A PATHOLOGICAL EXAMPLE IN NONLINEAR SPECTRAL THEORY

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ABSTRACT. We construct an open set $\Omega \subset \mathbb{R}^N$ on which an eigenvalue problem for the $p-$Laplacian has not isolated first eigenvalue and the spectrum is not discrete. The same example shows that the usual Lusternik-Schnirelmann minimax construction does not exhaust the whole spectrum of this eigenvalue problem.

CONTENTS

1. INTRODUCTION

1.1. **Framework.** For an open set $\Omega \subset \mathbb{R}^N$, we pick an exponent $1 < p < \infty$ and consider the $p-$Laplace operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\, \nabla u),$$

acting on the homogeneous Sobolev space $\mathcal{D}^{1,p}_0(\Omega)$. The latter is defined as the completion of $C^\infty_0(\Omega)$ with respect to the norm

$$u \mapsto \left(\int_\Omega |\nabla u|^p\, dx\right)^{\frac{1}{p}}, \qquad \text{for } u \in C^\infty_0(\Omega).$$

The usual eigenvalue problem for the $p-$Laplace operator with homogeneous Dirichlet boundary condition is the following: find the numbers $\lambda \in \mathbb{R}$ such that the boundary value problem

$$-\Delta_p u = \lambda\, |u|^{p-2}\, u, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \tag{1.1}$$

admits a solution $u \in \mathcal{D}^{1,p}_0(\Omega) \setminus \{0\}$, see for example [5].

In this note we want to consider the following variant

$$-\Delta_p u = \lambda\, \|u\|^{p-q}_{L^q(\Omega)}\, |u|^{q-2}\, u, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \tag{1.2}$$

where $1 < q < p$. This problem has already been studied by the second author and Lamberti in [4]. At a first glance, equation (1.2) could seem a bit weird, due to the presence of the L^q norm on

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the right-hand side. We observe that this term guarantees that both sides of the equation share the same homogeneity, exactly like in the standard case (1.1).

Though the introduction of this term containing the L^q norm may looks artificial, nevertheless it is easily seen that (1.2) is a natural extension of (1.1). Indeed, eigenvalues of the $p-$Laplacian can be seen as critical points of the functional $u \mapsto \int_\Omega |\nabla u|^p\,dx$ restricted to the manifold

$$\mathcal{S}_p(\Omega) = \{u \in \mathcal{D}^{1,p}_0(\Omega) \,:\, \|u\|_{L^p(\Omega)} = 1\}.$$

In a similar fashion, eigenvalues of (1.2) correspond to critical points of the same functional, this time restricted to the manifold

$$\mathcal{S}_{p,q}(\Omega) = \{u \in \mathcal{D}^{1,p}_0(\Omega) \,:\, \|u\|_{L^q(\Omega)} = 1\}.$$

We define the $(p,q)-$*spectrum of* Ω as follows

$$\mathrm{Spec}(\Omega;p,q) = \{\lambda \in \mathbb{R} \,:\, \text{equation (1.2) admits a solution in } \mathcal{D}^{1,p}_0(\Omega)\setminus\{0\}\},$$

and we call every element of this set a $(p,q)-$*eigenvalue of* Ω.

Let us assume that the open set $\Omega \subset \mathbb{R}^N$ is such that the embedding $\mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ is compact. It is known that $\mathrm{Spec}(\Omega;p,q)$ is a closed set, see [4, Theorem 5.1]. It is not difficult to see that

$$\lambda \ge \lambda^1_{p,q}(\Omega) > 0, \qquad \text{for every } \lambda \in \mathrm{Spec}(\Omega;p,q),$$

where $\lambda^1_{p,q}(\Omega)$ is the *first* $(p,q)-$*eigenvalue of* Ω, defined by

$$\lambda^1_{p,q}(\Omega) = \min_{u \in \mathcal{S}_{p,q}(\Omega)} \int_\Omega |\nabla u|^p\,dx.$$

Moreover, it is known that $\mathrm{Spec}(\Omega;p,q)$ contains an increasing unbounded sequence of eigenvalues $\{\lambda^k_{p,q}(\Omega)\}_{k\in\mathbb{N}\setminus\{0\}}$, defined through a variational procedure analogous to the so-called *Courant minimax principle* used for the spectrum of the Laplacian.

