A DISCRETE COAREA-TYPE FORMULA FOR THE MUMFORD-SHAH FUNCTIONAL IN DIMENSION ONE

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Abstract. We study the convex lift of Mumford-Shah type functionals in the space of rectifiable currents and we prove a generalized coarea formula in dimension one, for finite linear combinations of SBV graphs. We use this result to prove the equivalence between the minimum problems for the Mumford-Shah functional and the lifted one and, as a consequence, we obtain a weak existence result for calibrations in one dimension.

1. Introduction

The Mumford-Shah functional is one of the most important variational model for image segmentation. It was introduced in the late 80's by Mumford and Shah ([20],[19]) and it can be defined in its general form as

$$J(u,K)=\int_{\Omega\setminus K}|\nabla u|^2\,dx+\beta\mathcal{H}^{n-1}(K)+\alpha\int_{\Omega\setminus K}|u-g|^2\,dx, \tag{1}$$

where $\Omega\in\mathbb{R}^n$ is open, $K\subset\Omega$ is closed and such that $\mathcal{H}^{n-1}(K)<\infty$, $g\in L^\infty(\Omega)$, $u\in W^{1,2}(\Omega\setminus K)$ and β and α are tuning parameters.
The idea of the model is that given g representing the level of gray of an image, it is possible to get a "smoother" version of it, "close" to the starting one in the L^2 norm, by finding a minimizer of (1). The gain of smoothness for the minimizers comes from penalizing the oscillation of the competitors (i.e. the Dirichlet energy) and the length of the contour, in order to avoid fractal behaviour of the boundary of the processed image.
The existence of minimizers for (1) was proved in [15] introducing a weak formulation obtained considering $u\in SBV(\Omega)$ and replacing the set K with S_u, i.e. the singular set of u:

$$F(u)=\int_{\Omega}|\nabla u|^2\,dx+\beta\mathcal{H}^{n-1}(S_u)+\alpha\int_{\Omega}|u-g|^2\,dx. \tag{2}$$

It is worth to remark that when $\alpha=0$ and $\beta=1$, F is called homogeneous Mumford-Shah functional.
In the following years there have been a huge effort in understanding the regularity properties of the functional defined above. We can cite some relevant papers in this direction like [3], [4], [5], [10]. However, despite all the effort, the main conjecture proposed by Mumford and Shah in their seminal paper still remains open in its full generality.

Conjecture 1.1 (Mumford, Shah)**.** *Let (u,K) be a pair minimizing* (2)*. Then K is locally union of finitely many $C^{1,1}$ embedded arcs.*

As pointed out for the first time in [5], a blow up limit of appropriate sequences of minimizers of (2) is a local minimizer of the homogeneous Mumford-Shah functional; for this reason the characterization of these minimizers is directly related to the solution of the conjecture stated above. For example it is known that harmonic functions are local minimizers of (2) (for $\alpha=0$ and $\beta=1$) in small domains and that the same result holds for step functions and triple junctions ([1]). Moreover the main achievement in this direction is contained in [6] and it answers affirmatively to a conjecture proposed by De Giorgi in [14]:

$$u(\rho,\theta)=\sqrt{\frac{2\rho}{\pi}}\sin\left(\frac{\theta}{2}\right)\qquad\rho>0,\ -\pi<\theta<\pi \tag{3}$$

is a global minimizer of the homogeneous Mumford-Shah functional. (3) is usually called crack-Tip.
In [1] Alberti, Bouchitté and Dal Maso introduced the notion of calibration for the Mumford-Shah functional that resembles closely the classical theory for minimal surfaces by Harvey and Lawson ([16]). With this technique in [1] they were able to prove the minimality of some candidates for the homogeneous Mumford-Shah functional like the triple junction or reproving the minimality of harmonic functions in a very elegant way. However it remains open the problem of finding a calibration for the crack-tip and for general minimums in higher dimensions. It is therefore a relevant issue to understand if, given u a minimum for the Mumford-Shah functional, then there exists a calibration for u.
This is the question we are going to address in this paper. Existence of calibration is a common issue also in the field of minimal surfaces and also there it is not completely solved. One can refer to the work of Federer [12] for the classical results in this theory.
As for the Mumford-Shah the main result in this direction was obtained by Chambolle in [8]. He proved the existence of a calibration in dimension one in a weak asymptotic sense using the following representation formula introduced in [1]:

$$F(u) = \sup_{\phi\in K} \int_{\Gamma_u} \langle \phi, \nu_{\Gamma_u}\rangle \, d\mathcal{H}^n = \sup_{\phi\in K} \int_{\Omega\times\mathbb{R}} \langle \phi, D\mathbf{1}_{\{u>t\}}\rangle,$$

where K is the set of Borel vector fields $\phi : \Omega\times\mathbb{R}\to\mathbb{R}^{n+1}$ such that

$$\begin{cases} \phi^t(x,t) \geq \dfrac{|\phi^x(x,t)|^2}{4} - \beta(t-g)^2 & \forall x, t \\ \left|\displaystyle\int_{t_1}^{t_2} \phi^x(x,s)\, ds\right| \leq \alpha & \forall x, t_1, t_2. \end{cases} \tag{4}$$

More precisely this representation formula is the particular case of a general one for "local" functionals in BV presented by Bouchitté in [7].
In particular one can lift F to higher dimension to obtain a convex functional $\mathcal{F}$ defined as

$$\mathcal{F}(w) = \sup_{\phi\in K\cap C_0} \int_{\Omega\times\mathbb{R}} \langle \phi, Dw\rangle$$

for $w \in SBV(\Omega\times\mathbb{R})$ decreasing in the last variable. If one is able to prove that given u a minimizer of F, then $\mathbf{1}_{\{u>t\}}$ is a minimizer of $\mathcal{F}$, then this would imply the existence of a calibration in a weak asymptotic sense by argument of convex analysis. Moreover another important consequence is that one can compute the minimum F using the functional $\mathcal{F}$ that, being convex, allows for a efficient gradient descent method ([21]). Chambolle in [8] was able to prove these facts in dimension one and he pointed out that the same results could be obtained building up a coarea-type formula for the previous functional generalising the classical coarea formula for functionals ([9], [22])

$$\mathcal{F}(w) = \int_0^1 \mathcal{F}(\mathbf{1}_{\{w(x,t)>s\}})\, ds,$$

that is false for $\mathcal{F}$ as the example below shows:

$$u_1(x) = \begin{cases} 0 & \text{if } x \leq 1/2 \\ x & \text{if } x > 1/2, \end{cases} \qquad u_2(x) = \begin{cases} x & \text{if } x \leq 1/2 \\ 1 & \text{if } x > 1/2 \end{cases}$$

and $w(x,t) = (1/2)\mathbf{1}_{\{u_1(x)>t\}} + (1/2)\mathbf{1}_{\{u_2(x)>t\}}$.

In this article we use an alternative representation of the Mumford-Shah functional by rectifiable currents of the type

$$G(T) = \sup_{\phi\in K} \int_{\mathcal{M}} \theta\langle \phi, \nu_T\rangle d\mathcal{H}^n,$$

where $T = (\mathcal{M}, \xi, \theta)$ is a rectifiable current and ν_T is the normal to $\mathcal{M}$, and we start to exploit the validity of a general coarea-type formula for the functional G. In Section 3 we study the structure of the functional and we prove the following decomposition for a finite linear combination of graphs in dimension one.

Theorem (Coarea-type formula). *Let I be open interval. Given $T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i}$ with $u_i \in SBV(I)$ and $\lambda_i > 0$ such that $|\bigcup S_{u_i}| < +\infty$ there exists $k' \in \mathbb{N}$, $\{\mu_i\}_{i=1\dots k'} > 0$ and $\{w_i\}_{i=1\dots k'} \subset SBV(I)$ such that $T = \sum_{i=1}^{k'} \mu_i \Gamma_{w_i}$ and*

$$G(T) = \sum_{i=1}^{k'} \mu_i G(\Gamma_{w_i}).$$

In the next sections we will often refer to it by the denomination *discrete coarea formula*, stressing that it holds for finite linear combination of SBV graphs.
The immediate consequence of this result is the following theorem that links the minimizers of (2) with the minimizers of G

Theorem. *Given $u \in SBV(I)$ a minimizer of the Mumford-Shah functional, Γ_u (i.e. the graph associated to u) is a minimizer of G among all the linear combinations of graphs of the form $T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i}$ with $\partial \Gamma_u = \partial T$.*

In Section 4, we use this theorem to prove the existence of calibration in a weak sense (see Definition 4.3) as a consequence of the Hahn-Banach theorem. The general idea of this proof follows closely Federer's approach to calibrations for minimal surfaces in [12] and it suggests that, at least in dimension one, it would be possible to produce the analogue result and to extract an L^∞ vector field playing the role of a calibration.
It is worth to notice that the coarea-type formula presented in this paper relies on the one dimensional structure of the domain. In particular in Proposition 3.21 it is necessary that the singular points of an SBV function disconnect the domain; this is clearly peculiar of the dimension one, but it is likely that similar decomposition can be found in higher dimension and similar results could be obtained.
Moreover, even if all the proof of this paper are carried on for the functional (2) the results can be extended with minor modifications to more general Mumford-Shah type functionals. We refer to Remark 3.1 for further details in this direction.

