Solvability of a class of phase field systems related to a sliding mode control problem

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Abstract

We consider a phase-field system of Caginalp type perturbed by the presence of an additional maximal monotone nonlinearity. Such a system arises from a recent study of a sliding mode control problem. We prove existence of strong solutions. Moreover, under further assumptions, we show the continuous dependence on the initial data and the uniqueness of the solution.

Key words: Phase transition problem; phase field system; nonlinear parabolic boundary value problem; existence; continuous dependence.

AMS (MOS) subject classification: 35K61, 35K25, 35B25, 35D30, 80A22.

1 Introduction

In the present contribution we consider the phase-field system

$$\partial_t(\vartheta + \ell\varphi) - k\Delta\vartheta + \zeta = f \quad \text{a.e. in } Q := (0, T) \times \Omega, \tag{1.1}$$

$$\partial_t\varphi - \nu\Delta\varphi + \xi + \pi(\varphi) = \gamma\vartheta \quad \text{a.e. in } Q, \tag{1.2}$$

$$\zeta(t) \in A(\vartheta(t) + \alpha\varphi(t) - \eta^*) \text{ for a.e. } t \in (0, T), \tag{1.3}$$

$$\xi \in \beta(\varphi) \text{ a.e. in } Q, \tag{1.4}$$

where Ω is the domain in which the evolution takes place, T is some final time, ϑ denotes the relative temperature around some critical value that is taken to be 0 without loss of generality, and φ is the order parameter. Moreover, ℓ, k, ν, γ and α are positive constants, η^* is a function in $H^2(\Omega)$ with null outward normal derivative on the boundary of Ω and f is a source term. The above system is complemented by homogeneous Neumann boundary conditions for both ϑ and φ, that is,

$$\partial_N\vartheta = 0, \qquad \partial_N\varphi = 0 \quad \text{on } \Sigma := (0, T) \times \Gamma, \tag{1.5}$$

where Γ is the boundary of Ω and ∂_N is the outward normal derivative, and by the initial conditions

$$\vartheta(0) = \vartheta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{1.6}$$

The term $\xi + \pi(\varphi)$, appearing in (1.2), represents the derivative of a double-well potential F defined as the sum

$$F = \widehat{\beta} + \widehat{\pi}, \tag{1.7}$$

where

$$\widehat{\beta} : \mathbb{R} \longrightarrow [0, +\infty] \text{ is proper, l.s.c. and convex with } \widehat{\beta}(0) = 0, \tag{1.8}$$

$$\widehat{\pi} : \mathbb{R} \to \mathbb{R}, \ \widehat{\pi} \in C^1(\mathbb{R}) \text{ with } \pi := \widehat{\pi}' \text{ Lipschitz continuous.} \tag{1.9}$$

Since $\widehat{\beta}$ is proper, l.s.c. and convex, the subdifferential $\partial\widehat{\beta} =: \beta$ is well defined and is a maximal monotone graph. For a comprehensive discussion of the theory of maximal monotone operators, we refer, e.g., to [1,3,20]. In our problem we also consider a maximal monotone operator

$$A : \ H := L^2(\Omega) \longrightarrow 2^H \tag{1.10}$$

such that $0 \in A(0)$ and

$$\|v\|_H \leq C(1 + \|x\|_H) \qquad \text{for all } x \in H, \ v \in Ax, \tag{1.11}$$

for some constant $C > 0$.

The problem (1.1)–(1.6) under study is an interesting development of the following simple version of the phase-field system of Caginalp type (see [5]):

$$\partial_t(\vartheta + \ell\varphi) - k\Delta\vartheta = f \quad \text{in } Q, \tag{1.12}$$

$$\partial_t\varphi - \nu\Delta\varphi + F'(\varphi) = \gamma\vartheta \quad \text{in } Q. \tag{1.13}$$

As already noticed, $F' \cong \xi + \pi$ is related to a double-well potential F. Typical examples for F are

$$F_{reg}(r) = \frac{1}{4}(r^2 - 1)^2, \qquad r \in \mathbb{R}, \tag{1.14}$$

$$F_{log}(r) = ((1 + r)\ln(1 + r) + (1 - r)\ln(1 - r)) - c_0 r^2, \quad r \in (-1, 1), \tag{1.15}$$

$$F_{obs}(r) = I(r) - c_0 r^2, \qquad r \in \mathbb{R}, \tag{1.16}$$

where $c_0 > 1$ in (1.15) in order to produce a double well, while c_0 is an arbitrary positive number in (1.16), and the function I in (1.16) is the indicator function of $[-1, 1]$, i.e., it takes the values 0 or $+\infty$ according to whether or not r belongs to $[-1, 1]$. The potentials (1.14) and (1.15) are the usual classical regular potential and the so-called logarithmic potential, respectively.

The well-posedness, the long-time behavior of solutions, and also the related optimal control problems concerning Caginalp-type systems have been widely studied in the literature. We refer, without any sake of completeness, e.g., to [4, 12, 14, 18, 19] and references therein for the well-posedness and long time behavior results and to [6–8, 15, 16] for the treatment of optimal control problems.

The paper [2] is related to control problems, but it goes in the direction of designing sliding mode controls (SMC) for a particular phase-field system. The main objective of the authors is to find some state-feedback control laws $(\vartheta, \varphi) \mapsto u(\vartheta, \varphi)$ that can be that, once inserted into the equations, force the solution to reach some submanifold of the phase space, in finite time, then slide along it. The first analytical difficulty consists in deriving the equations governing the sliding modes and the conditions for this motion to exist.

The problem needs the development of special methods, since the conventional theorems regarding existence and uniqueness of solutions are not directly applicable. Moreover, the authors need to manipulate the system through the control in order to constrain the evolution on the desired sliding manifold.

In particular, in the paper [2] the authors consider the operator Sign : $H \longrightarrow 2^H$ defined as $\mathrm{Sign}(v) = \frac{v}{\|v\|}$, if $v \neq 0$ and $\mathrm{Sign}(0) = B_1(0)$, if $v = 0$, where $B_1(0)$ is the closed unit ball of H. Sign is a maximal monotone operator on H and is a nonlocal counterpart of the operator sign : $\mathbb{R} \longrightarrow 2^{\mathbb{R}}$ defined as $\mathrm{sign}(r) = \frac{r}{|r|}$, if $r \neq 0$ and $\mathrm{sign}(0) = [-1, 1]$, if $r = 0$. Then the authors of [2] deal with the system

$$\partial_t(\vartheta + \ell\varphi) - k\Delta\vartheta = f - \rho\sigma \quad \text{a.e. in } Q, \tag{1.17}$$

$$\partial_t\varphi - \nu\Delta\varphi + \xi + \pi(\varphi) = \gamma\vartheta \quad \text{a.e. in } Q, \tag{1.18}$$

$$\sigma(t) \in \mathrm{Sign}(\vartheta(t) + \alpha\varphi(t) - \eta^*) \quad \text{for a.e. } t \in (0, T), \tag{1.19}$$

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q, \tag{1.20}$$

$$\partial_N\vartheta = 0, \qquad \partial_N\varphi = 0 \quad \text{on } \Sigma, \tag{1.21}$$

$$\vartheta(0) = \vartheta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega, \tag{1.22}$$

which turns out to be a particular case of (1.1)–(1.6) with $A = \rho\,\mathrm{Sign}$. The paper [2] is mostly concerned with the sliding mode property for (1.17)–(1.22). In this contribution we deal with (1.1)–(1.6), which turns out to be a particular generalization of the problem (1.17)–(1.22) since we only require (1.10)–(1.11) for the maximal monotone operator A. We prove existence and regularity of the solutions for the problem (1.1)–(1.6), as well as the uniqueness and the continuous dependence on the initial data in case $\alpha = \ell$. In order to obtain our results, we first make a change of variable. We set:

$$\eta = \vartheta + \alpha\varphi - \eta^*. \tag{1.23}$$

Consequently, the previous system (1.1)–(1.6) becomes

$$\partial_t(\eta + (\ell - \alpha)\varphi) - k\Delta\eta + k\alpha\Delta\varphi + \zeta = f - k\Delta\eta^* \quad \text{a.e. in } Q, \tag{1.24}$$

$$\partial_t\varphi - \nu\Delta\varphi + \xi + \pi(\varphi) = \gamma(\eta - \alpha\varphi + \eta^*) \quad \text{a.e. in } Q, \tag{1.25}$$

$$\zeta(t) \in A(\eta(t)) \quad \text{for a.e. } t \in (0, T), \tag{1.26}$$

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q. \tag{1.27}$$

$$\partial_N\eta = 0, \qquad \partial_N\varphi = 0 \quad \text{on } \Sigma, \tag{1.28}$$

$$\eta(0) = \eta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{1.29}$$

From now on, we refer to the initial and boundary value problem (1.24)–(1.29) as Problem (P). In order to prove the existence of solutions, we first consider the approximating problem (P_ε), obtained from problem (P) by approximating A and β by their Yosida regularizations. Then we construct a further approximating problem $(P_{\varepsilon,n})$, obtained from (P_ε) by a Faedo-Galerkin scheme based on a system of eigenfunctions $\{v_n\} \subseteq W$, where

$$W = \{u \in H^2(\Omega) : \ \partial_N u = 0 \text{ on } \partial\Omega\}. \tag{1.30}$$

Then, we prove the existence of a local solution for $(P_{\varepsilon,n})$ and, passing to the limit as $n \to +\infty$, we infer that the limit of some subsequence of solutions for $(P_{\varepsilon,n})$ yields a solution of (P_ε). Finally, we pass to the limit as $\varepsilon \searrow 0$ and show that some limit of a subsequence yields a solution of (P).

Next, we let $\alpha = \ell$ and write problem (P) for two different sets of initial data f_i, η_i^*, η_{0_i} and φ_{0_i}, $i = 1, 2$. By performing suitable contracting estimates for the difference of the corresponding solutions, we deduce the continuous dependence result whence the uniqueness property is also achieved.

