A VARIATIONAL APPROACH TO THE HELE–SHAW FLOW WITH INJECTION

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Abstract. We introduce a variational approach to the Hele-Shaw flow $\partial_t \chi = \triangle u + f\chi$, $f \geq 0$ in $\mathbb{R}^N$, where χ is the characteristic function of an open set $\Omega(t)$ in $\mathbb{R}^N$ and $u(t,\cdot) \in H_0^1(\Omega(t))$ solves $-\triangle u(t,\cdot) = f$ in $\Omega(t)$.

By choosing a time step $\tau_k = \tau/2^k$ and iteratively solving a variational problem in $\mathbb{R}^N$, we construct a staircase family of opens sets and a corresponding family of functions: as $k \to \infty$, both sets and functions converge increasingly, at fixed time, to a weak solution of the problem. When the latter is not unique, the solution thus obtained is characterized by a minimality property, with respect to set inclusion, at fixed time.

We also prove several monotonicity results of the solutions thus obtained, with respect to both the initial set and the forcing term f. In particular, these monotonicity properties imply that $\Omega(t)$ has finite perimeter for every t, provided that $\Omega(0)$ has finite perimeter.

Finally, under very mild assumptions, we prove that the number of connected components in non increasing, that $\Omega(t)$ is connected for large t and that it tends to fill the whole of $\mathbb{R}^N$.

1. Introduction

In this paper we deal with a one-phase Hele-Shaw problem when a non-zero forcing term is acting. This is the case e.g. of a compressible fluid in a Hele-Shaw cell; in this case the unknown is the pressure field of the fluid. Here we are not interested in surface tension effects, so we do not suppose any Gibbs-Thomson relation at the interface.

In its classical formulation, the problem can be described as follows. We are given a (smooth) open set $\Omega(0)$ in $\mathbb{R}^N$, $N \geq 2$, which represents the portion of space occupied by an isotropic compressible fluid at time $t = 0$, and a function $f \geq 0$ defined on $\mathbb{R}^N$, representing up to some physical constant the fluid's velocity divergence (the case $f \equiv 0$ modeling an incompressible fluid). We seek a family of (smooth open) sets $\Omega(t)$, whose boundary $\partial\Omega(t)$ moves outward as time increases, according to the following rule. At each time $t \geq 0$ one solves the Laplace equation

$$-\triangle u(t) = f \text{ in } \Omega(t), \qquad u(t) \in H_0^1(\Omega(t)). \tag{1}$$

When everything is smooth, $u(t)$ is non negative and superharmonic in $\Omega(t)$ and, if $x \in \partial\Omega(t)$ is a boundary point and ν is the inner normal to $\partial\Omega(x)$ at x, the normal derivative $\partial u(t)/\partial\nu$ is non negative. The motion of the boundary

$\partial\Omega_t$ is described by

$$\vec{v}(x) := -\nu \frac{\partial u(t)}{\partial \nu}(x) \quad \forall x \in \partial\Omega_t, \tag{2}$$

where $\vec{v}(x)$ denotes the velocity of the expanding boundary $\partial\Omega_t$ at each boundary point x. Note that only the restriction of f to $\Omega(t)$ is relevant at time t. In the case of a fluid, this means that the external force can act only in regions effectively filled by the fluid.

Due to (1), the motion equation (2) means that $\partial_t \chi_{\Omega(t)} = \triangle u(t) + f$ distributionally in space-time. This leads to the following definition.

A weak solution is a family of open sets $\Omega(t)$, each of finite measure, such that $\chi_{\Omega(t)} \in L^1_{\mathrm{loc}}([0,+\infty); L^1)$, satisfying

$$\begin{aligned}\int\limits_{\Omega(t_0)} \eta(t_0,x)\,dx - \int\limits_{\Omega(t_1)} \eta(t_1,x)\,dx + \int\limits_{t_0}^{t_1}\int\limits_{\Omega(t)} \partial_t\eta(t,x)\,dxdt = \\ = \int\limits_{t_0}^{t_1}\int \big(\nabla u(t,x)\nabla\eta(t,x) - f(x)\eta(t,x)\chi_{\Omega(t)}(x)\big)\,dxdt\end{aligned} \tag{3}$$

for every $\eta \in W^{1,\infty}((t_0,t_1)\times\mathbb{R}^N)$ and for every $t_1 > t_0 \geq 0$. The function $u(t,x)$ belongs to $L^1_{\mathrm{loc}}([0,+\infty); W^{1,1}(\mathbb{R}^N))$ and satisfies

$$\begin{cases} u(t,\cdot) \in H^1_0(\Omega(t)), \quad u(t,\cdot) \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega(t), \\ -\triangle u(t,\cdot) = f \text{ in } \Omega(t) \end{cases} \tag{4}$$

for almost every (a.e.) $t > 0$.

Given $\tau > 0$ and the dyadic time-step $\tau_k = \tau/2^k$, we define $\Omega^k_0 := \Omega(0)$ and, inductively for $n \geq 0$, $\Omega^k_{n+1} = \Omega^k_n \cup \{u^k_n > 0\}$, where u^k_n is the minimizer of

$$u \mapsto \int\limits_{\mathbb{R}^N} |\nabla u|^2 - 2\int\limits_{\Omega^k_n} fu + \frac{2}{\tau_k}\int\limits_{\mathbb{R}^N\setminus\Omega^k_n} |u|, \quad u \in W^{1,2}(\mathbb{R}^N). \tag{5}$$

It turns out that $\triangle u^k_n = -f\chi_{\Omega^k_n} + \frac{1}{\tau_k}(\chi_{\Omega^k_{n+1}} - \chi_{\Omega^k_n})$, $n \geq 0$ and, if we consider the (piecewise constant) family of sets $\Omega^k_{\lfloor t/\tau_k\rfloor}$ and functions $u^k_{\lfloor t/\tau_k\rfloor}$, letting $k \to \infty$ we obtain a weak solution in the limit, called the *variational solution*, which is independent of the initial τ. A surprising feature of this approach is that, as $k \to \infty$, convergence is monotone from below (both for sets and functions). Moreover, the variational solution enjoys several nice properties: it is the smallest weak solution when the latter is not unique (in fact, we will prove uniqueness of the weak solution when $f \in L^\infty$, and provide counterexamples for $f \in L^p$ for every $p \geq 1$), it is increasing w.r.to both $\Omega(0)$ and f, and, surprisingly enough, the perimeter of $\Omega(t)$ is finite for every t, whenever $\Omega(0)$ is a set of finite perimeter. Finally, the number of connected components of $\Omega(t)$ is non-increasing and, if $\Omega(0)$ has finitely many connected components, then $\Omega(t)$ is connected for large t.

Our approach is somewhat related to the classical idea of exact mild solutions for accretive operators (see [7]), yet we do not ask approximate solutions to be continuous w.r.to time with values in L^1, and the way we construct them is entirely variational. Moreover, we point out that the solutions we obtain are rather strong (for instance, $\Omega(t)$ is open for every t and χ is really the characteristic function of $\Omega(t)$, contrary to the accretive operators setting where merely $\chi \in H(u)$, the graph of the Heavyside function, and χ may take values in $[0,1)$ where u vanishes), under minimal assumptions on f (we do not ask $f \in L^\infty$ as in the accretive theory, and this should be compared with the way similar elliptic-parabolic problems are treated e.g. in [3], [8], neither do we suppose f being supported in $\Omega(0)$, see e.g. [5]). We also remark that we set the problem in the whole of $\mathbb{R}^{\mathrm{N}}$ and not in a bounded regular domain as somewhat usual in literature (see e.g. [3], [8]). We only assume that $\Omega(0)$ has finite measure (neither boundedness nor any smoothness assumption), and that f satisfies

$$f \geq 0 \text{ in } \mathbb{R}^{\mathrm{N}} \text{ and } f \in L^r(\Omega) \text{ for every } \Omega \subset \mathbb{R}^{\mathrm{N}} \text{ such that } |\Omega| < +\infty, \tag{6}$$

where

$$r = (2^*)' = 2\mathrm{N}/(\mathrm{N}+2) \text{ if } \mathrm{N} > 2, \quad r \in (1, +\infty] \text{ if } \mathrm{N} = 2. \tag{7}$$

Throughout, if $\Omega \subset \mathbb{R}^{\mathrm{N}}$, then $|\Omega|$ denotes its Lebesgue measure (as usual, $|\cdot|$ may also denote the length of a vector or the absolute value of a scalar). Moreover, $\int$ stands for $\int_{\mathbb{R}^{\mathrm{N}}}$ and $L^p, W^{1,p}$ etc. stand for $L^p(\mathbb{R}^{\mathrm{N}}), W^{1,p}(\mathbb{R}^{\mathrm{N}})$ etc.. Whenever a proper subdomain of $\mathbb{R}^{\mathrm{N}}$ is involved, it will always be explicitly mentioned, as in $\int_\Omega$ or $L^p(\Omega)$.

We point out that, due to Sobolev embedding, the summability condition on f is the weakest possible in order to solve (1) in any (a priori unknown) domain of finite measure.

2. Basic properties of weak solutions

It is well known that the summability condition (6) is equivalent to the splitting condition $f = a + b$, for some $a \in L^\infty$ and $b \in L^r$, as well as to the estimate $\int_\Omega f \leq C\left(1 + |\Omega|^{1-\frac{1}{r}}\right)$ for every $\Omega \subseteq \mathbb{R}^{\mathrm{N}}$, with C is independent of Ω. In particular, it follows that

$$\int_\Omega f \leq C\,(1 + |\Omega|) \quad \text{for every } \Omega \subseteq \mathbb{R}^{\mathrm{N}}. \tag{8}$$

Moreover, a solution to (4) can be defined pointwise in $\mathbb{R}^{\mathrm{N}}$: indeed, it vanishes outside $\Omega(t)$, whereas inside $\Omega(t)$ it is superharmonic.