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2. Preliminaries

Let Ω be an open, bounded, regular set of $\mathbb{R}^n$. Given $g \in L^\infty(\Omega)$ we consider the Mumford-Shah functional as stated in the introduction

$$\mathfrak{F}(u) = \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(S_u) + \int_\Omega |u - g|^2 \, dx \tag{5}$$

and the homogeneous version

$$F(u) = \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(S_u), \tag{6}$$

where $u \in SBV(\Omega)$ and S_u is the singular set of u. We refer to [13] for the basic properties of BV and SBV functions and to [11] for a comprehensive treatise on the Mumford-Shah functional.

Throughout this paper we consider the following notions of minimizers:

Definition 2.1 (Minimizer of $\mathfrak{F}$). *Given $g \in L^\infty(\Omega)$ we say that $u \in SBV(\Omega)$ is a minimizer of $\mathfrak{F}$ if $\mathfrak{F}(u) \le \mathfrak{F}(v)$ for all $v \in SBV(\Omega)$.*

Definition 2.2 (Dirichlet minimizers)**.** *We say that $u \in SBV(\Omega)$ is a Dirichlet minimizer of F (resp. $\mathfrak{F}$) if*

$$F(u) \leq F(v) \quad \forall v \in SBV(\Omega) \ s.t. \ v_{\partial\Omega} = u_{\partial\Omega}.$$
$$(resp. \ \mathfrak{F}(u) \leq \mathfrak{F}(v) \quad \forall v \in SBV(\Omega) \ s.t. \ v_{\partial\Omega} = u_{\partial\Omega}),$$

where we denote by $u_{\partial\Omega}$ and $v_{\partial\Omega}$ the trace of u and v on $\partial\Omega$.

We remark that the notion of Dirichlet minimizer of F is classically known as *local minimizer* in the literature ([11]).
Proving that a function $u \in SBV(\Omega)$ is a Dirichlet minimizer is not an easy question (in general); this is one of the main reasons why a calibration notion resembling the one of minimal surfaces by Harvey and Lawson ([16]) has turned out to be really useful. It was proposed by Alberti, Bouchittè and Dal Maso in [1] and developed among the others in [18] and [17]. In this next section we will give a brief introduction on this topic.

2.1. Calibration for the Mumford-Shah Functional.

Given $H : L^1(\Omega) \to \mathbb{R}$ let us define an abstract calibration in the following way:

Definition 2.3 (Abstract calibration)**.** *Given $u \in L^1(\Omega)$, an abstract calibration for u is a functional $G : L^1(\Omega) \to \mathbb{R}$ such that*

$$(i) \quad H(u) = G(u), \qquad (ii) \quad H(v) \geq G(v), \qquad (iii) \quad G(u) = G(v) \tag{7}$$

for all $v \in L^1(\Omega)$ such that $\{v \neq u\} \subset\subset \Omega$.

Remark 2.4. *If G is a calibration for u, then u is a Dirichlet minimizer in Ω for H, indeed*

$$H(u) \overset{(i)}{=} G(u) \overset{(iii)}{=} G(v) \overset{(ii)}{\leq} H(v)$$

for all $v \in L^1(\Omega)$ such that $\{v \neq u\} \subset\subset \Omega$.

In [1] Alberti, Bouchitté and Dal Maso introduced a stronger notion of calibration for the Mumford-Shah functional. Given $v \in SBV(\Omega)$, we denote by $v^-(x)$ and $v^+(x)$ the lower and the upper traces of v. Moreover let Γ_v be the extended graph of v defined as

$$\Gamma_v = \{(x,t) \in \Omega \times \mathbb{R} : v^-(x) \leq t \leq v^+(x)\}.$$

For standard theory on BV functions ([13]) Γ_v is rectifiable and then it admits a generalized normal that we are going to denote with ν_{Γ_v}.
The calibration proposed in [1] has the following form:

$$G(v) = \int_{\Gamma_v} \langle \phi, \nu_{\Gamma_v} \rangle \, d\mathcal{H}^n,$$

where $\phi : \Omega \times \mathbb{R} \to \mathbb{R}^{n+1}$ is a vector field to be determined. The regularity asked on ϕ is the least that guarantees the existence of a divergence theorem on $\Omega \times \mathbb{R}$. To be more precise we refer to [1] and for reader convenience we propose the definition of *approximately regular* vector field:

Definition 2.5 (Approximately regular vector field)**.** *Given $A \subset \mathbb{R}^{n+1}$, a vectorfield $\phi : A \to \mathbb{R}^{n+1}$ is approximately regular if it is bounded and for every Lipschitz hypersurface M in $\mathbb{R}^{n+1}$ there holds*

$$\lim_{r \to 0} \fint_{B_r(x_0) \cap A} |(\phi(x) - \phi(x_0)) \cdot \nu_M(x_0)| \, dx = 0 \tag{8}$$

for $\mathcal{H}^n$-a.e. $x_0 \in M \cap A$.

Comparing the functional G with F, it is possible to find sufficient conditions on ϕ such that G satisfies properties (i), (ii) and (iii) with respect to F for a given $u \in SBV(\Omega)$. Then the vector field satisfying these properties is called calibration for u.

Definition 2.6 (Calibration for the Mumford-Shah Functional). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u \in SBV(\Omega)$. Given $\phi = (\phi^x, \phi^t) : \Omega \times \mathbb{R} \to \mathbb{R}^{n+1}$ an approximately regular vector field, we say that it is a calibration for u if it is divergence free and*

a) $\phi^t(x,t) \geq \dfrac{|\phi^x(x,t)|^2}{4}$ *for $\mathscr{L}^n$-a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$,*

b) $\left| \displaystyle\int_{t_1}^{t_2} \phi^x(x,t)\, dt \right| \leq 1$ *for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$ and for all $t_1, t_2 \in \mathbb{R}$,*

c) $\phi^x(x, u(x)) = 2\nabla u(x), \qquad \phi^t(x, u(x)) = |\nabla u(x)|^2$ *for $\mathscr{L}^n$-a.e. $x \in \Omega$,*

d) $\displaystyle\int_{u^-(x)}^{u^+(x)} \phi^x(x,t)\, dt = \nu_u(x)$ *for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$,*

where ν_u is the approximate normal of S_u.

As properties $(a), (b), (c), (d)$ imply $(i), (ii)$ and (iii) for G we have the following theorem:

Theorem 2.7. *Given $u \in SBV(\Omega)$, suppose that there exists $\phi : \Omega \times \mathbb{R} \to \mathbb{R}^{n+1}$ a calibration for u. Then u is a Dirichlet minimizer in Ω of the homogeneous Mumford-Shah functional* (6)

In an analogous way a similar notion can be introduced in order to study minimizers of $\mathfrak{F}$. It is enough to replace conditions (a) and (c) with

a') $\phi^t(x,t) \geq \dfrac{|\phi^x(x,t)|^2}{4} - (t-g)^2$ for $\mathscr{L}^n$-a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$,

c') $\phi^x(x, u(x)) = 2\nabla u(x), \quad \phi^t(x, u(x)) = |\nabla u(x)|^2 - (u-g)^2$ for $\mathscr{L}^n$-a.e. $x \in \Omega$.

Theorem 2.8. *Given $u \in SBV(\Omega)$, suppose that there exists $\phi : \Omega \times \mathbb{R} \to \mathbb{R}^{n+1}$ a calibration for u with (a) and (c) replaced with (a') and (c'). Then u is a Dirichlet minimizer in Ω of the Mumford-Shah functional* (5).

As a consequence, in [1], the authors proposed the following alternative formulation of the Mumford-Shah functional

$$F(u) = \max_{\phi \in K} \int_{\Gamma_u} \langle \phi, \nu_{\Gamma_u} \rangle \, d\mathcal{H}^n = \max_{\phi \in K} \int_{\Omega \times \mathbb{R}} \langle \phi, D\mathbf{1}_{\{u>t\}} \rangle, \tag{9}$$

$$\mathfrak{F}(u) = \max_{\phi \in K'} \int_{\Gamma_u} \langle \phi, \nu_{\Gamma_u} \rangle \, d\mathcal{H}^n = \max_{\phi \in K'} \int_{\Omega \times \mathbb{R}} \langle \phi, D\mathbf{1}_{\{u>t\}} \rangle, \tag{10}$$

where

$$K = \{\phi : \Omega \times \mathbb{R} \to \mathbb{R}^{n+1},\ \textit{Borel} : (a) \text{ and } (b) \text{ hold pointwise}\} \tag{11}$$

and

$$K' = \{\phi : \Omega \times \mathbb{R} \to \mathbb{R}^{n+1},\ \textit{Borel} : (a') \text{ and } (b) \text{ hold pointwise}\}. \tag{12}$$

