof of the compactness result

In this section we begin the proof of the compactness result with respect to Γ-convergence, Theorem 3.5. We start with the following perturbation result, which, together with Theorem 4.3, provides a slightly weaker version of Theorem 3.5. Indeed it does not establish that the surface integrand g^0, defined in (5.2) below, satisfies properties $(g2)$, $(g5)$, and $(g6)$.

Theorem 5.1 (Perturbation result)**.** *Under the hypotheses of Theorem 3.5, let D be a countable subset of $(0, +\infty)$ with $0 \in \overline{D}$. Assume that for every $\varepsilon \in D$ there exists a functional $E^{\varepsilon,p}\colon L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m) \times \mathscr{A} \to [0, +\infty]$ such that for every $A \in \mathscr{A}$ the sequence $E_k^{\varepsilon,p}(\cdot, A)$ defined in (4.22) Γ-converges to $E^{\varepsilon,p}(\cdot, A)$ in $L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$. Let $f^{\varepsilon,p}$ and $g^{\varepsilon,p}$ be the functions defined by (4.24) and (4.25), and let $f^0\colon \mathbb{R}^n \times \mathbb{R}^{m\times n} \to [0, +\infty]$ and $g^0\colon \mathbb{R}^n \times \mathbb{R}^m_0 \times \mathbb{S}^{n-1} \to [0, +\infty]$ be the functions defined by*

$$f^0(x, \xi) := \inf_{\varepsilon \in D} f^{\varepsilon,p}(x, \xi) = \lim_{\substack{\varepsilon \to 0+ \\ \varepsilon \in D}} f^{\varepsilon,p}(x, \xi), \tag{5.1}$$

$$g^0(x, \zeta, \nu) := \inf_{\varepsilon \in D} g^{\varepsilon,p}(x, \zeta, \nu) = \lim_{\substack{\varepsilon \to 0+ \\ \varepsilon \in D}} g^{\varepsilon,p}(x, \zeta, \nu). \tag{5.2}$$

Then $f^0 \in \mathcal{F}$ and g^0 satisfies $(g1)$, $(g3)$, $(g4)$, and $(g7)$, with c_3 replaced by $\hat{c}_3 := \max\{c_2/c_1, c_3\}$.

Let E^0 and E_k be as in (3.4), with f and g replaced by f^0 and g^0 and by f_k and g_k, respectively, and let $E^{0,p}$ and E^p_k be the corresponding restrictions to $L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m) \times \mathscr{A}$. Then

$$\begin{aligned} &E_k(\cdot, A) \ \Gamma\text{-converges to } E^0(\cdot, A) \text{ in } L^0(\mathbb{R}^n, \mathbb{R}^m), \\ &E^p_k(\cdot, A) \ \Gamma\text{-converges to } E^{0,p}(\cdot, A) \text{ in } L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m), \end{aligned}$$

for every $A \in \mathscr{A}$.

Proof. By Theorem 4.3 $E^{\varepsilon,p}$ can be written in integral form as in (4.26), where $f^{\varepsilon,p}$ and $g^{\varepsilon,p}$ are defined by (4.24) and (4.25) and satisfy $(f1)$-$(f4)$ and $(g1)$, $(g3)$, $(g4)$, $(g7)$. It follows from (4.24) and (4.25) that $f^{\varepsilon_1,p} \le f^{\varepsilon_2,p}$ and $g^{\varepsilon_1,p} \le g^{\varepsilon_2,p}$ for $0 < \varepsilon_1 < \varepsilon_2$.

Properties $(f1)$-$(f4)$ for f^0 and properties $(g1)$, $(g3)$, $(g4)$, $(g7)$ for g^0 follow from (5.1) and (5.2) and from the corresponding properties for $f^{\varepsilon,p}$ and $g^{\varepsilon,p}$.

By the Monotone Convergence Theorem we have

$$E^{0,p}(u, A) = \lim_{\substack{\varepsilon \to 0+ \\ \varepsilon \in D}} E^{\varepsilon,p}(u, A) \tag{5.3}$$

for every $A \in \mathscr{A}$ and every $u \in L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$ with $u|_A \in SBV^p(A, \mathbb{R}^m)$.

Let E', $E''\colon L^0(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}\to[0,+\infty]$ and E'^p, $E''^p\colon L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}\to[0,+\infty]$ be defined by

$$E'(\cdot,A):=\Gamma\text{-}\liminf_{k\to+\infty}E_k(\cdot,A)\quad\text{and}\quad E''(\cdot,A):=\Gamma\text{-}\limsup_{k\to+\infty}E_k(\cdot,A),$$
$$E'^p(\cdot,A):=\Gamma\text{-}\liminf_{k\to+\infty}E^p_k(\cdot,A)\quad\text{and}\quad E''^p(\cdot,A):=\Gamma\text{-}\limsup_{k\to+\infty}E^p_k(\cdot,A),$$

where for E' and E'' we use the topology of $L^0(\mathbb{R}^n,\mathbb{R}^m)$, while for E'^p and E''^p we use the topology of $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$.

Then for every $u\in L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ and for every $\varepsilon\in D$ we have $E''(u,A)\le E''^p(u,A)\le E^{\varepsilon,p}(u,A)$, thus by (5.3)

$$E''(u,A)\le E''^p(u,A)\le E^{0,p}(u,A)=E^0(u,A)\tag{5.4}$$

for every $A\in\mathscr{A}$ and $u\in L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ with $u|_A\in SBV^p(A,\mathbb{R}^m)$.

We claim that

$$E^0(u,A)=E^{0,p}(u,A)\le E'(u,A)\le E'^p(u,A)\tag{5.5}$$

for every $A\in\mathscr{A}$ and every $u\in L^\infty(\mathbb{R}^n,\mathbb{R}^m)$. Let us fix A and u. The inequality $E'(u,A)\le E'^p(u,A)$ is trivial. By Γ-convergence there exists a sequence (u_k) converging to u in $L^0(\mathbb{R}^n,\mathbb{R}^m)$ such that

$$E'(u,A)=\liminf_{k\to+\infty}E_k(u_k,A).\tag{5.6}$$

Let us fix $\lambda>\|u\|_{L^\infty(\mathbb{R}^n,\,\mathbb{R}^m)}$ and $\varepsilon>0$. By Lemma 4.1 there exist $\mu>\lambda$, independent of k, and a sequence $(v_k)\subset L^\infty(\mathbb{R}^n,\mathbb{R}^m)$, converging to u in measure on bounded sets, such that for every k we have

$$\|v_k\|_{L^\infty(\mathbb{R}^n,\,\mathbb{R}^m)}\le\mu,\tag{5.7}$$
$$v_k=u_k\quad\mathcal{L}^n\text{-a.e. in }\{|u_k|\le\lambda\},\tag{5.8}$$
$$E_k(v_k,A)\le(1+\varepsilon)E_k(u_k,A)+c_2\mathcal{L}^n(A\cap\{|u_k|\ge\lambda\}).\tag{5.9}$$

It follows from (5.7) that $v_k\to u$ also in $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$. If $E_k(u_k,A)<+\infty$, by $(f3)$, $(g5)$, and (5.9) the function v_k belongs to $GSBV^p(A,\mathbb{R}^m)$ and

$$\mathcal{H}^{n-1}(S_{v_k}\cap A)\le(1/c_4)(1+\varepsilon)E_k(u_k,A)+(c_2/c_4)\mathcal{L}^n(A\cap\{|u_k|\ge\lambda\}).\tag{5.10}$$

By (4.22) and (5.7) this implies that

$$E^{\varepsilon,p}_k(v_k,A)\le E_k(v_k,A)+2\varepsilon\mu\mathcal{H}^{n-1}(S_{v_k}\cap A),$$

which, in its turn, by (5.9) and (5.10), leads to

$$E^{\varepsilon,p}_k(v_k,A)\le(1+\varepsilon)(1+(2\varepsilon\mu/c_4))E_k(u_k,A)+c_2(1+(2\varepsilon\mu/c_4))\mathcal{L}^n(A\cap\{|u_k|\ge\lambda\}).$$

Clearly this inequality holds also when $E_k(u_k,A)=+\infty$. Therefore, using (5.6) and the inequality $\|u\|_{L^\infty(\mathbb{R}^n,\,\mathbb{R}^m)}<\lambda$, by Γ-convergence we get

$$E^{\varepsilon,p}(u,A)\le(1+\varepsilon)(1+(2\varepsilon\mu/c_4))E'(u,A)$$

for every $\varepsilon\in D$. By (5.3), passing to the limit as $\varepsilon\to0+$ we obtain (5.5) whenever $u\in L^\infty(\mathbb{R}^n,\mathbb{R}^m)$.

We now prove that

$$E''(u,A)\le E^0(u,A)\quad\text{for every }u\in L^0(\mathbb{R}^n,\mathbb{R}^m)\text{ and every }A\in\mathscr{A}.\tag{5.11}$$

Let us fix u and A. It is enough to prove the inequality when $u|_A\in GSBV^p(A,\mathbb{R}^m)$. By Lemma 4.1 for every $\varepsilon>0$ and for every integer $k\ge1$ there exists $u_k\in L^\infty(\mathbb{R}^n,\mathbb{R}^m)$, with $u_k|_A\in SBV^p(A,\mathbb{R}^m)$, such that $u_k=u$ $\mathcal{L}^n$-a.e. in $\{|u|\le k\}$ and

$$E^0(u_k,A)\le(1+\varepsilon)E^0(u,A)+c_2\mathcal{L}^n(A\cap\{|u|\ge k\}).$$

By (5.4) we have $E''(u_k,A)\le E^0(u_k,A)$, hence

$$E''(u_k,A)\le(1+\varepsilon)E^0(u,A)+c_2\mathcal{L}^n(A\cap\{|u|\ge k\}).$$

Since $u_k\to u$ in measure on bounded sets, passing to the limit as $k\to+\infty$, by the lower semicontinuity of the Γ-limsup we deduce

$$E''(u,A)\le(1+\varepsilon)E^0(u,A).$$

Hence letting $\varepsilon\to0+$ we obtain (5.11). The same proof shows that

$$E''^p(u,A)\le E^{0,p}(u,A)\quad\text{for every }u\in L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\text{ and every }A\in\mathscr{A}.\tag{5.12}$$

We now prove that

$$E^0(u,A)\le E'(u,A)\quad\text{for every }u\in L^0(\mathbb{R}^n,\mathbb{R}^m)\text{ and every }A\in\mathscr{A}.\tag{5.13}$$

Let us fix u and A. It is enough to prove the inequality when $u|_A \in GSBV^p(A,\mathbb{R}^m)$, since otherwise $E'(u,A)=+\infty$ due to the lower bounds $(f3)$ and $(g5)$. By Lemma 4.2 for every $\varepsilon>0$ and every integer $k\geq 1$ there exists $u_k\in L^\infty(\mathbb{R}^n,\mathbb{R}^m)$, with $u_k|_A\in SBV^p(A,\mathbb{R}^m)$, such that $u_k=u$ $\mathcal{L}^n$-a.e. in $\{|u|\leq k\}$, $u_k^\pm=u^\pm$ $\mathcal{H}^{n-1}$-a.e. in $S_u\cap\{|u^\pm|\leq k\}$, and

$$E'(u_k,A)\leq(1+\varepsilon)E'(u,A)+c_2\mathcal{L}^n(A\cap\{|u|\geq k\}).$$

By (5.5) we have $E^0(u_k,A)\leq E'(u_k,A)$, hence

$$\int_{A\cap\{|u|\leq k\}} f^0(x,\nabla u)\,dx+\int_{S_u\cap A\cap\{|u^+|\leq k\}\cap\{|u^-|\leq k\}} g^0(x,[u],\nu_u)\,d\mathcal{H}^{n-1}\leq E^0(u_k,A)\leq(1+\varepsilon)E'(u,A)+c_2\mathcal{L}^n(A\cap\{|u|\geq k\}).$$

As $k\to+\infty$ we get

$$E^0(u,A)=\int_A f^0(x,\nabla u)\,dx+\int_{S_u\cap A} g^0(x,[u],\nu_u)\,d\mathcal{H}^{n-1}\leq(1+\varepsilon)E'(u,A),$$

and as $\varepsilon\to 0+$ we obtain (5.13). Since $E'(u,A)\leq E'^p(u,A)$ for every $u\in L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$, from (5.13) we also get

$$E^{0,p}(u,A)\leq E'^p(u,A)\quad\text{for every } u\in L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\text{ and every } A\in\mathscr{A}. \tag{5.14}$$

The Γ-convergence of $E_k(\cdot,A)$ to $E^0(\cdot,A)$ in $L^0(\mathbb{R}^n,\mathbb{R}^m)$ follows from (5.11) and (5.13), while the Γ-convergence of $E^p_k(\cdot,A)$ to $E^{0,p}(\cdot,A)$ in $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ follows from (5.12) and (5.14). □

To conclude the proof of Theorem 3.5 and to prepare the proof of Theorem 3.7, we now establish some relations between the functions f^0 and g^0 introduced in Theorem 5.1 and the functions f', f'', g', and g'' defined in (3.8)-(3.11).

Theorem 5.2. *Under the assumptions of Theorems 3.5 and 5.1, let f^0 and g^0 be defined by* (5.1) *and* (5.2) *and let f', f'', g', and g'' be defined by* (3.8)-(3.11). *Then*

(a) *for every $x\in\mathbb{R}^n$ and every $\xi\in\mathbb{R}^{m\times n}$ we have $f^0(x,\xi)\leq f'(x,\xi)$;*
(b) *for $\mathcal{L}^n$-a.e. $x\in\mathbb{R}^n$ we have $f''(x,\xi)\leq f^0(x,\xi)$ for every $\xi\in\mathbb{R}^{m\times n}$;*
(c) *for every $x\in\mathbb{R}^n$, every $\zeta\in\mathbb{R}^m_0$, and every $\nu\in\mathbb{S}^{n-1}$ we have $g^0(x,\zeta,\nu)\leq g'(x,\zeta,\nu)$;*
(d) *for every $A\in\mathscr{A}$ and every $u\in GSBV^p(A,\mathbb{R}^m)$ we have*

$$g''(x,[u](x),\nu_u(x))\leq g^0(x,[u](x),\nu_u(x)) \tag{5.15}$$

for $\mathcal{H}^{n-1}$-a.e. $x\in S_u\cap A$.

The proof of Theorem 5.2 is postponed to Sections 7 and 8.

Remark 5.3. Since by definition $f'\leq f''$ and $g'\leq g''$, Theorem 5.2 implies that for $\mathcal{L}^n$-a.e. $x\in\mathbb{R}^n$ we have $f'(x,\xi)=f''(x,\xi)=f^0(x,\xi)$ for every $\xi\in\mathbb{R}^{m\times n}$, and that for every $A\in\mathscr{A}$ and every $u\in GSBV^p(A,\mathbb{R}^m)$ we have

$$g'(x,[u](x),\nu_u(x))=g''(x,[u](x),\nu_u(x))=g^0(x,[u](x),\nu_u(x)),$$

for $\mathcal{H}^{n-1}$-a.e. $x\in S_u\cap A$.

Appealing to Theorem 5.2 we can now conclude the proof of the compactness result, Theorem 3.5.

Proof of Theorem 3.5. By combining Theorem 4.3 and a diagonal argument, we obtain a subsequence, not relabelled, and, for every $\varepsilon\in D$, a functional $E^{\varepsilon,p}\colon L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)\times\mathscr{A}\to[0,+\infty]$, such that for every $A\in\mathscr{A}$ the sequence $E^{\varepsilon,p}_k(\cdot,A)$ Γ-converges in $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ to $E^{\varepsilon,p}(\cdot,A)$ for every $\varepsilon\in D$. By Theorem 5.1 $E_k(\cdot,A)$ Γ-converges to $E^0(\cdot,A)$ in $L^0(\mathbb{R}^n,\mathbb{R}^m)$ for every $A\in\mathscr{A}$, and E^0 can be written as

$$E^0(u,A)=\int_A f^0(x,\nabla u)\,dx+\int_{S_u\cap A} g^0(x,[u],\nu_u)\,d\mathcal{H}^{n-1},$$

where f^0 and g^0 are defined as in (5.1) and (5.2) (note that f^0 and g^0 depend on the chosen subsequence). Note that $f^0\in\mathcal{F}$, but g^0 only satisfies $(g1)$, $(g3)$, $(g4)$, and $(g7)$, with c_3 replaced by $\hat{c}_3:=\max\{c_2/c_1,c_3\}$. To conclude the proof it remains to show that there exists $g\in\mathcal{G}$, possibly different from g^0, such that E^0 can still be represented as in (3.4) using f^0 and g.

Let now g' be defined as in (3.10) (note that also this function depends on the chosen subsequence). We can now apply Theorem 5.2 and Remark 5.3 to obtain

$$\begin{aligned}E^0(u,A)&=\int_A f^0(x,\nabla u)\,dx+\int_{S_u\cap A} g^0(x,[u],\nu_u)\,d\mathcal{H}^{n-1}\\&=\int_A f^0(x,\nabla u)\,dx+\int_{S_u\cap A} g'(x,[u],\nu_u)\,d\mathcal{H}^{n-1}.\end{aligned}$$

Since $g'\in\mathcal{G}$ by Lemma A.7, the theorem is proved. □

6. Identification of the Γ-limit and related results

In this section we prove Theorem 3.7 using Theorem 5.2, which will be proved in Sections 7 and 8. We also prove a result on the convergence of minimisers.

Proof of Theorem 3.7. To prove that (a1) and (a2) imply (3.12), we observe that, by the Urysohn property of Γ-convergence [12, Proposition 8.3], the sequence $E_k(\cdot, A)$ Γ-converges to $E_\infty(\cdot, A)$ in $L^0(\mathbb{R}^n, \mathbb{R}^m)$ for every $A \in \mathscr{A}$ if and only if for every $A \in \mathscr{A}$ every subsequence of $E_k(\cdot, A)$ has a sub-subsequence Γ-converging to $E_\infty(\cdot, A)$ in $L^0(\mathbb{R}^n, \mathbb{R}^m)$.

Let D be a countable subset of $(0, +\infty)$ with $0 \in \overline{D}$. By Theorem 4.3, using a diagonal argument, for every subsequence of (E_k) we obtain a sub-subsequence (E_{k_j}) which satisfies the assumptions of Theorem 5.1. Let f^0, g^0, and E^0 be defined as in Theorem 5.1, corresponding to the subsequence (E_{k_j}). Then $E_{k_j}(\cdot, A)$ Γ-converges to $E^0(\cdot, A)$ for every $A \in \mathscr{A}$. Thus, proving (3.12) is equivalent to showing that

$$E^0(u, A) = E_\infty(u, A) \quad \text{for every } u \in L^0(\mathbb{R}^n, \mathbb{R}^m) \text{ and every } A \in \mathscr{A}. \tag{6.1}$$

Let $\tilde{f}', \tilde{f}'', \tilde{g}', \tilde{g}''$ be the functions defined as in (3.8)-(3.11), corresponding to the subsequences F_{k_j} and G_{k_j}. Since

$$f' \le \tilde{f}' \le \tilde{f}'' \le f'' \quad \text{and} \quad g' \le \tilde{g}' \le \tilde{g}'' \le g'',$$

equalities (a1) give

$$f_\infty(x, \xi) = \tilde{f}'(x, \xi) = \tilde{f}''(x, \xi) \text{ for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n \text{ and every } \xi \in \mathbb{R}^{m \times n},$$

while (a2) implies that for every $A \in \mathscr{A}$ and every $u \in GSBV^p(A, \mathbb{R}^m)$ we have

$$g_\infty(x, [u](x), \nu_u(x)) = \tilde{g}'(x, [u](x), \nu_u(x)) = \tilde{g}''(x, [u](x), \nu_u(x))$$

for $\mathcal{H}^{n-1}$-a.e. $x \in S_u \cap A$.

