# MINIMIZING MOVEMENTS FOR MEAN CURVATURE FLOW OF PARTITIONS

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ABSTRACT. We prove the existence of a weak global in time mean curvature flow of a bounded partition of space using the method of minimizing movements. The result is extended to the case when suitable driving forces are present. We also show that the minimizing movement solution starting from a partition made by a union of bounded convex sets at a positive distance agrees with the classical mean curvature flow, and the motion is stable with respect to the Hausdorff convergence of the initial partition.

## 1. INTRODUCTION

Mean curvature evolution of partitions became popular in recent years because of its applications in material science and physics, especially evolutions of grain boundaries and motion of immiscible fluid systems, see e.g. [9, 5, 34, 28] and references therein. Behaviour of the motion in the two phase case, i.e. in the case of classical motion by mean curvature of a boundary as a gradient flow of the area functional, is rather well-understood, see for instance [25, 22, 17, 23, 33, 6, 11] and references therein.

Mean curvature evolution of interfaces in the multiphase case in general involves motion of surface junctions in $\mathbb{R}^n$, or triple and multiple points in the plane, an already nontrivial problem. We refer to the survey [34] and references therein for recent results on curvature evolution of planar networks.

Not much seems to be known in higher space dimensions; short time existence of the motion of subgraph-type partitions has been derived in [21, 20] and well-posedness and short time existence of the motion by mean curvature of three surface clusters have been recently shown in [16].

Even in the two phase case, the classical flow describes the motion only up to the appearance of the first singularity. In order to continue the motion through singularities, several notions of generalized solutions have been suggested: Brakke varifold-solution [9], the viscosity solution (see [23] and references therein), the Almgren-Taylor-Wang [1] and Luckhaus-Sturzenhecker [31] solutions, the minimal barrier solution (see [6] and references therein); we also refer to [19, 26] for other types of solutions. At the moment the lack of the comparison principle in the multiphase case results in a lot of difficulties to extend such notions as viscosity and barrier solutions, while besides Brakke solution, some other generalized solutions have been successfully extended to partitions. For example, the authors of [29] have proved the existence of a distributional solution of mean curvature evolution of partitions on the torus using the time thresholding method introduced in [35], see also [36, 18]; furthermore the authors of [27] showed that their solution is indeed a Brakke solution.

In [14] De Giorgi generalized the Almgren-Taylor-Wang and Luckhaus-Sturzenhecker approach to what he called the minimizing movements method. In the present paper, we prove the existence of a generalized minimizing movement solution in $\mathbb{P}_b(N+1)$, the collection of all partitions of $\mathbb{R}^n$, $n \geq 2$, having $N+1 \geq 2$ components, with the first $N$-components bounded. This is the multiphase generalization of the evolution of a compact boundary in the two-phase case ($N=1$), for which the generalized minimizing movement solution has been introduced and studied in [1, 31].

Let us recall the definition in [14] (see also [2, 4]).

**Definition 1.1 (Generalized minimizing movement ).** *Let $S$ be a topological space, $F : S \times S \times [1, +\infty) \times \mathbb{Z} \to [-\infty, +\infty]$ be a functional and $u : [0, +\infty) \to S$. We say that $u$ is a generalized minimizing movement associated to $F, S$ (shortly a $GMM$ associated to $F$) starting from $a \in S$ and we write $u \in GMM(F, a)$, if there exist $w : [1, +\infty) \times \mathbb{Z} \to S$ and a diverging sequence $\{\lambda_j\}$ such that*

$$\lim_{j\to+\infty} w(\lambda_j, [\lambda_j t]) = u(t) \quad \textit{for any } t \geq 0,$$

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*and the functions* $w(\lambda, k)$, $\lambda \geq 1$, $k \in \mathbb{Z}$, *are defined inductively as* $w(\lambda, k) = a$ *for* $k \leq 0$ *and*

$$F(\lambda, k, w(\lambda, k+1), w(\lambda, k)) = \min_{s \in S} F(\lambda, k, s, w(\lambda, k)) \qquad \forall k \geq 0.$$

When $GMM(F, a)$ is a singleton, it is called the minimizing movement starting from $a$ and denoted by $MM(F, a)$. In the present paper we apply this definition for $S = \mathbb{P}_b(N+1)$ endowed with the $L^1(\mathbb{R}^n)$-topology, and following [15],

$$F(\mathcal{A}, \mathcal{B}; \lambda) = \mathrm{Per}(\mathcal{A}) + \frac{\lambda}{2} \sum_{j=1}^{N+1} \int_{A_j \Delta B_j} d(x, \partial B_j) dx, \quad \mathcal{A}, \mathcal{B} \in \mathbb{P}_b(N+1),$$

where $\mathrm{Per}(\mathcal{A}) = \frac{1}{2} \sum_{j=1}^{N+1} P(A_j)$ is the perimeter of the partition $\mathcal{A} = (A_1, \ldots, A_{N+1})$, $d(\cdot, E)$ is the distance function from $E \subseteq \mathbb{R}^n$, and $a \in \mathcal{P}_b(N+1)$ is an initial partition. We shall also consider the functional

$$F_H(\mathcal{A}, \mathcal{B}; \lambda) = \mathrm{Per}(\mathcal{A}) + \sum_{j=1}^{N+1} \int_{A_j} H_j dx + \frac{\lambda}{2} \sum_{j=1}^{N+1} \int_{A_j \Delta B_j} d(x, \partial B_j) dx, \quad \mathcal{A}, \mathcal{B} \in \mathbb{P}_b(N+1)$$

for suitable driving forces $H_i$, $i = 1, \ldots, N+1$ (see Section 5).

Our main result is the following (see Theorems 4.10 and 5.1 for the precise statements):

**Theorem 1.2.** *For any* $\mathcal{G} \in \mathbb{P}_b(N+1)$, $GMM(F, \mathcal{G})$ *is nonempty, i.e. there exists a generalized minimizing movement starting from* $\mathcal{G}$. *Moreover,*

1) *any such movement* $\mathcal{M}(t) = (M_1(t), \ldots, M_{N+1}(t))$ *is locally* $\frac{1}{n+1}$*-Hölder continuous in time;*
2) $\bigcup_{j=1}^{N} M_j(t)$ *is contained in the closed convex envelope of the union* $\bigcup_{j=1}^{N} G_j$ *of the bounded components of* $\mathcal{G}$ *for any* $t > 0$.

*Finally, similar results are valid for* $F_H$.

To prove Theorem 1.2 we establish uniform density estimates for minimizers of $F$ and $F_H$. A lower-type density estimate for minimizers of $F$ could be proven using the slicing method for currents as in the thesis [10], or also using the infiltration technique of [30, Lemma 4.6] (see also [32, Section 30.2]). In Section 3 we prove that $(\Lambda, r_0)$-minimizers of Per in $\mathbb{R}^n$ (Definition 3.5) satisfy uniform density estimates using the method of cutting out and filling in with balls, an argument of [31].

In Theorems 6.5 and 6.7 we also show the following consistency and stability result.

**Theorem 1.3.** *Suppose that* $\mathcal{C} = (C_1, \ldots, C_{N+1}) \in \mathbb{P}_b(N+1)$, *where* $C_1, \ldots, C_N$ *are convex sets whose closures are disjoint. Then the generalized minimizing movement associated to* $F$ *and starting from* $\mathcal{C}$ *is a minimizing movement* $\{\mathcal{M}\} = MM(F, \mathcal{C})$ *and writing*

$$\mathcal{M}(t) = (M_1(t), \ldots, M_{N+1}(t)),$$

*we have that each* $M_i(t)$ *agrees with the classical mean curvature flow starting from* $C_i$, *up to the extinction time. Moreover, if a sequence* $\{\mathcal{G}^{(k)}\} \subset \mathbb{P}_b(N+1)$ *converges to* $\mathcal{C} \in \mathbb{P}_b(N+1)$ *in the Hausdorff distance, then any* $\mathcal{M}^{(k)} \in GMM(F, \mathcal{G}^{(k)})$ *converges to* $\{\mathcal{M}\} = MM(F, \mathcal{C})$ *in the Hausdorff distance.*

The proof of the consistency with the classical mean curvature flow relies on the results of [7], while for the stability in the Hausdorff distance we employ the comparison results from [8].

The plan of the paper is the following.

In Section 2 we set the notation and recall some results from the theory of finite perimeter sets. Section 3 is devoted to the definitions of partitions and density estimates for $(\Lambda, r_0)$-minimizers. In Section 4 we prove the existence of minimizers of $F$ in $\mathbb{P}_b(N+1)$ (Theorem 4.2), the density estimates (Theorem 4.6), and – one of our main results – the existence of $GMM$ for $F$ (Theorem 4.10). The existence of $GMM$ for $F_H$ is shown in Section 5. Finally in Section 6 we prove that the minimizers of $F(\cdot, \mathcal{G}; \lambda)$ with disjoint $\mathcal{G}$ (Definition 6.1) is also disjoint provided $\lambda$ is large enough (Theorem 6.5) and as a nontrivial application of this fact, we show Theorem 1.3.

## 2. NOTATION AND PRELIMINARIES

In this section we introduce the notation and collect some important properties of sets of locally finite perimeter. The standard references for $BV$-functions and sets of finite perimeter are [3, 24].

We use $\mathbb{N}_0$ to denote the set of all nonnegative integers. Given a finite subset $I \subset \mathbb{N}_0$, we write $|I|$ for the number of elements of $I$. The symbol $B_r(x)$ stands for the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ of radius $r > 0$. The characteristic function of a Lebesgue measurable set $F$ is denoted by $\chi_F$ and its Lebesgue measure by $|F|$; we set also $\omega_n := |B_1(0)|$. We denote by $E^c$ the complement of $E$ in $\mathbb{R}^n$.

$\mathrm{Op}(\mathbb{R}^n)$ (resp. $\mathrm{Op}_b(\mathbb{R}^n)$) is the collection of all open (resp. open and bounded) subsets of $\mathbb{R}^n$. The set of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-functions having locally bounded total variation in $\mathbb{R}^n$ is denoted by $BV_{\mathrm{loc}}(\mathbb{R}^n)$ and the elements of

$$BV_{\mathrm{loc}}(\mathbb{R}^n, \{0,1\}) := \{E \subseteq \mathbb{R}^n : \ \chi_E \in BV_{\mathrm{loc}}(\mathbb{R}^n)\}$$

are called locally finite perimeter sets. Given a $E \in BV_{\mathrm{loc}}(\mathbb{R}^n, \{0,1\})$ we denote by

a) $P(E,\Omega) := \int_\Omega |D\chi_E|$ the perimeter of $E$ in $\Omega \in \mathrm{Op}(\mathbb{R}^n)$;

b) $\partial E$ the measure-theoretic boundary of $E$:

$$\partial E := \{x \in \mathbb{R}^n : \ 0 < |B_\rho \cap E| < |B_\rho| \quad \forall \rho > 0\};$$

c) $\partial^* E$ the reduced boundary of $E$;

d) $\nu_E$ the outer generalized unit normal to $\partial^* E$.

For simplicity, we set $P(E) := P(E,\mathbb{R}^n)$ provided $E \in BV(\mathbb{R}^n, \{0,1\})$. Further, given a Lebesgue measurable set $E \subseteq \mathbb{R}^n$ and $\alpha \in [0,1]$ we define

$$E^{(\alpha)} := \left\{ x \in \mathbb{R}^n : \ \lim_{\rho \to 0^+} \frac{|B_\rho(x) \cap E|}{|B_\rho(x)|} = \alpha \right\}.$$

Unless otherwise stated, we always suppose that any locally finite perimeter set $E$ we consider coincides with $E^{(1)}$ (so that by [24, Proposition 3.1] $\partial E$ coincides with the topological boundary). We recall that $\overline{\partial^* E} = \partial E$ and $D\chi_E = \nu_E d\mathcal{H}^{n-1} \llcorner \partial^* E$, where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ and $\llcorner$ is the symbol of restriction. Given a nonempty set $E \subseteq \mathbb{R}^n$, $d(\cdot, E)$ stands for the distance function from $E$ and

$$\tilde{d}(x, \partial E) = d(x, E) - d(x, \mathbb{R}^n \setminus E)$$

is the signed distance function from $\partial E$, negative inside $E$.

**Theorem 2.1.** [13] *Let* $E \in BV_{\mathrm{loc}}(\mathbb{R}^n, \{0,1\})$. *Then for any* $x \in \partial^* E$

$$\lim_{\rho \to 0^+} \frac{|E \cap B_\rho(x)|}{|B_\rho(x)|} = \frac{1}{2}, \qquad \lim_{\rho \to 0^+} \frac{P(E, B_\rho(x))}{\omega_{n-1} r^{n-1}} = 1.$$

**Theorem 2.2.** [3, Theorem 3.61] *For every* $E \in BV_{\mathrm{loc}}(\mathbb{R}^n, \{0,1\})$

$$\mathcal{H}^{n-1}(\mathbb{R}^n \setminus (E^{(0)} \cup E \cup \partial^* E)) = 0.$$

*Moreover,* $\mathcal{H}^{n-1}(E^{(1/2)} \setminus \partial^* E) = 0$.

**Remark 2.3.** Given $E \in BV_{\mathrm{loc}}(\mathbb{R}^n, \{0,1\})$ the map $\Omega \in \mathrm{Op}(\mathbb{R}^n) \mapsto P(E,\Omega)$ extends to a Borel measure in $\mathbb{R}^n$, so that $P(E,B) = \mathcal{H}^{n-1}(B \cap \partial^* E)$ for every Borel set $B \subseteq \mathbb{R}^n$.

**Theorem 2.4.** [32, Theorem 16.3] *If* $E$ *and* $F$ *are Caccioppoli sets, and we let*

$$\{\nu_E = \nu_F\} = \{x \in \partial^* E \cap \partial^* F : \ \nu_E(x) = \nu_F(x)\},$$
$$\{\nu_E = -\nu_F\} = \{x \in \partial^* E \cap \partial^* F : \ \nu_E(x) = -\nu_F(x)\},$$

*then* $E \cap F$, $E \setminus F$ *and* $E \cup F$ *are locally finite perimeter sets with*

$$\partial^*(E \cap F) \approx \big(F \cap \partial^* E\big) \cup \big(E \cap \partial^* F\big) \cup \big\{\nu_E = \nu_F\big\}, \tag{2.1}$$

$$\partial^*(E \setminus F) \approx \big(F^{(0)} \cap \partial^* E\big) \cup \big(E \cap \partial^* F\big) \cup \big\{\nu_E = -\nu_F\big\}, \tag{2.2}$$

$$\partial^*(E \cup F) \approx \big(F^{(0)} \cap \partial^* E\big) \cup \big(E^{(0)} \cap \partial^* F\big) \cup \big\{\nu_E = \nu_F\big\}, \tag{2.3}$$

*where* $A \approx B$ *means* $\mathcal{H}^{n-1}(A \Delta B) = 0$. *Moreover, for every Borel set* $B \subseteq R^n$

$$P(E \cap F, B) = P(E, F \cap B) + P(F, E \cap B) + H^{n-1}\big(\{\nu_E = \nu_F\} \cap B\big), \tag{2.4}$$

$$P(E\setminus F,B)=P(E,F^{(0)}\cap B)+P(F,E\cap B)+H^{n-1}\big(\{\nu_E=-\nu_F\}\cap B\big), \tag{2.5}$$

$$P(E\cup F,B)=P(E,F^{(0)}\cap B)+P(F,E^{(0)}\cap B)+H^{n-1}\big(\{\nu_E=\nu_F\}\cap B\big). \tag{2.6}$$

Finally, recall that for every $E,F\in BV_{\rm loc}(\mathbb{R}^n,\{0,1\})$ and $\Omega\in \mathrm{Op}(\mathbb{R}^n)$

$$P(E\cap F,\Omega)+P(E\cup F,\Omega)\le P(E,\Omega)+P(F,\Omega). \tag{2.7}$$

## 3. Partitions

Now we give the notions of partition, $(\Lambda,r_0)$-minimizer and bounded partition. The main result of this section is represented by the density estimates for $(\Lambda,r_0)$-minimizers (Theorem 3.6).

