A non-local approximation of free discontinuity problems in SBV and SBD

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Abstract. We consider a class of functionals which are defined in the spaces SBV and SBD and which do not depend on the traces $u^\pm$ on the set of discontinuity points. In this work we prove that it is possible to approximate these energies, in the sense of Γ-convergence, by means of a family of non-local functionals defined in Sobolev spaces. Moreover we illustrate some applications for image processing and mechanics.

1 Introduction

There are many problems for which it is worth using a non-local approach. Often non-local operators are employed for their regularization effect. This property makes them particularly suitable dealing with problems whose solutions may have sharp gradients or discontinuities.

In mechanics non-local models have been introduced originally by Eringen [22] in the framework of elasticity theory and later were extensively applied for damage [33] (recently [17] and [7]), plasticity [23] and in general for those models where strain localizations or softening effects generate a loss of coerciveness (the interested reader can find in [26] an overview on non-local models in mechanics).

For partial differential equations a remarkable problem, where a non-local approach has been employed, is the Perona-Malik equation. It is well known that this equation is ill posed because it presents a backward diffusion effect. In order to prevent discontinuous solutions, preserving at the same time the qualitative behaviour, Cattè et al. proposed in [13] a possible regularization, based on a Gaussian convolution, which makes the problem to be well posed in Sobolev spaces (i.e. ensuring existence and uniqueness of the solution).

In this work we will focus our attention on the non-local approach for the approximation of free discontinuity problems by means of functionals defined in Sobolev spaces. The first result in this direction is the one of Braides and Dal Maso [11]. They considered non-local functionals of the form

$$\mathcal{F}_\varepsilon(u) = \int_\Omega f_\varepsilon \left(\fint_{B_\varepsilon(x)} |\nabla u(y)|^2 \, dy \right) dx \,, \tag{1}$$

where $B_\varepsilon(x)$ is the ball with center x and radius ε. Here $f_\varepsilon(t)$ denotes the rescaling $f(\varepsilon t)/\varepsilon$ of a function $f : [0,+\infty) \to [0,+\infty)$ which is non-decreasing, continuous and such that

$$\lim_{t\to 0^+} \frac{f(t)}{t} = 1 \qquad \lim_{t\to+\infty} f(t) = 1 \,. \tag{2}$$

They proved that the family $\mathcal{F}_\varepsilon$ Γ-converges (as $\varepsilon \to 0$) to the Mumford-Shah functional

$$\mathcal{F}(u) = \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(S_u) \, . \tag{3}$$

Later this result has been generalized by Cortesani and Toader [20] for more general non-local operators, using functionals of the form

$$\mathcal{F}_\varepsilon(u) = \int_\Omega f_\varepsilon \Big(\int_{\varepsilon K} |\nabla u(y)|^p \rho_\varepsilon(x-y) \, dy \Big) \, dx \, , \tag{4}$$

where ρ is a convolution kernel with support K while $\rho_\varepsilon(\xi)$ is the usual sequence of convolution kernels $\rho(\xi/\varepsilon)/\varepsilon^n$. They proved that these energies Γ-converge to the functional

$$\mathcal{F}(u) = \int_\Omega |\nabla u|^p \, dx + \int_{S_u} \phi_\rho(\nu) \, d\mathcal{H}^{n-1} \, , \tag{5}$$

where ϕ_ρ represents an anisotropy term depending on the geometry of the support K.

Note that in this case the non-local approach regularizes the functionals $\mathcal{F}_\varepsilon$, indeed by Fatou's lemma it is easy to check that the energies (1) and (4) turn out to be lower semicontinuous, with respect to the weak topology of $W^{1,p}(\Omega)$, without any convexity assumption on f, which would be required for local energies and which is not compatible with (2).

Another approach, based on a different type of non-locality, is the one of Gobbino [24]. Under similar assumptions on the integrand f he considers functionals of the form

$$\mathcal{F}_\varepsilon(u) = \int_\Omega \left(\int_{\mathbf{R}^n} f_\varepsilon \left(\frac{|u(x+\varepsilon\,\xi) - u(x)|^2}{|\varepsilon\,\xi|^2} \right) \rho_\varepsilon(\xi) \, d\xi \right) \, dx \tag{6}$$

(where the convolution term is outside the integrand f_ε) and proves that this family Γ-converges to the Mumford-Shah functional as well. (In the same spirit is a prior result for finite differences [14] due to Chambolle).

Here we restrict our attention to energies like (4) with the same conditions on the convolution kernel and more general hypotheses on the functions f_ε. We assume that $f_\varepsilon(t)$ is of the form $f(a_\varepsilon t)/a_\varepsilon$ (for a suitable choice of $a_\varepsilon > 0$) where $f : [0, +\infty) \to [0, +\infty)$ is non-decreasing, differentiable and such that

$$\lim_{t \to 0^+} \frac{f(t)}{t} = 1 \qquad \lim_{t \to +\infty} \frac{t\, f'(t)}{f(t)} = 0 \, . \tag{7}$$

We prove that the corresponding family $\mathcal{F}_\varepsilon$, defined as in (4), Γ-converges to the anisotropic functional (5). Our proof is different from the ones of [11] and [20]. In particular the lower estimate, which is the more delicate, is proved using the slicing technique, which is typical for functionals like (6). In this way we can by pass some technical lemmas on Sobolev functions and we are able to extend the lower estimate to the case of functionals like

$$\mathcal{G}_\varepsilon(u) = \int_\Omega f_\varepsilon \Big(\int_{\varepsilon K} |Eu(y)|^p \rho_\varepsilon(x-y) \, dy \Big) \, dx \tag{8}$$

where $Eu = (Du + Du^T)/2$ is the symmetrized gradient. Then, using a recent density result obtained by Chambolle in [15], we can give a complete approximation by Γ-convergence of the Griffith energy

$$\mathcal{G}(u) = \int_\Omega |\mathcal{E}u|^p \, dx + \int_{J_u} \phi_\rho(\nu) \, d\mathcal{H}^{n-1} \tag{9}$$

which is used for fracture propagation in brittle materials.

As for the Mumford-Shah functional a non-local approximation like (6) is also possible. Indeed Alicandro, Focardi and Gelli proved in [2] that a family of the form

$$\mathcal{G}_\varepsilon(u) = \int_\Omega \left(\int_{\mathbf{R}^n} f_\varepsilon \left(\frac{|\langle u(x+\varepsilon\,\xi) - u(x), \xi\rangle|}{|\varepsilon\,\xi|^2} + \frac{|\langle u(x-\varepsilon\,\xi) - u(x), \xi\rangle|}{|\varepsilon\,\xi|^2} \right) \rho_\varepsilon(\xi)\, d\xi \right) dx \quad (10)$$

Γ-converges to a Griffith energy as well.

For the sake of completeness we mention briefly some approximations for energies depending on the traces $u^\pm$. A general result has been proved in [18] using bounded integrands and by means of an integral representation theorem, which gives implicitly the surface energy in terms of a minimization problem. An explicit form of the limit functional has been given in [12] in the case $p = 2$ and $\Omega \subset \mathbf{R}$ and recently in [27] in the case $p = 1$ by means of a blow-up argument. For functionals like (6), as it is more natural to apply the slicing technique, it is "easier" to give explicit characterizations of the limit functionals. In this directions a general convergence result in contained in [25].

The non-local functional which are considered in this work have a couple of interesting applications, which are briefly explained in §8 and which motivated this investigation. The first concerns a relation between a non-local regularization of the Perona-Malik equation and the Mumford-Shah, while the second shows how a non-local damage model for concrete is related to the Griffith (local) energy. Moreover part of this work can be applied also for non-local functionals defined in finite element spaces. In particular it is possible to improve an approximation of the Griffith energy [30] giving a more complete theoretical background for the numerical results contained in [9], this result is going to appear in a forthcoming paper.

2 Notations and preliminaries

This section recalls briefly the main properties of the functional spaces which are used in this work. For a comprehensive treatise we refer to [5] and [4]. We will always assume that $\Omega \subset \mathbf{R}^n$ is an open bounded set with Lipschitz boundary.

2.1 The space $SBV(\Omega)$

The space $SBV(\Omega)$ is the set of functions $u \in BV(\Omega)$ such that the Cantor part $D^c u$ of the distributional derivative Du vanishes. Thus we will write

$$Du = \nabla u\, \mathcal{L}^n + (u^+ - u^-)\nu_u\, \mathcal{H}^{n-1} \llcorner S_u\,.$$

In particular when $\nabla u \in L^p(\Omega, \mathbf{R}^n)$ for $p > 1$ and $\mathcal{H}^{n-1}(S_u) < +\infty$ we will say that $u \in SBV^p(\Omega)$.

In the proof of the Γ-liminf we will need the slicing properties for the functions in $SBV(\Omega)$. For $\xi \in \boldsymbol{S}^{n-1}$ let $\xi^\perp = \{x \in \mathbf{R}^n \,:\, \langle x, \xi\rangle = 0\}$ and $\langle \xi \rangle = \{x \in \mathbf{R}^n \,:\, x = t\xi \text{ for } t \in \mathbf{R}\}$. Then for $\xi \in \boldsymbol{S}^{n-1}$ and $z \in \xi^\perp$ we define

$$\Omega_{z,\xi} = \{t \in \mathbf{R} \,:\, (z + t\xi) \in \Omega\}$$

and for $u : \Omega \to \mathbf{R}$ we denote by $u_{z,\xi} : \Omega_{z,\xi} \to \mathbf{R}$ the function

$$u_{z,\xi}(t) = u(z + t\xi)\,.$$

Proposition 2.1 *Let $u \in SBV(\Omega)$. For $\xi \in \boldsymbol{S}^{n-1}$ and a.e. $z \in \xi^\perp$ the function $u_{z,\xi}$ belongs to $SBV(\Omega_{z,\xi})$. Moreover*

$$\langle \nabla u(z+t\xi), \xi \rangle = u'_{z,\xi}(t) \qquad \textit{for a.e.}\, t \in \Omega_{z,\xi}$$

$$S_{u_{z,\xi}} = (S_u)_{z,\xi} = \{t \in \mathbf{R} : (z+t\xi) \in S_u\}$$

$$\int_{\xi^\perp} \Big(\#(S_u)_{z,\xi}\Big)\, d\mathcal{H}^{n-1}(z) = \int_{S_u} |\langle \nu_u, \xi \rangle| d\mathcal{H}^{n-1}\,.$$

Conversely, for $u \in L^1(\Omega)$ let $\{\xi_1, \dots, \xi_n\}$ be a basis in $\mathbf{R}^n$ and assume that

$$u_{z,\xi_i} \in SBV(\Omega_{z,\xi_i}) \quad \textit{for a.e. } z \in \xi_i^\perp \qquad \int_{\xi_i^\perp} |Du_{z,\xi_i}|(\Omega_{z,\xi_i})\, d\mathcal{H}^{n-1}(z) < +\infty\,,$$

then $u \in SBV(\Omega)$.

The Γ-limsup inequality is based on the following approximation result which is proved (under more general hypotheses) in [19]. We denote by $\mathcal{W}(\Omega)$ be the set of functions $u \in SBV(\Omega)$ such that S_u is a finite union of $(n-1)$-simplexes and $u \in W^{k,\infty}(\Omega \setminus S_u)$ for every k.

Proposition 2.2 *Let $p > 1$ and ϕ be a norm in $\mathbf{R}^n$. Then for every $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$ there exists a sequence $u_n \in \mathcal{W}(\Omega)$ such that $\|u_n\|_\infty \le \|u\|_\infty$ and*

$$u_n \to u \textit{ in } L^1(\Omega) \qquad \nabla u_n \to \nabla u \textit{ in } L^p(\Omega, \mathbf{R}^n)$$

$$\lim_{n\to+\infty} \int_{S_{u_n}} \phi(\nu)\, d\mathcal{H}^1 \le \int_{S_u} \phi(\nu)\, d\mathcal{H}^1\,.$$

2.2 The space $SBD(\Omega)$

Let $\Omega \subset \mathbf{R}^n$. The space $SBD(\Omega)$ is the set of functions $u \in BD(\Omega)$ such that the Cantor part $E^c u$ of the symmetrized distributional derivative $Eu = \frac{1}{2}(Du + Du^T)$ vanishes. So we will write

$$Eu = \mathcal{E}u\, \mathcal{L}^n + (u^+ - u^-) \odot \nu_u\, \mathcal{H}^{n-1} \llcorner J_u\,.$$

In particular when $\mathcal{E}u \in L^p(\Omega, \mathbf{R}^{n\times n})$ for $p > 1$ and $\mathcal{H}^{n-1}(J_u) < +\infty$ we will say that $u \in SBD^p(\Omega)$.