Theorem 2.1. *Assume that $\Omega(t)$ (coupled with $u(t,x)$) is a weak solution. Then,*

a) *For every $t_1 > t_0 \geq 0$, $\chi_{\Omega(t_0)} \leq \chi_{\Omega(t_1)}$ a.e. in $\mathbb{R}^N$, and moreover*

$$|\Omega(t_1)| - |\Omega(t_0)| = \int_{t_0}^{t_1} \int_{\Omega(t)} f \, dxdt. \tag{9}$$

b) *For a.e. $t > 0$, $\triangle u(t,\cdot)$ is a signed measure and its total variation satisfies $|\triangle u(t,\cdot)| \leq C_f(1 + |\Omega(t)|)$. Moreover, $-\triangle u(t,\cdot) - f\chi_{\Omega(t)} \leq 0$ as Radon measures.*

c) *$\chi_{\Omega(t)} \in W^{1,\infty}_{loc}([0,+\infty); L^1)$, and $u(t,\cdot) \in L^\infty_{loc}([0,+\infty); W^{1,2})$.*

d) *$u(t_1,\cdot) \geq u(t_0,\cdot) \geq 0$ in $\mathbb{R}^N$, for a.e. $t_0 > 0$ and a.e. $t_1 > t_0$.*

e) *For a.e. $t > 0$, $\chi_{\Omega(0)\cup\{u(t,\cdot)>0\}} \leq \chi_{\Omega(t)}$ a.e. in $\mathbb{R}^N$.*

f) *In (3), one can choose $\eta(t,x)$ belonging to $L^1((t_0,t_1); W^{1,2})$, provided that $\partial_t \eta$ belongs to $L^1((t_0,t_1) \times \mathbb{R}^N)$.*

Proof. We first prove b). Pick $t > 0$ such that (4) holds true and choose bounded open sets Ω_n, such that $\partial\Omega_n \in C^\infty$, $\Omega_n \subseteq \Omega_{n+1}$ and $\Omega(t) = \bigcup \Omega_n$, and choose $f_n \in C_0^\infty(\Omega_n)$, such that $f_n \geq 0$ and $f_n \to f$ in $H^{-1}(\Omega(t)) \cap L^1(\Omega(t))$. Due to (4), if $u_n \in H_0^1(\Omega_n)$ solves (classically) $-\triangle u_n = f_n$ in Ω_n and $u_n \equiv 0$ outside Ω_n, then clearly $u_n \to u(t,\cdot)$ weakly in $H_0^1(\Omega(t))$. Moreover,

$$-\triangle u_n = f_n - D_\nu u_n \mathcal{H}^{N-1} \llcorner \partial\Omega_n \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

where $D_\nu u_n$ is the (inner) normal derivative at the boundary. By the maximum principle, $u_n \geq 0$ and hence $D_\nu u_n \geq 0$. Therefore, $\triangle u_n$ is a measure with negative part given by f_n and positive part given by $D_\nu u_n \mathcal{H}^{N-1} \llcorner \partial\Omega_n$. Since Ω_n is bounded, $\triangle u_n(\mathbb{R}^N) = 0$ and hence the total variation satisfies

$$|\triangle u_n|(\mathbb{R}^N) = 2\left|(\triangle u_n)^-\right|(\mathbb{R}^N) = 2\int_{\Omega_n} f_n \to 2\int_{\Omega(t)} f \overset{(8)}{\leq} 2C_f(1 + |\Omega(t)|)$$

as $n \to \infty$, according to (8). Hence also $\triangle u_n \rightharpoonup \triangle u(t,\cdot)$ weakly-* as Radon measures, and b) follows.

Now choose $\eta \in C_0^\infty(\mathbb{R}^N)$, $\eta \geq 0$ and independent of t, in (3). Using b), the right hand side of (3) is non-positive, hence

$$\int \chi_{\Omega(t_0)}(x)\eta(x)\,dx \leq \int \chi_{\Omega(t_1)}(x)\eta(x)\,dx, \quad \forall \eta \in C_0^\infty, \quad \eta \geq 0,$$

and the first claim in a) is proved. Then (9) follows choosing $\eta \equiv 1$ in (3). Using a), we find

$$\begin{aligned} \|\chi_{\Omega(t_1)} - \chi_{\Omega(t_1)}\|_{L^1} &= |\Omega(t_1)| - |\Omega(t_0)| \leq (t_1 - t_0)\int_{\Omega(t_1)} f \\ &\overset{(8)}{\leq} (t_1 - t_0)C_f(1 + |\Omega(t_1)|) \leq (t_1 - t_0)C_f(1 + |\Omega(T)|), \quad 0 \leq t_0 < t_1 \leq T \end{aligned} \tag{10}$$

for some fixed $T \geq t_1$, and the first part of c) follows. For the second part, since $|\Omega(t)| < +\infty$, standard energy estimates for (4) (separately handling the cases $N = 2$ and $N > 2$) reveal that

$$\|u(t,\cdot)\|_{W^{1,2}} \leq C_t\|f\|_{L^r(\Omega(t))} \leq C_T\|f\|_{L^r(\Omega(T))}, \quad t \leq T,$$

since C_t depends only on N, r and, increasingly, on $|\Omega(t)|$. Thus c) is established.

To prove d), note that $u(t,\cdot) \geq 0$ follows from (4) by the minimum principle. Now pick $t_1 > t_0 > 0$ such that (4) is satisfied, and let $\xi(t,\cdot) = \Big(u(t_0,\cdot) - u(t,\cdot)\Big)^+$. By c), a) and (4), $\xi(t,\cdot)$ belongs to $L^\infty((t_0,t_1); H^1_0(\Omega(t_0)))$. Hence it can be approximated, e.g. in $L^2((t_0,t_1); W^{1,2})$, by smooth functions $\eta_\varepsilon(t,x)$ which vanish outside the cylinder $\mathbb{R}^+ \times \Omega(t_0)$. Since by a) $\chi_{\Omega(t_0)} \leq \chi_{\Omega(t)}$ when $t \geq t_0$, and $\eta_\varepsilon(t,\cdot) \equiv 0$ outside $\Omega(t_0)$, the l.h.s. of (3) vanishes if $\eta = \eta_\varepsilon$, i.e.

$$0 = \int_{t_0}^{t_1} \int_{\Omega(t_0)} \nabla u(t,x) \nabla \eta_\varepsilon(t,x)\, dx dt - \int_{t_0}^{t_1} \left(\int_{\Omega(t_0)} f(x) \eta_\varepsilon(t,x)\, dx \right) dt.$$

Letting $\varepsilon \to 0$, we find

$$\begin{aligned} 0 &= \int_{t_0}^{t_1} \int_{\Omega(t_0)} \nabla u(t,x) \nabla \xi(t,x)\, dx dt - \int_{t_0}^{t_1} \left(\int_{\Omega(t_0)} f(x) \xi(t,x)\, dx \right) dt \\ &\overset{(4)}{=} \int_{t_0}^{t_1} \int_{\Omega(t_0)} \Big(\nabla u(t,x) - \nabla u(t_0,x)\Big) \nabla \xi(t,x)\, dx dt = - \int_{t_0}^{t_1} \int_{\Omega(t_0)} |\nabla \xi(t,x)|^2\, dx dt, \end{aligned}$$

having used integration by parts in $\Omega(t_0)$ for a.e. t with $\xi(t,\cdot) \in H^1_0(\Omega(t_0))$ as test function against $u(t_0,\cdot)$ (recall that (4) holds with $t = t_0$). Thus $\xi \equiv 0$, i.e. $u(t,\cdot) \geq u(t_0,\cdot)$ for a.e. $t \geq t_0$.

To prove e), choose t_1 and note that, for almost every $x \in \mathbb{R}^{\mathrm{N}}$, if $u(t_1,x) > 0$ (respectively, $x \in \Omega(0)$), then $x \in \Omega(t_1)$ by (4) (resp., by a)). Then $\chi_{\Omega(0) \cup \{u(t_1,\cdot)>0\}} \leq \chi_{\Omega(t_1)}$ a.e. in $\mathbb{R}^{\mathrm{N}}$.

Finally, f) follows from (3) with $\eta \in W^{1,\infty}$ by a standard density and duality argument, relying on c). □

3. A detailed study of the functional

In this section we study the minimizers of the functional

$$F_\tau(u,\Omega,f) = \int_{\mathbb{R}^{\mathrm{N}}} |\nabla u|^2 - 2\int_\Omega fu + \frac{2}{\tau} \int_{\mathbb{R}^{\mathrm{N}} \setminus \Omega} |u|, \quad u \in W^{1,2}(\mathbb{R}^{\mathrm{N}}), \tag{11}$$

where $\Omega \subset \mathbb{R}^{\mathrm{N}}$ has finite measure, $\tau \in (0,1]$ and f satisfies (6), (7).

We remark that, since in this section Ω is fixed and only the restriction of f to Ω plays a role in (11), the only relevant assumption on f in this section are $f \geq 0$ and $f \in L^r(\Omega)$, which of course is guaranteed by (6).

We recall the Nash inequality:

$$\|u\|_{L^2} \leq C(\mathrm{N}) \|u\|_{L^1}^{\frac{2}{\mathrm{N}+2}} \|\nabla u\|_{L^2}^{\frac{\mathrm{N}}{\mathrm{N}+2}}, \tag{12}$$

which holds true for every $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{\mathrm{N}})$.

Lemma 3.1. *Let $\lambda > 0$ satisfy $|\Omega| \le \lambda$ and $\|f\|_{L^r(\Omega)} \le \lambda$. Then*

$$F_\tau(u,\Omega,f) \ge C^{-1}(\|\nabla u\|_{L^2} + \|u\|_{L^1}) - C, \quad \forall u \in W^{1,2}, \tag{13}$$

where $C > 0$ depends only on N, λ and r.