Let us be more precise on this point. For every $k \in \mathbb{N}\setminus\{0\}$, we define

$$\Sigma^k_{p,q}(\Omega) = \Big\{A \subset \mathcal{S}_{p,q}(\Omega) \,:\, A \text{ compact and symmetric, with } \gamma(A) \ge k\Big\},$$

where $\gamma(\cdot)$ denotes the *Krasnosel'skiĭ genus* of a closed set, defined by

$$\gamma(A) = \inf\Big\{k \in \mathbb{N} \,:\, \exists \text{ a continuous odd map } \phi : A \to \mathbb{S}^{k-1}\Big\},$$

with the convention that $\gamma(A) = +\infty$, if no such an integer k exists. Then for every $k \in \mathbb{N}\setminus\{0\}$, one can define the number

$$\lambda^k_{p,q}(\Omega) = \inf_{A \in \Sigma^k_{p,q}(\Omega)} \max_{u \in A} \int_\Omega |\nabla u|^p\,dx.$$

By [4, Theorem 5.2] we have

$$\{\lambda^k_{p,q}(\Omega)\}_{k\in\mathbb{N}\setminus\{0\}} \subset \mathrm{Spec}(\Omega;p,q) \qquad \text{and} \qquad \lim_{k\to\infty} \lambda^k_{p,q}(\Omega) = +\infty.$$

We will use the notation

$$\mathrm{Spec}_{LS}(\Omega;p,q) := \{\lambda^k_{p,q}(\Omega)\}_{k\in\mathbb{N}\setminus\{0\}}$$

for the *Lusternik-Schnirelmann* $(p,q)-$*spectrum of* Ω.

We recall that when $p = q = 2$ then the Lusternik-Schnirelmann spectrum coincides with the whole spectrum of the Dirichlet-Laplacian, see for example [1, Theorem A.2]. In all the other cases, it is not known whether $\mathrm{Spec}_{LS}(\Omega;p,q)$ and $\mathrm{Spec}(\Omega;p,q)$ coincide or not.

1.2. **The content of the paper.** The humble aim of this small note is to shed some light on the relation between the two spectra. More precisely, in Theorem 3.1 below we construct an example of an open set $\mathcal{B} \subset \mathbb{R}^N$ such that for $1 < q < p$

- the embedding $\mathcal{D}^{1,p}_0(\mathcal{B}) \hookrightarrow L^q(\mathcal{B})$ is compact (the set $\mathcal{B}$ is indeed bounded);
- $\mathrm{Spec}_{LS}(\mathcal{B}; p, q) \neq \mathrm{Spec}(\mathcal{B}; p, q)$;
- $\mathrm{Spec}(\mathcal{B}; p, q)$ has (at least) countably many accumulation points.

Actually, by using the same idea, in Theorem 3.2 below we present an even worse example, i.e. an open set $\mathcal{T} \subset \mathbb{R}^N$ such that for $1 < q < p$

- the embedding $\mathcal{D}^{1,p}_0(\mathcal{T}) \hookrightarrow L^q(\mathcal{T})$ is compact;
- $\mathrm{Spec}_{LS}(\mathcal{T}; p, q) \neq \mathrm{Spec}(\mathcal{T}; p, q)$.
- $\mathrm{Spec}(\mathcal{T}; p, q)$ has (at least) countably many accumulation points;
- the first eigenvalue $\lambda^1_{p,q}(\mathcal{T})$ is *not* isolated, i.e. there exists $\{\lambda_n\}_n \subset \mathrm{Spec}(\mathcal{T}; p, q)$ such that
$$\lambda^1_{p,q}(\mathcal{T}) = \lim_{n\to\infty} \lambda_n.$$

Although we agree that our examples are quite pathological (in particular $\mathcal{T}$ could be bounded, but made of infinitely many connected components) and strongly based on the fact that $q/p < 1$, we believe them to have their own interest in abstract Critical Point Theory.

