

MULTIPLE VALUED JACOBI FIELDS

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ABSTRACT. We develop a multivalued theory for the stability operator of (a constant multiple of) a minimally immersed submanifold Σ of a Riemannian manifold \mathcal{M} . We define the multiple valued counterpart of the classical Jacobi fields as the minimizers of the second variation functional defined on a Sobolev space of multiple valued sections of the normal bundle of Σ in \mathcal{M} , and we study existence and regularity of such minimizers. Finally, we prove that any Q -valued Jacobi field can be written as the superposition of Q classical Jacobi fields everywhere except for a relatively closed singular set having codimension at least two in the domain.

KEYWORDS: Almgren's Q -valued functions; second variation; stability operator; Jacobi fields; existence and regularity.

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0. INTRODUCTION

Given an m -dimensional area minimizing integer rectifiable current T in \mathbb{R}^{m+n} and any point $x \in \text{spt}(T) \setminus \text{spt}(\partial T)$, it is a by now well known consequence of the monotonicity of the function $r \mapsto \frac{\|T\|(B_r(x))}{\omega_m r^m}$ (cf. [All72, Section 5]) that for any sequence of radii $\{r_j\}_{j=1}^\infty$ with $r_j \downarrow 0$ there exists a subsequence $r_{j'}$ such that the corresponding blow-ups $T_{x,r_{j'}} := (\eta_{x,r_{j'}})_\# T$ (where $\eta_{x,r}(y) := \frac{y-x}{r}$) converge to a (locally) area minimizing m -dimensional current C which is invariant with respect to homotheties about the origin: such a limit current is called a *tangent cone* to T at x . If x is a regular point, and thus $\text{spt}(T)$ is a classical m -dimensional minimal submanifold in a neighborhood of x , then the cone C is certainly unique, and in fact $C = Q[\pi]$, where $\pi = T_x(\text{spt}(T))$ is the tangent space to $\text{spt}(T)$ at x and $Q = \Theta(\|T\|, x)$ is the m -dimensional density of the measure $\|T\|$ at x . On the other hand, singularities do occur for area minimizing currents of arbitrary codimension as soon as the dimension of the current is $m \geq 2$: indeed, by the regularity theory developed by F. Almgren in his monumental Big Regularity Paper [Alm00] and recently revisited by C. De Lellis and E. Spadaro in [DLS14, DLS16a, DLS16b], we know that area minimizing m -currents in \mathbb{R}^{m+n} may exhibit a singular set of Hausdorff dimension at most $m - 2$, and that this result is sharp when $n \geq 2$ ([Fed65]). Now, if x happens to be singular, then not only we have no information about the limit cone, but in fact it is still an open question whether in general such a limit cone is unique (that is, independent of the approximating sequence) or not. The problem of uniqueness of tangent cones at the singular points of area minimizing currents of general dimension and codimension stands still today as one of the most celebrated of the unsolved problems in Geometric Measure Theory (cf. [ope86, Problem 5.2]), and only a few partial answers corresponding to a limited number of particular cases are available in the literature. In [Whi83], B. White showed such uniqueness for two-dimensional area minimizing currents in any codimension, building on a characterization of two-dimensional area minimizing cones proved earlier on by F. Morgan in [Mor82]. In general dimension, W. Allard and F. Almgren [AA81] were able to prove that uniqueness holds under some additional requirements on the limit cone. Specifically, they have the following theorem, which is valid in the larger class of stationary integral varifolds.

Theorem 0.1 ([AA81]). *Let V be an m -dimensional stationary integral varifold in \mathbb{R}^{m+n} , and let $x \in \text{spt}(V)$ be an isolated singular point. Assume that there exists a tangent cone C to V at x satisfying the following hypotheses:*

- (H1) *C is the cone over an $(m - 1)$ -dimensional minimal submanifold Σ of \mathbb{S}^{m+n-1} , and thus C has an isolated singularity at 0 and $\Theta(\|C\|, x) = 1$ for every $x \in \text{spt}(C) \setminus \{0\}$;*
- (H2) *all normal Jacobi fields N of Σ in \mathbb{S}^{m+n-1} are integrable, that is for every normal Jacobi field N there exists a one-parameter family of minimal submanifolds of \mathbb{S}^{m+n-1} having velocity N at Σ .*

Then, C is the unique tangent cone to V at x . Furthermore, the blow-up sequence $V_{x,r}$ converges to C as $r \downarrow 0$ with rate r^μ for some $\mu > 0$.

The hypotheses (H1) and (H2) are however quite restrictive. Allard and Almgren were able to show that (H2) holds in case Σ is the product of two lower dimensional standard spheres (of appropriate radii to ensure minimality), since in this case all normal Jacobi fields of Σ in \mathbb{S}^{m+n-1} arise from isometric motions of \mathbb{S}^{m+n-1} . It seems however rather unlikely that the condition can hold for any general Σ admitting normal Jacobi fields other than those generated by rigid motions of the sphere. In [Sim83a], L. Simon was able to prove Theorem

0.1 dropping the hypothesis (H2), with a quite different approach with respect to [AA81] and purely PDE-based techniques. Not much has been done, instead, in the direction of removing the hypothesis (H1): to our knowledge, indeed, the only result concerning the case when a tangent cone C has more than one isolated singularity at the origin is contained in L. Simon's work [Sim94], where the author proves uniqueness of tangent cones to any *codimension one* area minimizing m -current T whenever one limit cone C is of the form $C = C_0 \times \mathbb{R}$, with C_0 a *strictly stable, strictly minimizing* $(m - 1)$ -dimensional cone in \mathbb{R}^m with an isolated singularity at the origin, and under additional assumptions on the Jacobi fields of C and on the spectrum of the Jacobi normal operator of C_0 .

However, all the results discussed above do not cover the cases when a tangent cone has higher multiplicity: it is remarkable that uniqueness is still open even under the strong assumption that *all* tangent cones to an area minimizing m -current T ($m > 2$) at an interior singular point x are of the form $C = Q[\llbracket\pi\rrbracket]$, where $[\llbracket\pi\rrbracket]$ is the rectifiable current associated with an oriented m -dimensional linear subspace of \mathbb{R}^{m+n} .

The purpose of this work is to present a *multivalued* theory of the Jacobi normal operator: we believe that such a theory may facilitate the understanding of the qualitative behaviour of the area functional near a minimal submanifold *with multiplicity*, and eventually lead to a generalization of Theorem 0.1 (and neighbouring results) to relevant cases when the condition that $\Theta(\|C\|, x) = 1$ for every $x \in \text{spt}(C) \setminus \{0\}$ fails to hold.

In our investigation, we will make use of tools and techniques coming from the theory of *multiple valued functions* minimizing the Dirichlet energy, developed by Almgren in [Alm00] and revisited by De Lellis and Spadaro in [DLS11]. A quick tutorial on the theory of multiple valued functions is contained in § 1.2, in order to ease the reading of the remaining part of the paper. As a byproduct, the theory of multiple valued Jacobi fields will show that the regularity theory for Dir-minimizing Q -valued functions is robust enough to allow one to produce analogous regularity results for minimizers of functionals defined on Sobolev spaces of Q -valued functions other than the Dirichlet energy (see also [DLFS11] for a discussion about general integral functionals defined on spaces of multiple valued functions and their semi-continuity properties, and [Hir16] for a regularity theory for multiple valued energy minimizing maps with values into a Riemannian manifold).

0.1. Main results. Let us first recall what is classically meant by Jacobi operator and Jacobi fields. Let Σ be an n -dimensional compact oriented submanifold (with or without boundary) of an m -dimensional Riemannian manifold $\mathcal{M} \subset \mathbb{R}^M$, and assume that Σ is stationary with respect to the n -dimensional area functional. Then, a one-parameter family of normal variations of Σ in \mathcal{M} can be defined by setting $\Sigma_t := F_t(\Sigma)$, where F_t is the flow generated by a smooth cross-section N of the normal bundle $\mathcal{N}\Sigma$ of Σ in \mathcal{M} which has compact support in Σ . It is known that the second variation formula corresponding to such a family of variations can be expressed in terms of an elliptic differential operator \mathcal{L} defined on the space $\Gamma(\mathcal{N}\Sigma)$ of the cross-sections of the normal bundle. This operator, usually called the *Jacobi normal operator*, is given by $\mathcal{L} = -\Delta_{\Sigma}^{\perp} - \mathcal{A} - \mathcal{R}$, where Δ_{Σ}^{\perp} is the Laplacian on $\mathcal{N}\Sigma$, and \mathcal{A} and \mathcal{R} are linear transformations of $\mathcal{N}\Sigma$ defined in terms of the second fundamental form of the immersion $\iota: \Sigma \rightarrow \mathcal{M}$ and of a partial Ricci tensor of the ambient manifold \mathcal{M} , respectively. The notions of *Morse index*, *stability* and *Jacobi fields*, central in the analysis of the properties of the class of minimal submanifolds of a given Riemannian manifold, are all defined by means of the Jacobi normal operator and its spectral properties (see Section 2 for the precise definitions and for a discussion about the most relevant literature related to the topic). In particular,

Jacobi fields are defined as those sections $N \in \Gamma(\mathcal{N}\Sigma)$ lying in the kernel of the operator \mathcal{L} , and thus solving the system of partial differential equations $\mathcal{L}(N) = 0$.

In this work, we consider instead *multivalued* normal variations in the following sense. Let Σ and \mathcal{M} be as above, and consider, for a fixed integer $Q > 1$, a Lipschitz multiple valued vector field $N: \Sigma \rightarrow \mathcal{A}_Q(\mathbb{R}^M)$ vanishing at $\partial\Sigma$ and having the form $N = \sum_{l=1}^Q \llbracket N^l \rrbracket$, where $N^l(x)$ is tangent to \mathcal{M} and orthogonal to Σ at every point $x \in \Sigma$ and for every $l = 1, \dots, Q$. The flow of such a multiple valued vector field generates a one-parameter family Σ_t of n -dimensional integer rectifiable currents in \mathcal{M} such that $\Sigma_0 = Q\llbracket \Sigma \rrbracket$ and $\partial\Sigma_t = Q\llbracket \partial\Sigma \rrbracket$ for every t . The second variation

$$\delta^2\llbracket \Sigma \rrbracket(N) := \frac{d^2}{dt^2} \mathbf{M}(\Sigma_t)|_{t=0}$$

is a well-defined functional on the space $\Gamma_Q^{1,2}(\mathcal{N}\Sigma)$ of Q -valued $W^{1,2}$ sections of the normal bundle $\mathcal{N}\Sigma$ of Σ in \mathcal{M} . Such a Jacobi functional is denoted Jac , and it is given by

$$\text{Jac}(N, \Sigma) := \int_{\Sigma} \sum_{l=1}^Q \left(|\nabla^{\perp} N^l|^2 - |A \cdot N^l|^2 - \mathcal{R}(N^l, N^l) \right) d\mathcal{H}^n, \quad (0.1)$$

where ∇^{\perp} is the projection of the Levi-Civita connection of \mathcal{M} onto $\mathcal{N}\Sigma$, $|A \cdot N^l|$ is the Hilbert-Schmidt norm of the projection of the second fundamental form of the embedding $\Sigma \hookrightarrow \mathcal{M}$ onto N^l and $\mathcal{R}(N^l, N^l)$ is a partial Ricci tensor of the ambient manifold \mathcal{M} in the direction of N^l .

Unlike the classical case, it is not possible to characterize the stationary maps of the Jac functional as the solutions of a certain Euler-Lagrange equation, and no PDE techniques are allowed to study their regularity. Therefore, the theory of multiple valued Jacobi fields must be completely variational. Hence, we give the following definition.

Definition 0.2. Let $\Omega \subset \Sigma \hookrightarrow \mathcal{M}$ be a Lipschitz open set. A map $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ is said to be a *Jac-minimizer*, or a *Jacobi Q -field* in Ω , if it minimizes the Jacobi functional among all Q -valued $W^{1,2}$ sections of the normal bundle of Ω in \mathcal{M} having the same trace at the boundary, that is

$$\text{Jac}(N, \Omega) \leq \text{Jac}(u, \Omega) \quad \text{for all } u \in \Gamma_Q^{1,2}(\mathcal{N}\Omega) \text{ such that } u|_{\partial\Omega} = N|_{\partial\Omega}. \quad (0.2)$$

We are now ready to state the main theorems of this paper. They develop the theory of Jacobi Q -fields along three main directions, concerning *existence*, *regularity* and *estimate of the singular set*.

Theorem 0.3 (Conditional existence). *Let Ω be an open and connected subset of $\Sigma \hookrightarrow \mathcal{M}$ with $C^{2,\beta}$ boundary. Assume that the following strict stability condition is satisfied: the only Q -valued Jacobi field N in Ω such that $N|_{\partial\Omega} = Q\llbracket 0 \rrbracket$ is the null field $N_0 \equiv Q\llbracket 0 \rrbracket$. Then, for any $g \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ such that $g|_{\partial\Omega} \in W^{1,2}(\partial\Omega; \mathcal{A}_Q(\mathbb{R}^M))$ there is a Jacobi Q -field \bar{N} such that $\bar{N}|_{\partial\Omega} = g|_{\partial\Omega}$.*

Remark 0.4. Notice that the above result strongly resembles the classical ‘‘Fredholm alternative’’ condition for solving linear elliptic boundary value problems: the solvability of the minimum problem for the Jac functional in $\Gamma_Q^{1,2}(\mathcal{N}\Omega)$ for any given boundary datum g as in the statement is guaranteed whenever Ω does not admit any non-trivial Jacobi Q -field vanishing at the boundary.

Theorem 0.5 (Regularity). *Let $\Omega \subset \Sigma$ be an open subset, with $\Sigma \hookrightarrow \mathcal{M}$ as above. There exists a universal constant $\alpha = \alpha(n, Q) \in (0, 1)$ such that if $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ is Jac-minimizing then $N \in C_{loc}^{0,\alpha}(\Omega; \mathcal{A}_Q(\mathbb{R}^M))$.*

The statement of the next theorem requires the definition of regular and singular points of a Jacobi Q -field.

Definition 0.6 (Regular and singular set). Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be Jac-minimizing. A point $p \in \Omega$ is *regular* for N (and we write $p \in \text{reg}(N)$) if there exists a neighborhood B of p in Ω and Q classical Jacobi fields $N^l: B \rightarrow \mathbb{R}^M$ such that

$$N(x) = \sum_{l=1}^Q [N^l(x)] \quad \forall x \in B$$

and either $N^l \equiv N^{l'}$ or $N^l(x) \neq N^{l'}(x)$ for all $x \in B$, for any $l, l' \in \{1, \dots, Q\}$. The *singular set* of N is defined by

$$\text{sing}(N) := \Omega \setminus \text{reg}(N).$$

Theorem 0.7 (Estimate of the singular set). *Let N be a Q -valued Jacobi field in $\Omega \subset \Sigma^n$. Then, the singular set $\text{sing}(N)$ is relatively closed in Ω . Furthermore, if $n = 2$, then $\text{sing}(N)$ is at most countable; if $n \geq 3$, then the Hausdorff dimension $\dim_{\mathcal{H}} \text{sing}(N)$ does not exceed $n - 2$.*

Theorems 0.3, 0.5 and 0.7 have a counterpart in Almgren's theory of Dir-minimizing multiple valued functions (cf. Theorems 1.24 and 1.26 below). The existence result for Jacobi Q -field is naturally more difficult than its Dirichlet counterpart, because in general the space of Q -valued $W^{1,2}$ sections of $\mathcal{N}\Sigma$ with bounded Jacobi energy is not weakly compact. Therefore, the proof of Theorem 0.3 requires the development of an extension result (cf. Corollary 4.3) for multiple valued Sobolev functions defined on the boundary of an open subset of Σ to a tubular neighborhood, which eventually allows one to exploit the strict stability condition in order to gain the desired compactness. In turn, such an extension theorem is obtained as a corollary of a multivalued version of the celebrated Luckhaus' Lemma, cf. Proposition 4.1. The proof of Theorem 0.5 is obtained from the Hölder regularity of Dir-minimizing Q -valued functions by means of a perturbation argument. Finally, the estimate of the Hausdorff dimension of the singular set of a Jac-minimizer, Theorem 0.7, relies on its Dirichlet counterpart once that we have shown that the blow-ups of a Jacobi Q -field at a collapsed point are non-trivial homogeneous Dir-minimizing functions, see Theorem 6.21. In turn, the proof of the Blow-up Theorem is based on a delicate asymptotic analysis of an Almgren's type frequency function, which is shown to be almost monotone and bounded at every collapsed point. This is done by providing fairly general first variation integral identities satisfied by the Jac-minimizers.

This note is organized as follows. In Section 1 we fix the terminology and notation that will be used throughout the paper and we recap the main results of the theory of multiple valued functions. Section 2 contains the derivation of the second variation formula generated by a Q -valued section of $\mathcal{N}\Sigma$ which leads to the definition of the Jac functional. In section 3 we investigate the first elementary properties of the Jac functional, we show that it is lower semi-continuous with respect to $W^{1,2}$ weak convergence (cf. Proposition 3.3) and we study the strict stability condition mentioned in the statement of Theorem 0.3 (cf. Lemma 3.4). Section 4 contains the proof of Theorem 0.3. The proof of Theorem 0.5 (and actually of a

quantitative version of it including an estimate of the α -Hölder seminorm, cf. Theorem 5.1) is contained in Section 5, whereas Section 6 is dedicated to the proof of Theorem 0.7.

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1. NOTATION AND PRELIMINARIES

1.1. The geometric setting. We start immediately specifying the geometric environment and fixing the notation that will be used throughout the paper. We will denote by \mathcal{M} a closed (i.e. compact with empty boundary) m -dimensional Riemannian manifold, and by Σ a compact oriented minimal submanifold (possibly with boundary) of the ambient manifold \mathcal{M} of dimension $n = m - k$. We will assume $C^{3,\beta}$ regularity for both \mathcal{M} and Σ , for some $\beta \in (0, 1)$.

Without loss of generality, we will regard \mathcal{M} as an isometrically embedded submanifold of some Euclidean space \mathbb{R}^M , with $K := M - m$ being the codimension of \mathcal{M} in \mathbb{R}^M .

The Euclidean scalar product in \mathbb{R}^M is denoted $\langle \cdot, \cdot \rangle$. The metric on \mathcal{M} and Σ is induced by the flat metric in \mathbb{R}^M : therefore, the same symbol will also denote the scalar product between tangent vectors to \mathcal{M} or to Σ .

The tangent space to \mathcal{M} at a point z will be denoted $T_z\mathcal{M}$. The maps $\mathbf{p}_z^{\mathcal{M}}: \mathbb{R}^M \rightarrow T_z\mathcal{M}$ and $\mathbf{p}_z^{\mathcal{M}\perp}: \mathbb{R}^M \rightarrow T_z^\perp\mathcal{M}$ denote orthogonal projections of \mathbb{R}^M onto the tangent space to \mathcal{M} at z and its orthogonal complement in \mathbb{R}^M respectively. If $x \in \Sigma$, the tangent space $T_x\mathcal{M}$ can be decomposed into the direct sum

$$T_x\mathcal{M} = T_x\Sigma \oplus T_x^\perp\Sigma,$$

where $T_x^\perp\Sigma$ is the orthogonal complement of $T_x\Sigma$ in $T_x\mathcal{M}$. At each point $x \in \Sigma$, we define orthogonal projections $\mathbf{p}_x: T_x\mathcal{M} \rightarrow T_x\Sigma$ and $\mathbf{p}_x^\perp: T_x\mathcal{M} \rightarrow T_x^\perp\Sigma$.

This decomposition at the level of the tangent spaces induces an orthogonal decomposition at the level of the tangent bundle, namely

$$\mathcal{T}\mathcal{M} = \mathcal{T}\Sigma \oplus \mathcal{N}\Sigma,$$

where $\mathcal{N}\Sigma$ denotes the normal bundle of Σ in \mathcal{M} .

If $f: \Sigma \rightarrow \mathbb{R}^q$ is a C^1 map and ξ is a vector field tangent to Σ , the symbol $D_\xi f$ will denote the directional derivative of f along ξ , that is

$$D_\xi f(x) := \left. \frac{d}{dt} (f \circ \gamma) \right|_{t=0}$$

whenever $\gamma = \gamma(t)$ is a C^1 curve on Σ with $\gamma(0) = x$ and $\dot{\gamma}(0) = \xi(x)$. The differential of f at $x \in \Sigma$ will be denoted $Df|_x$: we recall that this is the linear operator $Df|_x: T_x\Sigma \rightarrow \mathbb{R}^q$ such that $Df|_x \cdot \xi(x) = D_\xi f(x)$ for any tangent vector field ξ . The notation $Df(x)$ will sometimes be used in place of $Df|_x$. Moreover, the derivative along ξ of a scalar function $f: \Sigma \rightarrow \mathbb{R}$ will be sometimes simply denoted by $\xi(f)$.

The symbol ∇ , instead, will identify the Levi-Civita connection on \mathcal{M} . If ξ and X are tangent vector fields to Σ , then for every $x \in \Sigma$ we have

$$\nabla_\xi X(x) = \mathbf{p}_x \cdot \nabla_\xi X(x) + \mathbf{p}_x^\perp \cdot \nabla_\xi X(x) =: \nabla_\xi^\Sigma X(x) + A_x(\xi(x), X(x)),$$

where ∇^Σ is the Levi-Civita connection on Σ and A is the 2-covariant tensor with values in $\mathcal{N}\Sigma$ defined by $A_x(X, Y) := \mathbf{p}_x^\perp \cdot \nabla_X Y$ for any $x \in \Sigma$, for any $X, Y \in T_x \Sigma$. A is called the *second fundamental form* of the embedding $\Sigma \hookrightarrow \mathcal{M}$ by some authors (cf. [Sim83b, Section 7], where the tensor is denoted B , or [Lee97, Chapter 8], where the author uses the notation Π) and we will use the same terminology, although in the literature in differential geometry (above all when working with embedded hypersurfaces, that is in case the codimension of the submanifold is $k = 1$) it is sometimes more customary to call A “shape operator” and to use “second fundamental form” for scalar products $h(X, Y) = \langle A(X, Y), \eta \rangle$ with a fixed normal vector field η (cf. [dC92, Chapter 6, Section 2]).

Observe that, since we have assumed Σ to be minimal in \mathcal{M} , the mean curvature $H := \text{tr}(A)$ is everywhere vanishing on Σ .

The curvature endomorphism of the ambient manifold \mathcal{M} is denoted by R : we recall that this is a tensor field on \mathcal{M} of type $(3, 1)$, whose action on vector fields is defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where $[X, Y]$ is the Lie bracket of the vector fields X and Y .

Recall also that the Riemann tensor can be defined by setting

$$\text{Rm}(X, Y, Z, W) := \langle R(X, Y)Z, W \rangle$$

for any choice of the vector fields X, Y, Z, W , and that the Ricci tensor is the trace of the curvature endomorphism with respect to its first and last indices, that is $\text{Ric}(X, Y)$ is the trace of the linear map

$$Z \mapsto R(Z, X)Y.$$

Observe that Σ has a natural structure of metric measure space: for any pair of points $x, y \in \Sigma$, $\mathbf{d}(x, y)$ will be their Riemannian geodesic distance, while measures and integrals will be computed with respect to the n -dimensional Hausdorff measure \mathcal{H}^n defined in the ambient space \mathbb{R}^M . Boldface characters will always be used to denote quantities which are related to the Riemannian geodesic distance: for instance, if $x \in \Sigma$ and r is a positive number, $\mathbf{B}_r(x)$ is the geodesic ball with center x and radius r , namely the set of points $y \in \Sigma$ such that $\mathbf{d}(y, x) < r$. In the same fashion, if U and V are two subsets of Σ we will set

$$\mathbf{dist}(U, V) := \inf\{\mathbf{d}(x, y) : x \in U, y \in V\}.$$

Finally, constants will be usually denoted by C . The precise value of C may change from line to line throughout a computation. Moreover, we will write $C(a, b, \dots)$ or $C_{a, b, \dots}$ to specify that C depends on previously introduced quantities a, b, \dots .

1.2. Multiple valued functions. In this subsection, we briefly recall the relevant definitions and properties concerning Q -valued functions. First introduced by Almgren in his groundbreaking Big Regularity Paper [Alm00], multiple valued functions have proved themselves to be a fundamental tool to tackle the problem of the interior regularity of area minimizing integral currents in codimension higher than one. The interested reader can see [DLS11] for a simple, complete and self-contained reference for Almgren’s theory of multiple valued functions, [DS15] for a nice presentation of their link with integral currents, and [DL16, De 15] for a nice survey of the strategy adopted in [DLS14, DLS16a, DLS16b] to revisit Almgren’s program and obtain a much shorter proof of his celebrated partial regularity result for area minimizing currents in higher codimension. Other remarkable references where the theory of Dir-minimizing multiple valued functions plays a major role include the papers [DSS15a, DSS15b, DSS15c], where the

authors investigate the regularity of suitable classes of almost-minimizing two-dimensional integral currents.

1.2.1. *The metric space of Q -points.* From now on, let $Q \geq 1$ be a fixed positive integer.

Definition 1.1 (Q -points). The space of Q -points in the Euclidean space \mathbb{R}^M is denoted $\mathcal{A}_Q(\mathbb{R}^M)$ and defined as follows:

$$\mathcal{A}_Q(\mathbb{R}^M) := \left\{ T = \sum_{l=1}^Q \llbracket T_l \rrbracket : T_l \in \mathbb{R}^M \text{ for every } l = 1, \dots, Q \right\}, \quad (1.1)$$

where $\llbracket T_l \rrbracket$ is the Dirac mass δ_{T_l} centered at the point $T_l \in \mathbb{R}^M$. Hence, every Q -point T is in fact a purely atomic measure of mass Q in \mathbb{R}^M .

For the sake of notational simplicity, we will sometimes write \mathcal{A}_Q instead of $\mathcal{A}_Q(\mathbb{R}^M)$ if there is no chance of ambiguity.

The space $\mathcal{A}_Q(\mathbb{R}^M)$ has a natural structure of complete separable metric space.

Definition 1.2. If $T = \sum \llbracket T_l \rrbracket$ and $S = \sum \llbracket S_l \rrbracket$, then the distance between T and S is denoted $\mathcal{G}(T, S)$ and given by

$$\mathcal{G}(T, S)^2 := \min_{\sigma \in \mathcal{P}_Q} \sum_{l=1}^Q |T_l - S_{\sigma(l)}|^2, \quad (1.2)$$

where \mathcal{P}_Q is the group of permutations of $\{1, \dots, Q\}$.

1.2.2. *Q -valued maps.* Given an open subset $\Omega \subset \Sigma$, continuous, Lipschitz, Hölder and measurable functions $u: \Omega \rightarrow \mathcal{A}_Q(\mathbb{R}^M)$ can be straightforwardly defined taking advantage of the metric space structure of both the domain and the target. As for the spaces $L^p(\Omega; \mathcal{A}_Q)$, $1 \leq p \leq \infty$, they consist of those measurable maps $u: \Omega \rightarrow \mathcal{A}_Q(\mathbb{R}^M)$ such that $\|u\|_{L^p} := \|\mathcal{G}(u, Q\llbracket 0 \rrbracket)\|_{L^p(\Omega)}$ is finite. We will systematically use the notation $|u| := \mathcal{G}(u, Q\llbracket 0 \rrbracket)$, so that

$$\|u\|_{L^p}^p = \int_{\Omega} |u|^p d\mathcal{H}^n$$

for $1 \leq p < \infty$ and

$$\|u\|_{L^\infty} = \operatorname{ess\,sup}_{\Omega} |u|.$$

In spite of this notation, we remark here that, when $Q > 1$, $\mathcal{A}_Q(\mathbb{R}^M)$ is not a linear space: thus, in particular, the map $T \mapsto |T|$ is not a norm.

Any measurable Q -valued function can be thought as coming together with a measurable selection, as specified in the following proposition.

Proposition 1.3 (Measurable selection, cf. [DLS11, Proposition 0.4]). *Let $B \subset \Sigma$ be a \mathcal{H}^n -measurable set and $u: B \rightarrow \mathcal{A}_Q(\mathbb{R}^M)$ be a measurable function. Then, there exist measurable functions $u_1, \dots, u_Q: B \rightarrow \mathbb{R}^M$ such that*

$$u(x) = \sum_{l=1}^Q \llbracket u_l(x) \rrbracket \quad \text{for a.e. } x \in B. \quad (1.3)$$

It is possible to introduce a notion of differentiability for multiple valued maps.

Definition 1.4 (Differentiable Q -valued functions). A map $u: \Omega \rightarrow \mathcal{A}_Q(\mathbb{R}^M)$ is said to be *differentiable* at $x \in \Omega$ if there exist Q linear maps $\lambda_l: T_x \Sigma \rightarrow \mathbb{R}^M$ satisfying:

- (i) $\mathcal{G}(u(\exp_x(\xi)), T_x u(\xi)) = o(|\xi|)$ as $|\xi| \rightarrow 0$ for any $\xi \in T_x \Sigma$, where \exp is the exponential map on Σ and

$$T_x u(\xi) := \sum_{l=1}^Q \llbracket u_l(x) + \lambda_l \cdot \xi \rrbracket; \quad (1.4)$$

- (ii) $\lambda_l = \lambda_{l'}$ if $u_l(x) = u_{l'}(x)$.