Proof. First we establish the inequality

$$\|u\|_{L^1} \le 2\|u\|_{L^1(\mathbb{R}^{\mathrm{N}}\setminus\Omega)} + C(\mathrm{N},\lambda)\|\nabla u\|_{L^2}, \quad \forall u \in W^{1,2}. \tag{14}$$

We have, using Hölder, (12) and the Young inequality:

$$\int_\Omega |u| \le |\Omega|^{\frac{1}{2}}\|u\|_{L^2} \le \frac{1}{2}\|u\|_{L^1} + C(\mathrm{N},\lambda)\|\nabla u\|_{L^2}.$$

Then (14) follows on adding $\int_{\mathbb{R}^{\mathrm{N}}\setminus\Omega}|u|$ to both sides and moving $(1/2)\|u\|_{L^1}$ to the left. Moreover, from (12) we have for every $\varepsilon > 0$

$$\|u\|_{W^{1,2}} = \|u\|_{L^2} + \|\nabla u\|_{L^2} \le \varepsilon\|u\|_{L^1} + C(\varepsilon,\mathrm{N})\|\nabla u\|_{L^2}$$

and, using (14), we find

$$\|u\|_{W^{1,2}} \le \varepsilon\|u\|_{L^1(\mathbb{R}^{\mathrm{N}}\setminus\Omega)} + C(\varepsilon,\mathrm{N},\lambda)\|\nabla u\|_{L^2}. \tag{15}$$

Then, since $\tau \in (0,1]$ we have

$$F_\tau(u,\Omega,f) \ge \|\nabla u\|_{L^2}^2 + 2\|u\|_{L^1(\mathbb{R}^{\mathrm{N}}\setminus\Omega)} - \|f\|_{L^r(\Omega)}\|u\|_{L^{r'}(\Omega)},$$

where $r' = 2^*$ if $\mathrm{N} > 2$ and $r' \in [1,\infty)$ if $\mathrm{N} = 2$. In the latter case, if $r' < 2$, we can further bound $\|u\|_{L^{r'}(\Omega)}$ by $C(\lambda)\|u\|_{L^2}$ using the Hölder inequality. In any case, using the Sobolev immersion theorem we find

$$F_\tau(u,\Omega,f) \ge \|\nabla u\|_{L^2}^2 + 2\|u\|_{L^1(\mathbb{R}^{\mathrm{N}}\setminus\Omega)} - C(\mathrm{N},\lambda,r)\|u\|_{W^{1,2}}.$$

Choosing $\varepsilon := C(\lambda,\mathrm{N},r)^{-1}$ in (15) and plugging it into the last inequality, we find

$$F_\tau(u,\Omega,f) \ge \|\nabla u\|_{L^2}^2 + \|u\|_{L^1(\mathbb{R}^{\mathrm{N}}\setminus\Omega)} - C(\mathrm{N},\lambda,r)\|\nabla u\|_{L^2}.$$

Finally, using (14) once more, we have

$$F_\tau(u,\Omega,f) \ge \|\nabla u\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^1} - C(\mathrm{N},\lambda,r)\|\nabla u\|_{L^2}$$

and (13) follows. □

Theorem 3.2. *The functional $u \mapsto F_\tau(u,\Omega,f)$ has a unique minimizer on $W^{1,2}(\mathbb{R}^{\mathrm{N}})$, which satisfies*

$$-\int \nabla u \nabla\varphi = -\int_\Omega f\varphi + \frac{1}{\tau}\int_{\{u>0\}\setminus\Omega}\varphi, \quad \forall\varphi \in W^{1,p}(\mathbb{R}^{\mathrm{N}}), \quad p \in [2,+\infty]. \tag{16}$$

In particular, u is a distributional solution of

$$\triangle u = -f\chi_\Omega + \frac{1}{\tau}\chi_{\{u>0\}\setminus\Omega}. \tag{17}$$

Moreover, $u \geq 0$, $u \in W^{2,p}(\mathbb{R}^{\mathrm{N}})$ *for every* $p \in [1, r]$ ($p \in [1, r)$ *if* $\mathrm{N} = 2$ *and* $r = \infty$)*, and*

$$|\{u > 0\} \setminus \Omega| = \tau \int_{\Omega} f \tag{18}$$

(in particular, $\{u > 0\}$ *has finite measure). Finally, if* $|\Omega| \leq \lambda$ *and* $\|f\|_{L^r(\Omega)} \leq \lambda$*, then*

$$\|u\|_{L_1} + \|u\|_{L^2} + \|\nabla u\|_{L^2} \leq C(\lambda, \mathrm{N}, r). \tag{19}$$

We point out that (16) is somewhat stronger than (17) in the sense of distributions. Indeed, since a generic $\varphi \in W^{1,p}(\mathbb{R}^{\mathrm{N}})$ ($p \geq 2$) may fail to be L^1, the last integral in (16) being meaningful is part of the claim of the theorem.

Proof. Let $\{u_n\} \subset W^{1,2}$ be a minimizing sequence. Since $F_\tau(0, \Omega, f) = 0$, we may assume that $F_\tau(u_n, \Omega, f) \leq 0$, and (13) provides a bound for u_n in L^1 and for ∇u_n in L^2. Using (12), we can also bound u_n in L^2. Then, possibly passing to a subsequence, $u_n \to u$ weakly in $W^{1,2}$ and pointwise a.e. in $\mathbb{R}^{\mathrm{N}}$, for some $u \in W^{1,2}$. Then, by semicontinuity, u is a minimizer: it is unique by strict convexity of F_τ and non-negative by comparison with u^+.

Now let $\eta \in C_0^1(\mathbb{R}^{\mathrm{N}})$ be a test function. The minimality of u then yields

$$\begin{aligned} 0 \leq \lim_{\varepsilon \downarrow 0^+} \frac{F_\tau(u + \varepsilon\eta, \Omega, f) - F_\tau(u, \Omega, f)}{2\varepsilon} &= \\ = \int \nabla u \nabla \eta - \int_{\Omega} f\eta + \frac{1}{\tau} \lim_{\varepsilon \downarrow 0^+} \int_{\mathbb{R}^{\mathrm{N}} \setminus \Omega} \frac{|u + \varepsilon\eta| - |u|}{\varepsilon} \end{aligned} \tag{20}$$

provided that the last limit exists. Indeed, the integrand is dominated by $|\eta|$, and the pointwise limit is $|\eta(x)|$ if $u(x) = 0$, and $\eta(x)$ if $u(x) > 0$. Hence (20) simplifies to

$$0 \leq \int_{\mathbb{R}^{\mathrm{N}}} \nabla u \nabla \eta - \int_{\Omega} f\eta + \frac{1}{\tau} \int_{\{u=0\} \setminus \Omega} |\eta| + \frac{1}{\tau} \int_{\{u>0\} \setminus \Omega} \eta, \quad \forall \eta \in C_0^1(\mathbb{R}^{\mathrm{N}}).$$

If we choose $\xi \in C_0^1(\mathbb{R}^{\mathrm{N}})$, $\xi \geq 0$ and we separately write the last equation with $\eta = \xi$ and $\eta = -\xi$, we obtain, by the arbitrariness of ξ,

$$-f\chi_\Omega - \frac{1}{\tau}\chi_{\{u=0\}\setminus\Omega} + \frac{1}{\tau}\chi_{\{u>0\}\setminus\Omega} \quad \leq \quad \triangle u \quad \leq \quad -f\chi_\Omega + \frac{1}{\tau}\chi_{\mathbb{R}^n\setminus\Omega} \tag{21}$$

in the sense of Radon measures. Since the two measures on the left and on the right hand side are both absolutely continuous with respect to Lebesgue measure (indeed, they are L^p_{loc}), by the Radon–Nikodym theorem we obtain that $\triangle u \in L^r_{\mathrm{loc}}(\mathbb{R}^{\mathrm{N}})$ and hence $u \in W^{2,r}_{\mathrm{loc}}(\mathbb{R}^{\mathrm{N}})$. Since $\nabla u = 0$ a.e. in the level set $\{u = 0\}$ and $\nabla u \in W^{1,r}_{\mathrm{loc}}(\mathbb{R}^{\mathrm{N}})$, we have $D^2 u = 0$ a.e. in $\{u = 0\}$: in particular, $\triangle u = 0$ a.e. in $\{u = 0\}$. This fact, combined with (21), yields (17), and hence also (16) at least when $\varphi \in C_0^\infty(\mathbb{R}^{\mathrm{N}})$.

Now we prove that $|\{u > 0\}|$ is finite. Given $n > 0$, pick $\varphi \in C_0^\infty(B_{n+1})$ such that $\varphi \equiv 1$ in B_n, $|\varphi| \leq 1$ and $|\nabla\varphi| \leq 2$, where B_r is the ball of radius

r. After simple computations, we find from (16) (recall that $\varphi \in C_0^\infty$)

$$\Big|\{u>0\}\cap B_n \setminus \Omega\Big| \le \tau \int_\Omega f + 2\tau \int_{\{u>0\}\cap B_{n+1}} |\nabla u|.$$

Adding $|\Omega|$ to both sides and using Hölder on the last term since $\nabla u \in L^2$, we obtain $a_n \le \alpha(1+\sqrt{a_{n+1}})$, where $a_n := |\{u>0\}\cap B_n|$ and α does not depend on n. If $a_n \le 1$ for all n then we are done; otherwise, $1 < \sqrt{a_{n+1}}$ for large n, and letting $\beta = 2\alpha$ we find $a_n \le \beta\sqrt{a_{n+1}}$ for large n. Iterating n times this estimate, we find

$$a_n \le \beta^{1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}}(a_{2n})^{\frac{1}{2^n}}, \quad n \text{ large}.$$

Since clearly $a_{2n} \le \omega_{\mathrm{N}}(2n)^{\mathrm{N}}$, we find $\lim_n a_n \le \beta^2$ and hence $|\{u>0\}|$ is finite. Since $u \ge 0$ and $u \in W^{1,2}$, this reveals that $u \in W^{1,p}$ for $1 \le p \le 2$: in particular, $u \in L^r$ (indeed, if $\mathrm{N} > 2$ then $r < 2$, whereas if $\mathrm{N} = 2$, then $u \in L^p$ for every $p \ge 1$). Moreover, $\triangle u \in L^p$ for $p \in [1,r]$ by (17), hence $u \in W^{2,p}$ for every $p \in (1,r]$ ($p \in (1,r)$ if $\mathrm{N} = 2$ and $r = \infty$) by elliptic estimates. Finally, $u \in W^{2,1}$ since $|\{u>0\}|$ is finite. Therefore, (16) follows by approximating φ by compactly supported smooth functions, in the weak-$*$ topology of $W^{2,p}$.

Now (18) follows immediately from (16) choosing $\varphi(x) \equiv 1$.

