An integral-representation result for continuum limits of discrete energies with multi-body interactions

Andrea Braides
Dipartimento di Matematica, Università di Roma Tor Vergata
via della ricerca scientifica 1, 00133 Roma, Italy
e-mail: braides@mat.uniroma2.it

Leonard Kreutz
Gran Sasso Science Institute
Viale F. Crispi 7, 67100 L'Aquila, Italy
e-mail: leonard.kreutz@gmail.com

Abstract

We prove a compactness and integral-representation theorem for families of lattice energies describing atomistic interactions defined on lattices with vanishing lattice spacing. The densities of these energies may depend on interactions between all points of the corresponding lattice contained in a reference set. We give conditions that ensure that the limit is an integral defined on a Sobolev space. A homogenization theorem is also proved. The result is applied to multi-body interactions corresponding to discrete Jacobian determinants and to linearizations of Lennard-Jones energies with mixtures of convex and concave quadratic pair-potentials.

Keywords: lattice energies, discrete-to-continuum, multi-body interactions, homogenization, Lennard-Jones energies

1 Introduction

This paper focuses on the passage from lattice theories to continuum ones in the framework of variational problems, such as for atomistic systems in Computational Materials Science (see e.g. [8, 17]). For notational convenience we will state our results for energies defined on functions u parameterized on a portion of $\mathbb{Z}^N$ (with values in $\mathbb{R}^n$), but our assumptions may be immediately extended to more general lattices. For central interactions such energies may be written as

$$E(u) = \sum_{i,j} \psi_{ij}(u_i - u_j), \tag{1}$$

where i, j are points in the domain under consideration. We are interested in the behaviour of such an energy when the dimensions of the domain are much larger than the lattice spacing. In the discrete-to-continuum approach this can be done by approximation with a continuum energy obtained as a limit after a scaling argument. To that end, we introduce a small parameter ε (which, for the unscaled energy E is the inverse of the linear dimension of the domain) and scale the energies as

$$E_\varepsilon(u) = \sum_{i,j} \varepsilon^N \psi^\varepsilon_{ij}\Big(\frac{u_i - u_j}{\varepsilon}\Big), \tag{2}$$

where now i, j belong to a domain Ω that is independent of ε, and the domain of u is $\Omega \cap \varepsilon\mathbb{Z}^N$; accordingly, we set $\psi^\varepsilon_{ij} = \psi_{i/\varepsilon\, j/\varepsilon}$. Both scalings, ε^N of the energy, and u_i/ε of the function, are important in this process and highlight that in this case we are regarding the energy as a volume integral (ε^N being the volume element of a lattice cell) depending on a gradient ($(u_i - u_j)/\varepsilon$ being interpreted as a scaled difference quotient or discrete gradient). Other scalings are possible and give rise to different types of energies, depending on the form of ψ^ε_{ij}, highlighting the multiscale nature of the problem. In the present context we focus on this particular "bulk" scaling (for an account of other scaling limits see [3, 11, 12]).

The continuum approximation of E_ε is obtained by taking a limit as $\varepsilon \to 0$. This has been done in different ways, using a pointwise limit in [7] (where lattice functions are considered as restrictions of a smooth function to $\mathbb{Z}^N$) or a Γ-limit in [2] (in this case lattice functions are extended as piecewise-constant functions and embedded in some common Lebesgue space) to obtain an energy of the form

$$F(u) = \int_\Omega f(x, \nabla u)\mathrm{d}x \tag{3}$$

with domain a Sobolev space (for energies in the surface scalings with spin parameters see [5]). We focus on the result of [2], which relies on the localization methods of Γ-convergence (see [10] Chapter 12, [22]) envisaged by De Giorgi to deduce the integral form of the Γ-limit from its behaviour both as a function of u and Ω. Conditions that allow to apply those methods are

(i) (*coerciveness*) growth conditions from below that allow to deduce that the limit is defined on some Sobolev space; e.g. that $\psi^\varepsilon_{ij}(w) \geq c(|w|^p - 1)$ for nearest-neighbours and $\psi^\varepsilon_{ij} \geq 0$ for all i, j;

(ii) (*finiteness*) growth conditions from above that allow to deduce that the limit is finite on the same Sobolev space; e.g. that $\psi^\varepsilon_{ij}(w) \leq c^\varepsilon_{ij}(|w|^p + 1)$ for all ij, with some summability conditions on c^ε_{ij} uniformly in ε;

(iii) (*vanishing non-locality*) conditions that allow to deduce that the Γ-limit is a measure in its dependence on Ω. This is again obtained from some uniform decay conditions on the coefficients c^ε_{ij}.

Hypotheses (i)–(iii) are sharp, in the sense that failure of any of these conditions may result in a Γ-limit that cannot be represented as in (3). The result in [2] has been successful in many applications, among which the computation of optimal bounds for conducting networks [16], the derivation of nonlinear elastic energies from atomistic systems [2, 28], of their linear counterpart [20], and of Q-tensor theories from spin interactions [14], numerical homogenization [27], the analysis of the pile-up of dislocations [26], and others. Moreover, it has been extended to cover stochastic lattices [4] and dimension-reduction problems [1]. However, its range of applicability is restricted to pairwise interactions, which implies constraints on the possible energy densities. The main motivation of the present work is to overcome some of those limitations. More precisely, we focus on two issues:

• *the extension to the result to many-body interactions.* In principle, a point in the lattice may interact with all other points in the domain Ω. As a particular case, we may think of k-body interactions corresponding to the minors of the lattice transformation (which is affine at the lattice level), such as the discrete determinant in two dimensions, which can be viewed as a three-point interaction. Some works in this direction are already present in the literature [4, 21, 30, 31];

• *the use of averaged growth conditions on the energy densities.* Some lattice energies are obtained as an approximation of non-convex long-range interactions. As such, even when considering pair interactions, they may fail to satisfy coerciveness conditions for some ψ_{ij}. As an example we can think of the linearization of Lennard-Jones interactions, which gives concave quadratic energies for distant i and j. The coerciveness of the energy can nevertheless be recovered using the fast decay of the potential so that short-range convex interactions dominate long-range concave ones. In general, coerciveness can be obtained by substituting a growth conditions on each of the interactions with an averaged growth condition. Another example is the mixture of interactions corresponding to elastic and brittle materials. See the Example 6.3.

In order to achieve the greatest generality, we assume that energy densities may indeed depend on all points in $\Omega \cap \varepsilon\mathbb{Z}^N$. An energy density ϕ_i^ε will describe the interaction of a point $i \in \Omega \cap \varepsilon\mathbb{Z}^N$ with all other points in the domain. This standpoint, already used in [13] for surface energies in a simpler setting (see also [19] in a one-dimensional setting), brings some notational complications (except for the case $\Omega = \mathbb{R}^N$) since it is convenient to regard each such function as defined on a different set $(\Omega - i) \cap \varepsilon\mathbb{Z}^N$. This complication is anyhow present each time that we consider more-than-two-body interactions. The energies are then defined as

$$F_\varepsilon(u) = \sum_{i \in \Omega \cap \varepsilon\mathbb{Z}^N} \varepsilon^N \phi_i^\varepsilon(\{u_{j+i}\}_{j \in (\Omega - i) \cap \varepsilon\mathbb{Z}^N}). \tag{4}$$

An important remark to make is that there are many ways to define energy densities giving the same F_ε. Note for example that for central interactions as above ϕ_i^ε may be simply given by

$$\phi_i^\varepsilon(\{z_j\}) = \sum_{j \in (\Omega - i) \cap \varepsilon\mathbb{Z}^N} \psi_{ij}^\varepsilon\Big(\frac{z_j - z_0}{\varepsilon}\Big) = \sum_{j \in (\Omega - i) \cap \varepsilon\mathbb{Z}^N} \psi_{i/\varepsilon\, j/\varepsilon}\Big(\frac{z_j - z_0}{\varepsilon}\Big), \tag{5}$$

but the interactions may also be regrouped differently and in principle ϕ_i^ε may include some ψ_{kj}^ε with $k \neq i$. This is important in order to allow that some ψ_{ij}^ε be unbounded from below, up to satisfying a lower bound when considered together with the other interactions.

The set of hypotheses we are going to list for ϕ_{ij}^ε will allow to treat a larger class of energies than those of the form (2), but they must be stated with some care. The precise statements are given in Section 3. Here we give a simplified description as follows:

(o) (*translational invariance in the codomain*) $\phi_i^\varepsilon(\{z_j+w\}) = \phi_i^\varepsilon(\{z_j\})$ for all i, $\{z_j\}$ and vector w. This condition is automatically satisfied for interactions depending on differences $z_i - z_j$;

(i) (*coerciveness*) the energy must be estimated from below by a nearest-neighbour pair energy and $\phi_i^\varepsilon \geq 0$ for all i. This condition is less restrictive than the corresponding one for pair interactions since it refers to an already averaged energy density;

(ii) (*Cauchy-Born hypothesis*) we assume a polynomial upper bound for $F_\varepsilon(u)$ only when u is linear. For energy densities as in (5) this in general rewritten in terms of ψ_{ij} as

$$\Psi(M) := \sum_j \psi_{i\,i+j}(Mj) \leq C(1+|M|^p), \tag{6}$$

for all $i \in \mathbb{Z}^N$, and all $n \times N$ matrices M. This condition is in principle weaker than the finiteness property (ii) for pair interactions. The Cauchy-Born rule (see [23, 25]) relates the macroscopic and the microscopic deformation gradient of monoatomic crystals. It states that if such a material subjected to a small linear displacement on its boundary, then all the atoms follow this displacement. Here we only assume that the energy of equilibrium displacement and the energy of the linear deformation are of the same scale. Examining this condition separately goes in the direction of analyzing first pointwise convergence (as in [7]) and then Γ-convergence;

(iii) (*vanishing non-locality*) we assume that if $u = v$ on a square of centre i and side-length δ then

$$\phi_i^\varepsilon(\{u_{j+i}\}_{j\in(\Omega-i)\cap\varepsilon\mathbb{Z}^N}) \leq \phi_i^\varepsilon(\{v_{j+i}\}_{j\in(\Omega-i)\cap\varepsilon\mathbb{Z}^N}) + r(\varepsilon,\delta,\|\nabla u\|_p)$$

(u is identified with a piecewise-affine interpolation), where the rest r is negligible as $\varepsilon \to 0$ for $\|\nabla u\|_p$ bounded. Note that this condition is automatically satisfied with $r = 0$ if the range of the interactions is finite, and can be deduced from the corresponding condition (iii) for central interactions;

(iv) (*controlled non-convexity*) a final condition must be added to ensure that the limit be a measure as a function of Ω. For central interactions, this condition is hidden in the previous (i) and (ii), which imply a convex growth condition on Ψ; more precisely a polynomial growth of the form

$$c(|M|^p - 1) \leq \Psi(M) \leq C(1+|M|^p).$$

This double inequality allows to use classical convex-combination arguments with cut-off functions even though Ψ may not be convex. In our case this compatibility with convex arguments must be required separately, and is formalized in condition (H5) in Section 3.1.

Under the conditions above we again deduce that Γ-limits of energies F_ε are integral functionals F as in (3) defined on a Sobolev space. The integrand f can be described by a derivation formula, which is allowed by the study of suitably defined boundary-value problems. This derivation formula can also be used to prove a periodic-homogenization result. In the generality of energies possibly depending on the interaction of all points in Ω some care must be used to define periodicity for the energy densities. In the case of finite-range interactions we require that in the interior of Ω we have $\phi_i^\varepsilon = \phi_{\varepsilon/i}$, where ϕ_k

is periodic in k. For infinite-range interactions the definition is given by approximation with periodic energy densities with finite-range interactions.

The paper is organized as follows. After some notation, in Section 3 we rigorously state the hypotheses outlined above and prove the main compactness and integral-representation theorem. Section 4 is devoted to formalizing and proving the convergence of Dirichlet boundary-value problems, which is used in the following Section 5 to state and derive a homogenization formula. Finally, Section 6 is devoted to examples. More precisely, we show how our hypotheses are satisfied by functions depending on discrete determinants and by a linearization of Lennard-Jones energies mixing convex and concave quadratic pair energy densities. Finally, in the same section we recover the result in [2] as a particular case of our main theorem. As a last example we discuss the mixing of elastic and brittle interactions as used to described damaged materials.

2 Notation and preliminaries

We denote by Ω an open and bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. We set Q to be the unit cube with sides orthogonal to the canonical orthonormal basis $\{e_1, \dots, e_N\}$, $Q = \{x \in \mathbb{R}^N : |\langle x, e_i\rangle| \leq \frac{1}{2}$, for all $i = 1, \dots, N\}$ and for $\delta > 0$ we define $Q_\delta = \delta Q$. Moreover, for $x \in \mathbb{R}^N$ we set $Q(x) = Q + x$ and $Q_\delta(x) = Q_\delta + x$. We set $\mathcal{A}(\Omega) = \{A \subset \Omega : A \text{ open}\}$, $\mathcal{A}^{reg}(\Omega) = \{A \in \mathcal{A}(\Omega) : \partial A \text{ Lipschitz}\}$, and for $\delta > 0$ set $A_\delta = \{x \in \Omega : \mathrm{dist}_\infty(x, A) < \delta\}$ and $A^\delta = \{x \in A : \mathrm{dist}_\infty(x, A^c) > \delta\}$. For $B \subset \mathbb{R}^N$ we write $|B|$ for the N-dimensional Lebesgue measure of B. For a vector $x \in \mathbb{R}^N$ we set

$$\lfloor x \rfloor = (\lfloor x_1 \rfloor, \dots, \lfloor x_N \rfloor).$$

We define for $u : \mathbb{R}^N \to \mathbb{R}^n, \xi \in \mathbb{Z}^N, x \in \mathbb{R}^N$ and $\varepsilon > 0$

$$D^\xi_\varepsilon u(x) := \frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon|\xi|}$$

the discrete difference quotient of u at x in direction ξ.

For a function u we set $C(u)$ to be a constant depending on u, the dimension and its domain of definition and which may vary from line to line.

Slicing. We recall the standard notation for slicing arguments (see [6]). Let $\xi \in S^{N-1}$, and let $\Pi_\xi = \{y \in \mathbb{R}^N : \langle y, \xi\rangle = 0\}$ be the linear hyperplane orthogonal to ξ. If $y \in \Pi_\xi$ and $E \subset \mathbb{R}^N$ we define $E_\xi = \{y \in \Pi_\xi \text{ such that } \exists t \in \mathbb{R} : y + t\xi \in E\}$ and $E^\xi_y = \{t \in \mathbb{R} : y + t\xi \in E\}$. Moreover, if $u : E \to \mathbb{R}^n$ we set $u_{\xi,y} : E^\xi_y \to \mathbb{R}^n$ to $u_{\xi,y}(t) = u(y + t\xi)$.

Γ-convergence. A sequence of functionals $F_n : L^p(\Omega; \mathbb{R}^n) \to [0, +\infty]$ is said to Γ-converge to a functional $F : L^p(\Omega; \mathbb{R}^n) \to [0, +\infty]$ at $u \in L^p(\Omega; \mathbb{R}^n)$ as $n \to \infty$ and we write $F(u) = \Gamma\text{-}\lim_{n\to\infty} F_n(u)$ if the following two conditions are satisfied:

(i) For every u_n converging to u in $L^p(\Omega; \mathbb{R}^n)$ we have $\liminf_{n\to\infty} F_n(u_n) \geq F(u)$.

(ii) There exists a sequence $\{u_n\}_n \subset L^p(\Omega; \mathbb{R}^n)$ converging to u in $L^p(\Omega; \mathbb{R}^n)$ such that $\limsup_{n\to\infty} F_n(u_n) \leq F(u)$.

We say that F_n Γ-converges to F if $F(u) = \Gamma\text{-}\lim_{n\to\infty} F_n(u)$ for all $u \in L^p(\Omega;\mathbb{R}^n)$.

If $\{F_\varepsilon\}_{\varepsilon>0}$ is a family of functionals indexed by a continuous parameter $\varepsilon > 0$ we say that F_ε Γ-converges to F as $\varepsilon \to 0^+$ if for all $\varepsilon_n \to 0$ we have that F_{ε_n} Γ-converges to F. We define the Γ-$\liminf$ $F' : L^p(\Omega;\mathbb{R}^n) \to [0,\infty]$ and the Γ-$\limsup$ $F'' : L^p(\Omega;\mathbb{R}^n) \to [0,\infty]$ respectively by

$$F'(u) = \Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u) = \inf\Big\{ \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) : u_\varepsilon \to u \Big\},$$
$$F''(u) = \Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u) = \inf\Big\{ \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) : u_\varepsilon \to u \Big\}.$$

Note that the functionals F',F'' are lower semicontinuous and F_ε Γ-converges to F as $\varepsilon \to 0^+$ if and only if $F = F' = F''$.

Lattice functions. For $A \in \mathcal{A}(\Omega)$, we set $Z_\varepsilon(A) = \varepsilon\mathbb{Z}^N \cap A$. We set $\mathcal{A}_\varepsilon(A,\mathbb{R}^n) := \{u : Z_\varepsilon(A) \to \mathbb{R}^n\}$.

Definition 2.1. (Convergence of discrete functions) Functions $u \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^n)$ can be interpreted by functions belonging to the space $L^p(\Omega;\mathbb{R}^n)$ by setting (with slight abuse of notation) $u(z) = 0$ for all $z \in Z_\varepsilon(\Omega^c)$ and

$$u(x) = u(z_x^\varepsilon)$$

where z_x^ε is the closest point of $Z_\varepsilon(\mathbb{R}^N)$ to x (which is uniquely defined up to a set of measure 0). We then say that $u_\varepsilon \to u$ in $L^p(\Omega;\mathbb{R}^n)$ if the interpolations of u_ε converge to u in $L^p(\Omega;\mathbb{R}^n)$.

Integral representation. We will use the following integral representation result (see [15]).

Theorem 2.2. *Let $F : W^{1,p}(\Omega;\mathbb{R}^n) \times \mathcal{A}(\Omega) \to [0,+\infty]$ satisfy the following properties*

i) (measure property) For every $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ we have that $F(u,\cdot)$ is the restriction of a Radon measure to the open sets.

ii) (lower semicontinuity) For every $A \in \mathcal{A}(\Omega)$ we have that $F(\cdot,A)$ is weakly-$W^{1,1}(\Omega;\mathbb{R}^n)$ lower semicontinuous.

iii) (bounds) For every $(u,A) \in W^{1,p}(\Omega;\mathbb{R}^n) \times \mathcal{A}(\Omega)$ it holds that

$$0 \le F(u,A) \le C\left(\int_A |\nabla u|^p \,\mathrm{d}x + |A| \right)$$

iv) (translational invariance) For every $(u,A) \in W^{1,p}(\Omega;\mathbb{R}^n) \times \mathcal{A}(\Omega)$ and for every $c \in \mathbb{R}^n$ it holds $F(u,A) = F(u+c,A)$.

v) (locality) For every $A \in \mathcal{A}(\Omega)$ and every $u,v \in W^{1,p}(\Omega;\mathbb{R}^n)$ such that $u = v$ a.e. in A, we have that $F(u,A) = F(v,A)$.