2 Main results

2.1 Preliminary assumptions

We assume $\Omega \subseteq \mathbb{R}^3$ to be a bounded domain of class C^1 and we write $|\Omega|$ for its Lebesgue measure. Moreover, Γ and ∂_N still stand for the boundary of Ω and the outward normal derivative, respectively. Given a finite final time $T > 0$, for every $t \in (0, T]$ we set

$$Q_t = (0,t) \times \Omega, \quad Q = Q_T, \tag{2.1}$$

$$\Sigma_t = (0,t) \times \Gamma, \quad \Sigma = \Sigma_T. \tag{2.2}$$

In the following, we set for brevity:

$$H = L^2(\Omega), \qquad V = H^1(\Omega), \qquad V_0 = H_0^1(\Omega), \tag{2.3}$$

$$W = \{u \in H^2(\Omega) : \ \partial_N u = 0 \text{ on } \partial\Omega\}, \tag{2.4}$$

with usual norms $\|\cdot\|_H$, $\|\cdot\|_V$ and inner products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_V$, respectively. Now we describe the problem under consideration. We assume that

$$\ell, \ \alpha, \ k, \ \nu, \ \gamma \in (0, +\infty), \tag{2.5}$$

$$f \in L^2(Q), \tag{2.6}$$

$$\eta^* \in W, \tag{2.7}$$

$$\eta_0, \ \varphi_0 \in V, \tag{2.8}$$

$$\widehat{\beta}(\varphi_0) \in L^1(\Omega). \tag{2.9}$$

We introduce the double-well potential F as the sum

$$F = \widehat{\beta} + \widehat{\pi}, \tag{2.10}$$

where

$$\widehat{\beta} : \mathbb{R} \longrightarrow [0, +\infty] \text{ is proper, l.s.c. and convex with } \widehat{\beta}(0) = 0, \tag{2.11}$$

$$\widehat{\pi} : \mathbb{R} \to \mathbb{R}, \ \widehat{\pi} \in C^1(\mathbb{R}) \text{ with } \pi := \widehat{\pi}' \text{ Lipschitz continuous.} \tag{2.12}$$

Since $\widehat{\beta}$ is proper, lower semicontinuous and convex, the subdifferential $\partial\widehat{\beta} =: \beta$ is well defined. We denote by $D(\beta)$ and $D(\widehat{\beta})$ the effective domains of β and $\widehat{\beta}$, respectively. Thanks to these assumptions, β is a maximal monotone graph. Moreover, as $\widehat{\beta}$ takes on its minimum in 0, we have that $0 \in \beta(0)$.

Remark 2.1 We introduce the operator $\mathcal{B}$ induced by β on $L^2(Q)$ in the following way:

$$\mathcal{B} : L^2(Q) \longrightarrow L^2(Q) \tag{2.13}$$

$$\xi \in \mathcal{B}(\varphi) \Longleftrightarrow \xi(x,t) \in \beta(\varphi(x,t)) \quad \text{for a.e. } (x,t) \in Q. \tag{2.14}$$

We notice that

$$\beta = \partial\widehat{\beta}, \qquad \mathcal{B} = \partial\Phi, \tag{2.15}$$

where

$$\Phi : \ L^2(Q) \longrightarrow (-\infty, +\infty] \tag{2.16}$$

$$\Phi(u) = \begin{cases} \int_Q \widehat{\beta}(u) & \text{if } u \in L^2(Q) \text{ and } \widehat{\beta}(u) \in L^1(Q), \\ +\infty & \text{elsewhere, with } u \in L^2(Q). \end{cases} \tag{2.17}$$

The maximal monotone operator A. In our problem a maximal monotone operator

$$A : H \longrightarrow H \tag{2.18}$$

also appears. We assume that

$$0 \in A(0) \tag{2.19}$$

and that there exists a constant $C > 0$ such that

$$\|v\|_H \leq C(1 + \|\eta\|_H) \qquad \text{for all } \eta \in H, \ v \in A\eta. \tag{2.20}$$

Remark 2.2 We introduce the operator $\mathcal{A}$ induced by A on $L^2(0,T;H)$ in the following way

$$\mathcal{A} : L^2(0,T;H) \longrightarrow L^2(0,T;H) \tag{2.21}$$

$$\zeta \in \mathcal{A}(\eta) \Longleftrightarrow \zeta(t) \in A(\eta(t)) \quad \text{for a.e. } t \in (0,T). \tag{2.22}$$

We notice that $\mathcal{A}$ is a maximal monotone operator.

2.2 Examples of operators A

Now, we provide some examples of maximal monotone operators fulfilling our assumptions.

Example 1. We consider the operator

$$\text{sign} \ : \mathbb{R} \longrightarrow 2^{\mathbb{R}} \tag{2.23}$$

$$\text{sign}(r) = \begin{cases} \frac{r}{|r|} & \text{if } r \neq 0, \\ [-1,1] & \text{if } r = 0. \end{cases} \tag{2.24}$$

Notice that sign induces a maximal monotone operator on H.

Example 2. We define the operator Sign as the nonlocal counterpart of the operator sign (see (2.23)–(2.24)):

$$\mathrm{Sign}: \qquad H \longrightarrow 2^H \tag{2.25}$$

$$\mathrm{Sign}(v) = \begin{cases} \frac{v}{\|v\|} & \text{if } v \neq 0, \\ B_1(0) & \text{if } v = 0, \end{cases} \tag{2.26}$$

where $B_1(0)$ is the closed unit ball of H. Sign is the subdifferential of the map $\|\cdot\| : H \to \mathbb{R}$ and is a maximal monotone operator on H which satisfies (2.19)–(2.20).

Example 3. We consider the operator

$$A_1 : \ \mathbb{R} \longrightarrow \mathbb{R} \tag{2.27}$$

$$A_1(r) = \begin{cases} \alpha_1 r & \text{if } r < 0, \\ 0 & \text{if } 0 \le r \le 1, \\ \alpha_2 r & \text{if } r > 1, \end{cases} \tag{2.28}$$

where α_1 and α_2 are positive coefficients. We observe that A_1 is a maximal monotone operator on $\mathbb{R}$, whose graph consists of an horizontal line segment and two rays of slope α_1, α_2. Moreover, $0 \in A_1(0)$ and

$$|v| \le C(1 + |r|) \qquad \text{for all } r \in \mathbb{R}, \ v \in A_1(r), \tag{2.29}$$

with $C = \max(\alpha_1, \alpha_2)$. Then A_1 satisfies (2.19)–(2.20). We notice that A_1 corresponds to the graph which correlates the enthalpy to the temperature in the Stefan problem (see, e.g., [9, 11, 13]).

Example 4. We consider the operator

$$A_2 : \ H \longrightarrow H \tag{2.30}$$

$$A_2(v) = \alpha |v|^{q-1} v, \tag{2.31}$$

where $0 < q < 1$ and α is a function in $L^\infty(\Omega)$ with $\alpha(x) \ge 0$ for a.e. $x \in \Omega$. We observe that A_2 induces a (nonlocal) multivalued maximal monotone operator on H, with $0 \in A_2(0)$. Moreover, A_2 can be considered a weighted perturbation of the operator appearing in the porous media equation and in the fast diffusion equation (see, e.g., [10, 17, 22]).

2.3 Setting of the problem and results

Now, we state the problem under consideration. We look for a pair (η, φ) satisfying at least the regularity requirements

$$\eta, \ \varphi \ \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \tag{2.32}$$

and solving the problem (P):

$$\partial_t(\eta + (\ell - \alpha)\varphi) - k\Delta\eta + k\alpha\Delta\varphi + \zeta = f - k\Delta\eta^* \ \text{ a.e. in } Q, \tag{2.33}$$

$$\partial_t\varphi - \nu\Delta\varphi + \xi + \pi(\varphi) = \gamma(\eta - \alpha\varphi + \eta^*) \ \text{ a.e. in } Q, \tag{2.34}$$

$$\zeta(t) \in A(\eta(t)) \ \text{ for a.e. } t \in (0,T), \tag{2.35}$$

$$\xi \in \beta(\varphi) \ \text{ a.e. in } Q, \tag{2.36}$$

$$\partial_N\vartheta = 0, \qquad \partial_N\varphi = 0 \quad \text{on } \Sigma, \tag{2.37}$$

$$\eta(0) = \eta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{2.38}$$

We notice that the homogeneous Neumann boundary conditions for both η and φ required by (2.37) follow from (2.32), due to the definition of W (see (2.4)).

Theorem - (Existence) 2.3 *Assume* (2.5)–(2.9), (2.11)–(2.12) *and* (2.18)–(2.20). *Then problem* (P) *(see* (2.33)–(2.38)*) has at least a solution* (η,φ) *satisfying the regularity requirements* (2.32).

Theorem - (Uniqueness and continuous dependence) 2.4 *Assume* (2.5)–(2.9), (2.11)–(2.12) *and* (2.18)–(2.20). *If* $\alpha = \ell$, *the solution* (φ,η) *of problem* (P) *(see* (2.33)–(2.38)*) is unique. Moreover, if* f_i, η_i^*, η_{0_i}, φ_{0_i}, $i = 1,2$, *are given as in* (2.6)–(2.8) *and* (φ_i,η_i), $i = 1,2$, *are the corresponding solutions, then the estimate*

$$\|\eta_1 - \eta_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\varphi_1 - \varphi_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}$$
$$\leq C(\|f_1 - f_2\|_{L^2(Q)} + \|\eta_1^* - \eta_2^*\|_W + \|\eta_{0_1} - \eta_{0_2}\|_H + \|\varphi_{0_1} - \varphi_{0_2}\|_H) \tag{2.39}$$

holds true for some constant C *that depends only on* Ω, T *and the parameters* ℓ, α, k, ν, γ.

3 Proof of the existence theorem

This section is devoted to the proof of Theorem 2.3.