By Theorem 5.2 and Remark 5.3 we have

$$f^0(x, \xi) = \tilde{f}'(x, \xi) = \tilde{f}''(x, \xi) \text{ for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n \text{ and every } \xi \in \mathbb{R}^{m \times n},$$
$$\int_{S_u \cap A} g^0(x, [u], \nu_u) d\mathcal{H}^{n-1} = \int_{S_u \cap A} \tilde{g}'(x, [u], \nu_u) d\mathcal{H}^{n-1} = \int_{S_u \cap A} \tilde{g}''(x, [u], \nu_u) d\mathcal{H}^{n-1}$$

for every $A \in \mathscr{A}$ and every $u \in GSBV^p(A, \mathbb{R}^m)$.

Therefore

$$f^0(x, \xi) = f_\infty(x, \xi) \text{ for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n \text{ and every } \xi \in \mathbb{R}^{m \times n},$$
$$\int_{S_u \cap A} g^0(x, [u], \nu_u) d\mathcal{H}^{n-1} = \int_{S_u \cap A} g_\infty(x, [u], \nu_u) d\mathcal{H}^{n-1}$$

for every $A \in \mathscr{A}$ and every $u \in GSBV^p(A, \mathbb{R}^m)$. By the definition of E_∞ this implies (6.1), and hence (3.12).

The same arguments also give (3.13). □

We now show that Theorem 3.7 implies the convergence of the solutions to some minimisation problems involving E_k. Other minimisation problems can be treated in a similar way.

Corollary 6.1 (Convergence of minimisers)**.** *Under the hypotheses of Theorem 3.5, assume that conditions* (3.12) *and* (3.13) *of Theorem 3.7 are satisfied for some $A \in \mathscr{A}$, and let $h \in L^p(A, \mathbb{R}^m)$. Then*

$$\inf_{v \in L^p(A, \mathbb{R}^m)} \Big(E_k^p(v, A) + \|v - h\|^p_{L^p(A, \mathbb{R}^m)} \Big) \longrightarrow \min_{v \in L^p(A, \mathbb{R}^m)} \Big(E_\infty^p(v, A) + \|v - h\|^p_{L^p(A, \mathbb{R}^m)} \Big) \tag{6.2}$$

as $k \to +\infty$. Moreover, if (u_k) is a sequence in $L^p(A, \mathbb{R}^m)$ such that

$$E_k^p(u_k, A) + \|u_k - h\|^p_{L^p(A, \mathbb{R}^m)} \le \inf_{v \in L^p(A, \mathbb{R}^m)} \Big(E_k^p(v, A) + \|v - h\|^p_{L^p(A, \mathbb{R}^m)} \Big) + \varepsilon_k \tag{6.3}$$

for some $\varepsilon_k \to 0+$, then there exists a subsequence of (u_k) which converges in $L^p(A, \mathbb{R}^m)$ to a solution of the minimisation problem

$$\min_{v \in L^p(A, \mathbb{R}^m)} \Big(E_\infty^p(v, A) + \|v - h\|^p_{L^p(A, \mathbb{R}^m)} \Big). \tag{6.4}$$

Proof. Let us fix a sequence (ε_k) of positive numbers, with $\varepsilon_k \to 0+$, and let (u_k) be a sequence in $L^p(A, \mathbb{R}^m)$ satisfying (6.3). By the lower bounds $(f3)$ and $(g5)$ we have that $u_k \in GSBV^p(A, \mathbb{R}^m)$ and we can apply [3, Theorem 4.36] to deduce that there exist a subsequence of (u_k), not relabelled, and a

function $u \in GSBV^p(A,\mathbb{R}^m)$ such that $u_k \to u$ in $L^0(A,\mathbb{R}^m)$ and $\mathcal{L}^n$-a.e. in A. Hence by the Fatou Lemma we deduce that

$$\|u-h\|^p_{L^p(A,\mathbb{R}^m)} \le \liminf_{k\to+\infty} \|u_k-h\|^p_{L^p(A,\mathbb{R}^m)}. \tag{6.5}$$

This inequality, combined with the fact that (6.3) also ensures that $\sup_k \|u_k\|_{L^p(A,\mathbb{R}^m)} < +\infty$, immediately gives $u \in L^p(A,\mathbb{R}^m)$.

Let us extend u_k by setting $u_k = u$ on $\mathbb{R}^n \setminus A$. Since $E_k(\cdot,A)$ Γ-converges to $E_\infty(\cdot,A)$ in $L^0(\mathbb{R}^n,\mathbb{R}^m)$, we have

$$E^p_\infty(u,A) = E_\infty(u,A) \le \liminf_{k\to+\infty} E_k(u_k,A) = \liminf_{k\to+\infty} E^p_k(u_k,A).$$

This inequality, together with (6.3) and (6.5), gives

$$\begin{aligned} E^p_\infty(u,A) + \|u-h\|^p_{L^p(A,\mathbb{R}^m)} &\le \liminf_{k\to+\infty} \Big(E^p_k(u_k,A) + \|u_k-h\|^p_{L^p(A,\mathbb{R}^m)} \Big) \\ &= \liminf_{k\to+\infty} \inf_{v\in L^p(A,\mathbb{R}^m)} \Big(E^p_k(v,A) + \|v-h\|^p_{L^p(A,\mathbb{R}^m)} \Big). \end{aligned} \tag{6.6}$$

Let us fix $w \in L^p(A,\mathbb{R}^m)$, that we can extend to a function $w \in L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$. By (3.13) we can find a sequence (w_k) in $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ such that

$$w_k \to w \quad \text{in } L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m) \quad \text{and} \quad \lim_{k\to+\infty} E^p_k(w_k,A) = E^p_\infty(w,A),$$

hence

$$\begin{aligned} &\limsup_{k\to+\infty} \inf_{v\in L^p(A,\mathbb{R}^m)} \Big(E^p_k(v,A) + \|v-h\|^p_{L^p(A,\mathbb{R}^m)} \Big) \le \\ &\qquad \le \lim_{k\to+\infty} \Big(E^p_k(w_k,A) + \|w_k-h\|^p_{L^p(A,\mathbb{R}^m)} \Big) = E^p_\infty(w,A) + \|w-h\|^p_{L^p(A,\mathbb{R}^m)}. \end{aligned} \tag{6.7}$$

Gathering (6.6) and (6.7) gives

$$\begin{aligned} E^p_\infty(u,A) + \|u-h\|^p_{L^p(A,\mathbb{R}^m)} &\le \liminf_{k\to+\infty} \inf_{v\in L^p(A,\mathbb{R}^m)} \Big(E^p_k(v,A) + \|v-h\|^p_{L^p(A,\mathbb{R}^m)} \Big) \\ &\le \limsup_{k\to+\infty} \inf_{v\in L^p(A,\mathbb{R}^m)} \Big(E^p_k(v,A) + \|v-h\|^p_{L^p(A,\mathbb{R}^m)} \Big) \le E^p_\infty(w,A) + \|w-h\|^p_{L^p(A,\mathbb{R}^m)}. \end{aligned}$$

Since this holds for every $w \in L^p(A,\mathbb{R}^m)$, we deduce that u is a solution of the minimisation problem (6.4).

Taking $w = u$ in the previous chain of inequalities gives (6.2) for the subsequence selected at the beginning of the proof. Since the limit does not depend on the subsequence, (6.2) holds for the whole sequence (E^p_k). □

7. Proof of Theorem 5.2 (a) and (b)

We start by proving the inequality $f^0 \le f'$.

Proof of Theorem 5.2 (a). Fix $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m\times n}$, $\rho > 0$, and $\varepsilon \in D \cap (0,1)$, where D is as in Theorem 5.1. By (3.5) for every k there exists $v_k \in L^0(\mathbb{R}^n,\mathbb{R}^m)$, with $v_k|_{Q_\rho(x)} \in W^{1,p}(Q_\rho(x),\mathbb{R}^m)$, such that $v_k - \ell_\xi \in W^{1,p}_0(Q_\rho(x),\mathbb{R}^m)$ and

$$E^{\varepsilon,p}_k(v_k,Q_\rho(x)) = F_k(v_k,Q_\rho(x)) \le m^{1,p}_{F_k}(\ell_\xi,Q_\rho(x)) + \varepsilon\,\rho^n. \tag{7.1}$$

Let k_j be a strictly increasing sequence of integers such that

$$\lim_{j\to+\infty} E^{\varepsilon,p}_{k_j}(v_{k_j},Q_\rho(x)) = \liminf_{k\to+\infty} E^{\varepsilon,p}_k(v_k,Q_\rho(x)).$$

From $(f3)$, $(f4)$, and (7.1) we obtain

$$c_1 \|\nabla v_k\|^p_{L^p(Q_\rho(x),\,\mathbb{R}^{m\times n})} \le (c_2(1+|\xi|^p)+\varepsilon)\rho^n.$$

By the Poincaré Inequality we deduce that the sequence (v_k) is bounded in $W^{1,p}(Q_\rho(x),\mathbb{R}^m)$. Therefore, up to a subsequence, $v_k \rightharpoonup v$ weakly in $W^{1,p}(Q_\rho(x),\mathbb{R}^m)$ for some $v \in W^{1,p}(Q_\rho(x),\mathbb{R}^m)$ such that $v - \ell_\xi \in W^{1,p}_0(Q_\rho(x),\mathbb{R}^m)$. Let w_k, $w \in W^{1,p}_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ be defined by

$$w_k := \begin{cases} v_k & \text{in } Q_\rho(x), \\ \ell_\xi & \text{in } \mathbb{R}^n \setminus Q_\rho(x), \end{cases} \qquad \text{and} \qquad w := \begin{cases} v & \text{in } Q_\rho(x), \\ \ell_\xi & \text{in } \mathbb{R}^n \setminus Q_\rho(x). \end{cases} \tag{7.2}$$

By the Rellich Theorem $w_k \to w$ in $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$, hence

$$E^{\varepsilon,p}(w,Q_\rho(x)) \le \liminf_{k\to+\infty} E^{\varepsilon,p}_k(w_k,Q_\rho(x)) = \liminf_{k\to+\infty} E^{\varepsilon,p}_k(v_k,Q_\rho(x))$$

by the Γ-convergence of $E_k^{\varepsilon,p}(\cdot,Q_\rho(x))$ to $E^{\varepsilon,p}(\cdot,Q_\rho(x))$. Using this inequality, together with $(f4)$, (7.1), and (7.2), we get

$$\begin{aligned} m_{E^{\varepsilon,p}}(\ell_\xi,Q_{(1+\varepsilon)\rho}(x)) &\le E^{\varepsilon,p}(w,Q_\rho(x)) + c_2(1+|\xi|^p)((1+\varepsilon)^n-1)\rho^n \\ &\le \liminf_{k\to+\infty} E_k^{\varepsilon,p}(v_k,Q_\rho(x)) + \varepsilon n 2^{n-1} c_2(1+|\xi|^p)\rho^n \\ &\le \liminf_{k\to+\infty} m_{F_k}^{1,p}(\ell_\xi,Q_\rho(x)) + \varepsilon C_\xi \rho^n \end{aligned}$$

where $C_\xi := 1 + n2^{n-1}c_2(1+|\xi|^p)$. Dividing by ρ^n and taking the limsup as $\rho\to 0+$, we obtain from (3.8) and (4.24)

$$\begin{aligned} (1+\varepsilon)^n f^{\varepsilon,p}(x,\xi) &= \limsup_{\rho\to 0+} \frac{m_{E^{\varepsilon,p}}(\ell_\xi,Q_{(1+\varepsilon)\rho}(x))}{\rho^n} \\ &\le \limsup_{\rho\to 0+}\liminf_{k\to+\infty} \frac{m_{F_k}^{1,p}(\ell_\xi,Q_\rho(x))}{\rho^n} + \varepsilon C_\xi = f'(x,\xi) + \varepsilon C_\xi. \end{aligned}$$

Letting $\varepsilon\to 0+$, from (5.1) we obtain that $f^0(x,\xi)\le f'(x,\xi)$. □

We now prove (b). Namely, we show that $f''\le f^0$.

Proof of Theorem 5.2 (b). In view of Lemma A.6 we have $f''\in\mathcal{F}$, while by Theorem 5.1 $f^0\in\mathcal{F}$, hence in particular f^0 and f'' are continuous with respect to ξ by $(f2)$. Therefore it is enough to prove that for every $\xi\in\mathbb{R}^{m\times n}$ we have $f''(x,\xi)\le f^0(x,\xi)$ for $\mathcal{L}^n$-a.e. $x\in\mathbb{R}^n$.

We may assume that the set D considered in Theorem 5.1 is contained in $(0,1)$. Let us fix $\xi\in\mathbb{R}^{m\times n}$. Since for every $\varepsilon\in D$

$$E^{\varepsilon,p}(\ell_\xi,A) = \int_A f^{\varepsilon,p}(x,\xi)\,dx \quad \text{for every } A\in\mathscr{A}, \tag{7.3}$$

by the Lebesgue Differentiation Theorem for every $\varepsilon\in D$ and for $\mathcal{L}^n$-a.e. $x\in\mathbb{R}^n$ we have

$$\lim_{\rho\to 0+}\frac{E^{\varepsilon,p}(\ell_\xi,Q_\rho(x))}{\rho^n} = f^{\varepsilon,p}(x,\xi)\le c_2(1+|\xi|^p), \tag{7.4}$$

where the last inequality follows from the fact that $f^{\varepsilon,p}\in\mathcal{F}$ by Theorem 4.3.

Let $x\in\mathbb{R}^n$ be fixed and such that (7.4) holds for every $\varepsilon\in D$. It follows that for every $\varepsilon\in D$ there exists $\rho_0(\varepsilon)\in(0,1)$ such that

$$\frac{E^{\varepsilon,p}(\ell_\xi,Q_\rho(x))}{\rho^n}\le c_2(2+|\xi|^p) \tag{7.5}$$

for every $0<\rho<\rho_0(\varepsilon)$.

Let $\varepsilon\in D$ be fixed. Since $E_k^{\varepsilon,p}(\cdot,Q(x))$ Γ-converges to $E^{\varepsilon,p}(\cdot,Q(x))$ in $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$, there exists $(u_k)\subset L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$, with $u_k|_{Q(x)}\in SBV^p(Q(x),\mathbb{R}^m)\cap L^p(Q(x),\mathbb{R}^m)$, such that

$$u_k\to\ell_\xi \ \text{ in } \ L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m) \quad\text{and}\quad \lim_{k\to+\infty} E_k^{\varepsilon,p}(u_k,Q(x)) = E^{\varepsilon,p}(\ell_\xi,Q(x)). \tag{7.6}$$

By (7.3) we have $E^{\varepsilon,p}(\ell_\xi,Q(x)) = E^{\varepsilon,p}(\ell_\xi,Q_\rho(x)) + E^{\varepsilon,p}(\ell_\xi,Q(x)\setminus\overline{Q}_\rho(x))$ for all $\rho\in(0,1)$. By Γ-convergence we have also

$$\begin{aligned} \liminf_{k\to+\infty} E_k^{\varepsilon,p}(u_k,Q_\rho(x)) &\ge E^{\varepsilon,p}(\ell_\xi,Q_\rho(x)) \\ \liminf_{k\to+\infty} E_k^{\varepsilon,p}(u_k,Q(x)\setminus\overline{Q}_\rho(x)) &\ge E^{\varepsilon,p}(\ell_\xi,Q(x)\setminus\overline{Q}_\rho(x)). \end{aligned}$$

From these inequalities and from (7.6) it follows that

$$\lim_{k\to+\infty} E_k^{\varepsilon,p}(u_k,Q_\rho(x)) = E^{\varepsilon,p}(\ell_\xi,Q_\rho(x)).$$

This yields the existence of $k_0(\varepsilon,\rho)>0$ such that $|E^{\varepsilon,p}(\ell_\xi,Q_\rho(x)) - E_k^{\varepsilon,p}(u_k,Q_\rho(x))|<\varepsilon\rho^n$ whenever $k\ge k_0(\varepsilon,\rho)$, hence

$$\frac{E_k^{\varepsilon,p}(u_k,Q_\rho(x))}{\rho^n} < \frac{E^{\varepsilon,p}(\ell_\xi,Q_\rho(x))}{\rho^n} + \varepsilon. \tag{7.7}$$

In the remaining part of the proof we modify the sequence (u_k) to construct a competitor for the minimisation problem $m_{F_k}^{1,p}(\ell_\xi,Q_\rho(x))$, which appears in the definition of f''. To this end, for every

$y \in Q := Q(0)$ we set

$$u_k^\rho(y) := \frac{u_k(x+\rho y) - u_k(x)}{\rho},$$
$$f_k^\rho(y,\cdot) := f_k(x+\rho y,\cdot).$$

Note that $u_k^\rho \in SBV^p(Q,\mathbb{R}^m) \cap L^p(Q,\mathbb{R}^m)$ and $f_k^\rho \in \mathcal{F}$.

We fix $\lambda > |\xi|\sqrt{n}/2$ and h, α, $\psi_1, \dots, \psi_h$, and μ as in Lemma 4.1 with $\eta = \varepsilon$. By (4.12) for every k there exists $i_k \in \{1, \dots, h\}$ such that

$$F_k^\rho(\psi_{i_k}(u_k^\rho), Q) \le (1+\varepsilon) F_k^\rho(u_k^\rho, Q) + c_2 \mathcal{L}^n(Q \cap \{|u_k^\rho| \ge \lambda\}), \tag{7.8}$$

where F_k^ρ is defined as in (3.2), with f replaced by f_k^ρ.