Remark 1.1 (More general index theories)**.** For simplicity, in this paper we consider the Lusternik-Schnirelmann spectrum defined by means of the Krasnosel'skiĭ genus. We recall that it is possible to define diverging sequences of eigenvalues in a similar fashion, by using another index in place of the genus. For example, one could use the $\mathbb{Z}_2-$*cohomological index* [3] or the *Lusternik-Schnirelmann Category* [6, Chapter 2]. Our examples still apply in each of these cases, since they are independent of the choice of the index.

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2. SPECTRUM OF DISCONNECTED SETS

2.1. **General eigenvalues.** For the standard eigenvalue problem (1.1), i.e. when $q = p$, it is well-known that the spectrum of a disconnected open set Ω is made of the collection of the eigenvalues of its connected components. For $1 < q < p$ this only gives a part of the spectrum, the general formula is contained in the following result.

Proposition 2.1. *Let $1 < q < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open set such that $\mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ is compact. Let us suppose that*
$$\Omega = \Omega_1 \cup \Omega_2,$$

with $\Omega_i\subset\mathbb{R}^N$ open set, such that $\mathrm{dist}(\Omega_1,\Omega_2)>0$*. Then λ is a $(p,q)-$eigenvalue of Ω if and only if it is of the form*

$$\lambda=\left[\left(\frac{\delta_1}{\lambda_1}\right)^{\frac{q}{p-q}}+\left(\frac{\delta_2}{\lambda_2}\right)^{\frac{q}{p-q}}\right]^{\frac{q-p}{q}}\qquad \textit{for some eigenvalue } \lambda_i \textit{ of } \Omega_i, \tag{2.1}$$

where the coefficients δ_1 and δ_2 are such that

$$\delta_i\in\{0,1\}\qquad \textit{and}\qquad \delta_1+\delta_2\neq 0.$$

Moreover, if we set

$$|\alpha_i|=\left(\frac{\lambda}{\lambda_i}\right)^{\frac{1}{p-q}},\qquad i=1,2,$$

each $(p,q)-$eigenfunction U of Ω corresponding to (2.1) takes the form

$$U=C\Big(\delta_1\,\alpha_1\,u_1+\delta_2\,\alpha_2\,u_2\Big), \tag{2.2}$$

where $C\in\mathbb{R}$ and $u_i\in\mathcal{D}^{1,p}_0(\Omega_i)$ is a $(p,q)-$eigenfunction of Ω_i with unitary L^q norm corresponding to λ_i, for $i=1,2$.

Proof. Let us suppose that λ is an eigenvalue and let $U\in\mathcal{D}^{1,p}_0(\Omega)$ be a corresponding eigenfunction. For simplicity, we take U with unitary L^q norm. Let us set

$$u_i=U\cdot 1_{\Omega_i}\in\mathcal{D}^{1,p}_0(\Omega_i),\qquad i=1,2,$$

then these two functions are weak solutions of

$$-\Delta_p u_i=\lambda\,|u_i|^{q-2}\,u_i,\qquad \text{in } \Omega_i,\ i=1,2.$$

We have to distinguish two situations: either both u_1 and u_2 are not identically zero; or at least one of the two identically vanishes.