3.1 The approximating problem (P_ε)

Yosida regularization of A. We introduce the Yosida regularization of A. For $\varepsilon > 0$ we define

$$A_\varepsilon : H \longrightarrow H, \qquad A_\varepsilon = \frac{I - (I + \varepsilon A)^{-1}}{\varepsilon}, \tag{3.1}$$

where I denotes the identity operator. Note that A_ε is is Lipschitz-continuous (with Lipschitz constant $\frac{1}{\varepsilon}$), maximal monotone, and satisfies the following properties. Denoting by $J_\varepsilon = (I + \varepsilon A)^{-1}$ the resolvent operator, for all $\delta > 0$ we have that

$$A_\varepsilon\eta \in A(J_\varepsilon\eta), \tag{3.2}$$

$$(A_\varepsilon)_\delta = A_{\varepsilon+\delta}, \tag{3.3}$$

$$\|A_\varepsilon\eta\|_H \leq \|A^0\eta\|_H, \tag{3.4}$$

$$\lim_{\varepsilon\to 0}\|A_\varepsilon\eta\|_H = \|A^0\eta\|_H, \tag{3.5}$$

where $A^0\eta$ is the element of the image of A having minimal norm.

Remark 3.1 We point out a key property of A_ε, which is a consequence of (2.20). There exists a positive constant C, independent of ε, such that

$$\|A_\varepsilon\eta\|_H \le C(1+\|\eta\|_H) \qquad \text{for all } \eta\in H,\ v\in A\eta. \tag{3.6}$$

Indeed, notice that $0\in A(0)$ and $0\in I(0)$: consequently, for every $\varepsilon>0$, $0\in(I+\varepsilon A)(0)$. This fact implies that $J_\varepsilon(0)=0$. Moreover, since A is maximal monotone, J_ε is a contraction. Then, from (2.20) and (3.2), it follows that

$$\begin{aligned}\|A_\varepsilon\eta\|_H &\le C(\|J_\varepsilon\eta\|_H+1)\\ &\le C(\|J_\varepsilon\eta-J_\varepsilon 0\|_H+\|J_\varepsilon 0\|_H+1)\\ &\le C(\|\eta\|_H+1).\end{aligned}$$

Yosida regularization of β. We introduce the Yosida regularization of β. For $\varepsilon>0$ we define

$$\beta_\varepsilon:\mathbb{R}\longrightarrow\mathbb{R}, \qquad \beta_\varepsilon=\frac{I-(I+\varepsilon\beta)^{-1}}{\varepsilon}. \tag{3.7}$$

We remark that β_ε is Lipschitz continuous (with Lipschitz constant $\frac{1}{\varepsilon}$) and satisfies the following properties. Denoting by $R_\varepsilon=(I+\varepsilon\beta)^{-1}$ the resolvent operator, for all $\delta>0$ and for every $\varphi\in D(\beta)$ we have that

$$\beta_\varepsilon(\varphi)\in\beta(R_\varepsilon\varphi), \tag{3.8}$$

$$(\beta_\varepsilon)_\delta=\beta_{\varepsilon+\delta}, \tag{3.9}$$

$$|\beta_\varepsilon(\varphi)|\le|\beta^0(\varphi)|, \tag{3.10}$$

$$\lim_{\varepsilon\to0}\beta_\varepsilon(\varphi)=\beta^0(\varphi), \tag{3.11}$$

where $\beta^0(\varphi)$ is the element of the image of β having minimal modulus.

Regularization of $\widehat{\beta}$. We introduce the Moreau-Yosida regularization of $\widehat{\beta}$. For $\varepsilon>0$ we define

$$\widehat{\beta}_\varepsilon:\mathbb{R}\longrightarrow[0,+\infty], \qquad \widehat{\beta}_\varepsilon=\frac{I-(I+\varepsilon\widehat{\beta})^{-1}}{\varepsilon}. \tag{3.12}$$

We recall that

$$\widehat{\beta}_\varepsilon(\varphi)\le\widehat{\beta}(\varphi) \quad \text{for every } \varphi\in D(\widehat{\beta}). \tag{3.13}$$

We also observe that β_ε is the Fréchet derivative of $\widehat{\beta}_\varepsilon$. Then, for every $\varphi_1,\varphi_2\in D(\widehat{\beta})$, we have that

$$\widehat{\beta}_\varepsilon(\varphi_2)=\widehat{\beta}_\varepsilon(\varphi_1)+\int_{\varphi_1}^{\varphi_2}\beta_\varepsilon(s)\,ds. \tag{3.14}$$

Approximating problem (P_ε). We denote by f_ε a regularization of f constructed in such a way that

$$f_\varepsilon\in C^1([0,T];H) \text{ for all } \varepsilon>0, \qquad \lim_{\varepsilon\to0}\|f_\varepsilon-f\|_{L^2(0,T;H)}=0. \tag{3.15}$$

Then, we look for a pair $(\eta_\varepsilon, \varphi_\varepsilon)$ satisfying at least the regularity requirements

$$\eta_\varepsilon,\ \varphi_\varepsilon\ \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \tag{3.16}$$

and solving the approximating problem (P_ε):

$$\partial_t(\eta_\varepsilon + (\ell - \alpha)\varphi_\varepsilon) - k\Delta\eta_\varepsilon + k\alpha\Delta\varphi_\varepsilon + \zeta_\varepsilon = f_\varepsilon - k\Delta\eta^* \ \text{ a.e. in } Q, \tag{3.17}$$

$$\partial_t\varphi_\varepsilon - \nu\Delta\varphi_\varepsilon + \xi_\varepsilon + \pi(\varphi_\varepsilon) = \gamma(\eta_\varepsilon - \alpha\varphi_\varepsilon + \eta^*) \ \text{ a.e. in } Q, \tag{3.18}$$

$$\zeta_\varepsilon(t) = A_\varepsilon\eta_\varepsilon(t) \ \text{ for a.e. } t \in (0,T), \tag{3.19}$$

$$\xi_\varepsilon = \beta_\varepsilon(\varphi_\varepsilon) \ \text{ a.e. in } Q, \tag{3.20}$$

$$\partial_N\eta_\varepsilon = 0, \qquad \partial_N\varphi_\varepsilon = 0 \quad \text{on } \Sigma, \tag{3.21}$$

$$\eta_\varepsilon(0) = \eta_0, \qquad \varphi_\varepsilon(0) = \varphi_0 \quad \text{in } \Omega, \tag{3.22}$$

where A_ε and β_ε are the Yosida regularizations of A and β defined in (3.1) and (3.7), respectively. We notice that the homogeneous Neumann boundary conditions for both η_ε and φ_ε required by (3.21) follow from (3.16) due to the definition of W (see (2.4)).

Remark 3.2 We can define f_ε as the regularization of f obtained solving

$$\begin{cases} -\varepsilon f_\varepsilon''(t) + f_\varepsilon(t) = f(t), \quad t \in (0,T), \\ f_\varepsilon(0) = f_\varepsilon(T) = 0. \end{cases} \tag{3.23}$$

Thanks to Sobolev immersions and elliptic regularity, we obtain that

$$f_\varepsilon \in C^1([0,T];H) \quad \text{and} \quad \lim_{\varepsilon\to 0} \|f_\varepsilon - f\|_{L^2(0,T;H)} = 0. \tag{3.24}$$

3.2 The approximating problem $(P_{\varepsilon,n})$

Now, we apply the Faedo-Galerkin method to the approximating problem (P_ε). We consider the orthonormal basis $\{v_i\}_{\{i\geq 1\}}$ of V formed by the normalized eigenfunctions of the Laplace operator with homogeneous Neumann boundary condition, that is

$$\begin{cases} -\Delta v_i = \lambda_i v_i & \text{in } \Omega, \\ \partial_N v_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

Note that, owing to the regularity of Ω, $v_i \in W$ for all $i \geq 1$. Then, for any integer $n \geq 1$, we denote by V_n the n-dimentional subspace of V spanned by $\{v_1, \cdots, v_n\}$. Hence, $\{V_n\}$ is a sequence of finite dimensional subspaces such that $\bigcup_{n=1}^{+\infty} V_n$ is dense in V and $V_k \subseteq V_n$ for all $k \leq n$.

Definition of the approximating problem $(P_{\varepsilon,n})$**.** We first approximate the initial data η_0 and φ_0. We set

$$\eta_{0,n} = P_{V_n}\eta_0, \qquad \varphi_{0,n} = P_{V_n}\varphi_0. \tag{3.26}$$

We notice that

$$\lim_{n\to+\infty} \|\eta_{0,n} - \eta_0\|_V = 0 \quad \text{and} \quad \lim_{n\to+\infty} \|\varphi_{0,n} - \varphi_0\|_V = 0. \tag{3.27}$$

Note that the convergence provided by (3.27) assures that $\eta_{0,n}$ and $\varphi_{0,n}$ are bounded in V. Now, we introduce the new approximating problem $(P_{\varepsilon,n})$. We look for $t_n \in]0, T]$ and a pair $(\eta_{\varepsilon,n}, \varphi_{\varepsilon,n})$ (in the following we will write (η_n, φ_n) instead of $(\eta_{\varepsilon,n}, \varphi_{\varepsilon,n})$) such that

$$\eta_n \in C^1([0, t_n]; V_n), \qquad \varphi_n \in C^1([0, t_n]; V_n), \tag{3.28}$$

and, for every $v \in V_n$ and for every $t \in [0, t_n]$, solving the approximating problem $(P_{\varepsilon,n})$:

$$(\partial_t[\eta_n(t) + (\ell - \alpha)\varphi_n(t)] - k\Delta\eta_n(t) + k\alpha\Delta\varphi_n(t) + A_\varepsilon\eta_n(t), v)_H$$

$$= (f_\varepsilon(t) - k\Delta\eta^*, v)_H, \tag{3.29}$$

$$(\partial_t\varphi_n(t) - \nu\Delta\varphi_n(t) + \beta_\varepsilon(\varphi_n(t)) + \pi(\varphi_n(t)), v)_H$$

$$= (\gamma[\eta_n(t) - \alpha\varphi_n(t) + \eta^*], v)_H, \tag{3.30}$$

$$\partial_N\eta_n = 0, \qquad \partial_N\varphi_n = 0 \quad \text{on } \Sigma, \tag{3.31}$$

$$\eta_n(0) = \eta_{0,n}, \qquad \varphi_n(0) = \varphi_{0,n} \quad \text{in } \Omega. \tag{3.32}$$

This is a Cauchy problem for a system of nonlinear ordinary differential equations. In the next section we will show by a change of variable that this system admits a local solution (η_n, φ_n), which is of the form

$$\varphi_n(t) = \sum_{i=1}^{n} a_{in}(t)v_i, \tag{3.33}$$

$$\eta_n(t) = \sum_{i=1}^{n} b_{in}(t)v_i, \tag{3.34}$$

for some $a_{in} \in C^1([0, t_n])$ and $b_{in} \in C^1([0, t_n])$.