We define

$$v_k^\rho := \psi_{i_k}(u_k^\rho). \tag{7.9}$$

Then $v_k^\rho = u_k^\rho$ in $Q \cap \{|u_k^\rho| < \lambda\}$ and $|v_k^\rho| \le \mu$ in Q. Since $u_k \to \ell_\xi$ in $L^p(Q_\rho(x), \mathbb{R}^m)$, we have $u_k^\rho \to \ell_\xi$ in $L^p(Q, \mathbb{R}^m)$, and since $|\ell_\xi| \le |\xi|\sqrt{n}/2 < \lambda$ in Q, it follows that $v_k^\rho \to \ell_\xi$ in $L^p(Q, \mathbb{R}^m)$ and that $\mathcal{L}^n(Q \cap \{|u_k^\rho| \ge \lambda\}) \to 0$ as $k \to +\infty$. Therefore, there exist $k_1(\varepsilon,\rho) \ge k_0(\varepsilon,\rho)$ such that

$$\|v_k^\rho - \ell_\xi\|_{L^p(Q,\mathbb{R}^m)} < \rho \quad \text{and} \quad \mathcal{L}^n(Q \cap \{|u_k^\rho| \ge \lambda\}) < \rho \quad \text{for every } k \ge k_1(\varepsilon,\rho). \tag{7.10}$$

Using $(f3)$, $(g5)$, (7.8)-(7.10), and a change of variables we obtain the two following estimates

$$c_1 \int_Q |\nabla v_k^\rho(y)|^p dy \le \int_Q f_k(x+\rho y, \nabla v_k^\rho(y)) dy \le \frac{1+\varepsilon}{\rho^n} \int_{Q_\rho(x)} f_k(y, \nabla u_k(y)) dy + c_2 \rho, \tag{7.11}$$

$$\frac{c_4}{\rho} \mathcal{H}^{n-1}(S_{v_k^\rho} \cap Q) \le \frac{c_4}{\rho^n} \mathcal{H}^{n-1}(S_{u_k} \cap Q_\rho(x)) \le \frac{1}{\rho^n} \int_{S_{u_k} \cap Q_\rho(x)} g_k^\varepsilon(y, [u_k], \nu_{u_k}) d\mathcal{H}^{n-1}, \tag{7.12}$$

for every $k \ge k_1(\varepsilon,\rho)$, where g_k^ε is defined in (4.23).

From (7.5), (7.7), and (7.12), we deduce that there exists $M > 0$, independent of k, ρ, and ε, such that

$$\|\nabla v_k^\rho\|_{L^p(Q,\mathbb{R}^{m\times n})} \le M \quad \text{and} \quad \mathcal{H}^{n-1}(S_{v_k^\rho} \cap Q) \le M\rho, \tag{7.13}$$

whenever $\varepsilon \in D$, $0 < \rho < \rho_0(\varepsilon)$, and $k \ge k_1(\varepsilon,\rho)$. Since $|[v_k^\rho]| \le 2\mu$ $\mathcal{H}^{n-1}$-a.e. on $S_{v_k^\rho}$ by (7.9), from (7.13) we obtain also that

$$|D^s v_k^\rho|(Q) \le 2\mu M \rho. \tag{7.14}$$

We now regularise v_k^ρ in order to obtain a function $w_k^\rho \in W^{1,p}(Q,\mathbb{R}^m)$ such that

$$\int_Q f_k(x+\rho y, \nabla w_k^\rho(y)) dy \le \int_Q f_k(x+\rho y, \nabla v_k^\rho(y)) dy + \varepsilon$$

for a suitable choice of ρ and k. We follow the procedure introduced in [23, Lemma 2.1], which we now illustrate in detail for the readers' convenience.

Step 1: Regularisation of v_k^ρ. Let $t > 0$; we define the sets

$$R_k^t := \Big\{ y \in Q : \frac{|Dv_k^\rho|(\overline{B_r(y)})}{\mathcal{L}^n(B_r(y))} \le t \ \text{ for every } r > 0 \text{ with } \overline{B_r(y)} \subset Q \Big\},$$
$$S_k^t := S_{v_k^\rho} \cup \Big\{ y \in Q : |\nabla v_k^\rho(y)| \ge \frac{t}{2} \Big\}.$$

For every k, by the Vitali Covering Lemma (see, e.g., [16, Section 1.5.1]), there exists a sequence of disjoint closed balls $\overline{B_{r_j}(y_j)} \subset Q$, with centres y_j in $Q \setminus R_k^t$, such that

$$\frac{|Dv_k^\rho|(\overline{B_{r_j}(y_j)})}{\mathcal{L}^n(B_{r_j}(y_j))} > t \quad \text{for every } j \text{ and } Q \setminus R_k^t \subset \bigcup_{j=1}^\infty \overline{B_{5r_j}(y_j)}. \tag{7.15}$$

Hence

$$t\mathcal{L}^n\Big(\bigcup_{j=1}^\infty B_{r_j}(y_j)\Big) = t\sum_{j=1}^\infty \mathcal{L}^n(B_{r_j}(y_j)) < \sum_{j=1}^\infty |Dv_k^\rho|(\overline{B_{r_j}(y_j)}) = |Dv_k^\rho|\Big(\bigcup_{j=1}^\infty \overline{B_{r_j}(y_j)}\Big). \tag{7.16}$$

On the other hand

$$|Dv_k^\rho|\Big(\bigcup_{j=1}^\infty \overline{B_{r_j}(y_j)}\Big) = |Dv_k^\rho|\Big(S_k^t \cap \bigcup_{j=1}^\infty \overline{B_{r_j}(y_j)}\Big) + |Dv_k^\rho|\Big((Q \setminus S_k^t) \cap \bigcup_{j=1}^\infty \overline{B_{r_j}(y_j)}\Big). \tag{7.17}$$

We are going to estimate the two terms in the right-hand side of (7.17) separately. We observe that

$$|Dv_k^\rho|\Big((Q\setminus S_k^t)\cap\bigcup_{j=1}^\infty \overline{B_{r_j}(y_j)}\Big)=\int_{(Q\setminus S_k^t)\cap\cup_{j=1}^\infty B_{r_j}(y_j)}|\nabla v_k^\rho|dy\le\frac{t}{2}\mathcal{L}^n\Big(\bigcup_{j=1}^\infty B_{r_j}(y_j)\Big). \tag{7.18}$$

By (7.16) we have, using also (7.17) and (7.18),

$$t\mathcal{L}^n\Big(\bigcup_{j=1}^\infty B_{r_j}(y_j)\Big)<|Dv_k^\rho|\Big(S_k^t\cap\bigcup_{j=1}^\infty \overline{B_{r_j}(y_j)}\Big)+\frac{t}{2}\mathcal{L}^n\Big(\bigcup_{j=1}^\infty B_{r_j}(y_j)\Big).$$

This implies that

$$\mathcal{L}^n\Big(\bigcup_{j=1}^\infty B_{r_j}(y_j)\Big)\le\frac{2}{t}|Dv_k^\rho|\Big(S_k^t\cap\bigcup_{j=1}^\infty \overline{B_{r_j}(y_j)}\Big). \tag{7.19}$$

By (7.15) and (7.19) we have

$$\begin{aligned}\mathcal{L}^n(Q\setminus R_k^t)&\le\sum_{j=1}^\infty\mathcal{L}^n(B_{5r_j}(y_j))=5^n\sum_{j=1}^\infty\mathcal{L}^n(B_{r_j}(y_j))=5^n\mathcal{L}^n\Big(\bigcup_{j=1}^\infty B_{r_j}(y_j)\Big)\\&\le\frac{2\cdot5^n}{t}|Dv_k^\rho|\Big(S_k^t\cap\bigcup_{j=1}^\infty \overline{B_{r_j}(y_j)}\Big)\le\frac{2\cdot5^n}{t}\Big(|D^sv_k^\rho|(Q)+\int_{S_k^t}|\nabla v_k^\rho|dy\Big)\\&\le\frac{2\cdot5^n}{t}\Big(|D^sv_k^\rho|(Q)+\Big(\int_{S_k^t}|\nabla v_k^\rho|^pdy\Big)^{\frac1p}(\mathcal{L}^n(S_k^t))^{1-\frac1p}\Big).\end{aligned} \tag{7.20}$$

Now, by the definition of S_k^t and by (7.13) we have that

$$\mathcal{L}^n(S_k^t)\Big(\frac{t}{2}\Big)^p\le\int_{S_k^t}|\nabla v_k^\rho|^pdy\le M^p$$

whenever $\varepsilon\in D$, $0<\rho<\rho_0(\varepsilon)$, and $k\ge k_1(\varepsilon,\rho)$. It then follows that $\mathcal{L}^n(S_k^t)\le 2^pM^p/t^p$, which, combined with (7.13) and (7.20), gives

$$\mathcal{L}^n(Q\setminus R_k^t)\le\frac{2\cdot5^n}{t}\Big(|D^sv_k^\rho|(Q)+\Big(\int_{S_k^t}|\nabla v_k^\rho|^pdy\Big)^{\frac1p}\frac{2^{p-1}M^{p-1}}{t^{p-1}}\Big)\le\frac{2\cdot5^n}{t}|D^sv_k^\rho|(Q)+\frac{2^p5^nM^p}{t^p}.$$

Hence we can conclude that

$$t^p\mathcal{L}^n(Q\setminus R_k^t)\le2\cdot5^nt^{p-1}|D^sv_k^\rho|(Q)+2^p5^nM^p \tag{7.21}$$

whenever $\varepsilon\in D$, $0<\rho<\rho_0(\varepsilon)$, and $k\ge k_1(\varepsilon,\rho)$.

Now we choose $t_{k,\rho}>0$ such that $t_{k,\rho}^{p-1}|D^sv_k^\rho|(Q)=1$. By (7.14) this implies

$$t_{k,\rho}^{p-1}=\frac{1}{|D^sv_k^\rho|(Q)}\ge\frac{1}{2\mu M\rho},$$

whenever $\varepsilon\in D$, $0<\rho<\rho_0(\varepsilon)$, and $k\ge k_1(\varepsilon,\rho)$. Then, from (7.21) we obtain

$$t_{k,\rho}^p\mathcal{L}^n(Q\setminus R_k^{t_{k,\rho}})\le2\cdot5^n+2^p5^nM^p=:M_1,$$

which gives in particular that

$$\mathcal{L}^n(Q\setminus R_k^{t_{k,\rho}})\le\frac{M_1}{t_{k,\rho}^p}\le M_2\rho^q, \tag{7.22}$$

with $q:=p/(p-1)$ and $M_2:=M_1(2\mu M)^q$.

By [16, Section 3.1.1 (Theorem 1) and Section 6.6.2 (Claim #2 of Theorem 2)] there exist a constant c_n, depending only on n, and Lipschitz functions z_k^ρ on Q, with Lipschitz constant bounded by $c_nt_{k,\rho}$, such that that $z_k^\rho=v_k^\rho$ $\mathcal{L}^n$-a.e. in $R_k^{t_{k,\rho}}$. Note that, since $|v_k^\rho|\le\mu$ $\mathcal{L}^n$-a.e. in Q, it is not restrictive to assume that $|z_k^\rho|\le\mu$ in Q. By (7.13) and (7.22) we have also

$$\int_Q|\nabla z_k^\rho|^pdy\le\int_{R_k^{t_{k,\rho}}}|\nabla v_k^\rho|^pdy+c_n^pt_{k,\rho}^p\mathcal{L}^n(Q\setminus R_k^{t_{k,\rho}})\le M^p+c_n^pM_1.$$

Therefore the sequence $(z_k^\rho)_k$ is bounded in $W^{1,p}(Q,\mathbb{R}^m)$.

By (3.9) there exists a decreasing sequence $\rho_j\to0+$, with $0<\rho_j<\rho_0(\varepsilon)$, such that

$$f''(x,\xi)=\lim_{j\to+\infty}\limsup_{k\to+\infty}\frac{1}{\rho_j^n}m_{F_k}^{1,p}(\ell_\xi,Q_{\rho_j}(x)). \tag{7.23}$$

By applying [18, Lemma 1.2] to the double sequence $(z_k^{\rho_j})_{j,k}$ we find a double sequence $(w_k^{\rho_j})_{j,k}$ in $W^{1,p}(Q,\mathbb{R}^m)$ such that $|\nabla w_k^{\rho_j}|^p$ is equi-integrable, uniformly with respect to j and k, and

$$\mathcal{L}^n(\{w_k^{\rho_j} \neq z_k^{\rho_j}\}) \to 0 \quad \text{as } k+j \to +\infty.$$

Note that, since $|z_k^{\rho_j}| \leq \mu$ in Q, it is not restrictive to assume that $|w_k^{\rho_j}| \leq \mu$ $\mathcal{L}^n$-a.e. in Q. By (7.10) and (7.22) these properties imply that for every j there exists $k_2(\varepsilon,j) \geq k_1(\varepsilon,\rho_j)$ such that for every $k \geq k_2(\varepsilon,j)$ we have

$$\mathcal{L}^n(\{w_k^{\rho_j} \neq v_k^{\rho_j}\}) \leq M_2\rho_j^q \quad \text{and} \quad \|w_k^{\rho_j} - \ell_\xi\|_{L^p(Q,\mathbb{R}^m)} \leq \rho_j + 4\mu M_2^{1/p}\rho_j^{q/p} =: r_j. \tag{7.24}$$

Moreover,

$$\int_Q f_k(x+\rho_j y, \nabla w_k^{\rho_j}(y))dy \leq \int_Q f_k(x+\rho_j y, \nabla v_k^{\rho_j}(y))dy + \int_{\{w_k^{\rho_j} \neq z_k^{\rho_j}\}} f_k(x+\rho_j y, \nabla w_k^{\rho_j}(y))dy.$$

By the equi-integrability of $|\nabla w_k^{\rho_j}|^p$, by the upper bound $(f4)$, and by (7.24) we can conclude that for every $\varepsilon \in D$ there exists $j_0(\varepsilon)$, with $\rho_{j_0(\varepsilon)} \leq \rho_0(\varepsilon)$, such that

$$\int_{\{w_k^{\rho_j} \neq z_k^{\rho_j}\}} f_k(x+\rho_j y, \nabla w_k^{\rho_j}(y))dy < \varepsilon$$

for every $j \geq j_0(\varepsilon)$ and every k, hence

$$\int_Q f_k(x+\rho_j y, \nabla w_k^{\rho_j}(y))dy \leq \int_Q f_k(x+\rho_j y, \nabla v_k^{\rho_j}(y))dy + \varepsilon, \tag{7.25}$$

for every $j \geq j_0(\varepsilon)$ and every $k \geq k_2(\varepsilon,j)$.

Step 2: Attainment of the boundary datum. We now modify $w_k^{\rho_j}$ so that it attains the linear boundary datum ℓ_ξ, which appears in the definition of $f''(x,\xi)$. To this end, we will apply the Fundamental Estimate to the functionals $F_k^{\rho_j}$ corresponding to the integrands $f_k^{\rho_j}(y,\cdot) := f_k(x+\rho_j y,\cdot)$. Let $Q_{1-\varepsilon} := Q_{1-\varepsilon}(0)$. By [12, Theorem 19.1] there exists a constant $C_\varepsilon > 0$ and a *finite* family of cut-off functions $(\varphi_i)_{1\leq i\leq N} \subset C_c^\infty(Q)$, with $0 \leq \varphi_i \leq 1$ in Q and $\varphi_i = 1$ in $Q_{1-\varepsilon}$, such that

$$F_k^{\rho_j}(\tilde{w}_k^{\rho_j}, Q) \leq (1+\varepsilon)\big(F_k^{\rho_j}(w_k^{\rho_j}, Q) + F_k^{\rho_j}(\ell_\xi, Q \setminus \overline{Q}_{1-\varepsilon})\big) + C_\varepsilon \|w_k^{\rho_j} - \ell_\xi\|_{L^p(Q)}^p + \varepsilon,$$

where $\tilde{w}_k^{\rho_j} := \varphi_{i_{k,j}} w_k^{\rho_j} + (1-\varphi_{i_{k,j}})\ell_\xi$ for a suitable $i_{k,j} \in \{1,\dots,N\}$. Clearly $\tilde{w}_k^{\rho_j}$ attains the boundary datum ℓ_ξ in a neighbourhood of ∂Q. Since $\mathcal{L}^n(Q \setminus Q_{1-\varepsilon}) < n\varepsilon$, by $(f4)$ and (7.24) it follows that

$$F_k^{\rho_j}(\tilde{w}_k^{\rho_j}, Q) \leq (1+\varepsilon)F_k^{\rho_j}(w_k^{\rho_j}, Q) + \varepsilon(1+\varepsilon)nc_2(1+|\xi|^p) + C_\varepsilon r_j^p + \varepsilon. \tag{7.26}$$

Combining (7.7), (7.11), (7.25), and (7.26), and setting $B_\xi := 7 + 2nc_2(1+|\xi|^p)$, we have the bound

$$\limsup_{k\to+\infty} \int_Q f_k(x+\rho_j y, \nabla \tilde{w}_k^{\rho_j}(y))dy \leq (1+\varepsilon)^2 \frac{E^{\varepsilon,p}(\ell_\xi, Q_{\rho_j}(x))}{{\rho_j}^n} + B_\xi\varepsilon + C_\varepsilon r_j^p + 2c_2\rho_j, \tag{7.27}$$

whenever $\varepsilon \in D$, $j \geq j_0(\varepsilon)$, and $k \geq k_2(\varepsilon,j)$.

Finally, we perform a change of variables in order to relate the left-hand side of (7.27) with the minimisation problems on $Q_{\rho_j}(x)$, appearing in (7.23). For $y \in Q_{\rho_j}(x)$, define

$$\tilde{v}_k^{\rho_j}(y) := \rho_j\, \tilde{w}_k^{\rho_j}\Big(\frac{y-x}{\rho_j}\Big) + \ell_\xi(x).$$

Clearly $\tilde{v}_k^{\rho_j} \in W^{1,p}(Q_{\rho_j}(x))$, $\tilde{v}_k^{\rho_j} = \ell_\xi$ in a neighbourhood of $\partial Q_{\rho_j}(x)$, and

$$\int_Q f_k(x+\rho_j y, \nabla \tilde{w}_k^{\rho_j}(y))dy = \frac{1}{{\rho_j}^n} \int_{Q_{\rho_j}(x)} f_k\left(y, \nabla \tilde{v}_k^{\rho_j}(y)\right) dy \geq \frac{1}{{\rho_j}^n} m_{F_k}^{1,p}\big(\ell_\xi, Q_{\rho_j}(x)\big).$$

Therefore, from (7.27) we conclude that

$$\limsup_{k\to+\infty} \frac{1}{{\rho_j}^n} m_{F_k}^{1,p}\big(\ell_\xi, Q_{\rho_j}(x)\big) \leq (1+\varepsilon)^2 \frac{E^{\varepsilon,p}(\ell_\xi, Q_{\rho_j}(x))}{{\rho_j}^n} + B_\xi\varepsilon + C_\varepsilon r_j^p + 2c_2\rho_j.$$

Since $r_j \to 0$ by (7.24), taking the limit as $j \to +\infty$, by (7.4) and (7.23) we obtain the estimate

$$f''(x,\xi) \leq (1+\varepsilon)^2 f^{\varepsilon,p}(x,\xi) + B_\xi\varepsilon$$

for every $\varepsilon \in D$. Taking the limit as $\varepsilon \to 0+$, from (5.1) we obtain $f''(x,\xi) \leq f^0(x,\xi)$. □

8. Proof of Theorem 5.2 (c) and (d)

We start by proving the inequality $g^0 \le g'$.