To prove (19), note that since u minimizes $F_\tau(\cdot,\Omega,f)$, then $F_\tau(u,\Omega,f) \le F_\tau(0,\Omega,f) = 0$, hence using (13) and (12) we obtain (19). ☐

Remark 3.1. *From* (17) *and* $f \ge 0$, *it follows that the function* $u(x) - |x|^2/(2\mathrm{N}\tau)$ *is superharmonic, hence it admits a lower semicontinuous representative defined everywhere as the increasing limit of averages over balls. Therefore, also u has a semicontinuous representative defined everywhere. We will always regard u as everywhere defined and semicontinuous. Thus, in particular, the set* $\{u>0\}$ *is open.*

Theorem 3.3. *Suppose that*

$$\Omega_1 \subseteq \Omega_2 \subset \mathbb{R}^{\mathrm{N}}, \quad 0 < \tau_1 \le \tau_2 \le 1, \quad \textit{and} \quad f_1 \le f_2, \tag{22}$$

where Ω_2 *has finite measure and* f_1, f_2 *satisfy assumptions analogue to* (6), (7). *If* u_i *is the minimizer of* $F_{\tau_i}(\cdot,\Omega_i,f_i)$, $i=1,2$, *then* $u_1 \le u_2$.

Proof. By the previous theorem, each u_i satisfies a relation analogue to (16). Let $\varphi := (u_1-u_2)^+$. Since $\varphi \in W^{1,2}(\mathbb{R}^{\mathrm{N}})$, we have using (16)

$$\begin{aligned}\int |\nabla\varphi|^2 &= \int \nabla\varphi(\nabla u_1 - \nabla u_2) = \int (-f_2\chi_{\Omega_2} + f_1\chi_{\Omega_1})\varphi \\ &+ \int (\tau_2^{-1}\chi_{\{u_2>0\}\setminus\Omega_2} - \tau_1^{-1}\chi_{\{u_1>0\}\setminus\Omega_1})\varphi \le \tau_1^{-1}\int(\chi_{\{u_2>0\}\setminus\Omega_1} - \chi_{\{u_1>0\}\setminus\Omega_1})\varphi\end{aligned}$$

having used (22) and $\varphi \ge 0$. But the last integrand is nowhere strictly positive: indeed, for it to be positive at some point x, we should have $u_2(x) > 0$, $x \notin \Omega_1$ and $u_1(x) = 0$, but in this case we would have $\varphi(x) = 0$, a contradiction. Thus, $\int |\nabla\varphi|^2 \le 0$ and hence $u_1 \le u_2$. ☐

4. Construction of the variational solution

From now on, $\Omega(0)$ is a given open set of finite measure, and f satisfies (6), (7).

Definition 4.1. *A pair (Ω, u) is called a* state *if $\Omega \subset \mathbb{R}^N$ has finite measure and $u \in W^{1,2}(\mathbb{R}^N)$ is non negative. States are partially ordered by*

$$(\Omega, u) \prec (\Omega', u') \qquad \Longleftrightarrow \qquad \Omega \subseteq \Omega' \text{ and } u \leq u'.$$

Given $\tau \in (0,1]$ and a state (u, Ω), we can construct a new state (Ω', u') as follows. We take u' to be the minimizer of $F_\tau(\cdot, \Omega, f)$ and we define

$$\Omega' := \Omega \cup \{u' > 0\}.$$

In this case, we say that (Ω', u') is obtained from (Ω, u) by one iteration with step τ, *and we abbreviate this by*

$$(\Omega, u) \xrightarrow{\tau} (\Omega', u').$$

The fact that (Ω', u') is still a state follows from Theorem 3.2.

On passing from (Ω, u) to (Ω', u') as above, the u plays no role (indeed, only Ω is relevant). However, the advantage of dealing with states rather than sets pairs will soon be apparent.

Now we are in a position to construct the approximate solutions to the Hele-Shaw flow.

Construction 4.1. *Fix a base time step $\tau \in (0,1]$ and, for every integer $k \geq 0$, set $\tau_k = \tau/2^k$ and inductively define the sequence of states*

$$\begin{cases} (\Omega_0^k, u_0^k) := (\Omega(0), 0), \\ (\Omega_n^k, u_n^k) \xrightarrow{\tau_k} (\Omega_{n+1}^k, u_{n+1}^k), \quad n \geq 0. \end{cases} \tag{23}$$

According to Definition 4.1 and Theorem 3.2, we have

$$\Omega_{n+1}^{(k)} = \Omega_n^{(k)} \cup \left\{ u_{n+1}^{(k)} > 0 \right\}, \qquad \triangle u_{n+1}^{(k)} = -f\chi_{\Omega_n^{(k)}} + \frac{1}{\tau_k}\chi_{\left\{u_n^{(k)}>0\right\}\setminus\Omega_n^{(k)}}. \tag{24}$$

Heuristically, the meaning of $(\Omega_n^k, u_n^k) \xrightarrow{\tau_k} (\Omega_{n+1}^k, u_{n+1}^k)$ is that, if (Ω_n^k, u_n^k) is the status of a Hele-Shaw flow at time $t = n\tau_k$, then $(\Omega_{n+1}^k, u_{n+1}^k)$ is an approximation of the status at time $(n+1)\tau_k$. This process can be iterated several times, thus constructing a sort of discretized Hele-Shaw flow with time step τ_k. Then better and better approximations can be obtained, repeating this construction, as k increases. It turns out that the approximation is monotone increasing and, in the limit, we obtain a weak solution, without passing to any subsequence.

Indeed, denoting by $\lfloor t/\tau_k \rfloor$ the integer part of t/τ_k, we have the following result.

Theorem 4.1. *For every fixed $t > 0$ and $x \in \mathbb{R}^N$, $u_{\lfloor t/\tau_k \rfloor}^{(k)}(x)$ converge increasingly, as $k \to \infty$, to a pointwise limit $u(t,x)$, and $u_{\lfloor t/\tau_k \rfloor}^{(k)} \to u(t, \cdot)$ strongly* *in* $W^{1,2}(\mathbb{R}^N)$ *for a.e. fixed* t *as well as strongly in* $L^2_{loc}([0,+\infty);W^{1,2})$. *Similarly, for every* $t>0$, *the family of open sets* $\Omega^{(k)}_{\lfloor t/\tau_k\rfloor}$ *is increasing and, letting* $\Omega(t)=\bigcup_{k=1}^\infty \Omega^{(k)}_{\lfloor t/\tau_k\rfloor}$, $\chi_{\Omega^{(k)}_{\lfloor t/\tau_k\rfloor}}\uparrow\chi_{\Omega(t)}$, *pointwise everywhere as well as strongly in* $L^\infty_{loc}((0,+\infty);L^1)$.

The pair $\Omega(t)$, $u(t,x)$ *is a weak solution to the Hele-Shaw with initial set* $\Omega(0)$. *Moreover, the functions* $\chi_{\Omega(t)}$ *and* $u(t,x)$ *are independent of the initial time step* τ, *up to negligible sets in* $\mathbb{R}^+\times\mathbb{R}^N$.

Finally, if $\Omega'(t)$, $u'(t,x)$ *is any other weak solution, then for a.e.* $t>0$ *we have*

$$u(t,x)\le u'(t,x),\qquad \chi_{\Omega(t)}\le\chi_{\Omega'(t)} \tag{25}$$

up to negligible sets in $\mathbb{R}^N$.

The proof of this theorem requires several steps. However, the main tool in proving convergence is the following monotonicity lemma, which can be resumed saying that "two iterations with time step $\tau/2$ bring further than one single iteration with time step τ".

Lemma 4.2. *Let* (Ω,u) *be a state, and let*

$$(\Omega,u)\xrightarrow{d}(\Omega_1^d,u_1^d),\qquad(\Omega,u)\xrightarrow{d/2}(\Omega_1^{d/2},u_1^{d/2})\xrightarrow{d/2}(\Omega_2^{d/2},u_2^{d/2})$$

for some time step $d\in(0,1]$. *Then*

$$(\Omega_1^d,u_1^d)\prec(\Omega_2^{d/2},u_2^{d/2}). \tag{26}$$

Proof. Recalling Def. 4.1 and Thm. 3.2, each of u_1^d, $u_1^{d/2}$ and $u_2^{d/2}$ satisfies an Euler equation of the kind (16), (17), namely

$$\begin{aligned}&\triangle u_1^d=-f\chi_\Omega+\frac1d\chi_{\{u_1^d>0\}\setminus\Omega},\\ &\qquad\triangle u_1^{d/2}=-f\chi_\Omega+\frac2d\chi_{\{u_1^{d/2}>0\}\setminus\Omega},\quad \triangle u_2^{d/2}=-f\chi_{\Omega_1^{d/2}}+\frac2d\chi_{\{u_2^{d/2}>0\}\setminus\Omega_1^{d/2}}\end{aligned}$$

Set $\varphi:=(2u_1^d-u_1^{d/2}-u_2^{d/2})^+$. Since $\varphi\in W^{1,2}$, we have integrating by parts

$$\begin{aligned}\int|\nabla\varphi|^2&=\int\nabla\varphi\cdot\nabla(2u_1^d-u_1^{d/2}-u_2^{d/2})=2\int_\Omega f\varphi-\frac2d\int_{\{u_1^d>0\}\setminus\Omega}\varphi\\ &-\int_\Omega f\varphi+\frac2d\int_{\{u_1^{d/2}>0\}\setminus\Omega}\varphi-\int_{\Omega_1^{d/2}}f\varphi+\frac2d\int_{\{u_2^{d/2}>0\}\setminus\Omega_1^{d/2}}\varphi.\end{aligned} \tag{27}$$