Remark 3.2 We point out that

$$\int_\Omega \widehat{\beta}_\varepsilon(\varphi_{0,n}) \le C + \frac{1}{2\varepsilon}\|\varphi_0 - \varphi_{0,n}\|_H(\|\varphi_0\|_H + \|\varphi_{0,n}\|_H), \tag{3.35}$$

where

$$C = \|\widehat{\beta}(\varphi_0)\|_{L^1(\Omega)}. \tag{3.36}$$

Indeed, for every $\varepsilon \in (0, 1]$, thanks to the property (3.13) of $\widehat{\beta}_\varepsilon$, we have that

$$0 \le \widehat{\beta}_\varepsilon(\varphi_0) \le \widehat{\beta}(\varphi_0). \tag{3.37}$$

Since $\widehat{\beta}(\varphi_0) \in L^1(\Omega)$ (see (2.9)), we obtain that

$$\int_\Omega \widehat{\beta}_\varepsilon(\varphi_0) \leq C, \tag{3.38}$$

where $C = \|\widehat{\beta}(\varphi_0)\|_{L^1(\Omega)}$. From (3.14), using the Lipschitz continuity of β_ε, we have

$$\begin{aligned} \widehat{\beta}_\varepsilon(\varphi_{0,n}) &\leq \widehat{\beta}_\varepsilon(\varphi_0) + \Big| \int_{\varphi_0}^{\varphi_{0,n}} \beta_\varepsilon(s)\, ds \Big| \\ &\leq \widehat{\beta}_\varepsilon(\varphi_0) + \frac{1}{\varepsilon} \int_{\varphi_0}^{\varphi_{0,n}} |s|\, ds \\ &\leq \widehat{\beta}_\varepsilon(\varphi_0) + \frac{1}{2\varepsilon} |\varphi_0 - \varphi_{0,n}|(|\varphi_0| + |\varphi_{0,n}|). \end{aligned} \tag{3.39}$$

By integrating (3.39) over Ω, we obtain that

$$\int_\Omega \widehat{\beta}_\varepsilon(\varphi_{0,n}) \leq Q_\varepsilon(n), \tag{3.40}$$

where

$$Q_\varepsilon(n) = C + \frac{1}{2\varepsilon}\|\varphi_0 - \varphi_{0,n}\|_H (\|\varphi_0\|_H + \|\varphi_{0,n}\|_H).$$

Remark 3.3 Thanks to (3.34) and the Lipschitz continuity of A_ε, we obtain that

$$A_\varepsilon(\eta_n) \in C^{0,1}([0, t_n]; H). \tag{3.41}$$

Indeed, $\|v_i\|_H \leq \|v_i\|_V = 1$, for all $i \in \mathbb{N}$. Then we choose $t, t' \in [0, t_n]$ and we have the following inequality:

$$\begin{aligned} \|A_\varepsilon(\eta_n(t)) - A_\varepsilon(\eta_n(t'))\|_H &= \|A_\varepsilon(\textstyle\sum_{i=1}^n b_{in}(t)v_i) - A_\varepsilon(\sum_{i=1}^n b_{in}(t')v_i)\|_H \\ &\leq \tfrac{1}{\varepsilon}\|\textstyle\sum_{i=1}^n (b_{in}(t) - b_{in}(t'))v_i\|_H \\ &\leq \tfrac{1}{\varepsilon}\textstyle\sum_{i=1}^n |b_{in}(t) - b_{in}(t')|\, \|v_i\|_H \\ &= \tfrac{1}{\varepsilon}\textstyle\sum_{i=1}^n |b_{in}(t) - b_{in}(t')|. \end{aligned}$$

Since b_{in} are continuous, we obtain (3.41).

Existence of a local solution for $(P_{\varepsilon,n})$. In order to prove the existence of a local solution (η_n, φ_n) for the approximating problem $(P_{\varepsilon,n})$, we make a change of variable. We set

$$\vartheta_n = \eta_n + (\ell - \alpha)\varphi_n, \qquad \vartheta_{0,n} = \eta_{0,n} + (\ell - \alpha)\varphi_{0,n}, \tag{3.42}$$

and we prove that there exists a local solution (ϑ_n, φ_n) of the problem

$$\begin{aligned} &(\partial_t\vartheta_n - k\Delta\vartheta_n + k\ell\Delta\varphi_n + A_\varepsilon(\vartheta_n - (\ell-\alpha)\varphi_n), v)_H = (f_\varepsilon - k\Delta\eta^*, v)_H, \\ &(\partial_t\varphi_n - \nu\Delta\varphi_n + \beta_\varepsilon(\varphi_n) + \pi(\varphi_n), v)_H = (\gamma[\vartheta_n - \ell\varphi_n + \eta^*], v)_H, \\ &\varphi_n(0) = \varphi_{0,n}, \qquad \vartheta_n(0) = \vartheta_{0,n}, \end{aligned} \tag{3.43}$$

whenever $v \in V_n$. Re-arranging the above system in explicit form, we have

$$\begin{aligned} &(\partial_t\vartheta_n, v)_H = (k\Delta\vartheta_n - k\ell\Delta\varphi_n - A_\varepsilon(\vartheta_n - (\ell-\alpha)\varphi_n) + f_\varepsilon - k\Delta\eta^*, v)_H, \\ &(\partial_t\varphi_n, v)_H = (\nu\Delta\varphi_n - \beta_\varepsilon(\varphi_n) - \pi(\varphi_n) + \gamma[\vartheta_n - \ell\varphi_n + \eta^*], v)_H, \\ &\varphi_n(0) = \varphi_{0,n}, \qquad \vartheta_n(0) = \vartheta_{0,n}, \end{aligned} \tag{3.44}$$

whenever $v \in V_n$. Thanks to the initial hypotheses (2.5)–(2.8), (2.11)–(2.12) and to the regularity of A_ε shown in (3.41), the right-hand side of (3.44) is a Lipschitz continuous function from $[0, t_n]$ to $\mathbb{R}^n$. Consequently, there exists a local solution for the approximating problem $(P_{\varepsilon,n})$.

3.3 Global a priori estimates

In this section we obtain four a priori estimates inferred from the main equations of the approximating problem $(P_{\varepsilon,n})$ (see (3.29)–(3.32)).

In the remainder of the paper we often use the Hölder inequality and to the elementary Young inequalities in performing our a priori estimates. In particular, let us recall that, for every $a, b > 0$, $\alpha \in (0,1)$ and $\delta > 0$, we have that

$$ab \leq \alpha a^{\frac{1}{\alpha}} + (1-\alpha) b^{\frac{1}{1-\alpha}}, \tag{3.45}$$

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2. \tag{3.46}$$

In the following the small-case symbol c stands for different constants which depend only on Ω, on the final time T, on the shape of the nonlinearities and on the constants and the norms of the functions involved in the assumptions of our statements.

First a priori estimate. We add $\nu\varphi_n$ to both sides of (3.30) and we test (3.29) by η_n and (3.30) by $\partial_t\varphi_n$, respectively. Then we sum up and integrate over Q_t, $t \in (0,T]$. We obtain that

$$\begin{aligned}
&\tfrac{1}{2}\textstyle\int_\Omega |\eta_n(t)|^2 + (\ell - \alpha)\int_{Q_t} \partial_t\varphi_n\eta_n + k\int_{Q_t} |\nabla\eta_n|^2 - k\alpha\int_{Q_t} \nabla\varphi_n \cdot \nabla\eta_n + \int_{Q_t} A_\varepsilon\eta_n\eta_n \\
&+ \textstyle\int_{Q_t} |\partial_t\varphi_n|^2 + \frac{\nu}{2}\int_\Omega |\varphi_n(t)|^2 + \frac{\nu}{2}\int_\Omega |\nabla\varphi_n(t)|^2 + \int_{Q_t} \partial_t\widehat{\beta}_\varepsilon(\varphi_n) \\
&= \tfrac{1}{2}\textstyle\int_\Omega |\eta_{0,n}|^2 + \frac{\nu}{2}\int_\Omega |\varphi_{0,n}|^2 + \frac{\nu}{2}\int_\Omega |\nabla\varphi_{0,n}|^2 + \int_{Q_t} (f_\varepsilon - k\Delta\eta^*)\eta_n \\
&+ \textstyle\int_{Q_t} [\gamma\eta_n + (\nu - \alpha\gamma)\varphi_n + \gamma\eta^*]\partial_t\varphi_n - \int_{Q_t} \pi(\varphi_n)\partial_t\varphi_n.
\end{aligned} \tag{3.47}$$

To estimate the last integral on the right-hand side of (3.47), we observe that π is a Lipschitz continuous function with Lipschitz constant C_π. Consequently we have that

$$\begin{aligned}
|\pi(\varphi_n)| &\leq |\pi(\varphi_n) - \pi(0)| + |\pi(0)| \\
&\leq C_\pi|\varphi_n| + |\pi(0)| \\
&\leq C_1(|\varphi_n| + 1),
\end{aligned} \tag{3.48}$$

where $C_1 = \max\{C_\pi; |\pi(0)|\}$. Due to (3.46) and (3.48), we obtain that

$$\begin{aligned}
-\textstyle\int_{Q_t} \pi(\varphi_n)\partial_t\varphi_n &\leq \textstyle\int_{Q_t} |\pi(\varphi_n)\partial_t\varphi_n| \\
&\leq \textstyle\int_{Q_t} C_1(|\varphi_n| + 1)|\partial_t\varphi_n| \\
&\leq \tfrac{1}{8}\textstyle\int_{Q_t} |\partial_t\varphi_n|^2 + 2C_1^2\int_{Q_t} (|\varphi_n| + 1)^2 \\
&= \tfrac{1}{8}\textstyle\int_{Q_t} |\partial_t\varphi_n|^2 + 4C_1^2\int_{Q_t} |\varphi_n|^2 + c.
\end{aligned} \tag{3.49}$$

