Geodesics in infinite dimensional Stiefel and Grassmann manifolds / Géodesiques sur des variétiés de Stiefel et de Grassmann de dimension infinie

Philipp Harms[*], Andrea Mennucci[†]

November 15, 2010

Abstract

Let V be a separable Hilbert space, possibly infinite dimensional. Let $\mathbf{St}(p, V)$ be the Stiefel manifold of orthonormal frames of p vectors in V, and let $\mathbf{Gr}(p, V)$ be the Grassmann manifold of p dimensional subspaces of V. We study the distance and the geodesics in these manifolds, by reducing the matter to the finite dimensional case. We then prove that any two points in those manifolds can be connected by a minimal geodesic, and characterize the cut locus.

RÉSUMÉ. Soit V un espace de Hilbert séparable, éventuellement de dimension infinie. Soient $\mathbf{St}(p, V)$ l'ensemble des systèmes orthonormés de p vecteurs de V, appelé la variété de Stiefel, et $\mathbf{Gr}(p, V)$ l'ensemble des sous-espaces vectoriels de V de dimension p, appelé la variété Grassmannienne. En réduisant le problème en dimension finie, nous montrons que dans ces espaces il existe des géodésiques minimales entre chaque paire de points et nous caractérisons le cut-locus.

1 Introduction

1.1 Stiefel and Grassmann manifolds

Let V be a separable Hilbert space, let p be a positive natural number. We assume that $\dim(V) \geq (2p)$. We are interested in the Stiefel manifold $\mathbf{St}(p, V)$ and the Grassmann manifold $\mathbf{Gr}(p, V)$.

$\mathbf{St}(p, V)$ is the set of orthonormal frames of p vectors in V. Equivalently, we consider $\mathbf{St}(p, V)$ to be the set of all linear isometric immersions of $\mathbb{R}^p$ into V,

$$\mathbf{St}(p, V) = \{x \in L(\mathbb{R}^p, V) : x^\top \circ x = \mathrm{Id}_{\mathbb{R}^p}\}.$$

Here $x^\top \in L(V, \mathbb{R}^p)$ is the transpose with respect to the metrics on V and $\mathbb{R}^p$, i.e.

$$\langle x^\top(v), r \rangle_{\mathbb{R}^p} = \langle v, x(r) \rangle_V \text{ for all } v \in V, r \in \mathbb{R}^p.$$

The induced Riemannian metric on $\mathbf{St}(p, V)$ is $\langle x, y \rangle = \mathrm{tr}(x^\top y)$. $\mathbf{St}(p, V)$ is a smooth embedded submanifold in V^p, and it is a complete Riemannian manifold with the induced metric.

[*]Faculty of Mathematics of the University of Vienna, Austria (philipp.harms@univie.ac.at)
[†]Scuola Normale Superiore, Pisa, Italy (a.mennucci@sns.it)

$\mathbf{Gr}(p, V)$ is the manifold of p-dimensional linear subspaces of V and equals the orbit space $\mathbf{St}(p, V)/O(p)$ with respect to $O(p)$ acting on $\mathbf{St}(p, V)$ by composition from the right.

Our interest in the Stiefel and Grassmann manifolds is due the fact that $\mathbf{St}(2, V)$ with $V = L^2([0, 1])$ is isometric to the space of planar closed curves up to translation and scaling, endowed with a Sobolev metric of order one. The $O(2)$-action on $\mathbf{St}(2, V)$ corresponds to rotations of the curves. Thus $\mathbf{Gr}(2, V)$ with $V = L^2([0, 1])$ is isometric to the space of planar closed curves up to translations, scalings and rotations. See [7], [8], [5] and [6]. Any results that are proven about the Stiefel or Grassmannian immediately carry over to the corresponding space of curves.

When V is finite dimensional, then $\mathbf{St}(p, V)$ is a compact manifold, and by the Hopf–Rinow theorem any two points in $\mathbf{St}(p, V)$ can be connected by a minimal geodesic. Furthermore, the diameter of $\mathbf{St}(p, V)$ is finite.