The functions in $SBD(\Omega)$ have slicing properties similar to the ones of $SBV(\Omega)$. For $u : \Omega \to \mathbf{R}^n$ we denote by $u_{z,\xi} : \Omega_{z,\xi} \to \mathbf{R}$ the function

$$u_{z,\xi}(t) = \langle u(z+t\xi), \xi \rangle\,,$$

where we consider $\xi \in \boldsymbol{S}^{n-1}$ and $z \in \xi^\perp$. The following Proposition is proved in [4].

Proposition 2.3 *Let $u \in SBD(\Omega)$. For a.e. $\xi \in \boldsymbol{S}^{n-1}$ and a.e. $z \in \xi^\perp$ the function $u_{z,\xi}$ belongs to $SBV(\Omega_{z,\xi})$ and*

$$\langle \mathcal{E}u(z+t\xi)\xi, \xi \rangle = u'_{z,\xi}(t) \qquad \textit{for a.e. } t \in \Omega_{z,\xi}$$

$$J_{u_{z,\xi}} = (J_u)_{z,\xi} = \{t \in \mathbf{R} : (z+t\xi) \in J_u\}$$

$$\int_{\xi^\perp} \Big(\#(J_u)_{z,\xi}\Big)\, d\mathcal{H}^{n-1}(z) = \int_{J_u} |\langle \nu_u, \xi\rangle| d\mathcal{H}^{n-1}\,.$$

Conversely, for $u \in L^1(\Omega, \mathbf{R}^n)$ *let* $\{\xi_1, \ldots, \xi_n\}$ *be a basis in* $\mathbf{R}^n$ *and assume that for every vector* $\xi = \xi_i + \xi_j$ *we have*

$$u_{z,\xi} \in SBV(\Omega_{z,\xi}) \quad \text{for a.e. } z \in \xi^\perp \qquad \int_{\xi^\perp} |Du_{z,\xi}|(\Omega_{z,\xi})\, d\mathcal{H}^{n-1}(z) < +\infty\,,$$

then $u \in SBD(\Omega)$.

The Γ-limsup inequality is based on the following density argument. (The proof in the two dimensional isotropic case follows from Proposition 2.2 and a recent result [15] due to Chambolle. The extension to the n-dimensional case is contained in [16]. The proof for anisotropic energies is due to Dal Maso and is unpublished).

Proposition 2.4 *Let* ϕ *be a norm in* $\mathbf{R}^n$. *Then for every* $u \in SBD^2(\Omega) \cap L^\infty(\Omega, \mathbf{R}^n)$ *there exists a sequence* $u_n \in \mathcal{W}(\Omega, \mathbf{R}^n)$ *such that* $\|u_n\|_\infty \le \|u\|_\infty$ *and*

$$u_n \to u \text{ in } L^1(\Omega, \mathbf{R}^n) \qquad Eu_n \to Eu \text{ in } L^2(\Omega, \mathbf{R}^{n\times n})$$

$$\lim_{n\to+\infty} \int_{J_{u_n}} \phi(\nu)\, d\mathcal{H}^1 \le \int_{J_u} \phi(\nu)\, d\mathcal{H}^1\,.$$

2.3 The supremum of a family of measures

Proposition 2.5 *We denote by* $\mathcal{A}(\Omega)$ *the collections of open sets contained in* Ω. *Let* $\mu : \mathcal{A}(\Omega) \to [0, +\infty)$ *be super-additive on open sets with disjoint compact closures and let* λ *be a positive Borel measure in* Ω. *Let* ψ_i *be a family of positive Borel functions such that*

$$\int_A \psi_i\, d\lambda \le \mu(A) \quad \text{for all } A \in \mathcal{A}(\Omega)$$

Then

$$\int_A \sup_i \psi_i(x)\, d\lambda \le \mu(A) \quad \text{for all } A \in \mathcal{A}(\Omega)\,.$$

3 Statement of the main results

3.1 Convergence result in *SBV*

Let Ω be an open, bounded, Lipschitz set in $\mathbf{R}^n$ for $n > 1$. Let $\rho \in L^\infty(\mathbf{R}^n, [0, +\infty))$ be a convolution kernel. As in [20] we assume that ρ is Riemann integrable and that $C_\rho = \{x : \rho(x) \neq 0\}$ is bounded, convex and symmetrical, i.e. $C_\rho = -C_\rho$. Moreover for $\varepsilon > 0$ we consider the rescaled kernels $\rho_\varepsilon(x) = \rho\,(x/\varepsilon)/\varepsilon^n$. Finally, for $v \in \mathbf{R}^n$ let $\phi_\rho(v)$ be the projection of C_ρ along v, i.e. $\phi_\rho(v) = 2\sup\{\,|\langle y, v\rangle| \,:\, y \in C_\rho\}$. Note that, ϕ_ρ is 1-homogenous, $\phi_\rho(v) = \phi_\rho(-v)$ (as C_ρ is symmetrical) and ϕ_ρ is convex (as it is the supremum of a family of convex functions). Hence ϕ_ρ is norm in $\mathbf{R}^n$.

The convergence result, which is proved in §5 and §6, is the following.

Theorem 3.1 *Let $p > 1$ and let $f : [0,+\infty) \to [0,+\infty)$ be a differentiable, non-decreasing function such that*

$$\lim_{t\to 0^+} \frac{f(t)}{t} = 1 \qquad \lim_{t\to+\infty} \frac{t\, f'(t)}{f(t)} = 0\,. \tag{11}$$

Then there exists a family of rescaled functions $f_\varepsilon(t) = f(a_\varepsilon t)/a_\varepsilon$ such that the functionals

$$\mathcal{F}_\varepsilon(u) = \int_\Omega f_\varepsilon\Big(|\nabla u|^p * \rho_\varepsilon(x)\Big)\, dx \qquad \text{if } u \in W^{1,p}(\Omega) \tag{12}$$

$$\mathcal{F}_\varepsilon(u) = +\infty \qquad \text{if } u \in L^1(\Omega) \setminus W^{1,p}(\Omega)$$

Γ-converge, with respect to the strong topology of $L^1(\Omega)$, to the functional

$$\mathcal{F}(u) = \int_\Omega |\nabla u|^p\, dx + \int_{S_u} \phi_\rho(\nu)\, d\mathcal{H}^{n-1} \qquad \text{if } u \in SBV^p(\Omega) \tag{13}$$

$$\mathcal{F}(u) = +\infty \qquad \text{if } u \in L^1(\Omega) \setminus SBV^p(\Omega)\,.$$

Moreover we will prove in §7 the following compactness result.

Proposition 3.2 *Let $\{u_\varepsilon\}$ be a family in $W^{1,p}(\Omega)$ equibounded in $L^\infty(\Omega)$ and such that $\mathcal{F}_\varepsilon(u_\varepsilon) \le C < +\infty$. Then $\{u_\varepsilon\}$ is precompact in $L^1(\Omega)$ (and thus in $L^q(\Omega)$ for $1 \le q < +\infty$).*

Remark 3.3 *For the sake of simplicity in (12) we consider ∇u to be defined in the whole $\mathbf{R}^n$, assuming that $\nabla u = 0$ in $\mathbf{R}^n \setminus \Omega$, in such a way that the convolution*

$$|\nabla u|^p * \rho_\varepsilon(x) = \int_{\mathbf{R}^n} |\nabla u|^p(x-y)\rho_\varepsilon(y)\, dy$$

makes sence for every $x \in \Omega$. Alternatively (without substantial changes in the proofs) the non-local operator could be written as

$$\int_\Omega |\nabla u|^p(y)\rho_{\varepsilon,x}(y)\, dy\,,$$

where the convolution kernel $\rho_{\varepsilon,x}$ is supported in $\varepsilon C_\rho(x) \cap \Omega$. A possible choice, which is commonly employed, is

$$\rho_{\varepsilon,x}(y) = \chi_\Omega(y)\, \rho_\varepsilon(y-x)/c_{\varepsilon,x} \qquad \text{for} \qquad c_{\varepsilon,x} = \int_\Omega \rho_\varepsilon(y-x)\, dy\,.$$

From Theorem 3.1 it follows easily that the functionals

$$\mathcal{F}_\varepsilon(u) = \int_\Omega b f_\varepsilon\Big(\frac{a}{b}|\nabla u|^p * \rho_\varepsilon(x)\Big)\, dx$$

Γ-converge to

$$\mathcal{F}(u) = a\int_\Omega |\nabla u|^p\, dx + b\int_{S_u} \phi_\rho(\nu)\, d\mathcal{H}^{n-1}\,.$$

In the one dimensional setting we can clearly prove an analogous result. Assume that I is an open bounded interval and that $\rho \in L^\infty(\mathbf{R})$ is a convolution kernel whose support is a compact and symmetrical interval J_ρ. If f satisfies the assumptions of Theorem 3.1 then the functionals

$$\mathcal{F}_\varepsilon(u) = \int_I f_\varepsilon\Big(|u'|^p * \rho_\varepsilon(x)\Big)\,dx \qquad \text{if } u \in W^{1,p}(I) \tag{14}$$

$$\mathcal{F}_\varepsilon(u) = +\infty \qquad \text{if } u \in L^1(I) \setminus W^{1,p}(I)$$

Γ-converge, with respect to the strong topology of $L^1(I)$, to the functional

$$\mathcal{F}(u) = \int_I |u'|^p\,dx + |J_\rho|\,\#(S_u) \qquad \text{if } u \in SBV^p(I) \tag{15}$$

$$\mathcal{F}(u) = +\infty \qquad \text{if } u \in L^1(I) \setminus SBV^p(I)$$

(the symbol # denotes the counting measure).

In particular when f is bounded we can drop the assumption on $tf'(t)/f(t)$ (by a simple approximation argument) and we can choose the rescaling factors $a_\varepsilon = \varepsilon$, recovering the following result which is contained in [11] and [20].

Theorem 3.4 *Let $p > 1$ and let $f : [0, +\infty) \to [0, +\infty)$ be a continuous, non-decreasing function such that*

$$\lim_{t\to 0^+} \frac{f(t)}{t} = 1 \qquad \lim_{t\to+\infty} f(t) = 1\,. \tag{16}$$

Let $f_\varepsilon(t) = f(\varepsilon t)/\varepsilon$. Then the functionals

$$\mathcal{F}_\varepsilon(u) = \int_\Omega f_\varepsilon\Big(|\nabla u|^p * \rho_\varepsilon(x)\Big)\,dx \qquad \textit{if } u \in W^{1,p}(\Omega) \tag{17}$$

$$\mathcal{F}_\varepsilon(u) = +\infty \qquad \textit{if } u \in L^1(\Omega) \setminus W^{1,p}(\Omega)$$

Γ-converge, with respect to the strong topology of $L^1(\Omega)$, to the functional

$$\mathcal{F}(u) = \int_\Omega |\nabla u|^p\,dx + \int_{S_u} \phi_\rho(\nu)\,d\mathcal{H}^{n-1} \qquad \textit{if } u \in SBV^p(\Omega) \tag{18}$$

$$\mathcal{F}(u) = +\infty \qquad \textit{if } u \in L^1(\Omega) \setminus SBV^p(\Omega)\,.$$

Example 3.5 *Common choices for the function $f(t)$, which has already been used in the literature [11] and which are suggested by some applications [17], [32], [26], could be*

$$f(t) = \tanh(t) \qquad f(t) = 1 - e^{-t} \qquad f(t) = \log(1 + t)\,,$$

while typical kernels are the "characteristic functions"

$$\rho(x) = \begin{cases} |C_\rho| & \textit{for } x \in C_\rho \\ 0 & \textit{otherwise} \end{cases}$$

and the "bell shaped" functions

$$\rho(x) = \begin{cases} c_n(1 - |x|^2) & \textit{for } |x| \le 1 \\ 0 & \textit{otherwise,} \end{cases}$$

where the normalization constant c_n depends only on the dimension n.

Moreover it is clear that when C_ρ is the ball B_r then $\phi_\rho(v) = 2r|v|$ where $|v|$ is the euclidean norm. Thus we get in the limit the isotropic energy

$$\mathcal{F}(u) = \int_\Omega |\nabla u|^p\,dx + 2r\,\mathcal{H}^{n-1}(S_u)\,.$$

Remark 3.6 *Consider a function $g(t)$ satisfying the hypothesis of Theorem 3.4 and a family c_ε such that $\lim_{\varepsilon\to 0^+} c_\varepsilon/\varepsilon = 0$. Let $g_\varepsilon(t) = g(\varepsilon\, t)/\varepsilon$. As shown in [11], the convergence result of Theorem 3.4 remains valid using a family $f_\varepsilon(t)$ such that*

$$g_\varepsilon(t) \leq f_\varepsilon(t) \leq g_\varepsilon(t) + c_\varepsilon t\,. \tag{19}$$

Since the upper bound $f_\varepsilon(t) + c_\varepsilon t$ is linear for $t \gg 1$ it may seem that Theorem 3.1 can be reduced to Theorem 3.4 (and thus to [11]). Actually this is not possible because of the different rescalings. For instance, consider

$$g(t) = \min\{t, 1\} \qquad f(t) = \log(1+t)\,.$$

For the second function a suitable rescaling is given by $a_\varepsilon = -\varepsilon \log \varepsilon$ (see Example 4.6 or [28]). Then (19) can be written as

$$g(s) \leq \frac{1}{a'_\varepsilon} \log\left(1 + a'_\varepsilon s\right) \leq g(s) + c_\varepsilon s\,,$$

where $a'_\varepsilon = a_\varepsilon/\varepsilon = -\log\varepsilon$. It is not difficult to see that in this case both the inequalities are not satisfied. Indeed, the upper inequality implies $c_\varepsilon/\varepsilon \geq e^{(c_\varepsilon - 1)} \geq e^{-1}$ (and thus $\lim_{\varepsilon\to 0^+} c_\varepsilon/\varepsilon \neq 0$) while $\log\left(1 + a'_\varepsilon s\right)/a'_\varepsilon \to 0$ for every $s \geq 0$.