We will use the notation $Du_l(x)$ for λ_l , and formally set $Du(x) = \sum_l \llbracket Du_l(x) \rrbracket$: observe that one can regard $Du(x)$ as an element of $\mathcal{A}_Q(\mathbb{R}^{M \times n})$ as soon as a basis of $T_x \Sigma$ has been fixed. For any $\xi \in T_x \Sigma$, we define the directional derivative of u along ξ to be $D_\xi u(x) := \sum_l \llbracket Du_l(x) \cdot \xi \rrbracket$, and establish the notation $D_\xi u = \sum_l \llbracket D_\xi u_l \rrbracket$.

Differentiable functions enjoy a chain rule formula.

Proposition 1.5 (Chain rules, cf. [DLS11, Proposition 1.12]). *Let $u: \Omega \rightarrow \mathcal{A}_Q(\mathbb{R}^M)$ be differentiable at x_0 .*

- (i) *Consider $\Phi: \tilde{\Omega} \rightarrow \Omega$ such that $\Phi(y_0) = x_0$, and assume that Φ is differentiable at y_0 . Then, $u \circ \Phi$ is differentiable at y_0 and*

$$D(u \circ \Phi)(y_0) = \sum_{l=1}^Q \llbracket Du_l(x_0) \cdot D\Phi(y_0) \rrbracket. \quad (1.5)$$

- (ii) *Consider $\Psi: \Omega_x \times \mathbb{R}_p^M \rightarrow \mathbb{R}^q$ such that Ψ is differentiable at the point $(x_0, u_l(x_0))$ for every l . Then, the map $\Psi(x, u): x \in \Omega \mapsto \sum_{l=1}^Q \llbracket \Psi(x, u_l(x)) \rrbracket \in \mathcal{A}_Q(\mathbb{R}^q)$ fulfills (i) of Definition 1.4. Moreover, if also (ii) holds, then*

$$D\Psi(x, u)(x_0) = \sum_{l=1}^Q \llbracket D_x \Psi(x_0, u_l(x_0)) + D_p \Psi(x_0, u_l(x_0)) \cdot Du_l(x_0) \rrbracket. \quad (1.6)$$

- (iii) *Consider a map $F: (\mathbb{R}^M)^Q \rightarrow \mathbb{R}^q$ with the property that for any choice of Q points $(y_1, \dots, y_Q) \in (\mathbb{R}^M)^Q$ then*

$$F(y_1, \dots, y_Q) = F(y_{\sigma(1)}, \dots, y_{\sigma(Q)})$$

for any permutation $\sigma \in \mathcal{P}_Q$. Then, if F is differentiable at $(u_1(x_0), \dots, u_Q(x_0))$ the composition $F \circ u$ (which is a well defined function $\Omega \rightarrow \mathbb{R}^q$ because, thanks to the above property, F is a well defined map on the quotient $\mathcal{A}_Q(\mathbb{R}^M) = (\mathbb{R}^M)^Q / \mathcal{P}_Q$) is differentiable at x_0 and

$$D(F \circ u)(x_0) = \sum_{l=1}^Q D_{y_l} F(u_1(x_0), \dots, u_Q(x_0)) \cdot Du_l(x_0). \quad (1.7)$$

Rademacher's theorem extends to the Q -valued setting, as shown in [DLS11, Theorem 1.13]: Lipschitz Q -valued functions are differentiable \mathcal{H}^n -almost everywhere in the sense of Definition 1.4. Moreover, for a Lipschitz Q -valued function the decomposition result stated in Proposition 1.3 can be improved as follows.

Proposition 1.6 (Lipschitz selection, cf. [DS15, Lemma 1.1]). *Let $B \subset \Sigma$ be measurable, and assume $u: B \rightarrow \mathcal{A}_Q(\mathbb{R}^M)$ is Lipschitz. Then, there are a countable partition of B in measurable subsets B_i ($i \in \mathbb{N}$) and Lipschitz functions $u_i^l: B_i \rightarrow \mathbb{R}^M$ ($l \in \{1, \dots, Q\}$) such that*

- (a) $u|_{B_i} = \sum_{l=1}^Q \llbracket u_i^l \rrbracket$ for every $i \in \mathbb{N}$, and $\text{Lip}(u_i^l) \leq \text{Lip}(u)$ for every i, l ;
 (b) for every $i \in \mathbb{N}$ and $l, l' \in \{1, \dots, Q\}$, either $u_i^l \equiv u_i^{l'}$ or $u_i^l(x) \neq u_i^{l'}(x) \forall x \in B_i$;

(c) for every i one has $Du(x) = \sum_{l=1}^Q \llbracket Du_i^l(x) \rrbracket$ for a.e. $x \in B_i$.

1.2.3. *Push-forward through multiple valued functions of C^1 submanifolds.* A useful fact, which will indeed be the starting point of our analysis of multivalued normal variations of Σ in \mathcal{M} , is that it is possible to push-forward C^1 submanifolds of the Euclidean space through Q -valued Lipschitz functions. Before giving the rigorous definition of a Q -valued push-forward, it will be useful to introduce some further notation. We will assume the reader to be familiar with the basic concepts and notions related to the theory of currents: standard references on this topic include the textbooks [Sim83b] and [KP08], and the more complete treatise [Fed69]. The space of smooth and compactly supported differential n -forms in \mathbb{R}^M will be denoted $\mathcal{D}^n(\mathbb{R}^M)$, and $T(\omega)$ will be the action of the n -current T on $\omega \in \mathcal{D}^n(\mathbb{R}^M)$. If T is a current, then ∂T and $\mathbf{M}(T)$ are its boundary and its mass respectively. If $B \subset \mathbb{R}^M$ is n -rectifiable with orientation $\vec{\xi}$ and multiplicity $\Theta \in L^1(B; \mathbb{Z})$, then the integer rectifiable current T associated to the triple $(B, \vec{\xi}, \Theta)$ will be denoted $T = \llbracket B, \vec{\xi}, \Theta \rrbracket$. In particular, if $\Sigma \subset \mathbb{R}^M$ is an n -dimensional C^1 oriented submanifold with finite \mathcal{H}^n -measure and orientation $\vec{\xi} = \xi_1 \wedge \cdots \wedge \xi_n$ (thus $\vec{\xi}(x)$ is a continuous unit n -vector field on Σ with $(\xi_i)_{i=1}^n$ an orthonormal frame of the tangent bundle $\mathcal{T}\Sigma$), and $B \subset \Sigma$ is a measurable subset, then we will simply write $\llbracket B \rrbracket$ instead of the more cumbersome $\llbracket B, \vec{\xi}, 1 \rrbracket$ to denote the current associated to B . We remark that the action of $\llbracket B \rrbracket$ on a form $\omega \in \mathcal{D}^n(\mathbb{R}^M)$ is given by

$$\llbracket B \rrbracket(\omega) := \int_B \langle \omega(x), \vec{\xi}(x) \rangle d\mathcal{H}^n(x).$$

In particular, the n -current $\llbracket \Sigma \rrbracket$ is obtained by integration of n -forms over Σ in the usual sense of differential geometry: $\llbracket \Sigma \rrbracket(\omega) = \int_{\Sigma} \omega$.¹ Since we will always deal with compact manifolds, we continue to assume that Σ is compact, in order to avoid some technicalities which are instead necessary when dealing with the non-compact case (see [DS15, Definition 1.2]).

Definition 1.7 (Q -valued push-forward, cf. [DS15, Definition 1.3]). Let Σ be as above, $B \subset \Sigma$ a measurable subset and $u: B \rightarrow \mathcal{A}_Q(\mathbb{R}^M)$ a Lipschitz map. Then, the *push-forward* of B through u is the current $\mathbf{T}_u := \sum_{i,l} (u_i^l)_\# \llbracket B_i \rrbracket$, where B_i and u_i^l are as in Proposition 1.6: that is,

$$\mathbf{T}_u(\omega) := \sum_{i \in \mathbb{N}} \sum_{l=1}^Q \int_{B_i} \langle \omega(u_i^l(x)), Du_i^l(x)_\# \vec{\xi}(x) \rangle d\mathcal{H}^n(x) \quad \forall \omega \in \mathcal{D}^n(\mathbb{R}^M), \quad (1.8)$$

where $Du_i^l(x)_\# \vec{\xi}(x) := D_{\xi_l} u_i^l(x) \wedge \cdots \wedge D_{\xi_n} u_i^l(x)$ for a.e. $x \in B_i$.

It is straightforward, using the properties of the Lipschitz decomposition outlined in Proposition 1.6 and recalling the standard theory of rectifiable currents (cf. [Sim83b, Section 27]) and the area formula (cf. [Sim83b, Section 8]), to conclude the following proposition.

Proposition 1.8 (Representation of the push-forward, cf. [DS15, Proposition 1.4]). *The definition of the action of \mathbf{T}_u in (1.8) does not depend on the chosen partition B_i , nor on the chosen decomposition $\{u_i^l\}$. If $u = \sum_l \llbracket u^l \rrbracket$, we are allowed to write*

$$\mathbf{T}_u(\omega) = \int_B \sum_{l=1}^Q \langle \omega(u^l(x)), Du^l(x)_\# \vec{\xi}(x) \rangle d\mathcal{H}^n(x) \quad \forall \omega \in \mathcal{D}^n(\mathbb{R}^M). \quad (1.9)$$

¹Observe that this convention is coherent with the use of $\llbracket P \rrbracket$, $P \in \mathbb{R}^M$, to denote the Dirac delta δ_P , conceived as 0-dimensional current in \mathbb{R}^M .

Thus, \mathbf{T}_u is a well-defined integer rectifiable n -current in \mathbb{R}^M given by $\mathbf{T}_u = \llbracket \text{Im}(u), \vec{\tau}, \Theta \rrbracket$, where:

- (R1) $\text{Im}(u) = \bigcup_{x \in B} \text{spt}(u(x)) = \bigcup_{i \in \mathbb{N}} \bigcup_{l=1}^Q u_i^l(B_i)$ is an n -rectifiable set in \mathbb{R}^M ;
- (R2) $\vec{\tau}$ is a Borel unit n -vector field orienting $\text{Im}(u)$; moreover, for \mathcal{H}^n -a.e. $p \in \text{Im}(u)$, we have $Du_i^l(x) \# \vec{\xi}(x) \neq 0$ for every i, l, x such that $u_i^l(x) = p$ and

$$\frac{Du_i^l(x) \# \vec{\xi}(x)}{|Du_i^l(x) \# \vec{\xi}(x)|} = \pm \vec{\tau}(p); \quad (1.10)$$

- (R3) for \mathcal{H}^n -a.e. $p \in \text{Im}(u)$, the (Borel) multiplicity function Θ equals

$$\Theta(p) = \sum_{i,l,x : u_i^l(x)=p} \left\langle \vec{\tau}(p), \frac{Du_i^l(x) \# \vec{\xi}(x)}{|Du_i^l(x) \# \vec{\xi}(x)|} \right\rangle. \quad (1.11)$$

Remark 1.9. The definition of a Q -valued push-forward can be extended to more general objects than C^1 submanifolds of the Euclidean space. Already in [DS15] it is indeed observed that, using standard methods in measure theory, it is possible to define a multiple valued push forward of Lipschitz manifolds. Furthermore, a simple application of the polyhedral approximation theorem [Fed69, Theorem 4.2.22] allows one to actually give a definition of the Q -valued push-forward of any k -dimensional flat chain with compact support in \mathbb{R}^n : the interested reader can refer to our note [Stu17] for the details.

The next proposition is the key tool to compute explicitly the mass of the current \mathbf{T}_u . Following standard notation, we will denote by $\mathbf{J}u^l(x)$ the Jacobian determinant of Du^l , i.e. the number

$$\mathbf{J}u^l(x) := |Du^l(x) \# \vec{\xi}(x)| = \sqrt{\det((Du^l(x))^T \cdot Du^l(x))}. \quad (1.12)$$

Proposition 1.10 (Q -valued area formula, cf. [DS15, Lemma 1.9]). *Let $B \subset \Sigma$ be as above, and $u = \sum_l \llbracket u^l \rrbracket$ a Lipschitz Q -function. Then, for every Borel function $h: \mathbb{R}^M \rightarrow [0, \infty)$, we have*

$$\int h(p) d\|\mathbf{T}_u\|(p) \leq \int_B \sum_{l=1}^Q h(u^l(x)) \mathbf{J}u^l(x) d\mathcal{H}^n(x). \quad (1.13)$$

Equality holds in (1.13) if there is a set $B' \subset B$ of full \mathcal{H}^n -measure for which

$$\langle Du^l(x) \# \vec{\xi}(x), Du^h(y) \# \vec{\xi}(y) \rangle \geq 0 \quad \forall x, y \in B' \text{ and } l, h \text{ with } u^l(x) = u^h(y). \quad (1.14)$$

1.2.4. Q -valued Sobolev functions and their properties. Next, we study the Sobolev spaces $W^{1,p}(\Omega; \mathcal{A}_Q)$. The definition that we use here was proposed by C. De Lellis and E. Spadaro (cf. [DLS11, Definition 0.5 and Proposition 4.1]), and allowed the authors to develop an alternative, intrinsic approach to the study of Q -valued Sobolev mappings minimizing a suitable generalization of the Dirichlet energy (Dir-minimizing multiple valued maps), which does not rely on Almgren's embedding of the space $\mathcal{A}_Q(\mathbb{R}^M)$ in a larger Euclidean space (cf. [Alm00] and [DLS11, Chapter 2]). Such an approach is close in spirit to the general theory of Sobolev maps taking values in abstract metric spaces and started in the works of Ambrosio [Amb90] and Reshetnyak [Res97, Res04, Res06].

Definition 1.11 (Sobolev Q -valued functions). A measurable function $u: \Omega \rightarrow \mathcal{A}_Q(\mathbb{R}^M)$ is in the Sobolev class $W^{1,p}$, $1 \leq p \leq \infty$ if and only if there exists a non-negative function $\psi \in L^p(\Omega)$ such that, for every Lipschitz function $\phi: \mathcal{A}_Q(\mathbb{R}^M) \rightarrow \mathbb{R}$, the following two properties hold:

- (i) $\phi \circ u \in W^{1,p}(\Omega)$ ²;
- (ii) $|D(\phi \circ u)(x)| \leq \text{Lip}(\phi)\psi(x)$ for almost every $x \in \Omega$.

We also recall (cf. [DLS11, Proposition 4.2]) that if $u \in W^{1,p}(\Omega; \mathcal{A}_Q(\mathbb{R}^M))$ and ξ is a tangent vector field defined in Ω , there exists a non-negative function $g_\xi \in L^p(\Omega)$ with the following two properties:

- (i) $|D_\xi \mathcal{G}(u, T)| \leq g_\xi$ a.e. in Ω for all $T \in \mathcal{A}_Q$;
- (ii) if $h_\xi \in L^p(\Omega)$ satisfies $|D_\xi \mathcal{G}(u, T)| \leq h_\xi$ for all $T \in \mathcal{A}_Q$, then $g_\xi \leq h_\xi$ a.e.

Such a function is clearly unique, and will be denoted by $|D_\xi u|$. Moreover, chosen a countable dense subset $\{T_i\}_{i=0}^\infty \subset \mathcal{A}_Q$, it satisfies

$$|D_\xi u| = \sup_{i \in \mathbb{N}} |D_\xi \mathcal{G}(u, T_i)| \quad (1.15)$$

almost everywhere in Ω .

As in the classical theory, Sobolev Q -valued maps can be approximated by Lipschitz maps.

Proposition 1.12 (Lipschitz approximation, cf. [DLS11, Proposition 4.4]). *Let u be a function in $W^{1,p}(\Omega; \mathcal{A}_Q)$. For every $\lambda > 0$, there exists a Lipschitz Q -function u_λ such that $\text{Lip}(u_\lambda) \leq C\lambda$ and*

$$\mathcal{H}^n(\{x \in \Omega : u_\lambda(x) \neq u(x)\}) \leq \frac{C}{\lambda^p} \int_\Omega |Du|^p d\mathcal{H}^n, \quad (1.16)$$

where the constant C depends only on Q , n and Ω .

As a corollary, Proposition 1.12 allows to prove that Sobolev Q -valued maps are approximately differentiable almost everywhere.

Corollary 1.13 (cf. [DLS11, Corollary 2.7]). *Let $u \in W^{1,p}(\Omega; \mathcal{A}_Q)$. Then, u is approximately differentiable \mathcal{H}^n -a.e. in Ω : precisely, for \mathcal{H}^n -a.e. $x \in \Omega$ there exists a measurable set $\tilde{\Omega} \subset \Omega$ containing x such that $\tilde{\Omega}$ has density 1 at x and $u|_{\tilde{\Omega}}$ is differentiable at x .*

The next proposition explores the link between the metric derivative defined in (1.15) and the approximate differential of a Q -valued Sobolev function.

Proposition 1.14 (cf. [DLS11, Proposition 2.17]). *Let u be a map in $W^{1,2}(\Omega; \mathcal{A}_Q(\mathbb{R}^M))$. Then, for any vector field ξ defined in Ω and tangent to Σ the metric derivative $|D_\xi u|$ defined in (1.15) satisfies*

$$|D_\xi u|^2 = \sum_{l=1}^Q |D_\xi u^l|^2 \quad \mathcal{H}^n\text{-a.e. in } \Omega, \quad (1.17)$$

²Here, the Sobolev space $W^{1,p}(\Omega)$ is classically defined as the completion of $C^1(\Omega)$ with respect to the $W^{1,p}$ -norm

$$\|f\|_{W^{1,p}(\Omega)}^p := \int_\Omega (|f(x)|^p + |Df(x)|^p) d\mathcal{H}^n(x)$$

for $1 \leq p < \infty$ and

$$\|f\|_{W^{1,\infty}(\Omega)} := \text{ess sup}_\Omega (|f(x)| + |Df(x)|).$$

where $\sum_l |D_\xi u^l|^2 = \mathcal{G}(D_\xi u, Q[[0]])^2$ and $D_\xi u(x) \in \mathcal{A}_Q(\mathbb{R}^M)$ is the approximate directional derivative of u along ξ at the point $x \in \Omega$. In particular, we will set

$$|Du|^2(x) := \sum_{i=1}^n |D_{\xi_i} u|^2(x) = \sum_{i=1}^n \sum_{l=1}^Q |D_{\xi_i} u^l|^2(x), \quad (1.18)$$

with $(\xi_i)_{i=1}^n$ any orthonormal frame of $\mathcal{T}\Sigma$, at all points x of approximate differentiability for u in Ω .

Remark 1.15. Observe that the definition in (1.18) is independent of the choice of the frame (ξ_i) , as in fact one has

$$|Du|^2(x) = \sum_{l=1}^Q |Du^l(x)|^2,$$

where $|Du^l(x)|$ is the Hilbert-Schmidt norm of the linear map $Du^l(x): T_x\Sigma \rightarrow \mathbb{R}^M$ at every point of approximate differentiability for u .

The main consequence of the above proposition is that essentially all the conclusions of the usual Sobolev space theory for single-valued functions can be recovered in the multivalued setting modulo routine modifications of the usual arguments. Some of these conclusions will be useful in the coming sections, thus we will list them here, again referring the interested reader to [DLS11] for their proofs and other useful considerations. In what follows, $\Omega \subset \Sigma$ is an open set with Lipschitz boundary.

Definition 1.16 (Trace of Sobolev Q -functions). Let $u \in W^{1,p}(\Omega; \mathcal{A}_Q(\mathbb{R}^M))$. A function g belonging to $L^p(\partial\Omega; \mathcal{A}_Q(\mathbb{R}^M))$ is said to be the *trace* of u at $\partial\Omega$ (and we write $g = u|_{\partial\Omega}$) if for any $T \in \mathcal{A}_Q$ the trace of the real-valued Sobolev function $\mathcal{G}(u, T)$ coincides with $\mathcal{G}(g, T)$.

Definition 1.17 (Weak convergence). Let $\{u_h\}_{h=1}^\infty$ be a sequence of maps in $W^{1,p}(\Omega; \mathcal{A}_Q)$. We say that u_h *converges weakly* to $u \in W^{1,p}(\Omega; \mathcal{A}_Q)$ for $h \rightarrow \infty$, and we write $u_h \rightharpoonup u$, if

- (i) $\lim_{h \rightarrow \infty} \int_\Omega \mathcal{G}(u_h, u)^p d\mathcal{H}^n = 0$;
- (ii) there exists a constant C such that $\sup_h \int_\Omega |Du_h|^p d\mathcal{H}^n \leq C$.

Proposition 1.18 (Weak sequential closure, cf. [DLS11, Proposition 2.10, Proposition 4.5]). Let $u \in W^{1,p}(\Omega; \mathcal{A}_Q)$. Then, there is a unique function $g \in L^p(\partial\Omega; \mathcal{A}_Q)$ such that $g = u|_{\partial\Omega}$ in the sense of Definition 1.16. Moreover, the set

$$W_g^{1,p}(\Omega; \mathcal{A}_Q) := \{u \in W^{1,p}(\Omega; \mathcal{A}_Q) : u|_{\partial\Omega} = g\}$$

is sequentially closed with respect to the notion of weak convergence introduced in Definition 1.17.

Proposition 1.19 (Sobolev embeddings, cf. [DLS11, Proposition 2.11, Proposition 4.6]). The following embeddings hold:

- (i) if $p < n$, then $W^{1,p}(\Omega; \mathcal{A}_Q) \subset L^q(\Omega; \mathcal{A}_Q)$ for every $q \in [1, p^*]$, $p^* := \frac{np}{n-p}$, and the inclusion is compact when $q < p^*$;
- (ii) if $p = n$, then $W^{1,p}(\Omega; \mathcal{A}_Q) \subset L^q(\Omega; \mathcal{A}_Q)$ for all $q \in [1, \infty)$, with compact inclusion;
- (iii) if $p > n$, then $W^{1,p}(\Omega; \mathcal{A}_Q) \subset C^{0,\alpha}(\Omega; \mathcal{A}_Q)$ for all $\alpha \in (0, 1 - \frac{n}{p}]$, with compact inclusion if $\alpha < 1 - \frac{n}{p}$.

Proposition 1.20 (Poincaré inequality, cf. [DLS11, Proposition 2.12, Proposition 4.9]). *Let Ω be a connected open subset of Σ with Lipschitz boundary, and let $p < n$. There exists a constant $C = C(p, n, M, Q, \Omega)$ with the following property: for every $u \in W^{1,p}(\Omega; \mathcal{A}_Q(\mathbb{R}^M))$ there exists a point $\bar{u} \in \mathcal{A}_Q(\mathbb{R}^M)$ such that*

$$\left(\int_{\Omega} \mathcal{G}(u, \bar{u})^{p^*} d\mathcal{H}^n \right)^{\frac{1}{p^*}} \leq C \left(\int_{\Omega} |Du|^p d\mathcal{H}^n \right)^{\frac{1}{p}}. \quad (1.19)$$

Proposition 1.21 (Campanato-Morrey estimates, cf. [DLS11, Proposition 2.14]). *Let u be a $W^{1,2}(B_1; \mathcal{A}_Q)$ function, with $B_1 = B_1(0) \subset \mathbb{R}^n$, and assume $\alpha \in (0, 1]$ is such that*

$$\int_{B_r(y)} |Du|^2 \leq Ar^{n-2+2\alpha} \quad \text{for every } y \in B_1 \text{ and a.e. } r \in (0, 1 - |y|).$$

Then, for every $0 < \delta < 1$ there is a constant $C = C(n, M, Q, \delta)$ such that

$$[u]_{C^{0,\alpha}(\bar{B}_\delta)} := \sup_{x,y \in \bar{B}_\delta} \frac{\mathcal{G}(u(x), u(y))}{|x-y|^\alpha} \leq C\sqrt{A}. \quad (1.20)$$

1.2.5. *The Dirichlet energy. Dir-minimizers.* A simple corollary of Proposition 1.14 and Remark 1.15 is that the *Dirichlet energy* of a map $u \in W^{1,2}(\Omega; \mathcal{A}_Q(\mathbb{R}^M))$ can be defined in a unique way by setting

$$\text{Dir}(u, \Omega) := \int_{\Omega} \sum_{i=1}^n |D_{\xi_i} u|^2 d\mathcal{H}^n = \int_{\Omega} \sum_{i=1}^n \sum_{l=1}^Q |D_{\xi_i} u^l|^2 d\mathcal{H}^n, \quad (1.21)$$

for any choice of a (local) orthonormal frame (ξ_1, \dots, ξ_n) of the tangent bundle of Σ .

Another interesting quantity that can be defined in our setting, the importance of which will become apparent in the sequel, is the *Dirichlet energy of a tangent vector field* to the manifold \mathcal{M} .

Definition 1.22 (Dirichlet energy of a tangent Q -field). *Let Ω be an open subset of $\Sigma \hookrightarrow \mathcal{M}$ as above. Let $u \in W^{1,2}(\Omega; \mathcal{A}_Q(\mathbb{R}^M))$ be a Sobolev Q -valued tangent vector field to \mathcal{M} : that is, assume that $\text{spt}(u(x)) \subset T_x \mathcal{M}$ for \mathcal{H}^n -a.e. $x \in \Omega$. Then, for any point x of approximate differentiability for u in Ω , and for any tangent vector field ξ , we set*

$$\nabla_{\xi} u(x) := \sum_{l=1}^Q [\mathbf{p}_x^{\mathcal{M}} \cdot D_{\xi} u^l(x)]. \quad (1.22)$$

The Dirichlet energy of the vector field u in Ω is thus given by

$$\text{Dir}^{\mathcal{T}\mathcal{M}}(u, \Omega) := \int_{\Omega} \sum_{i=1}^n |\nabla_{\xi_i} u|^2 d\mathcal{H}^n \quad (1.23)$$

for any orthonormal frame (ξ_1, \dots, ξ_n) of $\mathcal{T}\Sigma$.

Remark 1.23. Observe that, when u is Lipschitz continuous and $u|_{B_i} = \sum_{l=1}^Q \llbracket u_i^l \rrbracket$ is a local Lipschitz selection of u as in Proposition 1.6, one has

$$|\nabla_{\xi} u(x)|^2 = \sum_{l=1}^Q |\nabla_{\xi} u_i^l(x)|^2 \quad \text{for } \mathcal{H}^n - \text{a.e. } x \in B_i, \text{ for all vector fields } \xi,$$

where the ∇ on the right-hand side has to be intended as the classical covariant derivative.

The functional $\text{Dir}^{\mathcal{T}\mathcal{M}}$ defined in (1.23) is the “right” geometric quantity to consider when dealing with tangent vector fields, since it does not involve any geometric structure which is external to the manifold \mathcal{M} . In particular, it does not depend on the isometric embedding of the Riemannian manifold \mathcal{M} in the Euclidean space \mathbb{R}^M .

As already mentioned before, a theory concerning existence and regularity properties of minimizers of the Dirichlet energy in $W^{1,2}$ (the so called Dir-minimizers) has been extensively studied by Almgren in [Alm00] and revisited by De Lellis and Spadaro in [DLS11]. The theory can be summarized in three main theorems.

Theorem 1.24 (Existence and Hölder regularity, cf. [DLS11, Theorems 0.8 and 0.9]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with Lipschitz boundary. Let $g \in W^{1,2}(\Omega; \mathcal{A}_Q)$. Then, there exists a function $u \in W^{1,2}(\Omega; \mathcal{A}_Q)$ minimizing the Dirichlet energy (1.21) among all $W^{1,2}$ Q -valued functions v such that $v|_{\partial\Omega} = g|_{\partial\Omega}$. Furthermore, any Dir-minimizer u is in $C^{0,\alpha}(\Omega'; \mathcal{A}_Q)$ for every $\Omega' \Subset \Omega$, for some exponent $\alpha = \alpha(n, Q)$.*

The statement of the other two results requires the definition of regular and singular points of a Dir-minimizer u .

Definition 1.25 (Regular and singular points of a Dir-minimizing map). A Q -valued Dir-minimizer u is regular at a point $x \in \Omega$ if there exist a neighborhood B of x in Ω and Q harmonic functions $u_l: B \rightarrow \mathbb{R}^M$ such that

$$u(y) = \sum_{l=1}^Q \llbracket u_l(y) \rrbracket \quad \text{for almost every } y \in B$$

and either $u_l(y) \neq u_h(y)$ for every $y \in B$ or $u_l \equiv u_h$. We will write $x \in \text{reg}(u)$ if x is a regular point. The complement of $\text{reg}(u)$ in Ω is the *singular set*, and will be denoted $\text{sing}(u)$.