We will prove the same result about $\mathbf{St}(p, V)$ when V is infinite in Thm 3. This is not an obvious result, as we will recall in the next section. The theorem moreover shows that the geodesic moves in a finite dimensional subspace; this implies that the minimal geodesic can be numerically computed using a finite dimensional algorithm; see Sec. 3.3.3 in [6]. A corollary of Thm. 3 is also that the diameter of $\mathbf{St}(p, V)$ is equal to the diameter of $\mathbf{St}(p, \mathbb{R}^{2p})$. We also characterize the cut locus of of $\mathbf{St}(p, V)$ by the cut loci of $\mathbf{St}(p, W)$, where W is a $(2p)$-dimensional subspace of V. We then extend those results to the Grassmaniann $\mathbf{Gr}(p, V)$.

These results imply that in the spaces of curves mentioned above, any two curves can be connected by a minimizing geodesic. This is important whenever the distance between two curves is calculated by solving the boundary value problem for geodesics.

1.2 Geodesics in infinite dimensional Riemannian manifolds

Consider a smooth connected Riemannian manifold (M, g), modeled on a separable Hilbert space (possibly infinite dimensional); let ∇ be the covariant derivative.

Let $\gamma : [0, 1] \to M$ be a smooth path connecting x to y, where $x, y \in M$. We will say that γ is a **critical geodesic** if γ is a solution to the equation

$$\nabla_{\partial_t} \dot{\gamma} = 0 \quad ;$$

note that such γ is indeed a critical point for the action

$$\int_0^1 g(\dot{\gamma}, \dot{\gamma}) \, \mathrm{d}t \quad .$$

We will say that (M, g) is **geodesically complete** if the solution of the above equation exists for all times. This implies that the exponential map $\exp_x(v)$ is defined for all $(x, v) \in TM$.

We denote by d the **distance** induced by g. $d(x, y)$ is the infimum of the length of all paths connecting x to y. The infimum is computed in the family of all absolutely continuous paths connecting x to y. It coincides with the infimum in the family of all smooth paths connecting x to y [1].

We call γ a **minimal geodesic** if its length is equal to the distance $d(x, y)$. Up to a time reparameterization, a minimal geodesic is smooth and is a critical geodesic. We will always silently assume that minimal geodesics are reparameterized to be critical.

1. Lemma 6.1 in Chap. VIII in [4] can be used to convert any absolutely continuous path to a shorter piecewise smooth path.

It is well known that, for any point $x \in M$ there exists a neighborhood $\mathcal{U}$ of x_0 in M and a neighborhood $\mathcal{V}$ of $(x_0, 0)$ in TM such that

$$(x, v) \mapsto (x, \exp_x(v))$$

is a diffeomorphism from $\mathcal{V}$ to $\mathcal{U}^2$; see Cor. 5.2 and Th. 6.2 in Ch. VIII in [4], where the pair $\mathcal{U}, \mathcal{V}$ is called a **normal neighborhood**.

It is also trivially proved that, if the metric space (M, d) is metrically complete, then (M, g) is geodesically complete; see for example Prop. 6.5 in Ch. VIII in [4].

When M is finite dimensional, by the celebrated Hopf–Rinow theorem, metric completeness of (M, d) is equivalent to geodesic completeness of (M, g), and both imply that any two points $x, y \in M$ can be connected by a minimal geodesic.

When M is infinite dimensional, this result does not hold. Indeed, in [1] Atkin provided an example of an infinite dimensional metrically complete Hilbert smooth manifold M and $x, y \in M$ such that there is no *critical geodesic* connecting x to y. A simpler example, due to Grossman [3] (see also sec. VIII.§6 in [4]), is an infinite dimensional ellipsoid where the south and north pole can be connected by countably many critical geodesics of decreasing length, so that the distance between the poles is not attained by any minimal geodesic.

2 Paper

2.1 Critical geodesics

When $V = \mathbb{R}^n$, the frames in $\mathbf{St}(p, \mathbb{R}^n)$ are represented as $n \times p$ matrices. Geodesics in Stiefel manifolds $\mathbf{St}(p, \mathbb{R}^n)$ are described by a closed–form formula, as demonstrated by Edelman et al. [2]. [2]