Remark 2.9. *The previous representation formula is the starting point for the proof of existence of calibration in dimension one, due to Chambolle* [8]. *In particular one can introduce the following convex functional also called* lift *of F*

$$\mathcal{F}_K(w) = \sup_{\phi \in K \cap C_0(\Omega \times \mathbb{R}, \mathbb{R}^{n+1})} \int_{\Omega \times \mathbb{R}} \phi \cdot Dw,$$

with $w : I \times \mathbb{R} \to [0,1]$ decreasing in the second variable and of bounded variation. In [8] *Chambolle proves that if $u \in SBV(I)$ is a minimizer of the Mumford-Shah functional then $\mathbf{1}_{\{u(x)>t\}}$ is a minimizer of $\mathcal{F}_K$. Then by Hahn-Banach theorem it is possible to prove the existence of calibrations in a weak asymptotic sense.*

Remark 2.10. *It is interesting to notice that one can prove the same result in higher dimension if $\mathcal{F}_K$ satisfies a generalized coarea formula of the form*

$$\mathcal{F}_K(w) = \int_0^1 \mathcal{F}_K(\mathbf{1}_{\{w(x,t)>s\}})\, ds. \tag{13}$$

Unfortunately this is false even in dimension one. In fact it is enough to consider

$$u_1(x) = \begin{cases} 0 & \text{if } x \le 1/2 \\ x & \text{if } x > 1/2, \end{cases} \qquad u_2(x) = \begin{cases} x & \text{if } x \le 1/2 \\ 1 & \text{if } x > 1/2 \end{cases}$$

and $w(x,t) = (1/2)\mathbf{1}_{\{u_1(x)>t\}} + (1/2)\mathbf{1}_{\{u_2(x)>t\}}$ *to see that formula* (13) *does not hold.*

2.2. **A lifting of the Mumford-Shah functional in the space of rectifiable currents.** In this section we introduce a lifted functional that takes values in $\mathcal{R}_n(\Omega \times \mathbb{R})$ the n-dimensional rectifiable currents with real multiplicity. We briefly recall the basic theory of currents and we refer the reader to [13] for a more detailed overview.
Let U be an open subset of $\mathbb{R}^N$. A k-dimensional current on U is a linear continuous (see [13]) functional on the space of k-forms $\Lambda^k(U)$ with coefficients in $C_c^\infty(U)$.
In particular we define the space $\mathcal{R}_k(U)$ of k-dimensional rectifiable currents with real multiplicity as the triple $(\mathcal{M}, \theta, \xi)$ where $\mathcal{M} \subset U$ is a k-rectifiable set, $\theta : \mathcal{M} \to \mathbb{R}_+$ is a function called multiplicity and ξ is the k-vector giving an orientation of $\mathcal{M}$. We define the current $(\mathcal{M}, \theta, \xi)$ by its action on a k-diffential form $\omega \in \Lambda^k(U)$ in the following way:

$$(\mathcal{M}, \theta, \xi)(\omega) = \int_{\mathcal{M}} \langle \omega, \xi \rangle \theta \, d\mathcal{H}^k$$

where $\langle \cdot, \cdot \rangle$ denote the duality product between vectors and covectors. Moreover given $T = (\mathcal{M}, \theta, \xi)$ we define the total variation measure associate to T as

$$\|T\|(A) = \int_{\mathcal{M} \cap A} \theta \, d\mathcal{H}^k$$

for every $A \subset U$ measurable. We call $\|T\|(U) = M(T)$ the mass of T.
We define the restriction of a rectifiable current $T = (\mathcal{M}, \theta, \xi)$ on a measurable set as

$$T \llcorner A(\omega) = \int_{\mathcal{M} \cap A} \langle \omega, \xi \rangle \theta \, d\mathcal{H}^k$$

for every $A \subset U$ measurable. In addition given $\alpha \in \Lambda^h(U)$ with $h \le k$, we define the restriction of $T \in R_k(U)$ to α as the $(k-h)$-dimensional current $T \llcorner \alpha$ defined as

$$T \llcorner \alpha(\omega) = T(\alpha \wedge \omega)$$

for every $\omega \in \Lambda^{k-h}(U)$.
Moreover let $I^k(U)$ be the subset of $\mathcal{R}^k(U)$ such that the multiplicity θ is integer valued. Each element of $I^k(U)$ is called k-dimensional integer rectifiable current.

We introduce the lifting of the Mumford-Shah functional on the space of rectifiable currents for the functionals $\mathfrak{F}$ and F.

Definition 2.11 (Lifting to the space of rectifiable current)**.** *Given* $T = (\mathcal{M}, \theta, \xi) \in \mathcal{R}_n(\Omega \times \mathbb{R})$ *we define*

$$G_K(T) := \sup_{\phi \in K} \int_{\mathcal{M}} \langle \phi, \star(-\xi) \rangle d\|T\| = \sup_{\phi \in K} \int_{\mathcal{M}} \theta \langle \phi, \nu_T \rangle d\mathcal{H}^n \tag{14}$$

and

$$G_{K'}(T) := \sup_{\phi \in K'} \int_{\mathcal{M}} \langle \phi, \star(-\xi) \rangle d\|T\| = \sup_{\phi \in K} \int_{\mathcal{M}} \theta \langle \phi, \nu_T \rangle d\mathcal{H}^n \tag{15}$$

where $\nu_T := -(\star\xi)$, $\star$ *is the Hodge star and* K *and* K' *are defined as in* (11) *and in* (12).

Proposition 2.12. *The functionals G_K and $G_{K'}$ satisfy the following properties:*

(*i*) *They are convex on $\mathcal{R}_n(\Omega \times \mathbb{R})$.*
(*ii*) *They are lower semicontinous with respect to the mass bounded convergence.*
(*iii*) *Given $v \in SBV(\Omega)$, $G_K(\Gamma_v) = F(v)$ and $G_{K'}(\Gamma_v) = \mathfrak{F}(v)$.*

Proof. Statement (*i*) follows from the definition and (*iii*) is a consequence of the representation formulas (9) and (10). Moreover (*ii*) can be proved with an easy modification of the argument in [13] sec. 3.3.1.

□

3. A discrete coarea-type formula for the Mumford-Shah functional in dimension one

We restrict our analysis to the case $n = 1$. We can also assume $\Omega = I$ an open interval and consider the Mumford-Shah functional in its general form

$$F(u) := \int_I |u'(x)|^2\, dx + \beta \int_I |u - g|^2\, dx + \alpha \mathcal{H}^0(S_u) \tag{16}$$

where $\alpha > 0$, $\beta \geq 0$, $g \in L^\infty(I)$ and $u \in SBV(I)$. Notice that when $\beta = 0$ and $\alpha = 1$, F is the homogeneous version of the Mumford-Shah functional as defined in (6).

Remark 3.1. *Even if we restrict our attention to* (16) *it is important to remark that the results of this section and of the following one hold for a more general class of functionals with minor modification of the proofs. Functionals of the form*

$$W(u) = \int_I f(u'(x), u(x), x)\, dx + \sum_{x \in S_u} \psi(x, u^+(x), u^-(x))$$

with suitable hypothesis on f and ψ necessary to ensure the lower semicontinuity of W and the existence of minimizers can be treated by this theory. We refer to [2] *for the precise assumptions and we stress the fact that in our setting f need not to be assumed more regular as in* [8]*. For example in the case of the Mumford-Shah functional g can be taken in L^∞ without affecting the proof, while in* [8] *the function g needs to have a l.s.c. and a u.s.c. representatives in L^∞.*

If we consider the functional F as defined in (16), its convex lift defined in (14) on $\mathcal{R}_1(I \times \mathbb{R})$ reads

$$G(T) = \sup_{\phi \in K} \int_{\mathcal{M}} \theta \langle \phi, \nu_T \rangle d\mathcal{H}^1 \tag{17}$$

for every $T = (\mathcal{M}, \theta, \xi)$.
In particular K is the set of $\phi : I \times \mathbb{R} \to \mathbb{R}^2$, Borel, such that

I) $\phi^t(x,t) \geq \dfrac{|\phi^x(x,t)|^2}{4} - \beta(t-g)^2$ for all $x \in I$ and for all $t \in \mathbb{R}$,

II) $\left| \displaystyle\int_{t_1}^{t_2} \phi^x(x,t)\, dt \right| \leq \alpha$ for all $x \in I$ and for all $t_1, t_2 \in \mathbb{R}$.

We are going to consider as the domain of G the cone $C \subset \mathcal{R}_1(I \times \mathbb{R})$ made by finite linear combination of SBV graphs:

$$C := \left\{ T = \sum_{i=1}^k \lambda_i \Gamma_{u_i} : k \in \mathbb{N}, \lambda_i \in \mathbb{R}_+, u_i \in SBV(I) \right\}. \tag{18}$$

In order to avoid any confusion we stress that u^- is the trace of u from the left and u^+ is the trace of u from the right.
Moreover for every $T \in C$ we will assume implicitly that, being a rectifiable current, it is defined by the triple $T = (\mathcal{M}, \theta, \xi)$.

3.1. **Simplifying the cone C.** From the definition of the cone C in (18) one easily notices that for every current $T \in C$ there exists different combinations of SBV graphs $\{u_i\}$ that represent it. In particular there are some configurations we would like to avoid and this subsection is devoted to make this simplifications for C.