Theorem 1.26 (Estimate of the singular set, cf. [DLS11, Theorem 0.11]). *Let u be a Dir-minimizer. Then, the Hausdorff dimension of $\text{sing}(u)$ is at most $n - 2$. If $n = 2$, then $\text{sing}(u)$ is at most countable.*

Theorem 1.27 (Improved estimate of the singular set for $n = 2$, cf. [DLS11, Theorem 0.12]). *Let u be Dir-minimizing, and $n = 2$. Then, the singular set $\text{sing}(u)$ consists of isolated points.*

Remark 1.28. It is worth observing that here we have only recapped those results in the theory of Dir-minimizing multiple valued functions which will be useful for our purposes at a later stage of this paper, and therefore our summary is far from being complete. Among the results that we have not included in the above presentation, we mention the paper [Hir14], concerned with the problem of extending the Hölder regularity in Theorem 1.24 up to the boundary of Ω , and the recent result [DMSV16], where the authors prove that if u is Dir-minimizing then $\text{sing}(u)$ is actually countably $(n - 2)$ -rectifiable (and hence \mathcal{H}^{n-2} σ -finite), thus extending to general Q a previous result obtained for $Q = 2$ by Krummel and Wickramasekera in [KW13] and considerably improving Almgren’s original theory.

2. Q -VALUED SECOND VARIATION OF THE AREA FUNCTIONAL

In this section we define the admissible Q -valued normal variations of Σ in \mathcal{M} and we compute the associated second variation functional. In the following, we will denote by $\mathcal{A}_Q(\mathcal{M})$ the space of Q -points $T = \sum_l \llbracket T_l \rrbracket \in \mathcal{A}_Q(\mathbb{R}^M)$ with each T_l in \mathcal{M} .

Definition 2.1. An *admissible* variational Q -field of Σ in \mathcal{M} is a Lipschitz map

$$N := \sum_{l=1}^Q \llbracket N^l \rrbracket : \Sigma \rightarrow \mathcal{A}_Q(\mathbb{R}^M)$$

satisfying the following assumptions:

- (H1) $N^l(x) \in T_x^\perp \Sigma \subset T_x \mathcal{M}$ for every $l \in \{1, \dots, Q\}$, for every $x \in \Sigma$;
- (H2) N^l vanishes in a neighborhood of $\partial \Sigma$ for every $l \in \{1, \dots, Q\}$.

Definition 2.2. Given an admissible variational Q -field N , the one-parameter family of Q -valued deformations of Σ in \mathcal{M} induced by N is the map

$$F : \Sigma \times (-\delta, \delta) \rightarrow \mathcal{A}_Q(\mathcal{M})$$

defined by

$$F(x, t) := \sum_{l=1}^Q \llbracket \exp_x(tN^l(x)) \rrbracket, \quad (2.1)$$

where \exp denotes the exponential map on \mathcal{M} .

Observe that, for any given N as in Definition 2.1, the induced one-parameter family of Q -valued deformations is always well defined for a positive δ which depends on the L^∞ norm of N and on the injectivity radius of \mathcal{M} . Notice, furthermore, that $F(x, 0) = Q \llbracket x \rrbracket$ for every $x \in \Sigma$, and that $F(x, t) = Q \llbracket x \rrbracket$ for all t if $x \in \partial \Sigma$.

If F is an admissible one-parameter family of Q -valued deformations, we will often write $F_t(x)$ instead of $F(x, t)$. Moreover, we will set $F_t^l(x) := \exp_x(tN^l(x))$.

In what follows, we will always assume to have fixed an orthonormal frame (ξ_1, \dots, ξ_n) of the tangent bundle $\mathcal{T}\Sigma$, so that $\vec{\xi} = \xi_1 \wedge \dots \wedge \xi_n$ is a continuous simple unit n -vector field orienting Σ . Given any admissible variational Q -field N , we can now apply the results of the previous section, and consider the push-forward of Σ through the family F_t induced by N . An immediate consequence of Proposition 1.8 is that the resulting object is a one-parameter family of integer rectifiable n -currents, denoted $\Sigma_t := \mathbf{T}_{F_t}$ with $\text{spt}(\Sigma_t) \subset \mathcal{M}$. From (1.9), we have also the explicit representation formula

$$\Sigma_t(\omega) = \int_{\Sigma} \sum_{l=1}^Q \left\langle \omega \left(F_t^l(x) \right), DF_t^l(x) \sharp \vec{\xi}(x) \right\rangle d\mathcal{H}^n(x) \quad \forall \omega \in \mathcal{D}^n(\mathbb{R}^M). \quad (2.2)$$

We will denote $\mu(t)$ the *mass* of the current Σ_t , that is

$$\mu(t) := \mathbf{M}(\Sigma_t) := \sup \{ \Sigma_t(\omega) : \omega \in \mathcal{D}^n(\mathbb{R}^M), \|\omega(x)\|_c \leq 1 \text{ for every } x \in \mathbb{R}^M \}.^3 \quad (2.3)$$

Definition 2.3. Let $\Sigma \subset \mathcal{M}$ be as above, and let N be an admissible variational Q -field. For $k \geq 1$, the k^{th} order *variation* of Σ generated by N is the quantity

$$\delta^k \llbracket \Sigma \rrbracket (N) := \left. \frac{d^k \mu}{dt^k} \right|_{t=0}. \quad (2.4)$$

$\delta^1 \llbracket \Sigma \rrbracket$ is usually denoted $\delta \llbracket \Sigma \rrbracket$, and called first variation. $\delta^2 \llbracket \Sigma \rrbracket$ is called second variation.

³If $\omega \in \Lambda^n(\mathbb{R}^M)$ is an n -covector, then $\|\omega\|_c$ denotes its *comass norm* (see, for instance, [Sim83b, Section 26]).

For every k , $\delta^k[\Sigma]$ is a functional defined on the space of admissible variational Q -fields. Clearly, the first variation functional is identically zero under the hypothesis that Σ is minimal in \mathcal{M} ; our next goal, instead, is to provide an explicit representation formula for $\delta^2[\Sigma](N)$.

The key tool to obtain such a representation formula is the Q -valued area formula, Proposition 1.10, which yields an explicit formula for the function $\mu(t)$. Indeed, we may write

$$\mu(t) = \int_{\Sigma} \sum_{l=1}^Q \mathbf{J}F_t^l(x) d\mathcal{H}^n(x), \quad (2.5)$$

provided condition (1.14) is satisfied: that is, provided there is a set $B \subset \Sigma$ of full measure for which

$$\langle DF_t^l(x) \sharp \vec{\xi}(x), DF_t^h(y) \sharp \vec{\xi}(y) \rangle \geq 0 \quad \forall x, y \in B \text{ and } l, h \text{ with } F_t^l(x) = F_t^h(y). \quad (2.6)$$

Now, it is not difficult to show that in fact condition (2.6) holds with $B = \Sigma$: to see this, first observe that since Σ is compact there exists a number $\varepsilon > 0$ such that $\langle \vec{\xi}(x), \vec{\xi}(y) \rangle \geq \frac{1}{2}$ for all points $x, y \in \Sigma$ such that $\mathbf{d}(x, y) \leq \varepsilon$. On the other hand, the very definition of F implies that for any $x \in \Sigma$ one may write $F_t^l(x) = x + tN^l(x) + o(t)$ for $t \rightarrow 0$. Therefore, if $|t|$ is chosen small enough, depending on Σ , ε and on the L^∞ norm of N in Σ , the condition $F_t^l(x) = F_t^h(y)$ implies $\mathbf{d}(x, y) \leq \varepsilon$ and consequently the condition $\langle \vec{\xi}(x), \vec{\xi}(y) \rangle \geq \frac{1}{2}$. But now, since $DF_t^l(x) = \text{Id} + tDN^l(x) + o(t)$, we easily infer that $\langle DF_t^l(x) \sharp \vec{\xi}(x), DF_t^h(y) \sharp \vec{\xi}(y) \rangle \geq \frac{1}{4}$ for all $x, y \in \Sigma$ and l, h with $F_t^l(x) = F_t^h(y)$ provided $|t| \leq \delta_0$ for some $\delta_0 = \delta_0(\Sigma, \varepsilon, \|N\|_{L^\infty(\Sigma)}, \text{Lip}(N))$.

Thus, we can work on each component F^l of the decomposition of F separately: in the end, we will just apply (2.5) to obtain the desired variation formulas. Moreover, since the coming arguments are local, we will assume in what follows that the frame $\{\xi_i\}_{i=1}^n$ is C^2 in a neighborhood of any given point x .

With that being said, let us now consider a fixed value of $l \in \{1, \dots, Q\}$ and introduce the following quantities. For any x point of differentiability for N in Σ , let $Z^l(x) := \partial_{tt}F^l(x, 0)$ denote the initial acceleration of the l^{th} sheet at the point x , so that the second order Taylor expansion of $F^l(x, \cdot)$ around $t = 0$ is

$$F^l(x, t) = x + tN^l(x) + \frac{1}{2}t^2Z^l(x) + o(t^2)$$

in a suitable δ -neighborhood of $t = 0$. Then, for any $i \in \{1, \dots, n\}$, define

$$e_i^l = e_i^l(x, t) := D_{\xi_i}F_t^l(x) = \xi_i(x) + tD_{\xi_i}N^l(x) + \frac{1}{2}t^2D_{\xi_i}Z^l(x) + o(t^2) \quad (2.7)$$

and

$$V^l = V^l(x, t) := \partial_t F^l(x, t). \quad (2.8)$$

Observe that e_i^l and V^l are tangent vector fields to \mathcal{M} .

Next, for $i, j \in \{1, \dots, n\}$ denote

$$g_{ij}^l = g_{ij}^l(x, t) := \langle e_i^l(x, t), e_j^l(x, t) \rangle \quad (2.9)$$

and

$$g^l = g^l(x, t) := \det(g_{ij}^l(x, t)). \quad (2.10)$$

Using these notations, we readily see that the Jacobian determinant $\mathbf{J}F_t^l$ can be written as follows:

$$J^l = J^l(x, t) := \mathbf{J}F_t^l(x) = \sqrt{g^l(x, t)}, \quad (2.11)$$

so that, finally, the mass of the push-forward current is given by

$$\mu(t) = \sum_{l=1}^Q \mu^l(t), \quad (2.12)$$

where

$$\mu^l(t) := \int_{\Sigma} J^l(x, t) d\mathcal{H}^n(x). \quad (2.13)$$

Thus, we conclude that the first and second variation of Σ under the deformation generated by N can be represented in the following way:

$$\delta[\Sigma](N) = \sum_{l=1}^Q \frac{d\mu^l}{dt} \Big|_{t=0} \quad (2.14)$$

and

$$\delta^2[\Sigma](N) = \sum_{l=1}^Q \frac{d^2\mu^l}{dt^2} \Big|_{t=0}. \quad (2.15)$$

In what follows, in order to simplify the notation, we will drop the superscript l when carrying the computation on.

One has:

$$\frac{d\mu}{dt} \Big|_{t=0} = \int_{\Sigma} \partial_t J(x, 0) d\mathcal{H}^n(x). \quad (2.16)$$

Now, since

$$\partial_t J = \frac{1}{2J} \partial_t g,$$

and since $g_{ij} = \delta_{ij}$ at time $t = 0$, easy computations show that

$$\partial_t J \Big|_{t=0} = \frac{1}{2} \sum_{i=1}^n \partial_t g_{ii} \Big|_{t=0} = \sum_{i=1}^n \langle e_i, \partial_t e_i \rangle \Big|_{t=0}. \quad (2.17)$$

From this, we immediately recover the usual formulation of the *first variation formula* in terms of the divergence of the variational vector field:

$$\delta[\Sigma](N) = \int_{\Sigma} \sum_{l=1}^Q \sum_{i=1}^n \langle \xi_i, D_{\xi_i} N^l \rangle d\mathcal{H}^n = \int_{\Sigma} \sum_{l=1}^Q \sum_{i=1}^n \langle \xi_i, \nabla_{\xi_i} N^l \rangle d\mathcal{H}^n = \int_{\Sigma} \sum_{l=1}^Q \operatorname{div}_{\Sigma}(N^l) d\mathcal{H}^n. \quad (2.18)$$

Remark 2.4. Observe that, since by orthogonality the scalar product $\langle \xi_i, N^l \rangle$ is everywhere vanishing, we have that $\langle \xi_i, \nabla_{\xi_i} N^l \rangle = -\langle \nabla_{\xi_i} \xi_i, N^l \rangle = -\langle A(\xi_i, \xi_i), N^l \rangle$. Therefore, recalling the definition of the mean curvature vector H as the trace of the second fundamental form, one can also write

$$\delta[\Sigma](N) = - \int_{\Sigma} \sum_{l=1}^Q \langle H, N^l \rangle d\mathcal{H}^n. \quad (2.19)$$

Thus, as expected, $\delta[\Sigma](N) = 0$ if the submanifold Σ is minimal in \mathcal{M} .

Next, we go further and we compute the second variation of the mass. We first write, for every t and for every $x \in \Sigma$ of differentiability for the variational field:

$$\partial_t J = \frac{1}{2\sqrt{g}} \partial_t g = \frac{1}{2} J \frac{1}{g} \partial_t g = \frac{1}{2} J \partial_t (\log(g)) = \frac{1}{2} J g^{ij} \partial_t g_{ji},$$

where in the last identity we have used Jacobi's formula

$$\partial_t \log \det A(t) = \operatorname{tr} \left(A(t)^{-1} \cdot \partial_t A(t) \right)$$

for any invertible matrix $A(t)$ with positive determinant. Moreover, (g^{ij}) is the inverse matrix of (g_{ij}) , and Einstein's convention on the summation of repeated indices has been used. Now, since

$$\partial_t g_{ji} = \partial_t (\langle e_j, e_i \rangle) = \langle \partial_t e_j, e_i \rangle + \langle e_j, \partial_t e_i \rangle,$$

and using the fact that the matrix (g^{ij}) is symmetric, we can conclude the following identity:

$$\partial_t J = J g^{ij} \langle e_i, \partial_t e_j \rangle.$$

In turn, this produces:

$$\begin{aligned} \partial_{tt} J &= (\partial_t J) g^{ij} \langle e_i, \partial_t e_j \rangle \\ &+ J \left(\partial_t g^{ij} \right) \langle e_i, \partial_t e_j \rangle \\ &+ J g^{ij} \partial_t (\langle e_i, \partial_t e_j \rangle) =: I + II + III. \end{aligned} \quad (2.20)$$

Now, we evaluate equation (2.20) at time $t = 0$. Regarding the first term in the sum, we use (2.17), the orthonormality condition $g^{ij}|_{t=0} = \delta^{ij}$ and the fact that $e_i|_{t=0} = \xi_i$, $\partial_t e_i|_{t=0} = D_{\xi_i} N$ (here, of course, we are writing N instead of N^l) to conclude

$$I|_{t=0} = \left(\sum_{i=1}^n \langle \xi_i, \nabla_{\xi_i} N \rangle \right)^2. \quad (2.21)$$

Arguing as in Remark 2.4, one therefore obtains that

$$I|_{t=0} = (\langle H, N \rangle)^2 = 0 \quad (2.22)$$

due to the minimality of Σ .

In order to derive a formula for $II|_{t=0}$, we first differentiate the identity

$$g^{ij} g_{jh} = \delta_h^i$$

to obtain that

$$\partial_t g^{ij} = -g^{ik} (\partial_t g_{kh}) g^{hj},$$

whence

$$\partial_t g^{ij}|_{t=0} = -\partial_t g_{ij}|_{t=0} = - \left(\langle \nabla_{\xi_i} N, \xi_j \rangle + \langle \xi_i, \nabla_{\xi_j} N \rangle \right). \quad (2.23)$$

Since $\langle \nabla_{\xi_i} N, \xi_j \rangle = -\langle A(\xi_i, \xi_j), N \rangle$, the symmetry of the second fundamental form implies

$$\partial_t g^{ij}|_{t=0} = 2 \langle A(\xi_i, \xi_j), N \rangle. \quad (2.24)$$

Again, since

$$\langle e_i, \partial_t e_j \rangle|_{t=0} = \langle \xi_i, \nabla_{\xi_j} N \rangle = -\langle A(\xi_i, \xi_j), N \rangle,$$

we can finally obtain

$$II|_{t=0} = -2 \sum_{i,j=1}^n |\langle A(\xi_i, \xi_j), N \rangle|^2. \quad (2.25)$$

Finally, we compute $III|_{t=0}$. The simplest way to do it is to regard the operator ∂_t as the covariant derivative along the vector field $V = V^l$. One therefore has:

$$\begin{aligned}\partial_t(\langle e_i, \partial_t e_j \rangle) &= V\langle e_i, \nabla_V e_j \rangle \\ &= \langle \nabla_V e_i, \nabla_V e_j \rangle + \langle e_i, \nabla_V \nabla_V e_j \rangle \\ &= \langle \nabla_{e_i} V, \nabla_{e_j} V \rangle + \langle e_i, \nabla_V \nabla_{e_j} V \rangle,\end{aligned}$$

where in the last identity we have used the fact that the vector fields e_i and V commute, and, of course, that the Riemannian connection on \mathcal{M} is torsion-free. Now, using again that $[V, e_j] = 0$ and the definition of the curvature tensor R , we may write

$$\nabla_V \nabla_{e_j} V = \nabla_{e_j} \nabla_V V + R(V, e_j)V,$$

so that, finally, the evaluation of $\partial_t(\langle e_i, \partial_t e_j \rangle)$ at time $t = 0$ yields

$$\partial_t(\langle e_i, \partial_t e_j \rangle)|_{t=0} = \langle \nabla_{\xi_i} N, \nabla_{\xi_j} N \rangle + \langle \xi_i, \nabla_{\xi_j} Z \rangle + \langle \xi_i, R(N, \xi_j)N \rangle,$$

with $Z = Z^l$. Since $J|_{t=0} = 1$ and $g^{ij}|_{t=0} = \delta^{ij}$, we conclude the following identity:

$$III|_{t=0} = \sum_{i=1}^n |\nabla_{\xi_i} N|^2 + \operatorname{div}_{\Sigma} Z - \sum_{i=1}^n \langle R(N, \xi_i) \xi_i, N \rangle. \quad (2.26)$$

Observe that, in deriving formula (2.26), we have used that $\langle R(X, Y)Z, W \rangle = -\langle R(X, Y)W, Z \rangle$ for any choice of X, Y, Z, W vector fields on \mathcal{M} .

Now, we decompose

$$Z = \mathbf{p} \cdot Z + \mathbf{p}^{\perp} \cdot Z = Z^{\top} + Z^{\perp}, \quad (2.27)$$

and we see that, since $\langle \xi_i, Z^{\perp} \rangle = 0$ for all i ,

$$\operatorname{div}_{\Sigma}(Z^{\perp}) = \sum_{i=1}^n \langle \xi_i, \nabla_{\xi_i} Z^{\perp} \rangle = - \sum_{i=1}^n \langle A(\xi_i, \xi_i), Z \rangle = -\langle H, Z \rangle = 0. \quad (2.28)$$

On the other hand, Stokes' theorem and the fact that N is vanishing in a neighborhood of $\partial\Sigma$ readily imply that

$$\int_{\Sigma} \operatorname{div}_{\Sigma}(Z^{\top}) d\mathcal{H}^n = 0. \quad (2.29)$$

We have now all the tools to derive the *second variation formula* for the function $\mu(t)$: from the Q -valued area formula (2.5) follows that

$$\frac{d^2 \mu}{dt^2} \Big|_{t=0} = \int_{\Sigma} \sum_{l=1}^Q \partial_{tt} J^l(x, 0) d\mathcal{H}^n(x),$$

thus it suffices to plug equations (2.22), (2.25), (2.26) in (2.20) and use (2.28), (2.29) to deduce that

$$\delta^2[\Sigma](N) = \int_{\Sigma} \sum_{l=1}^Q \left(\sum_{i=1}^n |\nabla_{\xi_i} N^l|^2 - 2 \sum_{i,j=1}^n |\langle A(\xi_i, \xi_j), N^l \rangle|^2 - \sum_{i=1}^n \langle R(N^l, \xi_i) \xi_i, N^l \rangle \right) d\mathcal{H}^n. \quad (2.30)$$

Remark 2.5. Observe that formula (2.30) makes sense because the quantity on the right-hand side does not depend on the particular selection chosen for N , nor on the orthonormal frame chosen for the tangent bundle $\mathcal{T}\Sigma$.

The first addendum in the sum is the Dirichlet energy of the multivalued vector field N on the manifold \mathcal{M} as defined in (1.23).

The second term in the sum can as well be given an intrinsic formulation, once we observe that

$$\sum_{i,j=1}^n |\langle A(\xi_i, \xi_j), N^l \rangle|^2 = |A \cdot N^l|^2, \quad (2.31)$$

where $A \cdot N^l: \mathcal{T}\Sigma \times \mathcal{T}\Sigma \rightarrow \mathbb{R}$ is the symmetric bilinear form defined by $A \cdot N^l(\xi, \eta) := \langle A(\xi, \eta), N^l \rangle$ and $|A \cdot N^l|$ is its Hilbert-Schmidt norm.

Regarding the third term, the symmetry properties of the Riemann tensor allow to write

$$\langle R(N^l, \xi_i)\xi_i, N^l \rangle = \langle R(\xi_i, N^l)N^l, \xi_i \rangle = \langle \mathbf{p} \cdot R(\xi_i, N^l)N^l, \xi_i \rangle,$$

which in turn implies that $\sum_{i=1}^n \langle R(N^l, \xi_i)\xi_i, N^l \rangle$ coincides with the trace of the endomorphism

$$\xi \mapsto \mathbf{p} \cdot R(\xi, N^l)N^l$$

of the tangent bundle $\mathcal{T}\Sigma$. In other words, this term is a partial Ricci curvature in the direction of the vector field N^l . We will set

$$\mathcal{R}(N^l, N^l) := \sum_{i=1}^n \langle R(\xi_i, N^l)N^l, \xi_i \rangle \quad (2.32)$$

to denote the associated quadratic form.

We notice now that the quantity appearing on the right-hand side of formula (2.30) is in fact well defined for any Q -valued vector field tangent to \mathcal{M} and belonging to the class $W^{1,2}(\Sigma; \mathcal{A}_Q(\mathbb{R}^M))$. This motivates the following definitions.

Definition 2.6 ($W^{1,2}$ sections of the normal bundle). Let $\Sigma \hookrightarrow \mathcal{M}$ be as above, and let $\Omega \subset \Sigma$ be open. We define the class of $W^{1,2}$ sections of the normal bundle of Ω in \mathcal{M} , denoted $\Gamma_Q^{1,2}(\mathcal{N}\Omega)$, as follows:

$$\Gamma_Q^{1,2}(\mathcal{N}\Omega) := \left\{ N \in W^{1,2}(\Omega; \mathcal{A}_Q(\mathbb{R}^M)) : \text{spt}(N(x)) \subset T_x^\perp \Sigma \subset T_x \mathcal{M} \text{ for } \mathcal{H}^n\text{-a.e. } x \in \Omega \right\}. \quad (2.33)$$

Definition 2.7 (Jacobi functional). For a section $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$, the *Jacobi functional*, or *stability functional*, is defined by:

$$\text{Jac}(N, \Omega) := \text{Dir}^{\mathcal{T}\mathcal{M}}(N, \Omega) - 2 \int_\Omega \sum_{l=1}^Q |A \cdot N^l|^2 d\mathcal{H}^n - \int_\Omega \sum_{l=1}^Q \mathcal{R}(N^l, N^l) d\mathcal{H}^n. \quad (2.34)$$

We will soon show that the definitions (0.1) and (2.34) of the Jacobi functional are equivalent. Before coming to that, we remark that the classical theory of the Jacobi normal operator can be recovered within the above framework by simply setting $Q = 1$.

Remark 2.8. Consider the classical single-valued setting, corresponding to $Q = 1$, let $\Omega = \Sigma$ and recall that

$$\text{Dir}^{\mathcal{T}\mathcal{M}}(N, \Sigma) = \int_\Sigma \sum_{i=1}^n |\nabla_{\xi_i} N|^2 d\mathcal{H}^n$$

for any orthonormal frame (ξ_1, \dots, ξ_n) of $\mathcal{T}\Sigma$. Assume also that N is Lipschitz continuous for convenience. Let (ν_1, \dots, ν_k) be local sections of the normal bundle $\mathcal{N}\Sigma$ of Σ in \mathcal{M} such that,

at each point $x \in \Sigma$, the system $\left((\xi_j(x))_{j=1}^n, (\nu_\alpha(x))_{\alpha=1}^k \right)$ is an orthonormal basis of $T_x \mathcal{M}$. Then, for every point of differentiability for N and for every $i = 1, \dots, n$ we have:

$$|\nabla_{\xi_i} N|^2 = \sum_{j=1}^n |\langle \nabla_{\xi_i} N, \xi_j \rangle|^2 + \sum_{\alpha=1}^k |\langle \nabla_{\xi_i} N, \nu_\alpha \rangle|^2.$$

Now, the usual considerations about the orthogonality of N and ξ_j imply that $\langle \nabla_{\xi_i} N, \xi_j \rangle = -\langle A(\xi_i, \xi_j), N \rangle$. We may therefore obtain that

$$\int_{\Sigma} \left(\sum_{i=1}^n |\nabla_{\xi_i} N|^2 - \sum_{i,j=1}^n |\langle A(\xi_i, \xi_j), N \rangle|^2 \right) d\mathcal{H}^n = \int_{\Sigma} \sum_{i=1}^n \sum_{\alpha=1}^k |\langle \nabla_{\xi_i} N, \nu_\alpha \rangle|^2 d\mathcal{H}^n,$$

and finally conclude the identity

$$\text{Jac}(N, \Sigma) = \int_{\Sigma} \left(\sum_{i=1}^n \sum_{\alpha=1}^k |\langle \nabla_{\xi_i} N, \nu_\alpha \rangle|^2 - \sum_{i,j=1}^n |\langle A(\xi_i, \xi_j), N \rangle|^2 - \sum_{i=1}^n \langle R(N, \xi_i) \xi_i, N \rangle \right) d\mathcal{H}^n. \quad (2.35)$$

It is immediately seen that the Euler-Lagrange operator associated to the second variation functional (2.35) is the linear elliptic operator \mathcal{L} defined on the space of sections of $\mathcal{N}\Sigma$ and given by

$$\mathcal{L} := -\Delta_{\Sigma}^{\perp} - \mathcal{A} - \mathcal{R}, \quad (2.36)$$

where Δ_{Σ}^{\perp} is the Laplacian on the normal bundle of Σ , \mathcal{A} is Simons' operator, defined by

$$\mathcal{A}(N) := \sum_{i,j=1}^n \langle A(\xi_i, \xi_j), N \rangle A(\xi_i, \xi_j), \quad (2.37)$$

and \mathcal{R} is given by

$$\mathcal{R}(N) := \sum_{i=1}^n \mathbf{p}^{\perp} \cdot R(N, \xi_i) \xi_i. \quad (2.38)$$

As already anticipated in the Introduction, the operator \mathcal{L} is classically called *Jacobi normal operator*, and the solutions of the differential equation $\mathcal{L}(N) = 0$ (that is, the normal vector fields that are in its kernel) are known in the literature as *Jacobi fields*. The importance of studying the second variation operator of minimal submanifolds into Riemannian manifolds is well justified by the arguments given earlier on in this section: in the single valued case $Q = 1$, the Jacobi operator \mathcal{L} carries the information about the stability properties of the submanifold itself, when it is thought of as a stationary point for the n -dimensional volume. In particular, non-trivial Jacobi fields vanishing on $\partial\Sigma$ are, when they exist, the infinitesimal normal deformations of Σ which preserve the volume up to second order. From a functional analytic point of view, \mathcal{L} is a second-order strongly elliptic operator. When diagonalized on the space of sections of $\mathcal{N}\Sigma$ vanishing on $\partial\Sigma$ with respect to the standard inner product, it exhibits distinct, real eigenvalues $\{\lambda_h\}_{h=1}^{\infty}$ (counted with multiplicities) such that

$$\lambda_1 < \lambda_2 < \dots < \lambda_h < \dots \rightarrow +\infty.$$

Moreover, the dimension of each eigenspace is finite. The number of negative eigenvalues is called the *Morse index* of Σ : it accounts for the number of independent infinitesimal normal deformations of Σ which do decrease the volume at second order. We recall that Σ is called *stable* if its Morse index is 0, and *strictly stable* if there exist no non-trivial Jacobi fields vanishing at the boundary.

A systematic study of the Jacobi normal operator was initiated by J. Simons in [Sim68]. One of Simons' main results was to prove that if $\mathcal{M} = \mathbb{S}^m$ and Σ is a closed minimal hypersurface immersed in \mathbb{S}^m which is not totally geodesic then the first eigenvalue of the operator \mathcal{L} satisfies the upper bound $\lambda_1 \leq -(m-1)$. As a consequence of this, he was able to show that no non-trivial stable minimal cones exist in \mathbb{R}^{m+1} for $m \leq 6$. In turn, this led to the proof of the Bernstein conjecture, stating that the only entire solutions $u: \mathbb{R}^m \rightarrow \mathbb{R}$ of the minimal surface equation are linear, for every $m \leq 7$. The result is sharp, as the Bernstein conjecture was proved to be false for $m > 7$ by E. Bombieri, E. De Giorgi and E. Giusti in [BDGG69].