Proposition 1 (Critical geodesics in $\mathbf{St}(p, V)$) *Let $\mathbf{St}(p, V)$ be endowed with the induced metric from V^p. Let $\gamma : [0, 1] \to \mathbf{St}(p, V)$ be a path. Then the geodesic equation is $\ddot{\gamma} + \gamma(\dot{\gamma}^\top \dot{\gamma}) = 0$. Solutions to the geodesic equation exist for all time and are given by*

$$(\gamma(t)e^{At}, \dot{\gamma}(t)e^{At}) = (\gamma(0), \dot{\gamma}(0)) \exp t \begin{pmatrix} A & -S \\ Id & A \end{pmatrix} \tag{1}$$

where Id is the $p \times p$ identity matrix and $A = \gamma(0)^\top \dot{\gamma}(0)$, $S = \dot{\gamma}(0)^\top \dot{\gamma}(0)$, that is, A and S are the $p \times p$ matrices of components

$$A_{i,j} = \langle \gamma_i(0), \dot{\gamma}_j(0) \rangle_V \quad , \quad S_{i,j} = \langle \dot{\gamma}_i(0), \dot{\gamma}_j(0) \rangle_V$$

and γ_i are the columns of γ.

The proof and discussion of these results is in Section 2.2.2 in [2].

The solution in (1), while written for $\mathbf{St}(p, \mathbb{R}^n)$, extends to $\mathbf{St}(p, V)$, for a generic separable Hilbert space V. So, in analogy to the finite dimensional space, we will call **columns** the p orthonormal vectors that compose a frame in $\mathbf{St}(p, V)$.

We note this important fact.

2. [2] credits a personal communication by R. A. Lippert for the final closed form formula (1).

Proposition 2 *Equation* (1) *shows that the subspace of* V *spanned by the* $(2p)$ *columns of* $\gamma(t), \dot\gamma(t)$ *remains in the space spanned by the columns of* $\gamma(0), \dot\gamma(0)$ *for all* t.

This means that, if W *is the subspace of* V *spanned by the columns of* $\gamma(0), \dot\gamma(0)$*, then we can formulate the geodesic equation as an equation in* $\mathbf{St}(2, W)$*. Obviously,* $\dim(W) \le 2p$.

This also means that, if γ *is a critical geodesic connecting* x *to* y*, and the space* W *spanned by the columns of* x, y *is* $(2p)$ *dimensional, then, for any* t*, the columns of* $\gamma(t)$ *and of* $\dot\gamma(t)$ *must be contained in* W.

2.2 Minimal geodesics

2.2.1 Minimal geodesics in the Stiefel manifold

Theorem 3 *Let* V *be a Hilbert space. Consider a* $(2p)$ *dimensional Hilbert space* W *and an isometric linear embedding* $i : W \to V$*. Then* i *induces an isometric embedding*

$$i_* : \mathbf{St}(p, W) \to \mathbf{St}(p, V), \qquad x \mapsto i \circ x$$

(here we consider $x \in \mathbf{St}(p, W)$ *to be a linear isometric immersion of* $\mathbb{R}^p$ *into* W*).*

1. $i_*(\mathbf{St}(p, W))$ *is totally geodesic in* $\mathbf{St}(p, V)$.
2. *Let* d_W *be the distance in* $\mathbf{St}(p, W)$ *and similarly* d_V *in* $\mathbf{St}(p, V)$*, then*
$$d_W(x, y) = d_V(i_*(x), i_*(y)) \quad . \tag{2}$$
3. *Let* $x, y \in \mathbf{St}(p, W)$*, and a minimal geodesic* γ *connecting* x *to* y *in* $\mathbf{St}(p, W)$*: then* $i_* \circ \gamma$ *is a minimal geodesic connecting* $i_*(x)$ *to* $i_*(y)$ *in* $\mathbf{St}(p, V)$.
4. *The diameter of* $\mathbf{St}(p, V)$ *is equal to the diameter of* $\mathbf{St}(p, \mathbb{R}^{2p})$.
5. *Any two points* $x, y \in \mathbf{St}(p, V)$ *can be connected by a minimal geodesic* γ*. Any minimal geodesic lies in* $\mathbf{St}(p, U)$*, where* U *is a* $(2p)$ *dimensional subspace of* V *(dependent on* γ*).*
6. *Let* $x, y \in \mathbf{St}(p, V)$*. Then* y *is in the cut locus of* x *if and only if there is a* $(2p)$ *dimensional subspace* W *of* V *and* $\tilde x, \tilde y \in \mathbf{St}(p, W)$ *such that* $x = i_*(\tilde x)$*,* $y = i_*(\tilde y)$ *and* $i_*(y)$ *is in the cut locus of* $i_*(x)$.