Now, we recall that A_ε is a maximal monotone operator and $A_\varepsilon(0) = 0$. Hence we have that

$$\int_{Q_t} A_\varepsilon\eta_n\eta_n \geq 0. \tag{3.50}$$

Using (3.49)–(3.50), from (3.47) we obtain that

$$\frac{1}{2}\int_\Omega |\eta_n(t)|^2 + k\int_{Q_t}|\nabla\eta_n|^2 + \int_{Q_t}|\partial_t\varphi_n|^2 + \frac{\nu}{2}\int_\Omega|\varphi_n(t)|^2 + \frac{\nu}{2}\int_\Omega|\nabla\varphi_n(t)|^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t))$$
$$\leq c + \frac{1}{2}\int_\Omega|\eta_{0,n}|^2 + \frac{\nu}{2}\int_\Omega|\varphi_{0,n}|^2 + \frac{\nu}{2}\int_\Omega|\nabla\varphi_{0,n}|^2 + \int_\Omega\widehat{\beta}_\varepsilon(\varphi_{0,n})$$
$$-(\ell-\alpha)\int_{Q_t}\partial_t\varphi_n\eta_n + k\alpha\int_{Q_t}\nabla\varphi_n\cdot\nabla\eta_n + \frac{1}{8}\int_{Q_t}|\partial_t\varphi_n|^2 + 4C_1^2\int_{Q_t}|\varphi_n|^2$$
$$+\int_{Q_t}(f_\varepsilon - k\Delta\eta^*)\eta_n + \int_{Q_t}[\gamma\eta_n + (\nu-\alpha\gamma)\varphi_n + \gamma\eta^*]\partial_t\varphi_n. \tag{3.51}$$

We notice that the convergence provided by (3.27) assures that $\eta_{0,n}$ and $\varphi_{0,n}$ are bounded in V. Consequently, thanks to (3.40), the first four integrals on the right-hand side of (3.51) are estimated as follows:

$$\frac{1}{2}\int_\Omega|\eta_{0,n}|^2 + \frac{\nu}{2}\int_\Omega|\varphi_{0,n}|^2 + \frac{\nu}{2}\int_\Omega|\nabla\varphi_{0,n}|^2 + \int_\Omega\widehat{\beta}_\varepsilon(\varphi_{0,n}) \leq c + Q_\varepsilon(n). \tag{3.52}$$

We also notice that

$$\begin{aligned} k\alpha\int_{Q_t}\nabla\varphi_n\cdot\nabla\eta_n &= \frac{k}{2}\Big(2\alpha\int_{Q_t}\nabla\varphi_n\cdot\nabla\eta_n\Big) \\ &\leq \frac{k}{2}\int_{Q_t}|\nabla\eta_n|^2 + \frac{k\alpha^2}{2}\int_{Q_t}|\nabla\varphi_n|^2 \\ &= \frac{k}{2}\int_{Q_t}|\nabla\eta_n|^2 + \frac{k\alpha^2}{\nu}\int_{Q_t}\frac{\nu}{2}|\nabla\varphi_n|^2. \end{aligned} \tag{3.53}$$

We re-arrange the right-hand side of (3.51) using (3.46), (3.52) and (3.53). Then we have that

$$\frac{1}{2}\|\eta_n(t)\|_H^2 + k\int_{Q_t}|\nabla\eta_n|^2 + \int_{Q_t}|\partial_t\varphi_n|^2 + \frac{\nu}{2}\|\varphi_n(t)\|_V^2 + \int_\Omega\widehat{\beta}_\varepsilon(\varphi_n(t))$$
$$\leq c + Q_\varepsilon(n) + 2(\ell-\alpha)^2\int_{Q_t}|\eta_n|^2 + \frac{1}{8}\int_{Q_t}|\partial_t\varphi_n|^2 + \frac{k}{2}\int_{Q_t}|\nabla\eta_n|^2 + \frac{k\alpha^2}{\nu}\int_{Q_t}\frac{\nu}{2}|\nabla\varphi_n|^2$$
$$+\frac{1}{8}\int_{Q_t}|\partial_t\varphi_n|^2 + 4C_1^2\int_{Q_t}|\varphi_n|^2 + 2\int_{Q_t}|f_\varepsilon - k\Delta\eta^*|^2 + \frac{1}{8}\int_{Q_t}|\eta_n|^2$$
$$+2\int_{Q_t}|\gamma\eta_n + (\nu-\alpha\gamma)\varphi_n + \gamma\eta^*|^2 + \frac{1}{8}\int_{Q_t}|\partial_t\varphi_n|^2. \tag{3.54}$$

According to (3.15), f_ε is bounded in $L^2(0,T;H)$ uniformly with respect to ε. Consequently, due to (2.6)–(2.7), the seventh integral on the right-hand side of (3.54) is under control and similarly the third addendum in the ninth integral on the right-hand side. Then we infer that

$$\frac{1}{2}\|\eta_n(t)\|_H^2 + k\int_{Q_t}|\nabla\eta_n|^2 + \int_{Q_t}|\partial_t\varphi_n|^2 + \frac{\nu}{2}\|\varphi_n(t)\|_V^2 + \int_\Omega\widehat{\beta}_\varepsilon(\varphi_n(t))$$
$$\leq c + Q_\varepsilon(n) + \Big[2(\ell-\alpha)^2 + \frac{1}{8}\Big]\int_{Q_t}|\eta_n|^2 + \frac{1}{2}\int_{Q_t}|\partial_t\varphi_n|^2 + \frac{k\alpha^2}{\nu}\int_{Q_t}\frac{\nu}{2}|\nabla\varphi_n|^2$$

$$+4C_1^2 \int_{Q_t} |\varphi_n|^2 + 8\gamma^2 \int_{Q_t} |\eta_n|^2 + 8(\nu-\alpha\gamma)^2 \int_{Q_t} |\varphi_n|^2. \tag{3.55}$$

Now, we recollect the constants in (3.55) and obtain that

$$\frac{1}{2}\|\eta_n(t)\|_H^2 + k\int_{Q_t} |\nabla\eta_n|^2 + \int_{Q_t} |\partial_t\varphi_n|^2 + \frac{\nu}{2}\|\varphi_n(t)\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t))$$

$$\leq c + Q_\varepsilon(n) + C_2\frac{1}{2}\int_0^t \|\eta_n(s)\|_H^2\,ds + C_3\frac{\nu}{2}\int_0^t \|\nabla\varphi_n(s)\|_H^2\,ds + C_4\frac{\nu}{2}\int_0^t \|\varphi_n(s)\|_H^2\,ds, \tag{3.56}$$

where

$$C_2 = 2[2(\ell-\alpha)^2 + \frac{1}{8} + 8\gamma^2], \quad C_3 = \frac{k\alpha^2}{\nu}, \quad C_4 = \frac{2[4C_1^2 + 8(\nu-\alpha\gamma)^2]}{\nu}.$$

Consequently, from (3.56) we have that

$$\frac{1}{2}\|\eta_n(t)\|_H^2 + k\int_{Q_t} |\nabla\eta_n|^2 + \int_{Q_t} |\partial_t\varphi_n|^2 + \frac{\nu}{2}\|\varphi_n(t)\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t))$$

$$\leq c + Q_\varepsilon(n) + C_5\left(\frac{1}{2}\int_0^t \|\eta_n(s)\|_H^2\,ds + \frac{\nu}{2}\int_0^t \|\varphi_n(s)\|_V^2\,ds\right), \tag{3.57}$$

where

$$C_5 = \max(C_2,\ C_3,\ C_4).$$

Then, from (3.57) we conclude that

$$\frac{1}{2}\|\eta_n(t)\|_H^2 + k\int_{Q_t} |\nabla\eta_n|^2 + \int_{Q_t} |\partial_t\varphi_n|^2 + \frac{\nu}{2}\|\varphi_n(t)\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t))$$

$$\leq c_\varepsilon\left(1 + \frac{1}{2}\int_0^t \|\eta_n(s)\|_H^2\,ds + \frac{\nu}{2}\int_0^t \|\varphi_n(s)\|_V^2\,ds\right). \tag{3.58}$$

Now, we apply the Gronwall lemma to (3.58) and infer that

$$\frac{1}{2}\|\eta_n(t)\|_H^2 + k\int_{Q_t} |\nabla\eta_n|^2 + \int_{Q_t} |\partial_t\varphi_n|^2 + \frac{\nu}{2}\|\varphi_n(t)\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t)) \leq c_\varepsilon. \tag{3.59}$$

As (3.59) holds true for any $t \in [0, t_n)$, we conclude that

$$\|\varphi_n\|_{H^1(0,t_n;H)\cap L^\infty(0,t_n;V)} \leq c_\varepsilon, \tag{3.60}$$

$$\|\eta_n\|_{L^\infty(0,t_n;H)\cap L^2(0,t_n;V)} \leq c_\varepsilon, \tag{3.61}$$

$$\|\widehat{\beta}_\varepsilon(\varphi_n)\|_{L^\infty(0,t_n;L^1(\Omega))} \leq c_\varepsilon. \tag{3.62}$$

Second a priori estimate. First of all, we notice that $\pi(\varphi_n)$ is bounded in $L^2(0,t_n;H)$ owing to (2.12) and (3.60). Thanks to (3.60)–(3.62), we can rewrite (3.30) as

$$(-\nu\Delta\varphi_n + \beta_\varepsilon(\varphi_n), v)_H = (g_1, v)_H, \qquad \text{for all } v \in V_n, \tag{3.63}$$

with $\|g_1\|_{L^2(0,t_n;H)} \leq c_\varepsilon$. The choice of the basis v_i as in (3.25) allows us to test (3.63) by $-\Delta\varphi_n$. Integrating over $(0,t)$, we obtain that

$$\nu\int_{Q_t}|\Delta\varphi_n|^2 + \int_{Q_t}\nabla\varphi_n\cdot\nabla\beta_\varepsilon(\varphi_n) = -\int_{Q_t} g_1\Delta\varphi_n. \tag{3.64}$$