**Definition 3.1 (Partition).** *Given an integer $N\ge 2$, an $N$-tuple $\mathcal{C}=(C_1,\dots,C_N)$ of subsets of $\mathbb{R}^n$ is called an $N$* **-partition** *of $\mathbb{R}^n$ (a partition, for short) if*

(a) $C_i\in BV_{\rm loc}(\mathbb{R}^n,\{0,1\})$ *for every* $i=1,\dots,N$,

(b) $\sum_{i=1}^{N}|C_i\cap K|=|K|$ *for each compact* $K\subseteq\mathbb{R}^n$.

The collection of all $N$-partitions of $\mathbb{R}^n$ is denoted by $\mathbb{P}(N)$. Our assumptions $C_i=C_i^{(1)}$ implies $C_i\cap C_j=\emptyset$ for $i\ne j$. Notice also that we do not exclude the case $C_i=\emptyset$.

The elements of $\mathbb{P}(N)$ are denoted by calligraphic letters $\mathcal{A},\mathcal{B},\mathcal{C},\dots$ and the entries (also called components) of $\mathcal{A}\in\mathbb{P}(N)$ by the corresponding roman letters $(A_1,\dots,A_N)$. The functional

$$(\mathcal{A},\Omega)\in\mathbb{P}(N)\times\mathrm{Op}(\mathbb{R}^n)\mapsto \mathrm{Per}(\mathcal{A},\Omega):=\frac12\sum_{j=1}^{N}P(A_j,\Omega)$$

is called the perimeter of the partition $\mathcal{A}$ in $\Omega$. For simplicity, we write $\mathrm{Per}(\mathcal{A}):=\mathrm{Per}(\mathcal{A},\mathbb{R}^n)$. We set

$$\mathcal{A}\Delta\mathcal{B}:=\bigcup_{j=1}^{N}A_j\Delta B_j$$

and

$$|\mathcal{A}\Delta\mathcal{B}|:=\sum_{j=1}^{N}|A_j\Delta B_j|, \tag{3.1}$$

where $\Delta$ is the symmetric difference of sets, i.e. $E\Delta F=(E\setminus F)\cup(F\setminus E)$.

We say that the sequence $\{\mathcal{A}^{(k)}\}\subseteq\mathbb{P}(N)$ *converges* to $\mathcal{A}\in\mathbb{P}(N)$ in $L^1_{\rm loc}(\mathbb{R}^n)$ if

$$|(\mathcal{A}^{(k)}\Delta\mathcal{A})\cap K|:=\sum_{j=1}^{N}|(A_j^{(k)}\Delta A_j)\cap K|\to 0\qquad\text{as } k\to+\infty$$

for every compact set $K\subseteq\mathbb{R}^n$. Since $E\in BV_{\rm loc}(\mathbb{R}^n,\{0,1\})\mapsto P(E,\Omega)$ is $L^1_{\rm loc}(\mathbb{R}^n)$-lower semicontinuous for any $\Omega\in\mathrm{Op}(\mathbb{R}^n)$, the map $\mathcal{A}\in\mathbb{P}(N)\mapsto\mathrm{Per}(\mathcal{A},\Omega)$ is $L^1_{\rm loc}(\mathbb{R}^n)$-lower semicontinuous. The following compactness result can be proven using [3, Theorem 3.39] and a diagonal argument.

**Theorem 3.2 (Compactness).** *Let $\{\mathcal{A}^{(l)}\}\subset\mathbb{P}(N)$ be a sequence of partitions such that*

$$\sup_{l\ge1}\mathrm{Per}(\mathcal{A}^{(l)},\Omega)<+\infty\qquad\forall\Omega\in\mathrm{Op}_b(\mathbb{R}^n). \tag{3.2}$$

*Then there exist a partition $\mathcal{A}\in\mathbb{P}(N)$ and a subsequence $\{\mathcal{A}^{(l_k)}\}$ such that $\mathcal{A}^{(l_k)}$ converges to $\mathcal{A}$ in $L^1_{\rm loc}(\mathbb{R}^n)$ as $k\to+\infty$.*

The next result is proven for the convenience of the reader.

**Proposition 3.3 (Boundaries of "neighboring" sets).** *Let $\mathcal{A}\in\mathbb{P}(N)$. Then*

$$\mathcal{H}^{n-1}\Big(\partial^*A_i\setminus\bigcup_{j=1,\,j\ne i}^{N}\partial^*A_j\Big)=0\qquad\forall i=1,\dots,N.$$

*Proof.* The case $N = 2$ is classical, so we suppose $N \geq 3$. It is enough to consider $i = 1$. Set

$$\Sigma^{(r)} := \partial^* A_1 \cap \Big( \bigcup_{j=2}^{N} \partial^* A_j \Big), \quad \Sigma^{(s)} := \partial^* A_1 \setminus \Sigma^{(r)}.$$

We divide the proof into four steps.

*Step 1.* If $x \in \partial^* A_1$ then there exists at most one $j \in \{2, \dots, N\}$ such that $x \in \partial^* A_j$. Indeed, otherwise up to a relabelling we would have $x \in \partial^* A_1 \cap \partial^* A_2 \cap \partial^* A_3$ and hence $\sum_{j=1}^{3} \frac{|A_j \cap B_r|}{|B_r|} \leq 1$, where $B_r := B_r(x)$. Now by Theorem 2.1 we get $1 \geq \sum_{j=1}^{3} \lim_{r \to 0^+} \frac{|A_j \cap B_r|}{|B_r|} = \frac{3}{2}$, a contradiction.

*Step 2.* If there exists a unique $j \in \{2, \dots, N\}$ so that $x \in \partial^* A_1 \cap \partial A_j$, then $x \in \partial^* A_j$.

Indeed, since $\partial A_k$ is closed, and $x \notin \partial A_k$ there is $\rho > 0$ such that $\mathrm{dist}(x, \partial A_k) \geq \rho$ for every $k \neq 1, j$, Hence, for every $r \in (0, \rho)$ up to an $\mathcal{L}^n$-negligible set we have $B_r = (A_1 \cup A_j) \cap B_r$ (and $A_1 \cap A_j = \emptyset$). Thus, $P(A_1, B_r) = P(A_j, B_r)$ for all $r \in (0, \rho)$, and since $x \in \partial^* A_1$,

$$\nu_{A_j}(x) := -\lim_{r \to 0^+} \frac{D\chi_{A_j}(B_r)}{P(A_j, B_r)} = \lim_{r \to 0^+} \frac{D\chi_{A_1}(B_r)}{P(A_1, B_r)} = -\nu_{A_1}(x),$$

hence $|\nu_{A_j}(x)| = 1$. This yields $x \in \partial^* A_j$.

*Step 3.* If $x \in \Sigma^{(s)}$, there are at least two indices $2 \leq k < l \leq N$ such that $x \in \partial A_k \cap \partial A_l$. Indeed, since $x \in \partial A_1$, there exists at least one $k \in \{2, \dots, N\}$ such that $x \in \partial A_k$. If $k$ is unique with this property, by Step 2 $x \in \partial^* A_k$ and hence, by definition $x \in \Sigma^{(r)}$.

*Step 4.* Now we prove $\mathcal{H}^{n-1}(\Sigma^{(s)}) = 0$. We may suppose that $\Sigma^{(s)}$ is bounded, otherwise we consider $\Sigma^{(s)} \cap B_R(0)$ and then let $R \to +\infty$.

By Steps 2 and 3, $x \in \Sigma^{(s)}$ if and only if $x \in (\partial A_i \setminus \partial^* A_i) \cap (\partial A_j \setminus \partial^* A_j)$ for some $i > j > 1$, therefore $\Sigma^{(s)} \subseteq \bigcup_{j=2}^{N} (\partial A_j \setminus \partial^* A_j)$ and $\sum_{j=2}^{N} P(A_j, \Sigma^{(s)}) = 0$. Hence for every $\varepsilon > 0$ there exists an open set $U \subseteq \mathbb{R}^n$ such that $\Sigma^{(s)} \subseteq U$ and $\sum_{j=2}^{N} P(A_j, U) < \varepsilon$. Since $\Sigma^{(s)} \subseteq \partial^* A_1$, by Theorem 2.1 for every $x \in \Sigma^{(s)}$, $r^{1-n} P(A_1, B_r(x)) \to \omega_{n-1}$ as $r \to 0^+$, thus there exists $\rho(x) > 0$ such that

$$\frac{\omega_{n-1}}{2} \leq \frac{P(A_1, B_r(x))}{r^{n-1}} \leq 2\omega_{n-1} \qquad \forall r \in (0, \rho(x)). \tag{3.3}$$

Fix $\delta > 0$ and consider the collection of balls $\mathtt{F} := \{B_r(x) :\ x \in \Sigma^{(s)},\ r \in (0, \min\{\delta, \rho(x)\}),\, B_r(x) \subset U\}$. Clearly, this is a fine cover of $\Sigma^{(s)}$ and hence by Vitali Covering Lemma there exists an at most countable disjoint subfamily $\mathtt{F}' \subseteq \mathtt{F}$ with $\Sigma^{(s)} \subseteq \bigcup_{B_{r_k} \in \mathtt{F}'} B_{5r_k}$. Now using (3.3), the definition of partition and (2.7) for the Hausdorff premeasures we get

$$\begin{aligned}
\mathcal{H}^{n-1}_{10\delta}(\Sigma^{(s)}) &\leq \sum_{B_{r_k} \in \mathtt{F}'} \omega_{n-1} (5r_k)^{n-1} = 2 \cdot 5^{n-1} \sum_{B_{r_k} \in \mathtt{F}'} \frac{\omega_{n-1}}{2} r_k^{n-1} \leq 2 \cdot 5^{n-1} \sum_{B_{r_k} \in \mathtt{F}'} P(A_1, B_{r_k}) \\
&= 2 \cdot 5^{n-1} P\Big(A_1, \bigcup_{B_{r_k} \in \mathtt{F}'} B_{r_k}\Big) \leq 2 \cdot 5^{n-1} P(A_1, U) = 2 \cdot 5^{n-1} P\Big(\bigcup_{j=2}^{N} A_j, U\Big) \\
&\leq 2 \cdot 5^{n-1} \sum_{j=2}^{N} P(A_j, U) < 2 \cdot 5^{n-1} \varepsilon.
\end{aligned}$$

Thus, letting $\delta, \varepsilon \to 0^+$, we establish $\mathcal{H}^{n-1}(\Sigma^{(s)}) = 0$. □

**Remark 3.4.** From Proposition 3.3 it follows that

$$\mathrm{Per}(\mathcal{A},\Omega) = \frac{1}{2}\sum_{j=1}^{N}\mathcal{H}^{n-1}(\Omega\cap\partial^* A_j) = \frac{1}{2}\sum_{j=1}^{N}\sum_{i=1,\,i\neq j}^{N}\mathcal{H}^{n-1}(\Omega\cap\partial^* A_j\cap\partial^* A_i).$$

Since $\mathcal{H}^{n-1}(\Omega\cap\partial^* A_j\cap\partial^* A_i)$ is the area of the interface between the phases $A_i$ and $A_j$, $\mathrm{Per}(\mathcal{A},\Omega)$ measures the total perimeter of the interfaces in $\Omega$.

3.1. $(\Lambda, r_0)$ **-minimizers.** In order to prove Theorem 4.6 it is convenient to give the following definition.

**Definition 3.5** ( $(\Lambda, r_0)$ **-minimizers).** *Given $\Lambda\geq 0$ and $r_0\in(0,+\infty]$ we say that a partition $\mathcal{A}\in\mathbb{P}(N)$ is a $(\Lambda,r_0)$-minimizer of* Per *in $\mathbb{R}^n$ (a $(\Lambda,r_0)$-minimizer, for short) if*

$$\mathrm{Per}(\mathcal{A},B_r)\leq\mathrm{Per}(\mathcal{B},B_r)+\Lambda|\mathcal{A}\Delta\mathcal{B}|$$

*whenever $\mathcal{B}\in\mathbb{P}(N)$, $\mathcal{A}\Delta\mathcal{B}\subset\subset B_r$, and $r\in(0,r_0)$.*

The crucial technical tool is the following.

**Theorem 3.6 (Density estimates for $(\Lambda, r_0)$ - minimizers).** *Let $\mathcal{A}\in\mathbb{P}(N)$ be a $(\Lambda,r_0)$-minimizer and $i\in\{1,\dots,N\}$. Then either $\partial A_i=\emptyset$ (i.e. $A_i=\emptyset$ or $A_i=\mathbb{R}^n$ ) or there exists $c(N,n)\in(0,1)$ such that for any $x\in\partial A_i$ and $r\in(0,\hat r_0)$, where $\hat r_0:=\min\{r_0,\frac{n}{4(N-1)\Lambda}\}$ if $\Lambda>0$ and $\hat r_0:=r_0$ if $\Lambda=0$, the following density estimates hold:*

$$\Big(\frac{1}{2N}\Big)^n\leq\frac{|A_i\cap B_r(x)|}{|B_r(x)|}\leq 1-\frac{1}{2^n}\Big(1-\frac{1}{2(N-1)}\Big)^n, \tag{3.4}$$

$$c(N,n)\leq\frac{P(A_i,B_r(x))}{r^{n-1}}\leq\frac{2N-1}{2(N-1)}\,n\omega_n. \tag{3.5}$$

*Moreover,*

$$\sum_{i=1}^{N}\mathcal{H}^{n-1}(\partial A_i\setminus\partial^* A_i)=0. \tag{3.6}$$

*Proof.* We may suppose $i=1$ and $\partial A_i\neq\emptyset$. Moreover, since $\overline{\partial^* A_1}=\partial A_1$, it suffices to show (3.4)-(3.5) whenever $x\in\partial^* A_1$. Choose $r\in(0,\hat r_0)$ such that $B_r:=B_r(x)$ satisfies

$$\sum_{j=1}^{N}\mathcal{H}^{n-1}(\partial B_r\cap\partial^* A_j)=0 \tag{3.7}$$

and define the competitor $\mathcal{B}\in\mathbb{P}(N)$ as

$$\mathcal{B}:=(A_1\cup B_r,A_2\setminus B_r,\dots,A_N\setminus B_r).$$

Then $\mathcal{A}\Delta\mathcal{B}\subset\subset B_s$ for every $s\in(r,\hat r_0)$ and thus, by $(\Lambda,r_0)$-minimality,

$$\begin{aligned}0\leq& 2\,\mathrm{Per}(\mathcal{B},B_s)-2\,\mathrm{Per}(\mathcal{A},B_s)+2\Lambda|\mathcal{A}\Delta\mathcal{B}|=P(A_1\cup B_r,B_s)-P(A_1,B_s)\\ &+\sum_{j=2}^{N}\Big(P(A_j\setminus B_r,B_s)-P(A_j,B_s)\Big)+2\Lambda|B_r\setminus A_1|+2\Lambda\sum_{j=2}^{N}|A_j\cap B_r|.\end{aligned} \tag{3.8}$$

By the disjointness of the $A_j$'s we have

$$\sum_{j=2}^{N}|A_j\cap B_r|=|B_r\setminus A_1|. \tag{3.9}$$

Moreover, recalling that $A_j^{(1)}=A_j$, from (2.5), (3.7) and $\mathcal{H}^{n-1}(B_s\cap\{\nu_{A_j}=-\nu_{B_r}\})=0$, we get

$$P(A_j\setminus B_r,B_s)=P(A_j,B_s\setminus\overline{B_r})+\mathcal{H}^{n-1}(A_j\cap\partial B_r)\qquad\forall j\in\{2,\dots,N\}. \tag{3.10}$$

Thus,

$$\sum_{j=2}^{N}P(A_j\setminus B_r,B_s)=\sum_{j=2}^{N}P(A_j,B_s\setminus\overline{B_r})+\sum_{j=2}^{N}\mathcal{H}^{n-1}(A_j\cap\partial B_r).$$

By the disjointness of the $A_j$'s, Theorem 2.2 and the choice of $r$ in (3.7),

$$\sum_{j=2}^{N} \mathcal{H}^{n-1}(A_j \cap \partial B_r) = \mathcal{H}^{n-1}(A_1^{(0)} \cap \partial B_r) = \mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r).$$

Therefore,

$$\sum_{j=2}^{N} P(A_j \setminus B_r, B_s) = \sum_{j=2}^{N} P(A_j, B_s \setminus \overline{B_r}) + \mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r). \tag{3.11}$$

Finally, since $\mathcal{H}^{n-1}(B_s \cap \{\nu_{A_1} = \nu_{B_r}\}) = 0$ by (3.7), from (2.6) we deduce

$$P(A_1 \cup B_r, B_s) = P(A_1, B_s \setminus \overline{B_r}) + \mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r). \tag{3.12}$$