Note that point *(5)* in the above theorem implies that minimal geodesics can be numerically computed using a finite dimensional algorithm; see Sec. 3.3.4 in [6].

We will need two lemmas.

Lemma 4 *Given* $x \in \mathbf{St}(p, V)$*, the set of* $y \in \mathbf{St}(p, V)$ *such that the columns of* x, y *are linearly independent is dense in* $\mathbf{St}(p, V)$.

Proof. Let U be the linear space spanned by the columns of x, y; if this space is not $(2p)$ dimensional, then let $r_1, \dots r_k$ be orthonormal vectors that lie in $U^\perp$, with $k = 2p - \dim(U)$. Up to reindexing the columns of y, we can suppose that the columns $x_1, \dots, x_p, y_1, \dots, y_{p-k}$ are linearly independent. For $\varepsilon > 0$ small, we then define

$$\tilde y_i = \begin{cases} y_i & i = 1, \dots p - k \\ \cos(\varepsilon) y_i + \sin(\varepsilon) r_i & i = (p - k + 1), \dots p \end{cases} .$$

It is easy to verify that $\tilde y \in \mathbf{St}(p, V)$ and that the columns of $x, \tilde y$ are linearly independent. □

Lemma 5 *The theorem holds when V is a Hilbert space of finite dimension n with $n > 2p$.*

Proof. We prove point *(1)*. Let us consider the subgroup $G = O(i(W)^\perp)$ of $O(V)$ that keeps $i(W)$ fixed. Then G acts isometrically on $\mathbf{St}(p, V)$ as well, and its fixed point set is $i_*(\mathbf{St}(p, W))$. This proves that $i_*(\mathbf{St}(p, W))$ is totally geodesic in $\mathbf{St}(p, V)$.

To prove point *(2)* we first note that since $\mathbf{St}(p, W)$ is isometrically embedded in $\mathbf{St}(p, V)$, we have

$$d_W(x, y) \geq d_V(i_*(x), i_*(y)) \quad .$$

We will show the inverse inequality only for the case when the columns of x and y are linearly independent. The general case then follows because the set of y such that the columns of x and y are linearly independent is dense in W by lemma 4 and since distances are Lipschitz continuous.

Since V is finite dimensional, $\mathbf{St}(p, V)$ is compact, so by the Hopf–Rinow Theorem $i_*(x)$ and $i_*(y)$ can be connected by a minimizing geodesic in $\mathbf{St}(p, V)$. The columns of $i_*(x)$ and $i_*(y)$ together span the $(2p)$ dimensional space $i(W)$, so we can apply proposition 2. This allows us to write $\gamma = i_* \circ \tilde{\gamma}$ for a path $\tilde{\gamma}$ in $\mathbf{St}(p, W)$. Then

$$d_W(x, y) \leq \mathrm{len}(\tilde{\gamma}) = \mathrm{len}(i_* \circ \tilde{\gamma}) = \mathrm{len}(\gamma) = d_V(i_*(x), i_*(y)) \quad .$$

Point *(3)* follows from point *(2)* and the equality

$$\mathrm{len}(i_* \circ \gamma) = \mathrm{len}(\gamma) = d_W(x, y) = d_V(i_*(x), i_* y) \quad .$$

Point *(4)* follows from point *(3)*. Point *(5)* follows from the Hopf–Rinow theorem and the discussion in Prop. 2.

We now prove point *(6)*. By definition, y is in the cut locus of x if and only if there is a geodesic γ in $\mathbf{St}(p, V)$ with $\gamma(0) = x, \gamma(1) = y$ such that

$$\sup\{t : \mathrm{len}(\gamma|_{[0,t]}) = d_V(\gamma(0), \gamma(t))\} = 1.$$

(Recall that we write d_V for the distance in $\mathbf{St}(p, V)$.) Any such geodesic lies in $\mathbf{St}(p, W)$ for some $(2p)$ dimensional space W. Letting $i : W \to V$ denote the isometric embedding, we can write $\gamma = i_* \circ \tilde{\gamma}$ for a path $\tilde{\gamma}$ in $\mathbf{St}(p, W)$. Then one has by point *(2)* that

$$\sup\{t : \mathrm{len}(\tilde{\gamma}|_{[0,t]}) = d_W(\tilde{\gamma}(0), \tilde{\gamma}(t))\} = 1 \quad .$$

□

We now prove Theorem 3.