Definition 3.2. *Given $\{u_i\}_{i=1\dots k} \subset SBV(I)$. We say that the family $\{u_i\}_{i=1\dots k}$ has* cancellation on the jumps *if there exists l_1, l_2 and $x_0 \in S_{u_{l_1}} \cap S_{u_{l_2}}$ such that*

$$u_{l_1}^-(x_0) < u_{l_1}^+(x_0), \quad u_{l_2}^-(x_0) > u_{l_2}^+(x_0), \quad u_{l_1}^+(x_0) > u_{l_2}^+(x_0).$$

We need a lemma that ensures that we can rearrange the graphs in order not to have this cancellation.

Lemma 3.3. *Given $T = \sum_{i=1}^k \lambda_i \Gamma_{u_i} \in C$ there exists $l \in \mathbb{N}$, $w_i \in SBV(I)$ and $\mu_i \in \mathbb{R}^+$ for $i = 1 \dots l$ such that $T = \sum_{i=1}^l \mu_i \Gamma_{w_i}$ and there is no cancellation on the jumps.*

Proof. Given $T = \sum_{i=1}^k \lambda_i \Gamma_{u_i}$ let us suppose that we have cancellation between Γ_{u_1} and Γ_{u_2} in $A \subset S_{u_1} \cap S_{u_2}$ and $\lambda_1 \geq \lambda_2$ (without loss of generality). As A is countable we will denote it by the sequence $\{x_1, x_2, \dots\}$ possibly infinite. Given $I = (a,b)$ consider the new sequence $\{a = x_0, x_1, x_2, \dots\}$ and define two SBV functions in the following way:

$$w_1(x) = \begin{cases} u_1(x) & \text{for } x_{i-1} < x \leq x_i,\ i \geq 1 \text{ and odd} \\ u_2(x) & \text{for } x_{i-1} < x \leq x_i,\ i \geq 1 \text{ and even} \end{cases}$$

and

$$w_2(x) = \begin{cases} u_2(x) & \text{for } x_{i-1} < x \leq x_i,\ i \geq 1 \text{ and odd} \\ u_1(x) & \text{for } x_{i-1} < x \leq x_i,\ i \geq 1 \text{ and even.} \end{cases}$$

Then we have that $\lambda_2 \Gamma_{w_1} + \lambda_2 \Gamma_{w_2} + (\lambda_1 - \lambda_2)\Gamma_{u_1} = \lambda_1 \Gamma_{u_1} + \lambda_2 \Gamma_{u_2}$. Hence we produce a decomposition of $\lambda_1 \Gamma_{u_1} + \lambda_2 \Gamma_{u_2}$ that has no cancellation on the jumps. It is easy to check that one can repeat this operation for any pair of graphs that has cancellation on jumps and that this procedure ends in a finite number of steps.

□

From now on we will assume that given $T = \sum_i \lambda_i \Gamma_{u_i} \in C$, the graphs composing T have no cancellation on jumps.
In what follows we will need for technical reasons to have the graphs ordered. Clearly this is possible when we have superposition of graphs with the same multiplicity. In particular we need the following decomposition theorem ([3]) that we state for the reader convenience.

Theorem 3.4 (Ambrosio, Crippa, Le Floch)**.** *Let $T \in I^1(\mathbb{R}^2)$ be an integer rectifiable current satisfying the zero boundary condition $\partial T = 0$, the positivity condition $T \llcorner dx \geq 0$ and the cylindrical mass condition $\|T\|(B(0,R) \times \mathbb{R}) < \infty$ for every R. Then there exists a unique family of functions $w_i \in BV_{loc}(\mathbb{R})$ satisfying $w_1 \leq w_2 \leq \ldots \leq w_l$. Such that*

$$T = \sum_{i=1}^l \Gamma_{w_i} \quad \text{and} \quad \|T\| = \sum_{i=1}^l \|\Gamma_{w_i}\|.$$

Proposition 3.5. *Given $T = \sum_{i=1}^k \Gamma_{u_i} \in C$ there exists $w_1 \leq \ldots \leq w_l \in SBV(I)$ such that $T = \sum_{i=1}^l \Gamma_{w_i}$.*

Proof. Notice that the current T is integer rectifiable and $T \llcorner dx \geq 0$. As we have assumed that T does not have cancellation on jumps thanks to Lemma 3.3 we have

$$\|T\| = \sum_{i=1}^k \|\Gamma_{u_i}\|. \tag{19}$$

Moreover by extending each function u_i as a constant outside I we can apply Theorem 3.4 to T to get the following representation:

$$T = \sum_{i=1}^{l} \Gamma_{w_i}$$

where $w_i \in BV(I)$ and they are ordered in an increasing way.
It remains to show that $w_i \in SBV(I)$, $\forall i$. By Theorem 3.4 and (19) one has that for every measurable set $C \subset I$ with $\mathscr{L}^1(C) = 0$

$$\sum_{i=1}^{k} \|\Gamma_{u_i}\|(C \times \mathbb{R}) = \|T\|(C \times \mathbb{R}) = \sum_{i=1}^{l} \|\Gamma_{w_i}\|(C \times \mathbb{R}). \tag{20}$$

By standard results on the graph of BV functions (see [13]) one has

$$\|\Gamma_{w_i}\|(C \times \mathbb{R}) = |\mu(Dw_i)|(C) \tag{21}$$

where $\mu(Dw_i) = (Dw_i, -\mathscr{L}^1)$. So from (20) and (21) and the fact the C is negligible it follows that

$$\sum_{i=1}^{l} |Dw_i|(C) = \sum_{i=1}^{k} |Du_i|(C)$$

and thus

$$\sum_{i=1}^{l}(|D^j w_i|(C) + |D^c w_i|(C)) = \sum_{i=1}^{k}(|D^j u_i|(C) + |D^c u_i|(C)).$$

Choose $C = \bigcup_{i=1}^{k} S_{u_i} = \bigcup_{i=1}^{l} S_{w_i}$ a countable measurable set; as the Cantor part of the derivative of a BV function is a diffuse measure we have

$$\sum_{i=1}^{l} |D^j w_i| = \sum_{i=1}^{k} |D^j u_i|.$$

Hence

$$\sum_{i=1}^{l} |D^c w_i| = \sum_{i=1}^{k} |D^c u_i|,$$

that implies that $w_i \in SBV(I)$ for every $i = 1, \ldots, k$.

□

3.2. **Properties of the regular part of $G(T)$.**

Definition 3.6 (Regular part and singular part of T)**.** *We define the singular part of $T \in C$ as*

$$S_T := \bigcup S_{u_i} \tag{22}$$

and the regular part as $R_T := I \setminus S_T$.

Remark 3.7. *One can easily notice that if we assume that the graphs do not have cancellation according to Lemma 3.3, S_T is well defined, so it does not depend on the representation of T.*

Given a measurable set $A \subset I$ we define the localized version of G as

$$G(T, A) := \sup_{\phi \in K} \int_{\mathcal{M} \cap (A \times \mathbb{R})} \langle \phi, \nu_T \rangle d\|T\|.$$

Remark 3.8. *It is clear that given A_1, A_2 disjoint measurable sets we have*

$$G(T, A_1 \cup A_2) = G(T, A_1) + G(T, A_2)$$

so in particular

$$G(T) = G(T, S_T) + G(T, S_R). \tag{23}$$

Moreover when one computes the localized functional, it is possible to restrict the set K accordingly:

$$G(T,A) = \sup_{\phi \in K_A} \int_{\mathcal{M}\cap(A\times\mathbb{R})} \langle \phi, \nu_T \rangle d\|T\|.$$

where K_A is the set of $\phi : I \times \mathbb{R} \to \mathbb{R}$, Borel, such that

- $\phi^t(x,t) \geq \dfrac{|\phi^x(x,t)|^2}{4} - \beta(t-g)^2 \quad \forall x \in A$ *and* $\forall t \in \mathbb{R}$,
- $\left| \displaystyle\int_{t_1}^{t_2} \phi^x(x,t)\, dt \right| \leq \alpha$ *for every* $x \in A$ *and for all* $t_1, t_2 \in \mathbb{R}$.

We are presenting a proposition that allows us to split $G(T, R_T)$ as the sum of $\lambda_i G(\Gamma_{u_i}, R_T)$.

Proposition 3.9. *Given $T = \sum_{i=1}^k \lambda_i \Gamma_{u_i} \in C$, then*

$$G(T,R_T) = \sum_{i=1}^k \lambda_i G(\Gamma_{u_i}, R_T) = \sum_{i=1}^k \lambda_i \left(\alpha \int_I (u_i')^2\, dx + \beta \int_I |u_i - g|^2\, dx \right). \tag{24}$$

In order to give a proof of this fact we need some preliminary lemmas that simplifies the situation.