We claim that the last expression is non positive. Indeed, since $\Omega\subseteq\Omega_1^{d/2}$ by construction and $f\varphi\ge0$, the sum of the terms containing $f\varphi$ is non positive. The remaining terms, recalling that $\Omega_1^{d/2}=\{u_1^{d/2}>0\}\setminus\Omega$ by construction, reduce to

$$\frac{2}{d}\int \varphi\big(\chi_{\{u_1^{d/2}>0\}\setminus\Omega} + \chi_{\{u_2^{d/2}>0\}\setminus\{u_1^{d/2}>0\}\setminus\Omega} - \chi_{\{u_1^d>0\}\setminus\Omega}\big)$$

$$= \frac{2}{d}\int_{\mathbb{R}^N\setminus\Omega} \varphi\big(\chi_{\{u_1^{d/2}>0\}} + \chi_{\{u_2^{d/2}>0\}\setminus\{u_1^{d/2}>0\}} - \chi_{\{u_1^d>0\}}\big)$$

$$= \frac{2}{d}\int_{\mathbb{R}^N\setminus\Omega} \varphi\big(\chi_{\{u_1^{d/2}>0\}\cup\{u_2^{d/2}>0\}} - \chi_{\{u_1^d>0\}}\big)$$

which is non negative, since, if $\varphi(x) \neq 0$, then necessarily $u_1^d(x) > 0$ by the definition of φ. We have thus proved that $\int |\nabla\varphi|^2 \leq 0$, hence $\varphi \equiv 0$, that is $2u_1^d \leq u_1^{d/2} + u_2^{d/2}$. Now, since Theorem 3.3 implies that $u_1^{d/2} \leq u_1^d$, we obtain that $u_1^d \leq u_2^{d/2}$. To prove (26), it remains to prove that $\Omega_1^d \subseteq \Omega_2^{d/2}$, but this follows from $u_1^d \leq u_2^{d/2}$ since by construction

$$\Omega_1^d := \Omega \cup \{u_1^d > 0\} \subseteq \Omega_1^{d/2} \cup \{u_1^d > 0\} \subseteq \Omega_1^{d/2} \cup \{u_2^{d/2} > 0\} =: \Omega_2^{d/2}.$$

□

Proposition 4.1. *The family of sets $\Omega^{(k)}_{\lfloor t/\tau_k\rfloor}$ is increasing (w.r.to set inclusion) in t and k. Similarly, $u^{(k)}_{\lfloor t/\tau_k\rfloor}(x)$ is increasing in t and k, for every fixed x. Finally, we have the estimates*

$$|\Omega^{(k)}_{\lfloor t/\tau_k\rfloor}| \leq |\Omega(0)|\exp(Ct), \quad t \geq 0, \tag{28}$$

where $C > 0$ depends only on f and $|\Omega(0)|$ (but not on k), and

$$u^{(k)}_{\lfloor t/\tau_k\rfloor} \in H^1_0(\Omega^{(k)}_{\lfloor t/\tau_k\rfloor}), \quad t \geq \tau_k, \tag{29}$$

$$\|u^{(k)}_{\lfloor t/\tau_k\rfloor}\|_{L_1} + \|u^{(k)}_{\lfloor t/\tau_k\rfloor}\|_{W^{1,2}} \leq C = C(f, t, |\Omega(0)|), \quad t \geq \tau_k. \tag{30}$$

Proof. The monotonicity in t is immediate, since $\Omega^{(k)}_{n-1} \subseteq \Omega^{(k)}_n$ by construction and this, using Theorem 3.2 (monotonicity of the minimizers w.r.to Ω), reveals that $u^{(k)}_{n-1}(x) \leq u^{(k)}_n(x)$.

The monotonicity in k is more involved. We have to prove that

$$u^{(k)}_{\lfloor t/\tau_k\rfloor} \leq u^{(k+1)}_{\lfloor t/\tau_{k+1}\rfloor}, \qquad \Omega^{(k)}_{\lfloor t/\tau_k\rfloor} \subseteq \Omega^{(k+1)}_{\lfloor t/\tau_{k+1}\rfloor}, \quad t \geq 0.$$

The integer part $\lfloor t/\tau_k\rfloor$ jumps by one unity when t is a multiple of τ_k, whereas $\lfloor t/\tau_{k+1}\rfloor$ jumps when t is a multiple of $\tau_k/2$. Therefore, we can restrict to consider the cases where $2t = h\tau_k$ for some integer $h \geq 0$, i.e.

$$u^{(k)}_{\lfloor h/2\rfloor} \leq u^{(k+1)}_h, \qquad \Omega^{(k)}_{\lfloor h/2\rfloor} \subseteq \Omega^{(k+1)}_h, \quad h \geq 0, \quad h \text{ integer}.$$

Separating the cases where h is even or odd, we reduce to

$$u^{(k)}_n \leq u^{(k+1)}_{2n} \quad \text{and} \quad u^{(k)}_n \leq u^{(k+1)}_{2n+1} \quad \text{for every } n \geq 0, \tag{31}$$

$$\Omega^{(k)}_n \subseteq \Omega^{(k+1)}_{2n} \quad \text{and} \quad \Omega^{(k)}_n \subseteq \Omega^{(k+1)}_{2n+1} \quad \text{for every } n \geq 0. \tag{32}$$

Due to what we have just proved, the first inequality in (31) (or inclusion in (32)) is stronger than the second one, thus we reduce to prove that

$$(\Omega_n^{(k)}, u_n^{(k)}) \prec (\Omega_{2n}^{(k+1)}, u_{2n}^{(k+1)}), \tag{33}$$

for every $k, n \geq 0$. This is trivial, by construction, for $n = 0$ and arbitrary k. Now we use induction on n, for every fixed k. For, suppose that (33) is satisfied for some n, let $d = \tau_k$ and define the state (Ω', u') by

$$(\Omega_{2n}^{(k+1)}, u_{2n}^{(k+1)}) \xrightarrow{d} (\Omega', u'). \tag{34}$$

Since by construction

$$(\Omega_n^{(k)}, u_n^{(k)}) \xrightarrow{d} (\Omega_{n+1}^{(k)}, u_{n+1}^{(k)}),$$

by (33) and Theorem (3.3) we get $u_{n+1}^{(k)} \leq u'$, and hence using (33) again

$$\Omega_{n+1}^{(k)} := \Omega_n^{(k)} \cup \{u_{n+1}^{(k)} > 0\} \subseteq \Omega_n^{(k)} \cup \{u' > 0\} \subseteq \Omega_{2n}^{(k+1)} \cup \{u' > 0\} =: \Omega'.$$

The last inclusion, combined with $u_{n+1}^{(k)} \leq u'$, gives

$$(\Omega_{n+1}^{(k)}, u_{n+1}^{(k)}) \prec (\Omega', u'). \tag{35}$$

On the other hand, by construction we have

$$(\Omega_{2n}^{(k+1)}, u_{2n}^{(k+1)}) \xrightarrow{d/2} (\Omega_{2n+1}^{(k+1)}, u_{2n+1}^{(k+1)}) \xrightarrow{d/2} (\Omega_{2n+2}^{(k+1)}, u_{2n+2}^{(k+1)})$$

and, recalling (34), Lemma 4.2 gives

$$(\Omega', u') \leq (\Omega_{2n+2}^{(k+1)}, u_{2n+2}^{(k+1)}). \tag{36}$$

On combining (35) with (36), we see that (33) is also valid with $n+1$ in place of n, thus by induction it holds for every $n \geq 0$.

To prove (28), we may suppose that $|\Omega(0)| > 0$, otherwise $u_n^{(k)} \equiv 0$, $\Omega_n^{(k)} = \Omega(0)$, and (28) is trivial. We have from (24) and (18)

$$|\Omega_n^{(k)}| = |\Omega_{n-1}^{(k)}| + |\{u_n^{(k)} > 0\} \setminus \Omega_{n-1}^{(k)}| = |\Omega_{n-1}^{(k)}| + \tau_k \int_{\Omega_{n-1}^{(k)}} f.$$

Using (8) and letting $a_n = |\Omega_n^{(k)}|$ we have

$$a_n \leq a_{n-1} + \tau_k C_f (1 + a_{n-1}) \leq a_{n-1} + \tau_k C a_{n-1}, \quad n \geq 1,$$

having chosen, for instance, $C = C_f(1 + a_0)/a_0$ (note that $a_0 = |\Omega(0)|$ does not depend on k). Therefore, by induction we have $a_n \leq (1 + C\tau_k)^n a_0$ and hence letting $n = \lfloor t/\tau_k \rfloor$ we obtain

$$|\Omega_{\lfloor t/\tau_k \rfloor}^{(k)}| = a_n \leq (1 + C\tau_k)^n \, |\Omega(0)| \leq |\Omega(0)| \exp(Cn\tau_k) \leq |\Omega(0)| \exp(Ct)$$

and (28) is proved.

To prove (29), let $n = \lfloor t/\tau_k \rfloor$. By construction, when $t \geq \tau_k$ (i.e. $n \geq$) we have

$$\Omega_n^{(k)} := \Omega_{n-1}^{(k)} \cup \{u_n^{(k)} > 0\} \supseteq \{u_n^{(k)} > 0\},$$

hence $u_n^{(k)} \equiv 0$ outside $\Omega_n^{(k)}$. Then (29) follows from Thm. 4.5 in [6] (according to Remark 3.1, we identify $u_n^{(k)} \equiv 0$ with its fine representative which is pointwise defined).

Finally, $u_n^{(k)}$ is by construction the minimizer of $F_{\tau_k}(\cdot, \Omega_{n-1}^{(k)}, f)$, and (30) follows from (19), (8), (28). $\square$

Proof of Theorem 4.1. Since $\Omega(0)$ is open, every $\Omega_n^{(k)}$ is open by Remark 3.1 hence $\Omega(t) = \bigcup_k \Omega_{\lfloor t/\tau_k \rfloor}^{(k)}$ is open and $\chi_{\Omega_{\lfloor t/\tau_k \rfloor}^{(k)}} \to \chi_{\Omega(t)}$ pointwise increasingly by Prop. 4.1 and also in $L^1(\mathbb{R}^N)$ (for fixed t) by (28) and monotone convergence. Given $t > 0$, from (29) and $\Omega_{\lfloor t/\tau_k \rfloor}^{(k)} \subseteq \Omega(t)$, we obtain that $u_{\lfloor t/\tau_k \rfloor}^{(k)} \in H_0^1(\Omega(t))$. Moreover, from Prop. 4.1 we know that $\{u_{\lfloor t/\tau_k \rfloor}^{(k)}\}$ are increasing: then using the a priori estimate (30), we obtain by monotone convergence that $u_{\lfloor t/\tau_k \rfloor}^{(k)} \to u(t, \cdot)$ strongly in $L^1 \cap L^2$ and also weakly in $H_0^1(\Omega(t))$ (since any weak limit is a fortiori equal to $u(t,\cdot)$). In particular, $u(t,\cdot) \in H_0^1(\Omega(t))$ and $u_{\lfloor t/\tau_k \rfloor}^{(k)} \to u(t,\cdot)$ weakly in $L^2_{\mathrm{loc}}([0,+\infty); W^{1,p})$, $1 \le p \le 2$, since $\Omega(t) \subseteq \Omega(T)$ for $t < T$ and $|\Omega(T) \le \exp(CT)$ by (28).