Proof. The proof of points *(1), (3), (4), (6)* works as in the finite dimensional case. We will now prove point *(2)*. We have

$$d_W(x, y) \geq d_V(i_*(x), i_*(y)) \quad ,$$

since $\mathbf{St}(p, W)$ is isometrically embedded in $\mathbf{St}(p, V)$. It remains to show the inverse inequality.

Consider a smooth path ξ connecting $i_*(x)$ to $i_*(y)$ in $\mathbf{St}(p, V)$. We can find finitely many points $0 = t_0 < t_1 < \ldots < t_k = 1$ such that $\xi|_{[t_i, t_{i+1}]}$ is contained inside (the manifold part of) a normal neighborhood. So $\xi(t_i), \xi(t_{i+1})$ can be connected by a minimal geodesic. By joining all these minimal geodesics we obtain a piecewise smooth path η, with $\mathrm{len}(\eta) \leq \mathrm{len}(\xi)$. Then by repeated application of proposition 2 there is a finite dimensional subspace $\tilde{W}$ of V that contains the columns of $\eta(t)$ for $t \in [0, 1]$. When necessary we enlarge $\tilde{W}$ such that it also contains $i(W)$.

Now the finite dimensional version of this Lemma allows us to compare $\mathbf{St}(p,W)$ to $\mathbf{St}(p,\tilde{W})$, and we get:

$$d_W(x,y) = d_{\tilde{W}}(i_*(x), i_*(y)) \leq \mathrm{len}(\eta) \leq \mathrm{len}(\xi) \quad .$$

Since this holds for arbitrary paths ξ connecting $i_*(x)$ to $i_*(y)$ in $\mathbf{St}(p,V)$, we get

$$d_W(x,y) \leq d_V(i_*(x), i_*(y)) \quad .$$

Point *(5)* now follows by choosing any linear subspace W containing the columns of x, y. □

2.2.2 Minimal geodesics in the Grassmann manifold

The result on existence of minimal geodesics in the Stiefel manifold carries over to the Grassmannian.

Theorem 6 *Thm. 3 remains valid when Stiefels are replaced by Grassmannians. Most importantly, for any two points $x, y \in \mathbf{Gr}(p,V)$, there is a minimal geodesic γ connecting x to y. The same holds for the Grassmannian $\mathbf{Gr}_+(p,V)$ of oriented p spaces.*

We need a Lemma.

Lemma 7 (Existence of horizontal paths) *For any path $x : [0,1] \to \mathbf{St}(p,V)$ there is a path $g : [0,1] \to O(p)$ such that the path $x(t) \circ g(t)$ is horizontal, i.e. normal to the $O(p)$-orbits in $\mathbf{St}(p,V)$.*

Proof. We will look at the Stiefel manifold as

$$\mathbf{St}(p,V) = \{x \in L(\mathbb{R}^p, V) : x^\top \circ x = \mathrm{Id}_{\mathbb{R}^p}\}.$$

Here $x^\top \in L(V, \mathbb{R}^p)$ is the transpose with respect to the metrics on V and $\mathbb{R}^p$, i.e.

$$\langle x^\top(v), r\rangle_{\mathbb{R}^p} = \langle v, x(r)\rangle_V \text{ for all } v \in V, r \in \mathbb{R}^p.$$

Then the tangent space to the Stiefel at a point x is

$$T_x\mathbf{St}(p,V) = \{y \in L(\mathbb{R}^p, V) : x^\top \circ y + y^\top \circ x = 0\}.$$