In the first case, by setting $\alpha_i=\|u_i\|_{L^q(\Omega_i)}$, for $i=1,2$, we can rewrite the previous equation as

$$-\Delta_p u_i=\frac{\lambda}{\alpha_i^{p-q}}\,\|u_i\|^{p-q}_{L^q(\Omega_i)}\,|u_i|^{q-2}\,u_i,\qquad \text{in } \Omega_i,\ i=1,2,$$

which implies that $\lambda_i:=\lambda\,\alpha_i^{q-p}$ is an eigenvalue of Ω_i, $i=1,2$. By using that $\alpha_1^q+\alpha_2^q=1$, we can infer that

$$1=\alpha_1^q+\alpha_2^q=\lambda^{\frac{q}{p-q}}\,\left[\left(\frac{1}{\lambda_1}\right)^{\frac{q}{p-q}}+\left(\frac{1}{\lambda_2}\right)^{\frac{q}{p-q}}\right],$$

which implies that λ has the form (2.1), with $\delta_1=\delta_2=1$. Moreover, since $\lambda\,\alpha_i^{q-p}=\lambda_i$, this gives that the eigenfunction U has the form

$$\begin{aligned}U=u_1+u_2&=\alpha_1\,\frac{u_1}{\|u_1\|_{L^q(\Omega)}}+\alpha_2\,\frac{u_2}{\|u_2\|_{L^q(\Omega)}}\\&=\left(\frac{\lambda}{\lambda_1}\right)^{\frac{1}{p-q}}\,\frac{u_1}{\|u_1\|_{L^q(\Omega)}}+\left(\frac{\lambda}{\lambda_2}\right)^{\frac{1}{p-q}}\,\frac{u_2}{\|u_2\|_{L^q(\Omega)}},\end{aligned}$$

which is formula (2.2).

Let us now suppose that $u_2\equiv 0$, this implies that $U=u_1$ and u_1 has unitary L^q norm. This automatically gives that λ is an eigenvalue of Ω_1, i.e. we have formula (2.1) with $\delta_1=1$ and $\delta_2=0$.

Let us now suppose that λ_i is an eigenvalue of Ω_i with eigenfunction $u_i \in \mathcal{D}^{1,}_0(\Omega_i)$ normalized in L^q, for $i = 1, 2$. We first observe that we immediately get that λ_1 and λ_2 are eigenvalues of Ω, with eigenfunctions u_1 and u_2 extended by 0 on the other component.

Now we set

$$U = \beta_1\, u_1 + \beta_2\, u_2 \in \mathcal{D}^{1,p}_0(\Omega),$$

where $\beta_1, \beta_2 \geq 0$ has to be suitably chosen. By using the equations solved by u_1 and u_2 and using that these have disjoint supports, we get that

$$\begin{aligned} -\Delta_p U = -\beta_i^{p-1}\, \Delta_p u_i &= \beta_i^{p-1}\, \lambda_i\, |u_i|^{q-2}\, u_i \\ &= \beta_i^{p-q}\, \lambda_i\, |U|^{q-2}\, U, \qquad \text{in } \Omega_i, \quad i = 1, 2. \end{aligned}$$

The previous implies that if we want U to be an eigenfunction of Ω with eigenvalue λ given by formula (2.1) with $\delta_1 = \delta_2 = 1$, we need to choose β_1, β_2 in such a way that

$$\beta_1^{p-q}\, \lambda_1 = \lambda\, \|U\|^{p-q}_{L^q(\Omega)} = \beta_2^{p-q}\, \lambda_2.$$

Since we have

$$\|U\|^{p-q}_{L^q(\Omega)} = (\beta_1^q + \beta_2^q)^{\frac{p-q}{q}},$$

this is equivalent to require that

$$\beta_1^{p-q}\, \lambda_1 = \left[\left(\frac{1}{\lambda_1}\right)^{\frac{q}{p-q}} + \left(\frac{1}{\lambda_2}\right)^{\frac{q}{p-q}}\right]^{\frac{q-p}{q}} (\beta_1^q + \beta_2^q)^{\frac{p-q}{q}}, \tag{2.3}$$

and

$$\beta_2^{p-q}\, \lambda_2 = \left[\left(\frac{1}{\lambda_1}\right)^{\frac{q}{p-q}} + \left(\frac{1}{\lambda_2}\right)^{\frac{q}{p-q}}\right]^{\frac{q-p}{q}} (\beta_1^q + \beta_2^q)^{\frac{p-q}{q}}. \tag{2.4}$$