Now take $\eta \in W^{1,\infty}(\mathbb{R}^+ \times \mathbb{R}^N)$. By (24), $u_{\lfloor t/\tau_k \rfloor}^{(k)}$ satisfies the Euler equation

$$\triangle u_{\lfloor t/\tau_k \rfloor}^{(k)} = -f\chi_{\Omega_{\lfloor t/\tau_k \rfloor -1}^{(k)}} + \frac{1}{\tau_k}\chi_{\Omega_{\lfloor t/\tau_k \rfloor}^{(k)}} - \frac{1}{\tau_k}\chi_{\Omega_{\lfloor t/\tau_k \rfloor -1}^{(k)}} \tag{37}$$

for fixed $t \ge \tau_k$ and, using (16), partial integration in $\mathbb{R}^N$ gives

$$\int \nabla\eta(t,\cdot)\nabla u_{\lfloor t/\tau_k \rfloor}^{(k)} - \int f\eta(t,\cdot)\chi_{\Omega_{\lfloor t/\tau_k \rfloor -1}^{(k)}} + \\ + \frac{1}{\tau_k}\left(\int \eta(t,\cdot)\chi_{\Omega_{\lfloor t/\tau_k \rfloor}^{(k)}} - \int \eta(t,\cdot)\chi_{\Omega_{\lfloor t/\tau_k \rfloor -1}^{(k)}}\right) = 0.$$

If $0 \le t_0 < t_1$ (and k so large that $t_0 + \tau_k < t_1$), time integration on $(t_0+\tau_k, t_1)$ yields

$$\int_{t_0+\tau_k}^{t_1}\int \nabla\eta\nabla u_{\lfloor t/\tau_k \rfloor}^{(k)}\,dxdt - \int_{t_0}^{t_1-\tau_k}\int f\eta(t+\tau_k,x)\chi_{\Omega_{\lfloor t/\tau_k \rfloor}^{(k)}}(x)\,dxdt + \\ + \frac{1}{\tau_k}\int_{t_0+\tau_k}^{t_1-\tau_k}\int \left(\eta(t,x) - \eta(t+\tau_k,x)\right)\chi_{\Omega_{\lfloor t/\tau_k \rfloor}^{(k)}}(x)\,dxdt + \\ + \frac{1}{\tau_k}\left(\int_{t_1-\tau_k}^{t_1}\int \eta(t,x)\chi_{\Omega_{\lfloor t/\tau_k \rfloor}^{(k)}}(x)\,dxdt - \int_{t_0}^{t_0+\tau_k}\int \eta(t+\tau_k,x)\chi_{\Omega_{\lfloor t/\tau_k \rfloor}^{(k)}}(x)\,dxdt\right) = 0.$$

Letting $k \to \infty$, one immediately obtains (3).

It remains to prove (4). Take $t_1 > t_0 > 0$, and $\eta \in C_0^\infty(\Omega(t_0))$ independent of the t variable. Then from (3)

$$\int\limits_{\Omega(t_0)} \eta\, dx - \int\limits_{\Omega(t_1)} \eta\, dx = \int\limits_{t_0}^{t_1} \int\limits_{\Omega(t)} (\nabla u \nabla \eta - \eta f)\, dxdt.$$

Since $\operatorname{supp} \eta \subset \Omega(t_0) \subseteq \Omega(t_1)$, the left hand side is zero. Dividing by $t_1 - t_0$ and letting $t_1 \downarrow t_0$ we find (4), provided t_0 is a Lebesgue point for $\nabla u(\cdot, x)$.

Now we prove that, in fact, $u^{(k)}_{\lfloor t/\tau_k \rfloor} \to u(t,\cdot)$ strongly in $W^{1,2}$. From L^2 convergence, we get

$$\int |\nabla u(t,\cdot)|^2 \leq \liminf_k \int |\nabla u^{(k)}_{\lfloor t/\tau_k \rfloor}|^2. \tag{38}$$

On the other hand, we have from (37) and (16)

$$\begin{aligned}\int |\nabla u^{(k)}_{\lfloor t/\tau_k \rfloor}|^2 &= \int\limits_{\Omega^{(k)}_{\lfloor t/\tau_k \rfloor -1}} f u^{(k)}_{\lfloor t/\tau_k \rfloor} - \frac{1}{\tau_k} \int\limits_{\mathbb{R}^N \setminus \Omega^{(k)}_{\lfloor t/\tau_k \rfloor -1}} u^{(k)}_{\lfloor t/\tau_k \rfloor} \leq \int\limits_{\Omega^{(k)}_{\lfloor t/\tau_k \rfloor -1}} f u^{(k)}_{\lfloor t/\tau_k \rfloor} \\ &\leq \int\limits_{\Omega^{(k)}_{\lfloor t/\tau_k \rfloor}} f u^{(k)}_{\lfloor t/\tau_k \rfloor} \leq \int\limits_{\Omega(t)} f u^{(k)}_{\lfloor t/\tau_k \rfloor} \leq \int\limits_{\Omega(t)} f u(t,\cdot) = \int\limits_{\Omega(t)} |\nabla u(t,\cdot)|^2|\end{aligned}$$

(the last equality follows from (4) and hence is valid for a.e. $t > 0$). Hence, taking the limsup in the last estimate, we see that in (38) the reverse inequality holds true with liminf replaced by limsup, at least for a.e. t. Combining this with strong L^2 convergence, we see that $\|u^{(k)}_{\lfloor t/\tau_k \rfloor}\|_{H^1} \to \|u(t,\cdot)\|_{H^1}$ and, since weak converge takes place in H^1, we deduce strong convergence in H^1.

Now let $\Omega'(t)$, $u'(t,x)$ be any weak solution, fix $\tau \in (0,1]$, and let $\tau_k = \tau/2^k$, where $k \geq 0$ is fixed. Letting

$$v_n(x) := \int\limits_{(n-1)\tau_k}^{n\tau_k} u'(t,x)\, dt, \quad n \geq 1, \tag{39}$$

we wish to prove that

$$\tau_k u_n^{(k)} \leq v_n \quad \text{and} \quad \chi_{\Omega^{(k)}_{(n-1)}} \leq \chi_{\Omega'((n-1)\tau)} \quad \text{a.e. in } \mathbb{R}^N \tag{40}$$

for every $n \geq 1$. Letting $\eta_j(x) = (\tau_k u_j(x) - v_j(x))^+$, by c) of Thm. 2.1, $v_j, \eta_j \in W^{1,2}$, and from (24), (16) we have

$$\tau_k \int \nabla \eta_{j+1} \nabla u^{(k)}_{j+1} = \tau_k \int\limits_{\Omega_j^{(k)}} f \eta_{j+1} - \int\limits_{\{u^{(k)}_{j+1} > 0\} \setminus \Omega_j^{(k)}} \eta_{j+1}. \tag{41}$$

Moreover, since Ω', u' satisfy a weak equation analog to (3), by f) of Theorem 2.1 η_{j+1} is a legitimate test function and, since $\partial_t \eta_{j+1} = 0$, we obtain recalling

(39)

$$\int \nabla\eta_{j+1}\nabla v_{j+1} = \int\limits_{j\tau_k}^{(j+1)\tau_k}\int \nabla\eta_{j+1}\nabla u' = \int\limits_{j\tau_k}^{(j+1)\tau_k}\int\limits_{\Omega'(t)} f\eta_{j+1} - \int\limits_{\Omega'((j+1)\tau_k)} \eta_{j+1}$$
$$+ \int\limits_{\Omega'(j\tau_k)} \eta_{j+1} \geq \tau_k \int\limits_{\Omega'(j\tau_k)} f\eta_{j+1} - \int\limits_{\Omega'((j+1)\tau_k)} \eta_{j+1} + \int\limits_{\Omega'(j\tau_k)} \eta_{j+1},$$

having used the monotonicity of $\chi_{\Omega'(t)}$ (see a) of Thm. 2.1). Subtracting the last inequality from (41), we find

$$\int |\nabla\eta_{j+1}|^2 \leq \tau_k \int f\eta_{j+1}\left(\chi_{\Omega_j^{(k)}} - \chi_{\Omega'(j\tau_k)}\right)$$
$$+\int \eta_{j+1}\left(\chi_{\Omega'((j+1)\tau_k)} - \chi_{\Omega'(j\tau_k)} - \chi_{\{u_{j+1}^{(k)}>0\}\setminus\Omega_j^{(k)}}\right).$$

Now observe that $\eta_{j+1}(x) \neq 0 \Rightarrow u_{j+1}^{(k)}(x) > 0$ and that, on the set $\{u_{j+1}^{(k)} > 0\}$, $\chi_{\{u_{j+1}^{(k)}>0\}\setminus\Omega_j^{(k)}} = 1 - \chi_{\Omega_j^{(k)}}$. Hence we can write the last estimate as

(42) $$\int |\nabla\eta_{j+1}|^2 \leq \tau_k \int f\eta_{j+1}\left(\chi_{\Omega_j^{(k)}} - \chi_{\Omega'(j\tau_k)}\right)$$
$$+ \int\limits_{\{u_{j+1}^{(k)}>0\}} \eta_{j+1}\left(\chi_{\Omega'((j+1)\tau_k)} - \chi_{\Omega'(j\tau_k)} - 1 + \chi_{\Omega_j^{(k)}}\right), \quad j \geq 0.$$

Now set $j = 0$ in (42): since $\Omega_0^{(k)} = \Omega'(0) = \Omega(0)$, we find $\int |\nabla\eta_1|^2 \leq 0$, hence $\eta_1 \equiv 0$, which means that (40) is satisfied when $n = 1$.