Proof of Theorem 5.2 (c). Fix $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^m_0$, $\nu \in \mathbb{S}^{n-1}$, $\rho > 0$, and $\varepsilon \in D \cap (0,1)$, where D is as in Theorem 5.1. By the definition of $m^{\rm pc}_{G_k}$, for every k there exists $u_k \in L^0(\mathbb{R}^n, \mathbb{R}^m)$, with $u_k|_{Q^\nu_\rho(x)} \in SBV_{\rm pc}(Q^\nu_\rho(x), \mathbb{R}^m)$, such that $u_k = u_{x,\zeta,\nu}$ in a neighbourhood of $\partial Q^\nu_\rho(x)$ and

$$G_k(u_k, Q^\nu_\rho(x)) \le m^{\rm pc}_{G_k}(u_{x,\zeta,\nu}, Q^\nu_\rho(x)) + \varepsilon\, \rho^{n-1}. \tag{8.1}$$

Now fix $\lambda > |\zeta|$ and h, α, $\psi_1, \dots, \psi_h$, and μ as in Lemma 4.1. Then by (4.13) for every k there exists $i_k \in \{1, \dots, h\}$ such that

$$E_k(\psi_{i_k}(u_k), Q^\nu_\rho(x)) \le (1+\varepsilon) E_k(u_k, Q^\nu_\rho(x)) + c_2 \mathcal{L}^n(Q^\nu_\rho(x) \cap \{|u_k| \ge \lambda\}).$$

By (4.10) and (4.11) we have $\psi_{i_k}(u_k) = u_{x,\zeta,\nu}$ in a neighbourhood of $\partial Q^\nu_\rho(x)$ and $|\psi_{i_k}(u_k)| \le \mu$ in $\mathbb{R}^n$. Moreover, the chain rule gives $\nabla(\psi_{i_k}(u_k)) = 0$ $\mathcal{L}^n$-a.e. in $Q^\nu_\rho(x)$. Therefore the functions v_k defined as

$$v_k := \begin{cases} \psi_{i_k}(u_k) & \text{in } \ Q^\nu_\rho(x) \\ u_{x,\zeta,\nu} & \text{in } \ \mathbb{R}^n \setminus Q^\nu_\rho(x) \end{cases} \tag{8.2}$$

satisfy $v_k|_A \in SBV_{\rm pc}(A, \mathbb{R}^m)$ for every $A \in \mathscr{A}$.

By definition we also have

$$|v_k| \le \mu \quad \text{in } \mathbb{R}^n. \tag{8.3}$$

Since $\nu_{v_k} = \nu_{u_k}$ and, by (4.5), $|[v_k]| \le |[u_k]|$ $\mathcal{H}^{n-1}$-a.e. in $S_{v_k} \cap Q^\nu_\rho(x) \subset S_{u_k} \cap Q^\nu_\rho(x)$, by using $(g3)$, $(g5)$, and $(g6)$ we get

$$c_4 \mathcal{H}^{n-1}(S_{v_k} \cap Q^\nu_\rho(x)) \le G_k(v_k, Q^\nu_\rho(x)) \le c_3 G_k(u_k, Q^\nu_\rho(x)).$$

Therefore, appealing to (8.1) we conclude that for every k

$$\mathcal{H}^{n-1}(S_{v_k} \cap Q^\nu_\rho(x)) \le M_\zeta \rho^{n-1}, \tag{8.4}$$

where $M_\zeta := c_3(c_5(1+|\zeta|)+1)/c_4$.

Since $v_k \in SBV_{\rm pc}(Q^\nu_\rho(x), \mathbb{R}^m)$, by combining (8.3) and (8.4) we can invoke [3, Theorem 4.8] to deduce the existence of a function $v \in SBV_{\rm pc}(Q^\nu_\rho(x), \mathbb{R}^m) \cap L^\infty(Q^\nu_\rho(x), \mathbb{R}^m)$ and a subsequence, not relabelled, such that $v_k \to v$ in $L^0(Q^\nu_\rho(x), \mathbb{R}^m)$. We extend v to $\mathbb{R}^n$ by setting $v = u_{x,\zeta,\nu}$ in $\mathbb{R}^n \setminus Q^\nu_\rho(x)$ and observe that $v|_A \in SBV_{\rm pc}(A, \mathbb{R}^m)$ for every $A \in \mathscr{A}$. By the definitions of v_k and v and by (8.3), the convergence in $L^0(Q^\nu_\rho(x), \mathbb{R}^m)$ also implies that

$$v_k \to v \ \text{ in } \ L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m), \tag{8.5}$$

$$|v| \le \mu \ \ \mathcal{L}^n\text{-a.e. in } \ \mathbb{R}^n. \tag{8.6}$$

Since $v|_{Q^\nu_{(1+\varepsilon)\rho}(x)} \in SBV_{\rm pc}(Q^\nu_{(1+\varepsilon)\rho}(x), \mathbb{R}^m)$ and $v = u_{x,\zeta,\nu}$ in $Q^\nu_{(1+\varepsilon)\rho}(x) \setminus Q^\nu_\rho(x)$, we have

$$m_{E^{\varepsilon,p}}(u_{x,\zeta,\nu}, Q^\nu_{(1+\varepsilon)\rho}(x)) \le E^{\varepsilon,p}(v, Q^\nu_{(1+\varepsilon)\rho}(x)). \tag{8.7}$$

Using the Γ-convergence of $E^{\varepsilon,p}_k(\cdot, Q^\nu_{(1+\varepsilon)\rho}(x))$ to $E^{\varepsilon,p}(\cdot, Q^\nu_{(1+\varepsilon)\rho}(x))$ in $L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$, we deduce from (8.5) that

$$E^{\varepsilon,p}(v, Q^\nu_{(1+\varepsilon)\rho}(x)) \le \liminf_{k\to+\infty} E^{\varepsilon,p}_k(v_k, Q^\nu_{(1+\varepsilon)\rho}(x)).$$

Since $v_k = u_{x,\zeta,\nu}$ in a neighbourhood of $\partial Q^\nu_\rho(x)$, we have $\mathcal{H}^{n-1}(S_{v_k} \cap \partial Q^\nu_\rho(x)) = 0$. Therefore, from (8.2) and (8.4) we obtain

$$\mathcal{H}^{n-1}(S_{v_k} \cap Q^\nu_{(1+\varepsilon)\rho}(x)) \le M_\zeta \rho^{n-1} + ((1+\varepsilon)^{n-1} - 1)\rho^{n-1} \le N_\zeta \rho^{n-1},$$

where $N_\zeta := M_\zeta + 2^{n-1}$. By (4.22) and (8.3), this inequality leads to the estimate

$$E^{\varepsilon,p}_k(v_k, Q^\nu_{(1+\varepsilon)\rho}(x)) \le E_k(v_k, Q^\nu_{(1+\varepsilon)\rho}(x)) + 2\varepsilon\mu N_\zeta \rho^{n-1}. \tag{8.8}$$

Gathering $(f4)$, $(g6)$, (8.1)-(8.2), and (8.8) we obtain

$$\begin{aligned} E^{\varepsilon,p}_k(v_k, Q^\nu_{(1+\varepsilon)\rho}(x)) &\le E_k(v_k, Q^\nu_\rho(x)) + E_k(u_{x,\zeta,\nu}, Q^\nu_{(1+\varepsilon)\rho}(x) \setminus \overline{Q}^\nu_\rho(x)) + 2\varepsilon\mu N_\zeta \rho^{n-1} \\ &\le (1+\varepsilon) E_k(u_k, Q^\nu_\rho(x)) + (1+2^n) c_2 \rho^n + G_k(u_{x,\zeta,\nu}, Q^\nu_{(1+\varepsilon)\rho}(x) \setminus \overline{Q}^\nu_\rho(x)) + 2\varepsilon\mu N_\zeta \rho^{n-1} \\ &\le (1+\varepsilon) G_k(u_k, Q^\nu_\rho(x)) + (3+2^n) c_2 \rho^n + \varepsilon(C_\zeta + 2\mu N_\zeta)\rho^{n-1} \\ &\le (1+\varepsilon)\, m^{\rm pc}_{G_k}(u_{x,\zeta,\nu}, Q^\nu_\rho(x)) + (3+2^n) c_2 \rho^n + \varepsilon(2 + C_\zeta + 2\mu N_\zeta)\rho^{n-1} \end{aligned}$$

where $C_\zeta := c_5(1+|\zeta|)(n-1)2^{n-2}$. This inequality, together with (8.7)-(8.8), gives

$$m_{E^{\varepsilon,p}}(u_{x,\zeta,\nu}, Q^\nu_{(1+\varepsilon)\rho}(x)) \le (1+\varepsilon) \liminf_{k\to+\infty} m^{\rm pc}_{G_k}(u_{x,\zeta,\nu}, Q^\nu_\rho(x)) + (3+2^n) c_2 \rho^n + \varepsilon K_\zeta \rho^{n-1},$$

where $K_\zeta := 2 + C_\zeta + 2\mu N_\zeta$. Hence dividing by ρ^{n-1}, taking the limsup as $\rho \to 0+$, and recalling (3.10) and (4.25), we obtain

$$(1+\varepsilon)^{n-1} g^{\varepsilon,p}(x,\zeta,\nu) \le (1+\varepsilon) g'(x,\zeta,\nu) + \varepsilon K_\zeta.$$

Eventually, by taking the limit as $\varepsilon \to 0+$ and appealing to (5.2) we get

$$g^0(x,\zeta,\nu) \le g'(x,\zeta,\nu),$$

which concludes the proof. □

We are now ready to conclude the proof of Theorem 5.2.

Proof of Theorem 5.2 (d). We divide the proof into several intermediate steps. In the first four steps we prove the claimed inequality for functions u which belong to $SBV^p(A,\mathbb{R}^m) \cap L^\infty(A,\mathbb{R}^m)$, while the general case of functions in $GSBV^p(A,\mathbb{R}^m)$ is treated in Step 5.

We may assume that the set D introduced in Theorem 5.1 is contained in $(0,1)$. Let $A \in \mathscr{A}$, $u \in SBV^p(A,\mathbb{R}^m) \cap L^\infty(A,\mathbb{R}^m)$, and $\varepsilon \in D$ be fixed. For every $x \in \mathbb{R}^n$ and every $\rho > 0$ we set

$$Q_\rho^{\nu,\varepsilon}(x) := x + R_\nu\Big(\big(-\frac{\rho}{2},\frac{\rho}{2}\big)^{n-1} \times \big(-\frac{\varepsilon\rho}{2},\frac{\varepsilon\rho}{2}\big)\Big), \tag{8.9}$$

where R_ν is the orthogonal matrix introduced in (k) Section 2. We fix $x \in S_u$ such that, by setting $\zeta := [u](x)$ and $\nu := \nu_u(x)$, we have

$$\zeta \neq 0, \tag{8.10}$$

$$\lim_{\rho\to 0+} \frac{1}{\rho^n} \int_{Q_\rho^{\nu,\varepsilon}(x)} |u(y) - u_{x,\zeta,\nu}(y)|^p dy = 0, \tag{8.11}$$

$$g^{\varepsilon,p}(x,\zeta,\nu) = \lim_{\rho\to 0+} \frac{E^{\varepsilon,p}(u, Q_\rho^{\nu,\varepsilon}(x))}{\rho^{n-1}}. \tag{8.12}$$

Note that (8.10) and (8.11) are satisfied for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$ (see, e.g., [3, Definition 3.67 and Theorem 3.78]). The same property holds for (8.12), thanks to a generalized version of the Besicovitch Differentiation Theorem (see [24] and [17, Sections 1.2.1-1.2.2]).

We extend u to $\mathbb{R}^n$ by setting $u = 0$ on $\mathbb{R}^n \setminus A$. By the Γ-convergence of $E_k^{\varepsilon,p}(\cdot,A)$ to $E^{\varepsilon,p}(\cdot,A)$ there exists a sequence (u_k) converging to u in $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ such that

$$\lim_{k\to+\infty} E_k^{\varepsilon,p}(u_k,A) = E^{\varepsilon,p}(u,A).$$

Since $E^{\varepsilon,p}(u,\cdot)$ is a finite Radon measure, we have that $E^{\varepsilon,p}(u,\partial Q_\rho^{\nu,\varepsilon}(x)) = 0$ for all $\rho > 0$ such that $Q_\rho^{\nu,\varepsilon}(x) \subset A$, except for a countable set. As a consequence (u_k) is a recovery sequence for $E^{\varepsilon,p}(u,\cdot)$ also in $Q_\rho^{\nu,\varepsilon}(x)$; *i.e.,*

$$\lim_{k\to+\infty} E_k^{\varepsilon,p}(u_k, Q_\rho^{\nu,\varepsilon}(x)) = E^{\varepsilon,p}(u, Q_\rho^{\nu,\varepsilon}(x)), \tag{8.13}$$

for all $\rho > 0$ except for a countable set.

We now fix $\lambda > \max\{\|u\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^m)}, |\zeta|\}$ and h, α, $\psi_1,\dots,\psi_h$, and μ as in Lemma 4.1. We also fix ρ satisfying (8.13). By (4.13) for every k there exists $i_k \in \{1,\dots,h\}$ such that

$$E_k^{\varepsilon,p}(\psi_{i_k}(u_k), Q_\rho^{\nu,\varepsilon}(x)) \le (1+\varepsilon) E_k^{\varepsilon,p}(u_k, Q_\rho^{\nu,\varepsilon}(x)) + c_2 \mathcal{L}^n(Q_\rho^{\nu,\varepsilon}(x) \cap \{|u_k| \ge \lambda\}).$$

Let $v_k := \psi_{i_k}(u_k)$. By (4.10) and (4.11) we deduce that $v_k \to u$ in $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ as well as

$$|v_k| \le \mu \ \text{ in } \mathbb{R}^n, \quad \limsup_{k\to+\infty} E_k^{\varepsilon,p}(v_k, Q_\rho^{\nu,\varepsilon}(x)) \le (1+\varepsilon) E^{\varepsilon,p}(u, Q_\rho^{\nu,\varepsilon}(x)).$$

Hence there exists $k_0(\rho) > 0$ such that whenever $k \ge k_0(\rho)$

$$E_k^{\varepsilon,p}(v_k, Q_\rho^{\nu,\varepsilon}(x)) \le (1+\varepsilon) E^{\varepsilon,p}(u, Q_\rho^{\nu,\varepsilon}(x)) + \rho^n. \tag{8.14}$$

We now start a multi-step modification of v_k in order to obtain a function z_k which is an admissible competitor in the k-th minimisation problem defining $g''(x,\zeta,\nu)$.

Step 1. Attainment of the boundary datum for a blow-up of u_k. The blow-up function v_k^ρ at x is defined by

$$v_k^\rho(y) := v_k(x+\rho y) \quad \text{for } y \in Q^{\nu,\varepsilon} := Q_1^{\nu,\varepsilon}(0).$$

We now modify v_k^ρ so that it agrees with $u_{0,\zeta,\nu}$ in a neighbourhood of $\partial Q^{\nu,\varepsilon}$. To this end, we consider the class $\mathscr{A}(Q^{\nu,\varepsilon}) := \{A \in \mathscr{A} : A \subset Q^{\nu,\varepsilon}\}$ and apply the Fundamental Estimate to the functionals $E_{k,\rho}^{\varepsilon,p}\colon \big(SBV^p(Q^{\nu,\varepsilon},\mathbb{R}^m) \cap L^p(Q^{\nu,\varepsilon},\mathbb{R}^m)\big) \times \mathscr{A}(Q^{\nu,\varepsilon}) \to [0,+\infty)$ defined as

$$E_{k,\rho}^{\varepsilon,p}(v,A) := \int_A f_k(x+\rho y, \nabla v(y)) dy + \int_{S_v \cap A} g_k^\varepsilon(x+\rho y, [v](y), \nu_v(y)) d\mathcal{H}^{n-1}(y), \tag{8.15}$$

where g_k^ε is defined in (4.23).

Let $K_\varepsilon \subset Q^{\nu,\varepsilon}$ be a compact set such that

$$c_2\mathcal{L}^n(Q^{\nu,\varepsilon}\setminus K_\varepsilon) + (c_5(1+|\zeta|)+\varepsilon|\zeta|)\,\mathcal{H}^{n-1}(\Pi_0^\nu \cap (Q^{\nu,\varepsilon}\setminus K_\varepsilon)) < \varepsilon. \tag{8.16}$$

We can appeal to [8, Proposition 3.1] to deduce the existence of a constant $M_\varepsilon > 0$ and a *finite* family of cut-off functions $\phi_1,\dots,\phi_N \in C_c^\infty(Q^{\nu,\varepsilon})$ such that $0\le\phi_i\le 1$ in $Q^{\nu,\varepsilon}$, $\phi_i = 1$ in a neighbourhood of K_ε, and

$$\begin{aligned} E_{k,\rho}^{\varepsilon,p}(\hat v_k^\rho, Q^{\nu,\varepsilon}) \le (1+\varepsilon)\big(E_{k,\rho}^{\varepsilon,p}(v_k^\rho, Q^{\nu,\varepsilon}) + E_{k,\rho}^{\varepsilon,p}(u_{0,\zeta,\nu}, Q^{\nu,\varepsilon}\setminus K_\varepsilon)\big) \\ + M_\varepsilon\|v_k^\rho - u_{0,\zeta,\nu}\|^p_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^m)} + \varepsilon, \end{aligned} \tag{8.17}$$

where $\hat v_k^\rho := \phi_{i_k} v_k^\rho + (1-\phi_{i_k})u_{0,\zeta,\nu}$ for a suitable $i_k \in \{1,\dots,N\}$. Clearly

$$|\hat v_k^\rho| \le \mu \quad \text{in } Q^{\nu,\varepsilon} \tag{8.18}$$

and $\hat v_k^\rho = u_{0,\zeta,\nu}$ in a neighbourhood of $\partial Q^{\nu,\varepsilon}$. By $(f4)$ and $(g6)$ we have that

$$\begin{aligned} E_{k,\rho}^{\varepsilon,p}(u_{0,\zeta,\nu}, Q^{\nu,\varepsilon}\setminus K_\varepsilon) &= \int_{Q^{\nu,\varepsilon}\setminus K_\varepsilon} f_k(x+\rho y,0)dy + \int_{\Pi_0^\nu\cap(Q^{\nu,\varepsilon}\setminus K_\varepsilon)} g_k^\varepsilon(x+\rho y,\zeta,\nu)d\mathcal{H}^{n-1}(y) \\ &\le c_2\mathcal{L}^n(Q^{\nu,\varepsilon}\setminus K_\varepsilon) + (c_5(1+|\zeta|)+\varepsilon|\zeta|)\,\mathcal{H}^{n-1}(\Pi_0^\nu\cap(Q^{\nu,\varepsilon}\setminus K_\varepsilon)) < \varepsilon, \end{aligned}$$

where the last inequality follows from (8.16). Since $v_k \to u$ in $L^p(Q_\rho^{\nu,\varepsilon}(x),\mathbb{R}^m)$, it follows that

$$v_k^\rho(\cdot) = v_k(x+\rho\,\cdot) \to u(x+\rho\,\cdot) \quad \text{in } L^p(Q^{\nu,\varepsilon},\mathbb{R}^m) \text{ as } k\to+\infty. \tag{8.19}$$

Hence, from (8.17) and (8.19) we have

$$\begin{aligned} \limsup_{k\to+\infty} E_{k,\rho}^{\varepsilon,p}(\hat v_k^\rho, Q^{\nu,\varepsilon}) \le (1+\varepsilon)\Big(\limsup_{k\to+\infty} E_{k,\rho}^{\varepsilon,p}(v_k^\rho, Q^{\nu,\varepsilon}) + \varepsilon\Big) \\ + M_\varepsilon\|u(x+\rho\,\cdot) - u_{0,\zeta,\nu}(\cdot)\|^p_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^m)} + \varepsilon. \end{aligned} \tag{8.20}$$

Step 2. Estimate for $\nabla\hat v_k^\rho$. We now show that $\nabla\hat v_k^\rho$ is small in L^p-norm for k large and ρ small. By the definition of $\hat v_k^\rho$ we have

$$\begin{aligned} \|\nabla\hat v_k^\rho\|_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^{m\times n})} &\le \|\nabla\phi_{i_k}\|_{L^\infty(Q^{\nu,\varepsilon},\mathbb{R}^n)}\|v_k^\rho - u_{0,\zeta,\nu}\|_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^m)} \\ &\quad + \|\phi_{i_k}\|_{L^\infty(Q^{\nu,\varepsilon})}\|\nabla v_k^\rho\|_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^{m\times n})} \\ &\le C_\varepsilon\|v_k^\rho - u_{0,\zeta,\nu}\|_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^m)} + \|\nabla v_k^\rho\|_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^{m\times n})}, \end{aligned} \tag{8.21}$$

for a suitable constant $C_\varepsilon > 0$. We now estimate separately the two terms in the right-hand side of (8.21).