Lemma 3.10. *Given $T = \sum_{i=1}^k \lambda_i \Gamma_{u_i} \in C$ let $A \subset I$ be a measurable set such that $A \cap S_T = \emptyset$ and $\mathcal{H}^1(\Gamma_{u_i} \cap \Gamma_{u_j} \cap (A \times \mathbb{R})) = 0$ for every $i \neq j$. Then*

$$G(T,A) = \sum_i \lambda_i G(\Gamma_{u_i}, A).$$

Proof. By induction it is enough to show that given, $T_1 = \sum_{i=1}^{k-1} \lambda_i \Gamma_{u_i}$ and $T_2 = \lambda_k \Gamma_{u_k}$ one has

$$G(T_1 + T_2, A) = G(T_1, A) + G(T_2, A).$$

Fix $\varepsilon > 0$. For $i = 1, 2$ there exist $\phi_i \in K_A$ such that

$$\int_{\mathcal{M}_i \cap (A\times\mathbb{R})} \langle \phi_i, \nu_{T_i} \rangle\, d\|T_i\| \geq G(T_i, A) - \varepsilon.$$

Define the following vector field

$$\tilde{\phi} = \begin{cases} \phi_1 & (x,t) \in \mathcal{M}_1 \setminus \mathcal{M}_2 \\ \phi_2 & (x,t) \in \mathcal{M}_2 \setminus \mathcal{M}_1 \\ 0 & \text{otherwise.} \end{cases}$$

Let prove that $\tilde{\phi} \in K_A$.
For every $x \in A$ we have that $x \notin S_T$ by hypothesis, so that (II) is satisfied and (I) is trivial by definition. Moreover, as $\mathcal{H}^1(\mathcal{M}_1 \cap \mathcal{M}_2 \cap (A \times \mathbb{R})) = 0$, one has

$$\int_{(\mathcal{M}_1 \cup \mathcal{M}_2)\cap(A\times\mathbb{R})} \langle \tilde{\phi}, \nu_T \rangle\, d\mathcal{H}^1 = \int_{\mathcal{M}_1\cap(A\times\mathbb{R})} \langle \phi_1, \nu_{T_1} \rangle\, d\mathcal{H}^1 + \int_{\mathcal{M}_2\cap(A\times\mathbb{R})} \langle \phi_2, \nu_{T_2} \rangle\, d\mathcal{H}^1.$$

So

$$G(T_1,A)+G(T_2,A) \leq \int_{\mathcal{M}_1\cap(A\times\mathbb{R})} \langle \phi_1, \nu_{T_1} \rangle\, d\mathcal{H}^1 + \int_{\mathcal{M}_2\cap(A\times\mathbb{R})} \langle \phi_2, \nu_{T_2} \rangle\, d\mathcal{H}^1 + 2\varepsilon \leq G(T_1+T_2, A) + 2\varepsilon.$$

Sending ε to zero we obtain the first inequality. The opposite one comes directly from the convexity of G.

□

Lemma 3.11. *Given $T = \sum_{i=1}^k \lambda_i \Gamma_{u_i} \in C$ let $A \subset I$ be a measurable set such that $A \cap S_T = \emptyset$. Then*

$$G(T,A) = \sum_i \lambda_i G(\Gamma_{u_i}, A).$$

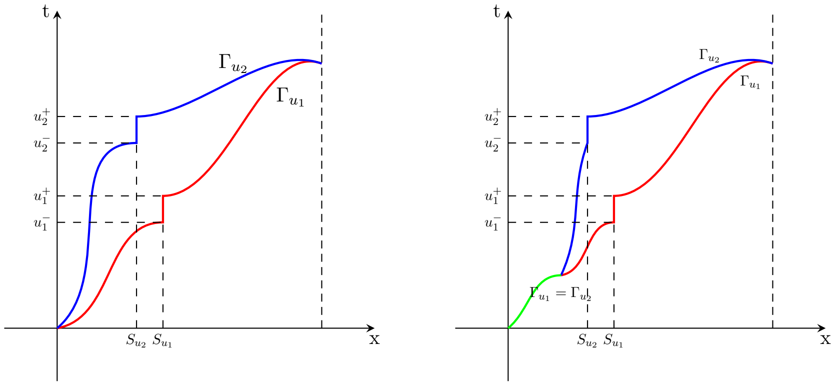

FIGURE 1. Configuration in Lemma 3.10 and 3.11

Proof. Given $T \in C$, let J be a set of indexes. Denote by $\Gamma = \bigcap_{i\in J} \Gamma_{u_i}$ an intersection of graphs and let $\theta = \sum_{i\in J} \lambda_i$ be the multiplicity on Γ. So

$$\sup_{\phi\in K} \int_{\Gamma\cap(A\times\mathbb{R})} \langle \phi, \nu_T\rangle \, d\|T\| = \sup_{\phi\in K} \int_{\Gamma\cap(A\times\mathbb{R})} \theta\langle \phi, \nu_T\rangle \, d\mathcal{H}^1 = \sup_{\phi\in K} \int_{\Gamma\cap(A\times\mathbb{R})} \sum_{i\in J} \lambda_i \langle \phi, \nu_T\rangle \, d\mathcal{H}^1$$
$$= \sum_{i\in J} \lambda_i \sup_{\phi\in K} \int_{\Gamma\cap(A\times\mathbb{R})} \langle \phi, \nu_T\rangle \, d\mathcal{H}^1.$$

Clearly this can be repeated for every intersection of an arbitrary number of graphs. Combining this result with Lemma 3.10 we have the thesis.

□

Proof of Proposition 3.9

Proposition 3.9 is a direct consequence of Lemma 3.11 choosing $A = S_R$ and the second equality in (24) follows from Proposition 2.12.

□

3.3. **Properties of the singular part of $G(T)$.** In this section we are going to study the properties of $\mathcal{G}(T) := G(T, S_T)$.
Given $T = \sum_{i=1}^k \lambda_i \Gamma_{u_i} \in C$ and calling $\nu_T = ((\nu_T)^x, (\nu_T)^t)$, by (17) we have

$$\mathcal{G}(T) = \sup_{\phi\in K} \int_{\mathcal{M}\cap(S_T\times\mathbb{R})} \theta\phi^x (\nu_T)^x \, d\mathcal{H}^1$$

and it is easy to see that

$$(\nu_T)^x(x,t) = \begin{cases} +1 & (x,t) \in S_{u_i} \times (u_i^-, u_i^+) \\ -1 & (x,t) \in S_{u_i} \times (u_i^+, u_i^-). \end{cases}$$

Hence

$$\mathcal{G}(T) = \sup_{\phi\in K} \sum_{i=1}^k \int_{S_{u_i}\times(u_i^-, u_i^+)} \theta\phi^x \, d\mathcal{H}^1.$$

From now on we will work with linear combinations of graphs with the same multiplicity. We will see later the reason why we can reduce to this situation. We want to prove that, given $T = \sum_i \Gamma_{u_i}$, $\mathcal{G}(T)$ can be written as the sum of $\mathcal{G}(\Gamma_{u_i})$ in all the configurations in which there is non-adjacency of the jumps of the graphs.

Theorem 3.12. *Consider $T \in C$ such that $T = \sum_{i=1}^k \Gamma_{u_i}$ and u_i are ordered in an increasing way. Suppose that for every $i = 1 \dots k$*

$$\{x \in S_{u_i} \cap S_{u_{i+1}} : u_i^+(x) = u_{i+1}^-(x) \text{ or } u_i^-(x) = u_{i+1}^+(x)\} = \emptyset.$$

Then

$$\mathcal{G}\left(\sum_{i=1}^{k}\Gamma_{u_i}\right)=\sum_{i=1}^{k}\mathcal{G}\left(\Gamma_{u_i}\right).$$

Remark 3.13. *The assumption of the ordering of the graphs is not essential as given $T \in C$ with graphs of the same multiplicity, by Proposition 3.5 is always possible to find an alternative representation by ordered graphs.*

Remark 3.14. *Notice that without loss of generality we can prove the previous statement restricting the functional $\mathcal{G}$ to every $x \in S_T$. So the lemmas needed to prove Theorem 3.12 will be stated for a fixed point $x \in S_T$.*

For sake of clarity we propose two lemmas (Lemma 3.15 and 3.16) that deals with a simple situation that is enough to explain the general strategy (See Figure 2). Then, in Proposition 3.17 and 3.18, we generalize this procedure and finally we prove the Theorem.

Lemma 3.15. *Consider $T = \sum_{i=1}^{k}\Gamma_{u_i} \in C$ such that u_i are ordered in an increasing way. Fix $x \in S_T$ and suppose that we have $u_i^-(x) \le u_i^+(x)$ for every $i = 1 \dots k$. Suppose in addition that*

$$u_i^+(x) < u_j^-(x) \quad \textit{for every} \quad i < j.$$

Then

$$\mathcal{G}(T,\{x\})=\sum_{i=1}^{k}\mathcal{G}(\Gamma_{u_i},\{x\})=\alpha|\{i : x \in S_{u_i}\}|.$$

In addition the the maximum is achieved and letting ϕ_T be the vector field realizing the maximum for T

$$\phi_T^x(x,t)=\alpha/(u_i^+-u_i^-) \quad \textit{for every} \quad t \in (u_i^-,u_i^+)$$

for every $i = 1 \dots k$ such that $x \in S_{u_i}$.