Now inserting (3.9), (3.11), (3.12) in (3.8) we get

$$P(A_1, B_r) + \sum_{j=2}^{N} P(A_j, B_r) \le 2\mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r) + 4\Lambda |(\mathbb{R}^n \setminus A_1) \cap B_r|. \tag{3.13}$$

Applying (2.7) and using the disjointness of the $A_j$'s we get

$$\sum_{j=2}^{N} P(A_j, B_r) \ge P\Big(\bigcup_{j=2}^{N} A_j, B_r\Big) = P(\mathbb{R}^n \setminus A_1, B_r) = P(A_1, B_r)$$

and thus from (3.13),

$$P(\mathbb{R}^n \setminus A_1, B_r) \le \mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r) + 2\Lambda |(\mathbb{R}^n \setminus A_1) \cap B_r|. \tag{3.14}$$

Adding $\mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r)$ to both sides of this inequality and using (3.7) we establish

$$P((\mathbb{R}^n \setminus A_1) \cap B_r) \le 2\mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r) + 2\Lambda |(\mathbb{R}^n \setminus A_1) \cap B_r|. \tag{3.15}$$

Now by the isoperimetric inequality [12],

$$n\omega_n^{1/n} |(\mathbb{R}^n \setminus A_1) \cap B_r|^{\frac{n-1}{n}} \le 2\mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r) + 2\Lambda |(\mathbb{R}^n \setminus A_1) \cap B_r|. \tag{3.16}$$

Since $r < \hat{r}_0 \le \frac{n}{4(N-1)\Lambda}$,

$$2\Lambda |(\mathbb{R}^n \setminus A_1) \cap B_r|^{\frac{1}{n}} \le 2\Lambda \omega_n^{1/n} \hat{r}_0 \le \frac{n\omega_n^{1/n}}{2(N-1)}.$$

As a result, from (3.16) we obtain

$$\frac{1}{2}\Big(1 - \frac{1}{2(N-1)}\Big) n\omega_n^{1/n} |(\mathbb{R}^n \setminus A_1) \cap B_r|^{\frac{n-1}{n}} \le \mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r).$$

Integrating this differential inequality we get

$$|(\mathbb{R}^n \setminus A_1) \cap B_r| \ge \frac{1}{2^n}\Big(1 - \frac{1}{2(N-1)}\Big)^n \omega_n r^n,$$

i.e.

$$\frac{|A_1 \cap B_r|}{|B_r|} \le 1 - \frac{1}{2^n}\Big(1 - \frac{1}{2(N-1)}\Big)^n,$$

which is the upper volume density estimate in (3.4). Moreover, since $2\Lambda r \le \frac{n}{2(N-1)}$, from (3.14) we obtain

$$P(A_1, B_r) \le \mathcal{H}^{n-1}(\partial B_r) + 2\Lambda |B_r| \le n\omega_n r^{n-1} + \frac{n\omega_n}{2(N-1)} r^{n-1} = \frac{2N-1}{2(N-1)} n\omega_n r^{n-1}$$

for a.e. $r \in (0, \hat{r}_0)$. Now the left-continuity of $\rho \mapsto P(A_1, B_\rho)$ implies the upper perimeter density estimate.

Let us prove the lower volume density estimate. As above we may suppose $i = 1$ and $\partial A_1 \neq \emptyset$. Take $x \in \partial^* A_1$ and set

$$I := \{j \in \{2, \dots, N\} :\ \mathcal{H}^{n-1}(B_{\hat{r}_0}(x) \cap \partial^* A_1 \cap \partial^* A_j) > 0\}.$$

Write as usual $B_\rho := B_\rho(x)$ and let $r \in (0, \hat r_0)$ satisfy (3.7). By virtue of Proposition 3.3 and Remark 3.4,

$$P(A_1, B_r) = \sum_{j=2}^{N} \mathcal{H}^{n-1}(B_r \cap \partial^* A_1 \cap \partial^* A_j) = \sum_{j\in I} \mathcal{H}^{n-1}(B_r \cap \partial^* A_1 \cap \partial^* A_j). \tag{3.17}$$

Since $x \in \partial A_1$, one has $I \neq \emptyset$. For every $j \in I$ let us define the competitor $\mathcal{B}^{(j)} \in \mathbb{P}(N)$ as

$$\mathcal{B}^{(j)} := (A_1 \setminus B_r, A_2, \dots, A_{j-1}, A_j \cup (A_1 \cap B_r), A_{j+1}, \dots, A_N).$$

By the $(\Lambda, r_0)$ -minimality of $\mathcal{A}$, for every $s \in (r, \hat r_0)$ one has

$$P(A_1, B_s) + P(A_j, B_s) \le P(A_1 \setminus B_r, B_s) + P(A_j \cup (A_1 \cap B_r), B_s) + 4\Lambda|A_1 \cap B_r|. \tag{3.18}$$

From (3.7) and (2.1)

$$\partial^*(A_1 \cap B_r) \approx (A_1 \cap \partial B_r) \cup (B_r \cap \partial^* A_1). \tag{3.19}$$

Moreover, $\mathcal{H}^{n-1}(B_s \cap \{\nu_{A_j} = \nu_{A_1\cap B_r}\}) = 0$ for any $j \in I$, therefore, from (2.6)

$$P(A_j \cup (A_1 \cap B_r), B_s) = \mathcal{H}^{n-1}((A_1 \cap B_r)^{(0)} \cap B_s \cap \partial^* A_j) + \mathcal{H}^{n-1}(A_j^{(0)} \cap B_s \cap \partial^*(A_1 \cap B_r)). \tag{3.20}$$

Now according to Theorem 2.2, (3.7), $\mathcal{H}^{n-1}(A_1 \cap B_r \cap \partial^* A_j) = 0$, and (3.19)

$$\begin{aligned}\mathcal{H}^{n-1}((A_1 \cap B_r)^{(0)} \cap B_s \cap \partial^* A_j) =&\mathcal{H}^{n-1}(B_s \cap \partial^* A_j) - \mathcal{H}^{n-1}(B_s \cap \partial^*(A_1 \cap B_r) \cap \partial^* A_j)\\ =&P(A_j, B_s) - \mathcal{H}^{n-1}(B_r \cap \partial^* A_1 \cap \partial^* A_j).\end{aligned}$$

Similarly, since $\mathcal{H}^{n-1}(A_j \cap B_r \cap \partial^*(A_1 \cap B_r)) = 0$, for any $j \in I$

$$\begin{aligned}\mathcal{H}^{n-1}(A_j^{(0)} \cap B_s \cap \partial^*(A_1 \cap B_r)) =&\mathcal{H}^{n-1}(B_s \cap \partial^*(A_1 \cap B_r)) - \mathcal{H}^{n-1}(B_s \cap \partial^*(A_1 \cap B_r) \cap \partial^* A_j)\\ =&\mathcal{H}^{n-1}(A_1 \cap \partial B_r) + P(A_1, B_r) - \mathcal{H}^{n-1}(B_r \cap \partial^* A_1 \cap \partial^* A_j).\end{aligned}$$

Therefore, from (3.20) we get

$$\begin{aligned}P(A_j \cup (A_1 \cap B_r), B_s) =&P(A_j, B_s) + \mathcal{H}^{n-1}(A_1 \cap \partial B_r)\\ &+ P(A_1, B_r) - 2\mathcal{H}^{n-1}(B_r \cap \partial^* A_1 \cap \partial^* A_j).\end{aligned} \tag{3.21}$$

Inserting this and

$$P(A_1 \setminus B_r, B_s) = P(A_1, B_s \setminus \overline{B_r}) + \mathcal{H}^{n-1}(A_1 \cap \partial B_r)$$

(whose proof is the same as (3.10)) in (3.18) and using (3.7) once more we get

$$\mathcal{H}^{n-1}(B_r \cap \partial^* A_1 \cap \partial^* A_j) \le \mathcal{H}^{n-1}(A_1 \cap \partial B_r) + 2\Lambda|A_1 \cap B_r|. \tag{3.22}$$

Summing these inequalities in $j \in I$ and using (3.17) and $|I| \le N - 1$, we obtain

$$P(A_1, B_r) \le (N-1)\mathcal{H}^{n-1}(A_1 \cap \partial B_r) + 2(N-1)\Lambda|A_1 \cap B_r|,$$

whence

$$P(A_1 \cap B_r) \le N\mathcal{H}^{n-1}(A_1 \cap \partial B_r) + 2(N-1)\Lambda|A_1 \cap B_r|. \tag{3.23}$$

Since $2(N-1)\Lambda|A_1 \cap B_r|^{1/n} \le \frac{n\omega_n^{1/n}}{2}$ for any $r < \hat r_0$, from the isoperimetric inequality we get

$$\frac{1}{2N} n\omega_n^{1/n}|A_1 \cap B_r|^{\frac{n-1}{n}} \le \mathcal{H}^{n-1}(A_1 \cap \partial B_r).$$

Now integrating this differential inequality we obtain the lower volume density estimate

$$|A_1 \cap B_r| \ge \left(\frac{1}{2N}\right)^n \omega_n r^n.$$

The lower perimeter density estimate in (3.5) follows from the volume density estimates and the relative isoperimetric inequality for the ball [3, page 152].

Finally, (3.6) is a consequence of a standard covering argument. □

**Remark 3.7.** Let $\alpha_1,\alpha_2>\frac{n-1}{n}$, $\Lambda_1\ge 0$, $\Lambda_2>0$, $r_0\in(0,+\infty]$. Suppose that $\mathcal{A}\in\mathbb{P}(N)$ satisfies

$$\mathrm{Per}(\mathcal{A},B_r)\le \mathrm{Per}(\mathcal{B},B_r)+\Lambda_1|\mathcal{A}\Delta\mathcal{B}|^{\alpha_1}+\Lambda_2|\mathcal{A}\Delta\mathcal{B}|^{\alpha_2}$$

whenever $\mathcal{B}\in\mathbb{P}(N)$, $\mathcal{A}\Delta\mathcal{B}\subset\subset B_r$ and $r\in(0,r_0)$. Then repeating the proof of Theorem 3.6, one obtains that (3.15) and (3.23) are replaced by

$$P((\mathbb{R}^n\setminus A_1)\cap B_r)\le 2\mathcal{H}^{n-1}((\mathbb{R}^n\setminus A_1)\cap\partial B_r)+\Lambda_1|(\mathbb{R}^n\setminus A_1)\cap B_r|^{\alpha_1}+\Lambda_2|(\mathbb{R}^n\setminus A_1)\cap B_r|^{\alpha_2}$$

and

$$P(A_1\cap B_r)\le N\mathcal{H}^{n-1}(A_1\cap\partial B_r)+2(N-1)\Lambda_1|A_1\cap B_r|^{\alpha_1}+2(N-1)\Lambda_2|A_1\cap B_r|^{\alpha_2}$$

respectively and, thus, that for every $i=1,\dots,N$ either $\partial A_i=\emptyset$ or for every $x\in\partial A_i$ and for any $r\in(0,\tilde r_0)$, the relations (3.4)-(3.6) hold, where

$$\tilde r_0=\begin{cases}\min\{r_0,\omega_n^{-1/n}\big(\frac{n\omega_n^{1/n}}{4(N-1)\Lambda_2}\big)^{\frac{1}{n\alpha_2-n+1}}\} & \text{if } \Lambda_1=0,\\ \min\{r_0,\omega_n^{-1/n}\big(\frac{n\omega_n^{1/n}}{8(N-1)\Lambda_1}\big)^{\frac{1}{n\alpha_1-n+1}},\omega_n^{-1/n}\big(\frac{n\omega_n^{1/n}}{8(N-1)\Lambda_2}\big)^{\frac{1}{n\alpha_2-n+1}}\} & \text{if } \Lambda_1>0.\end{cases}$$

This will be used in the proof of Theorem 5.1.

3.2. **Bounded partitions.** The multiphase analog of a bounded phase in $\mathbb{R}^n$ is the following.

**Definition 3.8 (Bounded partition).** *A partition $\mathcal{C}=(C_1,\dots,C_{N+1})\in\mathbb{P}(N+1)$ is called* **bounded** *if $C_i$ is bounded for each $i=1,\dots,N$.*

Therefore, $C_{N+1}$ is the only unbounded entry of $\mathcal{C}$. We denote by $\mathbb{P}_b(N+1)$ the collection of all bounded partitions of $\mathbb{R}^n$.

Given $\mathcal{A}\in\mathbb{P}_b(N+1)$, we denote by

$$\mathrm{co}(\mathcal{A})$$

the closed convex hull of $\bigcup_{i=1}^N A_i$. Since $\mathcal{A}\Delta\mathcal{B}\subset\subset\mathbb{R}^n$ for every $\mathcal{A},\mathcal{B}\in\mathbb{P}_b(N+1)$,

$$|\mathcal{A}\Delta\mathcal{B}|=\sum_{j=1}^{N+1}|A_j\Delta B_j|$$

is the $L^1(\mathbb{R}^n)$-distance in $\mathbb{P}_b(N+1)$.

The following compactness result can be proven similarly to Theorem 3.2.

**Theorem 3.9 (Compactness).** *Let $\mathcal{A}^{(k)}\in\mathbb{P}_b(N+1)$, $k=1,2,\dots$, and $\Omega\in\mathrm{Op}_b(\mathbb{R}^n)$ be such that*

$$\sup_{k\ge1}\mathrm{Per}(\mathcal{A}^{(k)})<+\infty,\qquad \mathrm{co}(\mathcal{A}^{(k)})\subseteq\Omega\qquad\forall k\ge1.$$

*Then there exist $\mathcal{A}\in\mathbb{P}_b(N+1)$ and a subsequence $\{\mathcal{A}^{(k_l)}\}$ converging to $\mathcal{A}$ in $L^1(\mathbb{R}^n)$ as $l\to+\infty$. Moreover, $\bigcup_{i=1}^N A_j\subseteq\overline{\Omega}$.*

## 4. Existence of generalized minimizing movements for bounded partitions

Given $E,F\subseteq\mathbb{R}^n$ set

$$\bar\sigma(E,F):=\int_{E\Delta F}d(x,\partial F)dx.$$

Note that $\bar\sigma(E,F)=0$ if $|E\Delta F|=0$ whereas $\bar\sigma(E,F)=+\infty$ if $\partial F=\emptyset$ and $|E\Delta F|>0$. Moreover, $X,Y\subseteq\mathbb{R}^n$ are measurable and $\partial Y\neq\emptyset$,

$$\begin{aligned}\int_{X\Delta Y}d(x,\partial Y)dx&=\int_X\tilde d(x,\partial Y)dx-\int_Y\tilde d(x,\partial Y)dx\quad\text{if } X\cap Y \text{ is bounded,}\\ \int_{X\Delta Y}d(x,\partial Y)dx&=\int_{Y^c}\tilde d(x,\partial Y)dx-\int_{X^c}\tilde d(x,\partial Y)dx\quad\text{if } X^c\cap Y^c \text{ is bounded.}\end{aligned}\tag{4.1}$$

Now the *nonsymmetric distance* between $\mathcal{A}, \mathcal{B} \in \mathbb{P}_b(N+1)$ is defined as

$$\sigma(\mathcal{A}, \mathcal{B}) := \sum_{i=1}^{N+1} \bar{\sigma}(A_i, B_i),$$

where $N+1 \geq 2$. Observe that for every $\mathcal{B} \in \mathbb{P}_b(N+1)$ the map $\sigma(\cdot, \mathcal{B})$ is $L^1(\mathbb{R}^n)$-lower semicontinuous.

**Definition 4.1 (The functional $F$).** *We let $F : \mathbb{P}_b(N+1) \times \mathbb{P}_b(N+1) \times [1, +\infty) \to [0, +\infty]$ be the functional defined as*

$$F(\mathcal{B}, \mathcal{A}; \lambda) = \mathrm{Per}(\mathcal{B}) + \frac{\lambda}{2}\sigma(\mathcal{B}, \mathcal{A}) = \frac{1}{2}\sum_{j=1}^{N+1} P(B_j) + \frac{\lambda}{2}\sum_{j=1}^{N+1}\int_{B_j \Delta A_j} d(x, \partial A_j)dx.$$

The domain of $F$ is independent of $\mathbb{Z}$, and $F$ is the natural generalization of the Almgren-Taylor-Wang functional [1] to the case of partitions [15, 10]. One can readily check that the map $\mathcal{B} \in \mathbb{P}_b(N+1) \mapsto F(\mathcal{B}, \mathcal{A}; \lambda)$ is $L^1(\mathbb{R}^n)$-lower semicontinuous.