Then there exists a Carathéodory function $f : \Omega \times \mathbb{R}^{n\times N} \to [0, +\infty]$ such that

$$F(u, A) = \int_A f(x, \nabla u)\mathrm{d}x$$

for every $(u, A) \in W^{1,p}(\Omega; \mathbb{R}^n) \times \mathcal{A}(\Omega)$.

vi) (translational invariance in x) if for every $M \in \mathbb{R}^{n\times N}$, $z, y \in \Omega$ and for every $\rho > 0$ such that $Q_\rho(z) \cup Q_\rho(y) \subset \Omega$ we have that

$$F(Mx, Q_\rho(y)) = F(Mx, Q_\rho(z)),$$

then f does not depend on x.

3 The main result

For all $i \in \Omega$, we denote by $\Omega_i = \Omega - i$ the translation of the set Ω with i at the origin, and we consider a function $\phi_i^\varepsilon : (\mathbb{R}^n)^{Z_\varepsilon(\Omega_i)} \to [0, +\infty)$. Let $F_\varepsilon : \mathcal{A}_\varepsilon(\Omega, \mathbb{R}^n) \times \mathcal{A}(\Omega) \to [0, +\infty)$ be defined by

$$F_\varepsilon(u, A) = \sum_{i \in Z_\varepsilon(A)} \varepsilon^N \phi_i^\varepsilon(\{u_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}). \quad (7)$$

In this section we give hypothesis on the energy densities ϕ_i^ε in order to ensure that the Γ-limits of the energies defined in (7) be finite only on $W^{1,p}(A, \mathbb{R}^n) \cap L^p(\Omega; \mathbb{R}^n)$ and there exists a Carathéodory function $f : \Omega \times \mathbb{R}^{n\times N} \to [0, \infty)$ such that

$$F(u, A) = \int_A f(x, \nabla u(x))\mathrm{d}x \quad (8)$$

for all $(u, A) \in W^{1,p}(A, \mathbb{R}^n) \cap L^p(\Omega; \mathbb{R}^n) \times \mathcal{A}(\Omega)$. A corresponding problem on the continuum is one of the first formalized in the theory of Γ-convergence, when F_ε themselves are integral energies. In that approach integral functionals are interpreted as depending on a pair (u, A) with u a Sobolev function and A a subset of Ω, when the integration is performed on A only. The compactness property of Γ-convergence then ensures that a Γ-converging subsequence exits on a dense family of open sets by a simple diagonal argument. Showing that the dependence of the limit on the set variable is that of a regular measure, the convergence is extended to a larger family of sets, and an integral representation result can be applied. The type of conditions singled out in that case can be adapted to the discrete setting, taking into account that discrete energies are “nonlocal” in nature since they depend on the interactions of points at a finite distance. The locality of the limit energy F must then be assured by a requirement of “vanishing nonlocality” as $\varepsilon \to 0$.

3.1 Hypotheses on the energy densities

A first requirement is that F_ε be invariant under addition of constants to u; namely

(H1) (*translational invariance*) for all $w \in \mathbb{R}^n$ we have

$$\phi_i^\varepsilon(\{z_j + w\}_{j\in Z_\varepsilon(\Omega_i)}) = \phi_i^\varepsilon(\{z_j\}_{j\in Z_\varepsilon(\Omega_i)}) \tag{9}$$

for all $\varepsilon > 0$, $i \in Z_\varepsilon(\Omega)$ and $z : Z_\varepsilon(\Omega) \to \mathbb{R}^n$.

A second requirement is that $F_\varepsilon(u_\varepsilon)$ be finite if $\widehat{u}_\varepsilon$ are a discretization of a $W^{1,p}$ function. In particular this should hold for affine functions.

(H2) (*upper bound for the Cauchy-Born deformation*) there exists $C > 0$, such that for every $M \in \mathbb{R}^{n\times N}$ and $Mx(i) = Mi$ we have

$$\phi_i^\varepsilon(\{(Mx)_j\}_{j\in Z_\varepsilon(\Omega_i)}) \le C(|M|^p + 1) \tag{10}$$

for all $\varepsilon > 0$ and all $i \in Z_\varepsilon(\Omega)$.

We then also require that the limit domain be exactly $W^{1,p}$ functions, with $p > 1$. To that end a coerciveness condition should be imposed.

(H3) (*equi-coerciveness*) there exists $c > 0$ such that

$$c\Big(\sum_{n=1}^{N} |D_\varepsilon^{e_n} z(0)|^p - 1\Big) \le \phi_i^\varepsilon(\{z_j\}_{j\in Z_\varepsilon(\Omega_i)}) \tag{11}$$

for all ε and i such that $i + \varepsilon e_n \in Z_\varepsilon(\Omega)$ for all $n \in \{1, \cdots, N\}$.

Next, we have to impose that the approximating continuum energy be local. Indeed, in principle discrete interactions are non-local, in that they take into account nodes of the lattice at a finite distance. This condition ensures that we can always find recovery sequences for a set $A \in \mathcal{A}(\Omega)$ that will not oscillate too much a finite distance away from A. We expect the limit to depend on ∇u if only the interactions for small distances are relevant, or, in other words, if the decay of interactions is fast enough. This can be formulated otherwise: we may require that the overall effect of long-range interactions at a point decay sufficiently fast as follows.

(H4) (*decaying non-locality*) There exist $\{C_{\varepsilon,\delta}^{j,\xi}\}_{\varepsilon>0,\delta>0,j\in\varepsilon\mathbb{Z}^N,\xi\in\mathbb{Z}^N}$, $C_{\varepsilon,\delta}^{j,\xi} \ge 0$ satisfying

$$\limsup_{\varepsilon\to 0} \sum_{j\in Z_\varepsilon(\mathbb{R}^N),\xi\in\mathbb{Z}^N} C_{\varepsilon,\delta}^{j,\xi} = 0 \quad \forall \delta > 0 \tag{12}$$

such that for all $\delta > 0$, $z, w \in \mathcal{A}_\varepsilon(\Omega, \mathbb{R}^n)$ satisfying $z(j) = w(j)$ for all $j \in Z_\varepsilon(Q_\delta(i))$ we have

$$\phi_i^\varepsilon(\{z_j\}_{j\in Z_\varepsilon(\Omega_i)}) \le \phi_i^\varepsilon(\{w_j\}_{j\in Z_\varepsilon(\Omega)}) + \sum_{\substack{j\in Z_\varepsilon(\Omega_i),\xi\in\mathbb{Z}^N \\ j+\varepsilon\xi\in Z_\varepsilon(\Omega_i)}} C_{\varepsilon,\delta}^{j,\xi} \left(|D_\varepsilon^\xi z(j)|^p + 1\right).$$

The final condition is the most technical and derives from our requirement that the limit can be expressed in terms of an integral. This is the most restrictive in the vectorial case $d > 1$ where convexity conditions have to be relaxed. A function $\psi : Z_\varepsilon(\Omega) \to \mathbb{R}$ is called a *cut-off function* if $0 \leq \psi \leq 1$.

(H5) (*controlled non-convexity*) There exist $C > 0$ and $\{C_\varepsilon^{j,\xi}\}_{\varepsilon>0, j\in\varepsilon\mathbb{Z}^N, \xi\in\mathbb{Z}^N}$, $C_\varepsilon^{j,\xi} \geq 0$ satisfying

$$\limsup_{\varepsilon\to 0} \sum_{j\in Z_\varepsilon(\mathbb{R}^N), \xi\in\mathbb{Z}^N} C_\varepsilon^{j,\xi} < +\infty, \quad \forall\, \delta > 0 \text{ we have } \limsup_{\varepsilon\to 0} \sum_{\max\{\varepsilon|\xi|, |j|\}>\delta} C_\varepsilon^{j,\xi} = 0 \tag{13}$$

such that for all $z, w \in \mathcal{A}_\varepsilon(\Omega, \mathbb{R}^n)$ and ψ cut-off functions we have

$$\begin{aligned} \phi_i^\varepsilon(\{\psi_j z_j + (1-\psi_j) w_j\}_{j\in Z_\varepsilon(\Omega_i)}) \leq & C \left(\phi_i^\varepsilon(\{z_j\}_{j\in Z_\varepsilon(\Omega_i)}) + \phi_i^\varepsilon(\{w_j\}_{j\in Z_\varepsilon(\Omega_i)}) \right) \\ & + R_i^\varepsilon(z, w, \psi) \end{aligned}$$

where

$$\begin{aligned} R_i^\varepsilon(z,w,\psi) = & \sum_{\substack{j\in Z_\varepsilon(\Omega_i), \xi\in\mathbb{Z}^N \\ j+\varepsilon\xi\in Z_\varepsilon(\Omega_i)}} C_\varepsilon^{j,\xi} \Big(\big(\sup_{\substack{k\in Z_\varepsilon(\Omega_i) \\ n\in\{1,\dots,N\}}} |D_\varepsilon^{e_n}\psi(k)|^p + 1 \big) |z(j+\varepsilon\xi) - w(j+\varepsilon\xi)|^p \Big) \\ & + \sum_{\substack{j\in Z_\varepsilon(\Omega_i), \xi\in\mathbb{Z}^N \\ j+\varepsilon\xi\in Z_\varepsilon(\Omega_i)}} C_\varepsilon^{j,\xi} \Big(|D_\varepsilon^\xi z(j)|^p + |D_\varepsilon^\xi w(j)|^p + 1 \Big). \end{aligned}$$

Remark 3.1. (observations on the assumptions) If condition (H1) fails we expect the limit not to be translational invariant anymore and if a integral representation exists it is expected to be of the form

$$F(u, A) = \int_A f(x, u, \nabla u) \mathrm{d}x.$$

However, integral-representation theorems for non-translation-invariant functionals in general require restrictive hypotheses that should be added to (H2)–(H5).

If condition (H2) fails the Γ-limit may not be finite on $W^{1,p}(\Omega; \mathbb{R}^n)$. Condition (H3) allows to estimate nearest-neighbour interactions centered in i in terms of ϕ_i^ε. Note that this estimate may still be true even if there are no interactions of the type $|D_\varepsilon^{e_n} u|^p$ taken into account by ϕ_i^ε. Indeed if $d = 1$ we may take $c_2, c_3 > 0$

$$\phi_i^\varepsilon(\{z_j\}_{j\in Z_\varepsilon(\Omega_i)}) = c_2 \left| \frac{z_{3\varepsilon} - z_\varepsilon}{2\varepsilon} \right|^2 + c_3 \left| \frac{z_{3\varepsilon} - z_0}{3\varepsilon} \right|^2.$$

If we assume a finite range R of interactions and assume that the potential ϕ_i^ε is well behaved in some sense condition (H4) is always satisfied and in the definition of R_i^ε the summation is only taken over $Q_R(i)$. If condition (H4) fails the Γ-limit may be non-local. Indeed there are examples (e.g. [9]) where functionals of the form

$$F(u) = \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_{\Omega\times\Omega} k(x,y) |u(x) - u(y)|^2 \mathrm{d}x$$

can be obtained as the Γ-limit of energies of the form

$$F_\varepsilon(u) = \sum_{i\in Z_\varepsilon(\Omega)} \sum_{\substack{\xi\in\mathbb{Z}^N \\ i+\varepsilon\xi\in Z_\varepsilon(\Omega)}} \varepsilon^N c^\varepsilon_{i,\xi} |D^\xi_\varepsilon u(i)|^2.$$

Note that (H1) is still satisfied. Condition (H5) mimics the so-called *fundamental estimate* in the continuum and ensures that the limit $F(u,\cdot)$ be subadditive as a set function. Note that this condition is satisfied for potentials with appropriate growth conditions; in particular, in Section 6.4 we show how the hypotheses above can be deduced from those in [2] in the case of pair potentials.

3.2 Compactness and integral representation

The goal of this section is to establish the proof of Theorem 3.2.

Theorem 3.2. (**Integral Representation**) *Let $F_\varepsilon : L^p(\Omega;\mathbb{R}^n) \to [0,+\infty]$ be defined by* (7)*, where $\phi^\varepsilon_i : (\mathbb{R}^n)^{Z_\varepsilon(\Omega)} \to [0,+\infty)$ satisfy* (H1)–(H5)*. Then for every sequence (ε_j) of positive numbers converging to 0, there exists a subsequence ε_{j_k} and a Carathéodory function $f : \Omega\times\mathbb{R}^{n\times N} \to [0,+\infty)$, quasiconvex in the second variable satisfying*

$$c(|\xi|^p - 1) \le f(x,\xi) \le C(|\xi|^p + 1)$$

with $0 < c < C$, such that $F_{\varepsilon_{j_k}}(\cdot)$ Γ-converges with respect to the $L^p(\Omega;\mathbb{R}^n)$-topology to the functional $F : L^p(\Omega;\mathbb{R}^n) \to [0,+\infty]$ defined by

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(x,\nabla u)\mathrm{d}x & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^n) \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, for any $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ and any $A \in \mathcal{A}(\Omega)$ we have

$$\Gamma\text{-}\lim_{k\to+\infty} F_{\varepsilon_{j_k}}(u,A) = \int_A f(x,\nabla u)\mathrm{d}x.$$

We will derive the proof of Theorem 3.2 as a consequence of some propositions and lemmas, which are fundamental in order to show that our limit functionals satisfy all the assumptions of Theorem 2.2. In the next two proposition we show with the use of (H1)–(H5) that assumption (ii) and assumption (iii) of Theorem 2.2 are satisfied. Note that property (14) below allows to deduce weak lower-semicontinuity in $W^{1,p}$ even though we prove the Γ-convergence of the discrete energies with respect to the strong $L^p(\Omega;\mathbb{R}^n)$-topology, so that assumption (ii) is satisfied.

Note that the proof of Proposition 3.3 is the same as the proof of Proposition 3.4 in [2]. We repeat it here only for completeness and the reader's convenience.

Proposition 3.3. *Let $\phi^\varepsilon_i : (\mathbb{R}^n)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ satisfy* (H3)*. If $u \in L^p(\Omega,\mathbb{R}^n)$ is such that $F'(u,A) < +\infty$, then $u \in W^{1,p}(A,\mathbb{R}^n)$ and*

$$F'(u,A) \ge c\left(||\nabla u||^p_{L^p(A;\mathbb{R}^{n\times N})} - |A|\right) \tag{14}$$

for some positive constant c independent on u and A.

Proof. Let $\varepsilon_n \to 0^+$ and let $u_n \to u$ in $L^p(\Omega;\mathbb{R}^n)$ be such that $\liminf_n F_{\varepsilon_n}(u_n, A) < +\infty$. By (H3) we get

$$F_{\varepsilon_n}(u_n, A) \geq c \sum_{i\in Z_\varepsilon(A)} \sum_{k=1}^{N} \varepsilon^N |D^{e_k}_{\varepsilon_n} u_n(i)|^p - cN|A|. \tag{15}$$

For any $k \in \{1,\cdots,N\}$, consider the sequence of piecewise-affine functions (v^k_n) defined as follows

$$v^k_n(x) = u_n(i) + D^{e_k}_{\varepsilon_n} u_n(i)(x_k - i_k) \quad x \in (i + [0,\varepsilon_n)^N) \cap \Omega,\ i \in Z_\varepsilon(A).$$

Note that v^k_n is a function of bounded variation and we will denote by $\frac{\partial v^k_n}{\partial x_k}$ the density of the absolutely continuous part of $D_{x_k} v^k_n$ with respect to the Lebesgue measure. Moreover, for $\mathcal{H}^{N-1}$-a.e. $y \in (A)^{e_k}$ the slices $(v^k_n)_{e_k,y}$ belong to $W^{1,p}((A)^{e_k}_y;\mathbb{R}^n)$. Note that, for any fixed $\eta > 0$, $v^k_n \to u$ in $L^p(A_\eta;\mathbb{R}^n)$ for every $k \in \{1,\cdots,N\}$. Moreover, since $\frac{\partial v^k_n}{\partial x_k}(x) = D^{e_k}_{\varepsilon_n} u_n(i)$ for $x \in i + [0,\varepsilon_n)^N$, we get

$$F_{\varepsilon_n}(u_n, A) \geq c \sum_{k=1}^{N} \int_{A^\eta} \left|\frac{\partial v^k_n}{\partial x_k}(x)\right|^p \mathrm{d}x - cN|A|.$$

We now apply a standard slicing argument. By Fubini's Theorem and Fatou's Lemma for any k we get

$$\liminf_n \int_{A^\eta} \left|\frac{\partial v^k_n}{\partial x_k}(x)\right|^p \mathrm{d}x \geq \int_{(A^\eta)^{e_k}} \liminf_n \int_{(A^\eta)^{e_k}_y} |(v^k_n)'_{e_k,y}(t)|^p \mathrm{d}t\mathrm{d}\mathcal{H}^{N-1}(y).$$

Since, up to passing to a subsequence, we may assume that, for $\mathcal{H}^{N-1}$-a.e. $y \in (A^\eta)^{e_k}$ $(v^k_n)_{e_k,y} \to u_{e_k,y}$ in $L^p((A^\eta)^{e_k}_y;\mathbb{R}^n)$, we deduce that $u_{e_k,y} \in W^{1,p}((A^\eta)^{e_k}_y;\mathbb{R}^n)$ for $\mathcal{H}^{N-1}$-a.e. $y \in (A^\eta)^{e_k}$ and

$$\liminf_n \int_{A^\eta} \left|\frac{\partial v^k_n}{\partial x_k}(x)\right|^p \mathrm{d}x \geq \int_{(A^\eta)^{e_k}} \int_{(A^\eta)^{e_k}_y} |u'_{e_k,y}(t)|^p \mathrm{d}t\mathrm{d}\mathcal{H}^{N-1}(y).$$

Then by (15), we have

$$\liminf_n F_{\varepsilon_n}(u_n, A) \geq c \sum_{k=1}^{N} \int_{(A^\eta)^{e_k}} \int_{(A^\eta)^{e_k}_y} |u'_{e_k,y}(t)|^p \mathrm{d}t\mathrm{d}\mathcal{H}^{N-1}(y) - cN|A|.$$

Since, in particular, the previous inequality implies that

$$\sum_{k=1}^{N} \int_{(A^\eta)^{e_k}} \int_{(A^\eta)^{e_k}_y} |u'_{e_k,y}(t)|^p \mathrm{d}t\mathrm{d}\mathcal{H}^{N-1}(y) < +\infty,$$

thanks to the characterization of $W^{1,p}$ by slicing we obtain that $u \in W^{1,p}(A^\eta,\mathbb{R}^n)$ and

$$\begin{aligned}\liminf_n F_{\varepsilon_n}(u_n, A) &\geq c \sum_{k=1}^{N} \int_{A^\eta} \left|\frac{\partial u}{\partial x_k}(x)\right|^p \mathrm{d}x - cN|A| \\ &\geq c\left(||\nabla u||^p_{L^p(A^\eta;\mathbb{R}^{n\times N})} - |A|\right)\end{aligned}$$

Letting $\eta \to 0^+$, we get the conclusion. □

Proposition 3.4. *Let* $\phi_i^\varepsilon : (\mathbb{R}^n)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ *satisfy* (H2),(H4) *and* (H5). *We then have*

$$F''(u,A) \leq C\left(||\nabla u||^p_{L^p(A;\mathbb{R}^{n\times N})} + |A|\right) \tag{16}$$

for some positive constant C independent on u and A.