3.2 Convergence result in *SBD*

Using the slicing and density properties of the space $SBD(\Omega)$ the previous convergence and compactness results can be easily proved also in this framework.

Theorem 3.7 *Let $f : [0, +\infty) \to [0, +\infty)$ be a differentiable, non-decreasing function such that*

$$\lim_{t\to 0^+} \frac{f(t)}{t} = 1 \qquad \lim_{t\to+\infty} \frac{t\, f'(t)}{f(t)} = 0\,.$$

Then there exists a family of rescaled functions $f_\varepsilon(t) = f(a_\varepsilon t)/a_\varepsilon$ such that the functionals

$$\mathcal{G}_\varepsilon(u) = \int_\Omega f_\varepsilon\Big(|Eu|^2 * \rho_\varepsilon(x)\Big)\, dx \qquad \text{if } u \in W^{1,2}(\Omega, \mathbf{R}^n) \tag{20}$$

$$\mathcal{G}_\varepsilon(u) = +\infty \qquad \text{if } u \in L^\infty(\Omega, \mathbf{R}^n) \setminus W^{1,2}(\Omega, \mathbf{R}^n)$$

Γ-converge, with respect to the strong topology of $L^1(\Omega, \mathbf{R}^n)$, to the functional

$$\mathcal{G}(u) = \int_\Omega |\mathcal{E}u|^2\, dx + \int_{J_u} \phi_\rho(\nu)\, d\mathcal{H}^1 \qquad \text{if } u \in SBD^2(\Omega) \tag{21}$$

$$\mathcal{G}(u) = +\infty \qquad \text{if } u \in L^\infty(\Omega, \mathbf{R}^n) \setminus SBD^2(\Omega)\,.$$

Proposition 3.8 *Let $\{u_\varepsilon\}$ be a family in $W^{1,p}(\Omega, \mathbf{R}^n)$ equibounded in $L^\infty(\Omega, \mathbf{R}^n)$ and such that $\mathcal{G}_\varepsilon(u_\varepsilon) \leq C < +\infty$. Then $\{u_\varepsilon\}$ is precompact in $L^1(\Omega, \mathbf{R}^n)$ (and thus in $L^q(\Omega, \mathbf{R}^n)$ for $1 \leq q < +\infty$).*

4 Some technical lemmas

Lemma 4.1 *Let f be as in Theorem 3.1. Then*

(a) $\lim_{t\to 0^+} tf'(t)/f(t) = 1$ *and* $\lim_{t\to+\infty} f(t)/t^\mu = 0$ *for every* $\mu > 0$*,*

(b) *for* $p > 1$ *the function* $f(t)/t^{1/p}$ *is increasing for* $t \ll 1$ *and decreasing for* $t \gg 1$*,*

(c) *for* $\lambda > 0$ *we have* $\lim_{t\to+\infty} f(\lambda t)/f(t) = 1$*.*

Proof. **(a)** The first limit follows by (11) and Lagrange theorem.

The second will follow (by l'Hôpital's rule) from $\lim_{t\to+\infty} tf'(t)/t^\mu = 0$. By (11) we can write that for $t \gg 1$

$$0 \le \frac{f'(t)}{f(t)} \le \frac{\mu}{t}.$$

It follows that $[\log f(t)]' \ge \mu/t$ and thus for $t' \gg 1$ and $t \ge t$ we get

$$\log(f(t)/f(t')) \le \log t^\mu,$$

which gives $f(t) \le ct^\mu$. Then from (11) we have $\lim_{t\to+\infty} tf'(t)/t^\mu = 0$

(b) Writing the derivative of $f(t)/t^{1/p}$ as

$$\frac{d}{dt}\left(\frac{f(t)}{t^{1/p}}\right) = \frac{tf'(t) - f(t)/p}{t^{1+1/p}},$$

it is sufficient to prove that

$$tf'(t) - f(t)/p < 0 \quad \text{for } t \gg 1 \qquad \text{and} \qquad tf'(t) - f(t)/p > 0 \quad \text{for } t \ll 1,$$

or, equivalently, that

$$tf'(t)/f(t) < 1/p \quad \text{for } t \gg 1 \qquad \text{and} \qquad tf'(t)/f(t) > 1/p \quad \text{for } t \ll 1,$$

which follow from (11) and **(a)** as $p > 1$.

(c) By a change of variable it's not restrictive to consider the case $\lambda > 1$. We will show that

$$\lim_{t\to+\infty} \log(f(\lambda t)/f(t)) = 0.$$

By Lagrange theorem there exists $\xi \in [t, \lambda t]$ such that

$$\log(f(\lambda t)/f(t)) = \log f(\lambda t) - \log f(t) = t(\lambda - 1)f'(\xi)/f(\xi) \le \xi(\lambda - 1)\ f'(\xi)/f(\xi).$$

The required limit comes from (11) as $\log(f(\lambda t)/f(t)) \ge 0$. ∎

Lemma 4.2 *Let $p > 1$ and f be as in Theorem 3.1. Then there exists $a_\varepsilon \to 0$ in such a way that the rescaled family $f_\varepsilon(t) = f(a_\varepsilon t)/a_\varepsilon$ satisfies*

$$f_\varepsilon(t) \to t \qquad \textit{for } t \in [0, +\infty) \tag{22}$$

$$\varepsilon f_\varepsilon(t/\varepsilon^p) \to 1 \qquad \textit{for } t \in (0, +\infty). \tag{23}$$

Moreover when f satisfies the hypotheses of Theorem 3.4 we can choose $a_\varepsilon = \varepsilon$.

Proof. By (11) we can find an increasing family $z_\varepsilon \to \infty$ such that $f(z_\varepsilon)/z_\varepsilon = \varepsilon^{p-1}$. We will show that for $a_\varepsilon = z_\varepsilon \varepsilon^p$ all the required properties are satisfied. Note that for $\mu = 1/p$

$$a_\varepsilon = \varepsilon f(z_\varepsilon) = \frac{f(z_\varepsilon)}{z_\varepsilon^\mu}\varepsilon z_\varepsilon^\mu = \frac{f(z_\varepsilon)}{z_\varepsilon^\mu} a_\varepsilon^\mu \,.$$

As $(1-\mu) > 0$ and z_ε is increasing it follows by (11) that $a_\varepsilon \to 0$ and thus again by (11) that

$$f_\varepsilon(t) = \frac{t f(a_\varepsilon t)}{a_\varepsilon t} \to t \qquad \text{for } t \in [0, +\infty)\,.$$

Now, let us write

$$f_\varepsilon\Big(\frac{t}{\varepsilon^p}\Big) = \frac{\varepsilon}{a_\varepsilon} f\left(\frac{a_\varepsilon t}{\varepsilon^p}\right) = \frac{\varepsilon f(z_\varepsilon t)}{\varepsilon f(z_\varepsilon)} = \frac{f(z_\varepsilon t)}{f(z_\varepsilon)}\,.$$

Since $z_\varepsilon \to +\infty$ the limit in (23) is proved from property **(c)** in Lemma 4.1. Finally, by (11) it is easy to check that when $\lim_{t\to+\infty} f(t) = 1$ we can choose $a_\varepsilon = \varepsilon$. ■

Lemma 4.3 *Assume that f satisfies the hypotheses of Theorem 3.1. Let $a \in (0,1)$ and $c \in (0,1)$. Let q be the conjugate exponent of p. Then there exists $b_\varepsilon \nearrow +\infty$ such that*

$$\begin{cases} f_\varepsilon\,(t) \ge at & \text{if } 0 \le t \le (b_\varepsilon/a)^q \\ \\ f_\varepsilon\,(t) \ge (b_\varepsilon\, t^{1/p}) \wedge (c/\varepsilon) & \text{if } (b_\varepsilon/a)^q \le t. \end{cases} \tag{24}$$

Proof. For $b \in (0, +\infty)$ we will show that for $\varepsilon \ll 1$ we have

$$\begin{cases} f_\varepsilon\,(t) \ge at & \text{if } 0 \le t \le (b/a)^q \\ \\ f_\varepsilon\,(t) \ge (b\, t^{1/p}) \wedge (c/\varepsilon) & \text{if } (b/a)^q \le t. \end{cases} \tag{25}$$

Considering the change of variable $z = a_\varepsilon t$ the previous system of inequalities can be written as

$$\begin{cases} f(z) \ge az & \text{if } 0 \le z \le (b/a)^q a_\varepsilon \\ \\ f\,(z) \ge (b a_\varepsilon^{1/q}\, z^{1/p}) \wedge (a_\varepsilon c/\varepsilon) & \text{if } (b/a)^q a_\varepsilon \le z. \end{cases} \tag{26}$$

Let us search for the values $z > 0$ such that

$$f(z) = (b\, a_\varepsilon^{1/q})\, z^{1/p}\,.$$

Note that $(f(z)/z^{1/p}) > 0$ for $z > 0$ and (by **(b)** in Lemma 4.1) that

$$f(z)/z^{1/p} \searrow 0 \quad \text{for } z \to 0\,, \qquad\qquad f(z)/z^{1/p} \searrow 0 \quad \text{for } z \to +\infty.$$

As $(b\, a_\varepsilon^{1/q}) \to 0$ we can say that for ε sufficiently small there are two values z'_ε and z''_ε such that $z'_\varepsilon \to 0$ and $z''_\varepsilon \to +\infty$ and

$$f\left(z'_\varepsilon\right) = (b\, a_\varepsilon^{1/q})\,(z'_\varepsilon)^{1/p} \qquad\qquad f\left(z''_\varepsilon\right) = (b\, a_\varepsilon^{1/q})\,(z''_\varepsilon)^{1/p}$$

$$f\,(z) \ge (b\, a_\varepsilon^{1/q})\, z^{1/p} \qquad \Leftrightarrow \qquad \text{if } z'_\varepsilon \le z \le z''_\varepsilon. \tag{27}$$

Now we show that when ε is small we have $z'_\varepsilon \le (b/a)^q a_\varepsilon$ and $az \le f(z)$ for $z \le (b/a)^q a_\varepsilon$. Since $z'_\varepsilon \to 0$ by (11) we can write

$$az'_\varepsilon \le f(z'_\varepsilon) = (b\, a_\varepsilon^{1/q})\, (z'_\varepsilon)^{1/p} \qquad \Rightarrow \qquad z'_\varepsilon \le (b/a)^q a_\varepsilon\,,$$

and again by (11) that

$$az \le f(z) \qquad \text{for } z \le (b/a)^q a_\varepsilon.$$

For $d = (c/b)^p$ let $\tilde{z}_\varepsilon = da_\varepsilon/\varepsilon^p$. Now we will show that $\tilde{z}_\varepsilon \le z''_\varepsilon$ and $f(z) \ge a_\varepsilon c/\varepsilon$ for $\tilde{z}_\varepsilon \le z$. By (27) we can write

$$\tilde{z}_\varepsilon \le z''_\varepsilon \quad \Leftarrow \quad f(\tilde{z}_\varepsilon) \ge (b\, a_\varepsilon^{1/q})\tilde{z}_\varepsilon^{1/p} \quad \Leftarrow \quad \varepsilon f_\varepsilon(d/\varepsilon^p) \ge c\,.$$

The last inequality is satisfied for $\varepsilon \ll 1$ since $c < 1$ and $\varepsilon f_\varepsilon(d/\varepsilon^p) \to 1$ by (23). Moreover since f is non-decreasing $f_\varepsilon(d/\varepsilon^p) \ge c$ implies easily that $f(z) \ge a_\varepsilon c/\varepsilon$ for $\tilde{z}_\varepsilon \le z$.