Proof. By induction it is enough to prove that for $T = T_1 + T_2$ where $T_1 = \sum_{i=1}^{k-1}\Gamma_{u_i}$ and $T_2 = \Gamma_{u_k}$ one has

$$\mathcal{G}(T_1+T_2,\{x\})=\mathcal{G}(T_1,\{x\})+\mathcal{G}(T_2,\{x\})$$

and

$$\phi_T^x(x,t)=\alpha/(u_k^+-u_k^-) \quad \text{for every} \quad t \in (u_k^-,u_k^+).$$

(We suppose $x \in S_{u_k}$ because if not, there is nothing to prove).
For the inductive hypothesis we have that for all $i = 1 \dots k-1$

$$\phi_{T_1}^x(x,t)=\alpha/(u_i^+-u_i^-) \quad \text{for every} \quad t \in (u_i^-,u_i^+).$$

For the general theory of calibration we have that, calling ϕ_{T_2} the vector field realizing the maximum in $\mathcal{G}(T_2,\{x\})$,

$$\phi_{T_2}^x(x,t)=\alpha/(u_k^+-u_k^-) \quad \text{for every} \quad t \in (u_k^-,u_k^+),$$

because

$$\int_{u_k^-}^{u_k^+}\phi_{T_2}^x(x)=\alpha \quad \text{for every } x \in S_{u_k}.$$

Define the following vector field on $\{x\} \times \mathbb{R}$:

$$\tilde{\phi}=\begin{cases}\phi_{T_1} & (x,t)\in\{x\}\times(u_1^-,u_{k-1}^+),\\ \phi_{T_2} & (x,t)\in\{x\}\times(u_k^-,u_k^+),\\ \{-\alpha/(u_k^- - u_{k-1}^+),\frac{(\tilde{\phi}^x)^2}{4}-\beta(t-g)^2\} & (x,t)\in\{x\}\times(u_{k-1}^+,u_k^-),\\ 0 & \text{otherwise.}\end{cases}$$

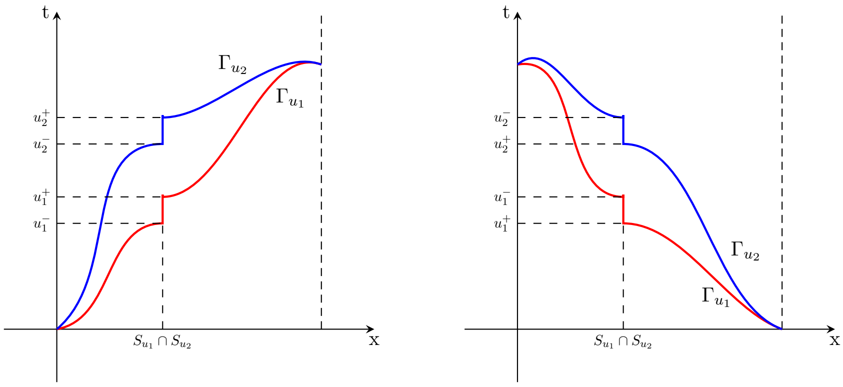

FIGURE 2. Configuration in Lemma 3.15 and in Lemma 3.16

Let prove that $\tilde{\phi} \in K_{\{x\}}$.

$$\begin{aligned}\left|\int_{t_1}^{t_2} \tilde{\phi}(x,t)\,dt\right| &= \left|\int_{t_1}^{u_{k-1}^-} \phi_{T_1}^x(x,t)\,dt - \alpha + \int_{u_k^+}^{t_2} \phi_{T_2}^x(x,t)\,dt\right| \\ &= \left|\alpha\frac{(u_{k-1}^- - t_1)}{(u_{k-1}^- - u_1^-)} - \alpha + \alpha\frac{(t_2 - u_k^+)}{(u_k^+ - u_k^-)}\right| \le \alpha\end{aligned}$$

for every $t_1 \le u_1^-$, $t_2 \ge u_k^+$. As in all the other cases the computation is similar, then $\tilde{\phi} \in K_{\{x\}}$. Therefore

$$\mathcal{G}(T_1,\{x\}) + \mathcal{G}(T_2,\{x\}) = \int_{\mathcal{M}\cap(\{x\}\times\mathbb{R})} \langle\tilde{\phi},\nu_T\rangle\theta\,d\mathcal{H}^1 \le \mathcal{G}(T,\{x\}).$$

On the other hand by convexity

$$\mathcal{G}(T,\{x\}) \le \mathcal{G}(T_1,\{x\}) + \mathcal{G}(T_2,\{x\}) = \int_{\mathcal{M}\cap(\{x\}\times\mathbb{R})} \langle\tilde{\phi},\nu_T\rangle\theta\,d\mathcal{H}^1.$$

So the thesis follows.

□

We can prove the analogue:

Lemma 3.16. *Given $T = \sum_{i=1}^k \Gamma_{u_i} \in C$ such that u_i are ordered in an increasing way. Fix $x \in S_T$ and suppose that we have $u_i^+(x) \le u_i^-(x)$ for every $i = 1\dots k$. Suppose in addition that*

$$u_i^+(x) > u_j^-(x) \quad \textit{for every} \quad i > j.$$

Then

$$\mathcal{G}(T,\{x\}) = \sum_{i=1}^k \mathcal{G}(\Gamma_{u_i},\{x\}) = \alpha|\{i : x \in S_{u_i}\}|.$$

In addition the the maximum is achieved and letting ϕ_T be the vector field realizing the maximum for T

$$\phi_T^x(x,t) = \alpha/(u_i^+ - u_i^-) \quad \textit{for every} \quad t \in (u_i^-, u_i^+)$$

for every $i = 1\dots k$ such that $x \in S_{u_i}$.

Proof. See Lemma 3.15.

□

We are now in position to prove two general statements that are generalizations of Lemmas 3.15 and 3.16.

Proposition 3.17. *Consider $T \in C$ such that $T = \sum_{i=1}^{k} \Gamma_{u_i}$ and u_i are ordered in an increasing way. Fix $x \in S_T$ and suppose that we have $u_i^-(x) \leq u_i^+(x)$ for every $i = 1 \dots k$. Moreover assume that $u_i^+(x) \neq u_{i+1}^-(x)$ for every i such that $x \in S_{u_i}$.*
Then

$$\mathcal{G}(T, \{x\}) = \sum_{i=1}^{k} \mathcal{G}(\Gamma_{u_i}, \{x\}).$$

Proof. We can assume without loss of generality that $x \in S_{u_i}$ for every $i = 1 \dots k$.
It is easy to see that $T \llcorner (\{x\} \times \mathbb{R}) = \sum_{i=1}^{k'} \lambda_i [\{x\} \times (a_i, a_{i+1})]$ for some $\lambda_i \in \mathbb{N}$ and $a_i \in \mathbb{R}$. Let denote by $\{\lambda_{M_j}\}$ the local maxima of the sequence $\{\lambda_i\}$ and let λ_{m_j} be the minimum multiplicity in $\{\lambda_{M_j}, \lambda_{M_j+1}, \dots, \lambda_{M_{j+1}-1}, \lambda_{M_{j+1}}\}$ for every j.
By the fact that the graphs are ordered, Lemma 3.3 and the current hypothesis we have

$$|\lambda_{i+1} - \lambda_i| = 1 \tag{25}$$

and

$$k = \sum_j \lambda_{M_j} - \sum_j \lambda_{m_j}. \tag{26}$$

Then the proof proceeds similarly to the proof of Lemma 3.15. One can build a vector field $\tilde{\phi}$ such that

$$\tilde{\phi}^x = \alpha / (a_{M_j+1} - a_{M_j}) \quad \text{in } \{x\} \times (a_{M_j+1}, a_{M_j}) \quad \forall j,$$
$$\tilde{\phi}^x = -\alpha / (a_{m_j+1} - a_{m_j}) \quad \text{in } \{x\} \times (a_{m_j+1}, a_{m_j}) \quad \forall j$$

and zero otherwise to get the thesis.

□

Proposition 3.18. *Consider $T \in C$ such that $T = \sum_{i=1}^{k} \Gamma_{u_i}$ and u_i are ordered in an increasing way. Fix $x \in S_T$ and suppose that we have $u_i^+(x) \leq u_i^-(x)$ for every $i = 1 \dots k$. Moreover assume that $u_i^-(x) \neq u_{i+1}^+(x)$ for every i such that $x \in S_{u_i}$.*
Then

$$\mathcal{G}(T, \{x\}) = \sum_{i=1}^{k} \mathcal{G}(\Gamma_{u_i}, \{x\}).$$

Proof. See Proposition 3.17.

□

Now Theorem 3.12 is an immediate consequence of the previous propositions.