Now we prove (40) by induction on n. Indeed, suppose (40) holds true for some $n \geq 1$. From (39) and d) of Thm. 2.1, it follows that $v_n/\tau_k \leq u'(t,\cdot)$ a.e. in $\mathbb{R}^N$, for a.e. $t > n\tau_k$. Therefore, using (40), we find

$$\chi_{\{u_n^{(k)}>0\}} \leq \chi_{\{v_n/\tau_k>0\}} \leq \chi_{\{u'(t,\cdot)>0\}} \quad \text{for a.e. } t > n\tau_k.$$

Hence, again using (40) and e), a) of Thm. 2.1, we find

$$\chi_{\Omega_n^{(k)}} = \chi_{\Omega_{n-1}^{(k)}\cup\{u_n^{(k)}>0\}} = \max\left\{\chi_{\Omega_{n-1}^{(k)}}, \chi_{\{u_n^{(k)}>0\}}\right\} \leq \max\left\{\chi_{\Omega_{n-1}^{(k)}}, \chi_{\{u'(t,\cdot)>0\}}\right\}$$
$$\leq \max\left\{\chi_{\Omega'((n-1)\tau_k)}, \chi_{\{u'(t,\cdot)>0\}}\right\} \leq \max\left\{\chi_{\Omega'((n-1)\tau_k)}, \chi_{\Omega'(t)}\right\} = \chi_{\Omega'(t)}$$

a.e. in $\mathbb{R}^N$, for a.e. $t > n\tau_k$. Taking the limit as $t \to n\tau_k$, from c) of Thm. 2.1 we see that $\chi_{\Omega_n^{(k)}} \leq \chi_{\Omega'(n\tau_k)}$, i.e., the second inequality in (40) is satisfied with $n+1$ in place of n. To prove also the first inequality with $n+1$ in place of n, it suffices to set $j = n$ in (42) and use $\chi_{\Omega_n^{(k)}} \leq \chi_{\Omega'(n\tau_k)}$, to find that $\int |\nabla\eta_{n+1}|^2 \leq 0$.

Now fix any $t > 0$ and write $t = n\tau_k + r$, with integer n and real $r \in [0,\tau_k)$, both depending on k (note that $n \geq 1$ for large k). Using (40) and d) of Thm. 2.1 we find

$$\tau_k u_{\lfloor t/\tau_k\rfloor}^{(k)} = \tau_k u_n^{(k)} \leq v_n = \int\limits_{(n-1)\tau_k}^{n\tau_k} u'(s,x)\,ds \leq \int\limits_{t}^{t+\tau_k} u'(s,x)\,ds.$$

Dividing by τ_k and letting $k \to \infty$, we obtain that $u(t,\cdot) \le u'(t,\cdot)$ a.e. in $\mathbb{R}^{\mathrm{N}}$, for a.e. $t > 0$. Similarly, we have

$$\chi_{\Omega^{(k)}_{\lfloor t/\tau_k \rfloor}} = \chi_{\Omega^{(k)}_n} \le \chi_{\Omega'(n\tau_k)} \le \chi_{\Omega'(t)}$$

and, letting $k \to \infty$, $\chi_{\Omega(t)} \le \chi_{\Omega'(t)}$ a.e. in $\mathbb{R}^{\mathrm{N}}$, for every $t > 0$.

From this minimality property of the variational solution, it easily follows that $\Omega(t)$ and $u(t,x)$ are independent of the initial $\tau \in (0,1]$, up to null sets in space-time. □

5. Properties of the variational solution

Theorem 5.1. *Suppose $f \in L^\infty(\mathbb{R}^{\mathrm{N}})$. Then the weak solution with fixed initial set is unique (up to negligible sets in $\mathbb{R}^+ \times \mathbb{R}^{\mathrm{N}}$), and it coincides with the variational solution.*

Proof. Let $\Omega'(t)$, coupled with $u'(t,x)$, be any weak solution, and let $\Omega(t)$ (with $u(t,x)$) be the variational solution. Then $\Omega(t)$ satisfies (9) and $\Omega'(t)$ satisfies an analogous relation. Subtracting these two identities, and recalling that $\chi_{\Omega(t)} \le \chi_{\Omega'(t)}$ by Theorem 4.1, we find

$$\left\|\chi_{\Omega(t)} - \chi_{\Omega'(t)}\right\|_{L^1} = \int_0^t \left(\int_{\Omega'(s)-\Omega(s)} f \right) ds \le \|f\|_{L^\infty} \int_0^t \left\|\chi_{\Omega(s)} - \chi_{\Omega'(s)}\right\|_{L^1} ds$$

and, by the Gronwall Lemma, $\chi_{\Omega(t)} = \chi_{\Omega'(t)}$ a.e. in $\mathbb{R}^{\mathrm{N}}$. Then, subtracting (3) from the corresponding weak equation with Ω', u', all terms involving Ω, Ω' cancel and we are left with

$$0 = \int_{t_0}^{t_1} \int \nabla(u(t,x) - u'(t,x)) \nabla\eta(t,x)\, dxdt \quad \text{for every } \eta \in W^{1,\infty}.$$

Choosing $\eta \in C_0^\infty$ independent of t, we see that $v_{t_0,t_1}(x) = \int_{t_0}^{t_1} (u(t,x) - u'(t,x))\, dt$ is harmonic for arbitrary t_0, t_1. Since $v_{t_0,t_1} \in L^1$, we find $v_{t_0,t_1} \equiv 0$ and hence $u(t,\cdot) = u'(t,\cdot)$ for a.e. t. □

Remark 5.1. *For $\mathrm{N} \ge 2$, let $f(x) = ((-\log|x|)^+)^2$, and $\Omega(0) = \emptyset$. Then the Hele-Shaw flow has the trivial solution $\Omega(t) = \emptyset$, $u \equiv 0$. However, another classical solution is given by $\Omega(t) = B_{r(t)}$, where*

$$r(t) = \begin{cases} \exp\big((1 + \tan(t/\mathrm{N}^2 - \pi/2))/\mathrm{N}\big) & \text{if } 0 \le t < \mathrm{N}^2\pi/4, \\ \left(\frac{2}{\mathrm{N}^2} t + 1 - \frac{\pi}{2}\right)^{\frac{1}{\mathrm{N}}} & \text{if } t > \mathrm{N}^2\pi/4. \end{cases}$$

The corresponding function $u(t,x)$ is given by

$$u(t,x) = \begin{cases} h(r) - h(|x|) & \text{if } |x| < r = r(t) < 1, \\ h(1) - h(|x|) + \int_1^r \frac{2\rho^{1-\mathrm{N}}}{\mathrm{N}^3}\, d\rho & \text{if } |x| < 1 < r = r(t), \\ \int_{|x|}^r \frac{2\rho^{1-\mathrm{N}}}{\mathrm{N}^3}\, d\rho & \text{if } 1 < |x| < r = r(t), \\ 0 & \text{otherwise,} \end{cases}$$

where $h(z) = \frac{(z\log z)^2}{2\mathrm{N}} + \frac{\mathrm{N}^2+2\mathrm{N}+4}{4\mathrm{N}^3} z^2 - \frac{\mathrm{N}+2}{2\mathrm{N}^2} z^2 \log z$. Note that $f \in L^p$ for every p.

The following is a monotonicity result, with respect to the initial set $\Omega(0)$ and the forcing term f.

Theorem 5.2. *Let $\Omega_i(t)$, $u_i(t,x)$ be the variational solution of a Hele-Shaw flow with forcing term f_i, $i=1,2$. If $\Omega_1(0)\subseteq\Omega_2(0)$ and $f_1\leq f_2$, then $\Omega_1(t)\subseteq\Omega_2(t)$ and $u_1(t,\cdot)\leq u_2(t,\cdot)$ for every $t\geq 0$.*

Proof. It suffices to apply Theorem 3.3 at every time step, in Construction 4.1. Then, all inclusions and inequalities are kept in the limit. □

Next we prove a stability theorem, with respect to the initial set $\Omega(0)$ (in measure) and the forcing term f (in the L^∞ norm).

Theorem 5.3. *Let $\Omega_i(t)$, $u_i(t,x)$ be the variational solution of a Hele-Shaw flow with forcing term f_i, $i=1,2$. Then for $t\geq 0$*

(43)
$$\|\chi_{\Omega_1(t)}-\chi_{\Omega_2(t)}\|_{L^1}\leq\Big(\|\chi_{\Omega_1(0)}-\chi_{\Omega_2(0)}\|_{L^1}+A\,|\Omega_1(0)\cap\Omega_2(0)|\,t\Big)\exp(Bt),$$

where $A=\|f_1-f_2\|_\infty$, and $B=\max\{\|f_1\|_\infty,\|f_2\|_\infty\}$.

Proof. Let $F=\max\{f_1,f_2\}$, $f=\min\{f_1,f_2\}$, $\Omega(0)=\Omega_1(0)\cup\Omega_2(0)$, $\omega(0)=\Omega_1(0)\cap\Omega_2(0)$, and let $\Omega(t)$ (respectively, $\omega(t)$) be the variational solution of the Hele-Shaw flow with forcing term F (resp., f) and initial set $\Omega(0)$ (resp., $\omega(0)$). Then we have by Theorem 5.2

$$\omega(t)\subseteq\Omega_i(t)\subseteq\Omega(t),\quad t\geq 0,\quad i=1,2. \tag{44}$$

We have from (44)

$$\begin{aligned}\|\chi_{\Omega_1(t)}-\chi_{\Omega_2(t)}\|_{L^1}&=|\Omega_1(t)\setminus\Omega_2(t)|+|\Omega_2(t)\setminus\Omega_1(t)|\leq\\ &\leq|\Omega(t)\setminus\omega(t)|+|\Omega(t)\setminus\omega(t)|=2|\Omega(t)|-2|\omega(t)|\end{aligned} \tag{45}$$

and, letting $d(t):=2|\Omega(t)|-2|\omega(t)|$, we find from (44), (9)

$$\begin{aligned}d(t)&=2|\Omega(0)|-2|\omega(0)|+2\int_0^t\left(\int_{\Omega(s)}F-\int_{\omega(s)}f\right)ds=\\ &=d(0)+2\int_0^t\left(\int_{\omega(s)}(F-f)+\int_{\Omega(s)\setminus\omega(s)}F\right)ds\leq\\ &\leq d(0)+2A\int_0^t|\omega(s)|\,ds+B\int_0^t d(s)\,ds.\end{aligned} \tag{46}$$

On the other hand we find from (9) $|\omega(s)|\leq|\omega(0)|+B\int_0^s|\omega(z)|\,dz$, and hence $|\omega(s)|\leq|\omega(0)|\exp(Bs)$ by the Gronwall Lemma. Plugging this estimate into (46), we find

$$d(t)\leq d(0)+2A\,|\omega(0)|\frac{e^{Bt}-1}{B}+B\int_0^t d(s)\,ds.$$

Solving this Gronwall inequality one finds $d(t) \le (d(0) + 2A|\omega(0)|t)\exp(Bt)$. Observing that $d(0) = 2\|\chi_{\Omega_1(0)} - \chi_{\Omega_2(0)}\|_{L^1}$, using (45) and recalling the definition of $d(t)$ and $\omega(0)$ we obtain (43). ☐

The following theorem concerns the perimeter of $\Omega(t)$. We recall that the perimeter of a set E in $\mathbb{R}^{\mathrm{N}}$ is defined by

$$P(E) = \sup \int_E \operatorname{div} g(x)\, dx$$

with $g \in C_0^\infty(\mathbb{R}^{\mathrm{N}};\mathbb{R}^{\mathrm{N}})$ and $|g(x)| \le 1$, and that $P(E)$ coincides with the total variation of $D\chi_E$ as a BV function.