Finally, for $a \in (0,1)$ and $c \in (0,1)$ it is not difficult to see that we can define a family $b_\varepsilon \nearrow +\infty$ in such a way that (24) holds. ■

Remark 4.4 *The function*

$$g_\varepsilon(t) = \begin{cases} at & \text{if } 0 \le t \le (b_\varepsilon/a)^q \\ \\ (b_\varepsilon\, t^{1/p}) \wedge (c/\varepsilon) & \text{if } (b_\varepsilon/a)^q \le t. \end{cases}$$

defined as in (24) is concave and it can be written as $g_\varepsilon(t) = (at) \wedge (b_\varepsilon\, t^{1/p}) \wedge (c/\varepsilon)$.

The following lemma will be useful in §6.

Lemma 4.5 *Let $C \subset \mathbf{R}^n$ be a bounded, convex and symmetrical set and let $\phi(v)$ be the projection $\phi(v) = 2\sup\{\, |\langle y, v\rangle| \,:\, y \in C\}$. Moreover let μ_ξ be the (one dimensional) measure of the slice $C_\xi = \{x \in C \,:\, x = t\xi$ for $t \in \mathbf{R}\}$. Then*

$$\phi(v) = \sup\{\mu_\xi |\langle v, \xi\rangle| \text{ for } \xi \in \boldsymbol{S}^{n-1}\}\,. \tag{28}$$

Proof. As C is compact it is clear that

$$\phi(v) = \sup\{|\langle v, x\rangle| \,:\, x \in C\} = \sup\{|\langle v, x\rangle| \,:\, x \in \partial C\}\,.$$

Since C is convex and symmetrical we can represent its boundary as

$$\partial C = \{x \in \mathbf{R}^n \,:\, x = \pm\xi\mu_\xi/2 \text{ for } \xi \in \boldsymbol{S}^{n-1}\}$$

from which follows (28). ■

Example 4.6 *For the functions of Example 3.5*

$$f(t) = \tanh(t) \qquad f(t) = 1 - e^{-t} \qquad f(t) = \log(1+t)\,,$$

it is not difficult to see that choosing respectively the rescaling factors

$$a_\varepsilon = \varepsilon \qquad a_\varepsilon = \varepsilon \qquad a_\varepsilon = -\varepsilon \log \varepsilon$$

then properties (22) and (23) are satisfied.

5 Γ-limsup inequality

In this section we will prove the following Proposition.

Proposition 5.1 *For $u \in SBV^p(\Omega)$ there exist $u_\varepsilon \in W^{1,p}(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$ and* $\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) \le \mathcal{F}(u)$.

Proof. By a standard density argument it is sufficient to consider the functions $u \in \mathcal{W}(\Omega) \cap L^\infty(\Omega)$. For $r > 0$ let $\mathcal{S}_r(C_\rho, S_u)$ be the (anisotropic) r-neighborhood of S_u with respect to C_ρ, i.e.

$$\mathcal{S}_r(C_\rho, S_u) = \bigcup_{x\in S_u} (x + r\, C_\rho)\,.$$

Since S_u is a finite union of $(n-1)$-simplexes, it is not difficult to see that

$$\lim_{r\to 0^+} |\mathcal{S}_r(C_\rho, S_u)|/r = \int_{S_u} \phi_\rho(\nu)\, d\mathcal{H}^{n-1}\,. \tag{29}$$

(The left hand side can be regarded as an anisotropic Minkowski content of S_u).

Let $d(x, S_u)$ denote the distance between x and S_u. Following [15], for $d_\varepsilon \searrow 0$ (which will be defined later) let

$$\psi_\varepsilon(x) = \begin{cases} 0 & \text{if } d(x,S_u) \le d_\varepsilon/2 \\ |2d(x,S_u)/d_\varepsilon - 1| & \text{if } d_\varepsilon/2 \le d(x,S_u) \le d_\varepsilon \\ 1 & \text{otherwise.} \end{cases}$$

We will show that (up to a suitable choice of d_ε) the family $u_\varepsilon = u\psi_\varepsilon$ satisfies all the required properties. It is easy to check that $u_\varepsilon \in W^{1,p}(\Omega)$ and $u_\varepsilon \to u$ in $L^1(\Omega)$ as $d_\varepsilon \to 0$. Note that

$$|\nabla u_\varepsilon| = \begin{cases} 0 & \text{if } d(x,S_u) \le d_\varepsilon/2 \\ |\nabla u| & \text{if } d(x,S_u) \ge d_\varepsilon\,, \end{cases}$$

while for $d_\varepsilon/2 \le d(x,S_u) \le d_\varepsilon$ we have

$$|\nabla u_\varepsilon| \le \frac{2}{d_\varepsilon}|u\nabla d(x,S_u)| + |\psi_\varepsilon \nabla u| \le \frac{2}{d_\varepsilon}|u| + |\nabla u| \le \frac{c}{d_\varepsilon}\,,$$

where c depends only on u, being $u \in L^\infty(\Omega)$ and $\nabla u \in L^\infty(\Omega, \mathbf{R}^n)$.

Let S_ε be the neighborhood $\mathcal{S}_{(d_\varepsilon+\varepsilon)}(C_\rho, S_u)$. Then for $x \in \Omega \setminus S_\varepsilon$ we have

$$|\nabla u_\varepsilon(y)|^p = |\nabla u(y)|^p \qquad \text{for every } y \in \varepsilon\, C_\rho(x),$$

while for $x \in S_\varepsilon$ we can simply write that

$$|\nabla u_\varepsilon(y)|^p \le c/d_\varepsilon^p \qquad \text{for every } y \in \varepsilon\, C_\rho(x).$$

Hence we get

$$|\nabla u_\varepsilon|^p * \rho_\varepsilon(x) \le c/d_\varepsilon^p \qquad \text{for } x \in S_\varepsilon$$
$$|\nabla u_\varepsilon|^p * \rho_\varepsilon(x) \le |\nabla u|^p * \rho_\varepsilon(x) \qquad \text{for } x \in \Omega \setminus S_\varepsilon.$$

As f_ε is non-decreasing we can write

$$\mathcal{F}_\varepsilon(u_\varepsilon) \le \int_{S_\varepsilon} f_\varepsilon\,(c/d_\varepsilon^p)\; dx + \int_{\Omega\setminus S_\varepsilon} f_\varepsilon\Big(|\nabla u|^p * \rho_\varepsilon(x)\Big)\, dx\,.$$

Now we will treat separately the two terms in (30), showing first that for a suitable choice of d_ε we have

$$\limsup_{\varepsilon\to 0^+} \int_{S_\varepsilon} f_\varepsilon\Big(c/d_\varepsilon^p\Big)\,dx = \int_{S_u} \phi_\rho(\nu)\,d\mathcal{H}^{n-1}\,. \tag{30}$$

From Lemma 4.1 we know that $\varepsilon f_\varepsilon(t/\varepsilon^p) \to 1$ for every $t \in (0,+\infty)$. Thus we can construct a sequence $t_\varepsilon \to +\infty$ such that $\varepsilon f_\varepsilon(t_\varepsilon/\varepsilon^p) \to 1$. Then for $d_\varepsilon = \varepsilon(c/t_\varepsilon)^{1/p}$ we have

$$\lim_{\varepsilon\to 0^+} (\varepsilon + d_\varepsilon) f_\varepsilon\left(c/d_\varepsilon^p\right) = \lim_{\varepsilon\to 0^+} \Big(1 + (c/t_\varepsilon)^{1/p}\Big)\,\varepsilon f_\varepsilon(t_\varepsilon/\varepsilon^p) = 1\,.$$

Now inequality (30) is easily proved, indeed by (29) we have

$$\begin{aligned}
\limsup_{\varepsilon\to 0^+} \int_{S_\varepsilon} f_\varepsilon\left(c/d_\varepsilon^p\right)\,dx &= \limsup_{\varepsilon\to 0^+} \frac{|S_\varepsilon|}{(\varepsilon + d_\varepsilon)}\,(\varepsilon + d_\varepsilon) f_\varepsilon\left(c/d_\varepsilon^p\right) \\
&= \limsup_{\varepsilon\to 0^+} \frac{|\mathcal{S}_{(d_\varepsilon+\varepsilon)}(C_\rho, S_u)|}{(\varepsilon + d_\varepsilon)} \\
&= \int_{S_u} \phi_\rho(\nu)\,d\mathcal{H}^{n-1}\,.
\end{aligned}$$

It remains to prove that

$$\limsup_{\varepsilon\to 0} \int_{\Omega\setminus S_\varepsilon} f_\varepsilon\Big(|\nabla u|^p * \rho_\varepsilon(x)\Big)\,dx \le \int_\Omega |\nabla u|^p\,dx\,.$$

Since $\nabla u \in C^\infty(\Omega\setminus S_u, \mathbf{R}^n)$ we have $|\nabla u|^p * \rho_\varepsilon \to |\nabla u|^p$ pointwise in Ω. Moreover $a_\varepsilon |\nabla u|^p * \rho_\varepsilon \to 0$ (as $a_\varepsilon \to 0$) and thus by (11) we get

$$\lim_{\varepsilon\to 0} f_\varepsilon\Big(|\nabla u|^p * \rho_\varepsilon(x)\Big) = \lim_{\varepsilon\to 0} \frac{1}{a_\varepsilon} f\Big(a_\varepsilon |\nabla u|^p * \rho_\varepsilon(x)\Big) = |\nabla u(x)|^p\,.$$

As $\nabla u \in L^\infty(\Omega \setminus S_u, \mathbf{R}^n)$, by dominated convergence we get the required inequality. ■

6 Γ-liminf inequality

The proof of the Γ-liminf inequality uses the slicing technique (for which we refer to §2 and § 4.1.1 in [10]).

6.1 A one-dimensional discrete inequality

Let $a, c \in (0,+\infty)$ and $b_\varepsilon \nearrow +\infty$. For $t \in [0,+\infty)$ we define a function $h_\varepsilon(t)$ as in Remark 4.4, namely

$$h_\varepsilon(t) = \begin{cases} at & \text{if } 0 \le t \le (b_\varepsilon/a)^q \\ \\ (b_\varepsilon\, t^{1/p}) \wedge (c/\varepsilon) & \text{if } (b_\varepsilon/a)^q \le t. \end{cases} \tag{31}$$

Let $m \in (2\mathbf{N}+1)$ and A an open interval in $\mathbf{R}$. Let $\boldsymbol{A}_\varepsilon = \{x \in \varepsilon\mathbf{Z} \,:\, x \in A\}$. For $\boldsymbol{x} \in \boldsymbol{A}_\varepsilon$ let $I_{m\varepsilon}(\boldsymbol{x}) = [\boldsymbol{x} - (m\varepsilon/2), \boldsymbol{x} + (m\varepsilon/2)]$. (In the sequel all the discrete variables will be written using bold symbols). We consider the discrete functionals

$$H_\varepsilon(u, A) = \sum_{\boldsymbol{x}\in\boldsymbol{A}_\varepsilon} \varepsilon h_\varepsilon\left(\fint_{I_{m\varepsilon}(\boldsymbol{x})} |u'|^p\right) \qquad \text{if } u \in W^{1,p}(A)\,, \tag{32}$$

$$H_\varepsilon(u,A) = +\infty \qquad \text{if } u \in L^1(A) \setminus W^{1,p}(A)\,,$$

and the limit functional

$$H(u,A) = a\int_A |u'|^p\,dx + (cm)\,\#(S_u) \qquad \text{if } u \in SBV^p(A)\,, \tag{33}$$

$$H(u) = +\infty \qquad \text{if } u \in L^1(A) \setminus SBV^p(A)\,.$$

Proposition 6.1 *Let $u_\varepsilon \to u$ in $L^1(A)$ then $\liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon, A) \geq H(u,A)$.*

Proof. Step 1. It is well known [10] that it is not restrictive to assume that $u \in L^\infty(A)$ and that $H_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty \leq C$.