If we now impose that U has unitary L^q norm, we now get that U must be of the form (2.2), in the case $\delta_1 = \delta_2 = 1$. □

We can iterate the previous result and get the following

Corollary 2.2. *Let $1 < q < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open set such that $\mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ is compact. Let us suppose that*

$$\Omega = \bigcup_{i=1}^{\#} \Omega_i,$$

with $\Omega_i \subset \mathbb{R}^N$ open set, such that $\mathrm{dist}(\Omega_i, \Omega_j) > 0$, for $i \neq j$. Then λ is a $(p,q)-$eigenvalue of Ω if and only if it is of the form

$$\lambda = \left[\sum_{i=1}^{\#} \left(\frac{\delta_i}{\lambda_i}\right)^{\frac{q}{p-q}}\right]^{\frac{q-p}{q}} \qquad \textit{for some } (p,q)-\textit{eigenvalue } \lambda_i \textit{ of } \Omega_i, \tag{2.5}$$

where the coefficients δ_i are such that

$$\delta_i \in \{0, 1\} \qquad \textit{and} \qquad \sum_{i=1}^{\#} \delta_i \neq 0.$$

Moreover, if we set

$$|\alpha_i|=\left(\frac{\lambda}{\lambda_i}\right)^{\frac{1}{p-q}},$$

each corresponding $(p,q)-$*eigenfunction* U *of* Ω *has the form*

$$U=C\,\left(\sum_{i=1}^{\#}\delta_i\,\alpha_i\,u_i\right), \tag{2.6}$$

where $C\in\mathbb{R}$ *and* $u_i\in\mathcal{D}^{1,p}_0(\Omega)$ *is* $(p,q)-$*eigenfunction of* Ω_i *with unitary* L^q *norm corresponding to* λ_i*.*

2.2. **The first eigenvalue.** Thanks to the formula of Proposition 2.1, we can now compute the first $(p,q)-$eigenvalue of a disconnected set. For ease of readability, we start as before with the case of two connected components.

Corollary 2.3. *Let* $1<q<p<\infty$ *and let* $\Omega\subset\mathbb{R}^N$ *be an open set such that* $\mathcal{D}^{1,p}_0(\Omega)\hookrightarrow L^q(\Omega)$ *is compact. Let us suppose that*

$$\Omega=\Omega_1\cup\Omega_2,$$

with $\Omega_i\subset\mathbb{R}^N$ *open connected set, such that* $\operatorname{dist}(\Omega_1,\Omega_2)>0$*. Then we have*

$$\lambda^1_{p,q}(\Omega)=\left[\left(\frac{1}{\lambda^1_{p,q}(\Omega_1)}\right)^{\frac{q}{p-q}}+\left(\frac{1}{\lambda^1_{p,q}(\Omega_2)}\right)^{\frac{q}{p-q}}\right]^{\frac{q-p}{q}}. \tag{2.7}$$

Moreover, each first $(p,q)-$*eigenfunction of* Ω *with unitary* L^q *norm has the form*

$$\alpha_1\,u_1+\alpha_2\,u_2,\qquad \text{where } |\alpha_i|=\left(\frac{\lambda^1_{p,q}(\Omega)}{\lambda^1_{p,q}(\Omega_i)}\right)^{\frac{1}{p-q}}, \tag{2.8}$$

and $u_i\in\mathcal{D}^{1,p}_0(\Omega)$ *is the first positive* $(p,q)-$*eigenfunction of* Ω_i *with unitary* L^q *norm, for* $i=1,2$*.*

Proof. From formula (2.1), we already know that we must have

$$\lambda^1_{p,q}(\Omega)=\left[\left(\frac{\delta_1}{\lambda_1}\right)^{\frac{q}{p-q}}+\left(\frac{\delta_2}{\lambda_2}\right)^{\frac{q}{p-q}}\right]^{\frac{q-p}{q}}\qquad\text{for some eigenvalue } \lambda_i \text{ of } \Omega_i. \tag{2.9}$$