Remark 2.9. Clearly, the considerations leading to formula (2.35) can be repeated in the Q -valued setting. In particular, if $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$, we can set, at any point $x \in \Omega$ where N is approximately differentiable, and for any tangent vector field ξ

$$\nabla_\xi^\perp N(x) := \sum_{l=1}^Q [\mathbf{p}_x^{\Sigma^\perp} \cdot D_\xi N^l(x)], \quad (2.39)$$

where $\mathbf{p}_x^{\Sigma^\perp} = \mathbf{p}_x^\perp \circ \mathbf{p}_x^{\mathcal{M}}$, and define

$$\text{Dir}^{\mathcal{N}\Sigma}(N, \Omega) := \int_\Omega \sum_{i=1}^n |\nabla_{\xi_i}^\perp N|^2 d\mathcal{H}^n. \quad (2.40)$$

The Jacobi functional can therefore be given the form

$$\begin{aligned} \text{Jac}(N, \Omega) &= \text{Dir}^{\mathcal{N}\Sigma}(N, \Omega) - \int_\Omega \sum_{l=1}^Q |A \cdot N^l|^2 d\mathcal{H}^n - \int_\Omega \sum_{l=1}^Q \mathcal{R}(N^l, N^l) d\mathcal{H}^n \\ &= \int_\Omega \sum_{l=1}^Q \left(\sum_{i=1}^n \sum_{\alpha=1}^k |\langle D_{\xi_i} N^l, \nu_\alpha \rangle|^2 - \sum_{i,j=1}^n |\langle A(\xi_i, \xi_j), N^l \rangle|^2 - \sum_{i=1}^n \langle R(\xi_i, N^l) N^l, \xi_i \rangle \right) d\mathcal{H}^n, \end{aligned} \quad (2.41)$$

which is indeed formula (0.1).

On the other hand, the lack of linear structure of $\Gamma_Q^{1,2}(\mathcal{N}\Omega)$ in the multivalued case $Q > 1$ does not allow one to associate an operator to the Jacobi functional, nor to characterize multiple valued Jacobi fields as the solutions of a certain (Euler-Lagrange) PDE. Nonetheless, the variational structure of the problem suggests that the minimizers of the Jacobi functional for a given boundary datum have the right to be considered the multivalued counterpart of the classical Jacobi fields.

3. JACOBI Q -FIELDS

The arguments contained in Remark 2.9 validate the definition of Jacobi Q -fields as it was proposed in Definition 0.2. The analysis of the first elementary properties of Jac-minimizers is the content of this section.

Remark 3.1. If $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ is a Jacobi Q -field with $N|_{\partial\Omega} = Q[0]$, then $\text{Jac}(N, \Omega) = 0$. This is a consequence of the homogeneity properties of the functional: in this case, indeed, for any $t \in \mathbb{R}$ the Q -field $tN := \sum_l [tN^l]$ is a suitable competitor, and

$$\text{Jac}(tN, \Omega) = t^2 \text{Jac}(N, \Omega).$$

Hence, were $\text{Jac}(N, \Omega) < 0$ ⁴, one would obtain that $\lim_{t \rightarrow \infty} \text{Jac}(tN, \Omega) = -\infty$, thus contradicting the definition of N .

We are then able to conclude that if the minimum problem for the Jacobi functional with $Q[[0]]$ boundary data does admit a solution, then for any minimizer N one has $\text{Jac}(N, \Omega) = 0$. In particular, $N_0 \equiv Q[[0]]$ is a minimizer.

Our next goal will be to address the question of the existence of Jacobi Q -fields N in Ω with prescribed boundary value $N|_{\partial\Omega} = g|_{\partial\Omega}$ for some fixed $g \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$. Of course, the idea here is to analyze the properties of *lower semi-continuity* and *compactness* of the stability functional in order to eventually use the direct method in the Calculus of Variations. The proof of the weak lower semi-continuity is rather simple, as shown in Proposition 3.3 below. Before that, it will be useful to make some further considerations about the structure of the Jacobi functional, in order to simplify the notation and to express it as a perturbation of the Dirichlet functional $\text{Dir}(u, \Omega)$.

Remark 3.2. Given any Q -valued Lipschitz map $u = \sum_l \llbracket u^l \rrbracket$ satisfying $u^l(x) \in T_x^\perp \Sigma \subset T_x \mathcal{M}$ for all $x \in \Omega$, the orthogonal decomposition

$$D_\xi u^l(x) = \mathbf{p}_x^{\mathcal{M}} \cdot D_\xi u^l(x) + \mathbf{p}_x^{\mathcal{M}^\perp} \cdot D_\xi u^l(x) = \nabla_\xi u^l(x) + \bar{A}_x(\xi(x), u^l(x))$$

holds for any tangent vector field ξ at any point x of differentiability for u , hence \mathcal{H}^n -a.e. in Ω . Here, \bar{A} denotes the second fundamental form of the embedding $\mathcal{M} \hookrightarrow \mathbb{R}^M$. Hence, at any such point we may write

$$|\nabla_\xi u^l|^2 = |D_\xi u^l|^2 - |\bar{A}(\xi, u^l)|^2.$$

These considerations are extended in the obvious way to all sections $u \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ at all points of approximate differentiability. Ultimately, we will write

$$\text{Jac}(u, \Omega) = \text{Dir}(u, \Omega) - \mathcal{B}_\Omega(u), \quad (3.1)$$

where \mathcal{B}_Ω is the functional on $\Gamma_Q^{1,2}(\mathcal{N}\Omega)$ defined by

$$\mathcal{B}_\Omega(u) := \int_\Omega \sum_{l=1}^Q \left(\sum_{i=1}^n |\bar{A}(\xi_i, u^l)|^2 + 2 \sum_{i,j=1}^n |\langle A(\xi_i, \xi_j), u^l \rangle|^2 + \sum_{i=1}^n \langle R(u^l, \xi_i) \xi_i, u^l \rangle \right) d\mathcal{H}^n. \quad (3.2)$$

Observe that \mathcal{B}_Ω satisfies an estimate of the form

$$|\mathcal{B}_\Omega(u)| \leq C_0 \|u\|_{L^2}^2, \quad (3.3)$$

where C_0 is a *geometric* constant, depending on $\mathbf{A} := \|A\|_{L^\infty}$, $\bar{\mathbf{A}} := \|\bar{A}\|_{L^\infty}$ and $\mathbf{R} := \|R\|_{L^\infty}$.

The weak lower semi-continuity of the Jacobi functional is the content of the following proposition.

Proposition 3.3. *The Jacobi functional is lower semi-continuous with respect to the weak topology of $\Gamma_Q^{1,2}(\mathcal{N}\Omega)$.*

Proof. Consider Q -valued sections $u_h, u \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ and assume that $u_h \rightharpoonup u$ weakly in $W^{1,2}(\Omega; \mathcal{A}_Q)$. Recall now that

$$\text{Jac}(u_h, \Omega) = \text{Dir}(u_h, \Omega) - \mathcal{B}_\Omega(u_h).$$

⁴Observe that if $N|_{\partial\Omega} = Q[[0]]$, then the null Q -field $N_0 \equiv Q[[0]]$ is a competitor, whence $\text{Jac}(N, \Omega) \leq \text{Jac}(N_0, \Omega) = 0$

The weak lower semi-continuity of the Dirichlet energy was proved by De Lellis and Spadaro in [DLS11, Proof of Theorem 0.8, p.30]. On the other hand, the condition

$$\lim_{h \rightarrow \infty} \int_{\Omega} \mathcal{G}(u_h, u)^2 d\mathcal{H}^n = 0$$

is enough to show that in fact

$$\lim_{h \rightarrow \infty} \mathcal{B}_{\Omega}(u_h) = \mathcal{B}_{\Omega}(u). \quad (3.4)$$

To see this, just write (3.4) as

$$\lim_h \int_{\Omega} b_h d\mathcal{H}^n = \int_{\Omega} b d\mathcal{H}^n, \quad (3.5)$$

with

$$b_h(x) = \sum_{l=1}^Q \left(\sum_{i=1}^n |\bar{A}(\xi_i, u_h^l)|^2 + 2 \sum_{i,j=1}^n |\langle A(\xi_i, \xi_j), u_h^l \rangle|^2 + \sum_{i=1}^n \langle R(u_h^l, \xi_i) \xi_i, u_h^l \rangle \right)$$

and

$$b(x) = \sum_{l=1}^Q \left(\sum_{i=1}^n |\bar{A}(\xi_i, u^l)|^2 + 2 \sum_{i,j=1}^n |\langle A(\xi_i, \xi_j), u^l \rangle|^2 + \sum_{i=1}^n \langle R(u^l, \xi_i) \xi_i, u^l \rangle \right),$$

and observe that the strong convergence $u_h \rightarrow u$ in $L^2(\Omega; \mathcal{A}_Q)$ suffices to prove that along a subsequence (not relabeled) $b_h(x) \rightarrow b(x)$ pointwise \mathcal{H}^n -a.e. in Ω . Equation (3.5) then follows by standard techniques in integration theory. \square

As for compactness, the situation is quite more involved. Indeed, as already anticipated, the existence of a solution of the minimum problem for the Jacobi functional with any boundary datum g is strictly related with showing that $N_0 \equiv Q[0]$ is in fact the *only* minimizer under $Q[0]$ boundary conditions. We first proceed providing an equivalent condition for this last assumption to be satisfied.

Lemma 3.4. *There exists a unique solution $N_0 \equiv Q[0]$ of the problem*

$$\min \left\{ \text{Jac}(u, \Omega) : u \in \Gamma_Q^{1,2}(\mathcal{N}\Omega) \text{ such that } u|_{\partial\Omega} = Q[0] \right\}$$

if and only if there exists a positive constant $c = c(\Omega)$ such that

$$\text{Jac}(u, \Omega) \geq c \int_{\Omega} |u|^2 d\mathcal{H}^n, \quad (3.6)$$

for all $u \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ with $u|_{\partial\Omega} = Q[0]$.

Remark 3.5. In the classical case, corresponding to $Q = 1$, the above condition of non-existence of non-trivial Jacobi fields vanishing at the boundary of a submanifold Σ characterizes, as we have already pointed out, the strictly stable minimal submanifolds. If Σ is strictly stable, then the largest positive constant $c(\Omega)$ such that (3.6) holds for every $W^{1,2}$ section u of $\mathcal{N}\Omega$ with $u|_{\partial\Omega} = 0$ is the first eigenvalue λ_1 of the Jacobi operator \mathcal{L} .

Proof. Assume first that (3.6) holds. Then, the Jacobi functional is non-negative on the space of $W^{1,2}$ sections of $\mathcal{N}\Omega$ with vanishing trace at the boundary. It is then clear that $N_0 \equiv Q[0]$

is a minimizer. Moreover, it is the only one. Indeed, if u is a minimizer, then $\text{Jac}(u, \Omega) = 0$, and therefore (3.6) forces

$$\int_{\Omega} \mathcal{G}(u, Q[[0]])^2 d\mathcal{H}^n = 0.$$

For the converse, assume that the minimum problem for the Jacobi functional with vanishing boundary condition admits $N_0 \equiv Q[[0]]$ as the only solution. In particular, this implies that $\text{Jac}(u, \Omega) \geq 0$ for all sections $u \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ such that $u|_{\partial\Omega} = Q[[0]]$, and in fact that $\text{Jac}(u, \Omega) > 0$ for all such sections except the trivial one N_0 . Now, assume by contradiction that (3.6) does not hold. Then, for any $h \in \mathbb{N}$ there is a competitor $u_h \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ such that $u_h|_{\partial\Omega} = Q[[0]]$, $\|u_h\|_{L^2} = 1$ and

$$\text{Jac}(u_h, \Omega) \leq \frac{1}{h}.$$

In particular, as a consequence of (3.3), we conclude that

$$\int_{\Omega} |Du_h|^2 d\mathcal{H}^n \leq C. \quad (3.7)$$

By the compact embedding theorem for Q -valued maps, Proposition 1.19, and by Proposition 1.18, we infer that there exist a subsequence $\{u_{h'}\}$ of $\{u_h\}$ and a section $u_{\infty} \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$, $u_{\infty}|_{\partial\Omega} = Q[[0]]$, such that

$$\lim_{h' \rightarrow \infty} \int_{\Omega} \mathcal{G}(u_{h'}, u_{\infty})^2 d\mathcal{H}^n = 0,$$

that is $u_{h'} \rightharpoonup u_{\infty}$ weakly in $W^{1,2}$. Then, from the semi-continuity of the Jacobi functional follows:

$$0 \leq \text{Jac}(u_{\infty}, \Omega) \leq \liminf_{h' \rightarrow \infty} \text{Jac}(u_{h'}, \Omega) = 0.$$

Hence, $\text{Jac}(u_{\infty}, \Omega) = 0$, and thus u_{∞} is a minimizer. By hypothesis, $u_{\infty} \equiv Q[[0]]$, which contradicts $\|u_{\infty}\|_{L^2} = 1$. \square

In the next section, we will show that this condition is indeed sufficient to prove existence for any (suitably regular) boundary datum, thus providing the proof of Theorem 0.3.

4. EXISTENCE OF JACOBI Q -FIELDS

4.1. Q -valued Luckhaus Lemma and the extension theorem. The first step to prove existence of Jacobi Q -fields is to derive an extension theorem for Q -valued $W^{1,2}$ maps. Such a theorem will be obtained as a simple corollary of the following result, which is interesting per se.

Proposition 4.1. *Let \mathcal{N} be a d -dimensional closed Riemannian manifold of class $C^{2,\beta}$ for some $\beta \in (0, 1)$. Assume $0 < \lambda < 1$ and $f^1, f^2: \mathcal{N} \rightarrow \mathcal{A}_Q(\mathbb{R}^q)$ are two maps in $W^{1,2}$. Then, there exist a constant $C = C(\mathcal{N}, d, q, Q)$ and a map $h \in W^{1,2}(\mathcal{N} \times [0, \lambda]; \mathcal{A}_Q(\mathbb{R}^q))$ such that*

$$h(\cdot, 0) \equiv f^1 \quad \text{and} \quad h(\cdot, \lambda) \equiv f^2 \quad \text{in } \mathcal{N}, \quad (4.1)$$

satisfying

$$\int_{\mathcal{N} \times [0, \lambda]} |h|^2 \leq C\lambda \int_{\mathcal{N}} (|f^1|^2 + |f^2|^2), \quad (4.2)$$

$$\int_{\mathcal{N} \times [0, \lambda]} |Dh|^2 \leq C\lambda \int_{\mathcal{N}} (|Df^1|^2 + |Df^2|^2) + \frac{C}{\lambda} \int_{\mathcal{N}} \mathcal{G}(f^1, f^2)^2. \quad (4.3)$$

Such a result adapts to the Q -valued setting a classical result by S. Luckhaus, concerning the extension of a Sobolev map defined on the boundary of an annulus $\partial(B_1 \setminus B_{1-\lambda})$ in its interior with control on the L^2 norm of the gradient of the extension map (for the precise statement and the proof, see the original paper [Luc88, Lemma 1], or the nice presentations given by L. Simon in [Sim96, Section 2.12.2] or by R. Moser in [Mos05, Lemma 4.4]).

A version of this result in the Q -valued setting was given by C. De Lellis in [DL13], where the author interpolates between two functions defined on flat cubes. The proof of the interpolation between maps defined over a closed Riemannian manifold here presented is based on this result and on a decomposition of the manifold that is bi-Lipschitz to a d -dimensional cubic complex, following ideas already contained in [Whi88] and [Han05]. We will make an extensive use of the following Lemma, which provides the elementary step in the construction of the interpolation.

Lemma 4.2. *There is a constant C depending only on j and Q with the following property. Assume that $0 < \lambda \leq 1$, $L = [0, \lambda]^j + z$ is a j -dimensional cube of side length λ , and $\varphi \in W^{1,2}(\partial L; \mathcal{A}_Q(\mathbb{R}^q))$. Then, there is an extension ψ of φ to L such that*

$$\int_L |\psi|^2 d\mathcal{H}^j \leq C\lambda \int_{\partial L} |\varphi|^2 d\mathcal{H}^{j-1} \tag{4.4}$$

and

$$\text{Dir}(\psi, L) \leq C\lambda \text{Dir}(\varphi, \partial L). \tag{4.5}$$

Proof. First observe that, since the inequalities (4.4) and (4.5) are invariant with respect to translations and dilations, it suffices to prove the lemma when $L = [0, 1]^j$. Moreover, since L is bi-Lipschitz equivalent to the unit ball, it is enough to show the claim for $L = \overline{B}_1 \subset \mathbb{R}^j$.

For reasons that will soon become clear, the proof when working in dimension $j = 2$ is different with respect to the one in the higher dimension case ($j \geq 3$): for this reason, we will distinguish between these two scenarios.

The higher dimension case ($j \geq 3$). This is the easiest situation: indeed, it suffices to define ψ as the zero-degree homogeneous extension of φ . That is, if $\varphi = \sum_l \llbracket \varphi^l \rrbracket$ on ∂B_1 , one just sets

$$\psi(x) := \sum_{l=1}^Q \left\| \left\| \varphi^l \left(\frac{x}{|x|} \right) \right\| \right\| \quad \text{for } x \in B_1 \setminus \{0\}. \tag{4.6}$$

A simple computation in radial coordinates shows that both estimates (4.4) and (4.5) hold with $C = \max\{j^{-1}, (j-2)^{-1}\} = (j-2)^{-1}$. Observe that this proof breaks down when $j = 2$, because the zero-degree homogeneous extension of φ has infinite Dirichlet energy in two dimensions.

The planar case ($j = 2$). For this proof, it will be useful to introduce a complex notation. We identify \mathbb{R}^2 with the complex plane \mathbb{C} , and the unit ball $B_1 \subset \mathbb{R}^2$ with the disk \mathbb{D} , as usual defined as

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\} = \{re^{i\theta} : 0 \leq r < 1, \theta \in \mathbb{R}\}.$$

The boundary of \mathbb{D} is the unit circle \mathbb{S}^1 , described by

$$\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}.$$

Consider now a function $\varphi \in W^{1,2}(\mathbb{S}^1; \mathcal{A}_Q)$. By [DLS11, Proposition 1.5], there exists a decomposition $\varphi = \sum_{l=1}^P \llbracket \varphi_l \rrbracket$, where each φ_l is an *irreducible* map in $W^{1,2}(\mathbb{S}^1; \mathcal{A}_{k_l})$. This

means that $\sum_{l=1}^P k_l = Q$, and furthermore that for every $l = 1, \dots, P$ there exists a $W^{1,2}$ map $\gamma_l: \mathbb{S}^1 \rightarrow \mathbb{R}^q$ such that

$$\varphi_l(x) = \sum_{z^{k_l}=x} \llbracket \gamma_l(z) \rrbracket. \quad (4.7)$$

In other words, if we identify the point $x = e^{i\theta} \in \mathbb{S}^1$ with the phase $\theta \in [0, 2\pi)$ we have that

$$\varphi_l(\theta) = \sum_{m=0}^{k_l-1} \left\llbracket \gamma_l \left(\frac{\theta + 2\pi m}{k_l} \right) \right\rrbracket. \quad (4.8)$$

The idea, now, is to consider the *harmonic extension* to the disk \mathbb{D} of the function γ_l : if

$$\gamma_l(\theta) = \frac{a_{l,0}}{2} + \sum_{n=1}^{\infty} (a_{l,n} \cos(n\theta) + b_{l,n} \sin(n\theta)) \quad (4.9)$$

is the Fourier series of γ_l , we let

$$\zeta_l(r, \theta) = \frac{a_{l,0}}{2} + \sum_{n=1}^{\infty} r^n (a_{l,n} \cos(n\theta) + b_{l,n} \sin(n\theta)) \quad (4.10)$$

denote its harmonic extension to the whole disk. Then, for each $l = 1, \dots, P$, we consider the k_l -valued function ψ_l obtained “rolling” back the ζ_l , that is

$$\psi_l(x) := \sum_{z^{k_l}=x} \llbracket \zeta_l(z) \rrbracket \quad (4.11)$$

for $x \in \mathbb{D}$, and, finally, we set $\psi := \sum_{l=1}^P \llbracket \psi_l \rrbracket$. We claim now that ψ is a $W^{1,2}(\mathbb{D}; \mathcal{A}_Q)$ extension of φ satisfying the estimates (4.4) and (4.5). To see this, fix $l \in \{1, \dots, P\}$ and define the following subsets of the unit disk,

$$\mathcal{D}_m := \left\{ r e^{i\theta} : 0 < r < 1, \frac{2\pi m}{k_l} < \theta < \frac{2\pi(m+1)}{k_l} \right\}$$

for $m = 0, \dots, k_l - 1$, and

$$\mathcal{C} := \{ r e^{i\theta} : 0 < r < 1, \theta \neq 0 \}.$$

One immediately sees that $\psi_l|_{\mathcal{C}} = \sum_{m=0}^{k_l-1} \llbracket \zeta_l \circ \sigma_m \rrbracket$, where $\sigma_m: \mathcal{C} \rightarrow \mathcal{D}_m$ are the k_l determinations of the k_l^{th} root, that is

$$\sigma_m(r e^{i\theta}) = r^{\frac{1}{k_l}} e^{i \left(\frac{\theta + 2\pi m}{k_l} \right)}.$$

Similarly, if the arcs \mathcal{S}_m are defined by

$$\mathcal{S}_m := \left\{ e^{i\theta} : \frac{2\pi m}{k_l} < \theta < \frac{2\pi(m+1)}{k_l} \right\},$$

we have that $\varphi_l|_{\mathbb{S}^1 \setminus \{1\}} = \sum_{m=0}^{k_l-1} \llbracket \gamma_l \circ \tau_m \rrbracket$, where $\tau_m: \mathbb{S}^1 \setminus \{1\} \rightarrow \mathcal{S}_m$ is given by $\tau_m := \sigma_m|_{\mathbb{S}^1}$. Thus, we can immediately compute

$$\begin{aligned} \int_{\mathbb{S}^1} |\varphi_l|^2 &= \sum_{m=0}^{k_l-1} \int_{\mathbb{S}^1} |\gamma_l \circ \tau_m|^2 \\ &= k_l \sum_{m=0}^{k_l-1} \int_{\mathcal{S}_m} |\gamma_l|^2 \\ &= k_l \int_{\mathbb{S}^1} |\gamma_l|^2 = k_l \pi \left(\frac{|a_{l,0}|^2}{2} + \sum_{n=1}^{\infty} (|a_{l,n}|^2 + |b_{l,n}|^2) \right) \end{aligned} \quad (4.12)$$

by Plancherel's theorem. On the other hand, we can use polar coordinates to compute the integral of the extension ψ_l to the disk and see that

$$\begin{aligned}
 \int_{\mathbb{D}} |\psi_l|^2 &= \sum_{m=0}^{k_l-1} \int_{\mathcal{C}} |\zeta_l \circ \sigma_m|^2 \\
 &= \sum_{m=0}^{k_l-1} \int_0^1 \left(\int_0^{2\pi} \left| \zeta_l \left(\rho^{\frac{1}{k_l}}, \frac{\alpha + 2\pi m}{k_l} \right) \right|^2 d\alpha \right) \rho d\rho \\
 &= k_l^2 \sum_{m=0}^{k_l-1} \int_0^1 \left(\int_{\frac{2\pi m}{k_l}}^{\frac{2\pi(m+1)}{k_l}} |\zeta_l(r, \theta)|^2 d\theta \right) r^{2k_l-1} dr \\
 &= k_l^2 \int_0^1 \left(\int_0^{2\pi} |\zeta_l(r, \theta)|^2 d\theta \right) r^{2k_l-1} dr \\
 &= k_l^2 \pi \int_0^1 \left(\frac{|a_{l,0}|^2}{2} + \sum_{n=1}^{\infty} r^{2n} (|a_{l,n}|^2 + |b_{l,n}|^2) \right) r^{2k_l-1} dr \\
 &\stackrel{(4.12)}{\leq} k_l \left(\int_{\mathbb{S}^1} |\varphi_l|^2 \right) \left(\int_0^1 r^{2k_l-1} dr \right) \\
 &= \frac{1}{2} \int_{\mathbb{S}^1} |\varphi_l|^2.
 \end{aligned} \tag{4.13}$$

Summing over $l \in \{1, \dots, P\}$, we finally conclude that

$$\int_{\mathbb{D}} |\psi|^2 \leq \frac{1}{2} \int_{\mathbb{S}^1} |\varphi|^2, \tag{4.14}$$

that is, (4.4) holds with $C = \frac{1}{2}$. Concerning (4.5), we exploit the invariance of the Dirichlet energy under conformal mappings in order to infer that, for any $l = 1, \dots, P$,

$$\text{Dir}(\psi_l, \mathcal{C}) = \sum_{m=0}^{k_l-1} \text{Dir}(\zeta_l \circ \sigma_m, \mathcal{C}) = \sum_{m=0}^{k_l-1} \text{Dir}(\zeta_l, \mathcal{D}_m) = \int_{\mathbb{D}} |D\zeta_l|^2. \tag{4.15}$$

Now, by a simple computation on planar harmonic functions, it is easy to see that

$$\int_{\mathbb{D}} |D\zeta_l|^2 \leq \int_{\mathbb{S}^1} |\partial_{\theta}\gamma_l|^2, \tag{4.16}$$

where ∂_{θ} is the tangential derivative along the circle. On the other hand, for every $l = 1, \dots, P$,

$$\begin{aligned}
 \text{Dir}(\varphi_l, \mathbb{S}^1) &= \sum_{m=0}^{k_l-1} \int_{\mathbb{S}^1} |\partial_{\theta}(\gamma_l \circ \tau_m)|^2 \\
 &= \sum_{m=0}^{k_l-1} \int_{\mathbb{S}^1} \frac{1}{k_l^2} |\partial_{\theta}\gamma_l \circ \tau_m|^2 \\
 &= \sum_{m=0}^{k_l-1} \int_{\mathcal{S}_m} \frac{1}{k_l} |\partial_{\theta}\gamma_l|^2 \\
 &= \frac{1}{k_l} \int_{\mathbb{S}^1} |\partial_{\theta}\gamma_l|.
 \end{aligned} \tag{4.17}$$

Finally, summing on l , the above arguments produce

$$\text{Dir}(\psi, \mathbb{D}) \stackrel{(4.15)}{=} \sum_{l=1}^P \int_{\mathbb{D}} |D\zeta_l|^2 \stackrel{(4.16)}{\leq} \sum_{l=1}^P \int_{\mathbb{S}^1} |\partial_\theta \gamma_l|^2 \stackrel{(4.17)}{=} \sum_{l=1}^P k_l \text{Dir}(\varphi_l, \mathbb{S}^1) \leq Q \text{Dir}(\varphi, \mathbb{S}^1), \quad (4.18)$$

whence (4.5) holds with $C = Q$. \square

Proof of Proposition 4.1. Without loss of generality, we assume that \mathcal{N} is an embedded submanifold of some Euclidean space \mathbb{R}^N . We shall divide the proof into steps.

Step 1. We first consider a Lipschitz cubic decomposition of the manifold \mathcal{N} , that is a pair (\mathcal{K}, σ) , where \mathcal{K} is a d -dimensional cubic complex, and $\sigma: |\mathcal{K}| \rightarrow \mathcal{N}$ is a bi-Lipschitz map, $|\mathcal{K}|$ denoting the union of all cells of \mathcal{K} . Recall that a d -dimensional cubic complex is a collection of cubes in \mathbb{R}^d (*cells*) such that the following conditions are satisfied:

- (1) Each face of a cell in \mathcal{K} is also in \mathcal{K} .
- (2) The intersection of two cells in \mathcal{K} is either empty or a face of each of them.
- (3) Each point of $|\mathcal{K}|$ has a neighborhood intersecting finitely many faces of \mathcal{K} .

Without loss of generality, we may assume that each cell in \mathcal{K} has unit d -dimensional volume. Set $m := \lfloor \frac{1}{\lambda} \rfloor + 1$. Using that $[0, 1] = \bigcup_{i=1}^m \left[\frac{i-1}{m}, \frac{i}{m} \right]$, we can divide each cell in \mathcal{K} into m^d smaller d -dimensional cubes, whose side length is at most λ . We will denote the resulting cubic complex by \mathcal{K}_m , and regard σ as a bi-Lipschitz map $\sigma: |\mathcal{K}_m| \rightarrow \mathcal{N}$: observe that if L is any cell in \mathcal{K}_m then the image $\sigma(L)$ is a domain in \mathcal{N} with diameter (computed with respect to the geodesic distance on \mathcal{N}) $\mathbf{diam}(\sigma(L)) \leq \sqrt{d} \text{Lip}(\sigma) \lambda$.

For each $j \in \{0, 1, \dots, d\}$, \mathcal{K}_m^j will denote the j -skeleton of the complex \mathcal{K}_m , that is the family of all j -dimensional faces, and $|\mathcal{K}_m^j|$ will be their union.