First let us introduce some useful notations: let $\boldsymbol{A}_{m\varepsilon} = \{x \in m\varepsilon\mathbf{Z} \,:\, x \in A\}$ and for $n = 0,\ldots,m-1$ let $\boldsymbol{A}_{m\varepsilon,n} = \{x \in (n\varepsilon + m\varepsilon\mathbf{Z}) \,:\, x \in A\}$. Finally we introduce the auxiliary functionals

$$H^n_\varepsilon(u,A) = \sum_{\boldsymbol{x}\,\in\,\boldsymbol{A}_{m\varepsilon,n}} (m\varepsilon) h_\varepsilon\left(\fint_{I_{m\varepsilon}(\boldsymbol{x})} |u'|^p\right)$$

As $\boldsymbol{A}_\varepsilon$ is the disjoint union of the sets $\boldsymbol{A}_{m\varepsilon,n}$ for $n = 0,\ldots,m-1$, we can write

$$\begin{aligned} H_\varepsilon(u,A) &\geq \frac{1}{m}\sum_{n=0}^{m-1}\ \sum_{\boldsymbol{x}\,\in\,\boldsymbol{A}_{m\varepsilon,n}} (m\varepsilon) h_\varepsilon\left(\fint_{I_{m\varepsilon}(\boldsymbol{x})} |u'|^p\right) \\ &\geq \frac{1}{m}\sum_{n=0}^{m-1} H^n_\varepsilon(u,A)\,. \end{aligned}$$

In the last step of the proof we will show that for $u_\varepsilon \to u$ we have

$$\liminf_{\varepsilon\to 0} H^n_\varepsilon(u_\varepsilon,A) \geq a\int_A |u'|^p\,dx + mc\,\#(S_u)\,, \tag{34}$$

from which it follows the Γ-liminf inequality

$$\liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon,A) \geq \frac{1}{m}\sum_{n=0}^{m-1} \liminf_{\varepsilon\to 0} H^n_\varepsilon(u_\varepsilon,A) \geq a\int_A |u'|^p\,dx + mc\,\#(S_u) = H(u,A)\,.$$

Step 2. We define for every $\varepsilon > 0$ the sets

$$\boldsymbol{A}^\flat_{m\varepsilon,n} = \{\boldsymbol{x} \in \boldsymbol{A}_{m\varepsilon,n} \,:\, \fint_{I_{m\varepsilon}(\boldsymbol{x})} |u'|^p \leq (b_\varepsilon/a)^q\}\,,$$

$$\boldsymbol{A}^\sharp_{m\varepsilon,n} = \{\boldsymbol{x} \in \boldsymbol{A}_{m\varepsilon,n} \,:\, \fint_{I_{m\varepsilon}(\boldsymbol{x})} |u'|^p > (b_\varepsilon/a)^q\}\,.$$

For $u \in W^{1,p}(A)$ we can define a function $\tilde{u} \in SBV(A)$ in the following way: for every interval $I_{m\varepsilon}(\boldsymbol{x})$ let

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } \boldsymbol{x} \in \boldsymbol{A}^\flat_{m\varepsilon,n} \\ \\ u\left(\boldsymbol{x} - (m\varepsilon/2)\right) & \text{if } \boldsymbol{x} \in \boldsymbol{A}^\sharp_{m\varepsilon,n}. \end{cases}$$

(Note that $\tilde u$ depends on the parameters a and b_ε and $\tilde u \in SBV(A)$). It is clear that for $\boldsymbol{x} \in \boldsymbol{A}^\sharp_{m\varepsilon,n}$ the function $\tilde u$ may have a discontinuity in the point $\boldsymbol{x} + (m\varepsilon/2)$ and then we can write

$$
\begin{aligned}
(m\varepsilon)^p \fint_{I_{m\varepsilon}(\boldsymbol{x})} |u'|^p &\geq \Big(u(\boldsymbol{x} + (m\varepsilon/2)) - u(\boldsymbol{x} - (m\varepsilon/2))\Big)^p \\
&\geq \Big(\tilde u^+(\boldsymbol{x} + (m\varepsilon/2)) - \tilde u^-(\boldsymbol{x} + (m\varepsilon/2))\Big)^p .
\end{aligned}
$$

Then by (24) we have

$$
\begin{aligned}
H^n_\varepsilon(u, A) &= \sum_{\boldsymbol{x} \in \boldsymbol{A}_{m\varepsilon,n}} (m\varepsilon) h_\varepsilon \left(\fint_{I_{m\varepsilon}(\boldsymbol{x})} |u'|^p \right) \\
&\geq \sum_{\boldsymbol{x} \in \boldsymbol{A}^\flat_{m\varepsilon,n}} (m\varepsilon)\, a \left(\fint_{I_{m\varepsilon}(\boldsymbol{x})} |u'|^p \right) + \sum_{\boldsymbol{x} \in \boldsymbol{A}^\sharp_{m\varepsilon,n}} (m\varepsilon) \left[b_\varepsilon \left(\fint_{I_{m\varepsilon}(\boldsymbol{x})} |u'|^p \right)^{1/p} \wedge (c/\varepsilon) \right] \\
&\geq \sum_{\boldsymbol{x} \in \boldsymbol{A}^\flat_{m\varepsilon,n}} a \int_{I_{m\varepsilon}(\boldsymbol{x})} |u'|^p \, dx + \sum_{\boldsymbol{x} \in \boldsymbol{A}^\sharp_{m\varepsilon,n}} b_\varepsilon \left((m\varepsilon)^p \fint_{I_{m\varepsilon}(\boldsymbol{x})} |u'|^p \right)^{1/p} \wedge (mc) \\
&\geq a \int_A |\tilde u'|^p \, dx + \sum_{S_{\tilde u}} b_\varepsilon |\tilde u^+ - \tilde u^-| \wedge mc . \qquad (35)
\end{aligned}
$$

Step 3. Now we can prove (34). Note that, being $H_\varepsilon(u_\varepsilon, A)$ equibounded the same holds for $H^n_\varepsilon(u_\varepsilon, A)$. First of all we will check that for $u_\varepsilon \to u$ in $L^1(A)$ we have $\tilde u_\varepsilon \to u$ in $L^1(A)$. As u_ε is equibounded in $L^\infty(A)$ it is sufficient to estimate the measure of set where $\tilde u_\varepsilon \neq u_\varepsilon$, which is given by $(m\varepsilon)\#(\boldsymbol{A}^\sharp_{m\varepsilon,n})$. Hence

$$
C \geq H^n_\varepsilon(u_\varepsilon, A) \geq \sum_{\boldsymbol{x} \in \boldsymbol{A}^\sharp_{m\varepsilon,n}} (m\varepsilon) h_\varepsilon \left(\fint_{I_{m\varepsilon}(\boldsymbol{x})} |u'|^p \right) \geq (m\varepsilon)\#(\boldsymbol{A}^\sharp_{m\varepsilon,n}) h_\varepsilon \Big((b_\varepsilon/a)^q \Big) . \qquad (36)
$$

As $b_\varepsilon \to +\infty$ we can say that $h_\varepsilon(b^q_\varepsilon/a^q) = a^{1-q} b^q_\varepsilon \to +\infty$ and thus $(m\varepsilon)\#(\boldsymbol{A}^\sharp_{m\varepsilon,n}) \to 0$.

For $b \in (0, +\infty)$ let $(a|t|^p \wedge b|t|)^{**} = \phi_{a,b}(t)$ be the convex envelope of $(a|t|^p \wedge b|t|)$. Thus by lower-semicontinuity [8] we have

$$
\begin{aligned}
\liminf_{\varepsilon \to 0} H^n_\varepsilon(u_\varepsilon, A) &\geq \liminf_{\varepsilon \to 0} \int_A \phi_{a,b}(\tilde u'_\varepsilon) \, dx + \sum_{S_{\tilde u_\varepsilon}} b |\tilde u^+_\varepsilon - \tilde u^-_\varepsilon| \wedge c \\
&\geq \int_A \phi_{a,b}(u') \, dx + \sum_{S_u} b |u^+ - u^-| \wedge c .
\end{aligned}
$$

The required inequality follows taking the supremum for $b \nearrow +\infty$. ■

6.2 The slicing estimates

Let us introduce some notations, related to the slicing technique, which will be used in this section (see Figure 1). For $\xi \in \boldsymbol{S}^{n-1}$ we denote by $Z = \xi^\perp$ the subspace orthogonal to ξ and by $T = \langle \xi \rangle$ the subspace generated by ξ. Obviously every $x \in \mathbf{R}^n$ can be written in a unique way as $x = z + t$ for $z \in Z$ and $t \in T$. The Lebesgue measures on Z and T will be denoted by

dz and dt. Let $\{\hat{e}_i\}$ (for $i = 1, \dots, n$) be the canonical basis in $\mathbf{R}^n$ and $\{\tilde{e}_i\}$ an orthonormal basis with $\tilde{e}_n = \xi$. We define the square

$$Q_\delta^\xi = \{x \in \mathbf{R}^n \,:\, |x \cdot \tilde{e}_i| \le \delta/2 \ \text{ for } i = 1, \dots, n\}$$

and the lattice

$$\mathbf{X}_\delta^\xi = \{x \in \mathbf{R}^n \,:\, (x \cdot \tilde{e}_i) \in \delta \mathbf{Z} \ \text{ for } i = 1, \dots, n\}.$$

Finally Z_δ^ξ and T_δ^ξ will denote the projections of Q_δ^ξ (on $Z = \xi^\perp$ and $T = \langle \xi \rangle$ respectively) while $\mathbf{Z}_\delta^\xi$ and $\mathbf{T}_\delta^\xi$ will denote the projections of $\mathbf{X}_\delta^\xi$. (In the sequel all the discrete variables will be written using bold symbols).

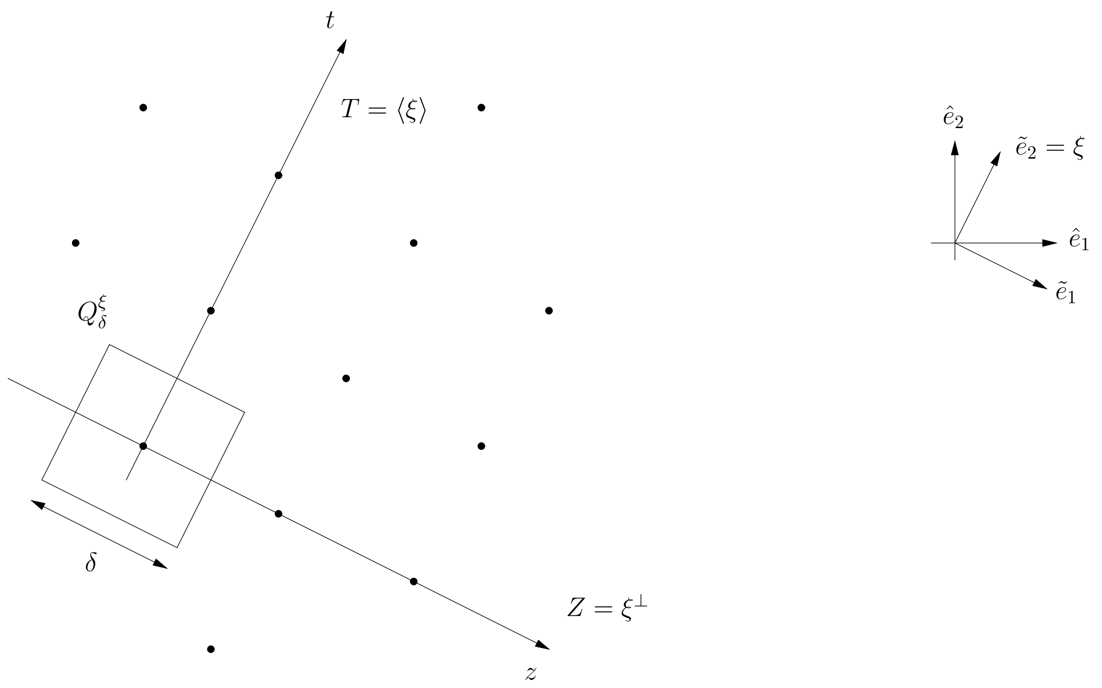

Figure 1: Some notations introduced for this section.

Let $Q_\delta^\xi(\boldsymbol{y}_i)$ (where $\boldsymbol{y}_i \in \mathbf{X}_\delta^\xi$ for $i = 1, \dots, m(\eta)$) be the squares contained in the support C_ρ. As the convolution kernel ρ is Riemann integrable, for every $\xi \in \boldsymbol{S}^{n-1}$ it possible to find a family of functions ρ_δ which are constant on the squares $Q_\delta^\xi(\boldsymbol{y}_i)$ and such that

$$0 < \rho_\delta(y) \le \rho(y) \qquad \int_{C_\rho} \rho_\delta(y)\, dy \to 1 \quad (\text{as } \delta \to 0). \tag{37}$$

Moreover, for every integrable function $g(y)$ we can write

$$⨍_{Q_\delta^\xi} g(y)\, dy = ⨍_{Z_\delta^\xi} ⨍_{T_\delta^\xi} g(z,t)\, dt\, dz\,. \tag{38}$$

Since the proof relies on Lemma 2.5 we need to define for every open set $A \subset \Omega$ the localized functional

$$F_\varepsilon(u, A) = \int_A f_\varepsilon\Big(|\nabla u|^p * \rho_\varepsilon(x)\Big)\, dx \qquad \text{if } u \in W^{1,p}(\Omega)$$

$$F_\varepsilon(u, A) = +\infty \qquad \text{if } u \in L^1(\Omega) \setminus W^{1,p}(\Omega)\,.$$

The following Lemma gives a lower bound of $F_\varepsilon(u, A)$ discretizing the energy on the lattice $\mathbf{X}^\xi_\delta$ and is a consequence of Lemma 4.2 in [11].