Proof. We first show that the inequality holds for a special subclass $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ piecewise affine and then we recover the inequality for any $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ by a density argument. To this end introduce for every $K \in \mathbb{N}, K > 1$ the set of functions

$$\begin{aligned}\mathcal{PA}_K(\Omega;\mathbb{R}^n) = \Big\{ & u \in C(\Omega;\mathbb{R}^n) : u \text{ piecewise affine and } \exists u_1, u_2 \in \mathcal{PA}_{K-1}(\Omega;\mathbb{R}^n) \\ & u = \chi_{\overline{H}^+} u_1 + \chi_{H^-} u_2, \Big\}.\end{aligned} \tag{17}$$

and $u_1 = u_2$ on H, where $H = \{x \cdot \nu = c\}, \nu \in S^{N-1}, c \in \mathbb{R}$ hyperplane and $H^\pm = \{\pm x \cdot \nu > c\}$. For $K = 0$ we set

$$\mathcal{PA}_0(\Omega;\mathbb{R}^n) = \Big\{ u = Mx + b, M \in \mathbb{R}^{n\times N}, b \in \mathbb{R}^n \Big\}.$$

Note that for a given simplex decomposition $\mathcal{T}$ and for a given piecewise affine function u such that $\nabla u = \text{const}$ for every $T \in \mathcal{T}$ we have that $u \in \mathcal{PA}_K(\Omega;\mathbb{R}^n)$ for some $K \in \mathbb{N}$. It therefore suffices to consider $u \in \mathcal{PA}_K(\Omega;\mathbb{R}^n)$ and then proceed by density. We now construct $u_\delta \in W^{1,p}(\Omega;\mathbb{R}^n)$ such that $u_\delta \to u$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ and in $L^\infty(\Omega;\mathbb{R}^n)$ as $\delta \to 0$ and

$$F''(u_\delta, A) \leq C\left(||\nabla u||^p_{L^p(A;\mathbb{R}^{n\times N})} + |A|\right) + O(\delta). \tag{18}$$

Fix $K \in \mathbb{N}$ and assume that $u \in \mathcal{PA}_K(\Omega;\mathbb{R}^n)$, i.e. $u = u_K$ is of the form (17) with H hyperplane, $u^1_{k-1}, u^2_{k-1} \in \mathcal{PA}_{K-1}(\Omega;\mathbb{R}^n)$. We construct u_δ inductively. To this end assume that $u^{1,\delta}_{k-1}, u^{2,\delta}_{k-1}$ have already been constructed, we then define u^δ_k by

$$u^\delta_k = \varphi^\delta_k u^{1,\delta}_{k-1} + (1 - \varphi^\delta_k) u^{2,\delta}_{k-1},$$

where $\varphi^\delta_k \in C^\infty(\Omega;\mathbb{R})$ is a cut-off function, that is $0 \leq \varphi^\delta_k \leq 1, \text{supp}(\varphi^\delta_k) \subset (\overline{H}^+)_\delta, (\overline{H}^+)^\delta \subset \{\varphi^\delta_k = 1\}$ and $||\nabla \varphi^\delta_k||_\infty \leq \frac{C}{\delta}$ where we have that (17) holds for u_k with u^1_{k-1}, u^2_{k-1} and H and $u^{1,\delta}_{k-1}, u^{2,\delta}_{k-1}$ are regularizations of u^1_{k-1}, u^2_{k-1} constructed in the previous step. For $k = 0$ we set $u^\delta_k = u_k$ and we set $u_\delta = u^\delta_K$. We now prove (18). To this end we define

$$u^\varepsilon_\delta(i) = u_\delta(i), \quad i \in Z_\varepsilon(\Omega).$$

We have that $u^\varepsilon_\delta \to u_\delta$ in $L^p(\Omega;\mathbb{R}^n)$ and therefore

$$F''(u_\delta, A) \leq \limsup_{\varepsilon\to 0} F_\varepsilon(u^\varepsilon_\delta, A).$$

It thus suffices to prove

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u^\varepsilon_\delta, A) \leq C\left(||\nabla u||^p_{L^p(A;\mathbb{R}^{n\times N})} + |A|\right) + o(1) \text{ as } \delta \to 0.$$

Noting that $u_\delta \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ with $||\nabla u_\delta||_\infty \leq C||\nabla u||_\infty$ and for every $x \in Z_\varepsilon(\Omega \setminus \bigcup(H)_{2\delta})$ we have that $u(z) = u_\delta(z) = Mz + b$ for all $z \in Q_\delta(x)$, for some $M \in \mathbb{R}^{n\times N}, b \in \mathbb{R}^n$, where the (finite) union is taken over all H that are the half spaces defining u^1_{k-1}, u^2_{k-1} for all k. Now by (H2) we have that for $i \in Z_\varepsilon(\Omega \setminus \bigcup(H)_{2\delta})$ there holds

$$\begin{aligned}\phi_i^\varepsilon(\{(u^\varepsilon_\delta)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) &\leq \phi_i^\varepsilon(\{(Mx+b)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) + \sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)}} C^{j-i,\xi}_{\varepsilon,\delta}(|D^\xi_\varepsilon u^\varepsilon_\delta(j)|^p + 1)\\ &\leq C(|\nabla u(i)|^p + 1) + C(u)\sum_{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N} C^{j-i,\xi}_{\varepsilon,\delta}.\end{aligned}$$

Summing over all $i \in Z_\varepsilon(A \setminus \bigcup(H)_{2\delta})$, taking the lim sup as $\varepsilon \to 0$, taking into account (12) and using the dominated convergence theorem we obtain

$$\limsup_{\varepsilon\to 0}\sum_{i\in Z_\varepsilon(\Omega\setminus\bigcup(H)_{2\delta})}\varepsilon^N\phi_i^\varepsilon(\{(u^\varepsilon_\delta)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) \leq C\left(||\nabla u||_{L^p(A;\mathbb{R}^{n\times N})} + |A|\right).$$

It remains to prove that

$$\limsup_{\varepsilon\to 0}\sum_{i\in Z_\varepsilon(\Omega\cap(\bigcup(H)_{2\delta}))}\varepsilon^N\phi_i^\varepsilon(\{(u^\varepsilon_\delta)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) = o(1) \text{ as } \delta\to 0.$$

To this end we prove that for every $i \in Z_\varepsilon(\Omega \cap (\bigcup(H)_{2\delta}))$ there holds

$$\phi_i^\varepsilon(\{(u^\varepsilon_\delta)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) \leq C(u) \tag{19}$$

for some constant $C(u)$ depending only on u. We prove (19) by induction. Assume that we proved already

$$\phi_i^\varepsilon(\{((u^{\alpha,\delta}_{k-1})^\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) \leq C(u), \alpha = 1, 2,$$

where $(u^{\alpha,\delta}_{k-1})^\varepsilon$ is the discretization of $u^{\alpha,\delta}_{k-1}$. Note that this claim follows directly from (H2) for $k = 0$, thus it suffices to prove only the induction step. Now assume that $\varphi^\delta_k \notin \{0, 1\}$ for some $x \in Q_\delta(i)$. Using (H5) with $u^{1,\delta}_{k-1}, u^{2,\delta}_{k-1}$ and with φ^δ_k as a cut-off function we obtain

$$\begin{aligned}\phi_i^\varepsilon(\{((u^\delta_k)^\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) \leq& C\left(\phi_i^\varepsilon(\{((u^{1,\delta}_{k-1})^\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) + \phi_i^\varepsilon(\{((u^{2,\delta}_{k-1})^\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) + 1\right)\\ &+ R^\varepsilon_i\left((u^{1,\delta}_{k-1})^\varepsilon, (u^{2,\delta}_{k-1})^\varepsilon, \varphi^\delta_k\right)\\ &\leq C(u) + R^\varepsilon_i\left((u^{1,\delta}_{k-1})^\varepsilon, (u^{2,\delta}_{k-1})^\varepsilon, \varphi^\delta_k\right).\end{aligned}$$

We have

$$\begin{aligned}R^\varepsilon_i\left((u^{1,\delta}_{k-1})^\varepsilon, (u^{2,\delta}_{k-1})^\varepsilon, \varphi^\delta_k\right) =& \left(\frac{1}{\delta^p}+1\right)\sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)}} C^{j-i,\xi}_\varepsilon|(u^{1,\delta}_{k-1})^\varepsilon(j+\varepsilon\xi) - (u^{2,\delta}_{k-1})^\varepsilon(j+\varepsilon\xi)|^p\\ &+ \sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)}} C^{j-i,\xi}_\varepsilon\left(|D^\xi_\varepsilon(u^{1,\delta}_{k-1})^\varepsilon(j)|^p + |D^\xi_\varepsilon(u^{2,\delta}_{k-1})^\varepsilon(j)|^p\right).\end{aligned}$$

Note that by construction we have that $||\nabla u_{k-1}^{\alpha,\delta}||_\infty \leq C||\nabla u||_\infty$ and therefore, using (13), we obtain

$$\sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)}} C_\varepsilon^{j-i,\xi}\left(|D_\varepsilon^\xi(u_{k-1}^{1,\delta})^\varepsilon(j)|^p + |D_\varepsilon^\xi(u_{k-1}^{2,\delta})^\varepsilon(j)|^p\right) \leq C(u).$$

Now for $x \in Q_\delta(i)$ we have that

$$|u_{k-1}^{1,\delta}(x) - u_{k-1}^{2,\delta}(x)| \leq |u_{k-1}^{1,\delta}(x) - u_{k-1}^{1,\delta}(x_H)| + |u_{k-1}^{1,\delta}(x_H) - u_{k-1}^{2,\delta}(x_H)| + |u_{k-1}^{2,\delta}(x) - u_{k-1}^{2,\delta}(x_H)|,$$

where $x_H \in Q_{2\delta}(x)\cap H$ so that $u_{k-1}^{2,\delta}(x_H) = u_{k-1}^{1,\delta}(x_H)$ and we therefore have that the third term on the right hand side is equal to 0. Since $||\nabla u_{k-1}^{\alpha,\delta}||_\infty \leq C||\nabla u||_\infty$ we have that

$$|u_{k-1}^{\alpha,\delta}(x) - u_{k-1}^{\alpha,\delta}(x_H)| \leq C(u)\delta$$

for $\alpha = 1, 2$. Therefore we obtain

$$|u_{k-1}^{1,\delta}(x) - u_{k-1}^{2,\delta}(x)| \leq C(u)\delta.$$

Splitting the sum into the summation over j, ξ such that $\max\{\varepsilon|\xi|, |j-i|\} > \delta$ and the complement and using that $||\nabla u_{k-1}^{\alpha,\delta}||_\infty \leq C||\nabla u||_\infty$ for all $k \in \{1,\dots,K\}, \alpha \in \{1,2\}, \delta > 0$ we obtain

$$\begin{aligned} R_i^\varepsilon\left((u_{k-1}^{1,\delta})^\varepsilon, (u_{k-1}^{2,\delta})^\varepsilon, \varphi_k^\delta\right) &\leq C(u)\Bigg((1+\delta^p)\sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)\\ \max\{\varepsilon|\xi|,|j-i|\}\leq\delta}} C_\varepsilon^{j-i,\xi}\Bigg) \\ &\quad + C(u)\Bigg(\Big(\frac{1}{\delta^p}+1\Big)\sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)\\ \max\{\varepsilon|\xi|,|j-i|\}>\delta}} C_\varepsilon^{j-i,\xi} + 1\Bigg) \\ &\leq C(u) \end{aligned}$$

for $\varepsilon > 0$ small enough. Now if $\varphi_k^\delta \notin \{0,1\}^C$ for all $x \in Q_\delta(i)$ we have without loss of generality that $\varphi_k^\delta = 1$ for all $x \in Q_\delta(i)$. Using (H4) we obtain and the same estimates as for $i \in Z_\varepsilon(A \setminus \bigcup(H)_{2\delta})$, that

$$\begin{aligned} \phi_i^\varepsilon(\{((u_k^\delta)^\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) &\leq C\phi_i^\varepsilon(\{((u_{k-1}^{1,\delta})^\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) + \sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)}} C_{\varepsilon,\delta}^{j-i,\xi}(|D_\varepsilon^\xi u_\delta^\varepsilon(j)|^p + 1) \\ &\leq C(u) \end{aligned}$$

and (19) follows. Summing over $i \in Z_\varepsilon(A \cap \bigcup(H)_{2\delta})$ we obtain

$$\sum_{i\in Z_\varepsilon(A\cap\bigcup(H)_{2\delta})} \varepsilon^N \phi_i^\varepsilon(\{(u_\delta^\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) \leq C(u)\varepsilon^N \# Z_\varepsilon(A\cap\bigcup(H)_{2\delta}) \leq C(u)\left|A\cap\bigcup(H)_{2\delta}\right|.$$

Hence it follows that

$$\limsup_{\varepsilon\to 0} \sum_{i\in Z_\varepsilon(A\cap\bigcup(H)_{2\delta})} \varepsilon^N \phi_i^\varepsilon(\{(u_\delta^\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) = o(1) \text{ as } \delta\to 0,$$

since $\lim_{\delta\to 0}\left|A\cap\bigcup(H)_{2\delta}\right| = 0$. and therefore (18) follows. Now by the lower semicontinuity of $F''(\cdot, A)$ we have

$$F''(u,A) \le \liminf_{\delta\to 0} F''(u_\delta, A) \le C\left(||\nabla u||^p_{L^p(A;\mathbb{R}^{n\times N})} + |A|\right).$$

Now for general $u\in W^{1,p}(\Omega;\mathbb{R}^n)$ take $\{u_n\}\subset W^{1,p}(\Omega;\mathbb{R}^n)$ piecewise affine such that $u_n\to u$ strongly in $W^{1,p}(\Omega;\mathbb{R}^n)$ and again by the lower semicontinuity of $F''(\cdot, A)$ we have

$$F''(u,A) \le \liminf_{n\to\infty} F''(u_n, A) \le \lim_{n\to\infty} C(||\nabla u_n||^p_{L^p(A;\mathbb{R}^{d\times N})} + |A|) = C(||\nabla u||^p_{L^p(A;\mathbb{R}^{n\times N})} + |A|)$$

and the statement is proven. □

Proposition 3.5. *Let $\phi_i^\varepsilon : (\mathbb{R}^n)^{Z_\varepsilon(\Omega)} \to [0,+\infty)$ satisfy* (H2)–(H5)*. Let $A, B\in\mathcal{A}(\Omega)$ and let $A', B'\in\mathcal{A}(\Omega)$ be such that $A'\subset\subset A$ and $B'\subset\subset B$. Then for any $u\in W^{1,p}(\Omega;\mathbb{R}^n)$ we have*

$$F''(u, A'\cup B') \le F''(u,A) + F''(u,B)$$

Proof. Without loss of generality, we may suppose $F''(u,A)$ and $F''(u,B)$ finite. Let $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ converge to u in $L^p(\Omega;\mathbb{R}^n)$ and be such that

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A) = F''(u,A), \quad \limsup_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon, B) = F''(u,B),$$

and therefore

$$\sup_{\varepsilon>0}\sum_{i\in Z_\varepsilon(A)} \varepsilon^N\phi_i^\varepsilon(\{(u_\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) < \infty, \tag{20}$$

$$\sup_{\varepsilon>0}\sum_{i\in Z_\varepsilon(B)} \varepsilon^N\phi_i^\varepsilon(\{(v_\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) < \infty. \tag{21}$$

By (H3) we have that

$$\sup_{n\in\{1,\dots,N\}}\sup_{\varepsilon>0}\sum_{i\in Z_\varepsilon(A'')} \varepsilon^N|D_\varepsilon^{e_n}u_\varepsilon(i)|^p < +\infty \tag{22}$$

$$\sup_{n\in\{1,\dots,N\}}\sup_{\varepsilon>0}\sum_{i\in Z_\varepsilon(B'')} \varepsilon^N|D_\varepsilon^{e_n}v_\varepsilon(i)|^p < +\infty \tag{23}$$

for all $A''\subset\subset A, B''\subset\subset B$. Since u_ε and v_ε converge to u in $L^p(\Omega;\mathbb{R}^n)$, we have that

$$\sum_{i\in Z_\varepsilon(\Omega)} \varepsilon^N\left(|u_\varepsilon(i)|^p + |v_\varepsilon(i)|^p\right) \le ||u_\varepsilon||^p_{L^p(\Omega;\mathbb{R}^n)} + ||v_\varepsilon||^p_{L^p(\Omega;\mathbb{R}^n)} \le C < \infty \tag{24}$$

$$\sum_{i\in Z_\varepsilon(\Omega)} \varepsilon^N\left(|u_\varepsilon(i) - v_\varepsilon(i)|^p\right) \le ||u_\varepsilon - v_\varepsilon||_{L^p(\Omega;\mathbb{R}^n)} \to 0. \tag{25}$$

Since $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ there exists $\tilde{u}_\varepsilon, \tilde{v}_\varepsilon$ such that $\tilde{u}_\varepsilon$ and $\tilde{v}_\varepsilon$ converge to u in $L^p(\Omega;\mathbb{R}^n)$ and

$$\sup_{n\in\{1,\dots,N\}} \sup_{\varepsilon>0} \sum_{i\in Z_\varepsilon(\Omega)} \varepsilon^N \left(|D_\varepsilon^{e_n}\tilde{u}_\varepsilon(i)|^p + |D_\varepsilon^{e_n}\tilde{v}_\varepsilon(i)|^p\right) < \infty. \tag{26}$$

Take $A'', A''', B'', B''' \in \mathcal{A}(\Omega)$,$\varphi_A, \varphi_B \in C^\infty(\Omega)$ such that $A' \subset\subset A'' \subset\subset A''' \subset\subset A$, $B' \subset\subset B'' \subset\subset B''' \subset\subset B$, $0 \le \varphi_A, \varphi_B \le 1$, $A''' \subset \{\varphi_A = 0\}$, $B''' \subset \{\varphi_B = 0\}$, $A'' \subset \{\varphi_A = 1\}$, $B'' \subset \{\varphi_B = 1\}$ and $||\nabla\varphi_A||_\infty, ||\nabla\varphi_B||_\infty \le C$, and define $u'_\varepsilon = \varphi_A u_\varepsilon + (1-\varphi_A)\tilde{u}_\varepsilon$,$v'_\varepsilon = \varphi_B v_\varepsilon + (1-\varphi_B)\tilde{v}_\varepsilon$. Now for $j \in Z_\varepsilon(\Omega)$, ψ cut-off function $z, w \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^n)$ $v = \psi z + (1-\psi)w$ we have

$$D_\varepsilon^{e_n}v(j) = \psi(j)D_\varepsilon^{e_n}z(j) + (1-\psi(j))D_\varepsilon^{e_n}w(j) + D_\varepsilon^{e_n}\psi(j)(z(j)-w(j)) \tag{27}$$