**Theorem 4.2 (Existence of minimizers of $F$).** *Given $\mathcal{A} \in \mathbb{P}_b(N+1)$ and $\lambda \geq 1$ the problem*

$$\inf_{\mathcal{B} \in \mathbb{P}_b(N+1)} F(\mathcal{B}, \mathcal{A}; \lambda) \tag{4.2}$$

*has a solution. Moreover, every minimizer $\mathcal{A}(\lambda) = (A_1(\lambda), \ldots, A_{N+1}(\lambda))$ satisfies the bound*

$$\bigcup_{i=1}^{N} A_i(\lambda) \subseteq \mathrm{co}(\mathcal{A}).$$

*Proof.* Given a partition $\mathcal{B} \in \mathbb{P}_b(N+1)$ define the competitor $\mathcal{B}' \in \mathbb{P}_b(N+1)$ as

$$\mathcal{B}' := \Big(B_1 \cap \mathrm{co}(\mathcal{A}), \ldots, B_N \cap \mathrm{co}(\mathcal{A}), \mathbb{R}^n \setminus \bigcup_{i=1}^{N}(B_i \cap \mathrm{co}(\mathcal{A}))\Big). \tag{4.3}$$

Since $\mathrm{co}(\mathcal{A})$ is convex and closed, by the comparison theorem of [2, page 152] we have $P(B_i) \geq P(B_i \cap \mathrm{co}(\mathcal{A}))$ for $i = 1, \ldots, N$, and

$$P(B_{N+1}) = P\Big(\bigcup_{i=1}^{N} B_i\Big) \geq P\Big(\Big(\bigcup_{i=1}^{N} B_i\Big) \cap \mathrm{co}(\mathcal{A})\Big) = P\Big(\bigcup_{i=1}^{N}(B_i \cap \mathrm{co}(\mathcal{A}))\Big) = P\Big(\mathbb{R}^n \setminus \bigcup_{i=1}^{N}(B_i \cap \mathrm{co}(\mathcal{A}))\Big),$$

with equality if and only if $|\bigcup_{i=1}^{N} B_i \setminus \mathrm{co}(\mathcal{A})| = 0$. In addition, for $i = 1, \ldots, N$

$$\begin{aligned}
\int_{B_i \Delta A_i} d(x, \partial A_i)dx &= \int_{B_i \setminus A_i} d(x, \partial A_i)dx + \int_{A_i \setminus B_i} d(x, \partial A_i)dx \\
&\geq \int_{(B_i \cap \mathrm{co}(\mathcal{A})) \setminus A_i} d(x, \partial A_i)dx + \int_{A_i \setminus (B_i \cap \mathrm{co}(\mathcal{A}))} d(x, \partial A_i)dx \\
&= \int_{(B_i \cap \mathrm{co}(\mathcal{A})) \Delta A_i} d(x, \partial A_i)dx,
\end{aligned} \tag{4.4}$$

where we used the nonnegativity of the distance function and $A_i \setminus B_i = A_i \setminus (B_i \cap \mathrm{co}(\mathcal{A}))$. The equality in (4.4) holds if and only if $\Big|\bigcup_{i=1}^{N} B_i \setminus \mathrm{co}(\mathcal{A})\Big| = 0$. For the same reason, since $A_{N+1}^c = \bigcup_{i=1}^{N} A_i \subseteq \mathrm{co}(\mathcal{A})$,

$$\begin{aligned}
\int_{B_{N+1} \Delta A_{N+1}} d(x, \partial A_{N+1})dx &= \int_{B_{N+1}^c \Delta A_{N+1}^c} d(x, \partial A_{N+1})dx \\
&\geq \int_{(B_{N+1}^c \cap \mathrm{co}(\mathcal{A})) \Delta A_{N+1}^c} d(x, \partial A_{N+1})dx.
\end{aligned}$$

So we have

$$F(\mathcal{B}, \mathcal{A}; \lambda) \geq F(\mathcal{B}', \mathcal{A}; \lambda) \qquad \forall \mathcal{B} \in \mathbb{P}_b(N+1)$$

and the inequality is strict whenever $\left|\bigcup_{i=1}^{N} B_i \setminus \mathrm{co}(\mathcal{A})\right| > 0.$

Let $\{\mathcal{B}^{(k)}\} \subseteq \mathbb{P}_b(N+1)$ be a minimizing sequence, which can be supposed so that $\mathrm{co}(\mathcal{B}^{(k)}) \subseteq \mathrm{co}(\mathcal{A})$ and $F(\mathcal{B}^{(k)}, \mathcal{A}; \lambda) \leq F(\mathcal{T}, \mathcal{A}; \lambda)$, $\mathcal{T} := (\emptyset, \dots, \emptyset, \mathbb{R}^n)$ being the trivial partition, so that

$$\mathrm{Per}(\mathcal{B}^{(k)}) \leq \frac{\lambda}{2} \sigma(\mathcal{T}, \mathcal{A}) = \frac{\lambda}{2} \sum_{j=1}^{N} \int_{A_j} \Big( d(x, \partial A_j) + d(x, \partial A_{N+1}) \Big)\, dx \qquad \forall k \geq 1.$$

By Proposition 3.9 there exists $\mathcal{A}(\lambda) \in \mathbb{P}_b(N+1)$ such that $\mathcal{B}^{(k)} \to \mathcal{A}(\lambda)$ in $L^1(\mathbb{R}^n)$ as $k \to +\infty$. Then the $L^1(\mathbb{R}^n)$ -lower semicontinuity of $F(\cdot, \mathcal{A}; \lambda)$ implies that $\mathcal{A}(\lambda)$ is a solution to (4.2).

Now let $\mathcal{A}(\lambda)$ be a minimizer of $F(\cdot, \mathcal{A}; \lambda)$. If $\left|\bigcup_{j=1}^{N} A_j(\lambda) \setminus \mathrm{co}(\mathcal{A})\right| > 0$, then, as shown above, $F(\mathcal{A}(\lambda), \mathcal{A}; \lambda) > F(\mathcal{A}(\lambda)', \mathcal{A}; \lambda)$, where $\mathcal{A}(\lambda)'$ is defined as in (4.3), which contradicts the minimality of $\mathcal{A}(\lambda)$. □

**Remark 4.3.** Let $C \subseteq \mathbb{R}^n$ be a compact convex set. Suppose that $\mathcal{G} \in \mathbb{P}_b(N+1)$ satisfies $\bigcup_{j=1}^{N} G_j \subseteq C$; from Theorem 4.2 it follows that every minimizer $\mathcal{A}(\lambda) \in \mathbb{P}_b(N+1)$ of $F(\cdot, \mathcal{G}; \lambda)$ satisfies $\mathrm{co}(\mathcal{A}(\lambda)) \subseteq C$. This property gives an a priori bound for minimizers of $F(\cdot, \mathcal{G}; \lambda)$ using only the bound for the initial partition and will be used in the proofs of Theorems 4.10 and 5.1.

**Remark 4.4.** Suppose that $\mathcal{G} \in \mathbb{P}_b(N+1)$ and $G_i = \emptyset$ for some $i \in \{1, \dots, N\}$. Then by definition of $\bar{\sigma}$ every minimizer $\mathcal{A}(\lambda) \in \mathbb{P}_b(N+1)$ of $F(\cdot, \mathcal{G}; \lambda)$ satisfies $A_i(\lambda) = \emptyset$. In particular, for $\mathcal{G} = (G, \emptyset, \dots, \emptyset, \mathbb{R}^n \setminus G)$, the $GMM$ problem for $F(\cdot, \mathcal{G}; \lambda)$ agrees with the $GMM$ problem of the Almgren-Taylor-Wang functional

$$E \in BV(\mathbb{R}^n) \mapsto P(E) + \lambda \int_{E \Delta G} d(x, \partial G) dx. \tag{4.5}$$

**Proposition 4.5 (Behaviour of $\mathcal{A}(\lambda)$ as time goes to $0$).** *Let $\mathcal{A} \in \mathbb{P}_b(N+1)$ be such that $\sum_{j=1}^{N+1} |\overline{A_j} \setminus A_j| = 0$, and $\mathcal{A}(\lambda)$ be a minimizer of $F(\cdot, \mathcal{A}; \lambda)$. Then:*

a) $\lim_{\lambda \to +\infty} |\mathcal{A}(\lambda) \Delta \mathcal{A}| = 0,$

b) $\lim_{\lambda \to +\infty} \mathrm{Per}(\mathcal{A}(\lambda)) = \mathrm{Per}(\mathcal{A}),$

c) $\lim_{\lambda \to +\infty} \lambda \sigma(\mathcal{A}(\lambda), \mathcal{A}) = 0.$

*Proof.* a) Choose any sequence $\lambda_k \to +\infty$. Since $F(\mathcal{A}(\lambda_k), \mathcal{A}; \lambda_k) \leq F(\mathcal{A}, \mathcal{A}; \lambda_k) = \mathrm{Per}(\mathcal{A})$, we have $\mathrm{Per}(\mathcal{A}(\lambda)) \leq \mathrm{Per}(\mathcal{A})$ and

$$\lim_{k \to +\infty} \sigma(\mathcal{A}(\lambda_k), \mathcal{A}) = 0. \tag{4.6}$$

Moreover, by Theorem 4.2 $\mathrm{co}(\mathcal{A}(\lambda)) \subseteq \mathrm{co}(\mathcal{A})$, therefore Proposition 3.9 yields the existence of a subsequence $\{\lambda_{k_l}\}_l$ and of $\mathcal{B} \in \mathbb{P}_b(N+1)$ such that $\mathcal{A}(\lambda_{k_l}) \to \mathcal{B}$ in $L^1(\mathbb{R}^n)$ as $l \to +\infty$. Now the lower semicontinuity of $\sigma(\cdot, \mathcal{A})$ and (4.6) imply $\sigma(\mathcal{B}, \mathcal{A}) = 0$. Then from the assumption on $\mathcal{A}$ we get $\mathcal{A} = \mathcal{B}$. Since $\lambda_k$ is arbitrary, a) follows.

b) Since $\mathrm{Per}(\mathcal{A}(\lambda)) \leq \mathrm{Per}(\mathcal{A})$, from a) we obtain

$$\mathrm{Per}(\mathcal{A}) \leq \liminf_{\lambda \to +\infty} \mathrm{Per}(\mathcal{A}(\lambda)) \leq \limsup_{\lambda \to +\infty} \mathrm{Per}(\mathcal{A}(\lambda)) \leq \mathrm{Per}(\mathcal{A}).$$

c) From b) we have

$$\limsup_{\lambda \to +\infty} \lambda \sigma(\mathcal{A}(\lambda), \mathcal{A}) \leq 2 \limsup_{\lambda \to +\infty} (\mathrm{Per}(\mathcal{A}) - \mathrm{Per}(\mathcal{A}(\lambda))) = 0.$$

□

**Theorem 4.6 (Density estimates).** *Suppose that* $\mathcal{A} \in \mathbb{P}_b(N+1)$ *and let* $\mathcal{A}(\lambda) \in \mathbb{P}_b(N+1)$ *be a minimizer of* $F(\cdot, \mathcal{A}; \lambda)$. *Then for every* $i \in \{1, \ldots, N+1\}$ *either* $\partial A_i(\lambda)$ *is empty or there exists* $c(N,n) \in (0,1)$ *such that*

$$\Big(\frac{1}{2(N+1)}\Big)^n \le \frac{|A_i(\lambda) \cap B_r(x)|}{|B_r(x)|} \le 1 - \frac{1}{2^n}\Big(1 - \frac{1}{2N}\Big)^n, \tag{4.7}$$

$$c(N,n) \le \frac{P(A_i(\lambda), B_r(x))}{r^{n-1}} \le \frac{2N+1}{2N} n\omega_n \tag{4.8}$$

*for any* $x \in \partial A_i(\lambda)$ *and* $r \in (0, \min\{1, \frac{n}{2\lambda N(\operatorname{diam}\operatorname{co}(\mathcal{A})+2)}\})$. *Moreover*

$$\sum_{j=1}^{N+1} \mathcal{H}^{n-1}(\partial A_j(\lambda) \setminus \partial^* A_j(\lambda)) = 0.$$

**Proof of Theorem 4.6.** Fix $r_0 > 0$. Then for every $x \in \mathbb{R}^n$ and $\mathcal{C} \in \mathbb{P}_b(N+1)$ such that $\mathcal{C}\Delta\mathcal{A}(\lambda) \subset\subset B_\rho(x)$ with $\rho \in (0, r_0)$, by Theorem 4.2 one has

$$d(z, \partial A_i) \le \operatorname{diam}\operatorname{co}(\mathcal{A}) + 2\rho \qquad \forall i = 1, \ldots, N+1,\ z \in \mathcal{C}\Delta\mathcal{A}(\lambda).$$

Therefore the minimality of $\mathcal{A}(\lambda)$ implies

$$\operatorname{Per}(\mathcal{A}(\lambda), B_\rho(x)) \le \operatorname{Per}(\mathcal{C}, B_\rho(x)) + \frac{\lambda}{2}\big(\operatorname{diam}\operatorname{co}(\mathcal{A}) + 2r_0\big)|\mathcal{C}\Delta\mathcal{A}(\lambda)|,$$

i.e.

$$\mathcal{A}(\lambda) \text{ is a } (\Lambda, r_0)\text{-minimizer with } \Lambda = \frac{\lambda}{2}\big(\operatorname{diam}\operatorname{co}(\mathcal{A}) + 2r_0\big).$$

Now application of Theorem 3.6 to $\mathcal{A}(\lambda)$ with $r_0 = 1$ finishes the proof. $\square$

**Remark 4.7.** The density estimates show that the entries of $\mathcal{A}(\lambda)$ are Lebesgue-equivalent to open sets. Indeed, since using $\overline{E} \setminus E \subset \partial E$, and $\overline{E} \setminus \overset{\circ}{E} \subset \partial E$ ($\overset{\circ}{E}$ being the interior of $E$), we have

$$\sum_{j=1}^{N+1} |A_j(\lambda)\Delta\overline{\overset{\circ}{A_j}(\lambda)}| \le \sum_{j=1}^{N+1} |\overline{A_j(\lambda)} \setminus A_j(\lambda)| + \sum_{j=1}^{N+1} |\overline{A_j(\lambda)} \setminus \overline{\overset{\circ}{A_j}(\lambda)}| \le 2\sum_{j=1}^{N+1} |\partial A_j(\lambda)|.$$

Now by the density estimates $\sum_{j=1}^{N+1} |\partial A_j(\lambda)| = 0$, and therefore $\sum_{j=1}^{N+1} |A_j(\lambda)\Delta\overline{\overset{\circ}{A_j}(\lambda)}| = 0$.

To prove the existence of $GMM$, we need the following volume-distance inequality from [1].

**Proposition 4.8.** *Suppose that* $C$ *is a compact subset of* $\mathbb{R}^n$, $A \subseteq \mathbb{R}^n$ *is Lebesgue measurable,* $\delta, \theta$ *and* $\gamma$ *are positive numbers such that*

$$\lambda \int_A d(x, C)dx \le \gamma, \tag{4.9}$$

*and*

$$\mathcal{H}^{n-1}(C \cap B_r(x)) \ge \theta r^{n-1}$$

*whenever* $x \in C$ *and* $0 < r \le \delta$. *Then for each* $\rho \in (\delta, +\infty)$

$$|A \setminus C| \le \Big[2\Gamma\Big(\frac{\rho}{\delta}\Big)^{n-1}\mathcal{H}^{n-1}(C)\Big]^{1/2}\lambda^{-1/2}\gamma^{1/2} + \frac{\gamma}{\rho\lambda},$$

*where*

$$\Gamma := 2^{2n+1} n\omega_n \beta(n)/\theta, \tag{4.10}$$

*and* $\beta(n)$ *is the Besicovitch constant.*

**Remark 4.9.** The assertion of Proposition 4.8 still holds for $\gamma = 0$. Indeed, $\gamma = 0$ in (4.9) implies that $|A \setminus C| = 0$.

One of the main results of the present paper reads as follows.