Lemma 6.2 *Let $u \in W^{1,p}(\Omega)$ and $A \subset \Omega$ an open set. For $\xi \in \boldsymbol{S}^{n-1}$ and $\delta > 0$ let $\boldsymbol{A}^\xi_\delta = \{\boldsymbol{x} \in \mathbf{X}^\xi_\delta : Q^\xi_\delta(\boldsymbol{x}) \subset A\}$. Then there exists $\tau \in Q^\xi_\delta$ such that*

$$F_\varepsilon(u, A) \geq \sum_{\boldsymbol{x} \in \boldsymbol{A}^\xi_\delta} \delta^n f_\varepsilon\Big(|\nabla u|^p * \rho_\varepsilon(\boldsymbol{x} + \tau)\Big)\,. \tag{39}$$

Note that for $\boldsymbol{x} \in \boldsymbol{A}^\xi_\delta$ we have $(\boldsymbol{x} + \tau) \in Q^\xi_\delta(\boldsymbol{x}) \subset A$.

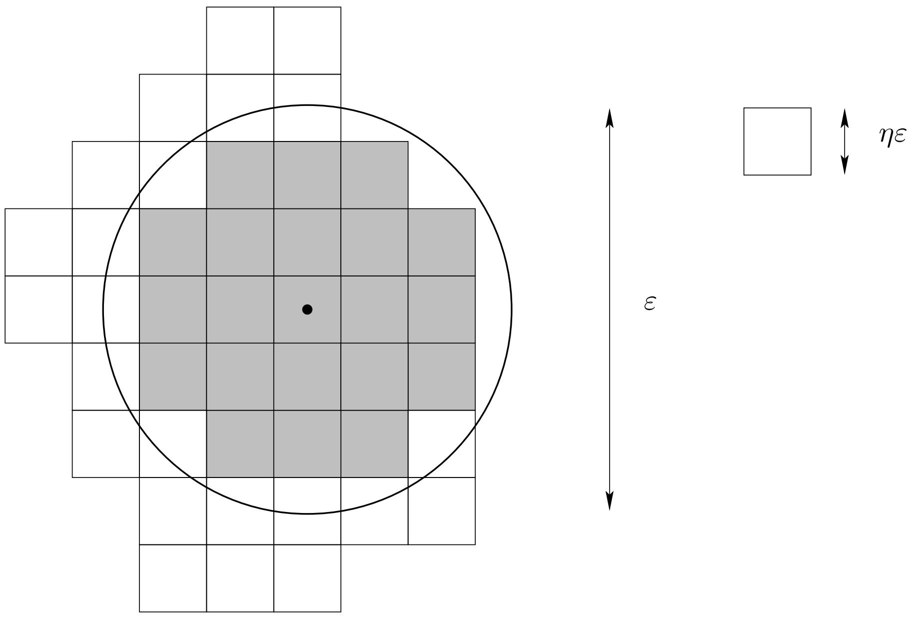

Figure 2: The set $P^\xi_{\eta\varepsilon}$ used for the bulk estimate when C is the unit ball.

Proposition 6.3 *Let $\delta \in (0, 1)$. Let $u_\varepsilon \to u$ in $L^1(\Omega)$ and let A be an open set in Ω. For $\xi \in \boldsymbol{S}^{n-1}$ there exists $\beta_\delta > 0$ such that*

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, A) \geq \int_A \delta |\nabla u(y) \cdot \xi|^p \, dy + \beta_\delta \int_{S_u \cap A} |\nu \cdot \xi| \, d\mathcal{H}^{n-1}\,. \tag{40}$$

Proof. For a value $\eta > 0$ (which will be defined later) let P^ξ_η be the union of the squares $Q^\xi_\eta(\boldsymbol{y}_i) \subset C_\rho$ where $y_i \in \mathbf{X}^\xi_\eta$ for $i = 1, \ldots, m(\eta, \xi)$. Let $\zeta = \rho_\eta$ be the piecewise constant function satisfying (37) (for $\delta = \eta$) and

$$c_i = \zeta(\boldsymbol{y}_i) \qquad c_\eta = \int_{C_\rho} \zeta(y)\, dy = \sum_{i=1}^m c_i \eta^n\,.$$

For every $\varepsilon > 0$ we will consider the lattice $\mathbf{X}^{\xi}_{\eta\varepsilon}$. We will denote by $P^{\xi}_{\eta\varepsilon}$ the union of the squares $Q^{\xi}_{\eta\varepsilon}(\boldsymbol{y}_i) \subset (\varepsilon C_\rho)$ (where $\boldsymbol{y}_i \in \mathbf{X}^{\xi}_{\eta\varepsilon}$ for $i = 1, \dots, m(\eta, \xi)$) and by ζ_ε the sequence of kernels $\zeta(y/\varepsilon)/\varepsilon^n$.

For every function u_ε let $\tau_{\eta\varepsilon}$ be the translation given by Lemma 6.2. Then

$$F_\varepsilon(u_\varepsilon, A) \geq \sum_{\boldsymbol{x} \in \boldsymbol{A}^{\xi}_{\eta\varepsilon}} (\eta\varepsilon)^n f_\varepsilon\Big(|\nabla u_\varepsilon|^p * \rho_\varepsilon(\boldsymbol{x} + \tau_{\eta\varepsilon})\Big).$$

Let $a, c \in (0, +\infty)$ such that $\delta < a = c < 1$ and let b_ε be the sequence given by Lemma (4.3). Then we define

$$g_\varepsilon(t) = \begin{cases} at & \text{if } 0 \leq t \leq (b_\varepsilon/a)^q \\ \\ (b_\varepsilon\, t^{1/p}) \wedge (c/\varepsilon) & \text{if } (b_\varepsilon/a)^q \leq t. \end{cases}$$

Then by Lemma (4.3) we can write that

$$F_\varepsilon(u_\varepsilon, A) \geq \sum_{\boldsymbol{x} \in \boldsymbol{A}^{\xi}_{\eta\varepsilon}} (\eta\varepsilon)^n g_\varepsilon\Big(|\nabla u_\varepsilon|^p * \rho_\varepsilon(\boldsymbol{x} + \tau_{\eta\varepsilon})\Big).$$

For convenience we introduce the functions $v_\varepsilon(x) = u_\varepsilon(x + \tau_{\eta\varepsilon})$. For $B \subset\subset A$ it is clear that $v_\varepsilon \in W^{1,p}(B)$ and $v_\varepsilon \to u$ in $L^1(B)$. Hence we can write

$$F_\varepsilon(u_\varepsilon, A) \geq \sum_{\boldsymbol{x} \in \mathbf{B}^{\xi}_{\eta\varepsilon}} (\eta\varepsilon)^n g_\varepsilon\Big(|\nabla v_\varepsilon|^p * \rho_\varepsilon(\boldsymbol{x})\Big),$$

where $\mathbf{B}^{\xi}_{\eta\varepsilon}$ is defined as in Lemma 6.2. Then we can write

$$\begin{aligned} |\nabla v_\varepsilon|^p * \rho_\varepsilon(\boldsymbol{x}) &= \int_{\varepsilon C_\rho} |\nabla v_\varepsilon(y - \boldsymbol{x})|^p \, \rho_\varepsilon(y)\, dy \\ &\geq \int_{P^{\xi}_{\eta\varepsilon}} |\nabla v_\varepsilon(y - \boldsymbol{x})|^p \, \zeta_\varepsilon(y)\, dy \\ &\geq \sum_{i=1}^{m} \frac{c_i}{\varepsilon^n} \int_{Q^{\xi}_{\eta\varepsilon}(\boldsymbol{y}_i)} |\nabla v_\varepsilon(y - \boldsymbol{x})|^p \, dy \\ &\geq \sum_{i=1}^{m} \frac{c_i \eta^n}{c_\eta} \fint_{Q^{\xi}_{\eta\varepsilon}(\boldsymbol{y}_i)} c_\eta |\nabla v_\varepsilon(y - \boldsymbol{x})|^p \, dy\,. \end{aligned} \tag{41}$$

Let $c_i' = c_i \eta^n / c_\eta$. Being $\sum_i c_i' = \sum_i c_i \eta^n / c_\eta = 1$ and g_ε concave, we have

$$\begin{aligned} g_\varepsilon\Big(|\nabla v_\varepsilon|^p * \rho_\varepsilon(\boldsymbol{x})\Big) &= g_\varepsilon\left(\sum_{i=1}^{m} c_i' \fint_{Q^{\xi}_{\eta\varepsilon}(\boldsymbol{y}_i)} c_\eta |\nabla v_\varepsilon(y - \boldsymbol{x})|^p \, dy\right) \\ &\geq \sum_{i=1}^{m} c_i' g_\varepsilon\left(\fint_{Q^{\xi}_{\eta\varepsilon}(\boldsymbol{y}_i)} c_\eta |\nabla v_\varepsilon(y - \boldsymbol{x})|^p \, dy\right). \end{aligned}$$

Thus

$$\begin{aligned} F_\varepsilon(u_\varepsilon, A) &\geq \sum_{\boldsymbol{x} \in \mathbf{B}^{\xi}_{\eta\varepsilon}} (\eta\varepsilon)^n \sum_{i=1}^{m} c_i' g_\varepsilon\left(\fint_{Q^{\xi}_{\eta\varepsilon}(\boldsymbol{y}_i)} c_\eta |\nabla v_\varepsilon(y - \boldsymbol{x})|^p \, dy\right) \\ &\geq \sum_{\boldsymbol{x} \in \mathbf{B}^{\xi}_{\eta\varepsilon}} (\eta\varepsilon)^n \sum_{i=1}^{m} c_i' g_\varepsilon\left(\fint_{Q^{\xi}_{\eta\varepsilon}(\boldsymbol{x} + \boldsymbol{y}_i)} c_\eta |\nabla v_\varepsilon(y)|^p \, dy\right). \end{aligned}$$

For $C \subset\subset B$ and ε sufficiently small, every square contained in C is counted exactly m times in the previous sum. Thus, reordering the terms and remembering that $\sum_i c'_i = 1$ we have

$$F_\varepsilon(u_\varepsilon, A) \geq \sum_{\boldsymbol{x} \in \boldsymbol{C}^\xi_{\eta\varepsilon}} (\eta\varepsilon)^n g_\varepsilon \left(\fint_{Q^\xi_{\eta\varepsilon}(\boldsymbol{x})} c_\eta |\nabla v_\varepsilon(y)|^p \, dy \right) .$$

Denoting by $C^\xi_{\eta\varepsilon}$ the union of the squares $Q^\xi_{\eta\varepsilon} \subset C$ we will consider, with a convenient abuse of notations, that $\nabla v_\varepsilon(x) = 0$ if $x \notin C^\xi_{\eta\varepsilon}$. As every $\boldsymbol{x} \in \mathbf{X}^\xi_{\eta\varepsilon}$ can be written as $\boldsymbol{x} = \boldsymbol{z} + \boldsymbol{t}$ for $\boldsymbol{z} \in \mathbf{Z}^\xi_{\eta\varepsilon}$ and $\boldsymbol{t} \in \mathbf{T}^\xi_{\eta\varepsilon}$, we will write

$$\begin{aligned} F_\varepsilon(u_\varepsilon, A) &\geq \sum_{\boldsymbol{x} \in \mathbf{X}^\xi_{\eta\varepsilon}} (\eta\varepsilon)^n g_\varepsilon \left(\fint_{Q^\xi_{\eta\varepsilon}(\boldsymbol{x})} c_\eta |\nabla v_\varepsilon(y)|^p \, dy \right) . \\ &\geq \sum_{\boldsymbol{z} \in \mathbf{Z}^\xi_{\eta\varepsilon}} \sum_{\boldsymbol{t} \in \mathbf{T}^\xi_{\eta\varepsilon}} (\eta\varepsilon)^n g_\varepsilon \left(\fint_{Q^\xi_{\eta\varepsilon}(\boldsymbol{z}+\boldsymbol{t})} c_\eta |\nabla v_\varepsilon(y)|^p \, dy \right) . \end{aligned}$$

Now, by (38), and remembering that g_ε is convex, using Jensen's inequality we get

$$\begin{aligned} F_\varepsilon(u_\varepsilon, A) &\geq \sum_{\boldsymbol{z} \in \mathbf{Z}^\xi_{\eta\varepsilon}} \sum_{\boldsymbol{t} \in \mathbf{T}^\xi_{\eta\varepsilon}} (\eta\varepsilon)^n g_\varepsilon \left(\fint_{Z^\xi_{\eta\varepsilon}(\boldsymbol{z})} \fint_{T^\xi_{\eta\varepsilon}(\boldsymbol{t})} c_\eta \, |\nabla v_\varepsilon(z+t) \cdot \xi|^p \; dt \, dz \right) \\ &\geq \sum_{\boldsymbol{z} \in \mathbf{Z}^\xi_{\eta\varepsilon}} \sum_{\boldsymbol{t} \in \mathbf{T}^\xi_{\eta\varepsilon}} (\eta\varepsilon)^n \fint_{Z^\xi_{\eta\varepsilon}(\boldsymbol{z})} g_\varepsilon \left(\fint_{T^\xi_{\eta\varepsilon}(\boldsymbol{t})} c_\eta \, |\nabla v_\varepsilon(z+t) \cdot \xi|^p \; dt \right) dz \, . \end{aligned} \tag{42}$$