As for the first term, note that by (8.19) we can find $k_1(\rho) \ge k_0(\rho)$ such that

$$\|v_k^\rho(\cdot) - u(x+\rho\,\cdot)\|_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^m)} \le \rho \quad \text{for} \quad k \ge k_1(\rho).$$

Hence from (8.11) we deduce that for $k \ge k_1(\rho)$

$$\begin{aligned} &\|v_k^\rho - u_{0,\zeta,\nu}\|_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^m)} \\ &\quad \le \|v_k^\rho(\cdot) - u(x+\rho\,\cdot)\|_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^m)} + \|u(x+\rho\,\cdot) - u_{0,\zeta,\nu}(\cdot)\|_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^m)} \le \omega_1(\rho), \end{aligned} \tag{8.22}$$

where $\omega_1(\rho)$ is independent of k and $\omega_1(\rho)\to 0$ as $\rho\to 0+$.

For the second term in (8.21), by the definition of v_k^ρ, $(f3)$, and the positivity of g_k, we have that

$$\begin{aligned} \int_{Q^{\nu,\varepsilon}} |\nabla v_k^\rho|^p dy &= \rho^{p-n}\int_{Q_\rho^{\nu,\varepsilon}(x)} |\nabla v_k|^p dy \le \frac{\rho^{p-n}}{c_1}\int_{Q_\rho^{\nu,\varepsilon}(x)} f_k(y,\nabla v_k)dy \\ &\le \frac{\rho^{p-1}}{c_1}\Big(\frac{1}{\rho^{n-1}}E_k^{\varepsilon,p}(v_k, Q_\rho^{\nu,\varepsilon}(x))\Big). \end{aligned} \tag{8.23}$$

By (8.12) there exists $\rho_0 > 0$ such that $E^{\varepsilon,p}(u,Q_\rho^{\nu,\varepsilon}(x))/\rho^{n-1} < g^{\varepsilon,p}(x,\zeta,\nu)+1$ for every $0<\rho<\rho_0$. Therefore, for every $0<\rho<\rho_0$ satisfying (8.13) there exits $k_2(\rho)\ge k_1(\rho)$ such that

$$\frac{1}{\rho^{n-1}}\,E_k^{\varepsilon,p}(u_k, Q_\rho^{\nu,\varepsilon}(x)) < g^{\varepsilon,p}(x,\zeta,\nu)+1,$$

for every $k \ge k_2(\rho)$. This inequality, together with (8.23), gives

$$\int_{Q^{\nu,\varepsilon}} |\nabla v_k^\rho|^p dy \le \frac{\rho^{p-1}}{c_1}(g^{\varepsilon,p}(x,\zeta,\nu)+1), \tag{8.24}$$

for every $k\ge k_2(\rho)$. Finally, putting together (8.21), (8.22), and (8.24) yields

$$\|\nabla\hat v_k^\rho\|_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^{m\times n})} \le \omega_2(\rho) \tag{8.25}$$

for every $0<\rho<\rho_0$ satisfying (8.13) and every $k\geq k_2(\rho)$, where $\omega_2(\rho)$ is independent of k and $\omega_2(\rho)\to 0$ as $\rho\to 0+$.

Step 3. Modification of $\hat v_k^\rho$ to make it piecewise constant. On account of estimate (8.25), we now further modify $\hat v_k^\rho$ using the same construction as in [8, page 332]. Let $\zeta_1,\dots,\zeta_m$ be the coordinates of ζ. By (8.10) for every $0<\rho<\rho_0$ satisfying (8.13) there exists an integer $N_\rho>0$, with $2\sqrt{m}/N_\rho<\mu$ and $1/N_\rho<|\zeta_i|$ for every i with $\zeta_i\neq 0$, such that,

$$N_\rho\to+\infty\quad\text{and}\quad\omega_2(\rho)\,N_\rho\to 0+\quad\text{as }\rho\to 0+. \tag{8.26}$$

Note that, by (8.18), we have $|\hat v_k^\rho|<2\mu-(1/N_\rho)$ in $Q^{\nu,\varepsilon}$. Let $\hat v_{k,1}^\rho,\dots,\hat v_{k,m}^\rho$ be the coordinates of $\hat v_k^\rho$. Since $\hat v_{k,i}^\rho\in SBV(Q^{\nu,\varepsilon})$ for $i=1,\dots,m$, by the Coarea Formula the set $\{\hat v_{k,i}^\rho>t\}$ has finite perimeter in $Q^{\nu,\varepsilon}$ for $\mathcal{L}^1$-a.e. $t\in\mathbb{R}$ and

$$\int_{Q^{\nu,\varepsilon}}|\nabla\hat v_{k,i}^\rho|dy=|D\hat v_{k,i}^\rho|(Q^{\nu,\varepsilon}\setminus S_{\hat v_k^\rho})=\int_{-2\mu}^{2\mu}\mathcal{H}^{n-1}\big((Q^{\nu,\varepsilon}\setminus S_{\hat v_k^\rho})\cap\partial^*\{\hat v_{k,i}^\rho>t\}\big)dt,$$

where ∂^* denotes the reduced boundary in $Q^{\nu,\varepsilon}$.

To simplify the exposition we assume that μ is an integer. From the Mean Value Theorem, for every integer ℓ, with $-2N_\rho\mu\leq\ell<2N_\rho\mu$, there exists $t_\ell^i\in\mathbb{R}$, with $\ell/N_\rho<t_\ell^i<(\ell+1)/N_\rho$, such that $\{\hat v_{k,i}^\rho>t_\ell^i\}$ has finite perimeter in $Q^{\nu,\varepsilon}$ and

$$\int_{Q^{\nu,\varepsilon}}|\nabla\hat v_{k,i}^\rho|dy\geq\frac{1}{N_\rho}\sum_{\ell=-2N_\rho\mu}^{2N_\rho\mu-1}\mathcal{H}^{n-1}\big((Q^{\nu,\varepsilon}\setminus S_{\hat v_k^\rho})\cap\partial^*\{\hat v_{k,i}^\rho>t_\ell^i\}\big). \tag{8.27}$$

We now define

$$Z_\ell^i:=\{y\in Q^{\nu,\varepsilon}:t_\ell^i\leq\hat v_{k,i}^\rho(y)<t_{\ell+1}^i\},$$

and note that Z_ℓ^i has finite perimeter in $Q^{\nu,\varepsilon}$. Moreover, since $|\hat v_k^\rho|<2\mu-(1/N_\rho)$ in $Q^{\nu,\varepsilon}$, the sets Z_ℓ^i, $-2N_\rho\mu\leq\ell<2N_\rho\mu$, form a partition of $Q^{\nu,\varepsilon}$.

We finally define the piecewise constant function $w_{k,i}^\rho\colon Q^{\nu,\varepsilon}\to\mathbb{R}$ as

$$w_{k,i}^\rho|_{Z_\ell}=\begin{cases}0 & \text{if } t_\ell^i\leq 0<t_{\ell+1}^i,\\ \zeta_i & \text{if } t_\ell^i\leq\zeta_i<t_{\ell+1}^i,\\ t_\ell^i & \text{otherwise.}\end{cases}$$

Note that $w_{k,i}^\rho$ is well defined, since $|\zeta_i|>1/N_\rho$ when $\zeta_i\neq 0$, and therefore in this case 0 and ζ_i cannot belong to the same interval $[t_\ell^i,t_{\ell+1}^i)$. Moreover, $w_{k,i}^\rho\in SBV_{\rm pc}(Q^{\nu,\varepsilon})$ since each set Z_ℓ^i has finite perimeter. Then the function $w_k^\rho:=(w_{k,1}^\rho,\dots,w_{k,m}^\rho)$ belongs to $SBV_{\rm pc}(Q^{\nu,\varepsilon},\mathbb{R}^m)$.

We now claim that for every $0<\rho<\rho_0$ satisfying (8.13) and for every $k\geq k_2(\rho)$ the following properties hold:

$$w_k^\rho=u_{0,\zeta,\nu}\text{ in a neighbourhood of }\partial Q^{\nu,\varepsilon}, \tag{8.28}$$

$$\|w_k^\rho-\hat v_k^\rho\|_{L^\infty(Q^{\nu,\varepsilon},\mathbb{R}^m)}\leq\frac{2\sqrt{m}}{N_\rho}<\mu, \tag{8.29}$$

$$\|w_k^\rho\|_{L^\infty(Q^{\nu,\varepsilon},\mathbb{R}^m)}\leq 2\mu, \tag{8.30}$$

$$\mathcal{H}^{n-1}((S_{w_k^\rho}\setminus S_{\hat v_k^\rho})\cap Q^{\nu,\varepsilon})\leq\omega_3(\rho), \tag{8.31}$$

where $\omega_3(\rho)$ is independent of k and $\omega_3(\rho)\to 0+$ as $\rho\to 0+$.

Property (8.28) follows from the definition of w_k^ρ. As for (8.29) we just note that $\|w_{k,i}^\rho-\hat v_{k,i}^\rho\|_{L^\infty(Q^{\nu,\varepsilon})}=\max_\ell\|w_{k,i}^\rho-\hat v_{k,i}^\rho\|_{L^\infty(Z_\ell^i)}\leq 2/N_\rho$. Inequality (8.30) follows from (8.18) and (8.29). To prove (8.31) we observe that, up to $\mathcal{H}^{n-1}$-negligible sets, $S_{w_k^\rho}\subset\cup_i\cup_\ell\partial^*Z_\ell^i$, and since $Z_\ell^i=\{\hat v_{k,i}^\rho>t_\ell^i\}\setminus\{\hat v_{k,i}^\rho>t_{\ell+1}^i\}$, it follows that $\partial^*Z_\ell^i\subset\partial^*\{\hat v_{k,i}^\rho>t_\ell^i\}\cup\partial^*\{\hat v_{k,i}^\rho>t_{\ell+1}^i\}$, and hence

$$S_{w_k^\rho}\cap Q^{\nu,\varepsilon}\subset\bigcup_{i=1}^m\bigcup_{\ell=-2N_\rho\mu}^{2N_\rho\mu-1}(\partial^*\{\hat v_{k,i}^\rho>t_\ell^i\}\cap Q^{\nu,\varepsilon}).$$

This inclusion implies that, by (8.25) and (8.27),

$$\mathcal{H}^{n-1}((S_{w_k^\rho}\setminus S_{\hat v_k^\rho})\cap Q^{\nu,\varepsilon}) \le \sum_{i=1}^m \sum_{\ell=-N_\rho\lambda}^{N_\rho\lambda-1} \mathcal{H}^{n-1}\Big((Q^{\nu,\varepsilon}\setminus S_{\hat v_k^\rho})\cap \partial^*\{\hat v_{k,i}^\rho > t_\ell^i\}\Big)$$
$$\le mN_\rho \int_{Q^{\nu,\varepsilon}} |\nabla \hat v_k^\rho| dy \le mN_\rho \|\nabla \hat v_k^\rho\|_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^{m\times n})} \le \omega_3(\rho)$$

where $\omega_3(\rho) := m\omega_2(\rho)N_\rho \to 0+$ as $\rho\to 0+$ by (8.26).

Step 4. Conclusion of the proof for bounded functions. We first note that by (8.15) and (8.20) we have

$$\limsup_{k\to+\infty} \int_{S_{\hat v_k^\rho}\cap Q^{\nu,\varepsilon}} g_k^\varepsilon(x+\rho y,[\hat v_k^\rho](y),\nu_{\hat v_k^\rho}(y))d\mathcal{H}^{n-1}(y)$$
$$\le (1+\varepsilon)\Big(\limsup_{k\to+\infty} E_{k,\rho}^{\varepsilon,p}(v_k^\rho, Q^{\nu,\varepsilon})+\varepsilon\Big) + M_\varepsilon \|u(x+\rho\,\cdot)-u_{0,\zeta,\nu}(\cdot)\|^p_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^m)} + \varepsilon. \tag{8.32}$$

Further, by $(f4)$ and (8.24), we can control the volume integral in (8.32) as follows:

$$\int_{Q^{\nu,\varepsilon}} f_k(x+\rho y,\nabla v_k^\rho(y))dy \le c_2\int_{Q^{\nu,\varepsilon}}(1+|\nabla v_k^\rho|^p)dy \le c_2\Big(\varepsilon+\frac{\rho^{p-1}}{c_1}(g^{\varepsilon,p}(x,\zeta,\nu)+1)\Big)$$

for every $0<\rho<\rho_0$ satisfying (8.13) and every $k\ge k_2(\rho)$.

By (8.15), this inequality and (8.32) imply in particular that

$$\limsup_{k\to+\infty} \int_{S_{\hat v_k^\rho}\cap Q^{\nu,\varepsilon}} g_k^\varepsilon(x+\rho y,[\hat v_k^\rho](y),\nu_{\hat v_k^\rho}(y))d\mathcal{H}^{n-1}(y)$$
$$\le (1+\varepsilon)\limsup_{k\to+\infty}\int_{S_{v_k^\rho}\cap Q^{\nu,\varepsilon}} g_k^\varepsilon(x+\rho y,[v_k^\rho](y),\nu_{v_k^\rho}(y))d\mathcal{H}^{n-1}(y)$$
$$+ 2c_2\Big(\varepsilon+\frac{\rho^{p-1}}{c_1}(g^{\varepsilon,p}(x,\zeta,\nu)+1)\Big) + M_\varepsilon\|u(x+\rho\,\cdot)-u_{0,\zeta,\nu}(\cdot)\|^p_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^m)} + 3\varepsilon. \tag{8.33}$$

Since

$$\int_{S_{v_k^\rho}\cap Q^{\nu,\varepsilon}} g_k^\varepsilon(x+\rho y,[v_k^\rho](y),\nu_{v_k^\rho}(y))\Big)d\mathcal{H}^{n-1}(y) = \frac{1}{\rho^{n-1}}\int_{S_{v_k}\cap Q_\rho^{\nu,\varepsilon}(x)} g_k^\varepsilon(y,[v_k](y),\nu_{v_k}(y))d\mathcal{H}^{n-1}(y),$$

gathering (8.14) and (8.33) gives

$$\limsup_{k\to+\infty} \int_{S_{\hat v_k^\rho}\cap Q^{\nu,\varepsilon}} g_k^\varepsilon(x+\rho y,[\hat v_k^\rho](y),\nu_{\hat v_k^\rho}(y))d\mathcal{H}^{n-1}(y)$$
$$\le (1+\varepsilon)^2\frac{1}{\rho^{n-1}}E^{\varepsilon,p}(u,Q_\rho^{\nu,\varepsilon}(x)) + 2\rho + 2c_2\Big(\varepsilon+\frac{\rho^{p-1}}{c_1}(g^{\varepsilon,p}(x,\zeta,\nu)+1)\Big) \tag{8.34}$$
$$+ M_\varepsilon\|u(x+\rho\,\cdot)-u_{0,\zeta,\nu}(\cdot)\|^p_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^m)} + 3\varepsilon.$$

We now estimate the left-hand side in (8.34). We have

$$\int_{S_{\hat v_k^\rho}\cap Q^{\nu,\varepsilon}} g_k^\varepsilon(x+\rho y,[\hat v_k^\rho](y),\nu_{\hat v_k^\rho}(y))d\mathcal{H}^{n-1}(y)$$
$$\ge \int_{(S_{\hat v_k^\rho}\cap S_{w_k^\rho})\cap Q^{\nu,\varepsilon}} g_k(x+\rho y,[\hat v_k^\rho](y),\nu_{\hat v_k^\rho}(y))d\mathcal{H}^{n-1}(y)$$
$$= \int_{S_{w_k^\rho}\cap Q^{\nu,\varepsilon}} g_k(x+\rho y,[w_k^\rho](y),\nu_{w_k^\rho}(y))d\mathcal{H}^{n-1}(y)$$
$$+ \int_{(S_{\hat v_k^\rho}\cap S_{w_k^\rho})\cap Q^{\nu,\varepsilon}} \left(g_k(x+\rho y,[\hat v_k^\rho](y),\nu_{\hat v_k^\rho}(y)) - g_k(x+\rho y,[w_k^\rho](y),\nu_{w_k^\rho}(y))\right)d\mathcal{H}^{n-1}(y)$$
$$- \int_{(S_{w_k^\rho}\setminus S_{\hat v_k^\rho})\cap Q^{\nu,\varepsilon}} g(x+\rho y,[w_k^\rho](y),\nu_{w_k^\rho}(y))d\mathcal{H}^{n-1}(y) =: I_1+I_2-I_3. \tag{8.35}$$

We now claim that

$$|I_2|\le\omega_4(\rho) \quad\text{and}\quad |I_3|\le\omega_5(\rho) \tag{8.36}$$

for $k\ge k_2(\rho)$, where $\omega_4(\rho)$ and $\omega_5(\rho)$ are independent of k and tend to $0+$ as $\rho\to0+$.