Remark 5.2. *Let E_h denote the translate of E by $h \in \mathbb{R}^{\mathrm{N}}$. Then it is well known (see [2]) that $P(E)$ is finite if and only if $\|\chi_E - \chi_{E_h}\|_{L^1} = O(|h|)$ as $h \to 0$ and, in any case,*

$$\limsup_{|h|\to 0^+} \frac{1}{|h|}\|\chi_E - \chi_{E_h}\|_{L^1} \le P(E) \le \limsup_{|h|\to 0^+} \frac{\sqrt{\mathrm{N}}}{|h|}\|\chi_E - \chi_{E_h}\|_{L^1}.$$

Theorem 5.4. *Assume $\Omega(0)$ has finite perimeter, and $f \in W^{1,\infty}(\mathbb{R}^{\mathrm{N}})$. Then*

$$P(\Omega(t)) \le \sqrt{\mathrm{N}}\Big(P(\Omega(0)) + \mathrm{Lip}(f)|\Omega(0)|t\Big)\exp(\|f\|_\infty t).$$

Proof. Choose $h \in \mathbb{R}^n$, and let $\Omega_h(t) = \Omega(t) + h$ be the translate of $\Omega(t)$. Clearly, $\Omega_h(t)$ is the variational solution of the Hele-Shaw flow with starting set $\Omega_h(0)$ and forcing term $f_h(x) = f(x-h)$. Then we have from Theorem 5.3

$$\|\chi_{\Omega_h(t)} - \chi_{\Omega(t)}\|_{L^1} \le \Big(\|\chi_{\Omega_h(0)} - \chi_{\Omega(0)}\|_{L^1} + |h|\,\mathrm{Lip}(f)|\Omega(0)|t\Big)\exp(\|f\|_\infty t).$$

Dividing by $|h|$ and letting $h \to 0$, we obtain our claim, according to Rem. 5.2. ☐

The next theorem shows that, under very reasonable assumptions on f, the variational solution tends to *fill* $\mathbb{R}^{\mathrm{N}}$ as time increases.

Theorem 5.5. *Let $\Omega(t)$ be the variational solution to a Hele-Shaw flow, and assume there exist $x_0 \in \Omega(0)$ such that $\liminf_{x\to x_0} f(x) > 0$. If $E \subset \mathbb{R}^{\mathrm{N}}$ is any bounded set, then $E \subseteq \Omega(t)$ for large t.*

Proof. We may assume that $x_0 = 0$. Our assumption means that there exist $r_0, c > 0$ such that $f \ge c\chi_{B_{r_0}}$ and $B_{r_0} \subseteq \Omega(0)$. If we consider the Hele-Shaw flow with initial set given by B_{r_0} and forcing term $c\chi_{B_{r_0}}$, one can check that the variational solution is given by a family of balls $B_{r(t)}$, where

$$r(t) = r_0(1+ct)^{\frac{1}{\mathrm{N}}}$$

(it suffices to check (9) and use Thm. 5.1). Then Thm. 5.2 gives $B_{r(t)} \subseteq \Omega(t)$ and the claim follows. ☐

Theorem 5.6. *Let $\Omega(t)$ be the variational solution to a Hele-Shaw flow, and assume that $\Omega(0)$ has finitely many connected components. Then the number of connected components of $\Omega(t)$ is non-increasing and, if there exist $x_0 \in \Omega(0)$ such that $\liminf_{x\to x_0} f(x) > 0$, then $\Omega(t)$ is connected for large t.*

Proof. Fix $\tau > 0$. With the same notation as in Construction 4.1, let $\mathcal{N}_k(t)$, $\mathcal{N}(t)$ denote the number of connected components (c.c. for brevity) of $\Omega^{(k)}_{\lfloor t/\tau_k\rfloor}$, $\Omega(t)$ (at this stage, we do not exclude that $\mathcal{N}_k(t) = +\infty$ and $\mathcal{N}(t) = +\infty$). Recalling that every $\Omega^{(k)}_n$ is open, we have:

(i) for every $n, k \geq 0$, each c.c. of $\Omega^{(k)}_n$ contains at least one c.c. of $\Omega^{(k)}_m$, for every m such that $0 \leq m \leq n$.

Indeed, suppose there is a c.c. A of $\Omega^{(k)}_{n+1}$ which contains no c.c. of $\Omega^{(k)}_n$. Since $\Omega^{(k)}_n \subseteq \Omega^{(k)}_{n+1}$, this means that $\overline{A} \cap \Omega^{(k)}_n = \emptyset$. Then (24) gives $u^{(k)}_{n+1} > 0$ in A, $\triangle u^{(k)}_{n+1} = 1/\tau_k > 0$ in A, and $u^{(k)}_{n+1} = 0$ along ∂A, which violates the maximum principle. As a consequence of (i),

(ii) $\mathcal{N}_k(t)$ is non increasing in t.

Similarly,

(iii) for every $n, k \geq 0$, each c.c. of $\Omega^{(k)}_n$ contains at least one c.c. of $\Omega^{(m)}_n$, for every m such that $0 \leq m \leq k$.

For, suppose $\Omega^{(k+1)}_{\lfloor t/\tau_{k+1}\rfloor}$ has a c.c. A which contains no c.c. of $\Omega^{(k)}_{\lfloor t/\tau_k\rfloor}$: since $\Omega^{(k)}_{\lfloor t/\tau_k\rfloor} \subseteq \Omega^{(k+1)}_{\lfloor t/\tau_{k+1}\rfloor}$ (Prop. 4.1), this means that $A \cap \Omega^{(k)}_{\lfloor t/\tau_k\rfloor} = \emptyset$. Due to (i), A must contain at least one c.c. of $\Omega(0)$, a contradiction since $\Omega(0) \subseteq \Omega^{(k)}_{\lfloor t/\tau_k\rfloor}$ and $A \cap \Omega^{(k)}_{\lfloor t/\tau_k\rfloor} = \emptyset$. As a consequence of (iii),

(iv) $\mathcal{N}_k(t)$ is non increasing in k, at fixed t.

Now we claim that

(v) $\mathcal{N}_k(t) \to \mathcal{N}(t)$ as $k \to +\infty$, for every $t \geq 0$.

Indeed, it is easy to see that the number of c.c. is a lower semicontinuous function, with respect to increasing convergence of open sets: since $\Omega(t) = \bigcup_k \Omega^{(k)}_{\lfloor t/\tau_k\rfloor}$, Prop. 4.1 implies that

$$\mathcal{N}(t) \leq \lim_k \mathcal{N}_k(t), \quad t \geq 0 \tag{47}$$

(the limit exists by (iv)). Let α denote the number of c.c. of $\Omega(0)$: by (ii), $\mathcal{N}_k(t) \leq \mathcal{N}_k(0) = \alpha$, since $\Omega^{(k)}_0 = \Omega(0)$ by construction; hence, $\mathcal{N}_k(t)$ is bounded. Now fix t and suppose that strict inequality occurs in (47): then $\mathcal{N}(t) < \mathcal{N}_k(t)$ for every k, by (iv). Since $\Omega(t) \supseteq \Omega^{(k)}_{\lfloor t/\tau_k\rfloor}$, this means that there is at least one c.c. of $\Omega(t)$ which contains at least two c.c. of infinitely many $\Omega^{(k)}_{\lfloor t/\tau_k\rfloor}$. Passing to a subsequence $\{k_j\}$ if necessary, we may assume that a given c.c. of $\Omega(t)$, say A, contains exactly a fixed number β of c.c. of $\Omega^{(k_j)}_{\lfloor t/\tau_{k_j}\rfloor}$, denoted by $A_{j,1}, \ldots, A_{j,\beta}$, with $2 \leq \beta \leq \alpha$. By (iii), we can arrange subscripts in such a way that $A_{j,i} \subseteq A_{j+1,i}$, $1 \leq i \leq \beta$. Letting $A_i = \bigcup_j A_{j,i}$, $1 \leq i \leq \beta$, the A_i's are open, nonempty and pairwise disjoint: since $\Omega(t) = \bigcup_j \Omega^{(k_j)}_{\lfloor t/\tau_{k_j} \rfloor}$, then $A = A_1 \cup \dots \cup A_\beta$, but this is a contradiction because A is connected. This proves that equality holds in (47) and (v) follows.

Now (ii) gives $\mathcal{N}_k(t_0) \geq \mathcal{N}_k(t_1)$ for $t_0 \leq t_1$ and, letting $k \to \infty$ and using (v), we find $\mathcal{N}(t_0) \geq \mathcal{N}(t_1)$, the first part of the claim.

For the last part, let E be a large open ball which shares at least one point with every c.c. of $\Omega(0)$: by Thm. 5.5, $\Omega(t) \supseteq E$ for large t and, since E is connected, at least one c.c. $A(t)$ of $\Omega(t)$ covers E, hence $A(t) \supseteq \Omega(0)$ by the choice of E. If $\Omega(t)$ has a c.c. other than $A(t)$, say $B(t)$, then $B(t) \cap \Omega(0) = \emptyset$. On the other hand, since $\Omega(t) = \bigcup \Omega^{(k)}_{\lfloor t/\tau_k \rfloor}$, $B(t)$ must contain at least one c.c. of some $\Omega^{(k)}_{\lfloor t/\tau_k \rfloor}$ which, by (i), must contain at least one c.c. of $\Omega(0)$. But this is a contradiction since $B(t) \cap \Omega(0) = \emptyset$, hence $\Omega(t)$ is connected. □

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