Step 2. Let now $\eta = \eta(\mathcal{N}) > 0$ be so small that the set $\mathcal{U} = \mathcal{U}_{2\eta}(\mathcal{N}) := \{x \in \mathbb{R}^N : \text{dist}(x, \mathcal{N}) < 2\eta\}$ is a tubular neighborhood of \mathcal{N} , with (unique) differentiable nearest point projection $\Pi: \mathcal{U}_{2\eta}(\mathcal{N}) \rightarrow \mathcal{N}$. For $i = 1, 2$, we extend f^i to a map $F^i: \mathcal{U} \rightarrow \mathcal{A}_Q(\mathbb{R}^q)$ by setting $F^i := f^i \circ \Pi$. One has that

$$\int_{\mathcal{U}} |F^i|^2 \leq c_1 \int_{\mathcal{N}} |f^i|^2, \quad (4.19)$$

$$\int_{\mathcal{U}} |DF^i|^2 \leq c_1 \int_{\mathcal{N}} |Df^i|^2, \quad (4.20)$$

and

$$\int_{\mathcal{U}} \mathcal{G}(F^1, F^2)^2 \leq c_1 \int_{\mathcal{N}} \mathcal{G}(f^1, f^2)^2, \quad (4.21)$$

where the constant c_1 depends only on the retraction Π (and, thus, on the dimensions d and N and on the width η of the tubular neighborhood).

Furthermore, for every $z \in |\mathcal{K}_m|$ and for every vector $v \in B_\eta^N$, we define $\sigma_v(z) := \Pi(\sigma(z) + v)$. Assume that η is so small that all σ_v 's are bi-Lipschitz maps $|\mathcal{K}_m| \rightarrow \mathcal{N}$, and set $f_v^i := f^i \circ \sigma_v$, that is $f_v^i(z) = F^i(\sigma(z) + v)$. By Fubini's theorem, for all $j = 1, \dots, d$ and for a.e. $v \in B_\eta^N$ one has that $f_v^i \in W^{1,2}(F; \mathcal{A}_Q(\mathbb{R}^q))$ for all faces $F \in \mathcal{K}_m^j$.

Consider now any non-negative function $\alpha \in L^1(\mathcal{U})$. It is easily seen that there exists a constant $c_2 = c_2(\mathcal{N}, d, N, \eta)$ such that for any $j = 0, \dots, d$ and for every $\theta \in (0, 1)$

$$\int_{|\mathcal{K}_m^j|} \alpha(\sigma(z) + v) d\mathcal{H}^j(z) \leq c_2 \theta^{-1} \lambda^{j-d} \int_{\mathcal{U}} \alpha \quad (4.22)$$

for all $v \in B_\eta^N$ with the exception of a set E of \mathcal{L}^N -measure $|E| \leq \theta|B_\eta^N|$. To prove this, first notice that

$$\int_{|\mathcal{K}_m^j|} \alpha(\sigma(z) + v) d\mathcal{H}^j(z) \leq \frac{1}{\theta|B_\eta^N|} \int_{B_\eta^N} \left(\int_{|\mathcal{K}_m^j|} \alpha(\sigma(z) + v) d\mathcal{H}^j(z) \right) dv \quad (4.23)$$

for all $v \in B_\eta^N \setminus E$, $|E| \leq \theta|B_\eta^N|$. Then, conclude by estimating:

$$\begin{aligned} \int_{B_\eta^N} \left(\int_{|\mathcal{K}_m^j|} \alpha(\sigma(z) + v) d\mathcal{H}^j(z) \right) dv &= \int_{|\mathcal{K}_m^j|} \left(\int_{B_\eta^N} \alpha(\sigma(z) + v) dv \right) d\mathcal{H}^j(z) \\ &= \int_{|\mathcal{K}_m^j|} \left(\int_{B_\eta^N(\sigma(z))} \alpha(w) dw \right) d\mathcal{H}^j(z) \\ &\leq \mathcal{H}^j(|\mathcal{K}_m^j|) \int_{\mathcal{U}} \alpha \\ &\leq Cm^d \lambda^j \int_{\mathcal{U}} \alpha \\ &\leq C\lambda^{j-d} \int_{\mathcal{U}} \alpha, \end{aligned} \quad (4.24)$$

where the constant C appearing in the last line depends only on the number of cells in the original cubic complex \mathcal{K} and on the dimension d .

Now, it suffices to apply (4.22) with $\alpha = |F^i|^2$, $\alpha = |DF^i|^2$ and $\alpha = \mathcal{G}(F^1, F^2)^2$, and, say, $\theta = \frac{1}{2}$, and to plug in equations (4.19), (4.20) and (4.21) to finally show the following: there exists $v \in B_\eta^N$ such that for all $j \in \{1, \dots, d\}$ the following inequalities

$$\int_{|\mathcal{K}_m^j|} (|f_v^1|^2 + |f_v^2|^2) \leq C\lambda^{j-d} \int_{\mathcal{N}} (|f^1|^2 + |f^2|^2), \quad (4.25)$$

and

$$\int_{|\mathcal{K}_m^j|} (|Df_v^1|^2 + |Df_v^2|^2 + \mathcal{G}(f_v^1, f_v^2)^2) \leq C\lambda^{j-d} \int_{\mathcal{N}} (|Df^1|^2 + |Df^2|^2 + \mathcal{G}(f^1, f^2)^2) \quad (4.26)$$

hold true with a constant $C = C(c_1, c_2, \text{Lip}(\sigma))$. Furthermore, for $j = 0$:

$$\sum_{z \in |\mathcal{K}_m^0|} (|f_v^1|^2(z) + |f_v^2|^2(z)) \leq C\lambda^{-d} \int_{\mathcal{N}} (|f^1|^2 + |f^2|^2), \quad (4.27)$$

$$\sum_{z \in |\mathcal{K}_m^0|} \mathcal{G}(f_v^1(z), f_v^2(z))^2 \leq C\lambda^{-d} \int_{\mathcal{N}} \mathcal{G}(f^1, f^2)^2. \quad (4.28)$$

From now on, we will then assume to have fixed a $v \in B_\eta^N$ such that the corresponding maps $f_v^i: |\mathcal{K}_m| \rightarrow \mathcal{A}_Q(\mathbb{R}^q)$ satisfy equations (4.25), (4.26), (4.27), (4.28) and the following condition: for every $j \geq 1$, for each $\tau \in \mathcal{K}_m^j$ and for all $\gamma \in \mathcal{K}_m^{j-1}$ with $\gamma \subset \tau$, the restrictions $f_v^i|_\tau$ and $f_v^i|_\gamma$ are all $W^{1,2}$, and moreover the trace of $f_v^i|_\tau$ at γ is precisely $f_v^i|_\gamma$.

Step 3. Consider now the $(d+1)$ -dimensional cubic complex $\bar{\mathcal{K}} := \mathcal{K}_m \times [0, \lambda]$ whose $(d+1)$ -dimensional cells are cubes of the form $L \times [0, \lambda]$ for some $L \in \mathcal{K}_m^d$. A face $\tau \in \bar{\mathcal{K}}^j$, $j < d+1$, is said to be *horizontal* if it is contained in $\mathcal{K}_m \times \{0\}$ (lower horizontal) or $\mathcal{K}_m \times \{\lambda\}$

(upper horizontal), *vertical* otherwise. The collection of j -dimensional faces of $\bar{\mathcal{K}}$ is hence given by

$$\bar{\mathcal{K}}^j = \mathcal{L}^j \cup \mathcal{U}^j \cup \mathcal{V}^j, \quad (4.29)$$

where \mathcal{L}^j , \mathcal{U}^j and \mathcal{V}^j are the lower horizontal, upper horizontal and vertical j -dimensional faces respectively. Observe that $\mathcal{V}^0 = \emptyset$, \mathcal{L}^0 consists of points $(z, 0)$, while \mathcal{U}^0 consists of points (z, λ) with $z \in \mathcal{K}_m^0$; notice, furthermore, that all $(d+1)$ -dimensional cells are vertical.

We are now in the position to define a map $\bar{h}: |\bar{\mathcal{K}}| \rightarrow \mathcal{A}_Q(\mathbb{R}^q)$. First of all, we set $\bar{h}|_\beta \equiv f_v^1|_\beta$ if β is a lower horizontal face, and $\bar{h}|_\tau \equiv f_v^2|_\tau$ if τ is an upper horizontal face. Consider next any vertical segment $\gamma \in \mathcal{V}^1$. Its two endpoints are given by $(z, 0)$ and (z, λ) for some $z \in \mathcal{K}_m^0$. Now, if $f_v^1(z) = \sum_l [(f_v^1)_l(z)]$ and $f_v^2(z) = \sum_l [(f_v^2)_l(z)]$ are ordered in such a way that $\mathcal{G}(f_v^1(z), f_v^2(z))^2 = \sum_l |(f_v^1)_l(z) - (f_v^2)_l(z)|^2$, then a natural extension is obtained by setting

$$\bar{h}(z, \theta) := \sum_{l=1}^Q \left[(f_v^1)_l(z) + \frac{\theta}{\lambda} \left((f_v^2)_l(z) - (f_v^1)_l(z) \right) \right], \quad (4.30)$$

for all $\theta \in [0, \lambda]$. In this way, we obtain the bounds

$$\int_\gamma |\bar{h}|^2 \leq 2\lambda \left(|f_v^1|^2(z) + |f_v^2|^2(z) \right) \quad (4.31)$$

and

$$\int_\gamma |D\bar{h}|^2 \leq \lambda^{-1} \mathcal{G}(f_v^1(z), f_v^2(z))^2. \quad (4.32)$$

If we carry on this procedure for all vertical segments, we obtain a well defined Q -valued map \bar{h} on all the vertical 1-skeleton \mathcal{V}^1 , which, thanks to (4.27) and (4.28), satisfies

$$\int_{|\mathcal{V}^1|} |\bar{h}|^2 \leq C\lambda^{1-d} \int_{\mathcal{N}} \left(|f^1|^2 + |f^2|^2 \right) \quad (4.33)$$

and

$$\int_{|\mathcal{V}^1|} |D\bar{h}|^2 \leq C\lambda^{-1-d} \int_{\mathcal{N}} \mathcal{G}(f^1, f^2)^2. \quad (4.34)$$

Pick next a vertical 2-dimensional face τ . Its boundary consists of two horizontal segments $\beta \in \mathcal{L}^1$ and $\delta \in \mathcal{U}^1$, and two vertical segments joining the points $(z, 0)$, $(w, 0)$ to the points (z, λ) , (w, λ) respectively. Using our assumptions on v , we can conclude that $\bar{h}|_{\partial\tau}$ is in $W^{1,2}$, whence Lemma 4.2 yields an extension of \bar{h} to τ with estimates

$$\int_\tau |\bar{h}|^2 \leq C\lambda \left(\int_\beta |f_v^1|^2 + \int_\delta |f_v^2|^2 \right) + C\lambda^2 \left((|f_v^1|^2 + |f_v^2|^2)(z) + (|f_v^1|^2 + |f_v^2|^2)(w) \right) \quad (4.35)$$

and

$$\int_\tau |D\bar{h}|^2 \leq C\lambda \left(\int_\beta |Df_v^1|^2 + \int_\delta |Df_v^2|^2 \right) + C \left(\mathcal{G}(f_v^1(z), f_v^2(z))^2 + \mathcal{G}(f_v^1(w), f_v^2(w))^2 \right). \quad (4.36)$$

Summing over the 2-skeleton \mathcal{V}^2 , from the estimates (4.25) and (4.27) we deduce

$$\int_{|\mathcal{V}^2|} |\bar{h}|^2 \leq C\lambda^{2-d} \int_{\mathcal{N}} \left(|f^1|^2 + |f^2|^2 \right), \quad (4.37)$$

whereas (4.26) and (4.28) imply

$$\int_{|\mathcal{V}^2|} |D\bar{h}|^2 \leq C\lambda^{2-d} \int_{\mathcal{N}} (|Df^1|^2 + |Df^2|^2) + C\lambda^{-d} \int_{\mathcal{N}} \mathcal{G}(f^1, f^2)^2. \quad (4.38)$$

We then proceed inductively over \mathcal{V}^j , iteratively applying Lemma 4.2 and using the inequalities (4.25) to (4.28) at each step. At the final iteration, namely for $j = d + 1$, we construct a map \bar{h} which is $W^{1,2}$ on each $(d + 1)$ -dimensional cell $L \times [0, \lambda]$, coinciding with f_v^1 on $L \times \{0\}$ and with f_v^2 on $L \times \{\lambda\}$. Furthermore, if two cells $H = L_1 \times [0, \lambda]$ and $K = L_2 \times [0, \lambda]$ have a common face $S \in \mathcal{V}^d$, the traces of $\bar{h}|_H$ and $\bar{h}|_K$ at S coincide. Thus, we can regard \bar{h} as a $W^{1,2}$ map defined on the whole cubic complex $\bar{\mathcal{K}}$. Moreover, since $|\bar{\mathcal{K}}| = \bigcup \mathcal{V}^{d+1} = |\mathcal{V}^{d+1}|$, the inductive step provides the following estimates:

$$\int_{|\bar{\mathcal{K}}|} |\bar{h}|^2 \leq C\lambda \int_{\mathcal{N}} (|f^1|^2 + |f^2|^2), \quad (4.39)$$

$$\int_{|\bar{\mathcal{K}}|} |D\bar{h}|^2 \leq C\lambda \int_{\mathcal{N}} (|Df^1|^2 + |Df^2|^2) + \frac{C}{\lambda} \int_{\mathcal{N}} \mathcal{G}(f^1, f^2)^2. \quad (4.40)$$

Step 4. Finally, we simply define a map $h \in W^{1,2}(\mathcal{N} \times [0, \lambda]; \mathcal{A}_Q(\mathbb{R}^q))$ by setting

$$h(x, \theta) := \bar{h}(\sigma_v^{-1}(x), \theta). \quad (4.41)$$

It is immediate to check that such a map indeed satisfies (4.1), (4.2) and (4.3) in the statement. \square

Corollary 4.3. *Let $\Sigma^n \hookrightarrow \mathbb{R}^M$ be a regular compact submanifold, and let $\lambda_0 := \text{inj}(\Sigma) > 0$ be the injectivity radius of Σ . Then, for any $\mathcal{V} \subsetneq \Sigma$ connected, open subset with C^2 boundary, for any $\tilde{g}_0 \in W^{1,2}(\partial\mathcal{V}; \mathcal{A}_Q(\mathbb{R}^M))$ and for all $0 < \lambda < \lambda_0$ there exist an open set $\mathcal{V}_\lambda \subset \Sigma$ with $\mathcal{V} \Subset \mathcal{V}_\lambda$, $\text{dist}(\mathcal{V}, \partial\mathcal{V}_\lambda) \leq \lambda$, and a map $\bar{g}_\lambda \in W^{1,2}(\mathcal{V}_\lambda \setminus \bar{\mathcal{V}}; \mathcal{A}_Q(\mathbb{R}^M))$ satisfying:*

$$\bar{g}_\lambda|_{\partial\mathcal{V}} = \tilde{g}_0 \quad \text{and} \quad \bar{g}_\lambda|_{\partial\mathcal{V}_\lambda} = Q[0], \quad (4.42)$$

$$\int_{\mathcal{V}_\lambda \setminus \bar{\mathcal{V}}} |\bar{g}_\lambda|^2 d\mathcal{H}^n \leq C\lambda \int_{\partial\mathcal{V}} |\tilde{g}_0|^2 d\mathcal{H}^{n-1}, \quad (4.43)$$

$$\text{Dir}(\bar{g}_\lambda, \mathcal{V}_\lambda \setminus \bar{\mathcal{V}}) \leq C\lambda \text{Dir}(\tilde{g}_0, \partial\mathcal{V}) + \frac{C}{\lambda} \int_{\partial\mathcal{V}} |\tilde{g}_0|^2 d\mathcal{H}^{n-1}, \quad (4.44)$$

for a constant $C = C(\mathcal{V}, n, M, Q)$.

Proof. Let \mathcal{V} and \tilde{g}_0 be as in the statement. Then, by the very definition of injectivity radius, for any $0 < \lambda < \lambda_0$ the exponential map, restricted to $\partial\mathcal{V}$, is injective in a ball of radius λ around the zero section of the normal bundle of $\partial\mathcal{V}$ in Σ . In turn, this allows one to define, for any such λ , a λ -tubular neighborhood \mathcal{U}_λ of $\partial\mathcal{V}$ in Σ by setting

$$\mathcal{U}_\lambda := \{\exp_\pi(\theta\eta(\pi)) : \pi \in \partial\mathcal{V}, |\theta| < \lambda\}, \quad (4.45)$$

where for every point $\pi \in \partial\mathcal{V}$ we have denoted $\eta(\pi) \in T_\pi\Sigma$ the unit outer conormal vector to $\partial\mathcal{V}$ at π .

Notice that it is well defined a differentiable parametrization $x \in \mathcal{U}_\lambda \mapsto (\pi(x), \theta(x)) \in \partial\mathcal{V} \times (-\lambda, \lambda)$ such that $\exp_{\pi(x)}(\theta(x)\eta(\pi(x))) = x$ for all $x \in \mathcal{U}_\lambda$.

Next, the *positive* and *negative* λ -tubular neighborhoods of $\partial\mathcal{V}$ in Σ are respectively defined by

$$\mathcal{U}_\lambda^+ := \{\exp_\pi(\theta\eta(\pi)) : \pi \in \partial\mathcal{V}, 0 < \theta < \lambda\}, \quad (4.46)$$

$$\mathcal{U}_\lambda^- := \{\exp_\pi(\theta\eta(\pi)) : \pi \in \partial\mathcal{V}, -\lambda < \theta < 0\}. \quad (4.47)$$

We set $\mathcal{V}_\lambda := \mathcal{V} \cup \mathcal{U}_\lambda$. The claimed result is then simply obtained by applying Proposition 4.1 with $\mathcal{N} = \partial\mathcal{V}$, $f^1 = \tilde{g}_0$, $f^2 = Q[[0]]$ and setting $\bar{g}_\lambda(x) := h(\pi(x), \theta(x))$ for $x \in \mathcal{V}_\lambda \setminus \bar{\mathcal{V}} = \mathcal{U}_\lambda^+$. \square

4.2. The compactness theorem. Back to our original setting, we let Ω be an open and connected subset of $\Sigma \hookrightarrow \mathcal{M}$ in which we wish to solve the minimum problem for the Jac functional. We will assume $C^{2,\beta}$ regularity for $\partial\Omega$, for some $\beta \in (0, 1)$. For $0 < \lambda < \lambda_0$, set $\mathcal{V} := \{x \in \Omega : \mathbf{dist}(x, \partial\Omega) > \lambda\}$, so that Ω coincides with the set $\mathcal{V}_\lambda = \mathcal{V} \cup \mathcal{U}_\lambda$ which was obtained in the proof of Corollary 4.3. Using the same notations introduced in the proof of Corollary 4.3, we parametrize \mathcal{U}_λ with coordinates $(\pi, \theta) \in \partial\mathcal{V} \times (-\lambda, \lambda)$.

Let us now define $\Phi_\lambda: \mathcal{V} \rightarrow \Omega$ to be the diffeomorphism given by:

$$\Phi_\lambda(x) := \begin{cases} \exp_{\pi(x)}(\varphi_\lambda(\theta(x))\eta(\pi(x))) & \text{if } x \in \mathcal{U}_\lambda^- \\ x & \text{otherwise,} \end{cases} \quad (4.48)$$

where φ_λ is any monotone increasing diffeomorphism $\varphi_\lambda: (-\lambda, 0) \rightarrow (-\lambda, \lambda)$ such that $\varphi_\lambda(\theta) = \theta$ for $\theta \in (-\lambda, -\frac{\lambda}{2})$. From this moment on, we will assume that such a family of diffeomorphisms φ_λ has been fixed, and satisfies a bound of the form

$$c^{-1} \leq |\varphi'_\lambda| \leq c \quad (4.49)$$

for a positive constant c which does not depend on λ .

Furthermore, if $u = \sum_l [[u_l]]$ is any map in $W^{1,2}(\Omega; \mathcal{A}_Q(\mathbb{R}^M))$, we set:

$$u^\perp(x) := \sum_{l=1}^Q [[\mathbf{p}_x^{\Sigma^\perp} \cdot u_l(x)]], \quad (4.50)$$

where $\mathbf{p}^{\Sigma^\perp}$ is the normal bundle projection defined in Remark 2.9. Observe that $u^\perp \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$. The following Lemma yields a useful formula to relate the Dirichlet energy of u with the Dirichlet energy of u^\perp .

Lemma 4.4. *For every $\varepsilon > 0$ there exists a positive constant C_ε such that the following estimate holds true:*

$$\text{Dir}(u^\perp, \Omega) \leq (1 + \varepsilon)\text{Dir}(u, \Omega) + C_\varepsilon \int_\Omega |u|^2 d\mathcal{H}^n. \quad (4.51)$$

Proof. Write $\mathbf{p}^{\Sigma^\perp}(x, v) := \mathbf{p}_x^{\Sigma^\perp} \cdot v$ for $x \in \Omega$, $v \in \mathbb{R}^M$. If $v = v(x)$ is a (single valued) Lipschitz map defined in Ω , then for any tangent vector field ξ one has

$$D_\xi \mathbf{p}^{\Sigma^\perp}(x, v(x)) = \partial_x \mathbf{p}^{\Sigma^\perp}(x, v(x)) \cdot \xi(x) + \partial_v \mathbf{p}^{\Sigma^\perp}(x, v(x)) \cdot D_\xi v(x).$$

Since, for fixed x , the map $v \in \mathbb{R}^M \mapsto \mathbf{p}^{\Sigma^\perp}(x, v) \in T_x^\perp \Sigma$ is linear with Lipschitz constant not bigger than 1, we conclude that for any $v: \Omega \rightarrow \mathbb{R}^M$ Lipschitz one has

$$\text{Dir}(v^\perp, \Omega) \leq \text{Dir}(v, \Omega) + C \int_\Omega |v|^2 d\mathcal{H}^n + C \int_\Omega |v| |Dv| d\mathcal{H}^n,$$

where C is a constant depending on $\max_{\bar{\Omega} \times \mathbb{S}^{M-1}} |\partial_x \mathbf{p}^{\Sigma^\perp}|$.

Formula (4.51) is then a consequence of Young's inequality. The formula is then extended to Lipschitz Q -valued maps via decomposition into Q Lipschitz functions (Proposition 1.6), and finally to Sobolev Q -maps via approximation (Proposition 1.12). \square

We are now ready to prove the proposition that will provide the key to show existence of Jacobi Q -fields with prescribed boundary datum g provided condition (3.6) is satisfied for every $u \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ such that $u|_{\partial\Omega} = Q[[0]]$.

Proposition 4.5. *Let $\Omega \subset \Sigma$ be open, connected with $C^{2,\beta}$ boundary. Assume the strict stability condition (3.6) holds for every $u \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ such that $u|_{\partial\Omega} = Q[[0]]$. Then, if $g \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ has boundary trace $g_0 := g|_{\partial\Omega} \in W^{1,2}(\partial\Omega; \mathcal{A}_Q(\mathbb{R}^M))$, the following estimate*

$$\text{Jac}(N, \Omega) \geq c(\Omega) \int_{\Omega} |N|^2 d\mathcal{H}^n - C(\Omega, g_0) \quad (4.52)$$

holds true for any $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ such that $N|_{\partial\Omega} = g_0$.

Proof. Fix $\lambda < \lambda_0$ to be chosen, and let $\mathcal{V} \Subset \Omega$ be such that $\Omega = \mathcal{V}_\lambda$ as above. For any $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ such that $N|_{\partial\Omega} = g_0$, consider the map $\tilde{N} := N \circ \Phi_\lambda \in W^{1,2}(\mathcal{V}; \mathcal{A}_Q(\mathbb{R}^M))$, and observe that $\tilde{N}|_{\partial\mathcal{V}} = \tilde{g}_0$, where $\tilde{g}_0(\pi) = g_0(\exp_\pi(\lambda\eta(\pi)))$ for $\pi \in \partial\mathcal{V}$.

Now, apply Corollary 4.3 with this choice of \mathcal{V} , \tilde{g}_0 and λ in order to extend \tilde{N} to the map $u \in W^{1,2}(\Omega; \mathcal{A}_Q(\mathbb{R}^M))$ given by

$$u(x) := \begin{cases} \tilde{N}(x) & \text{if } x \in \mathcal{V} \\ \bar{g}_\lambda(x) & \text{if } x \in \Omega \setminus \mathcal{V} = \mathcal{U}_\lambda^+, \end{cases} \quad (4.53)$$

Observe that the normal bundle projection u^\perp satisfies $u^\perp \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ and the boundary condition $u^\perp|_{\partial\Omega} = Q[[0]]$. From the hypothesis, we are therefore able to conclude that

$$\text{Jac}(u^\perp, \Omega) \geq c(\Omega) \int_{\Omega} |u^\perp|^2 d\mathcal{H}^n. \quad (4.54)$$

Now, notice that $u^\perp \equiv N$ in $\Omega \setminus \mathcal{U}_\lambda$. Combining this observation with (4.54), we trivially deduce

$$\text{Jac}(N, \Omega \setminus \mathcal{U}_\lambda) + \text{Jac}(u^\perp, \mathcal{U}_\lambda) \geq c(\Omega) \left(\int_{\Omega \setminus \mathcal{U}_\lambda} |N|^2 d\mathcal{H}^n + \int_{\mathcal{U}_\lambda} |u^\perp|^2 d\mathcal{H}^n \right). \quad (4.55)$$

In order to prove our result, we then clearly have to provide suitable estimates for $\text{Jac}(u^\perp, \mathcal{U}_\lambda)$ and $\int_{\mathcal{U}_\lambda} |u^\perp|^2$.

We observe first that

$$\int_{\mathcal{U}_\lambda} |u^\perp|^2 = \int_{\mathcal{U}_\lambda^-} |u^\perp|^2 + \int_{\mathcal{U}_\lambda^+} |u^\perp|^2 \geq \int_{\mathcal{U}_\lambda^-} |u^\perp|^2. \quad (4.56)$$

Recall that

$$u^\perp|_{\mathcal{U}_\lambda^-} = \sum_{l=1}^Q \llbracket \mathbf{p}^{\Sigma^\perp} \cdot (N_l \circ \Phi_\lambda) \rrbracket = \sum_{l=1}^Q \left\| \sum_{\beta=1}^k \langle N_l \circ \Phi_\lambda, \nu_\beta \rangle \nu_\beta \right\|,$$

whence

$$\int_{\mathcal{U}_\lambda^-} |u^\perp|^2 = \int_{\mathcal{U}_\lambda^-} \sum_{l=1}^Q \sum_{\beta=1}^k |\langle N_l(\Phi_\lambda(x)), \nu_\beta(x) \rangle|^2 d\mathcal{H}^n(x). \quad (4.57)$$

Now, changing variable $y = \Phi_\lambda(x)$, integrating along geodesics and using (4.49) one easily proves that from this follows

$$\int_{\mathcal{U}_\lambda^-} |u^\perp|^2 \geq C \left(\int_{\mathcal{U}_\lambda} |N|^2 d\mathcal{H}^n - \mathcal{E}_\lambda^{(1)} \right), \quad (4.58)$$

where the error term $\mathcal{E}_\lambda^{(1)}$ satisfies the estimate

$$|\mathcal{E}_\lambda^{(1)}| \leq \frac{1}{2} \int_{\mathcal{U}_\lambda} |N|^2 d\mathcal{H}^n, \quad (4.59)$$

provided λ satisfies some suitable smallness conditions which are *not* depending on N . Combining (4.55), (4.56), (4.58) and (4.59), we conclude that for suitably small λ

$$\text{Jac}(N, \Omega \setminus \mathcal{U}_\lambda) + \text{Jac}(u^\perp, \mathcal{U}_\lambda) \geq c(\Omega) \int_{\Omega} |N|^2 d\mathcal{H}^n, \quad (4.60)$$

up to possibly changing the value of $c(\Omega)$.

Now, we work on $\text{Jac}(u^\perp, \mathcal{U}_\lambda)$. As before, decompose

$$\text{Jac}(u^\perp, \mathcal{U}_\lambda) = \text{Jac}(u^\perp, \mathcal{U}_\lambda^-) + \text{Jac}(u^\perp, \mathcal{U}_\lambda^+). \quad (4.61)$$

Concerning the first addendum, one shows that

$$\text{Jac}(u^\perp, \mathcal{U}_\lambda^-) \leq C \text{Jac}(N, \mathcal{U}_\lambda) + \mathcal{E}_\lambda^{(2)}, \quad (4.62)$$

where the error $\mathcal{E}_\lambda^{(2)}$ satisfies

$$|\mathcal{E}_\lambda^{(2)}| \leq \varepsilon \left(\text{Dir}(N, \mathcal{U}_\lambda) + \int_{\mathcal{U}_\lambda} |N|^2 d\mathcal{H}^n \right) \quad (4.63)$$

for any choice of $\varepsilon > 0$, provided λ is smaller than some λ_* depending on ε and on the geometry of the problem, but, again, not on the map N . In particular, this allows to absorb the error term and conclude, under the previously considered smallness assumptions on λ , that

$$\text{Jac}(N, \Omega) \geq c(\Omega) \left(\int_{\Omega} |N|^2 d\mathcal{H}^n - \text{Jac}(u^\perp, \mathcal{U}_\lambda^+) \right) \quad (4.64)$$

after possibly having redefined $c(\Omega)$.