Using inequalities (3.45)–(3.46), from (3.64) we have that

$$\frac{\nu}{2}\int_{Q_t}|\Delta\varphi_n|^2 + \int_{Q_t}\beta'_\varepsilon(\varphi_n)|\nabla\varphi_n|^2 \leq \frac{1}{2\nu}\int_{Q_t}|g_1|^2. \tag{3.65}$$

Due to (3.60) and the monotonicity of β_ε, from (3.65) we obtain that

$$\|\Delta\varphi_n\|_{L^2(0,t;H)} \leq c_\varepsilon. \tag{3.66}$$

We observe that (3.66) holds true for any $t \in [0,t_n)$. Then, using elliptic regularity, from (3.60) and (3.66) we infer that

$$\|\varphi_n\|_{L^2(0,t_n;W)} \leq c_\varepsilon. \tag{3.67}$$

Third a priori estimate. Thanks to the previous a priori estimates, from (3.29) it follows that

$$(\partial_t\eta_n - k\Delta\eta_n + A_\varepsilon\eta_n, v)_H = (g_2, v)_H \qquad \text{for all } v \in V_n, \tag{3.68}$$

with $\|g_2\|_{L^2(0,t_n;H)} \leq c_\varepsilon$. We test (3.68) by $\partial_t\eta_n$ and integrate over $(0,t)$; we obtain that

$$\int_{Q_t}|\partial_t\eta_n|^2 + \frac{k}{2}\int_\Omega|\nabla\eta_n(t)|^2 + \int_{Q_t}A_\varepsilon\eta_n\partial_t\eta_n = \frac{k}{2}\int_\Omega|\nabla\eta_{0,n}|^2 + \int_{Q_t}g_2\partial_t\eta_n. \tag{3.69}$$

Then, using the property (3.6) of A_ε and inequalities (3.45)–(3.46), from (3.69) we infer that

$$\int_{Q_t}|\partial_t\eta_n|^2 + \frac{k}{2}\int_\Omega|\nabla\eta_n(t)|$$

$$\begin{aligned}
&\leq \frac{k}{2}\int_\Omega|\nabla\eta_{0,n}|^2 + \int_{Q_t}|A_\varepsilon\eta_n\partial_t\eta_n| + 2\int_{Q_t}|g_2|^2 + \frac{1}{8}\int_{Q_t}|\partial_t\eta_n|^2 \\
&\leq \frac{k}{2}\int_\Omega|\nabla\eta_{0,n}|^2 + 2\int_{Q_t}|A_\varepsilon\eta_n|^2 + \frac{1}{8}\int_{Q_t}|\partial_t\eta_n|^2 + 2\int_{Q_t}|g_2|^2 + \frac{1}{8}\int_{Q_t}|\partial_t\eta_n|^2 \\
&= \frac{k}{2}\int_\Omega|\nabla\eta_{0,n}|^2 + 2\int_0^t\|A_\varepsilon\eta_n(s)\|_H^2\,ds + \frac{1}{4}\int_{Q_t}|\partial_t\eta_n|^2 + 2\int_{Q_t}|g_2|^2 \\
&\leq \frac{k}{2}\int_\Omega|\nabla\eta_{0,n}|^2 + 2\int_0^t[C(\|\eta_n(s)\|_H + 1)]^2\,ds + \frac{1}{4}\int_{Q_t}|\partial_t\eta_n|^2 + 2\int_{Q_t}|g_2|^2 \\
&\leq c + \frac{k}{2}\int_\Omega|\nabla\eta_{0,n}|^2 + 4C^2\int_0^t\|\eta_n(s)\|_H^2\,ds + \frac{1}{2}\int_{Q_t}|\partial_t\eta_n|^2 + 2\int_{Q_t}|g_2|^2.
\end{aligned} \tag{3.70}$$

Due to (2.8), the first integral on the right-hand side of (3.70) is under control. Then, from (3.70) we infer that

$$\frac{1}{2}\int_{Q_t} |\partial_t \eta_n|^2 + \frac{k}{2}\int_\Omega |\nabla \eta_n(t)| \leq c + 4C^2 \int_0^t \|\eta_n(s)\|_H^2 \, ds + 2\int_{Q_t} |g_2|^2. \tag{3.71}$$

We observe that (3.71) holds true for any $t \in [0, t_n)$, Then, due to the previous estimates (3.60)–(3.61), we conclude that

$$\|\eta_n\|_{H^1(0,t_n;H)\cap L^\infty(0,t_n;V)} \leq c_\varepsilon. \tag{3.72}$$

Fourth a priori estimate. Due to the previous estimates (3.60)–(3.62), (3.67) and (3.72), by comparison in (3.68), we infer that

$$\|\Delta \eta_n\|_{L^2(0,t_n;H)} \leq c_\varepsilon. \tag{3.73}$$

Consequently, we conclude that

$$\|\eta_n\|_{L^2(0,t_n;W)} \leq c_\varepsilon. \tag{3.74}$$

Summary of the a priori estimates. Since the constants appearing in the a priori estimates are all independent of t_n, the local solution can be extended to a solution defined on the whole interval $[0, T]$, i.e., we can assume $t_n = T$ for any n. Hence, due to (3.60)–(3.62), (3.67), (3.72) and (3.74), we conclude that

$$\|\varphi_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq c_\varepsilon, \tag{3.75}$$
$$\|\eta_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq c_\varepsilon. \tag{3.76}$$

3.4 Passage to the limit as $n \to +\infty$

Now, we let $n \to +\infty$ and show that the limit of some subsequences of solutions for $(P_{\varepsilon,n})$ (see (3.29)–(3.32)) yields a solution of (P_ε) (see (3.17)–(3.22)). Estimates (3.75)–(3.76) for φ_n and η_n and the well-known weak or weak* compactness results ensure the existence of a pair $(\varphi_\varepsilon, \eta_\varepsilon)$ such that, at least for a subsequence,

$$\varphi_n \rightharpoonup \varphi_\varepsilon \quad \text{in} \quad H^1(0,T;H) \cap L^2(0,T;W), \tag{3.77}$$
$$\varphi_n \rightharpoonup^* \varphi_\varepsilon \quad \text{in} \quad L^\infty(0,T;V), \tag{3.78}$$
$$\eta_n \rightharpoonup \eta_\varepsilon \quad \text{in} \quad H^1(0,T;H) \cap L^2(0,T;W), \tag{3.79}$$
$$\eta_n \rightharpoonup^* \eta_\varepsilon \quad \text{in} \quad L^\infty(0,T;V), \tag{3.80}$$

as $n \to +\infty$. We notice that W, V, H are Banach spaces and $W \subset V \subset H$ with dense and compact embeddings. Then, we are under the assumptions of [21, Prop. 4, Sec. 8] and this fact implies the following strong convergences:

$$\varphi_n \to \varphi_\varepsilon \quad \text{in} \quad C^0([0,T];H) \cap L^2(0,T;V), \tag{3.81}$$
$$\eta_n \to \eta_\varepsilon \quad \text{in} \quad C^0([0,T];H) \cap L^2(0,T;V), \tag{3.82}$$

as $n \to +\infty$. Since π, A_ε and β_ε are Lipschitz continuous, we infer that

$$\begin{aligned} &|\pi(\varphi_n) - \pi(\varphi_\varepsilon)| \le C_\pi |\varphi_n - \varphi_\varepsilon| && \text{a.e. in } Q, \\ &\|A_\varepsilon \eta_n - A_\varepsilon \eta_\varepsilon\|_H \le \tfrac{1}{\varepsilon} \|\eta_n - \eta_\varepsilon\|_H && \text{a.e. in } [0,T], \\ &|\beta_\varepsilon(\varphi_n) - \beta_\varepsilon(\varphi_\varepsilon)| \le \tfrac{1}{\varepsilon} |\varphi_n - \varphi_\varepsilon| && \text{a.e. in } Q. \end{aligned} \tag{3.83}$$

Due to (3.83), we conclude that

$$\pi(\varphi_n) \to \pi(\varphi_\varepsilon) \quad \text{in} \quad C^0([0,T];H), \tag{3.84}$$

$$A_\varepsilon \eta_n \to A_\varepsilon \eta_\varepsilon \quad \text{in} \quad C^0([0,T];H), \tag{3.85}$$

$$\beta_\varepsilon(\varphi_n) \to \beta_\varepsilon(\varphi_\varepsilon) \quad \text{in} \quad C^0([0,T];H), \tag{3.86}$$

as $n \to +\infty$. Now, we fix $k \le n$ and we observe that, for every $v \in V_k$ and for every $t \in [0,T]$, the solution (η_n, φ_n) of problem $(P_{\varepsilon,n})$ satisfies

$$\begin{aligned} &(\partial_t[\eta_n(t) + (\ell - \alpha)\varphi_n(t)] - k\Delta\eta_n(t) + k\alpha\Delta\varphi_n(t) + A_\varepsilon\eta_n(t), v)_H \\ &\qquad = (f_\varepsilon(t) - k\Delta\eta^*, v)_H, \end{aligned} \tag{3.87}$$

$$\begin{aligned} &(\partial_t\varphi_n(t) - \nu\Delta\varphi_n(t) + \beta_\varepsilon(\varphi_n(t)) + \pi(\varphi_n(t)), v)_H \\ &\qquad = (\gamma[\eta_n(t) - \alpha\varphi_n(t) + \eta^*], v)_H. \end{aligned} \tag{3.88}$$

If k is fixed and $n \to +\infty$, we have the convergence of every term of (3.87)–(3.88) to the corresponding one with η_ε, φ_ε whenever $v \in V_k$, i.e.,

$$\begin{aligned} &(\partial_t[\eta_\varepsilon(t) + (\ell - \alpha)\varphi_\varepsilon(t)] - k\Delta\eta_\varepsilon(t) + k\alpha\Delta\varphi_\varepsilon(t) + A_\varepsilon\eta_\varepsilon(t), v)_H \\ &\qquad = (f_\varepsilon(t) - k\Delta\eta^*, v)_H, \end{aligned} \tag{3.89}$$

$$\begin{aligned} &(\partial_t\varphi_\varepsilon(t) - \nu\Delta\varphi_\varepsilon(t) + \beta_\varepsilon(\varphi_\varepsilon(t)) + \pi(\varphi_\varepsilon(t)), v)_H \\ &\qquad = (\gamma[\eta_\varepsilon(t) - \alpha\varphi_\varepsilon(t) + \eta^*], v)_H. \end{aligned} \tag{3.90}$$