Thanks to the symmetry condition $(g7)$, for the term I_2 we may choose the orientations of $\nu_{\hat v_k^\rho}$ and $\nu_{w_k^\rho}$ so that $\nu_{\hat v_k^\rho} = \nu_{w_k^\rho}$ $\mathcal{H}^{n-1}$-a.e. on $S_{\hat v_k^\rho} \cap S_{w_k^\rho}$. Thus, by assumptions $(g2)$ and $(g6)$,

$$\begin{aligned}
&|g_k(x+\rho y, [\hat v_k^\rho](y), \nu_{\hat v_k^\rho}(y)) - g_k(x+\rho y, [w_k^\rho](y), \nu_{w_k^\rho}(y))| \\
&\le \sigma_2(|[\hat v_k^\rho](y) - [w_k^\rho](y)|)\big(g_k(x+\rho y, [\hat v_k^\rho](y), \nu_{\hat v_k^\rho}(y)) + g_k(x+\rho y, [w_k^\rho](y), \nu_{w_k^\rho}(y))\big) \\
&\le 2c_5\sigma_2(2\|\hat v_k^\rho - w_k^\rho\|_{L^\infty(Q^{\nu,\varepsilon},\mathbb{R}^m)})(1 + \|\hat v_k^\rho\|_{L^\infty(Q^{\nu,\varepsilon},\mathbb{R}^m)} + \|w_k^\rho\|_{L^\infty(Q^{\nu,\varepsilon},\mathbb{R}^m)}),
\end{aligned}$$

for $\mathcal{H}^{n-1}$-a.e. $y \in S_{\hat v_k^\rho} \cap S_{w_k^\rho}$. Therefore, using (8.18), (8.29), and (8.30) we obtain

$$|I_2| \le 2c_5(1+3\mu)\,\sigma_2(4\sqrt{m}/N_\rho)\,\mathcal{H}^{n-1}(S_{\hat v_k^\rho} \cap Q^{\nu,\varepsilon})$$

for every $k \ge k_2(\rho)$.

Now recall that, by the definition of $\hat v_k^\rho$,

$$S_{\hat v_k^\rho} \cap Q^{\nu,\varepsilon} \subset \big(S_{v_k^\rho} \cap Q^{\nu,\varepsilon}\big) \cup \big(\Pi_0^\nu \cap (Q^{\nu,\varepsilon} \setminus K_\varepsilon)\big),$$

hence by (8.16),

$$\mathcal{H}^{n-1}(S_{\hat v_k^\rho} \cap Q^{\nu,\varepsilon}) \le \mathcal{H}^{n-1}(S_{v_k^\rho} \cap Q^{\nu,\varepsilon}) + \frac{\varepsilon}{c_5} = \frac{1}{\rho^{n-1}}\mathcal{H}^{n-1}(S_{v_k} \cap Q_\rho^{\nu,\varepsilon}(x)) + \frac{\varepsilon}{c_5}.$$

In terms of the functions v_k, by (8.14), this implies that

$$\mathcal{H}^{n-1}(S_{\hat v_k^\rho} \cap Q^{\nu,\varepsilon}) \le \frac{1+\varepsilon}{c_4}\frac{1}{\rho^{n-1}}E^{\varepsilon,p}(u, Q_\rho^{\nu,\varepsilon}(x)) + \frac{\rho}{c_4} + \frac{\varepsilon}{c_5}$$

for every $k \ge k_2(\rho)$. Hence, for the term I_2 we have

$$|I_2| \le 2c_5(1+3\mu)\,\sigma_2(4\sqrt{m}/N_\rho)\Big(\frac{1+\varepsilon}{c_4\,\rho^{n-1}}E^{\varepsilon,p}(u, Q_\rho^{\nu,\varepsilon}(x)) + \frac{\rho}{c_4} + \frac{\varepsilon}{c_5}\Big).$$

Since $\sigma_2(t) \to 0+$ as $t \to 0+$, by (8.12) we obtain that $|I_2| \le \omega_4(\rho)$ for every $k \ge k_2(\rho)$, where $\omega_4(\rho)$ is independent of k and $\omega_4(\rho) \to 0+$ as $\rho \to 0+$.

As for the term I_3, proceeding as above and using (8.30) we get

$$|I_3| \le c_5(1+4\mu)\,\mathcal{H}^{n-1}\big((S_{w_k^\rho} \setminus S_{\hat v_k^\rho}) \cap Q^{\nu,\varepsilon}\big),$$

which, by (8.31), implies that $|I_3| \le \omega_5(\rho)$ for every $k \ge k_2(\rho)$, where $\omega_5(\rho) := c_5(1+4\mu)\omega_3(\rho) \to 0+$ as $\rho \to 0+$. This concludes the proof of (8.36).

By combining (8.34), (8.35), and (8.36) we deduce that

$$\begin{aligned}
&\limsup_{k\to+\infty} \int_{S_{w_k^\rho} \cap Q^{\nu,\varepsilon}} g_k(x+\rho y, [w_k^\rho](y), \nu_{w_k^\rho}(y))\,d\mathcal{H}^{n-1}(y) \\
&\qquad \le (1+\varepsilon)^2\frac{1}{\rho^{n-1}}\,E^{\varepsilon,p}(u, Q_\rho^{\nu,\varepsilon}(x)) + 2\rho + \omega_4(\varepsilon,\rho) + \omega_5(\varepsilon,\rho) \\
&\qquad + 2c_2\Big(\varepsilon + \frac{\rho^{p-1}}{c_1}(g^{\varepsilon,p}(x,\zeta,\nu)+1)\Big) + M_\varepsilon\|u(x+\rho\,\cdot) - u_{0,\zeta,\nu}(\cdot)\|^p_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^m)} + 3\varepsilon.
\end{aligned}$$

We now define $z_k^\rho(y) := w_k^\rho((y-x)/\rho)$ for every $y \in Q_\rho^{\nu,\varepsilon}(x)$. Note that $z_k^\rho \in SBV_{\rm pc}(Q_\rho^{\nu,\varepsilon}(x), \mathbb{R}^m)$ and $z_k^\rho = u_{x,\zeta,\nu}$ in a neighbourhood of $\partial Q_\rho^{\nu,\varepsilon}(x)$. In terms of the functions z_k^ρ the previous estimate gives

$$\begin{aligned}
&\limsup_{k\to+\infty} \frac{1}{\rho^{n-1}} m^{\rm pc}_{G_k}(u_{x,\zeta,\nu}, Q_\rho^\nu(x)) \le \limsup_{k\to+\infty} \frac{1}{\rho^{n-1}} m^{\rm pc}_{G_k}(u_{x,\zeta,\nu}, Q_\rho^{\nu,\varepsilon}(x)) \\
&\qquad \le \limsup_{k\to+\infty} \frac{1}{\rho^{n-1}} \int_{S_{z_k^\rho} \cap Q_\rho^{\nu,\varepsilon}(x)} g_k(y, [z_k^\rho](y), \nu_{z_k^\rho}(y))\,d\mathcal{H}^{n-1}(y) \\
&\qquad \le (1+\varepsilon)^2\frac{1}{\rho^{n-1}}\,E^{\varepsilon,p}(u, Q_\rho^{\nu,\varepsilon}(x)) + 2\rho + \omega_4(\varepsilon,\rho) + \omega_5(\varepsilon,\rho) \\
&\qquad + 2c_2\Big(\varepsilon + \frac{\rho^{p-1}}{c_1}(g^{\varepsilon,p}(x,\zeta,\nu)+1)\Big) + M_\varepsilon\|u(x+\rho\,\cdot) - u_{0,\zeta,\nu}(\cdot)\|^p_{L^p(Q^{\nu,\varepsilon},\mathbb{R}^m)} + 3\varepsilon.
\end{aligned}$$

Finally, taking the limsup as $\rho \to 0+$ and invoking (3.11), (8.11), and (8.12), we obtain

$$g''(x,\zeta,\nu) \le (1+\varepsilon)^2 g^{\varepsilon,p}(x,\zeta,\nu) + C\varepsilon,$$

with $C := 2c_2 + 3$. Recalling the definition of ζ and ν, we obtain that

$$g''(x,[u](x),\nu_u(x)) \le (1+\varepsilon)^2 g^{\varepsilon,p}(x,[u](x),\nu_u(x)) + C\varepsilon$$

holds true for $\mathcal{H}^{n-1}$-a.e. $x \in S_u \cap A$. Taking the limit as $\varepsilon \to 0+$ and using (5.2) we get

$$g''(x,[u](x),\nu_u(x)) \le g^0(x,[u](x),\nu_u(x))$$

for $\mathcal{H}^{n-1}$-a.e. $x \in S_u \cap A$, thus proving (5.15) for $u \in SBV^p(A,\mathbb{R}^m) \cap L^\infty(A,\mathbb{R}^m)$.

Step 5. Extension to unbounded functions in $GSBV^p$. Let $A \in \mathscr{A}$ and $u \in GSBV^p(A,\mathbb{R}^m)$. For every integer $k \ge 1$ we define $z_k := \alpha_k(u)$, where $\alpha_k \in C^1_c(\mathbb{R}^m,\mathbb{R}^m)$ satisfies $\alpha_k(\zeta) = \zeta$ for every $\zeta \in \mathbb{R}^m$ with $|\zeta| \le k$. By (h) in Section 2 we have that $z_k \in SBV^p(A,\mathbb{R}^m) \cap L^\infty(A,\mathbb{R}^m)$. Let $\Sigma_k := \{x \in S_u \cap A : |u^\pm(x)| < k\}$. By the definition of $u^\pm(x)$ as approximate limits, it is easy to see that for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma_k$ we have either $z_k^\pm(x) = u^\pm(x)$ and $\nu_{z_k}(x) = \nu_u(x)$ or $z_k^\pm(x) = u^\mp(x)$ and $\nu_{z_k}(x) = -\nu_u(x)$ (see [3, Remark 4.32]). On the other hand, by the previous steps in the proof we have that

$$g''(x,[z_k](x),\nu_{z_k}(x)) \le g^0(x,[z_k](x),\nu_{z_k}(x))$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma_k$. By $(g7)$ this implies that

$$g''(x,[u](x),\nu_u(x)) \le g^0(x,[u](x),\nu_u(x)) \tag{8.37}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma_k$. Since the integer k is arbitrary, (8.37) holds for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$. □

Appendix

In this section we collect some technical results that we have used throughout the paper. We begin with an example of a family of orthogonal matrices R_ν satisfying all assumptions of (k) of Section 2.

Example A.1. Let $\phi_\pm \colon \mathbb{S}^{n-1} \setminus \{\pm e_n\} \to \mathbb{R}^{n-1}$ be the stereographic projection from $\pm e_n$ into the plane $x_n = 0$ and let $\psi_\pm \colon \mathbb{R}^{n-1} \to \mathbb{S}^{n-1} \setminus \{\pm e_n\}$ be its inverse function. For every $\nu \in \widehat{\mathbb{S}}^{n-1}_\pm$ we consider the vectors $\xi_i(\nu) := \partial_i \psi_\mp(\phi_\mp(\nu))$, $i = 1,\dots,n-1$, which are tangent to $\mathbb{S}^{n-1}$ at ν, and hence satisfy $\xi_i(\nu)\cdot\nu = 0$. Since $\psi_\mp$ are conformal maps, we have $\xi_i(\nu)\cdot\xi_j(\nu) = 0$ for $i \ne j$. Let $\nu_i(\nu) := \xi_i(\nu)/|\xi_i(\nu)|$. Then the vectors $\nu_1(\nu),\nu_2(\nu),\dots,\nu_{n-1}(\nu),\nu$ form an orthonormal basis of $\mathbb{R}^n$, therefore they are the columns of an orthogonal matrix, denoted by R_ν. It is clear from the construction that $R_\nu e_n = \nu$ and that the restriction of $\nu \mapsto R_\nu$ to $\widehat{\mathbb{S}}^{n-1}_\pm$ is continuous. Moreover, since $\phi_+(-\nu) = -\phi_-(\nu)$ for every $\nu \in \mathbb{S}^{n-1} \setminus \{e_n,-e_n\}$, we have $\psi_+(-y) = -\psi_-(y)$ for every $y \in \mathbb{R}^{n-1} \setminus \{0\}$. It follows that $\xi_i(-\nu) = \xi_i(\nu)$, hence $\nu_i(-\nu) = \nu_i(\nu)$ for every $\nu \in \mathbb{S}^{n-1} \setminus \{e_n,-e_n\}$. This property is clearly true also for $\nu = \pm e_n$, since $\nu_i(\pm e_n) = e_i$. It follows that $R_{-\nu}Q(0) = R_\nu Q(0)$ for every $\nu \in \mathbb{S}^{n-1}$.

The following remark will be used in [9].

Remark A.2. From the formulas defining the stereographic projections $\phi_\pm$ it follows that $\nu \in (\mathbb{S}^{n-1} \cap \mathbb{Q}^n) \setminus \{e_n,-e_n\}$ if and only if $\phi_\pm(\nu) \in \mathbb{Q}^{n-1} \setminus \{0\}$. Therefore $\mathbb{S}^{n-1} \cap \mathbb{Q}^n$ is dense in $\mathbb{S}^{n-1}$. Moreover, the explicit formulas for $\partial_i\psi_\pm$ show that $\nu_i(\nu) \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ for every $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$, hence $R_\nu \in \mathbb{Q}^{n\times n}$ for every $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$.

The rest of this section is devoted to some technical lemmas needed to prove some of the properties satisfied by the functions f', f'', g', and g'' introduced in (3.8)-(3.11) and by the functions $f^{\varepsilon,p}$ and $g^{\varepsilon,p}$ introduced in (4.24) and (4.25).

Lemma A.3 (Upper semicontinuity)**.** *Let X be either $L^0(\mathbb{R}^n,\mathbb{R}^m)$ or $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$, and let $H \colon X\times\mathscr{A} \to [0,+\infty]$ be a functional such that*

- *(h1) (locality) $H(u,A) = H(v,A)$ if u, $v \in X$, $A \in \mathscr{A}$, and $u = v$ $\mathcal{L}^n$-a.e. in A,*
- *(h2) (measure) for every $u \in X \cap SBV_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ the function $H(u,\cdot)$ is the restriction to $\mathscr{A}$ of a countably additive function defined on the σ-algebra of the Borel subsets of $\mathbb{R}^n$,*
- *(h3) (upper bound) for every $u \in X \cap SBV_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ and every $A \in \mathscr{A}$*

$$H(u,A) \le c_2 \int_A (1+|\nabla u|^p)dx + c_5 \int_{S_u\cap A} (1+|[u]|)\,d\mathcal{H}^{n-1}.$$

Let $m_H^{1,p}$, $m_H^{\rm pc}$, and m_H be as in (3.5)-(3.7), and let $\rho > 0$. Then

(a) *the functions*

$$(x,\xi) \mapsto m_H(\ell_\xi, Q_\rho(x)) \quad \text{and} \quad (x,\xi) \mapsto m_H^{1,p}(\ell_\xi, Q_\rho(x))$$

are upper semicontinuous in $\mathbb{R}^n\times\mathbb{R}^{m\times n}$;

(b) *the restrictions of the function*

$$(x,\zeta,\nu) \mapsto m_H(u_{x,\zeta,\nu}, Q^\nu_\rho(x))$$

to the sets $\mathbb{R}^n\times\mathbb{R}^m_0\times\widehat{\mathbb{S}}^{n-1}_+$ and $\mathbb{R}^n\times\mathbb{R}^m_0\times\widehat{\mathbb{S}}^{n-1}_-$ are upper semicontinuous;

(c) *for every* $\zeta_0 \in \mathbb{R}^m_0$ *the restrictions of the function*

$$(x,\nu) \mapsto m_H^{\rm pc}(u_{x,\zeta_0,\nu}, Q^\nu_\rho(x))$$

to the sets $\mathbb{R}^n \times \widehat{\mathbb{S}}^{n-1}_+$ *and* $\mathbb{R}^n \times \widehat{\mathbb{S}}^{n-1}_-$ *are upper semicontinuous.*

Proof. In the proof of (a) we only deal with m_H, the proof of the upper semicontinuity of $m_H^{1,p}$ being similar.

Fix $x_0 \in \mathbb{R}^n$, $\xi_0 \in \mathbb{R}^{m\times n}$, and $\varepsilon > 0$. By the definition of m_H there exist $u_0 \in X$, with $u_0|_{Q_\rho(x_0)} \in SBV^p(Q_\rho(x_0),\mathbb{R}^m)$, and $\delta_0 \in (0,\rho/3)$ such that

$$u_0 = \ell_{\xi_0} \quad \mathcal{L}^n\text{-a.e. in } Q_\rho(x_0) \setminus Q_{\rho-3\delta_0}(x_0), \tag{A.38}$$

$$H(u_0, Q_\rho(x_0)) < m_H(\ell_{\xi_0}, Q_\rho(x_0)) + \varepsilon. \tag{A.39}$$

Now fix $\delta \in (0,\delta_0)$, $x \in Q_\delta(x_0)$, $\xi \in \mathbb{R}^{m\times n}$ with $|\xi - \xi_0| < \delta$, and $\varphi \in C^\infty_c(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset Q_\rho(x)$, $\varphi = 1$ in $Q_{\rho-\delta}(x)$, $0 \le \varphi \le 1$ in $\mathbb{R}^n$, and $|\nabla\varphi| \le 3/\delta$ in $\mathbb{R}^n$. We define $u_1 \in SBV^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ by

$$u_1 := \begin{cases} u_0 & \text{in } Q_{\rho-\delta}(x), \\ \varphi\,\ell_{\xi_0} + (1-\varphi)\,\ell_\xi & \text{in } \mathbb{R}^n \setminus Q_{\rho-2\delta}(x). \end{cases}$$

Since $x \in Q_\delta(x_0)$, we have $Q_{\rho-\delta}(x) \setminus Q_{\rho-2\delta}(x) \subset\subset Q_\rho(x_0) \setminus Q_{\rho-3\delta_0}(x_0)$. Therefore u_1 is well defined, since, by (A.38), both formulas give the same value in the overlapping set $Q_{\rho-\delta}(x) \setminus Q_{\rho-2\delta}(x)$. Moreover $u_1 = \ell_\xi$ in a neighbourhood of $\partial Q_\rho(x)$, hence $m_H(\ell_\xi, Q_\rho(x)) \le H(u_1, Q_\rho(x))$. Therefore, using $(h1)$-$(h3)$, we obtain

$$m_H(\ell_\xi, Q_\rho(x)) \le H(u_0, Q_{\rho-\delta}(x)) + c_2 \int_{Q_\rho(x)\setminus Q_{\rho-2\delta}(x)} (1 + |\nabla u_1|^p)\,dy. \tag{A.40}$$

Since $\nabla u_1 = \varphi\xi_0 + (1-\varphi)\xi + (\ell_{\xi_0} - \ell_\xi)\otimes\nabla\varphi$ in $Q_\rho(x) \setminus Q_{\rho-2\delta}(x)$, by convexity we have $|\nabla u_1|^p \le 3^{p-1}(|\xi_0|^p + |\xi|^p + |\xi_0 - \xi|^p C_1 |\nabla\varphi|^p)$, where $C_1 := \sup\{|y|^p : y \in Q_{\rho+\delta_0}(x_0)\}$.