Proof of Theorem 3.12

Fix $x \in S_T$ and define

$$\mathcal{I} = \{i = 1 \dots k : u_i^-(x) \leq u_i^+(x)\} \qquad \mathcal{J} = \{i = 1 \dots k : u_i^-(x) > u_i^+(x)\}$$

and call $T_{\mathcal{I}} = \sum_{i \in \mathcal{I}} \Gamma_{u_i}$ and $T_{\mathcal{J}} = \sum_{i \in \mathcal{J}} \Gamma_{u_i}$. Moreover let $\phi_{\mathcal{I}}$ ($\phi_{\mathcal{J}}$) be the vector field realizing the maximum in $\mathcal{G}(T_{\mathcal{I}}, \{x\})$ ($\mathcal{G}(T_{\mathcal{J}}), \{x\}$). From Proposition 3.17 and 3.18 it is easy to see that $\phi_{\mathcal{I}}^x \leq 0$ outside the support of $T_{\mathcal{I}}$ restricted to $\{x\} \times \mathbb{R}$ and $\phi_{\mathcal{J}}^x \geq 0$ outside the support of $T_{\mathcal{I}}$ restricted to $\{x\} \times \mathbb{R}$. Therefore defining $\tilde{\phi} = \phi_{\mathcal{I}} + \phi_{\mathcal{J}}$, as we assumed that there is no cancellation on the jumps by Lemma 3.3, we have that $\tilde{\phi} \in K_{\{x\}}$ and

$$\mathcal{G}(T_{\mathcal{I}}, \{x\}) + \mathcal{G}(T_{\mathcal{J}}, \{x\}) = \int_{\{x\} \times \mathbb{R}} \langle \phi_{\mathcal{I}}^x + \phi_{\mathcal{J}}^x, \nu_T \rangle \, d\|T\| \leq \mathcal{G}(T, \{x\}).$$

So by convexity

$$\mathcal{G}(T_{\mathcal{I}}, \{x\}) + \mathcal{G}(T_{\mathcal{J}}, \{x\}) = \mathcal{G}(T, \{x\}).$$

Finally we apply Proposition 3.17 and 3.18 to $T_{\mathcal{I}}$ and $T_{\mathcal{J}}$ to get the thesis.

□

We conclude this section with a lemma that shows that we can reduce any combination of graphs belonging to C to a combination of graphs, all with the same multiplicity. We are going to use this property in the proof of the coarea formula in the next section.

Lemma 3.19. *Consider $T_1, T_2 \in C$ and $x \in S_{T_1} \cap S_{T_2}$. Suppose that $T_1 \llcorner (\{x\} \times \mathbb{R}) = \sum_{i=1}^{k} \lambda_i [\{x\} \times (a_i, a_{i+1})]$ with $a_i \le a_{i+1}$ and let $\{M_j\}_{j\in J}$ be the indexes of the maximums of the multiplicities. Assume in addition that $T_2 \llcorner (\{x\} \times \mathbb{R}) = \nu \sum_{j\in J} [\{x\} \times (a_{M_j}, a_{M_j+1})]$ for some $\nu > 0$. Then we have*

$$\mathcal{G}(T_1 + T_2, \{x\}) = \mathcal{G}(T_1, \{x\}) + \mathcal{G}(T_2, \{x\}). \tag{27}$$

Proof. Given $\phi \in K$ define

$$\Lambda_\phi(s) := \int_{a_1}^{s} \phi^x(x,t)\, dt - \frac{1}{2}$$

so that

$$\mathcal{G}(T) = \sup_{\phi\in K} \sum_{i=1}^{k} \lambda_i \int_{a_i}^{a_{i+1}} \phi^x\, dt = \sup_{\phi\in K} \sum_{i=1}^{k} \lambda_i (\Lambda_\phi(a_{i+1}) - \Lambda_\phi(a_i)) =: \sup_{\phi\in K} \tilde{\mathcal{G}}(\Lambda_\phi).$$

Observe that for every $\phi \in K$, $|\Lambda_\phi(a_i) - \Lambda_\phi(a_j)| \le 1$. Define then the following set:

$$H = \{\Lambda_\phi : \phi \in K, \text{ such that } |\Lambda_\phi(a_i)| \le 1/2 \quad \forall i = 1 \dots k\}.$$

As the value of the functional $\tilde{\mathcal{G}}$ depends only on the difference between $\Lambda_\phi(a_i)$ and $\Lambda_\phi(a_{i-1})$ we have that

$$\sup_{\phi\in K} \tilde{\mathcal{G}}(\Lambda_\phi) = \sup_{\Lambda_\phi \in H} \tilde{\mathcal{G}}(\Lambda_\phi). \tag{28}$$

Notice now that it is possible to rewrite the functional in the following form

$$\tilde{\mathcal{G}}(\Lambda_\phi) = -\lambda_1 \Lambda_\phi(a_1) + \sum_{i=2}^{k} (\lambda_{i-1} - \lambda_i) \Lambda_\phi(a_i) + \lambda_k \Lambda_\phi(a_{k+1}).$$

Hence the supremum in H is a maximum and thanks to (28) the maximum points in H are characterized by

$$\Lambda_\phi(a_1) = -1/2, \quad \Lambda_\phi(a_k) = 1/2, \quad \Lambda_\phi(a_i) = \frac{1}{2}\mathrm{sgn}(\lambda_{i-1} - \lambda_i). \tag{29}$$

Let us suppose without loss of generality that the maximums of the multiplicity $\{\lambda_{M_j}\}_{j\in J}$ correspond to intervals that are not adjacent (by changing a_i) and let Λ_ϕ be one of the maximum point in H of $\tilde{\mathcal{G}}$, then by (29) we get

$$1 = \Lambda_\phi(a_{M_j+1}) - \Lambda_\phi(a_{M_j}) = \int_{a_{M_j}}^{a_{M_j+1}} \phi^x(x,t)\, dt \quad \forall j \in J.$$

As the maximal multiplicities are located in the same interval both in T_1 and in $T_1 + T_2$, then the vector field realizing the maximum is the same and thus the thesis (27) follows.

□

Corollary 3.20. *Fix $1 \le k' < k$ and define $T_1, T_2 \in C$ such that $T_1 = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i}$ with λ_i ordered in an increasing way and $T_2 = \sum_{i=k'+1}^{k} \nu \Gamma_{u_i}$ with $\nu > 0$. Then $G(T_1 + T_2) = G(T_1) + G(T_2)$.*

Proof. Notice that by Lemma 3.11 it is enough to prove the thesis for every $x \in S_{T_2} \cap S_{T_1}$. Thanks to Lemma 3.19 one has

$$G(T_1 + T_2, \{x\}) = G(T_1, \{x\}) + G(T_2, \{x\}).$$

□

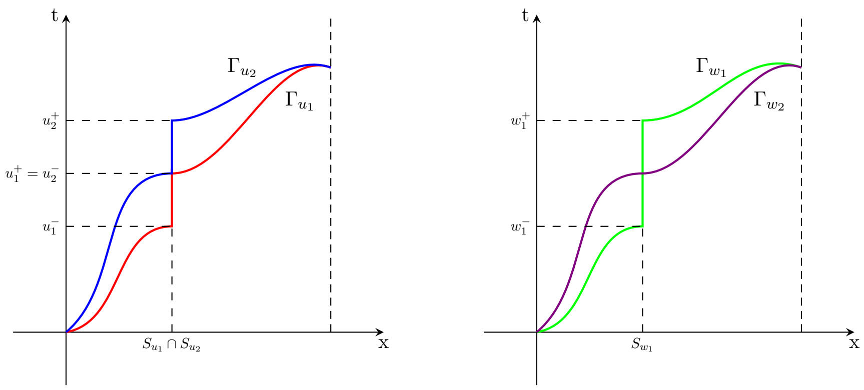

FIGURE 3. Coarea formula decomposition of two SBV graphs

3.4. **Coarea-type decomposition formula.** As anticipated in the introduction, this section is devoted to the proof of a decomposition formula for the Mumford-Shah functional in one dimension. This formula resembles closely a generalized coarea formula for functionals and it is performed for a finite combination of graphs with multiplicity. It is interesting to notice that the counterexample in the end of Remark 2.10 is "solved" by this decomposition, but it is difficult to generalize it to the continuous case. However it gives a strong indication on how this decomposition should be performed at least in dimension one. The higher dimensional case is a completely different issue, as the coarea-type formula we are going to present strongly relies on the one dimensional structure of the problem and cannot be extended in an easy way.

Proposition 3.21. *Given $T = \sum_{i=1}^k \Gamma_{u_i} \in C$ such that $|S_T| < +\infty$ there exists $\{w_i\}_{i=1\dots k} \subset SBV(I)$ such that $T = \sum_{i=1}^k \Gamma_{w_i}$ and*

$$G(T) = \sum_{i=1}^k G(\Gamma_{w_i}).$$

Proof. As a consequence of Proposition 3.5 we can suppose the graphs Γ_{u_i} ordered in an increasing way. Fix $x_0 \in S_T$ such that $x_0 \in \bigcap_{i=1}^l S_{u_i}$ with $l \leq k$. Thanks to Theorem 3.12 we can suppost wthout loss of generality that $u_i^+(x_0) = u_{i+1}^-(x_0)$ for every $i = 1 \dots l$ (the case $u_i^-(x_0) = u_{i+1}^+(x_0)$ is analogous). Define the following functions (See Figure 3):

$$w_1 = \begin{cases} u_1 & \text{for } x \leq x_0 \\ u_l & \text{for } x \geq x_0 \end{cases}$$

and

$$w_i = \begin{cases} u_i & \text{for } x \leq x_0 \\ u_{i-1} & \text{for } x \geq x_0 \end{cases} \qquad \forall i = 2 \dots l.$$