**Theorem 4.10 (Existence of $GMM$).** *Let $\mathcal{G} \in \mathbb{P}_b(N+1)$. Then $GMM(F,\mathcal{G})$ is non empty. Moreover, there exists a constant $\widehat{c} = \widehat{c}(N,n,\mathcal{G}) > 0$ such that for any $\mathcal{M} \in GMM(F,\mathcal{G})$,*

$$|\mathcal{M}(t)\Delta\mathcal{M}(t')| \le \widehat{c}\,|t-t'|^{\frac{1}{n+1}} \qquad \forall t,t' > 0,\ |t-t'| < 1 \tag{4.11}$$

*and*

$$\bigcup_{j=1}^{N} M_j(t) \subseteq \mathrm{co}(\mathcal{G}) \qquad \forall t \ge 0. \tag{4.12}$$

*In addition, if $\sum\limits_{j=1}^{N+1} |\overline{G_j} \setminus G_j| = 0$, then (4.11) holds for any $t,t' \ge 0$ and $|t-t'| < 1$.*

*Proof.* Set $2R := \mathrm{diam}\,\mathrm{co}(\mathcal{G})$. Let $\mathcal{L}(\lambda,k) = (L_1(\lambda,k),\dots,L_{N+1}(\lambda,k))$, $\lambda \ge 1$, $k \in \mathbb{N}_0$ be defined as follows: $\mathcal{L}(\lambda,0) := \mathcal{G}$, and for $k \ge 1$

$$F(\mathcal{L}(\lambda,k),\mathcal{L}(\lambda,k-1);\lambda) = \min_{\mathcal{A}\in\mathbb{P}_b(N+1)} F(\mathcal{A},\mathcal{L}(\lambda,k-1);\lambda);$$

recall that the existence of minimizers follows from Theorem 4.2 and also

$$\bigcup_{j=1}^{N} L_j(\lambda,k) \subseteq \mathrm{co}(G) \qquad \forall \lambda \ge 1,\ k \in \mathbb{N}_0. \tag{4.13}$$

Clearly, $F(\mathcal{L}(\lambda,k),\mathcal{L}(\lambda,k-1);\lambda) \le F(\mathcal{L}(\lambda,k-1),\mathcal{L}(\lambda,k-1);\lambda)$, hence

$$\lambda\sigma(\mathcal{L}(\lambda,k),\mathcal{L}(\lambda,k-1)) \le 2\big(\mathrm{Per}(\mathcal{L}(\lambda,k-1)) - \mathrm{Per}(\mathcal{L}(\lambda,k))\big) \qquad \forall k \ge 1. \tag{4.14}$$

Therefore, the sequence $k \in \mathbb{N}_0 \mapsto \mathrm{Per}(\mathcal{L}(\lambda,k))$ is nonincreasing, and $\mathrm{Per}(\mathcal{L}(\lambda,k)) \le \mathrm{Per}(\mathcal{G})$ for all $k \in \mathbb{N}_0$ and $\lambda \ge 1$ since $\mathcal{L}(\lambda,0) = \mathcal{G}$.

For every $t,t' > 0$, $0 < t-t' < 1$ let us prove

$$|\mathcal{L}(\lambda,[\lambda t])\Delta\mathcal{L}(\lambda,[\lambda t'])| \le \widehat{c}(N,n,\mathcal{G})|t-t'|^{\frac{1}{n+1}} + \widetilde{c}(N,n,\mathcal{G})|t-t'|^{-\frac{n-1}{2(n+1)}}\lambda^{-1/2} \tag{4.15}$$

provided that $\lambda$ is sufficiently large depending on $|t-t'|$, $n$, $N$ and $R$, where

$$\widehat{c}(N,n,\mathcal{G}) := \left(\sqrt{8\Gamma(N+1)} + \frac{8N(N+1)(R+1)}{n}\right)\mathrm{Per}(\mathcal{G}),$$

$$\widetilde{c}(N,n,\mathcal{G}) := \sqrt{8\Gamma(N+1)}\,\mathrm{Per}(\mathcal{G}),$$

and $\Gamma$ is given by (4.10) for the choice of $\theta = c(N,n)$ in (4.8).

Set $k_0 := [\lambda t']$, $m_0 := [\lambda t]$. Let $\lambda \ge \frac{n}{4(R+1)N}$ be so large that $m_0 \ge k_0 + 3 \ge 4$. We apply Proposition 4.8 as follows: for $i \in \{1,\dots,N+1\}$ and $k \ge k_0+1$ we take

$$A = L_i(\lambda,k)\Delta L_i(\lambda,k-1),$$

$$C = \partial L_i(\lambda,k-1).$$

$C$ satisfies the lower perimeter density estimate according to Theorem 4.6 with $\theta = c(N,n)$, and since it satisfies also the upper density estimates, we have $|A \setminus C| = |A|$. Thus (4.9) follows from (4.14) with

$$\gamma = 2\big(\mathrm{Per}(\mathcal{L}(\lambda,k-1)) - \mathrm{Per}(\mathcal{L}(\lambda,k))\big).$$

Now choose

$$\rho = \frac{n}{4N\lambda(R+1)}\,|t-t'|^{-1/(n+1)}, \qquad \delta = \frac{n}{4N\lambda(R+1)}.$$

From Proposition 4.8 for $k \geq k_0 + 1$ we get

$$|\mathcal{L}(\lambda, k)\Delta\mathcal{L}(\lambda, k-1)| = \sum_{j=1}^{N+1} |L_j(\lambda, k)\Delta L_j(\lambda, k-1)|$$

$$\leq \frac{\sqrt{4\Gamma}|t-t'|^{-\frac{n-1}{2(n+1)}}}{\lambda^{1/2}} \Big( \mathrm{Per}(\mathcal{L}(\lambda, k-1)) - \mathrm{Per}(\mathcal{L}(\lambda, k))\Big)^{\frac{1}{2}} \sum_{j=1}^{N+1} \big(P(L_j(\lambda, k-1))\big)^{\frac{1}{2}}$$

$$+ \frac{8N(N+1)(R+1)}{n} |t-t'|^{\frac{1}{n+1}} \Big( \mathrm{Per}(\mathcal{L}(\lambda, k-1)) - \mathrm{Per}(\mathcal{L}(\lambda, k))\Big).$$

From the inequality $\sum_{i=1}^{l} a_i \leq (l \sum_{i=1}^{l} a_i^2)^{1/2}$ we get

$$\sum_{j=1}^{N+1} \big(P(L_j(\lambda, k-1))\big)^{1/2} \leq \Big((N+1) \sum_{j=1}^{N+1} P(L_j(\lambda, k-1))\Big)^{1/2}$$

$$= \Big(2(N+1)\,\mathrm{Per}(\mathcal{L}(\lambda, k-1))\Big)^{1/2} \leq \Big(2(N+1)\,\mathrm{Per}(\mathcal{G})\Big)^{1/2}.$$

Now using

$$\sum_{k=k_0+1}^{m_0} \Big( \mathrm{Per}(\mathcal{L}(\lambda, k-1)) - \mathrm{Per}(\mathcal{L}(\lambda, k))\Big) \leq \mathrm{Per}(\mathcal{G}),$$

and

$$\sum_{k=k_0+1}^{m_0} \Big( \mathrm{Per}(\mathcal{L}(\lambda, k-1)) - \mathrm{Per}(\mathcal{L}(\lambda, k))\Big)^{\frac{1}{2}}$$

$$\leq (m_0 - k_0)^{\frac{1}{2}} \bigg( \sum_{k=k_0+1}^{m_0} \Big( \mathrm{Per}(\mathcal{L}(\lambda, k-1)) - \mathrm{Per}(\mathcal{L}(\lambda, k))\Big)\bigg)^{\frac{1}{2}} \leq (m_0 - k_0)^{\frac{1}{2}} \Big( \mathrm{Per}(\mathcal{G})\Big)^{1/2}$$

we get

$$|\mathcal{L}(\lambda, [\lambda t])\Delta\mathcal{L}(\lambda, [\lambda t'])| \leq \sum_{k=k_0+1}^{m_0} |\mathcal{L}(\lambda, k)\Delta\mathcal{L}(\lambda, k-1)|$$

$$\leq \frac{\Big(8\Gamma(N+1)\,\mathrm{Per}(\mathcal{G})\Big)^{\frac{1}{2}}}{\lambda^{1/2}|t-t'|^{\frac{n-1}{2(n+1)}}} \sum_{k=k_0+1}^{m_0} \Big( \mathrm{Per}(\mathcal{L}(\lambda, k-1)) - \mathrm{Per}(\mathcal{L}(\lambda, k))\Big)^{\frac{1}{2}}$$

$$+ \frac{8N(N+1)(R+1)}{n} |t-t'|^{\frac{1}{n+1}} \sum_{k=k_0+1}^{m_0} \Big( \mathrm{Per}(\mathcal{L}(\lambda, k-1)) - \mathrm{Per}(\mathcal{L}(\lambda, k))\Big)$$

$$\leq \frac{\sqrt{8\Gamma(N+1)}\,\mathrm{Per}(\mathcal{G})}{|t-t'|^{\frac{n-1}{2(n+1)}}} \Big(\frac{m_0 - k_0}{\lambda}\Big)^{\frac{1}{2}} + \frac{8N(N+1)(R+1)}{n}\,\mathrm{Per}(\mathcal{G})|t-t'|^{\frac{1}{n+1}}.$$

By the definition of $k_0$ and $m_0$ we have

$$\frac{m_0 - k_0}{\lambda} \leq |t-t'| + \frac{1}{\lambda},$$

hence, using $(a+b)^{1/2} \le a^{1/2}+b^{1/2}$,

$$
\begin{aligned}
|\mathcal{L}(\lambda,[\lambda t])\Delta\mathcal{L}(\lambda,[\lambda t'])| \le & \frac{\sqrt{8\Gamma(N+1)}\,\mathrm{Per}(\mathcal{G})}{|t-t'|^{\frac{n-1}{2(n+1)}}}\Big(|t-t'|+\frac{1}{\lambda}\Big)^{1/2} \\
& + \frac{8N(N+1)(R+1)}{n}\,\mathrm{Per}(\mathcal{G})|t-t'|^{\frac{1}{n+1}} \\
\le & \widehat{c}(N,n,\mathcal{G})|t-t'|^{\frac{1}{n+1}} + \widetilde{c}(N,n,\mathcal{G})|t-t'|^{-\frac{n-1}{2(n+1)}}\lambda^{-1/2},
\end{aligned}
$$

which is (4.15).

Now we prove the assertions of the theorem. Using (4.13), the inequality $\mathrm{Per}(\mathcal{L}(\lambda,k)) \le \mathrm{Per}(\mathcal{G})$, Proposition 3.9 and a diagonal argument we obtain the existence of a diverging sequence $\{\lambda_k\}$ and $\mathcal{M}(t) \in \mathbb{P}_b(N+1)$ such that

$$
\lim_{k\to+\infty} |\mathcal{L}(\lambda_k,[\lambda_k t])\Delta\mathcal{M}(t)| = 0 \tag{4.16}
$$

for every rational $t>0$ and also (4.12) holds. By (4.15) $\mathcal{M}(t)$ satisfies

$$
|\mathcal{M}(t)\Delta\mathcal{M}(t')| \le \widehat{c}(N,n,\mathcal{G})|t-t'|^{\frac{1}{n+1}} \qquad \forall t',t \in \mathbb{Q}\cap(0,+\infty),\ |t-t'|<1.
$$

Hence this map extends uniquely to a map $\{\mathcal{M}(t) : t>0\} \subseteq \mathbb{P}_b(N+1)$ satisfying (4.11) and (4.12).

It remains to show that $\mathcal{M} \in GMM(F,\mathcal{G})$. Since $\mathcal{L}(\lambda,0)=\mathcal{G}$, and we need just to prove (4.16) for any $t \ge 0$. Case $t=0$ is trivial: $\mathcal{M}(0)=\mathcal{G}$. Fix $t>0$. For every $\varepsilon \in (0,1)$ take $t_\varepsilon \in \mathbb{Q}\cap(0,+\infty)$ such that $|t-t_\varepsilon| < \varepsilon^{n+1}$. Since $\mathcal{M}$ satisfies (4.11), from (4.15) and (4.16) we deduce

$$
\begin{aligned}
\limsup_{k\to+\infty} |\mathcal{L}(\lambda_k,[\lambda_k t])\Delta\mathcal{M}(t)| \le & \limsup_{k\to+\infty} |\mathcal{L}(\lambda_k,[\lambda_k t])\Delta\mathcal{L}(\lambda_k,[\lambda_k t_\varepsilon])| \\
& + \limsup_{k\to+\infty} |\mathcal{L}(\lambda_k,[\lambda_k t_\varepsilon])\Delta\mathcal{M}(t_\varepsilon)| + \limsup_{k\to+\infty} |\mathcal{M}(t_\varepsilon)\Delta\mathcal{M}(t)| \\
\le & 2\widehat{c}(N,n,\mathcal{G})|t-t_\varepsilon|^{\frac{1}{n+1}} \le 2\widehat{c}(N,n,\mathcal{G})\varepsilon.
\end{aligned}
$$

Hence, (4.16) is obtained letting $\varepsilon \to 0^+$.

Finally, let $\sum\limits_{j=1}^{N+1} |\overline{G_j}\setminus G_j| = 0$. Given $t \in (0,1)$, choosing $\lambda$ sufficiently large, from (4.15) we get

$$
\begin{aligned}
|\mathcal{L}(\lambda,[\lambda t])\Delta\mathcal{L}(\lambda,0)| \le & \mathcal{L}(\lambda,[\lambda t])\Delta\mathcal{L}(\lambda,1)| + |\mathcal{L}(\lambda,1)\Delta\mathcal{G}| \\
\le & \widehat{c}(N,n,\mathcal{G})\Big|t-\frac{1}{\lambda}\Big|^{\frac{1}{n+1}} + \frac{\widetilde{c}(N,n,\mathcal{G})}{\lambda^{1/2}|t-\frac{1}{\lambda}|^{\frac{n-1}{2(n+1)}}} + |\mathcal{L}(\lambda,1)\Delta\mathcal{G}|.
\end{aligned}
$$

Now letting $\lambda \to +\infty$ and using Proposition 4.5 a) we establish

$$
|\mathcal{M}(t)\Delta\mathcal{M}(0)| \le \widehat{c}(N,n,\mathcal{G})\, t^{\frac{1}{n+1}}.
$$

□

In order to improve the Hölder exponent $\frac{1}{n+1}$ to the value $\frac{1}{2}$ in (4.11) we expect to be useful, for minimizers $\mathcal{A}(\lambda)$ of $F(\cdot,\mathcal{A};\lambda)$, an estimate of the form

$$
\sum_{i=1}^{N+1} \sup_{A_i(\lambda)\Delta A_i} d(\cdot,\partial A_i) \le O(\lambda^{-1/2}).
$$

We miss the proof of such an estimate; however, a partial result in this direction is given in Lemma 6.4.

## 5. Existence of $GMM$ for bounded partitions in the presence of external forces

In this section we consider the problem of the mean curvature evolution of bounded partitions with forcing terms. Given $\mathcal{A} \in \mathbb{P}_b(N+1)$ and measurable functions $H_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, N+1$, consider the functional

$$F_H(\mathcal{B}, \mathcal{A}; \lambda) = F(\mathcal{B}, \mathcal{A}; \lambda) + \sum_{i=1}^{N+1} \int_{B_i} H_i dx, \qquad \mathcal{B} \in \mathbb{P}_b(N+1).$$

When $N = 1$ and $H_2 = 0$, we get the Almgren-Taylor-Wang functional with an external force $H_1$ which is nonnegative outside a sufficiently large ball.

We suppose:

$$\begin{cases} H_i \in L^p_{\rm loc}(\mathbb{R}^n), i = 1, \dots, N+1, \text{ for some } p > n \text{ and } H_{N+1} \in L^1(\mathbb{R}^n); \\ \text{there exists } R > 0 \text{ such that } H_i \geq H_{N+1} \text{ a.e. in } \mathbb{R}^n \setminus B_R(0) \text{ for any } i = 1, \dots, N; \end{cases} \tag{5.1}$$

in particular $F_H(\cdot, \mathcal{A}; \lambda)$ is well-defined and $L^1(\mathbb{R}^n)$-lower semicontinuous.

The aim of this section is to prove the following result, generalizing Theorem 4.10.