We now observe that the function

$$\Phi(s,t)=\left[s^{\frac{q}{p-q}}+t^{\frac{q}{p-q}}\right]^{\frac{q-p}{q}},\qquad (s,t)\in\Big([0,+\infty)\times[0,+\infty)\Big)\setminus\{(0,0)\},$$

is decreasing in both variables (here we use that $q<p$). This implies that the right-hand side of (2.9) is minimal when

$$\delta_1=\delta_2=1,\qquad \lambda_1=\lambda^1_{p,q}(\Omega_1)\qquad\text{and}\qquad \lambda_2=\lambda^1_{p,q}(\Omega_2),$$

i.e. formula (2.7). The representation formula (2.8) now follows from that of Proposition 2.1. □

Remark 2.4. Under the assumptions of the previous result, we obtain in particular that $\Omega=\Omega_1\cup\Omega_2$ has exatcly 4 first $(p,q)-$eigenfunctions with unitary L^q norm, given by

$$|\alpha_1|\,u_1+|\alpha_2|\,u_2,\qquad |\alpha_1|\,u_1-|\alpha_2|\,u_2,\qquad -|\alpha_1|\,u_1+|\alpha_2|\,u_2\qquad\text{and}\qquad -\,|\alpha_1|\,u_1+|\alpha_2|\,u_2.$$

In particular, even if $\lambda^1_{p,q}(\Omega)$ is multiple in this situation, however the collection of the first eigenfunctions on $\mathcal{S}_{p,q}(\Omega)$ is a set of genus 1. This phenomenon disappears when $p = q$.

More generally, we get the following

Corollary 2.5. *Let $1 < q < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open set such that $\mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ is compact. Let us suppose that*

$$\Omega = \bigcup_{i=1}^{\#} \Omega_i,$$

with $\Omega_i \subset \mathbb{R}^N$ open set, such that $\operatorname{dist}(\Omega_i, \Omega_j) > 0$, for $i \neq j$. Then we have

$$\lambda^1_{p,q}(\Omega) = \left[\sum_{i=1}^{\#} \left(\frac{1}{\lambda^1_{p,q}(\Omega_i)}\right)^{\frac{q}{p-q}}\right]^{\frac{q-p}{q}}. \tag{2.10}$$

Moreover, each corresponding first $(p, q)-$eigenfunction of Ω with unitary L^q norm has the form

$$\sum_{i=1}^{\#} \alpha_i\, u_i, \qquad \textit{where } |\alpha_i| = \left(\frac{\lambda^1_{p,q}(\Omega)}{\lambda^1_{p,q}(\Omega_i)}\right)^{\frac{1}{p-q}}, \tag{2.11}$$

and $u_i \in \mathcal{D}^{1,p}_0(\Omega)$ is a first $(p, q)-$eigenfunction of Ω_i with unitary L^q norm corresponding to λ_i.

3. Construction of the examples

We are now ready for the main results of this note.

Theorem 3.1. *Let $1 < q < p < \infty$ and $0 < r \leq R$, we take the disjoint union of balls*

$$\mathcal{B} = B_R(x_0) \cup B_r(y_0), \qquad \textit{with } |x_0 - y_0| > R + r.$$

Then

$$\operatorname{Spec}_{LS}(\mathcal{B}; p, q) \neq \operatorname{Spec}(\mathcal{B}; p, q). \tag{3.1}$$

Moreover, the set $\operatorname{Spec}(\mathcal{B}; p, q)$ has (at least) countably many accumulation points.