Clearly $\sum_{i=1}^l \Gamma_{w_i} = \sum_{i=1}^l \Gamma_{u_i}$ and $w_i^+(x_0) \neq w_{i+1}^-(x_0)$ for every $i = 1 \dots l$.
Hence using Theorem 3.12 and repeating this procedure for every $x_0 \in S_T$ one obtains the thesis. □

Theorem 3.22 (Coarea-type formula)**.** *Given $T = \sum_{i=1}^k \lambda_i \Gamma_{u_i}$ such that $|S_T| < +\infty$ there exists $k' \in \mathbb{N}$, $\{\mu_i\}_{i=1\dots k'} \geq 0$ and $\{w_i\}_{i=1\dots k'} \subset SBV(I)$ such that $T = \sum_{i=1}^{k'} \mu_i \Gamma_{w_i}$ and*

$$G(T) = \sum_{i=1}^{k'} \mu_i G(\Gamma_{w_i}). \tag{30}$$

Proof. Consider $T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i} \in C$ with u_i ordered in an increasing way and suppose without loss of generality that also λ_i are ordered and λ_k is the maximum. Then T can be rewritten as

$$T = (\lambda_k - \lambda_{k-1})\Gamma_{u_k} + \lambda_{k-1}\Gamma_{u_k} + \sum_{i=1}^{k-1} \lambda_i \Gamma_{u_i}.$$

Hence by Corollary 3.20

$$G(T) = G((\lambda_k - \lambda_{k-1})\Gamma_{u_k}) + G\left(\lambda_{k-1}\Gamma_{u_k} + \sum_{i=1}^{k-1} \lambda_i \Gamma_{u_i}\right).$$

Then one can rewrite

$$\lambda_{k-1}\Gamma_{u_k} + \sum_{i=1}^{k-1} \lambda_i \Gamma_{u_i} = \lambda_{k-2}(\Gamma_{u_k} + \Gamma_{u_{k-1}}) + (\lambda_{k-1} - \lambda_{k-2})(\Gamma_{u_k} + \Gamma_{u_{k-1}}) + \sum_{i=1}^{k-2} \lambda_i \Gamma_{u_i}$$

and applying again Corollary 3.20

$$\begin{aligned} G\left(\lambda_{k-1}\Gamma_{u_k} + \sum_{i=1}^{k-1} \lambda_i \Gamma_{u_i}\right) &= G((\lambda_{k-1} - \lambda_{k-2})(\Gamma_{u_k} + \Gamma_{u_{k-1}})) \\ &+ G\left(\lambda_{k-2}(\Gamma_{u_k} + \Gamma_{u_{k-1}}) + \sum_{i=1}^{k-2} \lambda_i \Gamma_{u_i}\right). \end{aligned} \tag{31}$$

By Proposition 3.21 there exists u_k^2 and u_{k-1}^2 SBV functions such that $\Gamma_{u_k^2} + \Gamma_{u_{k-1}^2} = \Gamma_{u_k} + \Gamma_{u_{k-1}}$ and

$$(31) = G((\lambda_{k-1} - \lambda_{k-2})\Gamma_{u_k^2}) + G((\lambda_{k-1} - \lambda_{k-2})\Gamma_{u_{k-1}^2}) + G\left(\lambda_{k-2}(\Gamma_{u_k} + \Gamma_{u_{k-1}}) + \sum_{i=1}^{k-2} \lambda_i \Gamma_{u_i}\right)$$

and so on. Repeating this procedure k times one gets to

$$G(T) = \sum_{i=2}^{k} \sum_{j=i}^{k} (\lambda_i - \lambda_{i-1}) G(\Gamma_{u_j^{k-i+1}}) + G\left(\sum_{i=1}^{k} \lambda_1 \Gamma_{u_i}\right).$$

Hence, applying again Proposition 3.21 to the last term we obtain the desired decomposition (30).

□

4. Existence of calibration as a functional defined on currents

We now want to show an application of the coarea-type formula to the existence of calibration for the Mumford-Shah type functionals. Firstly we set the minimization problem associated to the previous functional G. Consider $S \in C$ and define

$$\psi_G(S) = \inf\{G(T) : T \in C,\ \partial T = \partial S\}.$$

Proposition 4.1. *The functional ψ_G is convex in C.*

Proof. As G is convex and the constraint is linear the proof is straightforward.

□

It is easy to see that by the coarea-type formula in Theorem 3.22 we have the following theorem:

Theorem 4.2. *If $u \in SBV(I)$ is a Dirichlet minimizer of F, then $\psi_G(\Gamma_u) = G(\Gamma_u) = F(u)$.*

Proof. Consider $T = \sum_{i=1}^k \lambda_i \Gamma_{u_i} \in C$ such that $\partial T = \partial \Gamma_u$. Without loss of generality we can suppose that $|S_T| < +\infty$. Then letting $I = (a, b)$ and $\pi : \mathbb{R}^2 \to \mathbb{R}$ the projection on the first component we have

$$\partial I = \pi^{\#}(\partial \Gamma_u) = \pi^{\#}(\partial T) = (\partial I) \sum_{i=1}^k \lambda_i. \tag{32}$$

Hence $\sum_{i=1}^k \lambda_i = 1$. By Theorem 3.22 there exist k' and $\{\mu_i\}_{i=1,\dots,k'} > 0$ such that

$$G(T) = G\left(\sum_{i=1}^k \lambda_i \Gamma_{u_i}\right) = \sum_{i=1}^{k'} \mu_i G(\Gamma_{w_i}) = \sum_{i=1}^{k'} \mu_i F(w_i).$$

and $\sum_{i=1}^k \lambda_i \Gamma_{u_i} = \sum_{i=1}^{k'} \mu_i \Gamma_{w_i}$. Moreover applying the push forward as in equation (32) we have also $\sum_{i=1}^{k'} \mu_i = 1$.
Thus, it remains to prove that $w_{i,\partial I} = u_{\partial I}$ for every $i = 1, \dots, k'$, where $w_{i,\partial I}$ denotes the trace of w_i on ∂I. This is an easy adaptation of the theory of cartesian currents; we refer to Section 3.2.5 in [13] for a proof in a more general setting.

□

This will imply the existence of a calibration in the following sense: let

$$\hat{C} = \left\{ T = \sum_{i=1}^k \lambda_i \Gamma_{u_i} : k \in \mathbb{N}, \lambda_i \in \mathbb{R}, u_i \in SBV(I) \right\}$$

be the double cone and define the following:

Definition 4.3 (Calibration for minimal graphs)**.** *Given $u \in SBV(I)$ and Γ_u its associated graph, we say that $\xi \in Hom(\hat{C})$ is a calibration for Γ_u with respect to G if*

i) $\xi(\Gamma_u) = G(\Gamma_u) = F(u)$,

ii) $\xi(T) = 0$ *for every* $T \in \hat{C}$ *such that* $\partial T = 0$,

iii) $\xi(T) \leq G(T)$ *for every* $T \in \hat{C}$.

Theorem 4.4. *Given $u \in SBV(I)$ a Dirichlet minimizer of F there exists a calibration for Γ_u with respect to G according to Definition 4.3.*

Proof. From Theorem 4.2 follows that

$$G(\Gamma_u) = \psi_G(\Gamma_u).$$

Consider the functional ψ_G defined on C and extend it to $+\infty$ for all the elements in $\hat{C} \setminus C$ (without renaming the extension). Clearly the extension is convex and $\psi_G(\Gamma_u) = G(\Gamma_u) > 0$.
Consider the vector subspace $L = \{a\Gamma_u : a \in \mathbb{R}\}$ and define $\psi : L \to \mathbb{R}$ as $\psi(a\Gamma_u) = a\psi_G(\Gamma_u)$ clearly linear. As we have that $\psi \leq \psi_G$ on L by Hahn-Banach theorem there exists $\xi \in Hom(\hat{C}, \mathbb{R})$ such that

$$\xi(\Gamma_u) = \psi(\Gamma_u) = \psi_G(\Gamma_u) \quad \text{and} \quad \xi(T) \leq \psi_G(T) \quad \forall T \in \hat{C}. \tag{33}$$

We want to prove that ξ is a calibration according to Definition 4.3. Let $T_0 \in \hat{C}$ be such that $\partial T_0 = 0$, then

$$\psi_G(T_0) = \inf\{G(S) : \partial S = \partial T_0 = 0\} \leq G(0) = 0.$$

In combination with (33) this implies $\xi(T) \leq 0$ for every $T_0 \in \hat{C}$ such that $\partial T_0 = 0$.
So, as ξ is an homeomorphism, one has also that $\xi(T_0) = 0$, so that (ii) holds. Moreover from (33), $\xi(\Gamma_u) = \psi_G(\Gamma_u) = F(u)$ that is (i).
Let us show that also (iii) is satisfied: if $T \in \hat{C} \setminus C$ then $G(T) = +\infty$ and so there is nothing

to prove. On the other hand given $T = \sum_{i=1}^k \lambda_i \Gamma_{u_i} \in C$ with $\lambda_i \in \mathbb{R}_+$ by (33) and using the definition of ψ_G

$$\xi(T) \leq \psi_G(T) \leq G(T).$$

Hence ξ is a calibration according to Definition 4.3. □

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