Since $\{\varphi_A > 0\} \subset\subset A$, by (22), (26) and (27) we have that

$$\sup_{n\in\{1,\dots,N\}} \sup_{\varepsilon>0} \sum_{j\in Z_\varepsilon(\Omega)} \varepsilon^N |D_\varepsilon^{e_n}u'_\varepsilon(j)|^p < \infty. \tag{28}$$

We can perform a similar construction for v'_ε and therefore assume that an analogous bound to (28) holds also for v'_ε. Moreover, since u'_ε and v'_ε converge to u in $L^p(\Omega;\mathbb{R}^n)$ we have that (24) and (25) hold with u'_ε and v'_ε. Now for $\delta > 0$, by (H4), it holds

$$\phi_i^\varepsilon(\{(u'_\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) \le \phi_i^\varepsilon(\{(u_\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) + \sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)}} C_{\varepsilon,\delta}^{j-i,\xi}(|D_\varepsilon^\xi u'_\varepsilon(j)|^p + 1) \tag{29}$$

as well as a similar estimate for v'_ε in B'. Set

$$\mathrm{d} := \mathrm{dist}_\infty(A', A^c) \qquad \text{and} \qquad A_k := (A')_{\frac{k}{3K}\mathrm{d}}$$

for any $k \in \{K,\dots,2K\}$. Let φ_k be a cut-off function between A_k and A_{k+1}, with $||\nabla\varphi_k||_\infty \le CK$. Then for any $k \in \{K,\dots,2K\}$ consider the family of functions $w_\varepsilon^k \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^n)$ converging to u in $L^p(\Omega;\mathbb{R}^n)$, defined as

$$w_\varepsilon^k(i) = \varphi_k(i)u'_\varepsilon(i) + (1-\varphi_k(i))v'_\varepsilon(i).$$

Given $i \in Z_\varepsilon(A' \cup B')$, then either $\mathrm{dist}_\infty(i, A_{k+1} \setminus \overline{A}_k) \ge \frac{\mathrm{d}}{3K}$, in which case either $w_\varepsilon^k(j) = u'_\varepsilon(j)$ for $j \in Z_\varepsilon(Q_{\frac{\mathrm{d}}{2K}}(i))$ and $i \in Z_\varepsilon(A_k)$ or $w_\varepsilon^k(j) = v'_\varepsilon(j)$ $j \in Z_\varepsilon(Q_{\frac{\mathrm{d}}{2K}}(i))$ and $i \in Z_\varepsilon((A' \cup B') \setminus A_{k+1}) \subset Z_\varepsilon(B')$, or $\mathrm{dist}_\infty(i, A_{k+1} \setminus \overline{A}_k) < \frac{\mathrm{d}}{6K}$. In the first case, using (H4), we estimate

$$\phi_i^\varepsilon(\{(w_\varepsilon^k)_{j+i}\}_{j\in Z_\varepsilon(\Omega)}) \le \phi_i^\varepsilon(\{(u'_\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega)}) + \sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)}} C_{\varepsilon,\frac{\mathrm{d}}{2K}}^{j-i,\xi}(|D_\varepsilon^\xi w_\varepsilon^k(j)|^p + 1). \tag{30}$$

In the second case, using (H4), we estimate

$$\phi_i^\varepsilon(\{(w_\varepsilon^k)_{j+i}\}_{j\in Z_\varepsilon(\Omega)}) \le \phi_i^\varepsilon(\{(v'_\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega)}) + \sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)}} C_{\varepsilon,\frac{\mathrm{d}}{2K}}^{j-i,\xi}(|D_\varepsilon^\xi w_\varepsilon^k(j)|^p + 1). \tag{31}$$

Using (27) and the convexity of $|\cdot|^p$ we have for $j \in Z_\varepsilon(\Omega)$ and $\xi \in \mathbb{Z}^N$

$$|D_\varepsilon^\xi w_\varepsilon^k(j)|^p \leq |D_\varepsilon^\xi u'_\varepsilon(j)|^p + |D_\varepsilon^\xi v'_\varepsilon(j)|^p + CK^p|u'_\varepsilon(j+\varepsilon\xi) - v'_\varepsilon(j+\varepsilon\xi)|^p. \tag{32}$$

Now if $\mathrm{dist}_\infty(i, A_{k+1} \setminus \overline{A}_k) < \frac{\mathrm{d}}{3K}$ we have that $i \in Z_\varepsilon(A_{k+2} \setminus \overline{A}_{k-1}) =: Z_\varepsilon(S_k)$ where $S_k \subset\subset A \cap B$. By (H5) we have that for such an i it holds

$$\phi_i^\varepsilon(\{(w_\varepsilon^k)_{j+i}\}_{j\in Z_\varepsilon(\Omega)}) \leq C(\phi_i^\varepsilon(\{(v'_\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega)}) + \phi_i^\varepsilon(\{(u'_\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega)})) + R_i^\varepsilon(u'_\varepsilon, v'_\varepsilon, \varphi_k) \tag{33}$$

where

$$\begin{aligned} R_i^\varepsilon(u'_\varepsilon, v'_\varepsilon, \varphi_k) = (CK^p+1) &\sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)}} C_\varepsilon^{j-i,\xi}|u_\varepsilon(j+\varepsilon\xi) - v_\varepsilon(j+\varepsilon\xi)|^p \\ &+ \sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)}} C_\varepsilon^{j-i,\xi}\left(|D_\varepsilon^\xi u'_\varepsilon(j)|^p + |D_\varepsilon^\xi v'_\varepsilon(j)|^p + 1\right). \end{aligned} \tag{34}$$

Summing over $i \in Z_\varepsilon(A' \cup B')$ and splitting into the two cases as described above, using (30)–(34), we have

$$\begin{aligned} F_\varepsilon(w_\varepsilon^k, A'\cup B') \leq &\sum_{\substack{i\in Z_\varepsilon(A'\cup B')\\ \mathrm{dist}_\infty(i,A_{k+1}\setminus\overline{A}_k)\geq\frac{\mathrm{d}}{3K}}} \varepsilon^N\phi_i^\varepsilon(\{(w_\varepsilon^k)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) + \sum_{i\in Z_\varepsilon(S_k)}\varepsilon^N\phi_i^\varepsilon(\{(w_\varepsilon^k)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) \\ \leq &F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) \\ &+ CK^p\sum_{i\in Z_\varepsilon(S_k)}\varepsilon^N \sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)}} C_\varepsilon^{j-i,\xi}|u'_\varepsilon(j+\varepsilon\xi) - v'_\varepsilon(j+\varepsilon\xi)|^p \\ &+ CK^p\sum_{i\in Z_\varepsilon(A'\cup B')}\varepsilon^N \sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)}} C_{\varepsilon,\frac{\mathrm{d}}{2K}}^{j-i,\xi}|u'_\varepsilon(j+\varepsilon\xi) - v'_\varepsilon(j+\varepsilon\xi)|^p \\ &+ \sum_{i\in Z_\varepsilon(A'\cup B')}\varepsilon^N \sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)}} C_{\varepsilon,\frac{\mathrm{d}}{2K}}^{j-i,\xi}\left(|D_\varepsilon^\xi u'_\varepsilon(j)|^p + |D_\varepsilon^\xi v'(j)|^p + 1\right) \\ &+ \sum_{i\in Z_\varepsilon(S_k)}\varepsilon^N \sum_{\substack{j\in Z_\varepsilon(\Omega),\xi\in\mathbb{Z}^N\\ j+\varepsilon\xi\in Z_\varepsilon(\Omega)}} C_\varepsilon^{j-i,\xi}\left(|D_\varepsilon^\xi u'_\varepsilon(j)|^p + |D_\varepsilon^\xi v'(j)|^p + 1\right) \\ &+ C\sum_{i\in Z_\varepsilon(S_k)}\varepsilon^N\left(\phi_i^\varepsilon(\{(v'_\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) + \phi_i^\varepsilon(\{(u'_\varepsilon)_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)})\right). \end{aligned}$$

Note that $\#\{j \neq k : S_k \cap S_j \neq \emptyset\} \leq 5$. Therefore summing over $k \in \{K, \dots, 2K-1\}$, averaging and taking into account (20)–(24), (28) and Lemma 3.6 in [2], we get

$$\frac{1}{K}\sum_{k=K}^{2K-1} F_\varepsilon(w_\varepsilon^k, A'\cup B') \leq F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) + \frac{C}{K} + (K^p+1)O(\varepsilon). \tag{35}$$

For any $\varepsilon > 0$ there exists $k(\varepsilon) \in \{K, \dots, 2K-1\}$ such that

$$F_\varepsilon(w_\varepsilon^{k(\varepsilon)}, A' \cup B') \leq \frac{1}{K} \sum_{k=K}^{2K-1} F_\varepsilon(w_\varepsilon^k, A' \cup B'). \tag{36}$$

Then, since $w_\varepsilon^{k(\varepsilon)}$ still converges to u in $L^p(\Omega; \mathbb{R}^n)$, by (35) and (36), letting $\varepsilon \to 0$ we get

$$F''(u, A' \cup B') \leq F''(u, A) + F(u, B) + \frac{C}{K}.$$

Letting $K \to \infty$ we obtain the claim. □

Proposition 3.6. *Let* $\phi_i^\varepsilon : (\mathbb{R}^n)^{Z_\varepsilon(\Omega)} \to [0, +\infty)$ *satisfy* (H2)–(H5). *Then for any* $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ *and any* $A \in \mathcal{A}(\Omega)$ *we have*

$$\sup_{A' \subset\subset A} F''(u, A') = F''(u, A).$$

Proof. Since $F''(u, \cdot)$ is an increasing set function, it suffices to prove

$$\sup_{A' \subset\subset A} F''(u, A') \geq F''(u, A).$$

In order to prove this, we define an extension of the functional F_ε to a functional $\tilde{F}_\varepsilon$ defined on a bounded, smooth, open set $\tilde{\Omega} \supset\supset \Omega$ such that

$$\tilde{F}_\varepsilon(\tilde{u}, A) = F_\varepsilon(u, A)$$

for all $A \in \mathcal{A}(\Omega)$ and all $\tilde{u} \in \mathcal{A}_\varepsilon(\tilde{\Omega}; \mathbb{R}^n)$ such that $\tilde{u} = u$ in $Z_\varepsilon(\Omega)$ and therefore

$$F''(u, A) = \tilde{F}''(\tilde{u}, A) \tag{37}$$

for all $A \in \mathcal{A}(\Omega)$, $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ and $\tilde{u} \in W^{1,p}(\Omega; \mathbb{R}^n)$ such that $\tilde{u} = u$ a.e. in Ω. To this end we define $F_\varepsilon : \mathcal{A}_\varepsilon(\tilde{\Omega}) \times \mathcal{A}(\tilde{\Omega}) \to [0, +\infty)$ by

$$\tilde{F}_\varepsilon(u, A) = \sum_{i \in Z_\varepsilon(A)} \varepsilon^N \tilde{\phi}_i^\varepsilon(\{u_{j+i}\}_{j \in Z_\varepsilon(\tilde{\Omega}_i)})$$

where $\tilde{\phi}_i^\varepsilon : (\mathbb{R}^n)^{Z_\varepsilon(\tilde{\Omega})} \to [0, +\infty)$ is defined by

$$\tilde{\phi}_i^\varepsilon(\{z_{j+i}\}_{j \in Z_\varepsilon(\Omega_i)}) := \begin{cases} \phi_i^\varepsilon(\{(z|_\Omega)_{j+i}\})_{j \in Z_\varepsilon(\Omega)} & i \in Z_\varepsilon(\Omega) \\ c \sum_{n=1}^N |D_\varepsilon^{e_n} z(i)|^p & i \in \tilde{\Omega} \setminus \Omega \end{cases}$$

with $c > 0$ as in (15). Note that $\tilde{\phi}_i^\varepsilon$ satisfies (H2)–(H5). Let $u \in W^{1,p}(\Omega; \mathbb{R}^n)$, extended to $\tilde{u} \in W^{1,p}(\tilde{\Omega}; \mathbb{R}^n)$. Let $A \in \mathcal{A}(\Omega)$; for $\delta > 0$ find $A^\delta, A_\delta, B_\delta$ such that $A^\delta \supset\supset A \supset\supset A_\delta \supset\supset A'_\delta \supset\supset B^\delta \supset\supset B_\delta$ and

$$|A^\delta \setminus B_\delta| + ||\nabla u||_{L^p(A^\delta \setminus B_\delta; \mathbb{R}^{n \times N})} \leq \delta.$$

Applying Proposition 3.5 with $U = A^\delta \setminus \overline{B}_\delta$, $V = A_\delta$, $U' = A \setminus \overline{B}^\delta$ and $V' = A'_\delta$ we have $U' \cup V' = A$ and therefore

$$\begin{aligned}\tilde{F}''(\tilde{u}, A) &\leq \tilde{F}''(\tilde{u}, U' \cup V') \leq \tilde{F}''(u, U) + \tilde{F}''(u, V) \leq \tilde{F}''(\tilde{u}, A_\delta) + \tilde{F}''(\tilde{u}, A^\delta \setminus \overline{B}_\delta)\\ &\leq \tilde{F}''(\tilde{u}, A_\delta) + C\left(|A^\delta \setminus B_\delta| + ||\nabla u||^p_{L^p(A^\delta \setminus B_\delta;\mathbb{R}^{d\times N})}\right)\\ &\leq \tilde{F}''(u, A_\delta) + C\delta \leq \sup_{A' \subset\subset A} \tilde{F}''(\tilde{u}, A') + C\delta\end{aligned}$$

Applying (37) to $u, \tilde{u}$ and A, A' we obtain

$$F''(u, A) \leq \sup_{A' \subset\subset A} F''(u, A') + C\delta.$$

The claim follows as $\delta \to 0^+$. □

Proposition 3.7. *Let* $\phi_i^\varepsilon : (\mathbb{R}^n)^{Z_\varepsilon(\Omega)} \to [0, +\infty)$ *satisfy* (H2)–(H5). *Then for any* $A \in \mathcal{A}(\Omega)$ *and for any* $u, v \in W^{1,p}(\Omega; \mathbb{R}^n)$*, such that* $u = v$ *a.e. in* A *we have*

$$F''(u, A) = F''(v, A)$$

Proof. Thanks to Proposition 3.6, we may assume that $A \subset\subset \Omega$. We first prove

$$F''(u, A) \geq F''(v, A)$$

Given $\delta > 0$ there exist $A_\delta \subset\subset A$ such that

$$|A \setminus \overline{A_\delta}| + ||\nabla u||^p_{L^p(\Omega;\mathbb{R}^{n\times N})} \leq \delta$$

Let $v_\varepsilon : Z_\varepsilon(\Omega) \to \mathbb{R}^n$, $u_\varepsilon : Z_\varepsilon(\Omega) \to \mathbb{R}^n$ be such that $v_\varepsilon \to v$ and $u_\varepsilon \to u$ in $L^p(\Omega; \mathbb{R}^n)$ and

$$\limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon, A) = F''(u, A)$$

$$\limsup_{\varepsilon \to 0^+} F_\varepsilon(v_\varepsilon, A \setminus \overline{A_\delta}) = F''(v, A \setminus \overline{A_\delta}) \leq C\left(|A \setminus \overline{A_\delta}| + ||\nabla u||^p_{L^p(\Omega;\mathbb{R}^{n\times N})}\right) \leq C\delta$$

Performing the same cut-off construction as in Proposition 3.5 we obtain a function w_ε converging to v in $L^p(\Omega; \mathbb{R}^n)$ such that for $\varepsilon > 0$ small enough we obtain

$$F_\varepsilon(w_\varepsilon, A') \leq F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, A \setminus \overline{A_\delta}) + \frac{C_\delta}{K} + K^p O(\varepsilon)$$

for some $A' \subset\subset A$. Taking $\varepsilon \to 0^+$ we obtain

$$F''(v, A') \leq F''(u, A) + \frac{C_\delta}{K} + C\delta$$

Letting $K \to +\infty$ and $\delta \to 0$ we obtain the desired inequality. Exchanging the roles of u and v we obtain the other inequality. □

Proof of Theorem 3.2. By the compactness property of Γ-convergence there exists a subsequence ε_{j_k} of ε_j such that for any $(u, A) \in W^{1,p}(\Omega;\mathbb{R}^n) \times \mathcal{A}(\Omega)$ there exists

$$\Gamma(L^p)\text{-}\lim_k F_{\varepsilon_{j_k}}(u, A) =: F(u, A)$$

(see [15] Theorem 10.3). Moreover, by Proposition 3.4 we have that

$$\Gamma(L^p)\text{-}\lim_k F_{\varepsilon_{j_k}}(u) = +\infty$$

for any $u \in L^p(\Omega;\mathbb{R}^n) \setminus W^{1,p}(\Omega;\mathbb{R}^n)$. So it suffices to check that for every $(u, A) \in W^{1,p}(\Omega;\mathbb{R}^n) \times \mathcal{A}(\Omega)$, $F(u, A)$ satisfies all the hypothesis of Theorem 2.2 in [2]. In fact the superaditivity property of $F_\varepsilon(u, \cdot)$ is conserved in the limit. Thus, as an consequence of Propositions (3.4)–(3.7) and thanks to De Giorgi-Letta Criterion (see [15]), hypotheses (i), (ii), (iii) hold true. Moreover, since $F_\varepsilon(u, A)$ is translationally invariant, hypothesis (iv) is satisfied and finally, by the lower semicontinuity property of Γ-limit, also hypothesis (v) is fulfilled. □

4 Treatment of Dirichlet boundary data

In order to recover the limiting energy density we will establish the next lemma which asserts that our energies still converge if we suitably assign affine boundary conditions. From this, one is able to recover the value of f in Theorem 3.2 by a blow-up argument. Given $M \in \mathbb{R}^{n\times N}$,$m \in \mathbb{N}$, $\varepsilon > 0$ and $A \in \mathcal{A}^{reg}(\Omega)$ set

$$\mathcal{A}^{M,m}_\varepsilon(A;\mathbb{R}^n) = \big\{u \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^n) : u(i) = Mi \text{ if } (i + [-m\varepsilon, m\varepsilon)^N) \cap A^c \neq \emptyset\big\} \tag{38}$$

For $M \in \mathbb{R}^{d\times N}$, $m \in \mathbb{N}$ we define $F^{M,m}_\varepsilon : L^p(\Omega;\mathbb{R}^n) \times \mathcal{A}^{reg}(\Omega) \to [0, +\infty]$ by

$$F^{M,m}_\varepsilon(u, A) = \begin{cases} F(u, A) & \text{if } u \in \mathcal{A}^{M,m}_\varepsilon(A;\mathbb{R}^n) \\ +\infty & \text{otherwise.} \end{cases}$$

Proposition 4.1. *Let $\phi^\varepsilon_i : (\mathbb{R}^n)^{Z_\varepsilon(\Omega)} \to [0, +\infty)$ satisfy* (H1)–(H5)*. Let ε_{j_k} and f be as in Theorem* 3.2*. For any $M \in \mathbb{R}^{d\times N}$ and $A \in \mathcal{A}^{reg}(\Omega)$ we set $F^M : L^p(\Omega;\mathbb{R}^n) \times \mathcal{A}^{reg}(\Omega) \to [0, +\infty]$ by*

$$F^M(u, A) = \begin{cases} \displaystyle\int_A f(x, \nabla u)\mathrm{d}x & \textit{if } u - Mx \in W^{1,p}_0(A;\mathbb{R}^n) \\ +\infty & \textit{otherwise.} \end{cases}$$

Then for any $M \in \mathbb{R}^{d\times N}$, $m \in \mathbb{N}$ and any $A \in \mathcal{A}^{reg}$ we have that $F^{M,m}_{\varepsilon_{j_k}}(\cdot, A)$ Γ-converges with respect to the strong $L^p(\Omega;\mathbb{R}^n)$-topology to the functional $F^M(\cdot, A)$.