At this point the previous sum becomes

$$\begin{aligned} F_\varepsilon(u_\varepsilon, A) &\geq \sum_{\boldsymbol{z} \in \mathbf{Z}^\xi_{\eta\varepsilon}} \sum_{\boldsymbol{t} \in \mathbf{T}^\xi_{\eta\varepsilon}} \int_{Z^\xi_{\eta\varepsilon}(\boldsymbol{z})} \eta\varepsilon \, g_\varepsilon \left(\fint_{T^\xi_{\eta\varepsilon}(\boldsymbol{t})} c_\eta \, |\nabla v_\varepsilon(z+t) \cdot \xi|^p \; dt \right) dz \\ &\geq \sum_{\boldsymbol{z} \in \mathbf{Z}^\xi_{\eta\varepsilon}} \int_{Z^\xi_{\eta\varepsilon}(\boldsymbol{z})} \sum_{\boldsymbol{t} \in \mathbf{T}^\xi_{\eta\varepsilon}} \eta\varepsilon \, g_\varepsilon \left(\fint_{T^\xi_{\eta\varepsilon}(\boldsymbol{t})} c_\eta \, |\nabla v_\varepsilon(z+t) \cdot \xi|^p \; dt \right) dz \\ &\geq \int_Z \sum_{\boldsymbol{t} \in \mathbf{T}^\xi_{\eta\varepsilon}} \eta\varepsilon \, g_\varepsilon \left(c_\eta \fint_{T^\xi_{\eta\varepsilon}(\boldsymbol{t})} |\nabla v_\varepsilon(z+t) \cdot \xi|^p \; dt \right) dz \, . \end{aligned} \tag{43}$$

As $\eta \in (0,1)$ it is easy to see that $g_\varepsilon(c_\eta t) \geq h_{\eta\varepsilon}(t)$ where

$$h_\varepsilon(t) = a' t \wedge b'_\varepsilon t^{1/p} \wedge (c'/\varepsilon) \qquad \text{for} \qquad a' = a c_\eta \quad b'_\varepsilon = b_{(\varepsilon/\eta)} c_\eta^{1/p} \quad c' = c\eta \, .$$

Thus

$$F_\varepsilon(u_\varepsilon, A) \geq \int_Z \sum_{\boldsymbol{t} \in \mathbf{T}^\xi_{\eta\varepsilon}} \eta\varepsilon \, h_{\eta\varepsilon} \left(\fint_{T^\xi_{\eta\varepsilon}(\boldsymbol{t})} |\nabla v_\varepsilon(z+t) \cdot \xi|^p \; dt \right) dz \, .$$

For $\gamma > 0$ let $C^\gamma = \{x \in C \, : \, d(x, \partial C) > \gamma\}$ and let $C^\gamma_z = \{x \in C^\gamma \, : \, x = z + t\xi\}$. For γ small and $\varepsilon \ll \gamma$ the functions $v_\varepsilon(z+t)$ belongs to $W^{1,p}(C^\gamma_z)$ (with respect to the variable t) for $a.e.\, z \in Z$. Thus we can write that for $a.e.\, z \in Z$

$$\sum_{\boldsymbol{t} \in \mathbf{T}^\xi_{\eta\varepsilon}} \eta\varepsilon \, h_{\eta\varepsilon} \left(\fint_{T^\xi_{\eta\varepsilon}(\boldsymbol{t})} |\nabla v_\varepsilon(z+t) \cdot \xi|^p \; dt \right) = H_{\eta\varepsilon}(v_\varepsilon, C^\gamma_z) \, ,$$

where $H_{\eta\varepsilon}$ is the one-dimensional discrete functional defined in (32) for $m = \eta$. As $v_\varepsilon \to u$ in $L^1(C_z^\gamma)$ for *a.e.* $z \in Z$, by Proposition 6.1 and Fatou's Lemma we can say that

$$\begin{aligned} \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, A) &\geq \int_Z \left(\int_{C_z^\gamma} ac_\eta |u'(z+t)|^p \, dt + c\eta \, \#\Big(S_u \cap C_z^\gamma\Big) \right) dz \\ &\geq \int_{C^\gamma} ac_\eta |\nabla u(y) \cdot \xi|^p \, dy + c\eta \int_{S_u \cap C^\gamma} |\nu \cdot \xi| \, d\mathcal{H}^{n-1} . \end{aligned} \tag{44}$$

Taking the supremum for $\gamma \searrow 0$ and $C \nearrow A$ we get

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, A) \geq \int_A ac_\eta |\nabla u(y) \cdot \xi|^p \, dy + c\eta \int_{S_u \cap A} |\nu \cdot \xi| \, d\mathcal{H}^{n-1} , \tag{45}$$

from which follows (40) considering that $c_\eta \to 1$ as $\eta \to 0$ and $a > \delta$. ■

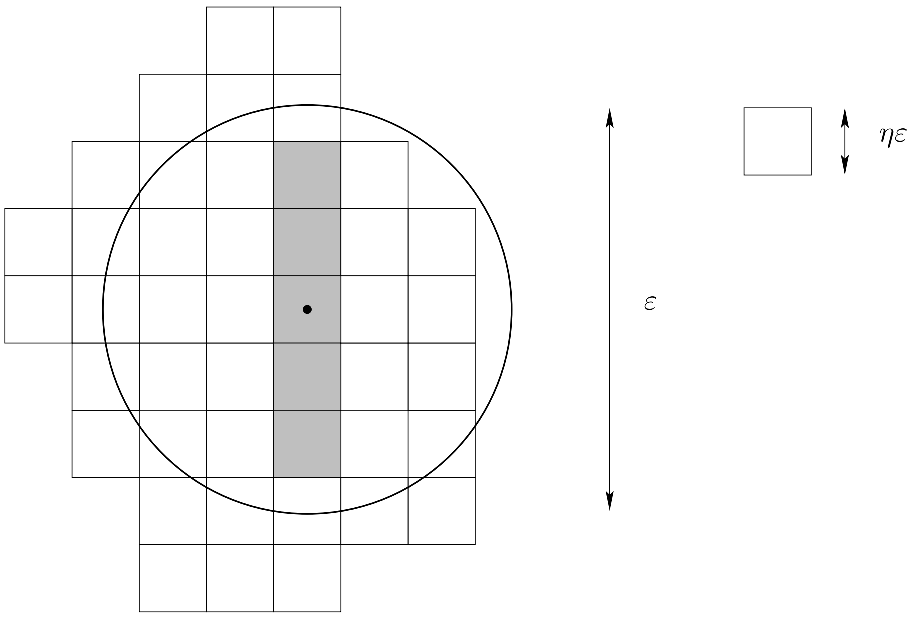

Figure 3: The set $S^\xi_{\eta\varepsilon}$ used for the surface estimate when C is the unit ball.

Proposition 6.4 *Let $u_\varepsilon \to u$ in $L^1(\Omega)$ and A an open set in Ω. For $\xi \in S^{n-1}$ let μ_ξ be defined as in Lemma 4.5. For every $\delta \in (0,1)$ there exists $\alpha_\delta > 0$ such that*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, A) \geq \int_A \alpha_\delta |\nabla u(y) \cdot \xi|^p \, dy + \delta \int_{S_u \cap A} \mu_\xi |\nu \cdot \xi| \, d\mathcal{H}^{n-1} . \tag{46}$$

Proof. For a value $\eta > 0$ (which will be defined later) let S^ξ_η be the union of the squares $Q^\xi_\eta(\boldsymbol{y}_i) \subset C_\rho$ where $\boldsymbol{y}_i \in \mathbf{T}^\xi_\eta$ for $i = 1, \ldots, m(\eta, \xi)$ (see Figure 3). Note that the projections of S^ξ_η on $Z = \xi^\perp$ and $T = \langle \xi \rangle$ can be written respectively as Z^ξ_η and T^ξ_μ where $\mu(\eta, \xi) = m(\eta, \xi)\eta$. In particular

$$|S^\xi_\eta| = \mu \, \eta^{n-1} \qquad \mu(\eta, \xi) \to \mu_\xi \quad \text{as } \eta \to 0 .$$

Let $\zeta = \rho_\eta$ be the piecewise constant function satisfying (37) and let

$$c_i = \zeta(\boldsymbol{y}_i) \qquad c_\eta = \mu\eta^{n-1} \min\{c_i\} .$$

Considering for every $\varepsilon > 0$ the lattice $\mathbf{X}^{\xi}_{\eta\varepsilon}$ we will denote by $S^{\xi}_{\eta\varepsilon}$ the union of the squares $Q^{\xi}_{\eta\varepsilon}(\boldsymbol{y}_i) \subset (\varepsilon C_\rho)$ (where $\boldsymbol{y}_i \in \mathbf{T}^{\xi}_{\eta\varepsilon}$ for $i = 1, \ldots, m(\eta, \xi)$) and by ζ_ε the sequence of kernels $\zeta(y/\varepsilon)/\varepsilon^n$.

Let again $a, c \in (0, +\infty)$ such that $\delta < a = c < 1$, arguing as in the proof of Proposition 6.3 we get

$$F_\varepsilon(u_\varepsilon, A) \geq \sum_{\boldsymbol{x} \in \mathbf{B}^{\xi}_{\eta\varepsilon}} (\eta\varepsilon)^n g_\varepsilon\Big(|\nabla v_\varepsilon|^p * \rho_\varepsilon(\boldsymbol{x})\Big),$$

for $B \subset\subset A$ and for a sequence $v_\varepsilon \in W^{1,p}(B)$ with $v_\varepsilon \to u$ in $L^1(B)$. Then

$$\begin{aligned}
|\nabla v_\varepsilon|^p * \rho_\varepsilon(\boldsymbol{x}) &= \int_{\varepsilon C_\rho} |\nabla v_\varepsilon(y - \boldsymbol{x})|^p \, \rho_\varepsilon(y)\, dy \\
&\geq \int_{S^{\xi}_{\eta\varepsilon}} |\nabla v_\varepsilon(y - \boldsymbol{x})|^p \, \zeta_\varepsilon(y)\, dy \\
&\geq \sum_{i=1}^{m} \frac{c_i}{\varepsilon^n} \int_{Q^{\xi}_{\eta\varepsilon}(x_i)} |\nabla v_\varepsilon(y - \boldsymbol{x})|^p \, dy \\
&\geq c_\eta \fint_{S^{\xi}_{\eta\varepsilon}(\boldsymbol{x})} |\nabla v_\varepsilon(y)|^p \, dy\,.
\end{aligned} \tag{47}$$

Considering again that $\nabla v_\varepsilon(x) = 0$ if $x \notin B^{\xi}_{\eta\varepsilon}$ we can write that

$$\begin{aligned}
F_\varepsilon(u_\varepsilon, A) &\geq \sum_{\boldsymbol{x} \in \mathbf{X}^{\xi}_{\eta\varepsilon}} (\eta\varepsilon)^n g_\varepsilon \left(\fint_{S^{\xi}_{\eta\varepsilon}(\boldsymbol{x})} c_\eta |\nabla v_\varepsilon(y)|^p \, dy \right) \\
&\geq \sum_{\boldsymbol{z} \in \mathbf{Z}^{\xi}_{\eta\varepsilon}} \sum_{\boldsymbol{t} \in \mathbf{T}^{\xi}_{\eta\varepsilon}} (\eta\varepsilon)^n g_\varepsilon \left(\fint_{S^{\xi}_{\eta\varepsilon}(\boldsymbol{z}+\boldsymbol{t})} c_\eta |\nabla v_\varepsilon(y)|^p \, dy \right).
\end{aligned}$$

Using Jenses's inequality and following the previous proof we get

$$F_\varepsilon(u_\varepsilon, A) \geq \int_Z \sum_{\boldsymbol{t} \in \mathbf{T}^{\xi}_{\eta\varepsilon}} \eta\varepsilon \, g_\varepsilon \left(\fint_{I^{\xi}_{\mu\varepsilon}(\boldsymbol{t})} c_\eta \, |\nabla u_\varepsilon(z+t) \cdot \xi|^p \ dt \right) dz\,.$$

It is easy to check that $\eta g_\varepsilon(c_\eta t) \geq h_\varepsilon(t)$ where

$$h_\varepsilon(t) = a' t \wedge b'_\varepsilon t^{1/p} \wedge (c'/\varepsilon) \qquad \text{for} \qquad a' = \eta c_\eta a \quad b'_\varepsilon = b_\varepsilon \eta c_\eta^{1/p} \quad c' = c\eta\,.$$