As k is arbitrary, the limit equalities hold true for every $v \in \bigcup_{k=1}^{\infty} V_k$, which is dense in V. Then the limit equalities actually hold for every $v \in V$, i.e.,

$$\partial_t(\eta_\varepsilon + (\ell - \alpha)\varphi_\varepsilon) - k\Delta\eta_\varepsilon + k\alpha\Delta\varphi_\varepsilon + A_\varepsilon\eta_\varepsilon = f_\varepsilon - k\Delta\eta^* \quad \text{a.e. in } Q, \tag{3.91}$$

$$\partial_t\varphi_\varepsilon - \nu\Delta\varphi_\varepsilon + \beta_\varepsilon(\varphi_\varepsilon) + \pi(\varphi_\varepsilon) = \gamma(\eta_\varepsilon - \alpha\varphi_\varepsilon + \eta^*) \quad \text{a.e. in } Q. \tag{3.92}$$

Now, we prove the convergence of the initial data. We recall that

$$\eta_{0,n} = P_{V_n}\eta_0, \qquad\qquad \varphi_{0,n} = P_{V_n}\varphi_0. \tag{3.93}$$

If ε is fixed, then

$$\lim_{n\to+\infty} \eta_{0,n} = \eta_0 \quad \text{in } V, \tag{3.94}$$

$$\lim_{n\to+\infty} \varphi_{0,n} = \varphi_0 \quad \text{in } V, \tag{3.95}$$

and then also in H. These observations and (3.81)–(3.82) show that the weak limit of some subsequences of solutions for $(P_{\varepsilon,n})$ (see (3.29)–(3.32)) yields a solution for (P_ε) (see (3.17)–(3.22)). We also notice that taking the limit as $n \to +\infty$ in (3.40) entails that $Q_\varepsilon(n) \to C$, with

$$\int_\Omega \widehat{\beta}_\varepsilon(\varphi_0) \le C. \tag{3.96}$$

Then, after the first passage to the limit, we conclude that estimates (3.75)–(3.76) still hold for the limiting functions with constants independent of ε, i.e.,

$$\|\varphi_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le c, \tag{3.97}$$

$$\|\eta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le c. \tag{3.98}$$

3.5 Passage to the limit as $\varepsilon \searrow 0$

Now, we let $\varepsilon \searrow 0$ and show that the limit of some subsequences of solutions for (P_ε) (see (3.17)–(3.22)) tends to a solution of the initial problem (P) (see (2.33)–(2.38)). First of all, due to (3.77)–(3.82), (3.86) and (3.96), we have that the constants in (3.97)–(3.98) do not depend on ε. Moreover, thanks to (3.97)–(3.98), by comparison in (3.92), we infer that

$$\|\beta_\varepsilon(\varphi_\varepsilon)\|_{L^2(Q)} \leq c. \tag{3.99}$$

The well-known weak or weak* compactness results and the useful theorem [21, Prop. 4, Sec. 8] ensure the existence of a pair (φ, η) such that, at least for a subsequence,

$$\varphi_\varepsilon \rightharpoonup^* \varphi \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{3.100}$$

$$\eta_\varepsilon \rightharpoonup^* \eta \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{3.101}$$

$$\varphi_\varepsilon \to \varphi \qquad \text{in } C^0([0,T];H) \cap L^2(0,T;V), \tag{3.102}$$

$$\eta_\varepsilon \to \eta \qquad \text{in } C^0([0,T];H) \cap L^2(0,T;V), \tag{3.103}$$

as $\varepsilon \searrow 0$. Now, we observe that (3.102) implies that

$$\varphi_\varepsilon \to \varphi \qquad \text{in } L^2(0,T;H) \equiv L^2(Q) \tag{3.104}$$

as $\varepsilon \searrow 0$. We set $\xi_\varepsilon = \beta_\varepsilon(\varphi_\varepsilon)$ and remark that

$$\|\xi_\varepsilon\|_{L^2(Q)} = \|\beta_\varepsilon(\varphi_\varepsilon)\|_{L^2(Q)} \leq c. \tag{3.105}$$

Thus, we may suppose that, as $\varepsilon \searrow 0$, at least for a subsequence,

$$\xi_\varepsilon \rightharpoonup \xi \qquad \text{in } L^2(Q), \tag{3.106}$$

for some $\xi \in L^2(Q)$. Now, we introduce the operator $\mathcal{B}_\varepsilon$ induced by β_ε on $L^2(Q)$ in the following way:

$$\mathcal{B}_\varepsilon : L^2(Q) \longrightarrow L^2(Q) \tag{3.107}$$

$$\xi_\varepsilon \in \mathcal{B}_\varepsilon(\varphi_\varepsilon) \Longleftrightarrow \xi_\varepsilon(x,t) \in \beta_\varepsilon(\varphi_\varepsilon(x,t)) \quad \text{for a.e. } (x,t) \in Q. \tag{3.108}$$

Due to (3.104) and (3.106), we have that

$$\begin{cases} \mathcal{B}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \xi & \text{in } L^2(Q), \\ \varphi_\varepsilon \to \varphi & \text{in } L^2(Q), \end{cases} \tag{3.109}$$

$$\limsup_{\varepsilon \searrow 0} \int_Q \xi_\varepsilon \varphi_\varepsilon = \int_Q \xi \varphi. \tag{3.110}$$

Thanks to (3.109)–(3.110) and to the useful results proved in [1, Prop. 2.2, p. 38], we conclude that

$$\xi \in \mathcal{B}(\varphi) \quad \text{in } L^2(Q), \tag{3.111}$$

where $\mathcal{B}$ is defined by (2.13)–(2.14). This is equivalent to say that

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q. \tag{3.112}$$

Moreover, we pass to the limit in A_ε by repeating the previous arguments and conclude that

$$\zeta \in \mathcal{A}(\eta) \quad \text{in } L^2(0,T;H), \tag{3.113}$$

with obvious definition for $\mathcal{A}$ (see (2.21)–(2.22)), and this is equivalent to say that

$$\zeta \in A(\eta) \quad \text{a.e. in } [0,T]. \tag{3.114}$$

Conclusion of the proof. Thanks to the previous steps, we conclude that, as $\varepsilon \searrow 0$, the limit of some subsequences of solutions $(\eta_\varepsilon, \varphi_\varepsilon)$ to (P_ε) (see (3.17)–(3.22)) yields a solution (η, φ) of the initial boundary value problem (P), i.e.,

$$\partial_t(\eta + (\ell - \alpha)\varphi) - k\Delta\eta + k\alpha\Delta\varphi + \zeta = f - k\Delta\eta^* \quad \text{a.e. in } Q, \tag{3.115}$$

$$\partial_t\varphi - \nu\Delta\varphi + \xi + \pi(\varphi) = \gamma(\eta - \alpha\varphi + \eta^*) \quad \text{a.e. in } Q, \tag{3.116}$$

$$\zeta(t) \in A(\eta(t)) \quad \text{for a.e. } t \in (0, T), \tag{3.117}$$

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q, \tag{3.118}$$

$$\partial_N\eta = 0, \qquad \partial_N\varphi = 0 \quad \text{on } \Sigma, \tag{3.119}$$

$$\eta(0) = \eta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{3.120}$$

We notice that the homogeneous Neumann boundary conditions for both η and φ follow from (2.32), due to the definition of W (see (2.4)).

4 Proof of the continuous dependence theorem

This section is devoted to the proof of Theorem 2.4.

Assume $\alpha = \ell$. If f_i, η_i^*, η_{0_i}, φ_{0_i}, $i = 1, 2$, are given as in (2.6)–(2.8) and (φ_i, η_i), $i = 1, 2$, are the corresponding solutions, we can write problem (2.33)–(2.38) for both (φ_i, η_i), $i = 1, 2$, obtaining

$$\partial_t\eta_i - k\Delta\eta_i + k\ell\Delta\varphi_i + \zeta_i = f_i - k\Delta\eta_i^* \quad \text{a.e. in } Q, \tag{4.1}$$

$$\partial_t\varphi_i - \nu\Delta\varphi_i + \xi_i + \pi(\varphi_i) = \gamma(\eta_i - \ell\varphi_i + \eta_i^*) \quad \text{a.e. in } Q, \tag{4.2}$$

$$\zeta_i(t) \in A(\eta_i(t)) \quad \text{for a.e. } t \in (0, T), \tag{4.3}$$

$$\xi_i \in \beta(\varphi_i) \quad \text{a.e. in } Q, \tag{4.4}$$

$$\partial_N\eta_i = 0, \qquad \partial_N\varphi_i = 0 \quad \text{on } \Sigma, \tag{4.5}$$

$$\eta_i(0) = \eta_{0_i}, \qquad \varphi_i(0) = \varphi_{0_i}. \tag{4.6}$$

First of all, we set

$$\varphi = \varphi_1 - \varphi_2, \quad \eta = \eta_1 - \eta_2, \tag{4.7}$$
$$f = f_1 - f_2, \quad \eta^* = \eta_1^* - \eta_2^*, \tag{4.8}$$
$$\varphi_0 = \varphi_{0_1} - \varphi_{0_2}, \quad \eta_0 = \eta_{0_1} - \eta_{0_2}. \tag{4.9}$$

We write (4.1) for both (φ_1, η_1) and (φ_2, η_2) and we take the difference. We obtain that

$$\partial_t\eta - k\Delta\eta + k\ell\Delta\varphi + \zeta_1 - \zeta_2 = f - k\Delta\eta^*. \tag{4.10}$$

We write (4.2) for both (φ_1, η_1) and (φ_2, η_2) and we take the difference. We obtain that

$$\partial_t\varphi - \nu\Delta\varphi + \xi_1 - \xi_2 + \pi(\varphi_1) - \pi(\varphi_2) = \gamma(\eta - \ell\varphi + \eta^*). \tag{4.11}$$