Proof. We only prove the statement for $m = 1$, the other cases being done analogously.

We first prove the Γ-lim inf inequality. Let $\{u_k\}_k \subset \mathcal{A}_{\varepsilon_{j_k}}(\Omega;\mathbb{R}^n)$ converge to u in the $L^p(\Omega;\mathbb{R}^n)$-topology and be such that

$$\liminf_{k\to\infty} F^{M,1}_{\varepsilon_{j_k}}(u_k, A) = \lim_{k\to\infty} F^M_{\varepsilon_{j_k}}(u_k, A) < +\infty.$$

Since $u_k \in \mathcal{A}^{M,m}_{\varepsilon_{j_k}}(A;\mathbb{R}^n)$ for all $k \in \mathbb{N}$, and by (H3), we have that $u_k \to Mx$ in $L^p(A\backslash\Omega;\mathbb{R}^n)$ and

$$\sup_{\varepsilon>0} \sum_{n=1}^{N} \sum_{i\in Z_\varepsilon(\Omega)} \varepsilon^N |D^{e_n}_\varepsilon u_k(i)|^p < +\infty.$$

By the same reasoning as in Proposition 3.4 $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ and $u - Mx \in W^{1,p}_0(A;\mathbb{R}^n)$. By Theorem 3.2 we therefore have

$$\liminf_{k\to\infty} F^{M,m}_{\varepsilon_{j_k}}(u_k, A) \geq \liminf_{k\to\infty} F_{\varepsilon_{j_k}}(u_k, A) = F^M(u, A).$$

To prove the Γ-lim sup inequality we may first suppose that $\operatorname{supp}(u - Mx) \subset\subset A$. Let $\{u_k\}_k \subset A_{\varepsilon_{j_k}}(\Omega;\mathbb{R}^n)$ converge to u in $L^p(\Omega;\mathbb{R}^n)$ and be such that

$$\limsup_{k\to\infty} F_{\varepsilon_{j_k}}(u_k, A) = F(u, A).$$

Then by reasoning as in the proof of Proposition 3.6 given $\delta > 0$ we can find $A_\delta \subset A$ and suitable cut-off functions φ_k with $\operatorname{supp}(u - Mx) \subset\subset \operatorname{supp}\varphi_k \subset\subset A_\delta$ and $|A \setminus A_\delta| < \delta$ such that for

$$w_k(i) := \varphi_k(i)u_k(i) + (1 - \varphi_k(i))Mi$$

we have that w_k converges to u in $L^p(\Omega;\mathbb{R}^n)$ and

$$\limsup_{k\to\infty} F_{\varepsilon_{j_k}}(w_k, A) \leq \limsup_{k\to\infty} F_{\varepsilon_{j_k}}(u_k, A) + \limsup_{k\to\infty} F_{\varepsilon_{j_k}}(Mx, A \setminus A_\delta) + \delta.$$

Using (H2) we have that for every $k \in \mathbb{N}$ it holds

$$F_{\varepsilon_{j_k}}(Mx, A \setminus A_\delta) \leq C(|M|^p + 1)|(A \setminus A_\delta)_\varepsilon| \leq C(|M|^p + 1)|\delta.$$

By the definition of the Γ-lim sup we have that

$$\Gamma\text{-}\limsup_{k\to\infty} F^{M,m}_{\varepsilon_{j_k}}(u, A) \leq F^M(u, A) + C\delta.$$

Letting $\delta \to 0$ we obtain the desired inequality. The general case follows by a density argument, approximating every function $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ such that $u - Mx \in W^{1,p}_0(A;\mathbb{R}^n)$ strongly in $W^{1,p}(\Omega;\mathbb{R}^n)$ by functions u_n such that $\operatorname{supp}(u_n - Mx) \subset\subset A$ and using the lower semicontinuity of the Γ-lim sup as well as the continuity of $F(\cdot, A)$ with respect to the strong convergence in $W^{1,p}(\Omega;\mathbb{R}^n)$. □

Remark 4.2. Let $\phi^\varepsilon_i : (\mathbb{R}^n)^{Z_\varepsilon(\Omega)} \to [0, +\infty)$ satisfy (H1)–(H5), and let ε_{j_k} be as in Theorem 3.2. For any $M \in \mathbb{R}^{d\times N}$, $m \in \mathbb{N}$ and $A \in \mathcal{A}^{reg}(\Omega)$ we have that

$$\lim_{k\to\infty} \inf \left\{ F_{\varepsilon_{j_k}}(u, A) : u \in \mathcal{A}^{M,m}_{\varepsilon_{j_k}}(A;\mathbb{R}^n) \right\} = \inf \left\{ F(u, A) : u - Mx \in W^{1,p}_0(A;\mathbb{R}^n) \right\},$$

since the functionals F^M_ε are coercive with respect to the strong $L^p(\Omega;\mathbb{R}^n)$-topology.

Note first that by extending the functional as in the proof of Proposition 3.6 we can assume that $A \subset\subset \Omega$. Moreover, by the boundary conditions and by (H3) any sequence $\{u_k\}_k$ satisfying

$$\sup_k F^{M,m}_{\varepsilon_{j_k}}(u_k, A) < +\infty$$

satisfies

$$\sup_{k\in\mathbb{N}} \sum_{n=1}^{N} \sum_{i\in Z_{\varepsilon_{j_k}}(\Omega)} \varepsilon^N |D^{e_n}_{\varepsilon_{j_k}} u_k(i)|^p < +\infty.$$

Then by the boundary conditions, Lemma 3.6 in [2] and the Riesz-Frechét-Kolmogorov Theorem there exists a function $u \in L^p(\Omega;\mathbb{R}^n)$ and a subsequence (not relabelled) that converges to u. By Proposition 3.4 we have that $u \in W^{1,p}(\Omega;\mathbb{R}^n)$. Moreover, $u_k \to Mx$ in $L^p(\Omega \setminus A;\mathbb{R}^n)$ and therefore $u - Mx \in W^{1,p}_0(A;\mathbb{R}^n)$. This implies the coercivity.

5 Homogenization

We now consider the case where $i \mapsto \phi^\varepsilon_i$ is periodic, though we have to explain what that means in our case, since the interaction energy at every point of the lattice may depend on the whole configuration of the state $\{z_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}$. This will be done by using a function $\phi_i : (\mathbb{R}^n)^{\mathbb{Z}^N} \to [0,+\infty)$, $i \in \mathbb{Z}^N$ defined on the entire lattice. In order to define the energy density inside Ω we assume that ϕ_i is approximated by finite-range interaction. More precisely, we suppose that there exist $\phi^k_i : (\mathbb{R}^n)^{\mathbb{Z}^N} \to [0,+\infty)$, $i \in \mathbb{Z}^N$ T-periodic, satisfying (H1)–(H3) uniformly in k and

(H_p4) (*locality*) For all $k \in \mathbb{N}$ and for all $z, w \in \mathcal{A}_1(\mathbb{R}^N,\mathbb{R}^n)$ satisfying $z(j) = w(j)$ for all $j \in \mathbb{Z}^N \cap Q_k(i)$ we have

$$\phi^k_i(\{z_j\}_{j\in\mathbb{Z}^N}) = \phi^k_i(\{w_j\}_{j\in\mathbb{Z}^N}).$$

(H_p5) (*controlled non-convexity*) There exist $C > 0$ and $\{C^{j,\xi}\}_{j\in\mathbb{Z}^N,\xi\in\mathbb{Z}^N}$, $C^{j,\xi} \geq 0$ satisfying

$$\sum_{j,\xi\in\mathbb{Z}^N} C^{j,\xi} < +\infty \text{ and we have } \limsup_{k\to\infty} \sum_{\max\{|\xi|,|j|\}>k} C^{j,\xi} = 0 \tag{39}$$

such that for all $k \in \mathbb{N}$, $z, w \in \mathcal{A}_1(\mathbb{R}^N,\mathbb{R}^n)$ and ψ cut-off functions we have

$$\begin{aligned}\phi^k_i(\{\psi_j z_j + (1-\psi_j)w_j\}_{j\in\mathbb{Z}^N}) \leq & C\left(\phi^k_i(\{z_j\}_{j\in\mathbb{Z}^N}) + \phi^k_i(\{w_j\}_{j\in\mathbb{Z}^N})\right) \\ & + R^k_i(z,w,\psi),\end{aligned}$$

where

$$\begin{aligned}R^k_i(z,w,\psi) = & \sum_{\substack{j,\xi\in\mathbb{Z}^N \\ j+\xi\in\mathbb{Z}^N\cap Q_k(0)}} C^{j,\xi}\Big(\big(\sup_{\substack{k\in\mathbb{Z}^N\cap Q_k(0) \\ n\in\{1,\dots,N\}}} |D^{e_n}_1\psi(k)|^p + 1\big)|z(j+\xi) - w(j+\xi)|^p\Big) \\ & + \sum_{\substack{j,\xi\in\mathbb{Z}^N \\ j+\xi\in\mathbb{Z}^N\cap Q_k(0)}} C^{j,\xi}\Big(|D^\xi_1 z(j)|^p + |D^\xi_1 w(j)|^p + 1\Big).\end{aligned}$$

(H_p6) (*closeness*) There exist $\{C_k^{j,\xi}\}_{k\in\mathbb{N},j\in\mathbb{Z}^N,\xi\in\mathbb{Z}^N}$, $C_k^{j,\xi}\geq C_{k+1}^{j,\xi}\geq 0$ satisfying

$$\limsup_{k\to\infty}\sum_{j,\xi\in\mathbb{Z}^N} C_k^{j,\xi}=0 \tag{40}$$

such that for all $z\in\mathcal{A}_1(\mathbb{R}^N;\mathbb{R}^n)$ and $k_1\leq k_2$ we have that

$$|\phi_i^{k_1}(\{z_j\}_{j\in\mathbb{Z}^N})-\phi_i^{k_2}(\{z_j\}_{j\in\mathbb{Z}^N})|\leq \sum_{\substack{j,\xi\in\mathbb{Z}^N\cap Q_{k_2}(0)\\ j+\xi\in\mathbb{Z}^N\cap Q_{k_2}(0)}} C_{k_1}^{j,\xi}\Big(|D_1^\xi z(j)|^p+1\Big).$$

(H_p7) (*monotonicity*) For every $k\in\mathbb{N}$, for every $i\in\mathbb{Z}^N$ and for every $z\in\mathcal{A}_1(\mathbb{R}^N;\mathbb{R}^n)$ we have

$$\phi_i^k(\{z_j\}_{j\in\mathbb{Z}^N})\leq\phi_i^{k+1}(\{z_j\}_{j\in\mathbb{Z}^N}),\quad \phi_i^k(\{z_j\}_{j\in\mathbb{Z}^N})\to\phi_i(\{z_j\}_{j\in\mathbb{Z}^N})\text{ as }k\to\infty. \tag{41}$$

The monotonicity property (H_p7) may seem restrictive at a first sight, but it is not since by the positivity of ϕ^k and ϕ respectively we may reorder the interactions in a way that we keep only adding positive interactions with increasing k.

For every $i\in Z_\varepsilon(\Omega)$ we define $\phi_i^\varepsilon:(\mathbb{R}^n)^{Z_\varepsilon(\Omega)}\to[0,+\infty)$ by

$$\phi_i^\varepsilon(\{z_j\}_{j\in Z_\varepsilon(\Omega_i)})=\phi_{\frac{i}{\varepsilon}}^{\lfloor\frac{d_i}{\varepsilon}\rfloor}(\{z_j^\varepsilon\}_{j\in\mathbb{Z}^N}), \tag{42}$$

where $\mathrm{dist}_\infty(\Omega^c,i)=d_i$ and

$$z^\varepsilon(j)=\begin{cases}\frac{z(\varepsilon j)}{\varepsilon} & j\in Q_{\lfloor\frac{d_i}{\varepsilon}\rfloor}(i)\cap\mathbb{Z}^N\\ 0 & \text{otherwise.}\end{cases}$$

Note that (42) is well defined due to the locality property (H_p4) and moreover, ϕ_i^ε satisfies (H1)–(H5). Those assumptions are made to avoid the dependence of ϕ_i^ε on Ω and still include infinite-range interactions.

Theorem 5.1. *Let $\phi_i^k:(\mathbb{R}^n)^{\mathbb{Z}^N}\to[0,+\infty)$ satisfy* (H1)–(H3) *and* (H_p4)–(H_p7) *and $\phi_i^\varepsilon:(\mathbb{R}^n)^{Z_\varepsilon(\Omega)}\to[0,+\infty)$ be defined by* (42)*. Then, $F_\varepsilon:L^p(\Omega;\mathbb{R}^n)\to[0,+\infty]$ Γ-converges with respect to the strong $L^p(\Omega;\mathbb{R}^n)$-topology to the functional $F:L^p(\Omega;\mathbb{R}^n)\to[0,+\infty]$ defined by*

$$F(u)=\begin{cases}\displaystyle\int_\Omega f_{\mathrm{hom}}(\nabla u)\mathrm{d}x & \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^n)\\ +\infty & \text{otherwise,}\end{cases}$$

where $f_{\mathrm{hom}}:\mathbb{R}^{d\times N}\to[0,\infty)$ is given by

$$f_{\mathrm{hom}}(M)=\lim_{L\to\infty}\frac{1}{L^N}\inf\Big\{\sum_{i\in\mathbb{Z}^N\cap Q_L}\phi_i(\{z_{j+i}\}_{j\in\mathbb{Z}^N}):z\in\mathcal{A}_1^{M,\lfloor\sqrt{L}\rfloor}(Q_L;\mathbb{R}^n)\Big\}, \tag{43}$$

where

$$\mathcal{A}_\varepsilon^{M,m}(Q_L;\mathbb{R}^n)=\big\{u\in\mathcal{A}_\varepsilon(\mathbb{R}^N;\mathbb{R}^n):u(i)=Mi \text{ if } (i+[-m\varepsilon,m\varepsilon)^N)\cap Q_L^c\neq\emptyset\big\}.$$

Remark 5.2. Note that in Theorem 5.1 we have that the whole sequence F_ε Γ-converges to the limit functional F. We fix the boundary conditions of the admissible test functions on a boundary layer of width $\lfloor\sqrt{L}\rfloor$ in order to have the boundary effects negligible while still being able to use a subadditivity argument in order to prove the existence of the limit in (43). Arguing as in the proof of Proposition 5.3 to show that the error goes to 0 when substituting ϕ_i^k with ϕ_i, and using the fact that the limit energy density is quasi-convex, we also have

$$f_{\hom}(M) = \lim_{L\to\infty} \frac{1}{L^N} \inf\Big\{ \sum_{i\in\mathbb{Z}^N\cap Q_L} \phi_i(\{z_{j+i}\}_{j\in\mathbb{Z}^N}) : z \in \mathcal{A}_1^{M,m}(Q_L;\mathbb{R}^n)\Big\}$$

for all $m \in \mathbb{N}$ and all $M \in \mathbb{R}^{d\times N}$.

Proof. By Theorem (3.2) for every sequence ε_j there exists a subsequence ε_{j_k} such that $F_{\varepsilon_{j_k}}$ Γ-converges to a functional F such that for any $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ and every $A \in \mathcal{A}(\Omega)$ we have

$$\Gamma\text{-}\lim_{k\to\infty} F_{\varepsilon_{j_k}}(u,A) = \int_A f(x,\nabla u)\mathrm{d}x.$$

By the Urysohn property of Γ-convergence the theorem is proved if we show that f does not depend on x and $f = f_{\hom}$. To prove the first claim it suffices to show that

$$F(Mx, Q_\rho(z)) = F(Mx, Q_\rho(y))$$

for all $M \in \mathbb{R}^{d\times N}$, $z, y \in \Omega$ and $\rho > 0$ such that $Q_\rho(z) \cup Q_\rho(y) \subset \Omega$. By symmetry it suffices to prove

$$F(Mx, Q_\rho(z)) \le F(Mx, Q_\rho(y)).$$

By the inner-regularity property it suffices to prove for any $\rho' < \rho$

$$F(Mx, Q_{\rho'}(z)) \le F(Mx, Q_\rho(y)).$$

Let $v_k \in \mathcal{A}_{\varepsilon_{j_k}}(\Omega;\mathbb{R}^n)$ be such that $v_k \to Mx$ in $L^p(\Omega;\mathbb{R}^n)$ and such that

$$\lim_{k\to\infty} F_{\varepsilon_{j_k}}(v_k, Q_\rho(y)) = F(Mx, Q_\rho(y)).$$

Let $\varphi \in C^\infty(\Omega)$ be a cut-off function such that $0 \le \varphi \le 1$

$$\mathrm{supp}(\varphi) \subset\subset Q_\rho(z), \quad Q_{\rho'}(z) \subset\subset \{\varphi = 1\} \text{ and } ||\nabla\varphi||_\infty \le \frac{C}{\rho-\rho'}.$$

For $k \in \mathbb{N}$ define $u_k \in \mathcal{A}_{\varepsilon_{j_k}}(\Omega;\mathbb{R}^n)$ by

$$u_k(i) = \varphi(i)\left(v_k\Big(i + \varepsilon_{j_k} T\lfloor\frac{y-z}{T\varepsilon_{j_k}}\rfloor\Big) + M(z-y)\right) + (1-\varphi(i))Mi.$$

Thus by the periodicity assumption and the locality property we have that

$$\sum_{i\in Z_{\varepsilon_{j_k}}(Q_{\rho'}(z))} \varepsilon_{j_k}^N \phi_i^{\varepsilon_{j_k}}(\{(u_k)_{j+i}\}_{j\in Z_{\varepsilon_{j_k}}(\Omega_i)}) \leq \sum_{i\in Z_{\varepsilon_{j_k}}(Q_{\rho}(y))} \varepsilon_{j_k}^N \phi_i^{\varepsilon_{j_k}}(\{(v_k)_{j+i}\}_{j\in Z_{\varepsilon_{j_k}}(\Omega_i)}) + O(\varepsilon_{j_k}).$$

Therefore, we obtain

$$F(Mx, Q_{\rho'}(z)) \leq \liminf_{k\to\infty} F_{\varepsilon_{j_k}}(u_k, Q_{\rho'}(z)) \leq \liminf_{k\to\infty} F_{\varepsilon_{j_k}}(u_k, Q_{\rho}(y)) = F(Mx, Q_{\rho}(y)).$$

In order to obtain that $f = f_{\mathrm{hom}}$ we note that by the lower semicontinuity with respect to the strong $L^p(\Omega;\mathbb{R}^n)$-topology and the coercivity of F we obtain that F is lower semicontinuous with respect to the weak $W^{1,p}(\Omega;\mathbb{R}^n)$-topology and hence f is quasiconvex. By the growth properties of f and Remark 4.2 we obtain for $Q = Q_\rho(x_0) \subset\subset \Omega$

$$\begin{aligned} f(M) &= \frac{1}{\rho^N}\inf\Big\{\int_Q f(\nabla u)\mathrm{d}x : u - Mx \in W_0^{1,p}(Q;\mathbb{R}^n)\Big\} \\ &= \frac{1}{\rho^N}\inf\Big\{F(u,Q) : u - Mx \in W_0^{1,p}(Q;\mathbb{R}^n)\Big\} \\ &= \lim_{m\to\infty}\lim_{k\to\infty}\frac{1}{\rho^N}\inf\Big\{F_{\varepsilon_{j_k}}(u,Q) : u \in \mathcal{A}^{M,m}_{\varepsilon_{j_k}}(Q;\mathbb{R}^n)\Big\} \\ &= f_{\mathrm{hom}}(M). \end{aligned}$$

Where the last inequality follows from the next proposition. □

Proposition 5.3. *Let* $\phi_i^k : (\mathbb{R}^n)^{\mathbb{Z}^N} \to [0,+\infty)$ *satisfy* (H1)–(H3) *and* (H$_p$4)–(H$_p$7), *and* $\phi_i^\varepsilon : (\mathbb{R}^n)^{Z_\varepsilon(\Omega)} \to [0,+\infty)$ *be defined by* (42). *Then*

$$f_{\mathrm{hom}}(M) = \lim_{m\to\infty}\lim_{k\to\infty}\frac{1}{\rho^N}\inf\Big\{F_{\varepsilon_{j_k}}(u,Q) : u \in \mathcal{A}^{M,m}_{\varepsilon_{j_k}}(Q;\mathbb{R}^n)\Big\}$$

for all $M \in \mathbb{R}^{n\times N}$.