**Theorem 5.1.** *Suppose that* $H_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, N+1$, *satisfy* (5.1) *and let* $\mathcal{G} \in \mathbb{P}_b(N+1)$. *Then* $GMM(F_H, \mathcal{G})$ *is non empty. Moreover, there exists a constant* $\mathrm{C} = \mathrm{C}(N, n, \mathcal{G}, p, H_1, \dots, H_{N+1}) > 0$ *such that for any* $\mathcal{M} \in GMM(F_H, \mathcal{G})$

$$|\mathcal{M}(t) \Delta \mathcal{M}(t')| \leq \mathrm{C}|t - t'|^{\frac{1}{n+1}}, \qquad \forall t, t' > 0, \ |t - t'| < 1 \tag{5.2}$$

*and*

$$\bigcup_{j=1}^{N} M_j(t) \subseteq \textit{closed convex hull of } \operatorname{co}(\mathcal{G}) \cup B_R(0) \qquad \forall t \geq 0. \tag{5.3}$$

*In addition, if* $\sum\limits_{j=1}^{N+1} |\overline{G_j} \setminus G_j| = 0$, *then* (5.2) *holds for any* $t, t' \geq 0$ *and* $|t - t'| < 1$.

Since the proof of this theorem is a minor modification of the proof of Theorem 4.10, we just sketch it.

*Proof. Step 1.* Given $\mathcal{A} \in \mathbb{P}_b(N+1)$, the problem

$$\inf_{\mathcal{B} \in \mathbb{P}_b(N+1)} F_H(\mathcal{B}, \mathcal{A}; \lambda)$$

has a solution. Let $D$ stand for the closed convex hull of $\operatorname{co}(\mathcal{A}) \cup B_R(0)$ and for every $\mathcal{B} \in \mathbb{P}_b(N+1)$ define the competitor $\mathcal{B}' \in \mathbb{P}_b(N+1)$ as

$$\mathcal{B}' := \Big( B_1 \cap D, \dots, B_N \cap D, \mathbb{R}^n \setminus \bigcup_{i=1}^{N} (B_i \cap D) \Big).$$

Observe that

$$F_H(\mathcal{B}, \mathcal{A}; \lambda) = F(\mathcal{B}, \mathcal{A}; \lambda) + \sum_{j=1}^{N} \int_{B_j} (H_j - H_{N+1}) dx + \int_{\mathbb{R}^n} H_{N+1} dx. \tag{5.4}$$

By Remark 4.3 we have $F(\mathcal{B}, \mathcal{A}; \lambda) \geq F(\mathcal{B}', \mathcal{A}; \lambda)$ with the equality if and only if $\Big| \bigcup\limits_{j=1}^{N} B_j \setminus D \Big| = 0$. Since $H_i \geq H_{N+1}$ a.e. in $\mathbb{R}^n \setminus D$, one has also

$$\sum_{j=1}^{N} \int_{B_j} (H_j - H_{N+1}) dx \geq \sum_{j=1}^{N} \int_{B_j \cap D} (H_j - H_{N+1}) dx.$$

Therefore, (5.4) implies $F_H(\mathcal{B},\mathcal{A};\lambda)\ge F_H(\mathcal{B}',\mathcal{A};\lambda)$ with the strict inequality when $\big|\bigcup_{j=1}^N B_j\setminus D\big|>0.$ Now proceeding as in the proof of Theorem 4.2 we can show that there exists a minimizer of $F_H(\cdot,\mathcal{A};\lambda)$. Moreover, every minimizer $\mathcal{A}(\lambda)$ satisfies

$$\mathrm{co}(\mathcal{A}(\lambda))\subseteq D. \tag{5.5}$$

Now we prove the density estimates for $\mathcal{A}(\lambda)$.

*Step 2.* Let us fix $r_0\in(0,R)$ and take any $\mathcal{B}\in\mathbb{P}_b(N+1)$ with $\mathcal{A}(\lambda)\Delta\mathcal{B}\subset\subset B_r$, $r\in(0,r_0)$. Then

$$\mathrm{Per}(\mathcal{A}(\lambda),B_r)\le \mathrm{Per}(\mathcal{B},B_r)+\Lambda_1|\mathcal{A}(\lambda)\Delta\mathcal{B}|^{1-1/p}+\Lambda_2|\mathcal{A}(\lambda)\Delta\mathcal{B}|, \tag{5.6}$$

where

$$\Lambda_1:=N^{1/p}\max_{i\le N}\|H_i-H_{N+1}\|_{L^p(D)},\qquad \Lambda_2:=\frac{\lambda}{2}\,(\mathrm{diam}\,D+2r_0). \tag{5.7}$$

Indeed, from (5.5) one has

$$d(z,\partial A_j)\le \mathrm{diam}\,D+2r,\qquad \forall j=1,\dots,N+1,\ z\in\mathcal{A}(\lambda)\Delta\mathcal{B},$$

hence using (4.1)

$$\Big|\sigma(\mathcal{B},\mathcal{A})-\sigma(\mathcal{A}(\lambda),\mathcal{A})\Big|\le\sum_{j=1}^{N+1}\int_{B_j\Delta A_j(\lambda)}d(z,\partial A_j)dz\le(\mathrm{diam}\,D+2r_0)|\mathcal{B}\Delta\mathcal{A}(\lambda)|,$$

since $\mathcal{B}\Delta\mathcal{A}(\lambda)\subset\subset B_{r_0}$. Moreover, from the Hölder inequality

$$\Big|\int_{A_i(\lambda)}(H_i-H_{N+1})dx-\int_{B_i}(H_i-H_{N+1})dx\Big|\le\int_{A_i(\lambda)\Delta B_i}|H_i-H_{N+1}|dx$$

$$\le|A_i(\lambda)\Delta B_i|^{1-1/p}\Big(\int_{A_i(\lambda)\Delta B_i}|H_i-H_{N+1}|^pdx\Big)^{1/p}\le\|H_i-H_{N+1}\|_{L^p(D)}|A_i(\lambda)\Delta B_i|^{1-1/p}.$$

Then the concavity of the function $t\in(0,+\infty)\mapsto t^{1-1/p}$ implies that

$$\Big|\sum_{i=1}^N\int_{A_i(\lambda)}(H_i-H_{N+1})dx-\int_{B_i}(H_i-H_{N+1})dx\Big|$$

$$\le N^{1/p}\max_{i\le N}\|H_i-H_{N+1}\|_{L^p(D)}|\mathcal{A}(\lambda)\Delta\mathcal{B}|^{1-1/p}.$$

Now minimality of $\mathcal{A}(\lambda)$ (Step 1) yields (5.6). Thus we can apply Remark 3.7 with $\alpha_1=1-1/p>1-1/n$, $\alpha_2=1$, $r_0\in(0,R)$ and

$$\tilde r_0=\begin{cases}\min\{r_0,\frac{n}{4\Lambda_2N}\} & \text{if } \Lambda_1=0,\\ \min\{r_0,\omega_n^{-1/n}\big(\frac{n\omega_n^{1/n}}{8\Lambda_1N}\big)^{\frac{p}{p-n}},\frac{n}{8\Lambda_2N}\} & \text{if } \Lambda_1>0,\end{cases}$$

to get that for every $i\in\{1,\dots,N+1\}$ either $\partial A_i(\lambda)=\emptyset$ or there exists $c(N,n)\in(0,1)$ such that (4.7)-(4.8) hold for any $x\in\partial A_i(\lambda)$ and $r\in(0,\tilde r_0)$. In particular, $\sum_{j=1}^{N+1}\mathcal{H}^{n-1}(\partial A_j(\lambda)\setminus\partial^*A_j(\lambda))=0.$

*Step 3.* Given $\mathcal{G}\in\mathbb{P}_b(N+1)$ let $K$ denote the closed convex hull of $\mathrm{co}(\mathcal{G})\cup B_R(0)$. Let $\mathcal{L}(\lambda,0):=\mathcal{G}$ and $\mathcal{L}(\lambda,k)$ be defined as

$$F_H(\mathcal{L}(\lambda,k),\mathcal{L}(\lambda,k-1);\lambda)=\min_{\mathcal{A}\in\mathbb{P}_b(N+1)}F_H(\mathcal{A},\mathcal{L}(\lambda,k-1);\lambda),\qquad k\ge1.$$

Notice that by Step 1 $F_H(\cdot,\mathcal{L}(\lambda,k-1);\lambda)$ has a minimizer $\mathcal{L}(\lambda,k)\in\mathbb{P}_b(N+1)$ and $\mathrm{co}(\mathcal{L}(\lambda,k))\subseteq K$ for any $\lambda\ge1$ and $k\ge0$. Observe that for any $\lambda\ge1$ the map

$$k\in\mathbb{N}_0\mapsto\Psi(\lambda,k):=\mathrm{Per}(\mathcal{L}(\lambda,k))+\sum_{j=1}^N\int_{L_j(\lambda,k)}(H_j-H_{N+1})dx$$

is nonincreasing. Indeed, since $F_H(\mathcal{L}(\lambda,k),\mathcal{L}(\lambda,k-1);\lambda) \le F_H(\mathcal{L}(\lambda,k-1),\mathcal{L}(\lambda,k-1);\lambda)$, recalling (5.4) one has

$$\lambda\sigma(\mathcal{L}(\lambda,k),\mathcal{L}(\lambda,k-1)) \le 2\big(\Psi(\lambda,k-1)-\Psi(\lambda,k)\big).$$

In particular,

$$\begin{aligned}\operatorname{Per}(\mathcal{L}(\lambda,k)) &\le \operatorname{Per}(\mathcal{G}) + \sum_{j=1}^{N}\int_{L_j(\lambda,k)\Delta G_j}|H_j-H_{N+1}|dx\\ &\le \operatorname{Per}(\mathcal{G}) + N\max_{j\le N}\|H_j-H_{N+1}\|_{L^1(K)} =: \kappa.\end{aligned} \tag{5.8}$$

We claim that for every $t,t'>0,\ 0<t-t'<1$,

$$|\mathcal{L}(\lambda,[\lambda t])\Delta\mathcal{L}(\lambda,[\lambda t'])| \le \mathrm{C}(N,n,\mathcal{G})|t-t'|^{\frac{1}{n+1}} + \tilde{\mathrm{C}}(N,n,\mathcal{G})|t-t'|^{-\frac{n-1}{2(n+1)}}\lambda^{-1/2}$$

provided that $\lambda\ge\max\{4/t',4/(t-t')\}$ is sufficiently large so that the density estimates (4.7)-(4.8) hold for $r\in(0,\delta)$, $\delta=\frac{n}{4N(\operatorname{diam}K+2r_0)\lambda}$, here

$$\mathrm{C}(N,n,\mathcal{G}) := \left(\sqrt{8\Gamma(N+1)}+\frac{8N(N+1)(\operatorname{diam}K+2r_0)}{n}\right)\kappa,$$

$$\tilde{\mathrm{C}}(N,n,\mathcal{G}) := \sqrt{8\Gamma(N+1)}\,\kappa,$$

$\Gamma$ is given by (4.10). Set $k_0:=[\lambda t']$, $m_0:=[\lambda t]$. By the choice of $\lambda$, $m_0\ge k_0+3\ge 4$. Applying Proposition 4.8 with

$$A = L_j(\lambda,k)\Delta L_j(\lambda,k-1),$$
$$C = \partial L_j(\lambda,k-1),$$

which satisfies the lower perimeter density estimate according to Step 2 with $\theta=c(N,n)$,

$$\gamma = \Psi(\lambda,k-1)-\Psi(\lambda,k)$$

and

$$\rho = \frac{n}{4N(\operatorname{diam}K+2r_0)\lambda}|t-t'|^{-1/(n+1)},\quad \delta=\frac{n}{4N(\operatorname{diam}K+2r_0)\lambda},$$

for any $k\ge k_0+1$ we establish

$$\begin{aligned}|\mathcal{L}(\lambda,k)\Delta\mathcal{L}(\lambda,k-1)| \le{}& \frac{\sqrt{4\Gamma}|t-t'|^{-\frac{n-1}{2(n+1)}}}{\lambda^{1/2}}\Big(\Psi(\lambda,k-1)-\Psi(\lambda,k)\Big)^{\frac12}\sum_{j=1}^{N+1}\big(P(L_j(\lambda,k-1))\big)^{\frac12}\\ &+\frac{8N(N+1)(\operatorname{diam}K+2r_0)}{n}|t-t'|^{\frac{1}{n+1}}\Big(\Psi(\lambda,k-1)-\Psi(\lambda,k)\Big).\end{aligned}$$

According to (5.8)

$$\sum_{j=1}^{N+1}\big(P(L_j(\lambda,k-1))\big)^{1/2} \le \Big(2(N+1)\operatorname{Per}(\mathcal{L}(\lambda,k-1))\Big)^{1/2} \le \Big(2(N+1)\kappa\Big)^{1/2}.$$

Now using

$$\begin{aligned}\sum_{k=k_0+1}^{m_0}\Big(\Psi(\lambda,k-1)-\Psi(\lambda,k)\Big) &\le \Psi(\lambda,0)-\Psi(\lambda,m_0) \le \operatorname{Per}(\mathcal{G})-\operatorname{Per}(\mathcal{L}(\lambda,m_0))\\ &\quad+\sum_{j=1}^{N}\Big(\int_{\mathcal{G}}(H_j-H_{N+1})dx-\int_{\mathcal{L}(\lambda,m_0)}(H_j-H_{N+1})dx\Big)\\ &\le \operatorname{Per}(\mathcal{G})+N\max_{j\le N}\|H_j-H_{N+1}\|_{L^1(K)}=\kappa\end{aligned}$$

and
$$\sum_{k=k_0+1}^{m_0}\Big(\Psi(\lambda,k-1)-\Psi(\lambda,k)\Big)^{\frac12}\le\Big((m_0-k_0)\sum_{k=k_0+1}^{m_0}\Big(\Psi(\lambda,k-1)-\Psi(\lambda,k)\Big)\Big)^{\frac12}\le(\kappa(m_0-k_0))^{\frac12},$$
we get
$$\begin{aligned}|\mathcal{L}(\lambda,[\lambda t])\Delta\mathcal{L}(\lambda,[\lambda t'])|\le&\sum_{k=k_0+1}^{m_0}|\mathcal{L}(\lambda,k)\Delta\mathcal{L}(\lambda,k-1)|\\ \le&\frac{\sqrt{8\Gamma(N+1)\kappa}}{\lambda^{1/2}|t-t'|^{\frac{n-1}{2(n+1)}}}\sum_{k=k_0+1}^{m_0}\Big(\Psi(\lambda,k-1)-\Psi(\lambda,k)\Big)^{1/2}\\ &+\frac{8N(N+1)(\operatorname{diam}K+2r_0)}{n}|t-t'|^{\frac{1}{n+1}}\sum_{k=k_0+1}^{m_0}\Big(\Psi(\lambda,k-1)-\Psi(\lambda,k)\Big)\\ \le&\frac{\sqrt{8\Gamma(N+1)\kappa}}{|t-t'|^{\frac{n-1}{2(n+1)}}}\Big(\frac{m_0-k_0}{\lambda}\Big)^{1/2}+\frac{8N(N+1)(\operatorname{diam}K+2r_0)}{n}\kappa|t-t'|^{\frac{1}{n+1}}.\end{aligned}$$
By the definition of $k_0$ and $m_0$ we have $\frac{m_0-k_0}{\lambda}\le|t-t'|+\frac{1}{\lambda}$, hence
$$\begin{aligned}|\mathcal{L}(\lambda,[\lambda t])\Delta\mathcal{L}(\lambda,[\lambda t'])|\le&\frac{\sqrt{8\Gamma(N+1)\kappa}}{|t-t'|^{\frac{n-1}{2(n+1)}}}\Big(|t-t'|+\frac{1}{\lambda}\Big)^{1/2}\\ &+\frac{8N(N+1)(\operatorname{diam}K+2r_0)}{n}\kappa|t-t'|^{\frac{1}{n+1}}\\ \le&\mathrm{C}(N,n,\mathcal{G})|t-t'|^{\frac{1}{n+1}}+\tilde{\mathrm{C}}(N,n,\mathcal{G})|t-t'|^{-\frac{n-1}{2(n+1)}}\lambda^{-1/2}.\end{aligned}$$
Now the proofs of (5.2) and (5.3) are exactly the same as in the proof of Theorem 4.10.