Therefore (A.40), together with the estimates for $|\xi_0 - \xi|$ and $|\nabla\varphi|$, yields

$$m_H(\ell_\xi, Q_\rho(x)) \le H(u_0, Q_\rho(x)) + C_2(\rho^n - (\rho - 2\delta)^n),$$

where $C_2 := c_2\big(1 + 3^{2p-1}(|\xi_0|^p + \delta_0^p + C_1)\big)$. Combining this inequality with (A.39) we get

$$m_H(\ell_\xi, Q_\rho(x)) \le m_H(\ell_{\xi_0}, Q_\rho(x_0)) + \varepsilon + 2nC_2\rho^{n-1}\delta.$$

Therefore, if $0 < \delta < \min\{\delta_0, \varepsilon/(2nC_2\rho^{n-1})\}$, $x \in Q_\delta(x_0)$, and $|\xi - \xi_0| < \delta$, then

$$m_H(\ell_\xi, Q_\rho(x)) \le m_H(\ell_{\xi_0}, Q_\rho(x_0)) + 2\varepsilon.$$

This proves the upper semicontinuity of $(x,\xi) \mapsto m_H(\ell_\xi, Q_\rho(x))$ at $(x,\xi) = (x_0,\xi_0)$.

To prove (b), we fix three points $x_0 \in \mathbb{R}^n$, $\zeta_0 \in \mathbb{R}^m_0$, $\nu_0 \in \widehat{\mathbb{S}}^{n-1}_+$, three sequences $(x_j) \subset \mathbb{R}^n$, $(\zeta_j) \subset \mathbb{R}^m_0$, $(\nu_j) \subset \widehat{\mathbb{S}}^{n-1}_+$, with $x_j \to x_0$, $\nu_j \to \nu_0$, $\zeta_j \to \zeta_0$, and a constant $\varepsilon > 0$. By definition there exist $v_0 \in X$, with $v_0|_{Q^{\nu_0}_\rho(x_0)} \in SBV^p(Q^{\nu_0}_\rho(x_0),\mathbb{R}^m)$, and $\delta_0 \in (0,\rho/3)$ such that

$$v_0 = u_{x_0,\zeta_0,\nu_0} \quad \mathcal{L}^n\text{-a.e. in } Q^{\nu_0}_\rho(x_0) \setminus Q^{\nu_0}_{\rho-3\delta_0}(x_0), \tag{A.41}$$

$$H(v_0, Q^\nu_\rho(x_0)) < m_H(u_{x_0,\zeta_0,\nu_0}, Q^{\nu_0}_\rho(x_0)) + \varepsilon. \tag{A.42}$$

Let us fix $\delta \in (0,\delta_0/2)$. There exists an integer i_δ such that $Q^{\nu_0}_{\rho-\delta}(x_j) \subset Q^{\nu_0}_\rho(x_0)$, $Q^{\nu_0}_{\rho+\delta}(x_j) \subset Q^{\nu_0}_{\rho+2\delta}(x_0)$, and $Q^{\nu_0}_{\rho-3\delta_0}(x_0) \subset Q^{\nu_0}_{\rho-5\delta}(x_0) \subset Q^{\nu_0}_{\rho-4\delta}(x_j)$ for every $j \ge i_\delta$.

By (k) in Section 2 the function $\nu \mapsto R_\nu$ is continuous on $\widehat{\mathbb{S}}^{n-1}_+$. Consequently there exists an integer $j_\delta \ge i_\delta$ such that $Q^{\nu_j}_{\rho-2\delta}(x) \subset Q^{\nu_0}_{\rho-\delta}(x)$, $Q^{\nu_j}_\rho(x) \subset Q^{\nu_0}_{\rho+\delta}(x)$, and $Q^{\nu_0}_{\rho-4\delta}(x) \subset Q^{\nu_j}_{\rho-3\delta}(x)$ for every $j \ge j_\delta$ and every $x \in \mathbb{R}^n$. Therefore the previous inclusions imply that

$$Q^{\nu_j}_{\rho-2\delta}(x_j) \setminus Q^{\nu_j}_{\rho-3\delta}(x_j) \subset Q^{\nu_0}_\rho(x_0) \setminus Q^{\nu_0}_{\rho-5\delta}(x_0) \subset Q^{\nu_0}_\rho(x_0) \setminus Q^{\nu_0}_{\rho-3\delta_0}(x_0), \tag{A.43}$$

$$Q^{\nu_j}_\rho(x_j) \setminus Q^{\nu_j}_{\rho-3\delta}(x_j) \subset Q^{\nu_0}_{\rho+2\delta}(x_0) \setminus Q^{\nu_0}_{\rho-5\delta}(x_0), \tag{A.44}$$

for every $j \ge j_\delta$.

Let $\psi_j \in C^\infty_c(\mathbb{R}^n)$ be such that $\operatorname{supp}\psi_j \subset Q^{\nu_j}_\rho(x_j)$, $\psi_j = 1$ in $Q^{\nu_j}_{\rho-\delta}(x_j)$, $0 \le \psi_j \le 1$ in $\mathbb{R}^n$, and $|\nabla\psi_j| \le 3/\delta$ in $\mathbb{R}^n$. We define $v_j \in SBV^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^m)$ by

$$v_j := \begin{cases} v_0 & \text{in } Q^{\nu_j}_{\rho-2\delta}(x_j), \\ \psi_j u_{x_0,\zeta_0,\nu_0} + (1-\psi_j)u_{x_j,\zeta_j,\nu_j} & \text{in } \mathbb{R}^n \setminus Q^{\nu_j}_{\rho-3\delta}(x_j). \end{cases}$$

By (A.41) and (A.43) the function v_j is well defined, since both formulas give the same value in the overlapping set $Q^{\nu_j}_{\rho-2\delta}(x_j) \setminus Q^{\nu_j}_{\rho-3\delta}(x_j)$. Moreover $v_j = u_{x_j,\zeta_j,\nu_j}$ in a neighbourhood of $\partial Q^{\nu_j}_\rho(x_j)$, hence

$m_H(u_{x_j,\zeta_j,\nu_j}, Q_\rho^{\nu_j}(x_j)) \le H(v_j, Q_\rho^{\nu_j}(x_j))$. So, using $(h1)$-$(h3)$ and setting $A_j := Q_\rho^{\nu_j}(x_j) \setminus Q_{\rho-3\delta}^{\nu_j}(x_j)$, we obtain

$$\begin{aligned} m_H(u_{x_j,\zeta_j,\nu_j}, Q_\rho^{\nu_j}(x_j)) &\le H(v_0, Q_{\rho-2\delta}^{\nu_j}(x_j)) \\ &\quad + c_2 \int_{A_j} (1+|\nabla v_j|^p)\, dy + c_5 \int_{S_{v_j}\cap A_j} (1+|[v_j]|)\, d\mathcal{H}^{n-1}. \end{aligned} \tag{A.45}$$

Since $|\nabla v_j| \le |\nabla \psi_j||u_{x_0,\zeta_0,\nu_0} - u_{x_j,\zeta_j,\nu_j}|$ on A_j, we have $|\nabla v_j| \le (3/\delta)|u_{x_0,\zeta_0,\nu_0} - u_{x_j,\zeta_j,\nu_j}|$ on A_j. It follows that

$$\int_{A_j} (1+|\nabla v_j|^p)\, dy \le \rho^n - (\rho-3\delta)^n + \frac{3^p}{\delta^p}\eta_j \le 3n\delta\rho^{n-1} + \frac{3^p}{\delta^p}\eta_j,$$

where $\eta_j := \int_{A_j} |u_{x_0,\zeta_0,\nu_0} - u_{x_j,\zeta_j,\nu_j}|^p dy \to 0+$, as $j \to +\infty$.

On the other hand by (A.44) we have $S_{v_j} \cap A_j \subset \big(\Pi_{x_0}^{\nu_0} \cap Q_{\rho+2\delta}^{\nu_0}(x_0) \setminus Q_{\rho-5\delta}^{\nu_0}(x_0)\big) \cup \big(\Pi_{x_j}^{\nu_j} \cap Q_\rho^{\nu_j}(x_j) \setminus Q_{\rho-3\delta}^{\nu_j}(x_j)\big)$. Moreover there exists a constant $M_1 > 0$ such that $|[v_j]| \le M_1$ $\mathcal{H}^{n-1}$-a.e. on $S_{v_j} \cap A_j$ for every $j \ge j_\delta$. Therefore

$$\int_{S_{v_j}\cap A_j} (1+|[v_j]|)\, d\mathcal{H}^{n-1} \le 2(1+M_1)\big((\rho+2\delta)^{n-1} - (\rho-5\delta)^{n-1}\big) \le 14\delta(1+M_1)(n-1)(2\rho)^{n-2}. \tag{A.46}$$

From (A.42) and (A.45)-(A.46) it follows that for every $j \ge j_\delta$

$$m_H(u_{x_j,\zeta_j,\nu_j}, Q_\rho^{\nu_j}(x_j)) \le m_H(u_{x_0,\zeta_0,\nu_0}, Q_\rho^{\nu_0}(x_0)) + \varepsilon + M_2\delta + c_2\frac{3^p}{\delta^p}\eta_j,$$

where $M_2 := 3nc_2\rho^{n-1} + 14c_5(1+M_1)(n-1)(2\rho)^{n-2}$. Taking the limit as $j \to +\infty$ we get

$$\limsup_{j\to+\infty} m_H(u_{x_j,\zeta_j,\nu_j}, Q_\rho^{\nu_j}(x_j)) \le m_H(u_{x_0,\zeta_0,\nu_0}, Q_\rho^{\nu_0}(x_0)) + \varepsilon + M_2\delta.$$

Since $\varepsilon > 0$ and $\delta \in (0, \delta_0/2)$ are arbitrary, we obtain

$$\limsup_{j\to+\infty} m_H(u_{x_j,\zeta_j,\nu_j}, Q_\rho^{\nu_j}(x_j)) \le m_H(u_{x_0,\zeta_0,\nu_0}, Q_\rho^{\nu_0}(x_0)),$$

which proves the upper semicontinuity of the restriction of $(x,\zeta,\nu) \mapsto m_H(u_{x,\zeta,\nu}, Q_\rho^\nu(x))$ to $\mathbb{R}^n \times \mathbb{R}_0^m \times \widehat{\mathbb{S}}_+^{n-1}$. The same proof holds for $\mathbb{R}^n \times \mathbb{R}_0^m \times \widehat{\mathbb{S}}_-^{n-1}$.

To prove (c), we fix x_0, ζ_0, ν_0, (x_j), (ν_j), and $\varepsilon > 0$ as in the proof of (b). By definition there exist $w_0 \in X$, with $w_0|_{Q_\rho^{\nu_0}(x_0)} \in SBV_{\rm pc}(Q_\rho^{\nu_0}(x_0), \mathbb{R}^m)$, and $\delta_0 \in (0, \rho/3)$ such that

$$w_0 = u_{x_0,\zeta_0,\nu_0} \quad \mathcal{L}^n\text{-a.e. in } Q_\rho^{\nu_0}(x_0) \setminus Q_{\rho-3\delta_0}^{\nu_0}(x_0), \tag{A.47}$$

$$H(w_0, Q_\rho^\nu(x_0)) < m_H^{\rm pc}(u_{x_0,\zeta_0,\nu_0}, Q_\rho^{\nu_0}(x_0)) + \varepsilon. \tag{A.48}$$

Fix $\delta \in (0, \delta_0/2)$ and let j_δ be an integer such that (A.43) and (A.44) are satisfied for every $j \ge j_\delta$. We define $w_j \in SBV_{\rm loc}^p(\mathbb{R}^n, \mathbb{R}^m)$ by

$$w_j := \begin{cases} w_0 & \text{in } Q_{\rho-2\delta}^{\nu_j}(x_j), \\ u_{x_j,\zeta_0,\nu_j} & \text{in } \mathbb{R}^n \setminus Q_{\rho-2\delta}^{\nu_j}(x_j). \end{cases} \tag{A.49}$$

Then $w_j|_{Q_\rho^{\nu_j}(x_j)} \in SBV_{\rm pc}(Q_\rho^{\nu_j}(x_j), \mathbb{R}^m)$ and $w_j = u_{x_j,\zeta_0,\nu_j}$ in a neighbourhood of $\partial Q_\rho^{\nu_j}(x_j)$, hence $m_H^{\rm pc}(u_{x_j,\zeta_0,\nu_j}, Q_\rho^{\nu_j}(x_j)) \le H(w_j, Q_\rho^{\nu_j}(x_j))$. Therefore, using $(h1)$-$(h3)$ and setting $A_j := Q_\rho^{\nu_j}(x_j) \setminus Q_{\rho-3\delta}^{\nu_j}(x_j)$, we obtain

$$m_H^{\rm pc}(u_{x_j,\zeta_0,\nu_j}, Q_\rho^{\nu_j}(x_j)) \le H(w_0, Q_{\rho-2\delta}^{\nu_j}(x_j)) + c_5 \int_{S_{w_j}\cap A_j} (1+|[w_j]|)\, d\mathcal{H}^{n-1}. \tag{A.50}$$

By (A.43) and (A.47) we have $w_j = u_{x_0,\zeta_0,\nu_0}$ on $Q_{\rho-2\delta}^{\nu_j}(x_j) \setminus Q_{\rho-3\delta}^{\nu_j}(x_j)$ for every $j \ge j_\delta$, while by (A.49) we have $w_j = u_{x_j,\zeta_0,\nu_j}$ in $Q_\rho^{\nu_j}(x_j) \setminus Q_{\rho-2\delta}^{\nu_j}(x_j)$. Therefore $S_{w_j} \cap A_j \subset \big(\Pi_{x_0}^{\nu_0} \cap Q_{\rho-2\delta}^{\nu_j}(x_j) \setminus Q_{\rho-3\delta}^{\nu_j}(x_j)\big) \cup \Sigma_j \cup \big(\Pi_{x_j}^{\nu_j} \cap Q_\rho^{\nu_j}(x_j) \setminus Q_{\rho-2\delta}^{\nu_j}(x_j)\big) \subset \big(\Pi_{x_0}^{\nu_0} \cap Q_\rho^{\nu_0}(x_0) \setminus Q_{\rho-5\delta}^{\nu_0}(x_0)\big) \cup \Sigma_j \cup \big(\Pi_{x_j}^{\nu_j} \cap Q_\rho^{\nu_j}(x_j) \setminus Q_{\rho-2\delta}^{\nu_j}(x_j)\big)$, where Σ_j is the set of points $y \in \partial Q_{\rho-2\delta}^{\nu_j}(x_j)$ such that $(y-x_j)\cdot\nu_j$ and $(y-x_0)\cdot\nu_0$ have opposite sign. Moreover $|[w_j]| = |\zeta_0|$ $\mathcal{H}^{n-1}$-a.e. on $S_{w_j} \cap A_j$ for every $j \ge j_\delta$ and $\sigma_j := \mathcal{H}^{n-1}(\Sigma_j) \to 0$ as $j \to +\infty$. Therefore

$$\int_{S_{w_j}\cap A_j} (1+|[w_j]|)\, d\mathcal{H}^{n-1} \le 2\big(1+|\zeta_0|\big)\big(\rho^{n-1} - (\rho-5\delta)^{n-1} + \sigma_j\big) \le 2(1+|\zeta_0|)\big((n-1)\rho^{n-2}\delta + \sigma_j\big). \tag{A.51}$$

From (A.48), (A.50), and (A.51) it follows that for every $j \ge j_\delta$

$$m_H(u_{x_j,\zeta_0,\nu_j}, Q_\rho^{\nu_j}(x_j)) \le m_H(u_{x_0,\zeta_0,\nu_0}, Q_\rho^{\nu_0}(x_0)) + \varepsilon + 2c_5(1+|\zeta_0|)\big((n-1)\rho^{n-2}\delta + \sigma_j\big).$$

Since $\varepsilon > 0$ and $\delta \in (0, \delta_0/2)$ are arbitrary and $\sigma_j \to 0$, we obtain

$$\limsup_{j\to+\infty} m_H(u_{x_j,\zeta_0,\nu_j}, Q_\rho^{\nu_j}(x_j)) \le m_H(u_{x_0,\zeta_0,\nu_0}, Q_\rho^{\nu_0}(x_0)),$$

which proves the upper semicontinuity of the restriction of $(x,\zeta,\nu) \mapsto m_H^{\rm pc}(u_{x,\zeta_0,\nu}, Q_\rho^\nu(x))$ to $\mathbb{R}^n \times \widehat{\mathbb{S}}_+^{n-1}$. The same proof holds for $\mathbb{R}^n \times \widehat{\mathbb{S}}_-^{n-1}$. □

Lemma A.4 (Monotonicity in ρ)**.** *Let $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m\times n}$, $\zeta \in \mathbb{R}_0^m$, and $\nu \in \mathbb{S}^{n-1}$. Under the assumptions of Lemma A.3 the functions*

$$\rho \mapsto m_H(\ell_\xi, Q_\rho(x)) - c_2(1+|\xi|^p)\rho^n \qquad \rho \mapsto m_H^{1,p}(\ell_\xi, Q_\rho(x)) - c_2(1+|\xi|^p)\rho^n,$$
$$\rho \mapsto m_H(u_{x,\zeta,\nu}, Q_\rho^\nu(x)) - c_5(1+|\zeta|)\rho^{n-1} \qquad \rho \mapsto m_H^{\rm pc}(u_{x,\zeta,\nu}, Q_\rho^\nu(x)) - c_5(1+|\zeta|)\rho^{n-1}$$

are nonincreasing in $(0,+\infty)$.

Proof. Let $\rho_2 > \rho_1 > 0$ and $\varepsilon > 0$ be fixed. By the definition of m_H there exist $u_1 \in X$, with $u_1|_{Q_{\rho_1}(x)} \in SBV^p(Q_{\rho_1}(x), \mathbb{R}^m)$, and $\rho' \in (0,\rho_1)$, such that $u_1 = \ell_\xi$ $\mathcal{L}^n$-a.e. in $Q_{\rho_1}(x) \setminus Q_{\rho'}(x)$ and

$$H(u_1, Q_{\rho_1}(x)) < m_H(\ell_\xi, Q_{\rho_1}(x)) + \varepsilon. \tag{A.52}$$

Let u_2 be defined by

$$u_2 := \begin{cases} u_1 & \text{in } Q_{\rho_1}(x), \\ \ell_\xi & \text{in } \mathbb{R}^n \setminus Q_{\rho_1}(x). \end{cases}$$

Then $u_2 = \ell_\xi$ in a neighbourhood of $\partial Q_{\rho_2}(x)$, hence $m_H(\ell_\xi, Q_{\rho_2}(x)) \le H(u_2, Q_{\rho_2}(x))$. Let us fix $\rho'' \in (\rho', \rho_1)$. Using $(h1)$-$(h3)$, from the previous inequality we obtain

$$m_H(\ell_\xi, Q_{\rho_2}(x)) \le H(u_1, Q_{\rho_1}(x)) + H(\ell_\xi, Q_{\rho_2}(x) \setminus Q_{\rho''}(x)) \le H(u_1, Q_{\rho_1}(x)) + c_2(1+|\xi|^p)(\rho_2^n - (\rho'')^n)$$

Taking the limit as $\rho'' \to \rho_1-$, from (A.52) we obtain

$$m_H(\ell_\xi, Q_{\rho_2}(x)) \le m_H(\ell_\xi, Q_{\rho_1}(x)) + \varepsilon + c_2(1+|\xi|^p)(\rho_2^n - \rho_1^n).$$

Taking the limit as $\varepsilon \to 0+$ we obtain

$$m_H(\ell_\xi, Q_{\rho_2}(x)) - c_5(1+|\xi|^p)\rho_2^n \le m_H(\ell_\xi, Q_{\rho_1}(x)) - c_2(1+|\xi|^p)\rho_1^n,$$

which proves the monotonicity of $\rho \mapsto m_H(\ell_\xi, Q_\rho(x)) - c_2(1+|\xi|^p)\rho^n$. The same proof holds for $\rho \mapsto m_H^{1,p}(\ell_\xi, Q_\rho(x)) - c_2(1+|\xi|^p)\rho^n$.