Proof. Without loss of generality, assume $x_0 = 0$. We perform a change of variables

$$i' = \frac{i}{\varepsilon_{j_k}}, \quad \tilde{u}(i') = \frac{1}{\varepsilon_{j_k}}u(\varepsilon_{j_k}i'), \quad L_k = \frac{\rho}{\varepsilon_{j_k}}.$$

Set $d_{i'}^k = \mathrm{dist}(\frac{1}{\varepsilon_{j_k}}\Omega^c, i')$. We obtain

$$\begin{aligned} &\lim_{m\to\infty}\lim_{k\to\infty}\frac{1}{\rho^N}\inf\Big\{F_{\varepsilon_{j_k}}(u,Q) : u \in \mathcal{A}^{M,m}_{\varepsilon_{j_k}}(Q;\mathbb{R}^n)\Big\} \\ &= \lim_{m\to\infty}\lim_{k\to\infty}\frac{1}{L_k^N}\inf\Big\{\sum_{i'\in\mathbb{Z}^N\cap Q_L}\phi_{i'}^{\lfloor d_{i'}^k\rfloor}(\{\tilde{u}_{j+i'}\}_{j\in\mathbb{Z}^N}) : \tilde{u} \in \mathcal{A}_1^{M,m}(Q_{L_k};\mathbb{R}^n)\Big\}. \end{aligned}$$

By the monotonicity property and (H2) we have that

$$\begin{aligned} C(|M|^p+1) &\geq \lim_{m\to\infty}\lim_{k\to\infty}\frac{1}{L_k^N}\inf\Big\{\sum_{i'\in\mathbb{Z}^N\cap Q_L}\phi_{i'}(\{\tilde{u}_{j+i'}\}_{j\in\mathbb{Z}^N}) : \tilde{u} \in \mathcal{A}_1^{M,m}(Q_L;\mathbb{R}^n)\Big\} \\ &\geq \lim_{m\to\infty}\lim_{k\to\infty}\frac{1}{L_k^N}\inf\Big\{\sum_{i'\in\mathbb{Z}^N\cap Q_L}\phi_{i'}^{\lfloor d_{i'}^k\rfloor}(\{\tilde{u}_{j+i'}\}_{j\in\mathbb{Z}^N}) : \tilde{u} \in \mathcal{A}_1^{M,m}(Q_L;\mathbb{R}^n)\Big\}. \end{aligned}$$

On the other hand, let $u_k \in \mathcal{A}_1^{M,m}(Q_L;\mathbb{R}^n)$ be such that

$$\sum_{i'\in\mathbb{Z}^N\cap Q_{L_k}} \phi_{i'}^{\lfloor d_{i'}^k \rfloor}(\{(u_k)_{j+i'}\}_{j\in\mathbb{Z}^N}) \leq \inf\Big\{ \sum_{i'\in\mathbb{Z}^N\cap Q_{L_k}} \phi_{i'}^{\lfloor d_{i'}^k \rfloor}(\{\tilde{u}_{j+i'}\}_{j\in\mathbb{Z}^N}) : \tilde{u} \in \mathcal{A}_1^{M,m}(Q_{L_k};\mathbb{R}^n)\Big\} + \frac{1}{k}$$

Now by (H$_p$6) and setting $d_k = \lfloor \dfrac{\mathrm{dist}(Q,\Omega^c)}{\varepsilon_{j_k}} \rfloor$ we obtain $d_k \to \infty$, since $Q \subset\subset \Omega$, and

$$\sum_{i'\in\mathbb{Z}^N\cap Q_{L_k}} \phi_{i'}(\{(u_k)_{j+i'}\}_{j\in\mathbb{Z}^N}) \leq \sum_{i'\in\mathbb{Z}^N\cap Q_{L_k}} \Big(\phi_{i'}^{\lfloor d_{i'}^k \rfloor}(\{(u_k)_{j+i'}\}_{j\in\mathbb{Z}^N}) + \sum_{j,\xi\in\mathbb{Z}^N} C_{d_k}^{j-i',\xi}(|D_1^\xi u_k(j)|^p + 1)\Big).$$

We have that either $j, j+\xi \in \mathbb{Z}^N \setminus Q_{L_k}(0)$ in which case $|D_1^\xi u_k|^p \leq |M|^p$ or $\{j, j+\xi\} \cap Q_{L_k}(0) \neq \emptyset$. Now if $j, j+\xi \in Q_{L_k}(0)$, by [[2],Lemma 3.6] and (H2), we have that

$$\begin{aligned}\sum_{\substack{j\in\mathbb{Z}^N\\ j,j+\xi\in Q_{L_k}(0)}} |D_1^\xi u_k(j)|^p &\leq C\sum_{n=1}^N \sum_{j\in\mathbb{Z}^N\cap Q_{L_k}(0)} |D_1^{e_n} u_k(j)|^p \\ &\leq C \sum_{j\in\mathbb{Z}^N\cap Q_{L_k}(0)} \phi_j^{d_k}(\{(u_k)_{j'+j}\}_{j'\in\mathbb{Z}^N}) \leq C(|M|^p+1)L_k^N. \qquad (44)\end{aligned}$$

Now either $j \in Q_{L_k}(0)$, $j+\xi \notin Q_{L_k}(0)$ or $j \notin Q_{L_k}(0)$, $j+\xi \in Q_{L_k}(0)$. We only deal with the first case, the second one being done analogously. Now if $|\xi|_\infty \leq L_k$, by (H2) and using the boundary conditions, we have that

$$\begin{aligned}\sum_{j\in\mathbb{Z}^N} |D_1^\xi u_k(j)|^p &\leq \sum_{\substack{j\in\mathbb{Z}^N\\ j,j+\xi\in Q_{2L_k}(0)}} |D_1^\xi u_k(j)|^p \leq C\sum_{n=1}^N \sum_{j\in\mathbb{Z}^N\cap Q_{2L_k}(0)} |D_1^{e_n} u_k(j)|^p \\ &\leq C \sum_{j\in\mathbb{Z}^N\cap Q_{L_k}(0)} \phi_j^{d_k}(\{(u_k)_{j'+j}\}_{j'\in\mathbb{Z}^N}) + \sum_{j\in\mathbb{Z}^N\cap Q_{2L_k}(0)\setminus Q_{L_k}(0)} |D_1^{e_n} u_k(j)|^p \\ &\leq C \sum_{j\in\mathbb{Z}^N\cap Q_{L_k}(0)} \phi_j^{d_k}(\{(u_k)_{j'+j}\}_{j'\in\mathbb{Z}^N}) + CL_k^N|M|^p \\ &\leq C(|M|^p+1)L_k^N. \qquad (45)\end{aligned}$$

If $|\xi|_\infty > L_k$ for every j we choose a path $\gamma_\xi^j = (j_h)_{h=1}^{||\xi||_1+1} \subset \mathbb{Z}^N$ by defining

$$j_{||\xi||_1+1} = j+\xi, j_1 = j, j_{h+1} = j_h + e_{n(h)}, e_{n(h)} = \mathrm{sign}(\xi_k)e_k \text{ if } 1 + \sum_{n=1}^{k-1}|\xi_n| \leq h \leq \sum_{n=1}^{k}|\xi_n|.$$

For this path it holds

$$|D_1^\xi u(j)|^p \leq \frac{C(p,N)}{||\xi||_1}\sum_{h=1}^{||\xi||_1} |D_1^{e_n(h)} u(j_h)|^p.$$

Now for every $i \in \mathbb{Z}^N$ and for every $n \in \{1, \dots, N\}$ we set

$$N_{i,n}^{\xi,k} = \Big\{ j \in Q_{L_k}(0) : \exists h \in \{1, \dots, |\xi_1|\}, n \in \{1, \dots, N\} \\ \text{such that } i = j_h \in \gamma_j^\xi \text{ and } e_{n(h)} = \operatorname{sign}(\xi_n) e_n \}.$$

We have that $\# N_{i,n}^{\xi,k} \leq L_k$ for $i \in \mathbb{Z}^N \cap Q_{L_k}(0)$, using $|D_1^{e_n} u_k(i)| \leq |M|$ for every $i \in \mathbb{Z}^N \setminus Q_{L_k}(0)$ and using Fubini's Theorem we obtain

$$\begin{aligned} \sum_{j \in \mathbb{Z}^N \cap Q_{L_k}(0)} |D_1^\xi u_k(j)|^p &\leq \frac{C}{||\xi||_1} \sum_{j \in \mathbb{Z}^N \cap Q_{L_k}(0)} \sum_{h=1}^{||\xi||_1} |D_1^{e_n(h)} u_k(j_h)|^p \\ &\leq \frac{C}{||\xi||_1} \sum_{n=1}^N \sum_{i \in \mathbb{Z}^N \cap Q_{L_k}(0)} \# N_{i,n}^{\xi,k} |D_1^{e_n} u_k(i)|^p + |M|^p L_k^N \\ &\leq C \sum_{n=1}^N \sum_{i \in \mathbb{Z}^N \cap Q_{L_k}(0)} |D_1^{e_n} u_k(i)|^p + |M|^p L_k^N \\ &\leq C \sum_{j \in \mathbb{Z}^N \cap Q_{L_k}(0)} \phi_i^{d_k}(\{(u_k)_{j+i}\}_{j \in \mathbb{Z}^N}) + |M|^p L_k^N \\ &\leq C(|M|^p + 1) L_k^N. \end{aligned} \tag{46}$$

Now if $j, j + \xi \in Q_{L_k}(0)$, using Fubini's Theorem and (44), we obtain

$$\begin{aligned} \sum_{i' \in \mathbb{Z}^N \cap Q_{L_k}(0)} \sum_{\substack{j,\xi \in \mathbb{Z}^N \\ j, j+\xi \in Q_{L_k}(0)}} C_{d_k}^{j-i',\xi} |D_1^\xi u_k(j)|^p &\leq \sum_{i',\xi \in \mathbb{Z}^N} C_{d_k}^{j-i',\xi} \sum_{\substack{j \in \mathbb{Z}^N \cap Q_{L_k}(0) \\ j+\xi \in Q_{L_k}(0)}} |D_1^\xi u_k(j)|^p \\ &\leq C L_k^N \sum_{i',\xi \in \mathbb{Z}^N} C_{d_k}^{j-i',\xi} (|M|^p + 1). \end{aligned} \tag{47}$$

Now if $j \in Q_{L_k}(0)$ $|\xi|_\infty \leq L_k$, using Fubini's Theorem and (45), we obtain

$$\begin{aligned} \sum_{i' \in \mathbb{Z}^N \cap Q_{L_k}(0)} \sum_{\substack{j,\xi \in \mathbb{Z}^N \\ j \in Q_{L_k}(0) \\ |\xi|_\infty \leq L_k}} C_{d_k}^{j-i',\xi} |D_1^\xi u_k(j)|^p &\leq \sum_{\substack{i',\xi \in \mathbb{Z}^N \\ |\xi|_\infty \leq L_k}} C_{d_k}^{j-i',\xi} \sum_{j \in \mathbb{Z}^N \cap Q_{L_k}(0)} |D_1^\xi u_k(j)|^p \\ &\leq C L_k^N \sum_{i',\xi \in \mathbb{Z}^N} C_{d_k}^{j-i',\xi} (|M|^p + 1). \end{aligned} \tag{48}$$

If $j \in Q_{L_k}(0)$ $|\xi|_\infty > L_k$, using Fubini's Theorem and (46), we obtain

$$\begin{aligned} \sum_{i' \in \mathbb{Z}^N \cap Q_{L_k}(0)} \sum_{\substack{j,\xi \in \mathbb{Z}^N \\ j \in Q_{L_k}(0) \\ |\xi|_\infty > L_k}} C_{d_k}^{j-i',\xi} |D_1^\xi u_k(j)|^p &\leq \sum_{\substack{i',\xi \in \mathbb{Z}^N \\ |\xi|_\infty > L_k}} C_{d_k}^{j-i',\xi} \sum_{j \in \mathbb{Z}^N \cap Q_{L_k}(0)} |D_1^\xi u_k(j)|^p \\ &\leq C L_k^N \sum_{i',\xi \in \mathbb{Z}^N} C_{d_k}^{j-i',\xi} (|M|^p + 1). \end{aligned} \tag{49}$$

Now, dividing by L_k^N, using (40),(47)–(49) and taking the limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} \frac{1}{L_k^N} \sum_{i'\in\mathbb{Z}^N\cap Q_{L_k}} \sum_{j,\xi\in\mathbb{Z}^N} C_{d_k}^{j-i',\xi}(|D_1^\xi u_k(j)|^p + 1) = 0$$

It remains to show that the limit (43) exists and

$$f_{\rm hom}(M) = \lim_{m\to\infty} \lim_{L\to\infty} \frac{1}{L^N} \inf\Big\{ \sum_{i\in\mathbb{Z}^N\cap Q_L(0)} \phi_i(\{z_{j+i}\}_{j\in\mathbb{Z}^N}) : z \in \mathcal{A}_1^{M,m}(Q_L;\mathbb{R}^n)\Big\}. \tag{50}$$

Since $\mathcal{A}_1^{M,\lfloor\sqrt{L}\rfloor}(Q_L;\mathbb{R}^n) \subset \mathcal{A}_1^{M,m}(Q_L;\mathbb{R}^n)$ we have that

$$f_{\rm hom}(M) \geq \lim_{m\to\infty} \lim_{L\to\infty} \frac{1}{L^N} \inf\Big\{ \sum_{i\in\mathbb{Z}^N\cap Q_L(0)} \phi_i(\{z_{j+i}\}_{j\in\mathbb{Z}^N}) : z \in \mathcal{A}_1^{M,m}(Q_L;\mathbb{R}^n)\Big\}.$$

On the other hand, for every $u_L \in \mathcal{A}_1^{M,m}(Q_L;\mathbb{R}^n)$, also $u_L \in \mathcal{A}_1^{M,\lfloor\sqrt{L+\sqrt{L}}\rfloor}(Q_{L+\lfloor\sqrt{L}\rfloor};\mathbb{R}^n)$, so that for $\tilde{L} = L + \lfloor\sqrt{L}\rfloor$ we have

$$\begin{aligned}\sum_{i\in\mathbb{Z}^N\cap Q_{\tilde{L}}(0)} \phi_i(\{(u_L)_{j+i}\}_{j\in\mathbb{Z}^N}) &= \sum_{i\in\mathbb{Z}^N\cap Q_L(0)} \phi_i(\{(u_L)_{j+i}\}_{j\in\mathbb{Z}^N}) \\ &\quad + \sum_{i\in\mathbb{Z}^N\cap(Q_{\tilde{L}}(0)\setminus Q_L(0))} \phi_i(\{(u_L)_{j+i}\}_{j\in\mathbb{Z}^N}).\end{aligned}$$

Note that $\lim_{L\to\infty}\frac{\tilde{L}}{L} = 1$ and therefore we are done if we can show that

$$\frac{1}{L^N} \sum_{i\in\mathbb{Z}^N\cap(Q_{\tilde{L}}(0)\setminus Q_L(0))} \phi_i(\{(u_L)_{j+i}\}_{j\in\mathbb{Z}^N}) \to 0$$

as $L \to \infty$ and then $m \to \infty$. By the locality property (H$_p$4) and the boundary conditions we have for all $i \in \mathbb{Z}^N \cap (Q_{\tilde{L}}(0) \setminus Q_L(0))$

$$\begin{aligned}\phi_i(\{(u_L)_{j+i}\}_{j\in\mathbb{Z}^N}) &\leq \phi_i(\{Mx_{j+i}\}_{j\in\mathbb{Z}^N}) + \sum_{j,\xi\in\mathbb{Z}^N} C_m^{j-i,\xi}(|D_1^\xi u_L(j)|^p + 1) \\ &\leq C(|M|^p + 1) + \sum_{j,\xi\in\mathbb{Z}^N} C_m^{j-i,\xi}(|D_1^\xi u_L(j)|^p + 1).\end{aligned}$$

Using similar arguments as for (47)–(49) we obtain

$$\frac{1}{L^N} \sum_{i\in\mathbb{Z}^N\cap(Q_{\tilde{L}}(0)\setminus Q_L(0))} \sum_{j,\xi\in\mathbb{Z}^N} C_m^{j-i,\xi}(|D_1^\xi u_L(j)|^p + 1) \to 0 \tag{51}$$

as $L \to \infty$ and then $m \to \infty$ and hence (50). We are done if we show that the limit in the definition of (43) exists. To this end set

$$F_L(M) = \frac{1}{L^N} \inf\Big\{ \sum_{i\in\mathbb{Z}^N\cap Q_L} \phi_i(\{z_{j+i}\}_{j\in\mathbb{Z}^N}) : z \in \mathcal{A}_1^{M,\sqrt{L}}(Q_L;\mathbb{R}^n)\Big\}.$$