$O(p)$ acts on $\mathbf{St}(p,V)$ by composition from the right. The Lie algebra of $O(p)$ is

$$\mathfrak{o}(p) = \{z \in L(\mathbb{R}^p, \mathbb{R}^p) : z^\top + z = 0.\}$$

Then the tangent space at x to the $O(p)$-orbit through x is

$$T_x(x \circ O(p)) = \{x \circ z : z \in \mathfrak{o}(p)\}.$$

The metric on the Stiefel is given by $\mathrm{tr}(x^\top \circ y)$. Tangent vectors that are orthogonal to the $O(p)$-orbits are called horizontal. They form a linear subspace of the tangent space which is given by

$$\begin{aligned}\Big(T_x(x.O(p))\Big)^\perp &= \{y \in T_x(x.O(p))) : \forall z \in \mathfrak{o}(p) : \mathrm{tr}(y^\top xz) = 0\} \\ &= \{y \in L(\mathbb{R}^p, \mathrm{range}(x)^\perp)\}\end{aligned}$$

The path $x(t) \circ g(t)$ is horizontal if and only if

$$\partial_t(x(t) \circ g(t)) = \dot{x}(t) \circ g(t) + x(t) \circ \dot{g}(t) \in L(\mathbb{R}^p, \operatorname{range}(x(t))^\perp).$$

This can be achieved by letting g be the solution to the ODE

$$\partial_t g(t) = -x(t)^\top \circ \dot{x}(t) \circ g(t). \qquad \square$$

Note that the length of $x(t) \circ g(t)$ is smaller than or equal to the length of $x(t)$, with equality if and only if $x(t)$ is already a horizontal path.

We are now able to prove Thm. 6.

Proof. $\mathbf{St}(p, V)$ is a principal fiber bundle with structure group $O(p)$ over $\mathbf{Gr}(p, V) = \mathbf{St}(p, V)/O(p)$. We prove the existence of minimizing geodesics connecting any two points in $\mathbf{Gr}(p, V)$. Take any point $\tilde{x} \in \mathbf{St}(p, V)$ in the fiber over x. The fiber over y is compact since $O(p)$ is compact. Therefore $d(x, \cdot)$ attains a minimum at some point $\tilde{y}$ in the fiber over y. By theorem 3 there is a minimal geodesic connecting $\tilde{x}$ to $\tilde{y}$. This geodesic is horizontal since otherwise it could be made shorter by making it horizontal. (We use lemma 7 here.) By the theory of Riemannian submersions it projects to a minimal geodesic in $\mathbf{Gr}(p, V)$.

The remaining statements simply follow from Thm. 3 by going to the quotient with respect to the $O(p)$-action. For the case of $\mathbf{Gr}_+(p, V)$, we use the group $SO(p)$ instead of $O(p)$. $\square$

Acknowledgements

This research was founded by `SNS09MENNB` of the Scuola Normale Superiore.

Contents

References

[1] C. J. Atkin. The Hopf-Rinow theorem is false in infinite dimensions. *Bull. London Math. Soc.*, 7(3):261–266, 1975. *DOI: 10.1112/blms/7.3.261.*

[2] A. Edelman, T. Arias, and S. Smith. The geometry of algorithms with orthogonality constraints. *SIAM J Matrix Analy Appl*, 20:303–353, 1998. *DOI: 10.1137/S0895479895290954.* arXiv:physics/9806030v1 (1998).

[3] Nathaniel Grossman. Hilbert manifolds without epiconjugate points. *Proc. Amer. Math. Soc.*, 16:1365–1371, 1965. ISSN 0002-9939. *DOI: 10.2307/2035933.*

[4] Serge Lang. *Fundamentals of differential geometry.* Springer–Verlag, 1999.

[5] Ganesh Sundaramoorthi, Andrea Mennucci, Stefano Soatto, and Anthony Yezzi. Tracking deforming objects by filtering and prediction in the space of curves. In *Conference on Decision and Control*, pages 2395 – 2401, 2009. ISBN 978-1-4244-3871-6. *DOI: 10.1109/CDC.2009.5400786.*

[6] Ganesh Sundaramoorthi, Andrea Mennucci, Stefano Soatto, and Anthony Yezzi. A new geometric metric in the space of curves, and applications to tracking deforming objects by prediction and filtering. *SIAM Journal on Imaging Sciences*, 2010.

[7] Laurent Younes. Computable elastic distances between shapes. *SIAM Journal of Applied Mathematics*, 58(2):565–586, 1998. *DOI: 10.1137/S0036139995287685.*

[8] Laurent Younes, Peter W. Michor, Jayant Shah, and David Mumford. A metric on shape space with explicit geodesics. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 19(1):25–57, 2008. ISSN 1120-6330. *DOI: 10.4171/RLM/506.*