Then, arguing as in the previous proof, we can say that for $a.e.\, z \in Z$ we have

$$\sum_{\boldsymbol{t} \in \mathbf{T}^{\xi}_{\eta\varepsilon}} \eta\varepsilon \, g_\varepsilon \left(\fint_{I^{\xi}_{\mu\varepsilon}(\boldsymbol{t})} c_\eta \, |\nabla u_\varepsilon(z+t) \cdot \xi|^p \ dt \right) \geq H_\varepsilon(u_\varepsilon, B^{\gamma}_z)\,.$$

where H_ε is the one-dimensional discrete functional defined in (32) for $m = \mu(\eta, \xi)$. Again by Fatou's Lemma and Proposition 6.1 we get

$$\begin{aligned}
\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, A) &\geq \int_Z \left(\int_{B^{\gamma}_z} a\eta c_\eta |u'(z+t)|^p \, dt + c\eta\mu(\eta, \xi)\, \#\Big(S_u \cap B^{\gamma}_z\Big) \right) dz \\
&\geq \int_{B^{\gamma}} a\eta c_\eta |\nabla u(y) \cdot \xi|^p \, dy + c\eta\mu(\eta, \xi) \int_{S_u \cap C^{\gamma}} |\nu \cdot \xi| \, d\mathcal{H}^{n-1}\,.
\end{aligned} \tag{48}$$

Taking the supremum for $\gamma \searrow 0$ and $B \nearrow A$ we get

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, A) \geq \int_A a\eta c_\eta |\nabla u(y)\cdot\xi|^p\, dy + c\eta\mu(\eta,\xi)\int_{S_u\cap A} |\nu\cdot\xi|\, d\mathcal{H}^{n-1}\,, \tag{49}$$

from which follows (46) taking the supremum for $\eta \to 1$ and remembering that $c > \delta$. ■

6.3 The lower inequality

Let $u_\varepsilon \to u$ in $L^1(\Omega)$. By Propositions 6.3 and 6.4 for every $\xi \in \boldsymbol{S}^{n-1}$ and for every open set $A \subset \Omega$ we have

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, A) \geq \delta \int_A |\nabla u\cdot\xi|^p\, dx\,,$$

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, A) \geq \delta \int_{S_u\cap A} \mu_\xi |\xi\cdot\nu|\, d\mathcal{H}^{n-1}\,.$$

Taking the supremum as $\delta \to 1$ we get

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, A) \geq \int_A |\nabla u\cdot\xi|^p\, dx\,,$$

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, A) \geq \int_{S_u\cap A} \mu_\xi |\xi\cdot\nu|\, d\mathcal{H}^{n-1}\,.$$

Now we apply Lemma 2.5 (taking the supremum for $\xi \in \boldsymbol{S}^{n-1}$) and by Lemma 4.5 we get the Γ-liminf inequality

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \int_\Omega |\nabla u|^p\, dx + \int_{S_u} \phi_\rho(\nu)\, d\mathcal{H}^{n-1} = F(u)\,. \tag{50}$$

7 Compactness

In this section we will prove the following compactness property, whose proof is similar to the one of Theorem 6.6 in [1].

Proposition 7.1 *Let $\{u_\varepsilon\}$ be a family in $W^{1,p}(\Omega)$, equibounded in $L^\infty(\Omega)$ and such that $\mathcal{F}_\varepsilon(u_\varepsilon) \leq C < +\infty$. Then $\{u_\varepsilon\}$ is precompact in $L^1(\Omega)$ (and thus in $L^q(\Omega)$ for $1 \leq q < +\infty$).*

Proof. Let $\xi \in \boldsymbol{S}^{n-1}$ and A an open set with $A \subset\subset \Omega$. We will prove that for $\delta > 0$ (arbitrarily small) there exists a family $\{v_\varepsilon\}$ equibounded in $L^\infty(\Omega)$ and contained in a δ-neighborhood of $\{u_\varepsilon\}$ (with respect to the $L^1(\Omega)$ norm) such that $\{v_\varepsilon(z+t\xi)\}$ is precompact in $L^1(A_z)$ (with respect to the variable t) for *a.e.* $z \in Z$. Then the precompactness of $\{u_\varepsilon\}$ will follow arguing as in [1].

For $\eta, a, c \in (0,1)$ and $b_\varepsilon \nearrow +\infty$, Proposition 6.3 gives a family $\{v_\varepsilon\}$ defined by $v_\varepsilon(x) = u_\varepsilon(x+\tau_{\eta\varepsilon})$ in such a way that

$$\mathcal{F}_\varepsilon(u_\varepsilon) \geq \int_Z H_\varepsilon(v_\varepsilon, A_z)\, dz\,.$$

As $\tau_{\eta\varepsilon} \to 0$ it is sufficient to prove that the family $\{v_\varepsilon\}$ is precompact in $L^1(A)$. For $k > 0$ (which will be defined later) let Z^k be the set of points $z \in Z$ such that $H_\varepsilon(v_\varepsilon, A_z) \geq k$. We introduce the functions w_ε which are defined by slicing in the following way:

$$w_\varepsilon(z+t) = \begin{cases} v_\varepsilon(z+t) & \text{if } z \in Z^m \\ 0 & \text{otherwise.} \end{cases}$$

As Ω is bounded and $\{v_\varepsilon\}$ equibounded in $L^\infty(\Omega)$ it is clear that

$$\int_A |w_\varepsilon - v_\varepsilon|\, dx \le c\,|Z^k|\,,$$

where c does not depend on ε and m. By Chebychev inequality

$$|Z^k| \le \frac{1}{k}\int_Z H_\varepsilon(v_\varepsilon, A_z)\, dz \le \frac{1}{k}\mathcal{F}_\varepsilon(u_\varepsilon,\Omega) \le \frac{C}{k}$$

and thus, if k is sufficiently large, $\{w_\varepsilon\}$ is arbitrarily close to $\{v_\varepsilon\}$.

From (35) it follows that there exists a family $\{\tilde{w}_\varepsilon\}$ contained in $SBV(A_z)$ and such that

$$H_\varepsilon(w_\varepsilon, A_z) \ge a\int_A |\tilde{w}_\varepsilon'|^p\, dx + \sum_{S_{\tilde{w}_\varepsilon}} b_\varepsilon|\tilde{w}_\varepsilon^+ - \tilde{w}_\varepsilon^-| \wedge c\,.$$

Since for every $z \in Z$ we have $F_\varepsilon(w_\varepsilon, A_z) \le k$ it turns out that there is only a finite number of discontinuity points where $|\tilde{w}_\varepsilon^+ - \tilde{w}_\varepsilon^-| \ge c/b$. As $\tilde{w}_\varepsilon$ is equibounded in $L^\infty(A)$ we can say that $\tilde{w}_\varepsilon$ is also bounded in $BV(A_z)$. It follows that $\{\tilde{w}_\varepsilon\}$ is precompact in $L^1(A_z)$. As $(\tilde{w}_\varepsilon - v_\varepsilon) \to 0$ in $L^1(A_z)$ it is clear that v_ε in precompact in $L^1(A_z)$. ■

8 Applications

8.1 Perona-Malik equation and Mumford-Shah functional

Mumford and Shah suggested [29] a variational approach for image segmentation. Their original (strong) formulation can be written in weak form as

$$\min\left\{\int_\Omega |\nabla u|^2\, dx + \mathcal{H}^1(S_u) + \int_\Omega |u-g|^2\, dx \;:\; u \in SBV^2(\Omega)\right\}. \tag{51}$$

The datum g represents the grey level map of the original image g which is often noisy, while the minimizers are considered as segmentations of g. (Roughly speaking, a minimizer u is a regularization of the datum g which should represent in the set S_u the contours of the original image).

Our convergence result gives a non-local approximation of this problem. Indeed from Theorem 3.1 we know that the sequence

$$\mathcal{F}_\varepsilon(u) = \int_\Omega f_\varepsilon\Big(|\nabla u|^2 * \rho_\varepsilon(x)\Big)\, dx + \int_\Omega |u-g|^2\, dx$$

Γ-converges, with respect to the topology of $L^2(\Omega)$, to the Mumford-Shah functional and from Theorem 3.2 we know that a family $\{u_\varepsilon\}$ of minimizers has at least an accumulation point, which is a minimizer of the limit functional.

An alternative approach for image segmentation has been used by Perona and Malik [32]. They employed a nonlinear diffusion equation like

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(\dfrac{2\nabla u}{1+|\nabla u|^2}\right) \\[2ex] u(0,x) = g(x) \qquad \dfrac{\partial u}{\partial n} = 0. \end{cases} \tag{52}$$

The original image is the initial condition g while the solution $u(T,x)$ at some time T (which is in general not known a priori) represents the segmentation. This equation has been widely used in the applications but from the mathematical viewpoint it is still not clear in which weak sense the solutions should be considered, because of a backward diffusion effect which creates discontinuities. A possibly remedy has been suggested by Cattè, Lions, Morel and Coll [13] by means of a non-local regularization of the form

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(2\nabla u\left[\frac{1}{1+|\rho_\varepsilon * \nabla u|^2}\right]\right).$$

This two approaches are quite different but they can be compared computing the gradient flow of the functionals $\mathcal{F}_\varepsilon(u)$ with $f(t) = \log(1+t)$ and $a_\varepsilon = -\varepsilon\log\varepsilon$. Indeed we can write that

$$\frac{\partial u}{\partial t} = -d\mathcal{F}_\varepsilon(u) = \operatorname{div}\left(2\nabla u\left[\rho_\varepsilon * \left(\frac{1}{1+a_\varepsilon\,\rho_\varepsilon * |\nabla u'|^2}\right)\right]\right) - 2(u-g)\,.$$

This equation can be interpreted as a non-local regularization of the biased anisotropic diffusion

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{2\nabla u}{1+|\nabla u|^2}\right) - 2(u-g)\,,$$

which was proposed by Nordström [31] in order to avoid the problem of finding a good stopping time T for (52).

8.2 From non-local damage to brittle fracture

Many materials, such as concrete, exhibits under loading a loss of stiffness which macroscopically is described by a diffuse damage mechanisms and microscopically is related to the presence of micro fractures.

An easy way to model this phenomenon (see for instance [26] or [17] and the references therein) consists in introducing a damage variable $d \in [0,1]$ which enters into the linear constitutive relation

$$\sigma = (1-d)\boldsymbol{C}\,Eu\,,$$

where σ is the stress, Eu is the strain and $\boldsymbol{C}$ is the elastic tangent modulus (in this case the damage mechanism is said to be isotropic). Then the variable d is related to the strain by a (nonlinear) relation which is determined by experimental tests. For instance, denoting by $|Eu|_{\boldsymbol{C}}$ the elastic norm $(Eu^T\boldsymbol{C}Eu)^{1/2}$ we can consider an exponential law given by

$$d = 1 - \exp(-|Eu|^2_{\boldsymbol{C}})\,.$$

The resulting stress-strain relation

$$\sigma = \exp(-|Eu|^2_{\boldsymbol{C}})\boldsymbol{C}\,E$$

has clearly a softening behaviour and the energy density

$$W(Eu) = \frac{1}{2}\Big(1 - \exp(-|Eu|^2_{\boldsymbol{C}})\Big)$$

has a convex-concave behaviour that can be related to the elastic and damaging regimes. As a consequence the problem turns out to be ill posed in Sobolev spaces because of the loss of coerciveness. A possible remedy is based on the regularization of the model by means of a non-local operator. As a matter of fact there many different ways to employ the non-locality (see [26] or [7]). Here we take into account only the one suggested by Theorem 3.7 leaving further investigations for a future work. Thus we will consider the energy

$$\int_\Omega W\Big(|Eu|^2_{\boldsymbol{C}} * \rho_\varepsilon(x)\Big)\, dx = \frac{1}{2}\int_\Omega \Big(1 - \exp\Big(|Eu|^2_{\boldsymbol{C}} * \rho_\varepsilon(x)\Big)\Big)\, dx\,,$$

where the support of the convolution kernel is the unit ball. As the non-locality radius ε tends to zero we will pass from a non-local to a local model and without a suitable rescaling the limit problem will be ill posed again. By Theorem 3.7 and 3.4 the family

$$\int_\Omega W_\varepsilon\Big(|Eu|^2_{\boldsymbol{C}} * \rho_\varepsilon(x)\Big)\, dx = \frac{1}{2}\int_\Omega \frac{1}{\varepsilon}\Big(1 - \exp\Big(\varepsilon\, |Eu|^2_{\boldsymbol{C}} * \rho_\varepsilon(x)\Big)\Big)\, dx\,,$$

which is well defined in the Sobolev space $W^{1,2}(\Omega, \mathbf{R}^2)$, Γ-converge with respect to the strong $L^p(\Omega)$ topology (for $1 \le p < +\infty$) to the Griffith energy

$$\int_\Omega |\mathcal{E}u|^2_{\boldsymbol{C}}\, dx + 2\mathcal{H}^{n-1}(J_u)\,,$$

which is defined in the space $SBD(\Omega)$. This result can be interpreted as a link between a nonlocal damage model and a fracture model for brittle materials.

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