Now we are able to conclude: following the same strategy as before, it's not difficult to estimate

$$|\text{Jac}(u^\perp, \mathcal{U}_\lambda^+)| \leq C \left(\text{Dir}(\bar{g}_\lambda, \mathcal{U}_\lambda^+) + \int_{\mathcal{U}_\lambda^+} |\bar{g}_\lambda|^2 d\mathcal{H}^n \right), \quad (4.65)$$

where λ is small, but fixed, and does not depend on N . Our result, equation (4.52), is then an immediate consequence of Corollary 4.3 and of the definition of \tilde{g}_0 . \square

We are now ready to prove the Conditional Existence Theorem 0.3.

Proof of Theorem 0.3. The proof is an application of the direct method in the Calculus of Variations. Fix any $g \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ with boundary trace $g_0 := g|_{\partial\Omega} \in W^{1,2}(\partial\Omega; \mathcal{A}_Q(\mathbb{R}^M))$. Then, the inequality (4.52) implies that for any $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ with $N|_{\partial\Omega} = g_0$ one has

$$\text{Jac}(N, \Omega) \geq -C(\Omega, g_0),$$

thus the Jacobi functional is bounded from below in the class of competitors.

Set

$$m := \inf\{\text{Jac}(N, \Omega) : N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega), N|_{\partial\Omega} = g_0\} > -\infty,$$

and consider a minimizing sequence $\{N_h\}_{h=1}^\infty \subset \Gamma_Q^{1,2}(\mathcal{N}\Omega)$, $N_h|_{\partial\Omega} = g_0$, $\lim_{h \rightarrow \infty} \text{Jac}(N_h, \Omega) = m$. Then, for $h \geq h_0$ sufficiently large, one has

$$\text{Jac}(N_h, \Omega) \leq m + 1,$$

from which we deduce

$$\text{Dir}(N_h, \Omega) \leq C \int_{\Omega} |N_h|^2 d\mathcal{H}^n + |m| + 1.$$

On the other hand, (4.52) immediately implies that

$$\int_{\Omega} |N_h|^2 d\mathcal{H}^n \leq C(|m|, \Omega, g_0).$$

Putting all together, we conclude that

$$\text{Dir}(N_h, \Omega) + \int_{\Omega} |N_h|^2 d\mathcal{H}^n \leq C, \quad (4.66)$$

where C is a constant depending only on $|m|$, Ω , g_0 and the geometry of the embeddings $\Sigma \hookrightarrow \mathcal{M} \hookrightarrow \mathbb{R}^M$. Hence, up to extracting a subsequence, N_h converges weakly in $W^{1,2}$, strongly in L^2 , to a map $\bar{N} \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ with $\bar{N}|_{\partial\Omega} = g_0$. The lower semi-continuity of the Jacobi functional with respect to weak convergence, Proposition 3.3, allows us to conclude that \bar{N} is the desired minimizer. \square

5. HÖLDER REGULARITY OF JACOBI Q -FIELDS

In this section, we present a proof of the following quantitative version of Theorem 0.5. As usual, Ω is an open subset of the compact n -dimensional manifold Σ minimally embedded in \mathcal{M} .

Theorem 5.1 (Hölder regularity of Jacobi multi-fields). *There exist universal constants $\alpha = \alpha(n, Q) \in (0, 1)$ and $\Lambda = \Lambda(n, Q) > 0$ and a radius $0 < r_0 = r_0(n, Q) < \text{inj}(\Sigma)$ with the following property. If $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ is Jac-minimizing, then for every $0 < \theta < 1$ there exists a constant $C = C(n, M, Q, \Sigma, \theta)$ such that*

$$\begin{aligned} [N]_{C^{0,\alpha}(\bar{\mathbf{B}}_{\theta r}(p))} &:= \sup_{x_1, x_2 \in \bar{\mathbf{B}}_{\theta r}(p)} \frac{\mathcal{G}(N(x_1), N(x_2))}{\mathbf{d}(x_1, x_2)^\alpha} \\ &\leq C \left(r^{2-n-2\alpha} \left(\text{Dir}(N, \mathbf{B}_r(p)) + \Lambda \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n \right) \right)^{\frac{1}{2}} \end{aligned} \quad (5.1)$$

for every $p \in \Omega$ and for every $r \leq \min\{r_0, \mathbf{dist}(p, \partial\Omega)\}$. In particular, $N \in C_{loc}^{0,\alpha}(\Omega; \mathcal{A}_Q(\mathbb{R}^M))$.

The proof of Theorem 5.1 is a fairly easy consequence of Proposition 5.2 below. Before stating it, we need to introduce some further notation.

Let us fix a point $p \in \Omega$, and a radius $r < \min\{\text{inj}(\Sigma), \mathbf{dist}(p, \partial\Omega)\}$, in such a way that the exponential map \exp_p defines a diffeomorphism

$$\exp_p: B_r(0) \subset T_p\Sigma \rightarrow \mathbf{B}_r(p) \subset \Omega.$$

Denote by $y = (y^1, \dots, y^n)$ coordinates in $T_p\Sigma$ corresponding to the choice of an orthonormal basis (e_1, \dots, e_n) , and set $u := N \circ \exp_p$. Observe that for any $y \in B_r$ the differential $d(\exp_p)|_y$

realizes a linear isomorphism between $T_p\Sigma$ and $T_{\exp_p(y)}\Sigma$. Fix an orthonormal frame (ξ_1, \dots, ξ_n) of the tangent bundle $\mathcal{T}\Sigma|_{\mathbf{B}_r(p)}$ extending (e_1, \dots, e_n) (i.e. such that $\xi_i|_p = e_i$ for $i = 1, \dots, n$), and define, for $y \in B_r$,

$$\varepsilon_i(y) := \left(d(\exp_p)|_y\right)^{-1} \cdot \xi_i(\exp_p(y)). \quad (5.2)$$

Then, an elementary computation shows that

$$\int_{\mathbf{B}_r(p)} |N(x)|^2 d\mathcal{H}^n(x) = \int_{B_r} |u(y)|^2 \mathbf{J} \exp_p(y) dy \quad (5.3)$$

and

$$\text{Dir}(N, \mathbf{B}_r(p)) = \int_{B_r} \sum_{i=1}^n |D_{\varepsilon_i} u(y)|^2 \mathbf{J} \exp_p(y) dy, \quad (5.4)$$

where $\mathbf{J} \exp_p$ is the Jacobian determinant of the exponential map. From this it is immediate to deduce that the following asymptotic behaviors are satisfied for $r \rightarrow 0$ uniformly in p :

$$\int_{\mathbf{B}_r(p)} |N(x)|^2 d\mathcal{H}^n(x) = (1 + O(r)) \int_{B_r} |u(y)|^2 dy, \quad (5.5)$$

$$\text{Dir}(N, \mathbf{B}_r(p)) = (1 + O(r)) \int_{B_r} \sum_{i=1}^n |D_{e_i} u(y)|^2 dy = (1 + O(r)) \text{Dir}(u, B_r). \quad (5.6)$$

We can now state the key result from which we will conclude the Hölder regularity of Jacobi Q -fields.

Proposition 5.2. *There exist a universal positive constant $\Lambda = \Lambda(n, Q)$ and a radius $0 < r_0 = r_0(n, Q) < \text{inj}(\Sigma)$ with the following property. Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be Jac-minimizing and $p \in \Omega$. Then, for a.e. radius $r \leq \min\{r_0, \text{dist}(p, \partial\Omega)\}$ one has*

$$\text{Dir}(u, B_r) + \Lambda \int_{B_r} |u|^2 dy \leq C(n)r \left(\text{Dir}(u, \partial B_r) + \Lambda \int_{\partial B_r} |u|^2 d\mathcal{H}^{n-1} \right), \quad (5.7)$$

where $u := N \circ \exp_p|_{B_r} \in W^{1,2}(B_r; \mathcal{A}_Q(\mathbb{R}^M))$ and $C(n) < (n-2)^{-1}$ when $n \geq 3$.

In order to prove Proposition 5.2, we will need the following simple result on classical Sobolev functions in the Euclidean space.

Lemma 5.3. *For every $\varepsilon > 0$ there exists a constant $C = C_\varepsilon > 0$ such that the inequality*

$$\int_{B_r} |g|^2 dy \leq \left(\frac{1}{n} + \varepsilon\right) r \int_{\partial B_r} |g|^2 d\mathcal{H}^{n-1} + C_\varepsilon r^2 \int_{B_r} |Dg|^2 dy \quad (5.8)$$

holds for any function $g \in W^{1,2}(B_r)$.

Proof. First observe that, by a simple scaling argument, it is enough to prove the lemma for $r = 1$. Assume the lemma is false: suppose, by contradiction, there exists $\varepsilon_0 > 0$ such that for any $h \in \mathbb{N}$ there is $g_h \in W^{1,2}(B_1)$, with $\|g_h\|_{L^2} = 1$, such that

$$1 > \left(\frac{1}{n} + \varepsilon_0\right) \int_{\partial B_1} |g_h|^2 d\mathcal{H}^{n-1} + h \int_{B_1} |Dg_h|^2 dy. \quad (5.9)$$

The inequality (5.9) readily implies that

$$\lim_{h \rightarrow \infty} \int_{B_1} |Dg_h|^2 dy = 0, \quad (5.10)$$

whence, by Rellich's compactness theorem, the sequence g_h converges up to a subsequence (not relabeled) weakly in $W^{1,2}$, strongly in L^2 , to a constant function $g \equiv c$. The condition $\|g\|_{L^2} = 1$ forces the constant to satisfy $|c|^2 = \omega_n^{-1}$, where ω_n is the volume of the unit ball in \mathbb{R}^n . Hence, it suffices to pass to the limit the inequality

$$1 > \left(\frac{1}{n} + \varepsilon_0\right) \int_{\partial B_1} |g_h|^2 d\mathcal{H}^{n-1} \quad (5.11)$$

to obtain the desired contradiction:

$$1 > \left(\frac{1}{n} + \varepsilon_0\right) n. \quad (5.12)$$

□

Corollary 5.4. *For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that for any function $v \in W^{1,2}(B_r; \mathcal{A}_Q(\mathbb{R}^M))$ one has:*

$$\int_{B_r} |v|^2 dy \leq \left(\frac{1}{n} + \varepsilon\right) r \int_{\partial B_r} |v|^2 d\mathcal{H}^{n-1} + C_\varepsilon r^2 \text{Dir}(v, B_r). \quad (5.13)$$

Proof. Fix $\varepsilon > 0$ and $v \in W^{1,2}(B_r; \mathcal{A}_Q(\mathbb{R}^M))$, and apply Lemma 5.3 to the function $g = |v| = \mathcal{G}(v, Q[[0]]) \in W^{1,2}(B_r)$. The inequality (5.13) then follows immediately, because $g|_{\partial B_r} = |v|_{\partial B_r}$ and $|\partial_j g| \leq |\partial_j v|$ for every $j = 1, \dots, n$. □

Proof of Proposition 5.2. Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be Jac-minimizing, and fix any point $p \in \Omega$. For every radius $r < \min\{\text{inj}(\Sigma), \mathbf{dist}(p, \partial\Omega)\}$ the exponential map \exp_p maps the Euclidean ball $B_r(0) \subset T_p\Sigma$ diffeomorphically onto the geodesic ball $\mathbf{B}_r(p) \subset \Sigma$, and the composition $u := N \circ \exp_p$ is a $W^{1,2}$ Q -valued map defined in B_r .

Let now $f \in W^{1,2}(B_r; \mathcal{A}_Q(\mathbb{R}^M))$ be Dir-minimizing in B_r such that $f|_{\partial B_r} = u|_{\partial B_r}$ ⁵, and set $h := f \circ \exp_p^{-1}$. Then, the normal bundle projection $h^\perp \in \Gamma_Q^{1,2}(\mathcal{N}\mathbf{B}_r(p))$ satisfies $h^\perp|_{\partial\mathbf{B}_r(p)} = N|_{\partial\mathbf{B}_r(p)}$ and is therefore a competitor for the Jacobi functional. Hence, using minimality, the definition of the Jacobi functional and (3.3), we deduce:

$$\text{Jac}(N, \mathbf{B}_r(p)) \leq \text{Jac}(h^\perp, \mathbf{B}_r(p)) \leq \text{Dir}(h^\perp, \mathbf{B}_r(p)) + C_0 \int_{\mathbf{B}_r(p)} |h|^2 d\mathcal{H}^n, \quad (5.14)$$

which in turn produces

$$\text{Dir}(N, \mathbf{B}_r(p)) \leq \text{Dir}(h^\perp, \mathbf{B}_r(p)) + C_0 \left(\int_{\mathbf{B}_r(p)} |h|^2 d\mathcal{H}^n + \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n \right). \quad (5.15)$$

Hence, combining Lemma 4.4 with (5.5) and (5.6), we can conclude that for any $\varepsilon_1 \in (0, 1)$ there exists a radius $0 < r_{\varepsilon_1} < \text{inj}(\Sigma)$ such that the estimate

$$\text{Dir}(u, B_r) \leq (1 + \varepsilon_1) \text{Dir}(f, B_r) + C_{\varepsilon_1} \left(\int_{B_r} |f|^2 dy + \int_{B_r} |u|^2 dy \right), \quad (5.16)$$

holds true whenever $r \leq r_{\varepsilon_1}$.

Now we apply [DLS11, Proposition 3.10]: since f is Dir-minimizing in B_r , the following estimate

$$\text{Dir}(f, B_r) \leq C(n)r \text{Dir}(u, \partial B_r), \quad (5.17)$$

⁵Recall that the existence of such a map f is guaranteed by Theorem 1.24.

holds with constants $C(2) = Q$ and $C(n) < (n-2)^{-1}$ for $n \geq 3$ whenever $\text{Dir}(u, \partial B_r)$ is finite, and thus for a.e. r . Combining (5.16) with (5.17), we deduce that we can choose $\varepsilon_1 = \varepsilon_1(n, Q)$ so small that the inequality

$$\text{Dir}(u, B_r) \leq C(n)r\text{Dir}(u, \partial B_r) + C \left(\int_{B_r} |f|^2 dy + \int_{B_r} |u|^2 dy \right) \quad (5.18)$$

holds with, say, $C(2) = 2Q$ and again $C(n) < (n-2)^{-1}$ when $n \geq 3$ for a.e. $r \leq r_{n,Q}$.

Now, fix $\varepsilon > 0$ and apply the result of Corollary 5.4 first with $v = f$ and then with $v = u$, and plug the resulting inequalities in (5.18). Using the fact that f and u have the same boundary value and that $\text{Dir}(f, B_r) \leq \text{Dir}(u, B_r)$, we obtain the following key inequality:

$$\text{Dir}(u, B_r) \leq C(n)r\text{Dir}(u, \partial B_r) + C \left(\frac{1}{n} + \varepsilon \right) r \int_{\partial B_r} |u|^2 d\mathcal{H}^{n-1} + C_\varepsilon r^2 \text{Dir}(u, B_r). \quad (5.19)$$

This implies the following: for every $\Lambda > 0$ one has

$$\begin{aligned} \text{Dir}(u, B_r) + \Lambda \int_{B_r} |u|^2 dy &\leq C(n)r\text{Dir}(u, \partial B_r) \\ &+ (C + \Lambda) \left(\frac{1}{n} + \varepsilon \right) r \int_{\partial B_r} |u|^2 d\mathcal{H}^{n-1} \\ &+ C_{\varepsilon, \Lambda} r^2 \text{Dir}(u, B_r). \end{aligned} \quad (5.20)$$

For a suitable choice of $\Lambda = \Lambda_{n, \varepsilon} \gg 1$ this yields:

$$\left(1 - C_{n, \varepsilon} r^2 \right) \text{Dir}(u, B_r) + \Lambda \int_{B_r} |u|^2 dy \leq C(n)r\text{Dir}(u, \partial B_r) + \Lambda \left(\frac{1}{n} + 2\varepsilon \right) r \int_{\partial B_r} |u|^2 d\mathcal{H}^{n-1}. \quad (5.21)$$

Finally, we divide the whole inequality by $1 - C_{n, \varepsilon} r^2$ and conclude that if r is sufficiently small, say $r \leq r_{n, \varepsilon, Q}$ then the inequality

$$\text{Dir}(u, B_r) + \Lambda \int_{B_r} |u|^2 dy \leq C(n)r\text{Dir}(u, \partial B_r) + \Lambda \left(\frac{1}{n} + 4\varepsilon \right) r \int_{\partial B_r} |u|^2 d\mathcal{H}^{n-1} \quad (5.22)$$

holds with a possible new choice of $C(n)$, say $C(2) = 4Q$ and still $C(n) < (n-2)^{-1}$ for $n \geq 3$. The conclusion immediately follows, by choosing $\varepsilon = \varepsilon(n, Q)$ in such a way that $\frac{1}{n} + 4\varepsilon < 4Q$ when $n = 2$ and $\frac{1}{n} + 4\varepsilon < \frac{1}{n-2}$ when $n \geq 3$. \square

We have now all the ingredients that are needed to prove Theorem 5.1: as announced at the beginning of the section, the proof can be easily achieved by combining our Proposition 5.2 with the Campanato-Morrey estimates 1.21.

Proof of Theorem 5.1. Let r_0 be the radius given in Proposition 5.2. Fix any point $p \in \Omega$, and assume without loss of generality that $\mathbf{B}_{r_0}(p) \Subset \Omega$. Consider the corresponding exponential map $\exp_p: B_{r_0}(0) \subset T_p \Sigma \rightarrow \mathbf{B}_{r_0}(p) \subset \Sigma$, and set $u := N \circ \exp_p$. By Proposition 5.2, for a.e. radius $r \leq r_0$ the inequality (5.7) is satisfied with universal constants Λ and $C(n)$, with $C(n) < (n-2)^{-1}$ when $n \geq 3$. We set:

$$\gamma(n) := \begin{cases} C(n)^{-1} & \text{if } n = 2 \\ C(n)^{-1} - n + 2 & \text{if } n \geq 3, \end{cases} \quad (5.23)$$

and we denote by $\phi = \phi(r)$ the absolutely continuous function

$$\phi(r) := \text{Dir}(u, B_r) + \Lambda \int_{B_r} |u|^2 dy \tag{5.24}$$

for $r \in (0, r_0]$. By (5.7), ϕ satisfies the differential inequality

$$(n - 2 + \gamma)\phi \leq r\phi' \tag{5.25}$$

almost everywhere in the interval $(0, r_0]$. Integrating (5.25) we obtain:

$$\text{Dir}(u, B_r) \leq \phi(r) \leq \frac{\phi(r_0)}{r_0^{n-2+\gamma}} r^{n-2+\gamma} =: Ar^{n-2+\gamma}. \tag{5.26}$$

As a consequence of the Campanato - Morrey estimates, Proposition 1.21, we conclude that u is Hölder continuous with exponent $\alpha := \frac{\gamma}{2}$, with

$$[u]_{C^{0,\alpha}(\overline{B_{\theta r_0}})} := \sup_{y_1, y_2 \in \overline{B_{\theta r_0}}} \frac{\mathcal{G}(u(y_1), u(y_2))}{|y_1 - y_2|^\alpha} \leq C\sqrt{A}, \tag{5.27}$$

for any $0 < \theta < 1$ and for a constant $C = C(n, M, Q, \theta)$.

The estimate (5.1) is an immediate consequence of (5.27) and the properties of the exponential map. \square

6. BLOW-UP ANALYSIS

In this final part of the paper we prove Theorem 0.7, thus showing that a Jacobi Q -field N is in fact the “superposition” of Q classical Jacobi fields in a neighborhood of all points of the domain Ω with the exception of those belonging to a *singular set* of small Hausdorff dimension.

Before proceeding, observe that it is an immediate consequence of Definition 0.6 that the regular set $\text{reg}(N)$ is open in Ω , whereas $\text{sing}(N)$ is relatively closed.

6.1. First variations. The first step towards the proof of Theorem 0.7 consists of deriving some Euler-Lagrange conditions for Jac-minimizing multivalued maps. We will assume, as usual, that N is a Q -valued section of the normal bundle of Σ in \mathcal{M} defined in an open set Ω , where it minimizes the Jacobi functional as specified in Definition 0.2.

Now, suppose that for some $\delta > 0$ we have a 1-parameter family $\{N_s\}_{s \in (-\delta, \delta)} \subset \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ such that $N_0 = N$ and $N_s \equiv N$ in a neighborhood of $\partial\Omega$ for all s . Then, the minimization property of N implies that the map $s \mapsto \text{Jac}(N_s, \Omega)$ takes its minimum at $s = 0$, and thus

$$\frac{d}{ds} \text{Jac}(N_s, \Omega)|_{s=0} = 0 \tag{6.1}$$

whenever the derivative on the left exists. The family $\{N_s\}$ is called an (admissible) 1-parameter family of variations of N in Ω , and formula (6.1) is the first variation formula corresponding to the given variation.

We will consider two natural types of variations in order to perturb the map N . The *inner variations* are generated by right compositions with diffeomorphisms of the domain and by a suitable “orthogonalization procedure”; the *outer variations* correspond instead to “left compositions”. The relevant definition is the following.

Definition 6.1. Let $N = \sum_l \llbracket N^l \rrbracket \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be Jac-minimizing in Ω .

- (OV) Given $\psi \in C^1(\Omega \times \mathbb{R}^M; \mathbb{R}^M)$ such that $\text{supp}(\psi) \subset \Omega' \times \mathbb{R}^M$ for some $\Omega' \Subset \Omega$ and $\psi(x, u) \in T_x^\perp \Sigma \subset T_x \mathcal{M}$ for all $(x, u) \in \Omega \times T_x^\perp \Sigma$, an admissible variation of N in Ω can be defined by $N_s(x) := \sum_{l=1}^Q \llbracket N^l(x) + s\psi(x, N^l(x)) \rrbracket$. Such a family is called outer variation (OV);
- (IV) Given a C^1 vector field φ of $\mathcal{T}\Sigma$ compactly supported in Ω , for s sufficiently small the map $x \mapsto \Phi_s(x) := \exp_x(s\varphi(x))$ is a diffeomorphism of Ω which leaves $\partial\Omega$ fixed. As a consequence, the family $\{N_s\}$ defined by $N_s := (N \circ \Phi_s)^\perp$ is an admissible variation of N in Ω , which we call inner variation (IV).

In the next proposition, we obtain an explicit formulation of (6.1) in the case of outer variations induced by maps ψ as above. Consistently with the notation introduced for multi-fields in Remark 2.9, given $(x, u) \in \Omega \times \mathbb{R}^M$ we will denote by $\nabla^\perp \psi(x, u)$ the linear operator $T_x \Sigma \rightarrow T_x^\perp \Sigma$ obtained by projecting $D_x \psi(x, u)$ onto $T_x^\perp \Sigma$ at every x . Also, recall the definitions of $A \cdot u$, u being a (single-valued) section of $\mathcal{N}\Omega$, and of the quadratic form \mathcal{R} . The symbol $\langle A : B \rangle$ will be used to denote the usual Hilbert-Schmidt scalar product of two matrices.

Proposition 6.2 (Outer variation formula). *Let ψ be as in (OV) and such that*

$$|D_u \psi| \leq C < \infty \quad \text{and} \quad |\psi| + |D_x \psi| \leq C(1 + |u|). \quad (6.2)$$

Then, the first variation formula corresponding to the outer variation N_s defined by ψ is

$$\int_{\Omega} \sum_{l=1}^Q \langle \nabla^\perp N^l(x) : (\nabla^\perp \psi(x, N^l(x)) + D_u \psi(x, N^l(x)) \cdot DN^l(x)) \rangle d\mathcal{H}^n(x) = \mathcal{E}_{\text{OV}}(\psi), \quad (6.3)$$

where

$$\mathcal{E}_{\text{OV}}(\psi) := \int_{\Omega} \sum_{l=1}^Q \left(\langle A \cdot N^l(x) : A \cdot \psi(x, N^l(x)) \rangle + \mathcal{R}(N^l(x), \psi(x, N^l(x))) \right) d\mathcal{H}^n(x). \quad (6.4)$$

Proof. The proof is straightforward: using (2.41) with N_s in place of u , it suffices to differentiate in s and recall that \mathcal{R} is a symmetric quadratic form on the normal bundle of Σ in \mathcal{M} (the hypotheses in (6.2) ensure the summability of the various integrands involved in the computation). \square

An explicit formula for (6.1) in the case of inner variations induced by vector fields φ as in Definition (2.18) is the content of the following proposition.

Proposition 6.3 (Inner variation formula). *Let φ be as in (IV). Then, the first variation formula corresponding to the inner variation N_s defined by the family Φ_s of diffeomorphisms induced by φ is*

$$- \int_{\Omega} |\nabla^\perp N|^2 \text{div}_{\Sigma}(\varphi) d\mathcal{H}^n + 2 \int_{\Omega} \sum_{l=1}^Q \langle \nabla^\perp N^l : \nabla^\perp N^l \cdot \nabla^{\Sigma} \varphi \rangle d\mathcal{H}^n = \mathcal{E}_{\text{IV}}(\varphi), \quad (6.5)$$

where

$$\mathcal{E}_{\text{IV}}(\varphi) = \mathcal{E}_{\text{IV}}^{(1)}(\varphi) + \mathcal{E}_{\text{IV}}^{(2)}(\varphi) + \mathcal{E}_{\text{IV}}^{(3)}(\varphi)$$

is defined by

$$\mathcal{E}_{\text{IV}}^{(1)}(\varphi) := 2 \int_{\Omega} \sum_{l=1}^Q \left(\text{tr}_{\Sigma}(\langle \bar{A}(\cdot, N^l), \bar{A}(\varphi, \nabla_{(\cdot)}^\perp N^l) \rangle) - \text{tr}_{\Sigma}(\langle \bar{A}(\varphi, N^l), \bar{A}(\cdot, \nabla_{(\cdot)}^\perp N^l) \rangle) \right) d\mathcal{H}^n, \quad (6.6)$$

$$\mathcal{E}_{\text{IV}}^{(2)}(\varphi) := 2 \int_{\Omega} \sum_{l=1}^Q \langle A \cdot N^l : A \cdot \nabla_{\varphi}^{\perp} N^l \rangle d\mathcal{H}^n, \quad (6.7)$$

and

$$\mathcal{E}_{\text{IV}}^{(3)}(\varphi) := 2 \int_{\Omega} \sum_{l=1}^Q \mathcal{R}(N^l, \nabla_{\varphi}^{\perp} N^l) d\mathcal{H}^n. \quad (6.8)$$

Proof. Fix the vector field φ , and consider the associated variation $\{N_s\}$, with $N_s = \sum_l \llbracket N_s^l \rrbracket$ defined by

$$N_s^l(x) = (N^l \circ \Phi_s)^{\perp}(x) = \sum_{\beta=1}^k \langle N^l(\Phi_s(x)), \nu_{\beta}(x) \rangle \nu_{\beta}(x), \quad (6.9)$$

$(\nu_{\beta})_{\beta=1}^k$ being a (local) orthonormal frame of $\mathcal{N}\Omega$. Recall that $\Phi_0 = \text{Id}_{\Omega}$ and that $\frac{\partial}{\partial s} \Phi_s(x)|_{s=0} = \varphi(x)$. Now, using (2.41), we write the first variation formula as

$$\begin{aligned} 0 &= \frac{d}{ds} \text{Jac}(N_s, \Omega)|_{s=0} \\ &= \frac{d}{ds} \Big|_{s=0} \text{Dir}^{\mathcal{N}\Sigma}(N_s, \Omega) - \frac{d}{ds} \Big|_{s=0} \int_{\Omega} \sum_{l=1}^Q |A \cdot N_s^l|^2 d\mathcal{H}^n - \frac{d}{ds} \Big|_{s=0} \int_{\Omega} \sum_{l=1}^Q \mathcal{R}(N_s^l, N_s^l) d\mathcal{H}^n \\ &=: I_1 + I_2 + I_3, \end{aligned} \quad (6.10)$$

and we will work on the three terms separately.