We multiply (4.10) by η and (4.11) by $\frac{kl^2}{\nu}\varphi$. Then we sum up and integrate over Q_t, $t \in (0,T]$. We have that

$$\frac{1}{2}\int_\Omega |\eta(t)|^2 + \frac{k\ell^2}{2\nu}\int_\Omega |\varphi(t)|^2 + k\int_{Q_t}(|\nabla\eta|^2 - \ell\nabla\varphi\nabla\eta + \ell^2|\nabla\varphi|^2)$$

$$+\int_{Q_t}(\zeta_1-\zeta_2)(\eta_1-\eta_2) + \frac{k\ell^2}{\nu}\int_{Q_t}(\xi_1-\xi_2)(\varphi_1-\varphi_2)$$

$$= \frac{1}{2}\|\eta_0\|_H^2 + \frac{k\ell^2}{2\nu}\|\varphi_0\|_H^2 - \frac{k\ell^2}{\nu}\int_{Q_t}[\pi(\varphi_1)-\pi(\varphi_2)](\varphi_1-\varphi_2)$$

$$+\int_{Q_t}(f-k\Delta\eta^*)\eta + \frac{\gamma k\ell^2}{\nu}\int_{Q_t}\eta\varphi - \frac{\gamma k\ell^3}{\nu}\int_{Q_t}|\varphi|^2 + \frac{\gamma k\ell^2}{\nu}\int_{Q_t}\eta^*\varphi. \tag{4.12}$$

Since A and β are maximal monotone, we have that

$$\int_{Q_t}(\zeta_1-\zeta_2)(\eta_1-\eta_2) \geq 0, \tag{4.13}$$

$$\int_{Q_t}(\xi_1-\xi_2)(\varphi_1-\varphi_2) \geq 0. \tag{4.14}$$

Moreover, thanks to the Lipschitz continuity of π, we infer that

$$\begin{aligned} -\frac{k\ell^2}{\nu}\int_{Q_t}[\pi(\varphi_1)-\pi(\varphi_2)](\varphi_1-\varphi_2) &\leq \frac{k\ell^2}{\nu}\int_{Q_t}|\pi(\varphi_1)-\pi(\varphi_2)||\varphi_1-\varphi_2| \\ &\leq \frac{k\ell^2C_\pi}{\nu}\int_{Q_t}|\varphi|^2. \end{aligned} \tag{4.15}$$

We notice that the integral involving the gradients in (4.12) is estimated from below in this way:

$$\int_{Q_t}(|\nabla\eta|^2 - \ell\nabla\varphi\nabla\eta + \ell^2|\nabla\varphi|^2) \geq \frac{1}{2}\int_{Q_t}(|\nabla\eta|^2 + \ell^2|\nabla\varphi|^2). \tag{4.16}$$

We also observe that

$$-\frac{\gamma k\ell^3}{\nu}\int_{Q_t}|\varphi|^2 \leq 0. \tag{4.17}$$

Then, due to (4.13)–(4.17), from (4.12) we infer that

$$\frac{1}{2}\int_\Omega |\eta(t)|^2 + \frac{k\ell^2}{2\nu}\int_\Omega |\varphi(t)|^2 + \frac{1}{2}\int_{Q_t}(|\nabla\eta|^2 + \ell^2|\nabla\varphi|^2)$$

$$\leq \frac{1}{2}\|\eta_0\|_H^2 + \frac{k\ell^2C_\pi}{\nu}\int_{Q_t}|\varphi|^2 + \frac{k\ell^2}{2\nu}\|\varphi_0\|_H^2 + \int_{Q_t}(f-k\Delta\eta^*)\eta + \frac{\gamma k\ell^2}{\nu}\int_{Q_t}\eta\varphi + \frac{\gamma k\ell^2}{\nu}\int_{Q_t}\eta^*\varphi.$$

By applying the inequality (3.45) to the last three terms of the right-hand side of the previous equation, we obtain that

$$\frac{1}{2}\|\eta(t)\|_H^2 + \frac{k\ell^2}{2\nu}\|\varphi(t)\|_H^2 + \frac{1}{2}\int_{Q_t}(|\nabla\eta|^2 + \ell^2|\nabla\varphi|^2)$$

$$
\leq \frac{1}{2}\|\eta_0\|_H^2 + \frac{k\ell^2}{2\nu}\|\varphi_0\|_H^2 + \frac{1}{8}\int_{Q_t}|\eta|^2 + 2\int_{Q_t}|f - k\Delta\eta^*|^2 + \frac{1}{8}\int_{Q_t}|\eta|^2
$$

$$
+2\Big(\frac{\gamma k\ell^2}{\nu}\Big)^2\int_{Q_t}|\varphi|^2 + \frac{1}{8}\int_{Q_t}|\eta^*|^2 + 2\Big(\frac{\gamma k\ell^2}{\nu}\Big)^2\int_{Q_t}|\varphi|^2 + \frac{k\ell^2 C_\pi}{\nu}\int_{Q_t}|\varphi|^2. \tag{4.18}
$$

From (4.18) we infer that

$$
\frac{1}{2}\|\eta(t)\|_H^2 + \frac{k\ell^2}{2\nu}\|\varphi(t)\|_H^2 + \frac{1}{2}\int_{Q_t}(|\nabla\eta|^2 + \ell^2|\nabla\varphi|^2)
$$

$$
\leq \frac{1}{2}\|\eta_0\|_H^2 + \frac{k\ell^2}{2\nu}\|\varphi_0\|_H^2 + 4\|f\|_{L^2(Q)}^2 + 4k^2T\|\eta^*\|_W^2 + \frac{1}{8}T\|\eta^*\|_H^2
$$

$$
+M\int_0^t\left(\frac{1}{2}\|\eta(s)\|_H^2 + \frac{k\ell^2}{2\nu}\|\varphi(s)\|_H^2 + \frac{1}{2}\int_{Q_s}(|\nabla\eta|^2 + \ell^2|\nabla\varphi|^2)\right)ds, \tag{4.19}
$$

where

$$
M = \max\left(\frac{4\gamma^2 k\ell^2 + 2\nu C_\pi}{\nu}; \frac{1}{2}\right).
$$

From (4.19), by applying the Gronwall lemma, we conclude that

$$
\frac{1}{2}\|\eta(t)\|_H^2 + \frac{k\ell^2}{2\nu}\|\varphi(t)\|_H^2 + \frac{1}{2}\|\nabla\eta\|_{L^2(0,t;H)}^2 + \frac{\ell^2}{2}\|\nabla\varphi\|_{L^2(0,t;H)}^2
$$

$$
\leq C_1\left[4\|f\|_{L^2(Q)}^2 + 4k^2T\|\eta^*\|_W^2 + \frac{1}{8}T\|\eta^*\|_W^2 + C_0\Big(\|\eta_0\|_H^2 + \|\varphi_0\|_H^2\Big)\right], \tag{4.20}
$$

where

$$
C_0 = \max\left(\frac{1}{2};\ \frac{k\ell^2}{2\nu}\right), \qquad C_1 = e^{TM}.
$$

From (4.20), we infer that

$$
C_3\Big(\|\eta(t)\|_H^2 + \|\varphi(t)\|_H^2 + \|\nabla\eta\|_{L^2(0,t;H)}^2 + \|\nabla\varphi\|_{L^2(0,t;H)}^2\Big)
$$

$$
\leq \frac{1}{2}\|\eta(t)\|_H^2 + \frac{k\ell^2}{2\nu}\|\varphi(t)\|_H^2 + \frac{1}{2}\|\nabla\eta\|_{L^2(0,t;H)}^2 + \frac{\ell^2}{2}\|\nabla\varphi\|_{L^2(0,t;H)}^2
$$

$$
\leq C_2\Big(\|f\|_{L^2(Q)}^2 + \|\eta^*\|_W^2 + \|\eta_0\|_H^2 + \|\varphi_0\|_H^2\Big)
$$

$$
\leq C_2\Big(\|f\|_{L^2(Q)} + \|\eta^*\|_W + \|\eta_0\|_H + \|\varphi_0\|_H\Big)^2, \tag{4.21}
$$

where

$$
C_2 = \max\left(4C_1;\ 4k^2TC_1;\ \frac{1}{8}TC_1;\ C_1C_0\right), \qquad C_3 = \min\left(\frac{1}{2};\ \frac{k\ell^2}{2\nu};\ \frac{\ell^2}{2}\right).
$$

From (4.21) we obtain that

$$
\|\eta(t)\|_H^2 + \|\varphi(t)\|_H^2 + \|\nabla\eta\|_{L^2(0,t;H)}^2 + \|\nabla\varphi\|_{L^2(0,t;H)}^2
$$

$$\leq C_4\bigg(\|f\|_{L^2(Q)} + \|\eta^*\|_W + \|\eta_0\|_H + \|\varphi_0\|_H\bigg)^2, \tag{4.22}$$

where $C_4 = \frac{C_2}{C_3}$. From (4.22) we conclude that there exists a constant $C > 0$ which depends only on Ω, T depends only on Ω, T and the parameters ℓ, α, k, ν, γ of the system, such that

$$\|\eta_1 - \eta_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\varphi_1 - \varphi_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}$$

$$\leq C\big(\|f_1 - f_2\|_{L^2(Q)} + \|\eta_1^* - \eta_2^*\|_W + \|\eta_{0_1} - \eta_{0_2}\|_H + \|\varphi_{0_1} - \varphi_{0_2}\|_H\big). \tag{4.23}$$

To infer the uniqueness of the solution, we choose $f_1 = f_2$, $\eta_1^* = \eta_2^*$, $\varphi_{0_1} = \varphi_{0_2}$, $\eta_{0_1} = \eta_{0_2}$. Then, replacing the corresponding values in (4.23), we obtain that

$$\|\eta_1 - \eta_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\varphi_1 - \varphi_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} = 0. \tag{4.24}$$

Hence $\eta_1 = \eta_2$ and $\varphi_1 = \varphi_2$. Then the solution of problem (P) (see (4.1)–(4.6)) is unique.

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