Let $L \in \mathbb{N}$ and let $k \in \mathbb{N}$ be such that $kT \le L \le (k+1)T$. For any $u \in \mathcal{A}_1^{M,\lfloor\sqrt{L}\rfloor}(Q_L;\mathbb{R}^n)$ we have that $u \in \mathcal{A}_1^{M,\lfloor\sqrt{(k+1)T}\rfloor}(Q_{(k+1)T};\mathbb{R}^n)$ and

$$\begin{aligned}\frac{1}{L^N}\sum_{i\in\mathbb{Z}^N\cap Q_{(k+1)T}(0)}\phi_i(\{u_{j+i}\}_{j\in\mathbb{Z}^N}) \le& \frac{1}{L^N}\sum_{i\in\mathbb{Z}^N\cap Q_L(0)}\phi_i(\{u_{j+i}\}_{j\in\mathbb{Z}^N})\\ &+\frac{1}{L^N}\sum_{i\in\mathbb{Z}^N\cap(Q_{(k+1)T}(0)\setminus Q_L(0)}\phi_i(\{u_{j+i}\}_{j\in\mathbb{Z}^N}),\end{aligned}$$

where the last term tends to 0 as $L\to\infty$, again using similar arguments as to prove (51). Noting that for every $k \in \mathbb{N}$ the function $u \in \mathcal{A}_1^{M,\lfloor\sqrt{kT}\rfloor}(Q_{kT};\mathbb{R}^n)$ can also be used as a test function $u \in \mathcal{A}_1^{M,\lfloor\sqrt{L}\rfloor}(Q_L;\mathbb{R}^n)$ in the minimum on Q_L we obtain that

$$\lim_{k\to\infty} F_{kT}(M) = \lim_{L\to\infty} F_L(M).$$

Hence, we can assume that $L, S \in T\mathbb{N}$, $1 << L << S$ and $u_L \in \mathcal{A}_1^{M,\lfloor\sqrt{L}\rfloor}(Q_L;\mathbb{R}^n)$ be such that

$$\frac{1}{L^N}\sum_{i\in\mathbb{Z}^N\cap Q_L(0)}\phi_i(\{(u_L)_{j+i}\}_{j\in\mathbb{Z}^N}) \le F_L(M)+\frac{1}{L}.$$

We define $v_S \in \mathcal{A}_1^{M,\lfloor\sqrt{S}\rfloor}(Q_S;\mathbb{R}^n)$ by

$$v_S(i)=\begin{cases} u_L(i-Lk)+LMk & \text{if } i\in Lk+Q_L(0), k\in\{-\frac{1}{2}\lfloor\frac{S-\sqrt{S}}{L}\rfloor,\dots,\frac{1}{2}\lfloor\frac{S-\sqrt{S}}{L}\rfloor\}^N\\ Mi & \text{otherwise.}\end{cases}$$

By the periodicity assumption and (H4) we have that

$$\begin{aligned}F_S(M) &\le \frac{1}{S^N}\sum_{i\in\mathbb{Z}^N\cap Q_S(0)}\phi_i(\{(v_S)_{j+i}\}_{j\in\mathbb{Z}^N})\\ &= \frac{L^N}{S^N}\sum_{k\in\{-\frac{1}{2}\lfloor\frac{S-\sqrt{S}}{L}\rfloor,\dots,\frac{1}{2}\lfloor\frac{S-\sqrt{S}}{L}\rfloor\}^N}\frac{1}{L^N}\sum_{i\in\mathbb{Z}^N\cap Q_L(0)}\phi_{i+kL}(\{(u_L)_{j+i-kL}\}_{j\in\mathbb{Z}^N})\\ &\le \frac{L^N}{S^N}\Big\lfloor\frac{S-\sqrt{S}}{L}\Big\rfloor^N\frac{1}{L^N}\sum_{i\in\mathbb{Z}^N\cap Q_L(0)}\phi_i(\{(u_L)_{j+i}\}_{j\in\mathbb{Z}^N})\\ &\qquad+\frac{1}{S^N}\sum_{i\in Q_S(0)}\sum_{j,\xi\in\mathbb{Z}^N}C^{j-i}_{\sqrt{L}}(|D_1^\xi v_S(j)|^p+1)\\ &\le \frac{L^N}{S^N}\Big\lfloor\frac{S-\sqrt{S}}{L}\Big\rfloor^N\frac{1}{L^N}F_L(M)+\frac{1}{S^N}\sum_{i\in Q_S(0)}\sum_{j,\xi\in\mathbb{Z}^N}C^{j-i}_{\sqrt{L}}(|D_1^\xi v_S(j)|^p+1).\end{aligned}$$

Now, again using the same arguments as for (47)–(49), we obtain

$$\limsup_{L\to\infty}\limsup_{S\to\infty}\frac{1}{S^N}\sum_{i\in Q_S(0)}\sum_{j,\xi\in\mathbb{Z}^N}C^{j-i}_{\sqrt{L}}(|D_1^\xi v_S(j)|^p+1)=0$$

and therefore, noting that $\lim_{L\to\infty}\lim_{S\to\infty}\frac{L^N}{S^N}\left\lfloor\frac{S-\sqrt{S}}{L}\right\rfloor^N = 1$, we get $\limsup_{S\to\infty} F_S(M) \leq \liminf_{L\to\infty} F_L(M)$ and the claim follows. □

6 Examples

6.1 The discrete determinant

An example of interactions that can be taken into account with our type of energies are discrete determinants. For $z \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^n)$ we define

$$\phi_i^\varepsilon(\{z_j\}_{j\in Z_\varepsilon(\Omega_i)}) = \sum_{\xi_1,\dots,\xi_n\in\mathbb{Z}^N} g^\varepsilon_{\xi_1,\dots,\xi_n}(\det(D_\varepsilon^{\xi_1}z(0),\dots,D_\varepsilon^{\xi_n}z(0))) + \sum_{n=1}^N |D_\varepsilon^{e_n}z(0)|^p,$$

where $g^\varepsilon_{\xi_1,\dots,\xi_n} : \mathbb{R}\to[0,\infty)$ satisfy

$$g^\varepsilon_{\xi_1,\dots,\xi_n}(z) \leq C_{\xi_1,\dots,\xi_n}(|z|^{\frac{p}{n}}+1)$$

and $C_{\xi_1,\dots,\xi_n} > 0$ satisfy

$$\sum_{\xi_1,\dots,\xi_n\in\mathbb{Z}^N} C_{\xi_1,\dots,\xi_n} < +\infty.$$

(H1) follows, since ϕ_i^ε does only depend on its difference quotients. Note that by Hadamard's Inequality, the Geometric-Arithmetic mean Inequality and convexity we have

$$|\det(D_\varepsilon^{\xi_1}z(0),\dots,D_\varepsilon^{\xi_n}z(0))|^{\frac{p}{n}} \leq \left|\prod_{j=1}^n |D_\varepsilon^{\xi_j}z(0)|^{\frac{1}{n}}\right|^p \leq \left|\frac{1}{n}\sum_{j=1}^n |D_\varepsilon^{\xi_j}z(0)|\right|^p \leq \frac{1}{n}\sum_{j=1}^n |D_\varepsilon^{\xi_j}z(0)|^p.$$

Recall $\left|\frac{M(i+\varepsilon\xi)-Mi}{\varepsilon|\xi|}\right| \leq |M|$ and therefore

$$|\det(D_\varepsilon^{\xi_1}z(0),\dots,D_\varepsilon^{\xi_n}z(0))|^{\frac{p}{n}} \leq |M|^p$$

and by summing over $\xi_1,\dots,\xi_n\in\mathbb{Z}^N$ (H2) follows. (H3) follows since we have exactly the coercivity term in the definition of ϕ_i^ε and the first term is positive. For $\delta > 0$ and $z(j) = w(j)$ in $Z_\varepsilon(Q_\delta(i))$ we have that

$$\phi_i^\varepsilon(\{z_j\}_{j\in Z_\varepsilon(\Omega_i)}) \leq \phi_i^\varepsilon(\{w_j\}_{j\in Z_\varepsilon(\Omega_i)}) + \sum_{\substack{\xi_1,\dots,\xi_n\in\mathbb{Z}^N\\ \varepsilon|\xi_i|_\infty>\delta}} C_{\xi_1,\dots,\xi_n}\frac{1}{n}\sum_{j=1}^n |D_\varepsilon^{\xi_j}z(0)|^p.$$

Hence, by choosing

$$C^{0,\xi}_{\varepsilon,\delta} = \sum_{\substack{\xi\in\{\xi_1,\dots,\xi_n\}\subset(\mathbb{Z}^N)^n\\ \varepsilon|\xi_i|_\infty>\delta \text{ for some } i}} \frac{1}{n}C_{\xi,\dots,\xi_n}, \quad C^{j,\xi}_{\varepsilon,\delta} = 0, j\neq 0$$

(H4) follows. Setting

$$C_\varepsilon^{0,\xi} = \sum_{\xi\in\{\xi_1,\dots,\xi_n\}\subset(\mathbb{Z}^N)^n} \frac{1}{d} C_{\xi,\dots,\xi_n}, \quad C_\varepsilon^{j,\xi} = 0, j \neq 0$$

we have that $C_\varepsilon^{j,\xi}$ satisfies (13) and we have

$$\phi_i^\varepsilon(\{z_j\}_{j\in Z_\varepsilon(\Omega_i)}) \leq \sum_{\xi\in\mathbb{Z}^N} C_\varepsilon^{0,\xi}(|D_\varepsilon^\xi z(0)|^p + 1).$$

Note that for all cut-off functions ψ and for all $z, w \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^n)$ we have

$$D_\varepsilon^\xi(\psi z + (1-\psi)w) = \psi(i) D_\varepsilon^\xi z(i) + (1-\psi(i)) D_\varepsilon^\xi w(i) + D_\varepsilon^\xi \psi(i)(z(i+\varepsilon\xi) - w(i+\varepsilon\xi)) \tag{52}$$

and hence (H5) follows by using the convexity of $|\cdot|^p$, $0 \leq \psi \leq 1$ and noting that

$$|D_\varepsilon^\xi \psi(i)| \leq \max_{n\in\{1,\dots,N\}} \sup_{k\in Z_\varepsilon(\Omega)} |D_\varepsilon^{e_n}\psi(k)|. \tag{53}$$

A particular example could be $g_{e_1,e_2}^\varepsilon(z) = |z|$ and $g_{\xi_1,\xi_2}^\varepsilon(z) = 0$ otherwise. More in general our Theorems also apply to the case where we take functions g of minors of $\big(D_\varepsilon^{\xi_1} z(0), \dots, D^{\xi_n} z(0)\big)$ as long as g satisfies appropriate bounds.

6.2 The linearization of the Lennard-Jones potential

We assume $N = d = 3$. Our result is applicable to show an integral representation if the potential ϕ_i^ε is the linearization of the Lennard-Jones potential, where the Lennard-Jones potential, pictured in Fig. 1, is defined by (up to renormalization)

$$V(r) = \frac{1}{r^{12}} - \frac{2}{r^6}.$$

For $\Omega \subset \mathbb{R}^3$ open and smooth we define $E_\varepsilon : L^2(\Omega;\mathbb{R}^3) \to [0, +\infty]$ by

$$E_\varepsilon(u) = \begin{cases} \sum_{i,j\in Z_\varepsilon(\Omega)} \varepsilon^3 V''\Big(\left|\frac{i-j}{\varepsilon}\right|\Big)\left|\frac{u_i-u_j}{\varepsilon}\right|^2 & \text{if } u \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^3) \\ +\infty & \text{otherwise.} \end{cases}$$

In fact heuristically E_ε can be obtained by linearizing the Lennard-Jones Energy defined by

$$E_\varepsilon^{LJ}(u) = \begin{cases} \sum_{i,j\in Z_\varepsilon(\Omega)} \varepsilon^3 V\Big(\left|\frac{u_i-u_j}{\varepsilon}\right|\Big) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^3) \\ +\infty & \text{otherwise,} \end{cases}$$

where the set of admissible deformations u should be close to the identity (neglecting the linear term in the expansion by the assumption that $u(i) = i$ is an equilibrium point). The term

$$\tilde\phi_i^\varepsilon(\{u_{j+i}\}_{j\in Z_\varepsilon(\Omega_i)}) = \sum_{j\in Z_\varepsilon(\Omega)} V''\Big(\left|\frac{i-j}{\varepsilon}\right|\Big)\left|\frac{u_i-u_j}{\varepsilon}\right|^2$$

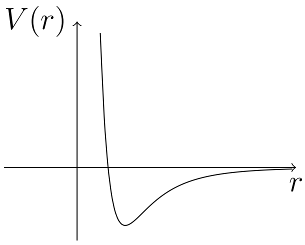

Figure 1: The Lennard-Jones potential

may not be positive in general, due to the long-range part of the potential. We may regroup the interactions of $\tilde{\phi}_i^{\varepsilon}$ such that we have a positive potential satisfying the assumptions of our main theorem. For every $\xi \in \mathbb{Z}^3, i \in \mathbb{Z}^N$ we choose a path $\gamma_\xi^i = (i_h)_{h=1}^{||\xi||_1+1} \subset \mathbb{Z}^3$ by defining

$$i^\xi_{||\xi||_1+1} = i + \xi, i_1^\xi = i, i_{h+1}^\xi = i_h^\xi + e_{n(h)}, e_{n(h)} = \mathrm{sign}(\xi_k)e_k \text{ if } 1 + \sum_{n=1}^{k-1} |\xi_n| \leq h \leq \sum_{n=1}^{k} |\xi_n|.$$

For this path it holds

$$|D_1^\xi u(i)|^2 \leq \frac{3}{||\xi||_1} \sum_{h=1}^{||\xi||_1} |D_1^{e_n(h)} u(i_h)|^2.$$

For $\xi \in \mathbb{Z}^3 \setminus \{\pm e_1, \pm e_2, \pm e_3\}$ we define $f_i^\xi : \mathcal{A}_1(\mathbb{R}^3;\mathbb{R}^3) \to [0,\infty)$ by

$$f_i^\xi(\{u_j\}_{j\in\mathbb{Z}^3}) = V''(|\xi|)\Big(- |D_1^\xi u(i)|^2 + \frac{3}{||\xi||_1} \sum_{h=1}^{||\xi||_1} |D_1^{e_{n(h)}} u(i_h)|^2\Big),$$

and for $v \in \{\pm e_1, \pm e_2, \pm e_3\}$ we define $f_i^v : \mathcal{A}_1(\mathbb{R}^3;\mathbb{R}^3) \to \mathbb{R}$

$$f_i^v(\{u_j\}_{j\in\mathbb{Z}^3}) = \Big(V''(1) - \sum_{j\in\mathbb{Z}^3} \sum_{\substack{\xi\in\mathbb{Z}^3 \\ i=i_h\in\gamma_j^\xi, e_{n(h)}=v}} \frac{3V''(|\xi|)}{||\xi||_1}\Big)|D_1^v u(i)|^2.$$

Moreover, we define $\phi_i^k : \mathcal{A}_1(\mathbb{R}^3;\mathbb{R}^3) \to \mathbb{R}$ by

$$\phi_i^k(\{u_j\}_{j\in\mathbb{Z}^3}) = \sum_{|\xi|_\infty \leq k} f_i^\xi(\{u_j\}_{j\in\mathbb{Z}^3})$$

and $\phi_i : \mathcal{A}_1(\mathbb{R}^3;\mathbb{R}^3) \to \mathbb{R}$ by

$$\phi_i(\{u_j\}_{j\in\mathbb{Z}^3}) = \sum_{\xi\in\mathbb{Z}^3} f_i^\xi(\{u_j\}_{j\in\mathbb{Z}^3}).$$

We need to check that

$$f_i^v(\{u_j\}_{j\in\mathbb{Z}^N}) \geq c|D_1^v u(i)|^2 \tag{54}$$

for some constant $c>0$, $v\in\{\pm e_1,\pm e_2,\pm e_3\}$ and that ϕ_i^k,ϕ_i satisfy (H1)–(H3) and (H$_p$4)–(H$_p$7). Note that for $u^\varepsilon(j)=\frac{u(\varepsilon j)}{\varepsilon}$ it holds

$$\sum_{i\in Z_\varepsilon(\mathbb{R}^3)}\phi_{\frac{i}{\varepsilon}}(\{u_j^\varepsilon\}_{j\in\mathbb{Z}^3})=\sum_{i\in Z_\varepsilon(\mathbb{R}^3)}\tilde{\phi}_i^\varepsilon(\{u_j\}_{j\in Z_\varepsilon(\mathbb{R}^3)}).$$

By the definition of ϕ_i^k,ϕ_i it is clear, that (H1), (H2) holds. To prove (H3) we have that $\phi_i^\xi\geq 0$ for all $\xi\in\mathbb{Z}^3\setminus\{\pm e_1,\pm e_2,\pm e_3\}$ and by Fubini's Theorem we have that

$$\sum_{j\in\mathbb{Z}^3}\sum_{\substack{\xi\in\mathbb{Z}^3,|\xi|>1\\ i=i_h\in\gamma_\xi^j,e_{n(h)}=v}}\frac{3V''(|\xi|)}{||\xi||_1}=\sum_{\xi\in\mathbb{Z}^3,|\xi|>1}\#N_{i,v}^\xi\frac{3V''(|\xi|)}{||\xi||_1},\tag{55}$$

where $N_{i,v}^\xi=\Big\{j\in\mathbb{Z}^3:\exists h\in\{1,\dots,|\xi_1|\}\text{ such that }i=j_h\in\gamma_j^\xi\text{ and }e_{n(h)}=v\Big\}$. Note that for $\xi\in\mathbb{Z}^3$ such that $\langle\xi,v\rangle>0$ we have $\#N_{i,v}^\xi\leq||\xi||_1$ and $\#N_{i,v}^\xi=0$ otherwise. Hence, using the monotonicity of $V''(r)$ for $r\geq\sqrt{2}$ and the fact that $||\xi||_\infty\leq||\xi||_2$ and using the fact that $\#\{\xi\in\mathbb{Z}^3:||\xi||_\infty=k\}=3k^2-3k+1$, we obtain

$$\begin{aligned}-\sum_{\substack{\xi\in\mathbb{Z}^3\\|\xi|>1}}\#N_{i,v}^\xi\frac{3V''(|\xi|)}{||\xi||_1}&\leq-\sum_{\substack{\xi\in\mathbb{Z}^3,|\xi|>1\\\langle\xi,v\rangle>0}}3V''(|\xi|)=-12V''(\sqrt{2})-3\sum_{k=2}^\infty\sum_{||\xi||_\infty=k}V''(|\xi|)(3k^2-3k+1)\\&\leq-12V''(\sqrt{2})-3\sum_{k=2}^\infty V''(k)(3k^2-3k+1)<V''(1).\end{aligned}\tag{56}$$