We now consider $m_H^{\rm pc}$. By definition there exist $v_1 \in X$, with $v_1|_{Q^\nu_{\rho_1}(x)} \in SBV_{\rm pc}(Q^\nu_{\rho_1}(x), \mathbb{R}^m)$, and $\rho' \in (0,\rho_1)$ such that $v_1 = u_{x,\zeta,\nu}$ $\mathcal{L}^n$-a.e. in $Q_{\rho_1}(x) \setminus Q_{\rho'}(x)$ and

$$H(v_1, Q^\nu_{\rho_1}(x)) < m_H^{\rm pc}(u_{x,\zeta,\nu}, Q^\nu_{\rho_1}(x)) + \varepsilon. \tag{A.53}$$

Let v_2 be defined by

$$v_2 := \begin{cases} v_1 & \text{in } Q^\nu_{\rho_1}(x), \\ u_{x,\zeta,\nu} & \text{in } \mathbb{R}^n \setminus Q^\nu_{\rho_1}(x). \end{cases}$$

Then $v_2 = u_{x,\zeta,\nu}$ in a neighbourhood of $\partial Q^\nu_{\rho_2}(x)$, hence $m_H^{\rm pc}(u_{x,\zeta,\nu}, Q^\nu_{\rho_2}(x)) \le H(v_2, Q^\nu_{\rho_2}(x))$. Let us fix $\rho'' \in (\rho', \rho_1)$. Using $(h1)$-$(h3)$, from the previous inequality we obtain

$$\begin{aligned} m_H^{\rm pc}(u_{x,\zeta,\nu}, Q^\nu_{\rho_2}(x_0)) &\le H(v_1, Q^\nu_{\rho_1}(x)) + H(u_{x,\zeta,\nu}, Q^\nu_{\rho_2}(x) \setminus Q^\nu_{\rho''}(x)) \\ &\le H(v_1, Q^\nu_{\rho_1}(x)) + c_5(1+|\zeta|)(\rho_2^{n-1} - (\rho'')^{n-1}). \end{aligned}$$

Taking the limit as $\rho'' \to \rho_1-$, from (A.53) we obtain

$$m_H^{\rm pc}(u_{x,\zeta,\nu}, Q^\nu_{\rho_2}(x)) \le m_H^{\rm pc}(u_{x,\zeta,\nu}, Q^\nu_{\rho_1}(x)) + \varepsilon + c_5(1+|\zeta|)(\rho_2^{n-1} - \rho_1^{n-1}).$$

Taking the limit as $\varepsilon \to 0+$ we obtain

$$m_H^{\rm pc}(u_{x,\zeta,\nu}, Q^\nu_{\rho_2}(x_0)) - c_5(1+|\zeta|)\rho_2^{n-1} \le m_H^{\rm pc}(u_{x,\zeta,\nu}, Q^\nu_{\rho_1}(x_0)) - c_5(1+|\zeta|)\rho_1^{n-1},$$

which proves the monotonicity of $\rho \mapsto m_H^{\rm pc}(u_{x,\zeta,\nu}, Q^\nu_\rho(x)) - c_5(1+|\zeta|)\rho^{n-1}$. The same proof holds for $\rho \mapsto m_H(u_{x,\zeta,\nu}, Q^\nu_\rho(x)) - c_5(1+|\zeta|)\rho^{n-1}$. □

Lemma A.5 (Borel measurability)**.** *Let (f_k) be a sequence in $\mathcal{F}$ and let (g_k) be a sequence in $\mathcal{G}$. Then for every $\varepsilon > 0$ the functions f', f'', $f^{\varepsilon,p}$, and $g^{\varepsilon,p}$ defined in* (3.8), (3.9), (4.24), *and* (4.25) *are Borel measurable. Moreover, for every $\zeta_0 \in \mathbb{R}_0^m$ the functions*

$$(x,\nu) \mapsto g'(x,\zeta_0,\nu) \quad \text{and} \quad (x,\nu) \mapsto g''(x,\zeta_0,\nu)$$

defined in (3.10) *and* (3.11) *are Borel measurable in $\mathbb{R}^n \times \mathbb{S}^{n-1}$.*

Proof. We prove the result only for f', the proof for f'', $f^{\varepsilon,p}$, $g^{\varepsilon,p}$, g', and g'' being analogous. For every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m\times n}$, and $\rho > 0$ we set

$$\psi(x,\xi,\rho) := \liminf_{k\to+\infty} m^{1,p}_{F_k}(\ell_\xi, Q_\rho(x)).$$

By Lemma A.4 for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^{m\times n}$ the function $\rho \mapsto \psi(x,\xi,\rho) - c_2(1+|\xi|^p)\rho^n$ is nonincreasing on $(0,+\infty)$. It follows that

$$\lim_{\rho'\to\rho-} \psi(x,\xi,\rho') \geq \psi(x,\xi,\rho) \geq \lim_{\rho'\to\rho+} \psi(x,\xi,\rho') \quad \text{for every } x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^{m\times n},\ \text{and } \rho > 0.$$

Therefore, if D is a countable dense subset of $(0,+\infty)$, we have

$$\limsup_{\rho\to0+} \frac{1}{\rho^{n-1}}\psi(x,\xi,\rho) = \limsup_{\rho\to0+,\,\rho\in D} \frac{1}{\rho^{n-1}}\psi(x,\xi,\rho),$$

hence

$$f'(x,\xi) = \limsup_{\rho\to0+,\,\rho\in D} \liminf_{k\to+\infty} \frac{1}{\rho^{n-1}} m^{1,p}_{F_k}(\ell_\xi, Q_\rho(x))$$

for every $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^{m\times n}$. The conclusion follows now from Lemma A.3. □

The next lemma provides all properties of the functions f' and f''.

Lemma A.6. *Let (f_k) be a sequence in $\mathcal{F}$ and let f' and f'' be as in* (3.8) *and* (3.9)*. Then f', $f'' \in \mathcal{F}$.*

Proof. Property $(f1)$ for f' and f'' is proved in Lemma A.5. The proof of $(f2)$ for f' and f'' can be easily obtained by adapting the proof of the same property for $f^{\varepsilon,p}$ established in Theorem 4.3. In fact it is enough to deduce from (4.37) that (4.38) holds, with $m_{E^{\varepsilon,p}}$ replaced by $m^{1,p}_{F_k}$. The conclusion then follows from (3.8) and (3.9), passing to the limit first as $k \to +\infty$ and then as $\rho \to 0+$.

We now prove $(f3)$ for f' and f''. Let x, $\xi \in \mathbb{R}^{m\times n}$ be fixed. By $(f3)$ for f_k for any $\rho > 0$ and $u \in W^{1,p}(Q_\rho(x),\mathbb{R}^m)$ with $u = \ell_\xi$ near $\partial Q_\rho(x)$ we have

$$\frac{1}{\rho^n} F_k(u, Q_\rho(x)) \geq \frac{c_1}{\rho^n} \int_{Q_\rho(x)} |\nabla u|^p \, dy \geq c_1 \Big| \frac{1}{\rho^n} \int_{Q_\rho(x)} \nabla u \, dy \Big|^p = c_1 |\xi|^p,$$

where we used Jensen's inequality and the boundary conditions for u. By letting $k \to +\infty$ and then $\rho \to 0+$, the lower bounds for f' and f'' follow from (3.8) and (3.9).

Since f_k satisfies $(f4)$, for any $\rho > 0$ we also have

$$\frac{1}{\rho^n} m^{1,p}_{F_k}(\ell_\xi, Q_\rho(x)) \leq \frac{1}{\rho^n} F_k(\ell_\xi, Q_\rho(x)) \leq c_2(1+|\xi|^p).$$

By letting $k \to +\infty$ and then $\rho \to 0+$ we obtain the upper bounds for f' and f''. □

The next lemma provides all properties of the functions g' and g''.

Lemma A.7. *Let (g_k) be a sequence in $\mathcal{G}$, and let g' and g'' be as in* (3.10) *and* (3.11)*. Then $g', g'' \in \mathcal{G}$.*

Proof. We prove $(g1)$–$(g7)$ only for g', the proof for g'' being similar.

We start by proving $(g2)$. To this end fix $x \in \mathbb{R}^n$, ζ_1, $\zeta_2 \in \mathbb{R}^m_0$, $\nu \in \mathbb{S}^{n-1}$, $k \in \mathbb{N}$, $\rho > 0$. There exists $u_1 \in L^0(\mathbb{R}^n,\mathbb{R}^m)$, with $u_1|_{Q^\nu_\rho(x)} \in SBV_{\rm pc}(Q^\nu_\rho(x),\mathbb{R}^m)$ and $u_1 = u_{x,\zeta_1,\nu}$ in a neighbourhood of $\partial Q^\nu_\rho(x)$, such that

$$G_k(u_1, Q^\nu_\rho(x)) \leq m^{\rm pc}_{G_k}(u_{x,\zeta_1,\nu}, Q^\nu_\rho(x)) + \varepsilon\,\rho^{n-1}. \tag{A.54}$$

Let $E := \{y \in Q^\nu_\rho(x) : u_1(y) = \zeta_1\}$ and let χ_E be its characteristic function. Then $\chi_E \in BV(Q^\nu_\rho(x))$ and $S_{\chi_E} \cap Q^\nu_\rho(x) \subset S_{u_1} \cap Q^\nu_\rho(x)$ (see [3, Theorem 4.23]).

Let $u_2 := u_1 + (\zeta_2 - \zeta_1)\chi_E$. Then $u_2|_{Q^\nu_\rho(x)} \in SBV_{\rm pc}(Q^\nu_\rho(x),\mathbb{R}^m)$ and $u_2 = u_{x,\zeta_2,\nu}$ in a neighbourhood of $\partial Q^\nu_\rho(x)$. Moreover $S_{u_2} \subset S_{u_1}$ and $[u_2] = [u_1]$ $\mathcal{H}^{n-1}$-a.e. on $S_{u_1} \setminus S_{\chi_E}$, while $[u_2] = [u_1] + \zeta_2 - \zeta_1$ $\mathcal{H}^{n-1}$-a.e. on $S_{\chi_E} \cap S_{u_2} \cap Q^\nu_\rho(x)$. By $(g2)$ we have

$$G_k(u_2, Q^\nu_\rho(x)) \leq G_k(u_1, Q^\nu_\rho(x)) + \sigma_2(|\zeta_1 - \zeta_2|)\big(G_k(u_1, Q^\nu_\rho(x)) + G_k(u_2, Q^\nu_\rho(x))\big)$$

hence

$$\big(1 - \sigma_2(|\zeta_1 - \zeta_2|)\big) G_k(u_2, Q^\nu_\rho(x)) \leq \big(1 + \sigma_2(|\zeta_1 - \zeta_2|)\big) G_k(u_1, Q^\nu_\rho(x)).$$

Assume that $\sigma_2(|\zeta_1 - \zeta_2|) < 1$. Then the previous inequality together with (A.54) yield

$$\big(1 - \sigma_2(|\zeta_1 - \zeta_2|)\big) m^{\rm pc}_{G_k}(u_{x,\zeta_2,\nu}, Q^\nu_\rho(x)) \leq \big(1 + \sigma_2(|\zeta_1 - \zeta_2|)\big)\big(m^{\rm pc}_{G_k}(u_{x,\zeta_1,\nu}, Q^\nu_\rho(x)) + \varepsilon\,\rho^{n-1}\big).$$

Dividing by ρ^{n-1} and taking the liminf as $k \to +\infty$, then the limsup as $\rho \to 0^+$, and finally the limit as $\varepsilon \to 0+$ we obtain

$$\big(1 - \sigma_2(|\zeta_1 - \zeta_2|)\big) g'(x,\zeta_2,\nu) \leq \big(1 + \sigma_2(|\zeta_1 - \zeta_2|)\big) g'(x,\zeta_1,\nu)$$

hence

$$g'(x,\zeta_2,\nu) \le g'(x,\zeta_1,\nu) + \sigma_2(|\zeta_1-\zeta_2|)\big(g'(x,\zeta_1,\nu)+g'(x,\zeta_2,\nu)\big). \tag{A.55}$$

Inequality (A.55) is trivial if $\sigma_2(|\zeta_1-\zeta_2|)\ge 1$. Then $(g2)$ can be obtained from (A.55) by interchanging the roles of ζ_1 and ζ_2.

We now observe that the Borel measurability of g' on $\mathbb{R}^n\times\mathbb{R}^m_0\times\mathbb{S}^{n-1}$ follows from Lemma A.5 and from the continuity estimate $(g2)$. This concludes the proof of $(g1)$.

To prove $(g3)$ for g', let us fix $x\in\mathbb{R}^n$, ζ_1, $\zeta_2\in\mathbb{R}^m_0$, with $|\zeta_1|\le|\zeta_2|$, $\nu\in\mathbb{S}^{n-1}$, and a rotation R on $\mathbb{R}^m$ such that $aR\zeta_2=\zeta_1$, where $a:=|\zeta_1|/|\zeta_2|\le 1$. For every k the functions g_k satisfy $(g3)$, thus for every $\rho>0$ and every $u\in SBV_{\rm pc}(Q^\nu_\rho(x),\mathbb{R}^m)$ we have

$$\int_{S_u\cap Q^\nu_\rho(x)} g_k(y,aR[u](y),\nu_u(y))d\mathcal{H}^{n-1}(y) \le c_3\int_{S_u\cap Q^\nu_\rho(x)} g_k(y,[u](y),\nu_u(y))d\mathcal{H}^{n-1}$$

Since $aR\zeta_2=\zeta_1$, this inequaliy implies that

$$m^{\rm pc}_{G_k}(u_{x,\zeta_1,\nu},Q^\nu_\rho(x)) = m^{\rm pc}_{G_k}(u_{x,aR\zeta_2,\nu},Q^\nu_\rho(x)) \le c_3 m^{\rm pc}_{G_k}(u_{x,\zeta_2,\nu},Q^\nu_\rho(x)).$$

Using (3.10) we obtain $g'(x,\zeta_1,\nu)\le c_3\,g'(x,\zeta_2,\nu)$, which proves $(g3)$.

To prove $(g4)$ for g', let us fix $x\in\mathbb{R}^n$, ζ_1, $\zeta_2\in\mathbb{R}^m_0$, with $c_3|\zeta_1|\le|\zeta_2|$, $\nu\in\mathbb{S}^{n-1}$, and a rotation R on $\mathbb{R}^m$ such that $aR\zeta_2=\zeta_1$, where $a:=|\zeta_1|/|\zeta_2|\le 1/c_3\le 1$. For every k the functions g_k satisfy $(g4)$, thus for every $\rho>0$ and every $u\in SBV_{\rm pc}(Q^\nu_\rho(x),\mathbb{R}^m)$ we have

$$\int_{S_u\cap Q^\nu_\rho(x)} g_k(y,aR[u](y),\nu_u(y))d\mathcal{H}^{n-1}(y) \le \int_{S_u\cap Q^\nu_\rho(x)} g_k(y,[u](y),\nu_u(y))d\mathcal{H}^{n-1}$$

Since $aR\zeta_2=\zeta_1$, this inequaliy implies that

$$m^{\rm pc}_{G_k}(u_{x,\zeta_1,\nu},Q^\nu_\rho(x)) = m^{\rm pc}_{G_k}(u_{x,aR\zeta_2,\nu},Q^\nu_\rho(x)) \le m^{\rm pc}_{G_k}(u_{x,\zeta_2,\nu},Q^\nu_\rho(x)).$$

Using (3.10) we obtain $g'(x,\zeta_1,\nu)\le g'(x,\zeta_2,\nu)$, which proves $(g4)$.

To prove $(g5)$ for g', let us fix $x\in\mathbb{R}^n$, $\zeta\in\mathbb{R}^m_0$, $\nu\in\mathbb{S}^{n-1}$, $k\in\mathbb{N}$, and $\rho>0$. Since $(g5)$ holds for g_k, for every $u\in L^0(\mathbb{R}^n,\mathbb{R}^m)$, with $u|_{Q^\nu_\rho(x)}\in SBV_{\rm pc}(Q^\nu_\rho(x),\mathbb{R}^m)$ we have $G_k(u,Q^\nu_\rho(x))\ge c_4\mathcal{H}^{n-1}(S_u)$. If u agrees with $u_{x,\zeta,\nu}$ in a neighbourhood of $\partial Q^\nu_\rho(x)$, each straight line intersecting $Q^\nu_\rho(x)$ and parallel to ν meets S_u (see [3, Theorem 3.108]). This implies that $\mathcal{H}^{n-1}(S_u)\ge\rho^{n-1}$, which, together with the previous estimate, gives $G_k(u,Q^\nu_\rho(x))\ge c_4\rho^{n-1}$. Taking the infimum with respect to u we obtain $m^{\rm pc}_{G_k}(u_{x,\zeta,\nu},Q^\nu_\rho(x))\ge c_4\rho^{n-1}$. By (3.10) this implies $(g5)$ for g'.

On the other hand, appealing to $(g6)$ for g_k we have

$$m^{\rm pc}_{G_k}(u_{x,\zeta,\nu},Q^\nu_\rho(x)) \le G_k(u_{x,\zeta,\nu},Q^\nu_\rho(x)) \le c_5(1+|\zeta|)\rho^{n-1}.$$

Then the latter leads to $(g6)$ for g' by (3.10).

To prove the symmetry condition $(g7)$, we observe that $u_{x,-\zeta,-\nu}=u_{x,\zeta,\nu}-\zeta$ for every $x\in\mathbb{R}^n$, $\zeta\in\mathbb{R}^m_0$, $\nu\in\mathbb{S}^{n-1}$, and $t>0$. Therefore $u\in SBV_{\rm pc}(Q^\nu_\rho(x),\mathbb{R}^m)$ satisfies $u=u_{x,-\zeta,-\nu}$ in a neighbourhood of $\partial Q^\nu_\rho(x)$ if and only if $u=v-\zeta$ for some $v\in SBV_{\rm pc}(Q^\nu_\rho(x),\mathbb{R}^m)$ satisfying $v=u_{x,\zeta,\nu}$ in a neighbourhood of $\partial Q^\nu_\rho(x)$. Since $Q^{-\nu}_\rho(x)=Q^\nu_\rho(x)$ by (k) and (l) in Section 2, it follows that $m^{\rm pc}_{G_k}(u_{x,-\zeta,-\nu},Q^{-\nu}_\rho(x))=m^{\rm pc}_{G_k}(u_{x,\zeta,\nu},Q^\nu_\rho(x))$ for every k. By (3.10) this implies that $g'(x,\zeta,\nu)=g'(x,-\zeta,-\nu)$, which proves $(g7)$ for g'. □

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