*Step 4.* Finally, let us show that if $\sum_{j=1}^{N+1}|\overline{G_j}\setminus G_j|=0,$ then (5.2) holds for any $t,t'\ge0,$ $|t-t'|<1.$ We need just to show that $|\mathcal{L}(\lambda,1)\Delta\mathcal{G}|\to0$ as $\lambda\to+\infty,$ and then we proceed as in the proof of the final assertion of Theorem 4.10.

Using minimality of $\mathcal{L}(\lambda,1)$ we have $F_H(\mathcal{L}(\lambda,1),\mathcal{G};\lambda)\le F_H(\mathcal{G},\mathcal{G};\lambda),$ i.e.
$$\frac{\lambda}{2}\sigma(\mathcal{L}(\lambda),\mathcal{G})\le\operatorname{Per}(\mathcal{G})-\operatorname{Per}(\mathcal{L}(\lambda,1))+N\max_{j\le N}\|H_j-H_{N+1}\|_{L^1(K)}\le\kappa. \tag{5.9}$$
Choose an arbitrary diverging sequence $\{\lambda_k\}$. By (5.8) it follows $\operatorname{Per}(\mathcal{L}(\lambda_k,1))\le\kappa$ for any $k\ge1$ and since $\bigcup_{j=1}^{N}L_j(\lambda_k,1)\subseteq K,$ by Theorem 3.9 there exists a (not relabelled) subsequence and $\mathcal{A}\in\mathbb{P}_b(N+1)$ such that $\mathcal{L}(\lambda_k,1)\to\mathcal{A}$ in $L^1(\mathbb{R}^n)$ as $k\to+\infty$. Then the $L^1(\mathbb{R}^n)$-lower semicontinuity of $\sigma$ and (5.9) yield
$$\sigma(\mathcal{A},\mathcal{G})\le\liminf_{k\to+\infty}\sigma(\mathcal{L}(\lambda_k,1),\mathcal{G})\le\liminf_{k\to+\infty}\frac{2\kappa}{\lambda_k}=0.$$
Hence $\sigma(\mathcal{A},\mathcal{G})=0$ and by the assumption of $\mathcal{G}$ we have $\mathcal{A}=\mathcal{G}$. Since $\{\lambda_k\}$ is arbitrary, $\mathcal{L}(\lambda,1)\to\mathcal{G}$ in $L^1(\mathbb{R}^n)$ as $\lambda\to+\infty$. □

## 6. UNIQUENESS AND CONSISTENCY OF $GMM$ FOR CONVEX DISJOINT PARTITIONS

**Definition 6.1 (Convex and disjoint partitions).** *A partition $\mathcal{A}\in\mathbb{P}_b(N+1)$ is called convex if the bounded components of $\mathcal{A}$ are convex and is called disjoint provided*
$$\min_{1\le i<j\le N}\operatorname{dist}(A_i,A_j)>0.$$

Notice that if $\mathcal{A} \in \mathbb{P}_b(N+1)$ is disjoint, then $\mathrm{Per}(\mathcal{A}) = \sum_{j=1}^{N} P(A_j)$. Moreover, if $\mathcal{A}$ and $\mathcal{G}$ are disjoint and satisfy

$$\bigcup_{j=1}^{N}(A_j \Delta G_j) = \Big(\bigcup_{j=1}^{N} A_j\Big)\Delta\Big(\bigcup_{j=1}^{N} G_j\Big), \tag{6.1}$$

then $\sigma(\mathcal{A},\mathcal{G}) = \sum_{j=1}^{N+1} \int_{A_j\Delta G_j} d(x,\partial G_j)dx$ and thus

$$F(\mathcal{A},\mathcal{G};\lambda) = \sum_{j=1}^{N}\Big(P(A_j) + \lambda\int_{A_j\Delta G_j} d(x,\partial G_j)dx\Big). \tag{6.2}$$

The aim of this section is to prove the following consistency result.

**Theorem 6.2 (Evolution of convex disjoint partitions).** *Assume that $\mathcal{C} \in \mathbb{P}_b(N+1)$ is disjoint and convex. Then*

$$GMM(F,\mathcal{C}) = \{\mathcal{M}\} = \{(M_1,\dots,M_{N+1})\}$$

*is a singleton. Moreover, for any $i = 1,\dots,N$, $M_i(\cdot)$ agrees with the classical mean curvature flow starting from $C_i$ up to its extinction time.*

In particular, for any $i,j \in \{1,\dots,N\}$, $i \neq j$, the function

$$t \in [0, \min\{t_i^\dagger, t_j^\dagger\}) \mapsto \mathrm{dist}(M_i(t), M_j(t)) \tag{6.3}$$

is nondecreasing, where $t_h^\dagger$ is the extinction time of $C_h$ [25].

We postpone the proof of this theorem after several auxiliary results. The proof of the following lemma is an adaptation of the proof of Theorem 3.6.

**Lemma 6.3.** *Given $\mathcal{G} \in \mathbb{P}_b(N+1)$ let $\mathcal{G}(\lambda) \in \mathbb{P}_b(N+1)$ be a minimizer of $F(\cdot,\mathcal{G};\lambda)$. Fix $i \in \{1,\dots,N+1\}$. If $x \in G_i(\lambda)^c \cap G_i$ and $d(x,\partial G_i) \geq \rho > 0$, then*

$$\frac{1}{2^n} \leq \frac{|B_\rho(x)\cap G_i(\lambda)^c|}{|B_\rho(x)|}. \tag{6.4}$$

*Proof.* Since the idea of the proof is the same for any $i$, we suppose $i = 1$. As usual, write $B_r := B_r(x)$ and set

$$I := \{j \in \{2,\dots,N+1\} : \mathcal{H}^{n-1}(B_\rho \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda)) > 0\}.$$

Clearly, if $I = \emptyset$, then by Remark 3.4 $B_\rho \subseteq G_1(\lambda)^c$ and (6.4) is satisfied, hence we can suppose $I \neq \emptyset$. Fix any $r \in (0,\rho)$ such that

$$\sum_{j=1}^{N+1} \mathcal{H}^{n-1}(\partial B_r \cap \partial^* G_j(\lambda)) = 0. \tag{6.5}$$

For each $j \in I$ define the competitor $\mathcal{B} \in \mathbb{P}_b(N+1)$ as

$$\mathcal{B} := (G_1(\lambda) \cup (G_j(\lambda)\cap B_r), G_2(\lambda)\dots, G_{j-1}(\lambda), G_j(\lambda)\setminus B_r, G_{j+1}(\lambda),\dots,G_{N+1}(\lambda)). \tag{6.6}$$

Fix $s \in (r,\rho)$. Recall that arguing as in the proofs of (3.21) and (3.10),

$$\begin{aligned} P(G_1(\lambda)\cup(G_j(\lambda)\cap B_r), B_s) =& P(G_1(\lambda),B_s) + \mathcal{H}^{n-1}(G_j(\lambda)\cap\partial B_r) + P(G_j(\lambda),B_r) \\ & - 2\mathcal{H}^{n-1}(B_r\cap\partial^* G_1(\lambda)\cap\partial^* G_j(\lambda)), \\ P(G_j(\lambda)\setminus B_r, B_s) =& P(G_j(\lambda), B_s\setminus\overline{B_r}) + \mathcal{H}^{n-1}(G_j(\lambda)\cap\partial B_r). \end{aligned}$$

Therefore from (6.5)

$$\begin{aligned} \lim_{s\to r^+}\Big(P(G_1(\lambda)\cup(G_j(\lambda)\cap B_r),B_s) + P(G_j(\lambda)\setminus B_r, B_s) - P(G_1(\lambda),B_s) - P(G_j(\lambda),B_s)\Big) \\ = 2\mathcal{H}^{n-1}(G_j(\lambda)\cap\partial B_r) - 2\mathcal{H}^{n-1}(B_r\cap\partial^* G_1(\lambda)\cap\partial^* G_j(\lambda)). \end{aligned}$$

Now the minimality of $\mathcal{G}(\lambda)$ and (4.1) imply

$$\begin{aligned}\mathcal{H}^{n-1}(G_j(\lambda)\cap\partial B_r)-&\mathcal{H}^{n-1}(B_r\cap\partial^*G_1(\lambda)\cap\partial^*G_j(\lambda))\\ &\ge\frac{\lambda}{2}\int_{G_j(\lambda)\cap B_r}\big(\tilde d(y,\partial G_j)-\tilde d(y,\partial G_1)\big)dy.\end{aligned}\tag{6.7}$$

Since $B_\rho\subseteq G_1$ (and hence $B_\rho\cap G_j=\emptyset$) we have

$$\tilde d(y,\partial G_j)-\tilde d(y,\partial G_1)=d(y,\partial G_j)+d(y,\partial G_1)\ge 0\qquad\forall y\in G_j(\lambda)\cap B_r,\tag{6.8}$$

and therefore

$$\mathcal{H}^{n-1}(B_r\cap\partial^*G_1(\lambda)\cap\partial^*G_j(\lambda))\le\mathcal{H}^{n-1}(G_j(\lambda)\cap\partial B_r).\tag{6.9}$$

Then summation of (6.9) over $j\in I$ and use of Remark 3.4 yield

$$P(G_1(\lambda)^c,B_r)\le\sum_{j\in I}\mathcal{H}^{n-1}(G_j(\lambda)\cap\partial B_r)\le\sum_{j=2}^{N+1}\mathcal{H}^{n-1}(G_j(\lambda)\cap\partial B_r)=\mathcal{H}^{n-1}(G_1(\lambda)^c\cap\partial B_r).$$

Now adding $\mathcal{H}^{n-1}(G_1(\lambda)^c\cap\partial B_r)$ to both sides we get

$$P(G_1(\lambda)^c\cap B_r)\le 2\mathcal{H}^{n-1}(G_1(\lambda)^c\cap\partial B_r).$$

From the isoperimetric inequality, for a.e. $r\in(0,\rho)$ we obtain

$$n\omega_n^{1/n}|G_1(\lambda)^c\cap B_r|^{\frac{n-1}{n}}\le 2\mathcal{H}^{n-1}(G_1(\lambda)^c\cap\partial B_r).\tag{6.10}$$

Since $x\in G_1(\lambda)^c$, one has $|G_1(\lambda)^c\cap B_r|>0$ for any $r>0$, therefore integrating (6.10) in $(0,\rho)$, we get (6.4). ☐

**Lemma 6.4.** *Given $\mathcal{G}\in\mathbb{P}_b(N+1)$ let $\mathcal{G}(\lambda)\in\mathbb{P}_b(N+1)$ be a minimizer of $F(\cdot,\mathcal{G};\lambda)$. Then for any $i\in\{1,\dots,N+1\}$,*

$$\sup_{x\in G_i(\lambda)^c\cap G_i}d(x,\partial G_i)\le\frac{\sqrt{2^{n+2}n}}{\sqrt{\lambda}}.$$

*Proof.* Without loss of generality we suppose $i=1$. By contradiction, let $x\in G_1(\lambda)^c\cap G_1$ be such that $d(x,\partial G_1)\ge\rho:=\frac{\sqrt{2^{n+2}n+\varepsilon}}{\sqrt{\lambda}}$ for some $\varepsilon>0$. Possibly decreasing $\varepsilon$ we may suppose that $x\in\partial G_1(\lambda)$, and $\rho$ satisfies (6.5) with $r=\rho$, so that the set

$$J:=\{j\in\{2,\dots,N+1\}:\ |B_{\rho/2}\cap G_j(\lambda)|>0\}$$

is nonempty, $B_{\rho/2}:=B_{\rho/2}(x)$. Moreover for every $y\in B_{\rho/2}$, the ball centered at $y$ of radius $\rho/2$ is contained in $G_1$ and hence

$$d(y,\partial G_j)\ge d(y,\partial G_1)\ge\rho/2\qquad\forall j\in J.$$

Therefore, for each $j\in J$ defining the competitor as in (6.6) with $r=\rho/2$, from the minimality of $\mathcal{G}(\lambda)$, (4.1) and (6.7) we get

$$\begin{aligned}\mathcal{H}^{n-1}&(G_j(\lambda)\cap\partial B_{\rho/2})-\mathcal{H}^{n-1}(B_{\rho/2}\cap\partial^*G_1(\lambda)\cap\partial^*G_j(\lambda))\\ &\ge\frac{\lambda}{2}\int_{G_j(\lambda)\cap B_{\rho/2}}\big(\tilde d(y,\partial G_j)-\tilde d(y,\partial G_1)\big)dy\ge\frac{\lambda\rho}{2}|G_j(\lambda)\cap B_{\rho/2}|,\end{aligned}$$

since $\tilde d(y,\partial G_j)=d(y,\partial G_j)$ and $\tilde d(y,\partial G_1)=-d(y,\partial G_1)$ for any $y\in B_{\rho/2}$. Summing these inequalities over $j\in J$ and using $\bigcup_{j=1}^{N+1}(G_j(\lambda)\cap B_{\rho/2})=\bigcup_{j\in J}(G_j(\lambda)\cap B_{\rho/2})=G_1(\lambda)^c\cap B_{\rho/2}$ (up to a negligible set), we get

$$\mathcal{H}^{n-1}(G_1(\lambda)^c\cap\partial B_{\rho/2})\ge\sum_{j\in J}\mathcal{H}^{n-1}(B_{\rho/2}\cap\partial^*G_1(\lambda)\cap\partial^*G_j(\lambda))+\frac{\lambda\rho}{2}|G_1(\lambda)^c\cap B_{\rho/2}|.$$

Now Lemma 6.3 yields

$$\Big(\frac{1}{2}\Big)^{n+1}\lambda\rho\omega_n\Big(\frac{\rho}{2}\Big)^n \le \mathcal{H}^{n-1}(G_1(\lambda)^c\cap\partial B_{\rho/2}) \le n\omega_n\Big(\frac{\rho}{2}\Big)^{n-1}.$$

But this implies $\rho = \frac{\sqrt{2^{n+2}n}+\varepsilon}{\sqrt{\lambda}} \le \frac{\sqrt{2^{n+2}n}}{\sqrt{\lambda}}$, a contradiction, since $\varepsilon > 0$. ☐

Given $A \subseteq \mathbb{R}^n$ and $\delta > 0$ set

$$A_\delta^+ := \{x \in \mathbb{R}^n : \ \mathrm{dist}(x, A) \le \delta\}.$$

The following theorem, valid without any convexity assumption on the components, shows that if the entries of the initial partition $\mathcal{G}$ are far from each other, then so are the entries of minimizers of $F(\cdot, \mathcal{G}; \lambda)$ provided $\lambda$ is large.

**Theorem 6.5 (Minimizers of $F$ for a disjoint initial partition).** *Suppose that $\mathcal{G} \in \mathbb{P}_b(N+1)$ is disjoint and set*

$$\min_{1\le i<j\le N} \mathrm{dist}(G_i, G_j) =: \varepsilon_0 > 0. \tag{6.11}$$

*Then for $\lambda \ge 2^{n+6}n\varepsilon_0^{-2}$ any minimizer $\mathcal{G}(\lambda)$ of $F(\cdot,\mathcal{G};\lambda)$ satisfies*

$$G_j(\lambda) \subseteq (G_j)^+_{\varepsilon_0/4}, \qquad j = 1,\dots,N. \tag{6.12}$$

*Proof.* We claim that the choice of $\lambda$ implies

$$G_{N+1}(\lambda)^c \subseteq (G^c_{N+1})^+_{\varepsilon_0/4}. \tag{6.13}$$

Indeed, obviously $G_{N+1}(\lambda)^c \cap G^c_{N+1} \subseteq (G^c_{N+1})^+_{\varepsilon_0/4}$. Now if $x \in G_{N+1}(\lambda)^c \cap G_{N+1}$, then $d(x, G^c_{N+1}) = d(x, \partial G_{N+1})$ and therefore by Lemma 6.4

$$d(x, G^c_{N+1}) \le \sup_{y\in G_{N+1}(\lambda)^c\cap G_{N+1}} d(y, \partial G_{N+1}) \le \frac{\sqrt{2^{n+2}n}}{\sqrt{\lambda}} \le \frac{\varepsilon_0}{4}.$$

Hence $x \in (G^c_{N+1})^+_{\varepsilon_0/4}$.