Hence, we obtain (54) and with that (H3). (H$_p$4) and (H$_p$7) follow from the definition of ϕ_i^k and ϕ_i. Setting

$$C^{j,e_n}=\begin{cases}V''(1)&\text{if }j=0\\\displaystyle\sum_{\substack{\xi\in\mathbb{Z}^3,|\xi|>1\\j=i_h,e_{n(h)}=e_n}}\frac{3V''(|\xi|)}{||\xi||_1}&\text{otherwise,}\end{cases}\tag{57}$$

and $C^{j,\xi}=0$ if $|\xi|>1$. Using (55) and (56) we obtain (40) and

$$\phi_i^k(\{\psi_jz_j+(1-\psi_j)w_j\}_{j\in\mathbb{Z}^N})\leq R_i^k(z,w,\psi),$$

with R_i^k defined in (H$_p$5) with $C^{j,\xi}$ defined by (57). By the non-negativity of ϕ_i^k it follows (H$_p$5). Setting

$$C_k^{j,e_n}=2\sum_{\substack{\xi\in\mathbb{Z}^3,||\xi||_\infty>k\\j=i_h,e_{n(h)}=e_n}}\frac{3V''(|\xi|)}{||\xi||_1}\tag{58}$$

and $C_k^{j,\xi}=0$ if $|\xi|>1$, using (55) and (56) we obtain (39). We have that

$$
\begin{aligned}
|\phi_i^{k_1}(\{z_j\}_{j\in\mathbb{Z}^N})-\phi_i^{k_2}(\{z_j\}_{j\in\mathbb{Z}^N})| &= \sum_{\xi\in\mathbb{Z}^3\cap(Q_{k_2}\setminus Q_{k_1})} f_i^\xi(\{z_j\}_{j\in\mathbb{Z}^3}) \\
&\leq \sum_{\xi\in\mathbb{Z}^3\cap(Q_{k_2}\setminus Q_{k_1})} V''(|\xi|)\frac{3}{||\xi||_1}\sum_{h=1}^{||\xi||_1}|D_1^{e_{n(h)}}z(i_h^\xi)|^2 \\
&\leq \sum_{n=1}^{3}\sum_{j\in\mathbb{Z}^3\cap Q_{k_2}}\sum_{\substack{\xi\in\mathbb{Z}^3,||\xi||_\infty>k_1\\ j=i_h,e_{n(h)}\in\{\pm e_n\}}}\frac{3V''(|\xi|)}{||\xi||_1}|D_1^{e_n}z(j)|^2 \\
&\leq \sum_{\substack{j,\xi\in\mathbb{Z}^3\cap Q_{k_2}\\ j+\xi\in\mathbb{Z}^3\cap Q_{k_2}}} C_{k_1}^{j,\xi}|D_1^\xi z(j)|^2
\end{aligned}
$$

and hence we obtain (H_p6). Applying Theorem 5.1 we obtain the Γ-convergence of E_ε to a functional $E:L^p(\Omega;\mathbb{R}^3)\times\mathcal{A}(\Omega)\to[0,+\infty]$ given by

$$E(u,A)=\int_A f_{\mathrm{hom}}(\nabla u)\mathrm{d}x,$$

where $f_{\mathrm{hom}}:\mathbb{R}^{3\times3}\to[0,+\infty)$ is given by

$$f_{\mathrm{hom}}(M)=\lim_{L\to\infty}\frac{1}{L^N}\inf\Big\{\sum_{i\in\mathbb{Z}^N\cap Q_L}\phi_i(\{z_{j+i}\}_{j\in\mathbb{Z}^N}):z\in\mathcal{A}_1^{M,m}(Q_L;\mathbb{R}^n)\Big\}.$$

6.3 "Damage energies" from microscopic oscillations

To conclude this section we give an example, where it is possible to compute the energy density explicitly highlighting microscopic oscillations due to non-convexity of the energy density. The resulting homogenized energies are similar to those used in the variational theory of damage (see e.g. [24, 29]). We consider the case $d=1$ and $\Omega=(0,1)$ and $u:Z_\varepsilon(\Omega)\to\mathbb{R}$. The energies we consider can be written as

$$F_\varepsilon(u)=\sum_{i=0}^{N_\varepsilon-1}\varepsilon W_1\left(\frac{u_{i+1}-u_i}{\varepsilon}\right)+\sum_{i=0}^{N_\varepsilon-2}\varepsilon W_2\left(\frac{u_{i+2}-u_i}{\varepsilon}\right),$$

where $u_i=u(\varepsilon i)$ and $N_\varepsilon=\lfloor\frac{1}{\varepsilon}\rfloor$. In our case we have $W_1(z)=|z|^2$ and $W_2(z)=|z|^2\wedge 1$.

One can check that

$$\phi_i^\varepsilon(\{z_j\}_{j\in Z_\varepsilon(\Omega_i)})=W_1\left(\frac{z_1-z_0}{\varepsilon}\right)+W_2\left(\frac{z_2-z_0}{\varepsilon}\right)$$

satisfies the hypothesis of Theorem 5.1 and we can therefore represent the Γ-limit with respect to the strong $L^2(\Omega)$-topology as a functional $F:L^2(\Omega)\to[0,+\infty]$ given by

$$F(u)=\int_\Omega f_{\mathrm{hom}}(u')\mathrm{d}x,$$

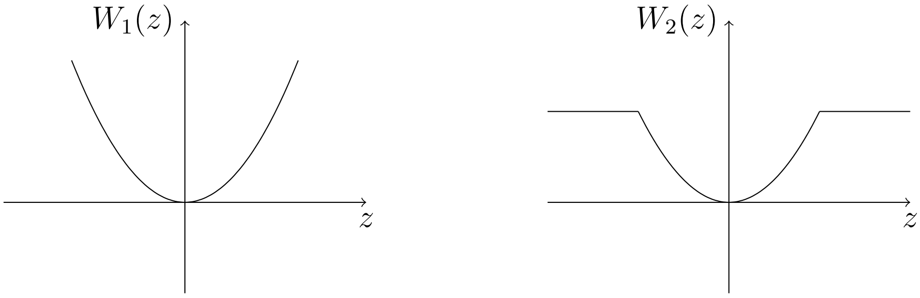

Figure 2: The potentials $W_1(z)$ and $W_2(z)$ in our explicit example

with $f_{\text{hom}} : \mathbb{R} \to \mathbb{R}$ given by (43). Moreover by [18], Theorem 3.2, we have that

$$f_{\text{hom}}(z) = W_{\text{eff}}^{**}(z),$$

where f^{**} denotes the convex envelope of a function f, that is

$$f^{**}(z) = \sup\{g(z) : g \leq f, g \text{ convex}\}$$

and $W_{\text{eff}} : \mathbb{R} \to \mathbb{R}$ is given by

$$W_{\text{eff}}(z) = \min\left\{\frac{1}{2}W_1(z_1) + \frac{1}{2}W_1(z_2) : \frac{z_1 + z_2}{2} = z\right\} + W_2(z).$$

Since W_1 is convex we have that

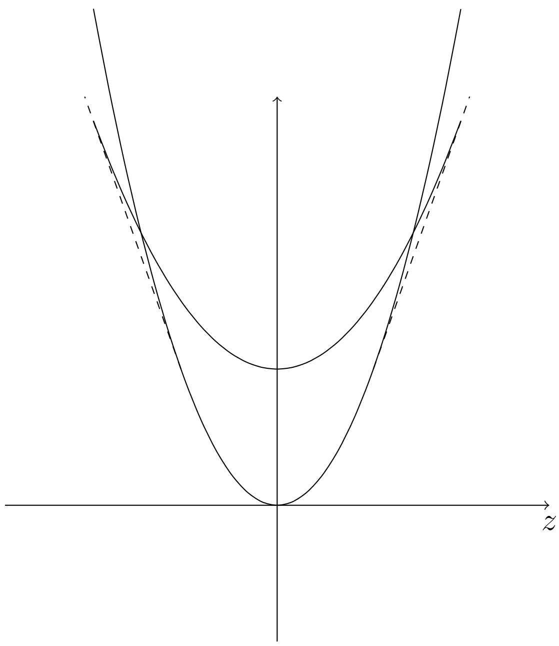

Figure 3: W_{eff} and W_{eff}^{**}

$$W_{\text{eff}}(z) = W_1(z) + W_2(z) = 2|z|^2 \wedge (|z|^2 + 1).$$

Note that W_{eff} is not convex and therefore one obtains after convexification

$$W_{\text{eff}}^{**}(z) = \begin{cases} 2|z|^2 & |z| \leq \frac{\sqrt{2}}{2} \\ 1 + \sqrt{2}(|z| - \frac{\sqrt{2}}{2}) & \frac{\sqrt{2}}{2} < |z| \leq \sqrt{2} \\ |z|^2 + 1 & \sqrt{2} < |z|. \end{cases}$$

This is illustrated in Fig. 3. W_{eff} is given by as the minimum of the two parabola. One can see, that the function is not convex and so it has to be convexified in order for the integral functional to be lower semicontinuous. The graph of W_{eff}^{**} is given by the two parabolas in the two regions where the two functions agree and by the dashed line where $W_{\text{eff}}^{**} < W_{\text{eff}}$.

Remark 6.1 (The relaxation formula)**.** Note that in this example the relaxation takes place on two different scales. The first relaxation is given by W_{eff} which captures the relaxation on a *microscopic scale*, a scale which is comparable to the typical length scale of the lattice (In this case at scale 2ε to be precise). Since the function obtained by this formula is not convex one has to relax another time in order to capture oscillations on a much larger scale, a so called *mesoscopic scale*. In fact the authors of [18] show with some interesting examples that both those relaxations are needed in order to capture the right asymptotic behaviour.

6.4 Pair interactions: the Alicandro-Cicalese theorem

The compactness theorem can be applied to the special case of pair potentials where ϕ_i^ε takes only into account the pair interactions of that point with every other point $j \in Z_\varepsilon(\Omega)$, that means it is of the form

$$\phi_i^\varepsilon(\{z_{j+i}\}_{j \in Z_\varepsilon(\Omega_i)}) = \sum_{\substack{\xi \in \mathbb{Z}^N \\ i+\varepsilon\xi \in Z_\varepsilon(\Omega)}} f_\varepsilon^\xi(i, D_\varepsilon^\xi z(i))$$

with $f_\varepsilon^\xi \geq 0$ satisfying

(i) $f_\varepsilon^{e_n}(i, z) \geq c(|z|^p - 1)$ for all $i \in Z_\varepsilon(\Omega)$, $z \in \mathbb{R}^n, \varepsilon > 0$ and $n \in \{1, \dots, N\}$.

(ii) $f_\varepsilon^\xi(i, z) \leq c_\varepsilon^\xi(|z|^p + 1)$ for all $i \in Z_\varepsilon(\Omega)$, $z \in \mathbb{R}^n, \varepsilon > 0$ and $\xi \in \mathbb{R}^N$, where

$$\limsup_{\varepsilon \to 0} \sum_{\xi \in \mathbb{Z}^N} c_\varepsilon^\xi < +\infty \tag{59}$$

$$\forall\, \delta > 0\, \exists M_\delta > 0 \text{ such that } \limsup_{\varepsilon \to 0} \sum_{|\xi| > M_\delta} c_\varepsilon^\xi < \delta. \tag{60}$$

(H1) follows since for each $\xi \in \mathbb{R}^N, i \in Z_\varepsilon(\Omega)$ the interaction depend only on $D_\varepsilon^\xi z$. (H2) follows from (59) and (ii). (H3) follows from (i). (H4) follows if we choose

$$C_{\varepsilon,\delta}^{i,\xi} = \begin{cases} c_\varepsilon^\xi & \varepsilon|\xi|_\infty \geq \delta, i = 0 \\ 0 & i \neq 0. \end{cases}$$

Let $z, w \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^n)$ such that $z(j) = w(j)$ in $Z_\varepsilon(Q_\delta(i))$. Then, using the positivity of f_ε^ξ and (ii), we obtain

$$
\begin{aligned}
\phi_i^\varepsilon(\{z_j\}_{j\in Z_\varepsilon(\Omega_i)}) &= \sum_{\substack{\xi\in\mathbb{Z}^N \\ i+\varepsilon\xi\in Z_\varepsilon(\Omega_i)}} f_\varepsilon^\xi(0, D_\varepsilon^\xi z(0)) = \sum_{\substack{\varepsilon|\xi|_\infty\leq\delta \\ \varepsilon\xi\in Z_\varepsilon(\Omega_i)}} f_\varepsilon^\xi(0, D_\varepsilon^\xi z(0)) + \sum_{\substack{\varepsilon|\xi|_\infty>\delta \\ \varepsilon\xi\in Z_\varepsilon(\Omega_i)}} f_\varepsilon^\xi(0, D_\varepsilon^\xi z(0)) \\
&\leq \sum_{\substack{\varepsilon|\xi|_\infty\leq\delta \\ \varepsilon\xi\in Z_\varepsilon(\Omega_i)}} f_\varepsilon^\xi(0, D_\varepsilon^\xi w(0)) + \sum_{\substack{\varepsilon|\xi|_\infty>\delta \\ \varepsilon\xi\in Z_\varepsilon(\Omega_i)}} c_\varepsilon^\xi(|D_\varepsilon^\xi z(0)|^p + 1) \\
&\leq \phi_i^\varepsilon(\{w_j\}_{j\in Z_\varepsilon(\Omega_i)}) + \sum_{j\in Z_\varepsilon(\Omega_i),\xi\in\mathbb{Z}^N} C_{\varepsilon,\delta}^{j,\xi}(|D_\varepsilon^\xi z(j)|^p + 1)
\end{aligned}
$$

and therefore (H4) follows. Setting

$$
C_\varepsilon^{i,\xi} = \begin{cases} c_\varepsilon^\xi & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}
$$

we have that

$$
\phi_i^\varepsilon(\{z_j\}_{j\in Z_\varepsilon(\Omega_i)}) \leq \sum_{\substack{j\in Z_\varepsilon(\Omega_i),\xi\in\mathbb{Z}^N \\ j+\varepsilon\xi\in Z_\varepsilon(\Omega_i)}} C_\varepsilon^{j,\xi}|D_\varepsilon^\xi z(j)|^p.
$$

and again for a cut-off function ψ and $z, w \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^n)$ (H5) follows by using (52), the convexity of $|\cdot|^p$ and (53).

References

[1] R. Alicandro, A. Braides and M. Cicalese. Continuum limits of discrete thin films with superlinear growth densities. Calculus of Variations and Partial Differential Equations **33.3** (2008): 267-297.

[2] R. Alicandro and M. Cicalese. A general integral representation result for continuum limits of discrete energies with superlinear growth. SIAM journal on mathematical analysis **36.1** (2004): 1-37.

[3] R. Alicandro and M. Cicalese. Variational analysis of the asymptotics of the XY model. Archive for rational mechanics and analysis **192.3** (2009): 501-536.

[4] R. Alicandro, M. Cicalese and A. Gloria. Integral representation results for energies defined on stochastic lattices and application to nonlinear elasticity. Archive for rational mechanics and analysis **200.3** (2011): 881-943.

[5] R. Alicandro and M.S. Gelli. Local and nonlocal continuum limits of Ising-type energies for spin systems. SIAM journal on mathematical analysis **49** (2016): 895–931.

[6] L. Ambrosio, N. Fusco and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* (Oxford: Clarendon Press, 2000).

[7] X. Blanc, C. Le Bris and P.L. Lions. From molecular models to continuum models. Archive for Rational Mechanics and Analysis **332** (2002): 949–956.

[8] X. Blanc, C. Le Bris and P.L. Lions. Atomistic to continuum limits for computational materials science. ESAIM: Mathematical Modelling and Numerical Analysis **41.2** (2007): 391-426.

[9] A. Braides. Non-local variational limits of discrete systems. Communications in Contemporary Mathematics **2.02** (2000): 285-297.

[10] A. Braides. *Γ-convergence for Beginners.* (Oxford University Press, Oxford, 2002).

[11] A. Braides. A handbook of Γ-convergence. In Handbook of Differential Equations. Stationary Partial Differential Equations, **3** (M. Chipot and P. Quittner, eds.)(Elsevier, 2006): 101–213.

[12] A. Braides. Discrete-to-continuum variational methods for lattice systems. *Proceedings of the International Congress of Mathematicians August* 13–21, 2014*, Seoul, Korea* (S. Jang, Y. Kim, D. Lee, and I. Yie, eds.) Kyung Moon Sa, Seoul, **4** (2014): 997–1015.

[13] A. Braides and M. Cicalese. Interfaces, modulated phases and textures in lattice systems. Archive for rational mechanics and analysis **223** (2017): 977-1017.

[14] A. Braides, M. Cicalese, and F. Solombrino. Q-tensor continuum energies as limits of head-to-tail symmetric spin systems. SIAM Journal on Mathematical Analysis **47.4** (2015): 2832-2867.

[15] A. Braides and A. Defranceschi. *Homogenization of Multiple Integrals.* (Oxford University Press, Oxford 1998).

[16] A. Braides and G. Francfort. Bounds on the effective behavior of a square conducting lattice. Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **460** (2004): 1755–1769.

[17] A. Braides and M.S. Gelli. From discrete systems to continuous variational problems: an introduction. In *Topics on concentration phenomena and problems with multiple scales* (A. Braides and V. Chiadò Piat, eds.) **2** (Springer, Berlin, 2006): 3–77.

[18] A. Braides, M.S. Gelli, and M. Sigalotti. The passage from nonconvex discrete systems to variational problems in Sobolev spaces: the one-dimensional case. Proceedings of the Steklov Inst.of Math. **236** (2002), 395-414.

[19] A. Braides, A.J. Lew and M. Ortiz. Effective cohesive behavior of layers of interatomic planes. Archive for rational Mechanics and analysis **180** (2006), 151–182.

[20] A. Braides, M. Solci, and E. Vitali. A derivation of linear elastic energies from pair-interaction atomistic systems. Networks and Heterogeneous Media **2.3** (2007): 551.

[21] J. Braun and B. Schmidt. On the passage from atomistic systems to nonlinear elasticity theory for general multi-body potentials with p-growth. Networks and Heterogeneous Media **8.4** (2013): 879-912.

[22] G. Dal Maso. *An Introduction to Gamma-convergence.* (Birkhäuser, Boston, 1993).

[23] J.L. Ericksen. On the Cauchy-Born rule. Mathematics and mechanics of solids **13** (2008):199–220.

[24] G. Francfort, J.J. Marigo. Stable damage evolution in a brittle continuous medium. European Journal of mechanics series a solids **12** (1993): 149–149.

[25] G. Friesecke, F. Theil. Validity and Failure of the Cauchy-Born Hypothesis in a Two-Dimensional Mass-Spring Lattice. Journal of nonlinear Science **12** (2002) 445-478 .

[26] M. Geers, R. Peerlings, M. Peletier, and L. Scardia. Asymptotic behaviour of a pile-up of infinite walls of edge dislocations. M.A. et al. Arch Rational Mech Anal **209** (2013) 495–539.

[27] A. Gloria. Numerical homogenization: survey, new results, and perspectives. ESAIM: Proceedings. **37**. EDP Sciences (2012).

[28] G. Kitavtsev, S. Luckhaus and A. Rüland. Surface energies arising in microscopic modeling of martensitic transformations. Mathematical Models and Methods in Applied Sciences **25.04** (2015): 647-683.

[29] R.V. Kohn, G. Strang Optimal design and relaxation of variational problems, I-III. Communications on Pure and Applied Mathematics. **39** (1986): 113–137,139–182,353-377.

[30] H. Le Dret and A. Raoult. Homogenization of hexagonal lattices. Networks and Heterogeneous Media **8** (2013): 541-572.

[31] N. Meunier, O. Pantz and A. Raoult. Elastic limit of square lattices with three point interactions Math. Models Methods Appl. Sci. **22** (2012): 1250032.

[32] A. Piatnitski and E. Remy. Homogenization of elliptic difference operators. SIAM journal on mathematical analysis **33.1** (2001): 53-83.