Proof. We observe that for every $k \geq 2$ there exists a sequence $\{\lambda_{n,k}\}_{n\in\mathbb{N}} \subset \operatorname{Spec}(\mathcal{B}; p, q)$ such that

$$\lambda^k_{p,q}(B_R(x_0)) = \lim_{n\to\infty} \lambda_{n,k}. \tag{3.2}$$

Namely,

$$\lambda_{n,k} = \left[\left(\frac{1}{\lambda^k_{p,q}(B_R(x_0))}\right)^{\frac{q}{p-q}} + \left(\frac{1}{\lambda^n_{p,q}(B_r(y_0))}\right)^{\frac{q}{p-q}}\right]^{\frac{q-p}{q}}$$

is a $(p, q)-$eigenvalue of $\mathcal{B}$ for all $n \geq 1$, thanks to formula (2.1), and we have that

$$\lim_{n\to\infty} \lambda^n_{p,q}(B_r(y_0)) = +\infty.$$

From (3.2) we immediately deduce the second part of the statement, since $\lambda^k_{p,q}(B_R(x_0))$ belongs to $\operatorname{Spec}(\mathcal{B}; p, q)$ by formula (2.1). Moreover, (3.2) implies (3.1) as well. Indeed, if the two spectra were the same then

$$\operatorname{Spec}(\mathcal{B}; p, q) = \{\lambda^k_{p,q}(\mathcal{B})\}_{k\in\mathbb{N}\setminus\{0\}}$$

would be an increasing sequence diverging to $+\infty$, with (infinitely many) accumulation points, which is impossibile. □

We can refine the previous construction and obtain that for our eigenvalue problem even the isolation of the first eigenvalue is not guaranteed, in general.

Theorem 3.2. *Let* $1<q<p$ *and let* $\{r_i\}_{i\in\mathbb{N}}\subset\mathbb{R}$ *be a sequence of strictly positive numbers, such that*

$$\sum_{i=0}^{\infty} r_i^{\frac{p\,q}{p-q}+N}<+\infty. \tag{3.3}$$

We then define the sequence of points $\{x_i\}_{i\in\mathbb{N}}\subset\mathbb{R}^N$ *by*

$$\left\{\begin{array}{rcl} x_0 & = & (0,\dots,0),\\ x_{i+1} & = & (2^{-i}+r_i+r_{i+1},0,\dots,0)+x_i, \end{array}\right.$$

and the disjoint union of balls

$$\mathcal{T}=\bigcup_{i=0}^{\infty} B_{r_i}(x_i). \tag{3.4}$$

Then

$$\mathrm{Spec}_{LS}(\mathcal{T};p,q)\neq \mathrm{Spec}(\mathcal{T};p,q).$$

and the set $\mathrm{Spec}(\mathcal{T};p,q)$ *has (at least) countably many accumulation points. Moreover, the first eigenvalue* $\lambda^1_{p,q}(\mathcal{T})$ *is not isolated.*

Proof. We first observe that the condition (3.3) guarantess compactness of $\mathcal{D}^{1,p}_0(\mathcal{T})\hookrightarrow L^q(\mathcal{T})$, see [2, Theorem 1.2 & Example 5.2]. The first statement follows as in the previous theorem.

In order to prove that $\lambda^1_{p,q}(\mathcal{T})$ is an accumulation point of the spectrum, we can now use Corollaries 2.5 and 2.2 to construct a sequence of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ such that λ_n converges to $\lambda^1_{p,q}(\mathcal{T})$. We just set

$$\lambda_n=\left[\sum_{i=1}^{n}\left(\frac{1}{\lambda^1_{p,q}(B_{r_i}(x_i))}\right)^{\frac{q}{p-q}}\right]^{\frac{q-p}{q}}.$$

This gives the desired sequence. □

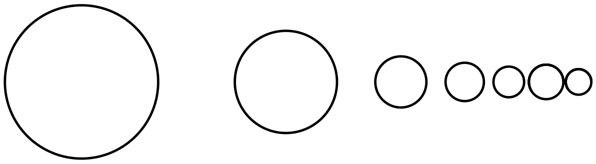

FIGURE 1. The set $\mathcal{T}$ is a disjoint union of countably many shrinking balls.

Remark 3.3. The examples above are given in terms of disjoint unions of balls just for simplicity. Actually, they still work with disjoint unions of more general bounded sets.

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