We prove the assertion of the theorem arguing by contradiction. Suppose for example $j = 1$ and $G_1(\lambda)$ is not contained in $(G_1)^+_{\varepsilon_0/4}$. In view of (6.13) and (6.11)

$$G_1(\lambda) \subseteq \bigcup_{j=1}^N G_j(\lambda) \subseteq \Big(\bigcup_{j=1}^N G_j\Big)^+_{\varepsilon_0/4} = \bigcup_{j=1}^N (G_j)^+_{\varepsilon_0/4}.$$

Since our assumption implies $G_1(\lambda)\cap(G_j)^+_{\varepsilon_0/4} \ne \emptyset$ for some $j \in \{2,\dots,N\}$, and by virtue of Remark 4.7 the set $G_1(\lambda)$ can be supposed to be open, there exists a ball $B_r$ of radius $r > 0$ whose closure is contained in $G_1(\lambda) \cap (G_j)^+_{\varepsilon_0/4}$. For shortness, let $j = 2$. Thus setting $\mathcal{B} := (G_1(\lambda) \setminus B_r, G_2(\lambda) \cup B_r, G_3(\lambda), \dots, G_{N+1}(\lambda))$, and using $P(G_1(\lambda)) - P(G_1(\lambda) \setminus B_r) = P(B_r)$, we obtain

$$\begin{aligned}2F(\mathcal{G}(\lambda), \mathcal{G}; \lambda) - 2F(\mathcal{B}, \mathcal{G}; \lambda) =& P(B_r) + P(G_2(\lambda)) - P(G_2(\lambda) \cup B_r)\\ &+ \lambda \int_{B_r} \big(\tilde{d}(x, \partial G_1) - \tilde{d}(x, \partial G_2)\big)dx.\end{aligned}$$

Since $B_r \cap G_2(\lambda) = \emptyset$, from (2.7) we get

$$P(B_r) + P(G_2(\lambda)) - P(B_r \cup G_2(\lambda)) \ge 0.$$

In addition, by the definition of $\varepsilon_0$, $d(B_r, G_1) \ge \frac{3\varepsilon_0}{4}$, (thus $\tilde{d}(\cdot, \partial G_1) = d(\cdot, \partial G_1)$ in $B_r$); moreover, since $B_r \subseteq (G_2)^+_{\varepsilon_0/4}$, one has

$$\tilde{d}(x, \partial G_1) - \tilde{d}(x, \partial G_2) \ge \frac{\varepsilon_0}{4} \qquad \forall x \in B_r$$

and therefore

$$F(\mathcal{G}(\lambda), \mathcal{G}; \lambda) - F(\mathcal{B}, \mathcal{G}; \lambda) \ge \frac{\lambda\varepsilon_0}{8}|B_r| > 0.$$

This implies that $\mathcal{G}(\lambda)$ is not a minimizer of $F(\cdot, \mathcal{G}; \lambda)$. ☐

**Corollary 6.6.** *Suppose that $\mathcal{G} \in \mathbb{P}_b(N+1)$ is disjoint. Then for sufficiently large $\lambda$, $\mathcal{G}(\lambda)$ is a minimizer of $F(\cdot,\mathcal{G};\lambda)$ if and only if each bounded component $G_j(\lambda)$, $j=1,\ldots,N$, of $\mathcal{G}(\lambda)$ is a minimizer of the Almgren-Taylor-Wang functional* (4.5) *with $G$ replaced by $G_j$.*

*Proof.* Let

$$\min_{1\le i<j\le N} \mathrm{dist}(G_i,G_j) =: \varepsilon_0 > 0. \tag{6.14}$$

Suppose that $A_j$, $j=1,\ldots,N$, minimizes (4.5) with $G$ replaced by $G_j$. By [31, Lemma 2.1] (see also [8, Proposition 5.5]) there exists $c(n)>0$ such that

$$\sup_{x\in A_j\Delta G_j} d(x,\partial G_j) \le \sqrt{\frac{c(n)}{\lambda}}.$$

Therefore, taking

$$\lambda \ge \tilde c(n)\varepsilon_0^{-2},\quad \tilde c(n) := \max\{2^{n+6}n, 16c(n)\}, \tag{6.15}$$

we deduce $A_j \subseteq (G_j)^+_{\varepsilon_0/4}$, $j=1,\ldots,N$. Set $\mathcal{A} = (A_1,\ldots,A_N,\mathbb{R}^n\setminus\bigcup_{j=1}^N A_j)$. Let us show that for $\lambda$ as in (6.15), $\mathcal{A}$ minimizes $F(\cdot,\mathcal{G};\lambda)$. Indeed, take any minimizer $\mathcal{G}(\lambda)$ of $F(\cdot,\mathcal{G};\lambda)$. By Theorem 6.5 we have $G_j(\lambda)\subseteq (G_j)^+_{\varepsilon_0/4}$, therefore both $(\mathcal{A},\mathcal{G})$ and $(\mathcal{G}(\lambda),\mathcal{G})$ satisfy (6.1). Hence, (6.2) and the minimality of $A_j$ yield

$$\begin{aligned} F(\mathcal{G}(\lambda),\mathcal{G};\lambda) =& \sum_{j=1}^N \Big(P(G_j(\lambda)) + \lambda\int_{G_j(\lambda)\Delta G_j} d(x,\partial G_j)dx\Big)\\ \ge& \sum_{j=1}^N \Big(P(A_j) + \lambda\int_{A_j\Delta G_j} d(x,\partial G_j)dx\Big) = F(\mathcal{A},\mathcal{G};\lambda). \end{aligned}$$

This implies that $\mathcal{A}$ is also a minimizer $F(\cdot,\mathcal{G};\lambda)$.

Conversely, suppose that $\lambda$ satisfies (6.15) and $\mathcal{G}(\lambda)$ minimizes $F(\cdot,\mathcal{G};\lambda)$ and let $A_j$, $j=1,\ldots,N$, be a minimizer (4.5) with $G$ replaced by $G_j$. Recall that $A_j\subseteq (G_j)^+_{\varepsilon_0/4}$, $j=1,\ldots,n$. Set $\mathcal{A} = (A_1,\ldots,A_N,\mathbb{R}^n\setminus\bigcup_{j=1}^N A_j)$. Then from the minimality of $A_j$ and $\mathcal{G}(\lambda)$, as well as (6.2), we deduce

$$\begin{aligned} F(\mathcal{G}(\lambda),\mathcal{G};\lambda) \le& F(\mathcal{A},\mathcal{G};\lambda) = \sum_{j=1}^N \Big(P(A_j) + \lambda\int_{A_j\Delta G_j} d(x,\partial G_j)dx\Big)\\ \le& \sum_{j=1}^N \Big(P(G_j(\lambda)) + \lambda\int_{G_j(\lambda)\Delta G_j} d(x,\partial G_j)dx\Big) = F(\mathcal{G}(\lambda),\mathcal{G};\lambda). \end{aligned}$$

Thus all inequalities are in fact equalities, which is possible if and only if

$$P(G_j(\lambda)) + \lambda\int_{G_j(\lambda)\Delta G_j} d(x,\partial G_j) = P(A_j) + \lambda\int_{A_j\Delta G_j} d(x,\partial G_j)dx,\quad j=1,\ldots,N.$$

Hence, $G_j(\lambda)$ is a minimizer of (4.5) with $G=G_j$. ☐

*Proof of Theorem 6.2.* Suppose that

$$\min_{1\le i<j\le N} \mathrm{dist}(C_i,C_j) \ge \varepsilon_0 > 0. \tag{6.16}$$

By [7, Corollary 5] the Amgren-Taylor-Wang solution $M_i(\cdot)$ starting from $C_i$ (i.e. $GMM$ starting from $C_i$ and associated with (4.5)), $i=1,\ldots,N$, is unique and agrees with the classical mean curvature flow starting from $C_i$ up to its extinction time. Moreover, since $M_i(\cdot)\subseteq C_i$, for any $t\ge0$ we have $\mathcal{M}(t) := (M_1(t),\ldots,M_N(t),\mathbb{R}^n\setminus\bigcup_{i=1}^N M_i(t)) \in \mathbb{P}_b(N+1)$.

We claim that $GMM(F,\mathcal{C}) = \{\mathcal{M}\}$.

Indeed, let $\mathcal{C}(\lambda)\in\mathbb{P}_b(N+1)$ be a minimizer of $F(\cdot,\mathcal{C};\lambda)$. By Corollary 6.6 if $\lambda$ satisfies (6.15), then $C_i(\lambda)$ minimizes the Almgren-Taylor-Wang functional (4.5) with $G=C_i$. By [7, Remark 8], $C_i(\lambda)\subseteq C_i$, $C_i(\lambda)$ is convex. Hence, $\mathcal{C}(\lambda)$ also satisfies (6.16).

Define $\mathcal{C}(\lambda,k)$ as $\mathcal{C}(\lambda,0)=\mathcal{C}$ and

$$F(\mathcal{C}(\lambda,k),\mathcal{C}(\lambda,k-1);\lambda)=\min_{\mathcal{A}\in\mathbb{P}_b(N+1)}F(\mathcal{A},\mathcal{C}(\lambda,k-1);\lambda).$$

From the previous observation, for $\lambda$ satisfies (6.15) and $k\geq 1$ each $C_i(\lambda,k)$, $i=1,\dots,N$, is a minimizer of (4.5) with $G=C_i(\lambda,k-1)$. Therefore, by [7, Corollary 5]

$$\lim_{\lambda\to+\infty}|C_i(\lambda,[\lambda t])\Delta M_i(t)|=0,\qquad\forall t\geq 0,\ i=1,\dots,N.\tag{6.17}$$

Since $C_i(\lambda,[\lambda t]),M_i(t)\subseteq C_i$, $i=1,\dots,N$, from (6.17) we deduce

$$\lim_{\lambda\to+\infty}|\mathcal{C}(\lambda,[\lambda t])\Delta\mathcal{M}(t)|=\lim_{\lambda\to+\infty}2\sum_{i=1}^N|C_i(\lambda,[\lambda t])\Delta M_i(t)|=0$$

for any $t\geq 0$. Thus, $GMM(F,\mathcal{C})=\{\mathcal{M}\}$.

□

**Theorem 6.7 (Stability of convex disjoint partitions).** *Under the hypotheses of Theorem 6.2, if the sequence* $\{\mathcal{G}^{(h)}\}\subset\mathbb{P}_b(N+1)$ *converges to* $\mathcal{C}$ *in the Hausdorff distance* $\mathbb{HD}$ *as* $h\to+\infty$, *then for any* $\mathcal{M}^{(h)}\in GMM(F,\mathcal{G}^{(h)})$,

$$\lim_{h\to+\infty}\mathbb{HD}(\mathcal{M}^{(h)}(t),\mathcal{M}(t)):=\lim_{h\to+\infty}\sum_{i=1}^N\mathbb{HD}(M_i^{(h)}(t),M_i(t))=0\qquad\forall t\in[0,\min_{i\leq N}t_i^\dagger),$$

*where* $t_i^\dagger$ *is the extinction time of* $C_i$.

*Proof.* Let us show first the following comparison principle:

*Claim 1.* If $\mathcal{C}\in\mathbb{P}_b(N+1)$ is convex and satisfies

$$\min_{1\leq i<j\leq N}\operatorname{dist}(C_i,C_j)\geq\varepsilon_0>0.\tag{6.18}$$

then for every $\mathcal{G}\in\mathbb{P}_b(N+1)$ with $G_i\subseteq C_i$, $i=1,\dots,N$, for every minimizer $\mathcal{G}(\lambda)$ of $F(\cdot,\mathcal{G};\lambda)$, the inclusion $G_i(\lambda)\subseteq C_i$ holds provided $\lambda\geq\tilde{c}(n)\varepsilon_0^{-2}$. In particular, $\mathcal{G}(\lambda)$ also satisfies (6.18) unless $G_i(\lambda)=\emptyset$.

Indeed, let $C_i(\lambda)^*$, $i=1,\dots,N$ be the maximal minimizer [8, Definition 6.4] of the Almgren-Taylor-Wang functional (4.5) with $G=C_i$. By [7, Remark 8] $C_i(\lambda)^*\subseteq C_i$, and from Corollary 6.6

$$\mathcal{C}(\lambda)=\Big(C_1(\lambda),\dots,C_N(\lambda),\mathbb{R}^n\setminus\bigcup_{i=1}^N C_i(\lambda)\Big)$$

is a minimizer of $F(\cdot,\mathcal{C};\lambda)$. Since $\mathcal{G}$ also satisfies (6.18), by Corollary 6.6 each $G_i(\lambda)$, $i=1,\dots,N$ is a minimizer of (4.5). Then by [8, Theorem 6.1] one has $G_i(\lambda)\subseteq C_i(\lambda)^*\subseteq C_i$ for any $i\leq N$.

Now we show the following stability property of convex sets.

*Claim 2.* Let $C\subset\mathbb{R}^n$ be a nonempty bounded convex set and a sequence of sets of finite perimeter $G^{(h)}$ converge to $C$ in Hausdorff distance as $h\to+\infty$. Then

$$G^{(h)}(t)\overset{\mathbb{HD}}{\to}C(t),\quad t\in[0,t_C^\dagger),\tag{6.19}$$

where $G^{(h)}(t)$ and $C(t)$ are Almgren-Taylor-Wang solutions starting from $G^{(h)}$ and $C$ respectively (recall that $C(\cdot)$ is unique by [7, Corollary 5]), and $t_C^\dagger$ is the extinction time of $C$.

Indeed, consider arbitrary sequences $\{A^{(l)}\}$, $\{B^{(l)}\}$ of convex sets such that $A^{(l)}\subset\subset C\subset\subset B^{(l)}$, $l\geq 1$, and $A^{(l)},B^{(l)}\overset{\mathbb{HD}}{\to}C$ as $l\to+\infty$. Then for any $l\geq 1$, there exists $h_l>0$ such that $A^{(l)}\subseteq C^{(h)}\subseteq B^{(l)}$ for any $h>h_l$. Let $A^{(l)}(t)$ (resp. $B^{(l)}(t)$ ) be the minimizing movements starting from $A^{(l)}$ (resp. $B^{(l)}$ ) for the Almgren-Taylor-Wang functional (4.5) and $G^{(h)}(t)^*$ and $G^{(h)}(t)_*$ be the maximal and minimal $GMM$s [8, Definition 7.2] for (4.5) starting from $G^{(h)}$ and so that ${G^{(h)}}_*(t) \subseteq G^{(h)}(t) \subseteq {G^{(h)}}^*(t)$ for all $t \geq 0$. By the comparison theorem [8, Theorem 7.3], $A^{(l)}(t) \subseteq {G^{(h)}}_*(t)$ and ${G^{(h)}}^*(t) \subseteq B^{(l)}(t)$ for any $t \geq 0$. Moreover, from [7, Theorem 12] we have $A^{(l)}(t), B^{(l)}(t) \overset{\mathbb{HD}}{\to} C(t)$ as $l \to +\infty$ for any $t \in [0, t_C)$, and since $h_l \to +\infty$, (6.19) follows.

Now we prove the assertion of the theorem. Let $\mathcal{A} \in \mathbb{P}_b(N+1)$ be a convex disjoint partition with $C_i \subset\subset A_i$, $i = 1, \dots, N$. Then for sufficiently large $h$, $G_i^{(h)} \subset A_i$. Let $\mathcal{G}^{(h)}(\lambda_{h,k}, [\lambda_{h,k}t])$ be the sequence chosen in the definition of $\mathcal{M}^{(h)}(t)$, i.e. $\mathcal{G}^{(h)}(\lambda_{h,k}, [\lambda_{h,k}t])$ minimizes $F(\cdot, \mathcal{G}^{(h)}(\lambda_{h,k}, [\lambda_{h,k}t] - 1); \lambda_{h,k})$ and $\mathcal{G}^{(h)}(\lambda_{h,k}, [\lambda_{h,k}t]) \to \mathcal{M}^{(h)}(t)$ in $L^1(\mathbb{R}^n)$ as $k \to +\infty$. By Claim 1 and Corollary 6.6, each $G_i^{(h)}(\lambda_{h,k}, [\lambda_{h,k}t])$, $i = 1, \dots, N$ minimizes (4.5) with $G = G_i^{(h)}(\lambda_{h,k}, [\lambda_{h,k}t] - 1)$, therefore, $M_i^{(h)}(\cdot)$ is an Almgren-Taylor-Wang solution starting from $G_i^{(h)}$. Now as $G_i^{(h)} \overset{\mathbb{HD}}{\to} C_i$, Claim 2 implies $M_i^{(h)}(t) \overset{\mathbb{HD}}{\to} M_i(t)$, $i = 1, \dots, N$ as $h \to +\infty$ for any $t \in [0, t_i^\dagger)$. □

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