Step 1: computing I_1 . Write $I_1 = \sum_l I_1^l$, where

$$I_1^l = \frac{d}{ds} \Big|_{s=0} \int_{\Omega} \sum_{i=1}^n \sum_{\alpha=1}^k |\langle D_{\xi_i} N_s^l, \nu_{\alpha} \rangle|^2 d\mathcal{H}^n. \quad (6.11)$$

Using the representation formula (6.9), one immediately computes

$$\begin{aligned} \langle D_{\xi_i} N_s^l, \nu_{\alpha} \rangle(x) &= \langle DN^l|_{\Phi_s(x)} \cdot D\Phi_s|_x \cdot \xi_i(x), \nu_{\alpha}(x) \rangle \\ &\quad + \langle N^l(\Phi_s(x)), D_{\xi_i} \nu_{\alpha}(x) \rangle \\ &\quad + \sum_{\beta=1}^k \langle N^l(\Phi_s(x)), \nu_{\beta}(x) \rangle \langle D_{\xi_i} \nu_{\beta}(x), \nu_{\alpha}(x) \rangle. \end{aligned} \quad (6.12)$$

Now, since $\langle \nu_{\alpha}, \nu_{\beta} \rangle = \delta_{\alpha\beta}$, we have that $\langle D_{\xi_i} \nu_{\beta}, \nu_{\alpha} \rangle = -\langle \nu_{\beta}, D_{\xi_i} \nu_{\alpha} \rangle$, so that the last term in formula (6.12) becomes

$$- \sum_{\beta=1}^k \langle N^l(\Phi_s(x)), \nu_{\beta}(x) \rangle \langle D_{\xi_i} \nu_{\alpha}(x), \nu_{\beta}(x) \rangle = -\langle N^l(\Phi_s(x)), \nabla_{\xi_i}^{\perp} \nu_{\alpha}(x) \rangle, \quad (6.13)$$

and we can write

$$\begin{aligned} \langle D_{\xi_i} N_s^l, \nu_{\alpha} \rangle(x) &= \langle DN^l|_{\Phi_s(x)} \cdot D\Phi_s|_x \cdot \xi_i(x), \nu_{\alpha}(x) \rangle \\ &\quad + \langle N^l(\Phi_s(x)), (D_{\xi_i} \nu_{\alpha} - \nabla_{\xi_i}^{\perp} \nu_{\alpha})(x) \rangle. \end{aligned} \quad (6.14)$$

For small values of the parameter s , the map Φ_s is a diffeomorphism of Ω , and we will denote by Φ_s^{-1} its inverse. Then, we can change variable $x = \Phi_s^{-1}(y)$ in the integral, and finally write

$$I_1^l = \frac{d}{ds} \Big|_{s=0} \int_{\Omega} \sum_{i=1}^n \sum_{\alpha=1}^k |g_{i\alpha}^l(s, y)|^2 \mathbf{J}\Phi_s^{-1}(y) d\mathcal{H}^n(y), \quad (6.15)$$

where $\mathbf{J}\Phi_s^{-1}$ is the Jacobian determinant of $D\Phi_s^{-1}$ and

$$g_{i\alpha}^l(s, y) = \langle DN^l|_y \cdot \zeta_i(s, y), \nu_{\alpha}(\Phi_s^{-1}(y)) \rangle + \langle N^l(y), (D_{\xi_i} \nu_{\alpha} - \nabla_{\xi_i}^{\perp} \nu_{\alpha})(\Phi_s^{-1}(y)) \rangle, \quad (6.16)$$

with $\zeta_i(s, y) := D\Phi_s|_{\Phi_s^{-1}(y)} \cdot \xi_i(\Phi_s^{-1}(y))$. Hence, we have:

$$\begin{aligned} I_1^l &= - \int_{\Omega} \sum_{i=1}^n \sum_{\alpha=1}^k |g_{i\alpha}^l(0, y)|^2 \operatorname{div}_{\Sigma}(\varphi) d\mathcal{H}^n + 2 \int_{\Omega} \sum_{i=1}^n \sum_{\alpha=1}^k g_{i\alpha}^l(0, y) \partial_s g_{i\alpha}^l(0, y) d\mathcal{H}^n \\ &= - \int_{\Omega} \sum_{i=1}^n \sum_{\alpha=1}^k |g_{i\alpha}^l(0, y)|^2 \operatorname{div}_{\Sigma}(\varphi) d\mathcal{H}^n + \int_{\Omega} \frac{\partial}{\partial s} \left(\sum_{i=1}^n \sum_{\alpha=1}^k |g_{i\alpha}^l(s, y)|^2 \right) \Big|_{s=0} d\mathcal{H}^n(y). \end{aligned} \quad (6.17)$$

Now, since

$$\sum_{i=1}^n \sum_{\alpha=1}^k |g_{i\alpha}^l(s, y)|^2 = |\nabla^{\perp} N_s^l|^2(\Phi_s^{-1}(y)),$$

its value is independent of the orthonormal frame chosen: thus, having fixed a point $y \in \Omega$, we can impose $\nabla \xi_i = \nabla \nu_{\alpha} = 0$ at y .

We can now proceed computing explicitly (6.17). Clearly, $g_{i\alpha}^l(0, y) = \langle D_{\xi_i} N^l, \nu_{\alpha} \rangle(y)$, so we are only left with the computation of $\partial_s g_{i\alpha}^l(0, y)$. We start observing that

$$\partial_s \zeta_i(0, y) = (D_{\xi_i} \varphi - D_{\varphi} \xi_i)(y) = -[\varphi, \xi_i](y), \quad (6.18)$$

from which we easily deduce

$$\begin{aligned} \partial_s|_{s=0} \left(\langle DN^l|_y \cdot \zeta_i(s, y), \nu_{\alpha}(\Phi_s^{-1}(y)) \rangle \right) &= -\langle D_{[\varphi, \xi_i]} N^l, \nu_{\alpha} \rangle - \langle D_{\xi_i} N^l, D_{\varphi} \nu_{\alpha} \rangle \\ &= \langle D_{\nabla_{\xi_i}^{\Sigma} \varphi} N^l, \nu_{\alpha} \rangle - \langle \bar{A}(\xi_i, N^l), \bar{A}(\varphi, \nu_{\alpha}) \rangle, \end{aligned} \quad (6.19)$$

where we have used that $\nabla_{\varphi} \xi_i = \nabla_{\varphi} \nu_{\alpha} = 0$ at y (and, therefore, $[\varphi, \xi_i](y) = -\nabla_{\xi_i}^{\Sigma} \varphi(y)$ and $D_{\varphi} \nu_{\alpha} = \bar{A}(\varphi, \nu_{\alpha})$).

On the other hand,

$$\begin{aligned} \partial_s|_{s=0} \left(\langle N^l(y), (D_{\xi_i} \nu_{\alpha} - \nabla_{\xi_i}^{\perp} \nu_{\alpha})(\Phi_s^{-1}(y)) \rangle \right) &= -\langle N^l, D_{\varphi} (D_{\xi_i} \nu_{\alpha} - \nabla_{\xi_i}^{\perp} \nu_{\alpha}) \rangle \\ &= \langle D_{\varphi} N^l, D_{\xi_i} \nu_{\alpha} - \nabla_{\xi_i}^{\perp} \nu_{\alpha} \rangle \\ &= \langle \bar{A}(\varphi, N^l), \bar{A}(\xi_i, \nu_{\alpha}) \rangle \end{aligned} \quad (6.20)$$

because the fields N^l and $D_{\xi_i} \nu_{\alpha} - \nabla_{\xi_i}^{\perp} \nu_{\alpha}$ are mutually orthogonal and, again, because $\nabla \nu_{\alpha} = 0$ at y .

This allows to conclude:

$$\begin{aligned}
 I_1 &= - \int_{\Omega} |\nabla^{\perp} N|^2 \operatorname{div}_{\Sigma}(\varphi) d\mathcal{H}^n + 2 \int_{\Omega} \sum_{l=1}^Q \langle \nabla^{\perp} N^l : \nabla^{\perp} N^l \cdot \nabla^{\Sigma} \varphi \rangle d\mathcal{H}^n \\
 &\quad + 2 \int_{\Omega} \sum_{l=1}^Q \left(\operatorname{tr}_{\Sigma}(\langle \bar{A}(\varphi, N^l), \bar{A}(\cdot, \nabla_{(\cdot)}^{\perp} N^l) \rangle) - \operatorname{tr}_{\Sigma}(\langle \bar{A}(\cdot, N^l), \bar{A}(\varphi, \nabla_{(\cdot)}^{\perp} N^l) \rangle) \right) d\mathcal{H}^n \quad (6.21) \\
 &= - \int_{\Omega} |\nabla^{\perp} N|^2 \operatorname{div}_{\Sigma}(\varphi) d\mathcal{H}^n + 2 \int_{\Omega} \sum_{l=1}^Q \langle \nabla^{\perp} N^l : \nabla^{\perp} N^l \cdot \nabla^{\Sigma} \varphi \rangle d\mathcal{H}^n - \mathcal{E}_{\text{IV}}^{(1)}(\varphi).
 \end{aligned}$$

Step 2: computing I_2 . Write $I_2 = \sum_l I_2^l$, where

$$I_2^l = - \frac{d}{ds} \Big|_{s=0} \int_{\Omega} |A \cdot N_s^l|^2 d\mathcal{H}^n. \quad (6.22)$$

Since the tensor A takes values in the normal bundle of Σ , clearly $A \cdot N_s^l = A \cdot (N^l \circ \Phi_s)$, whence

$$\int_{\Omega} |A \cdot N_s^l|^2 d\mathcal{H}^n = \int_{\Omega} \sum_{i,j=1}^n |A_x(\xi_i(x), \xi_j(x)), N^l(\Phi_s(x))|^2 d\mathcal{H}^n(x). \quad (6.23)$$

We can now differentiate in s and evaluate for $s = 0$ in formula (6.23) to obtain:

$$I_2^l = -2 \int_{\Omega} \sum_{i,j=1}^n \langle A_x(\xi_i(x), \xi_j(x)), N^l(x) \rangle \langle A_x(\xi_i(x), \xi_j(x)), D_{\varphi} N^l(x) \rangle, \quad (6.24)$$

which readily yields

$$I_2 = -2 \int_{\Omega} \sum_{l=1}^Q \langle A \cdot N^l : A \cdot \nabla_{\varphi}^{\perp} N^l \rangle d\mathcal{H}^n = -\mathcal{E}_{\text{IV}}^{(2)}(\varphi). \quad (6.25)$$

Step 3: computing I_3 . As before, write $I_3 = \sum_l I_3^l$, where

$$I_3^l = - \frac{d}{ds} \Big|_{s=0} \int_{\Omega} \mathcal{R}(N_s^l, N_s^l) d\mathcal{H}^n. \quad (6.26)$$

Now, it suffices to differentiate in s and evaluate at $s = 0$ inside the integral keeping in mind that \mathcal{R} is a symmetric 2-tensor to get

$$I_3 = -2 \int_{\Omega} \sum_{l=1}^Q \mathcal{R}(N^l, \nabla_{\varphi}^{\perp} N^l) d\mathcal{H}^n = -\mathcal{E}_{\text{IV}}^{(3)}(\varphi). \quad (6.27)$$

The statement, formula (6.5), is immediately obtained by plugging equations (6.21), (6.25) and (6.27) into (6.10). \square

The first variation formulas (6.3) and (6.5) will play a fundamental role in the next section to discuss the almost monotonicity properties of the frequency function. Before proceeding, we apply the outer variation formula to show that minimizers of the Jac functional enjoy a Caccioppoli type inequality.

Proposition 6.4 (Caccioppoli inequality). *There exists a geometric constant $C > 0$ such that for any Jac-minimizing Q -valued map $N = \sum_l \llbracket N^l \rrbracket \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ the inequality*

$$\int_{\Omega} \eta^2(x) |\nabla^\perp N|^2(x) d\mathcal{H}^n(x) \leq 4 \int_{\Omega} |D\eta|^2(x) |N|^2(x) d\mathcal{H}^n(x) + C \int_{\Omega} \eta^2(x) |N|^2(x) d\mathcal{H}^n(x) \quad (6.28)$$

holds for any choice of $\eta \in C_c^\infty(\Omega)$. In particular, for every $p \in \Omega$ and for every $r < \min\{\text{inj}(\Sigma), \text{dist}(p, \partial\Omega)\}$ one has

$$\int_{\mathbf{B}_{\frac{r}{2}}(p)} |\nabla^\perp N|^2 d\mathcal{H}^n \leq \frac{C}{r^2} \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n. \quad (6.29)$$

Proof. Fix N and η as in the statement, and apply the outer variation formula (6.3) with $\psi(x, u) := \eta(x)^2 u$. Since $D_x \psi(x, u) = 2\eta(x)u \otimes D\eta(x)$ and $D_u \psi(x, u) = \eta(x)^2 \text{Id}$, for this choice of ψ the outer variation formula reads

$$\int_{\Omega} \eta^2 |\nabla^\perp N|^2 + 2 \sum_{l=1}^Q \langle \eta \nabla^\perp N^l : N^l \otimes D\eta \rangle d\mathcal{H}^n = \int_{\Omega} \eta^2 \sum_{l=1}^Q (|A \cdot N^l|^2 + \mathcal{R}(N^l, N^l)) d\mathcal{H}^n. \quad (6.30)$$

Applying Young's inequality we immediately deduce that for any $\delta > 0$ one has

$$\int_{\Omega} \eta^2 |\nabla^\perp N|^2 d\mathcal{H}^n \leq \delta \int_{\Omega} \eta^2 |\nabla^\perp N|^2 d\mathcal{H}^n + \frac{1}{\delta} \int_{\Omega} |D\eta|^2 |N|^2 d\mathcal{H}^n + C \int_{\Omega} \eta^2 |N|^2 d\mathcal{H}^n, \quad (6.31)$$

where $C = C(\mathbf{A}, \mathbf{R})$. Choose $\delta = \frac{1}{2}$ to obtain (6.28). In order to deduce (6.29), apply (6.28) with $\eta(x) := \phi\left(\frac{d(x)}{r}\right)$, where $d(x) := \mathbf{d}(x, p)$ and ϕ is a cut-off function $0 \leq \phi \leq 1$ such that $\phi(t) = 1$ for $0 \leq t \leq \frac{1}{2}$, $\phi(t) = 0$ for $t \geq 1$ and $|\phi'| \leq 2$. \square

6.2. The frequency function. The next step towards the proof of Theorem 0.7 consists of a careful asymptotic analysis of the celebrated *frequency function*.

Definition 6.5 (Frequency function). Fix any point $p \in \Omega$. For any radius $0 < r < \min\{\text{inj}(\Sigma), \text{dist}(p, \partial\Omega)\}$, define the *energy function*

$$\mathbf{D}_{N,p}(r) := \int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2(x) d\mathcal{H}^n(x) \quad (6.32)$$

and the *height function*

$$\mathbf{H}_{N,p}(r) := \int_{\partial\mathbf{B}_r(p)} |N|^2(x) d\mathcal{H}^{n-1}(x). \quad (6.33)$$

The *frequency function* is then defined by

$$\mathbf{I}_{N,p}(r) := \frac{r\mathbf{D}_{N,p}(r)}{\mathbf{H}_{N,p}(r)} \quad (6.34)$$

for all r such that $\mathbf{H}_{N,p}(r) > 0$. When the Q -field N and the point p are fixed and there is no ambiguity, we will drop the subscripts and simply write $\mathbf{D}(r)$, $\mathbf{H}(r)$ and $\mathbf{I}(r)$.

Remark 6.6. It is an easy consequence of the Hölder regularity of N that the frequency function $\mathbf{I}(r)$ is well defined and bounded in a suitable neighborhood of every point $p \in \Omega$ such that $N(p) \neq Q\llbracket 0 \rrbracket$. Indeed, if such assumption is satisfied then

$$\lim_{r \rightarrow 0} \frac{1}{\mathcal{H}^{n-1}(\partial\mathbf{B}_r(p))} \mathbf{H}(r) = |N|^2(p) = \mathcal{G}(N(p), Q\llbracket 0 \rrbracket)^2 > 0,$$

which in turn implies that $\mathbf{H}(r) > 0$ for small values of r . Furthermore, from the proof of Theorem 5.1 (cf. in particular formula (5.26)) we can also infer that if r is sufficiently small then

$$\mathbf{D}(r) \leq Cr^{n-2+2\alpha},$$

where α is the Hölder exponent of N . In particular, from this one immediately concludes that there exists the limit

$$\lim_{r \rightarrow 0} \mathbf{I}(r) = 0$$

at every point p such that $N(p) \neq Q[[0]]$.

As we shall see, we will obtain as a byproduct of the improved regularity theory developed in this section that the frequency function is well defined and bounded also in a suitable neighborhood around every point p such that $N(p) = Q[[0]]$, and that also at such points the limit $\lim_{r \rightarrow 0} \mathbf{I}(r)$ exists, but it is strictly positive.

The main analytic feature of the frequency function is the following *almost monotonicity* property.

Theorem 6.7 (Almost monotonicity of the frequency). *There exist a geometric constant C_0 and a radius $0 < r_0 < \min\{\text{inj}(\Sigma), \text{dist}(p, \partial\Omega)\}$ such that for all $0 < s < t \leq r_0$ with $\mathbf{H}|_{[s,t]} > 0$ one has*

$$\mathbf{I}(s) \leq C_0 (1 + \mathbf{I}(t)). \quad (6.35)$$

The proof of Theorem 6.7 is a consequence of some estimates involving the functions \mathbf{D} and \mathbf{H} and their derivatives, which in turn can be obtained by testing the first variations formulas (6.3) and (6.5) with a suitable choice of the maps ψ and φ . The first step to perform this program is to define the following auxiliary functions.

Definition 6.8. We denote by $\frac{\partial}{\partial \hat{r}}$ the vector field which is tangent to geodesic arcs parametrized by arc length and emanating from p . We will set $\nabla_{\hat{r}} := \nabla_{\frac{\partial}{\partial \hat{r}}}$, the directional derivative along $\frac{\partial}{\partial \hat{r}}$, and we will let $\nabla_{\hat{r}}^\perp$ be its projection onto the normal bundle of Σ in \mathcal{M} . We set:

$$\mathbf{E}(r) = \mathbf{E}_{N,p}(r) := \int_{\partial \mathbf{B}_r(p)} \sum_{l=1}^Q \langle N^l(x), \nabla_{\hat{r}}^\perp N^l(x) \rangle d\mathcal{H}^{n-1}(x), \quad (6.36)$$

$$\mathbf{G}(r) = \mathbf{G}_{N,p}(r) := \int_{\partial \mathbf{B}_r(p)} |\nabla_{\hat{r}}^\perp N|^2(x) d\mathcal{H}^{n-1}(x), \quad (6.37)$$

and

$$\mathbf{F}(r) = \mathbf{F}_{N,p}(r) := \int_{\mathbf{B}_r(p)} |N|^2(x) d\mathcal{H}^n(x). \quad (6.38)$$

Remark 6.9. Observe that $\mathbf{D}(r) = \text{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p))$; \mathbf{D} is an absolutely continuous function with derivative

$$\mathbf{D}'(r) = \int_{\partial \mathbf{B}_r(p)} |\nabla^\perp N|^2 d\mathcal{H}^{n-1}$$

almost everywhere. Similarly, $\mathbf{F}(r) = \|N\|_{L^2(\mathbf{B}_r(p))}^2$ is absolutely continuous, and for a.e. r

$$\mathbf{F}'(r) = \int_{\partial \mathbf{B}_r(p)} |N|^2 d\mathcal{H}^{n-1} = \mathbf{H}(r). \quad (6.39)$$

As for $\mathbf{H}(r)$, notice that $|N|$ is the composition of N with the Lipschitz function $\mathcal{G}(\cdot, Q[[0]])$, thus it belongs to $W^{1,2}$. Hence, $|N|^2$ is a $W^{1,1}$ function, and also $\mathbf{H} \in W^{1,1}$.

Lemma 6.10 (First variation estimates). *There exist a geometric constant $C_0 > 0$ and a radius $0 < r_0 < \min\{\text{inj}(\Sigma), \mathbf{dist}(p, \partial\Omega)\}$ such that the following inequalities hold true for a.e. $0 < r \leq r_0$:*

$$|\mathbf{D}(r) - \mathbf{E}(r)| \leq C_0 \mathbf{F}(r), \quad (6.40)$$

$$|\mathbf{D}'(r) - 2\mathbf{G}(r) - \frac{n-2}{r}\mathbf{D}(r)| \leq C_0 \mathbf{D}(r) + C_0 (\mathbf{D}(r)\mathbf{F}(r))^{\frac{1}{2}}, \quad (6.41)$$

$$|\mathbf{H}'(r) - \frac{n-1}{r}\mathbf{H}(r) - 2\mathbf{E}(r)| \leq C_0 \mathbf{H}(r). \quad (6.42)$$

Furthermore, if $\mathbf{I}(r) \geq 1$ then

$$|\mathbf{D}(r) - \mathbf{E}(r)| \leq C_0 r^2 \mathbf{D}(r), \quad (6.43)$$

and

$$|\mathbf{D}'(r) - 2\mathbf{G}(r) - \frac{n-2}{r}\mathbf{D}(r)| \leq C_0 (1+r) \mathbf{D}(r). \quad (6.44)$$

Proof. Step 1: proof of (6.40). We test the outer variation formula (6.3) with the map ψ given by $\psi(x, u) := \phi\left(\frac{d(x)}{r}\right)u$, where $d(\cdot) := \mathbf{d}(\cdot, p)$, and $\phi = \phi(t)$ is a function in $C^\infty([0, \infty))$, $0 \leq \phi \leq 1$, with $\phi \equiv 1$ in a neighborhood of $t = 0$ and $\phi \equiv 0$ for $t \geq 1$. Observe first that this choice of ψ induces an admissible family of outer variations: indeed, one clearly sees that $\text{supp}(\psi) \subset \mathbf{B}_r(p)$, the geodesic ball centered at p and of radius r , which is compactly supported in Ω , and that the orthogonality conditions and the assumptions in (6.2) are satisfied. We compute:

$$D_x \psi(x, u) = r^{-1} \phi' \left(\frac{d(x)}{r} \right) u \otimes \nabla d,$$

which yields

$$\langle \nabla^\perp N^l(x) : \nabla^\perp \psi(x, N^l(x)) \rangle = r^{-1} \phi' \left(\frac{d(x)}{r} \right) \langle \nabla_{\hat{r}}^\perp N^l(x), N^l(x) \rangle. \quad (6.45)$$

On the other hand, $D_u \psi(x, u) = \phi\left(\frac{d(x)}{r}\right) \text{Id}$, whence

$$\langle \nabla^\perp N^l(x) : D_u \psi(x, N^l(x)) \cdot DN^l(x) \rangle = \phi\left(\frac{d(x)}{r}\right) |\nabla^\perp N^l|^2(x). \quad (6.46)$$

Analogously, we can compute explicitly the right-hand side of (6.3) corresponding to our choice of ψ and get:

$$\mathcal{E}_{\text{OV}}(\psi) = \int_\Sigma \phi\left(\frac{d(x)}{r}\right) \sum_{l=1}^Q \left(|A \cdot N^l|^2(x) + \mathcal{R}(N^l(x), N^l(x)) \right) d\mathcal{H}^n(x). \quad (6.47)$$

By a standard approximation procedure, the details of which are left to the reader, it is easy to see that we can test with

$$\phi(t) = \phi_h(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1 - \frac{1}{h} \\ h(1-t) & \text{for } 1 - \frac{1}{h} \leq t \leq 1 \\ 0 & \text{for } t \geq 1. \end{cases} \quad (6.48)$$

Inserting into (6.45), (6.46) and (6.47), the outer variation formula (6.3) becomes

$$\begin{aligned} -\frac{h}{r} \int_{\mathbf{B}_r(p) \setminus \mathbf{B}_{r-\frac{r}{h}}(p)} \sum_{l=1}^Q \langle \nabla_{\hat{r}}^\perp N^l(x), N^l(x) \rangle d\mathcal{H}^n(x) + \int_{\Sigma} \phi_h \left(\frac{d(x)}{r} \right) |\nabla^\perp N(x)|^2 d\mathcal{H}^n(x) \\ = \int_{\Sigma} \phi_h \left(\frac{d(x)}{r} \right) \sum_{l=1}^Q \left(|A \cdot N^l|^2(x) + \mathcal{R}(N^l(x), N^l(x)) \right) d\mathcal{H}^n(x). \end{aligned} \quad (6.49)$$

Now, let $h \uparrow \infty$. The left-hand side of (6.49) converges to $\mathbf{D}(r) - \mathbf{E}(r)$, whereas the right-hand side converges to

$$\int_{\mathbf{B}_r(p)} \sum_{l=1}^Q \left(|A \cdot N^l|^2 + \mathcal{R}(N^l, N^l) \right) d\mathcal{H}^n.$$

In particular, the inequality (6.40) readily follows with a constant $C_0 = C_0(\mathbf{A}, \mathbf{R})$.

Step 2: proof of (6.41). We test now the inner variation formula (6.5) with the vector field φ defined by

$$\begin{aligned} \varphi(x) &:= \frac{d(x)}{r} \phi \left(\frac{d(x)}{r} \right) \frac{\partial}{\partial \hat{r}} \\ &= \phi \left(\frac{d(x)}{r} \right) \frac{1}{2r} \nabla(d(x)^2), \end{aligned}$$

with ϕ as above.

Standard computations lead to

$$\begin{aligned} \nabla^\Sigma \varphi(x) &= \phi' \left(\frac{d(x)}{r} \right) \frac{d(x)}{r^2} \frac{\partial}{\partial \hat{r}} \otimes \frac{\partial}{\partial \hat{r}} + \phi \left(\frac{d(x)}{r} \right) \frac{1}{2r} \text{Hess}^\Sigma(d(x)^2) \\ &= \phi' \left(\frac{d(x)}{r} \right) \frac{d(x)}{r^2} \frac{\partial}{\partial \hat{r}} \otimes \frac{\partial}{\partial \hat{r}} + \phi \left(\frac{d(x)}{r} \right) \left(\frac{\text{Id}}{r} + O(1) \right) \end{aligned}$$

for $r \rightarrow 0$, and consequently

$$\begin{aligned} \text{div}_\Sigma \varphi(x) &= \phi' \left(\frac{d(x)}{r} \right) \frac{d(x)}{r^2} + \phi \left(\frac{d(x)}{r} \right) \frac{1}{2r} \Delta_\Sigma(d(x)^2) \\ &= \phi' \left(\frac{d(x)}{r} \right) \frac{d(x)}{r^2} + \phi \left(\frac{d(x)}{r} \right) \left(\frac{n}{r} + O(1) \right). \end{aligned}$$

Choosing again tests of the form $\phi = \phi_h$ as in (6.48), plugging into (6.5) and taking the limit $h \uparrow \infty$, we see that the left-hand side of the inner variation formula reads

$$\mathbf{D}'(r) - 2\mathbf{G}(r) - \frac{n-2}{r} \mathbf{D}(r) + O(1)\mathbf{D}(r) \quad (6.50)$$

for $r \rightarrow 0$.

We proceed with the analysis of the error term $\mathcal{E}_{\text{IV}}(\varphi)$. Straightforward computations imply the following estimates:

$$\begin{aligned} |\mathcal{E}_{\text{IV}}^{(1)}| &\leq C_1 \int_{\mathbf{B}_r(p)} |N(x)| |\nabla^\perp N(x)| d\mathcal{H}^n(x), \\ |\mathcal{E}_{\text{IV}}^{(2)}| + |\mathcal{E}_{\text{IV}}^{(3)}| &\leq C_{2,3} \int_{\mathbf{B}_r(p)} |N(x)| |\nabla_{\hat{r}}^\perp N(x)| d\mathcal{H}^n(x), \end{aligned}$$

where C_1 is a geometric constant depending on $\overline{\mathbf{A}}$, and $C_{2,3}$ depends on \mathbf{A} and \mathbf{R} . Applying the Cauchy-Schwarz inequality we conclude

$$|\mathcal{E}_{\text{IV}}(\varphi)| \leq C_0 (\mathbf{D}(r)\mathbf{F}(r))^{\frac{1}{2}}. \quad (6.51)$$

Combining (6.50) and (6.51), we deduce the inequality (6.41) whenever r is small enough.

Step 3: proof of (6.42). Let $\exp_p: \mathcal{V} \subset T_p\Sigma \rightarrow \Sigma$ be the exponential map with pole p . Since $B_r(0) \Subset \mathcal{V}$ for every $r < \text{inj}(\Sigma)$, we can use the change of coordinates $x = \exp_p(y)$ to write:

$$\begin{aligned} \mathbf{H}(r) &= \int_{\partial B_r} |N|^2(\exp_p(y)) \mathbf{J} \exp_p(y) d\mathcal{H}^{n-1}(y) \\ &= r^{n-1} \int_{\partial B_1} |N|^2(\exp_p(rz)) \mathbf{J} \exp_p(rz) d\mathcal{H}^{n-1}(z). \end{aligned}$$

Thus, we differentiate under the integral sign and compute

$$\begin{aligned} \mathbf{H}'(r) &= (n-1)r^{n-2} \int_{\partial B_1} |N|^2(\exp_p(rz)) \mathbf{J} \exp_p(rz) d\mathcal{H}^{n-1}(z) \\ &\quad + 2r^{n-1} \int_{\partial B_1} \sum_{l=1}^Q \langle N^l(\exp_p(rz)), \nabla_{\hat{r}}^\perp N^l(\exp_p(rz)) \rangle \mathbf{J} \exp_p(rz) d\mathcal{H}^{n-1}(z) \\ &\quad + r^{n-1} \int_{\partial B_1} |N|^2(\exp_p(rz)) \frac{d}{dr} \left(\mathbf{J} \exp_p(rz) \right) d\mathcal{H}^{n-1}(z). \end{aligned}$$

Since $\frac{d}{dr} \left(\mathbf{J} \exp_p(rz) \right) = O(1)$ for $r \rightarrow 0$, we are able to conclude

$$\mathbf{H}'(r) = \frac{n-1}{r} \mathbf{H}(r) + 2\mathbf{E}(r) + O(1)\mathbf{H}(r), \quad (6.52)$$

from which (6.42) readily follows.

Step 4: proof of (6.43) and (6.44). It suffices to exploit the inequality

$$\mathbf{F}(r) \leq C_0 r \mathbf{H}(r) + C_0 r^2 \mathbf{D}(r), \quad (6.53)$$

which can be easily deduced from the Poincaré inequality (notice also that the same inequality has been already proved in the Euclidean setting earlier on, cf. Corollary 5.13). In the regime $\mathbf{I}(r) \geq 1$, that is $\mathbf{H}(r) \leq r\mathbf{D}(r)$, (6.53) simply reads

$$\mathbf{F}(r) \leq C_0 r^2 \mathbf{D}(r). \quad (6.54)$$

Then, (6.43) and (6.44) are an immediate consequence of (6.40) and (6.41) respectively. \square

We can now proceed with the proof of the almost monotonicity property of the frequency.

Proof of Theorem 6.7. Set $\Omega(r) := \log(\max\{\mathbf{I}(r), 1\})$. In order to prove the theorem, it suffices to show that

$$\Omega(s) \leq C + \Omega(t) \quad (6.55)$$

for some positive geometric constant C . If $\Omega(s) = 0$ there is nothing to prove. Thus, we assume that $\Omega(s) > 0$. Define

$$\tau := \sup\{r \in (s, t] : \Omega(r) > 0 \text{ on } (s, r)\}.$$

If $\tau < t$, then by continuity it must be $\Omega(\tau) = 0$: hence, in this case we would have $\Omega(\tau) = 0 \leq \Omega(t)$, and therefore proving that $\Omega(s) \leq C + \Omega(\tau)$ would imply (6.55). Thus, we can assume without loss of generality that $\Omega(r) > 0$ in (s, t) : in other words, $\mathbf{I}(r) > 1$, and $\Omega(r) = \log(\mathbf{I}(r))$. Then, as a consequence of (6.43), if r_0 is taken small enough one has

$$\frac{\mathbf{D}(r)}{2} \leq \mathbf{E}(r) \leq 2\mathbf{D}(r), \quad (6.56)$$

that is the quantity $\mathbf{E}(r)$ is positive and comparable to $\mathbf{D}(r)$ at small scales.

Guided by this principle, we compute:

$$\begin{aligned} -\frac{d}{dr}(\log \mathbf{I}(r)) &= \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} - \frac{\mathbf{D}'(r)}{\mathbf{D}(r)} - \frac{1}{r} \\ &= \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} - \frac{\mathbf{D}'(r)}{\mathbf{E}(r)} - \mathbf{D}'(r)\mathbf{Z}(r) - \frac{1}{r}, \end{aligned} \quad (6.57)$$

where $\mathbf{Z}(r) := \frac{1}{\mathbf{D}(r)} - \frac{1}{\mathbf{E}(r)}$ satisfies

$$|\mathbf{Z}(r)| = \frac{|\mathbf{D}(r) - \mathbf{E}(r)|}{\mathbf{D}(r)\mathbf{E}(r)} \stackrel{(6.56)}{\leq} 2 \frac{|\mathbf{D}(r) - \mathbf{E}(r)|}{\mathbf{D}(r)^2} \stackrel{(6.40)}{\leq} C_0 \frac{\mathbf{F}(r)}{\mathbf{D}(r)^2}. \quad (6.58)$$

Now, by (6.42) one has that

$$\frac{\mathbf{H}'(r)}{\mathbf{H}(r)} \leq C + \frac{n-1}{r} + 2 \frac{\mathbf{E}(r)}{\mathbf{H}(r)}, \quad (6.59)$$

whereas the inner variation formula (6.44) yields

$$\begin{aligned} -\frac{\mathbf{D}'(r)}{\mathbf{E}(r)} &\leq C \frac{\mathbf{D}(r)}{\mathbf{E}(r)} + Cr \frac{\mathbf{D}(r)}{\mathbf{E}(r)} - 2 \frac{\mathbf{G}(r)}{\mathbf{E}(r)} - \frac{n-2}{r} \frac{\mathbf{D}(r)}{\mathbf{E}(r)} \\ &\stackrel{(6.56)}{\leq} C + Cr - 2 \frac{\mathbf{G}(r)}{\mathbf{E}(r)} - \frac{n-2}{r} (1 - \mathbf{D}(r)\mathbf{Z}(r)) \\ &\stackrel{(6.58)}{\leq} C + Cr - 2 \frac{\mathbf{G}(r)}{\mathbf{E}(r)} - \frac{n-2}{r} + Cr^{-1} \frac{\mathbf{F}(r)}{\mathbf{D}(r)} \\ &\leq C + Cr - 2 \frac{\mathbf{G}(r)}{\mathbf{E}(r)} - \frac{n-2}{r} \end{aligned} \quad (6.60)$$

because of (6.54).

Plugging (6.59) and (6.60) into (6.57), and using the estimate on the error term $\mathbf{Z}(r)$ given by (6.58), we obtain the following:

$$-\frac{d}{dr}(\log \mathbf{I}(r)) \leq C + Cr + 2 \left(\frac{\mathbf{E}(r)}{\mathbf{H}(r)} - \frac{\mathbf{G}(r)}{\mathbf{E}(r)} \right) + C \frac{\mathbf{D}'(r)}{\mathbf{D}(r)^2} \mathbf{F}(r). \quad (6.61)$$

Now, by the Cauchy-Schwarz inequality one has

$$\mathbf{E}(r)^2 \leq \mathbf{G}(r)\mathbf{H}(r),$$

whence the factor $\frac{\mathbf{E}(r)}{\mathbf{H}(r)} - \frac{\mathbf{G}(r)}{\mathbf{E}(r)}$ is non-positive and (6.61) yields

$$-\frac{d}{dr}(\log \mathbf{I}(r)) \leq C + Cr + C \frac{\mathbf{D}'(r)}{\mathbf{D}(r)^2} \mathbf{F}(r). \quad (6.62)$$

Integrating for $r \in (s, t)$, we obtain

$$\Omega(s) - \Omega(t) \leq C + C \left(\frac{\mathbf{F}(s)}{\mathbf{D}(s)} - \frac{\mathbf{F}(t)}{\mathbf{D}(t)} \right) + C \int_s^t \frac{\mathbf{F}'(r)}{\mathbf{D}(r)} dr \leq C, \quad (6.63)$$

where the last inequality follows from the above observation that, in the regime $\mathbf{I} \geq 1$, the inequalities

$$\mathbf{F}(r) \leq C_0 r^2 \mathbf{D}(r), \quad \mathbf{F}'(r) = \mathbf{H}(r) \leq r \mathbf{D}(r)$$

hold almost everywhere. This completes the proof. \square

6.3. Some fundamental consequences of the almost monotonicity of the frequency function. The most relevant consequences of Theorem 6.7 will be collected in this section. We keep the notation of the previous paragraphs, but many of the results here contained will be derived under the additional assumption that $p \in \Omega$ has been fixed in such a way that $N(p) = Q[[0]]$: the arguments contained in the next sections will illustrate the reason why an analysis of the Jacobi multi-field N in a neighborhood of such a point is crucial in order to obtain the proof of Theorem 0.7. It is useful to recall now that, at this stage, we are lacking a precise description of the behavior of the frequency function *precisely* at those points p such that $N(p) = Q[[0]]$ (see also Remark 6.6 above). Our first goal now is to fill this gap.

In order to do so, we will need the following version of the Poincaré inequality.

Lemma 6.11. *There exist a radius $0 < r_0 = r_0(n, Q) < \text{inj}(\Sigma)$ and a geometric constant $C > 0$ with the following property. Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be a multiple valued section of $\mathcal{N}\Sigma$ Jac-minimizing in Ω . Assume $p \in \Omega$ is such that $N(p) = Q[[0]]$. Then, the inequality*

$$\|N\|_{L^2(\mathbf{B}_r(p))}^2 \leq Cr^2 \text{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p)) \quad (6.64)$$

holds true for every $0 < r \leq \min\{r_0, \text{dist}(p, \partial\Omega)\}$.

Proof. Let $r_0 = r_0(n, Q)$ be the radius given by Theorem 5.1, and let $r \leq \min\{r_0, \text{dist}(p, \partial\Omega)\}$ be arbitrary. Let $\rho \in (0, \frac{r}{2}]$ be a radius to be chosen later and split $\|N\|_{L^2(\mathbf{B}_r(p))}^2$ into the sum

$$\int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n = \int_{\mathbf{B}_\rho(p)} |N|^2 d\mathcal{H}^n + \int_{\mathbf{B}_r(p) \setminus \mathbf{B}_\rho(p)} |N|^2 d\mathcal{H}^n. \quad (6.65)$$

In order to estimate the first term in the sum, we recall that $|N|^2(x) = \mathcal{G}(N(x), Q[[0]])^2 = \mathcal{G}(N(x), N(p))^2$ and exploit the α -Hölder continuity of N to conclude

$$\begin{aligned} \int_{\mathbf{B}_\rho(p)} |N|^2 d\mathcal{H}^n &\leq \rho^{2\alpha} [N]_{C^{0,\alpha}(\overline{\mathbf{B}_\rho(p)})}^2 \mathcal{H}^n(\mathbf{B}_\rho(p)) \\ &\stackrel{(5.1)}{\leq} C\rho^2 \left(\text{Dir}(N, \mathbf{B}_{2\rho}(p)) + \Lambda \int_{\mathbf{B}_{2\rho}(p)} |N|^2 d\mathcal{H}^n \right) \\ &\leq C\rho^2 \text{Dir}(N, \mathbf{B}_r(p)) + C\Lambda\rho^2 \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n \\ &\leq Cr^2 \text{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p)) + C(\Lambda + C_0)\rho^2 \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n =: I_1 + I_2, \end{aligned} \quad (6.66)$$

where C_0 depends on \mathbf{A} and $\overline{\mathbf{A}}$.

As for the second addendum in (6.65), we integrate by normal polar coordinates with pole p to write

$$\int_{\mathbf{B}_r(p) \setminus \mathbf{B}_\rho(p)} |N|^2 d\mathcal{H}^n = \int_\rho^r \left(\int_{\partial \mathbf{B}_\tau(p)} |N|^2 d\mathcal{H}^{n-1} \right) d\tau. \quad (6.67)$$

Now, fix any $\tau \in (\rho, r)$, and for every $x \in \partial \mathbf{B}_\tau(p)$ let $\gamma_x = \gamma_x(s)$, $s \in [0, \tau]$, be the unique geodesic parametrized by arclength joining p to x . Also denote by \bar{x} the point where γ_x intersects $\partial \mathbf{B}_\rho(p)$. Then, the fundamental theorem of calculus immediately yields

$$|N|^2(x) \leq |N|^2(\bar{x}) + 2 \int_\rho^\tau (|N| |\nabla^\perp N|)(\gamma_x(s)) ds. \quad (6.68)$$

Integrate the above inequality in $x \in \partial \mathbf{B}_\tau(p)$ to get

$$\int_{\partial \mathbf{B}_\tau(p)} |N|^2 d\mathcal{H}^{n-1} \leq C \left(\frac{\tau}{\rho} \right)^{n-1} \left(\int_{\partial \mathbf{B}_\rho(p)} |N|^2 d\mathcal{H}^{n-1} + 2 \int_{\mathbf{B}_\tau(p) \setminus \mathbf{B}_\rho(p)} |N| |\nabla^\perp N| d\mathcal{H}^n \right). \quad (6.69)$$

Using once again the Hölder estimate (5.1), we are able to control

$$\begin{aligned} \left(\frac{\tau}{\rho} \right)^{n-1} \int_{\partial \mathbf{B}_\tau(p)} |N|^2 d\mathcal{H}^{n-1} &\leq \left(\frac{\tau}{\rho} \right)^{n-1} \rho^{2\alpha} [N]_{C^{0,\alpha}(\overline{\mathbf{B}}_{\frac{\tau}{2}}(p))}^2 \mathcal{H}^{n-1}(\partial \mathbf{B}_\rho(p)) \\ &\leq C \tau^{n-1} \rho^{2\alpha} r^{2-n-2\alpha} \left(\text{Dir}(N, \mathbf{B}_r(p)) + \Lambda \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n \right). \end{aligned} \quad (6.70)$$

We can now integrate in $\tau \in (\rho, r)$, so that using the estimates in (6.69) and (6.70) we can easily deduce from (6.67) the following inequality:

$$\begin{aligned} \int_{\mathbf{B}_r(p) \setminus \mathbf{B}_\rho(p)} |N|^2 d\mathcal{H}^n &\leq C \rho^{2\alpha} r^{2-2\alpha} \text{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p)) \\ &\quad + C(\Lambda + C_0) \rho^{2\alpha} \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n \\ &\quad + C \left(\frac{r}{\rho} \right)^{n-1} r \int_{\mathbf{B}_r(p)} |N| |\nabla^\perp N| d\mathcal{H}^n =: J_1 + J_2 + J_3, \end{aligned} \quad (6.71)$$

where C and C_0 are geometric constants. Now we can sum up the contributions coming from the ball $\mathbf{B}_\rho(p)$ and from the annulus $\mathbf{B}_r(p) \setminus \mathbf{B}_\rho(p)$ and choose $\rho = \rho(\Lambda, C, C_0)$ so small that the terms J_2 and J_3 are absorbed in the left-hand side of the equation, thus ultimately providing

$$\int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n \leq C r^2 \text{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p)) + C r \int_{\mathbf{B}_r(p)} |N| |\nabla^\perp N| d\mathcal{H}^n. \quad (6.72)$$

Finally, use Young's inequality: for any choice of the parameter $\eta > 0$, (6.72) implies that

$$\int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n \leq C r^2 \text{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p)) + C r \left(\eta \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n + \frac{1}{\eta} \text{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p)) \right). \quad (6.73)$$

The conclusion immediately follows by choosing η such that $C r \eta = \frac{1}{2}$. \square

The first important result stemming from the almost monotonicity property of the frequency function is the following dichotomy: if $N(p) = Q[[0]]$, then either there exists a neighborhood of p where the map N is identically vanishing, and thus where the frequency function is not defined at all, or, conversely, there is a neighborhood of p where the frequency function is well defined everywhere and bounded.

Proposition 6.12. *Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be Jac-minimizing. Assume $p \in \Omega$ is such that $N(p) = Q[[0]]$. Then, the following dichotomy holds:*

- (i) *either $N \equiv Q[[0]]$ in a neighborhood of p ;*
- (ii) *or there exists a radius $r_0 > 0$ such that*

$$\mathbf{H}(r) > 0 \text{ for all } r \in (0, r_0] \quad \text{and} \quad \limsup_{r \rightarrow 0} \mathbf{I}(r) < \infty.$$

Proof. First observe that if N does not vanish identically in a neighborhood of p , then there exists $r_0 > 0$ such that $\mathbf{H}(r_0) > 0$. Clearly, without loss of generality we can suppose that (6.64) holds for every $0 < r \leq r_0$, and also that (6.35) holds in any interval $[s, t] \subset (0, r_0]$ such that $\mathbf{H}|_{[s,t]} > 0$. We claim that in fact $\mathbf{H}(r) > 0$ for all $0 < r \leq r_0$. Indeed, if this is not true, let $\rho > 0$ be given by $\rho := \sup\{r \in (0, r_0] : \mathbf{H}(r) = 0\}$. By definition $\mathbf{H}(r) > 0$ for $\rho < r \leq r_0$, whence for such r 's we can take advantage of Theorem 6.7 and write

$$\mathbf{I}(r) \leq C_0(1 + \mathbf{I}(r_0)).$$

By letting $r \downarrow \rho$ we conclude

$$\rho \mathbf{D}(\rho) \leq C_0(1 + \mathbf{I}(r_0))\mathbf{H}(\rho) = 0,$$

which in turn produces $\text{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_\rho(p)) = 0$. Then, by Lemma 6.11, N vanishes identically in $\mathbf{B}_\rho(p)$, contradiction.

It is now a simple consequence of Theorem 6.7 that

$$\limsup_{r \rightarrow 0} \mathbf{I}(r) \leq C_0(1 + \mathbf{I}(r_0)),$$

which completes the proof. □

As we shall see in the coming paragraphs, the most interesting situation is when condition (ii) in the above Proposition 6.12 is observed. As a first remark, we observe that the fact that the frequency function is bounded in a neighborhood of a point p allows to improve the almost monotonicity property itself.

Proposition 6.13 (Improved almost monotonicity of the frequency). *Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be Jac-minimizing. Assume $p \in \Omega$ is such that $N(p) = Q[[0]]$ but N does not vanish in a neighborhood of p . Then, there exist $r_0 > 0$ and a constant $\lambda = \lambda(\mathbf{I}(r_0)) > 0$ such that the function*

$$r \in (0, r_0] \mapsto e^{\lambda r} \mathbf{I}(r) \tag{6.74}$$

is monotone non-decreasing. The limit

$$\lim_{r \rightarrow 0} \mathbf{I}(r) =: I_0(p) \tag{6.75}$$

exists and is strictly positive.

Proof. Under the assumptions in the statement, case (ii) in Proposition 6.12 must hold, and thus the frequency function is well defined and bounded in an interval $(0, r_0]$. Moreover, the Poincaré inequality (6.64) implies that, modulo possibly taking a smaller value of r_0 , the first variation estimates of Lemma 6.10 can be again re-written as in (6.43) and (6.44), and that (6.56) holds. Thus, we can compute:

$$\begin{aligned} \mathbf{I}'(r) &= \frac{\mathbf{D}(r)}{\mathbf{H}(r)} + \frac{r\mathbf{D}'(r)}{\mathbf{H}(r)} - \frac{r\mathbf{D}(r)\mathbf{H}'(r)}{\mathbf{H}(r)^2} \\ &= \frac{\mathbf{D}(r)}{\mathbf{H}(r)} + \frac{r}{\mathbf{H}(r)} \left(2\mathbf{G}(r) + \frac{n-2}{r}\mathbf{D}(r) + \mathcal{E}_1(r) \right) - \frac{r\mathbf{D}(r)}{\mathbf{H}(r)^2} \left(\frac{n-1}{r}\mathbf{H}(r) + 2\mathbf{E}(r) + \mathcal{E}_2(r) \right) \\ &= \frac{2r}{\mathbf{H}(r)^2} \left(\mathbf{G}(r)\mathbf{H}(r) - \mathbf{E}(r)^2 \right) + \frac{r}{\mathbf{H}(r)} \mathcal{E}_1(r) - \frac{r\mathbf{D}(r)}{\mathbf{H}(r)^2} \mathcal{E}_2(r) + \mathcal{E}_3(r), \end{aligned} \quad (6.76)$$

where

$$|\mathcal{E}_1(r)| \stackrel{(6.44)}{\leq} C_0(1+r)\mathbf{D}(r), \quad (6.77)$$

$$|\mathcal{E}_2(r)| \stackrel{(6.42)}{\leq} C_0\mathbf{H}(r), \quad (6.78)$$

and

$$|\mathcal{E}_3(r)| = \frac{2r\mathbf{E}(r)}{\mathbf{H}(r)^2} |\mathbf{E}(r) - \mathbf{D}(r)| \stackrel{(6.43), (6.56)}{\leq} C_0 \frac{r^3\mathbf{D}(r)^2}{\mathbf{H}(r)^2}, \quad (6.79)$$

if r_0 is chosen small enough. Since $\mathbf{G}(r)\mathbf{H}(r) - \mathbf{E}(r)^2 \geq 0$ by the Cauchy-Schwartz inequality, the above arguments show the existence of a radius $r_0 > 0$ and a geometric constant $C_0 > 0$ such that

$$\mathbf{I}'(r) \geq -C_0(1+r)\mathbf{I}(r) - C_0r\mathbf{I}(r)^2 \quad (6.80)$$

for all $r \in (0, r_0]$. On the other hand, for such r 's one has $\mathbf{I}(r) \leq C_0(1 + \mathbf{I}(r_0))$ by Theorem 6.7. Thus, this allows to conclude that

$$\mathbf{I}'(r) \geq -\lambda\mathbf{I}(r), \quad (6.81)$$

for some positive λ depending only on r_0 and $\mathbf{I}(r_0)$. The monotonicity of the function $r \mapsto e^{\lambda r}\mathbf{I}(r)$ is now a simple consequence of (6.81).

Next, we conclude the proof showing that the limit

$$I_0 := \lim_{r \rightarrow 0} e^{\lambda r}\mathbf{I}(r) = \lim_{r \rightarrow 0} \mathbf{I}(r) \quad (6.82)$$

is positive. To see this, we show that the Poincaré inequality (6.64) allows to bound the frequency function from below with a positive constant. Indeed, arguing as in the proof of Lemma 6.11 (cf. in particular the equations (6.69) and (6.70)), it is easily seen that one can estimate

$$\begin{aligned} \mathbf{H}(r) &= \int_{\partial\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^{n-1} \leq C \left(\int_{\partial\mathbf{B}_{\frac{r}{2}}(p)} |N|^2 d\mathcal{H}^{n-1} + \int_{\mathbf{B}_r(p) \setminus \mathbf{B}_{\frac{r}{2}}(p)} |N| |\nabla^\perp N| d\mathcal{H}^n \right) \\ &\leq Cr \left(\text{Dir}(N, \mathbf{B}_r(p)) + \Lambda \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n \right) + C \int_{\mathbf{B}_r(p)} |N| |\nabla^\perp N| d\mathcal{H}^n \\ &\leq Cr\mathbf{D}(r) + Cr\mathbf{F}(r) + C \int_{\mathbf{B}_r(p)} |N| |\nabla^\perp N| d\mathcal{H}^n. \end{aligned} \quad (6.83)$$

In turn, applying Young's inequality to the last addendum in the right-hand side of (6.83) yields

$$\int_{\mathbf{B}_r(p)} |N| |\nabla^\perp N| d\mathcal{H}^n \leq \frac{r}{2} \mathbf{D}(r) + \frac{1}{2r} \mathbf{F}(r). \quad (6.84)$$

Plugging (6.84) in (6.83) and using the Poincaré inequality (6.64) finally gives

$$\mathbf{H}(r) \leq C(1 + r^2)r\mathbf{D}(r) \leq C(1 + r_0^2)r\mathbf{D}(r), \quad (6.85)$$

thus completing the proof. \square

In Proposition 6.4 we have shown that Jac-minimizers enjoy a Caccioppoli type inequality: the L^2 norm of a Jacobi Q -field N in a ball $\mathbf{B}_r(p)$ controls the Dirichlet energy in the ball with half the radius. As an immediate consequence of the boundedness of the frequency function, one can actually show that the Dirichlet energy in $\mathbf{B}_{\frac{r}{2}}(p)$ can be controlled with the L^2 norm of N in the annulus $\mathbf{B}_r(p) \setminus \mathbf{B}_{\frac{r}{2}}(p)$, provided that we allow the constant to depend on the value of the frequency at a given scale.

Proposition 6.14. *Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be Jac-minimizing. Then, there exists $r_0 > 0$ such that for any $r \in (0, r_0]$ one has*

$$\text{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_{\frac{r}{2}}(p)) \leq \frac{C}{r^2} \int_{\mathbf{B}_r(p) \setminus \mathbf{B}_{\frac{r}{2}}(p)} |N|^2 d\mathcal{H}^n \quad (6.86)$$

for some positive $C = C(\mathbf{I}(r_0))$.

Proof. If N is vanishing identically in a neighborhood of p there is nothing to prove. Therefore, we can assume that either $N(p) \neq Q[[0]]$ or $N(p) = Q[[0]]$ but N does not vanish identically in any neighborhood of p . In any of the two cases, there exists a positive radius r_0 such that the frequency function is well defined and bounded for all $r \in (0, r_0]$. Thus, there exists a positive constant $C = C(\mathbf{I}(r_0))$ such that, for fixed $r \leq r_0$, $\tau\mathbf{D}(\tau) \leq C\mathbf{H}(\tau)$ for τ in the interval $[\frac{r}{2}, r]$. Integrate with respect to τ to get (6.86):

$$\frac{3}{8}r^2 \text{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_{\frac{r}{2}}(p)) = \frac{3}{8}r^2 \mathbf{D}\left(\frac{r}{2}\right) \leq \int_{\frac{r}{2}}^r \tau \mathbf{D}(\tau) d\tau \leq C \int_{\frac{r}{2}}^r \mathbf{H}(\tau) d\tau = C \int_{\mathbf{B}_r(p) \setminus \mathbf{B}_{\frac{r}{2}}(p)} |N|^2 d\mathcal{H}^n. \quad \square$$

As shown in the next Proposition, the Caccioppoli inequality can be improved further under the assumption that $N(p) = Q[[0]]$: indeed, at small scales the inequality (6.29) holds without having to increase the support of the ball on the right-hand side. Again, for this to be true we need to allow the constant to depend on the value of the frequency at scale r_0 .

Proposition 6.15 (Reverse Poincaré Inequality). *Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be Jac-minimizing. Assume $N(p) = Q[[0]]$. Then, there exists $r_0 > 0$ such that for any $r \in (0, r_0]$ the following inequality*

$$\text{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p)) \leq \frac{C}{r^2} \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n \quad (6.87)$$

holds for some positive $C = C(\mathbf{I}(r_0))$.

Proof. Once again, we observe that (6.87) is trivial when $N \equiv Q[[0]]$ in a neighborhood of p . We assume then that case (ii) in Proposition 6.12 holds, and we let r_0 be the radius given in there. Since the frequency function is well defined and bounded in $(0, r_0]$, there exists $C = C(\mathbf{I}(r_0)) > 0$ such that

$$\int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2 d\mathcal{H}^n \leq \frac{C}{r} \int_{\partial\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^{n-1}, \quad (6.88)$$

for all r 's in the above interval. Arguing once again as in the proof of Lemma 6.11, we have that for every $\rho \in (0, \frac{r}{2}]$ it holds

$$\int_{\partial\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^{n-1} \leq C \left(\frac{r}{\rho}\right)^{n-1} \left(\int_{\partial\mathbf{B}_\rho(p)} |N|^2 d\mathcal{H}^{n-1} + 2 \int_{\mathbf{B}_r(p) \setminus \mathbf{B}_\rho(p)} |N| |\nabla^\perp N| d\mathcal{H}^n \right). \quad (6.89)$$

Furthermore, by the Hölder continuity of N we also have

$$\left(\frac{r}{\rho}\right)^{n-1} \int_{\partial\mathbf{B}_\rho(p)} |N|^2 d\mathcal{H}^{n-1} \stackrel{(6.64)}{\leq} C \left(\frac{\rho}{r}\right)^{2\alpha} r \left(\int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2 d\mathcal{H}^n + (C_0 + \Lambda) \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n \right). \quad (6.90)$$

Combining (6.88), (6.89) and (6.90) gives

$$\begin{aligned} \int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2 d\mathcal{H}^n &\leq C \left(\frac{\rho}{r}\right)^{2\alpha} \left(\int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2 d\mathcal{H}^n + \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n \right) \\ &\quad + \frac{C}{r} \left(\frac{r}{\rho}\right)^{n-1} \int_{\mathbf{B}_r(p)} |N| |\nabla^\perp N| d\mathcal{H}^n. \end{aligned} \quad (6.91)$$

Now, if we choose ρ so small that $C \left(\frac{\rho}{r}\right)^{2\alpha} \leq \frac{1}{2}$ then from (6.91) follows

$$\begin{aligned} \int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2 d\mathcal{H}^n &\leq \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n + \frac{C}{r} \int_{\mathbf{B}_r(p)} |N| |\nabla^\perp N| d\mathcal{H}^n \\ &\leq \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n + \frac{C}{2r} \eta \int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2 d\mathcal{H}^n + \frac{C}{2r\eta} \int_{\mathbf{B}_r(p)} |N|^2 d\mathcal{H}^n, \end{aligned} \quad (6.92)$$

by the Young's inequality. Choose $\eta = \frac{r}{C}$ to obtain

$$\int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2 d\mathcal{H}^n \leq \left(\frac{C}{r^2} + 2\right) \int_{\mathbf{B}_r(p)} |N|^2, \quad (6.93)$$

which immediately implies (6.87). \square

6.4. The top-multiplicity singular stratum. Blow-up. The final goal of this section is to perform the key step in the proof of Theorem 0.7, namely the blow-up procedure, see Theorem 6.21 below. In doing this, we will clarify the importance of the results obtained in the previous paragraph.

The main difficulty in the proof of Theorem 0.7 consists of estimating the Hausdorff dimension of the set of singular points with multiplicity exactly equal to Q . The proof of the general result then follows in a relatively easy way by an induction argument on Q . Therefore, it is fundamental to study the structure of the *top-multiplicity singular stratum* of N , denoted $\text{sing}_Q(N)$ and defined as follows.

Definition 6.16 (Top-multiplicity points). Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be Jac-minimizing. A point $p \in \Omega$ has multiplicity Q , or simply is a Q -point for N , and we will write $p \in D_Q(N)$, if there exists $v \in T_p^\perp \Sigma$ such that $N(p) = Q\llbracket v \rrbracket$. We will define the top-multiplicity regular and singular strata of N by

$$\text{reg}_Q(N) := \text{reg}(N) \cap D_Q(N), \quad \text{sing}_Q(N) := \text{sing}(N) \cap D_Q(N),$$

respectively.

From this point onward, we will assume to have fixed a point $p \in D_Q(N)$. The first step is to show that without loss of generality we can always assume that $N(p) = Q\llbracket 0 \rrbracket$. To see this, we introduce, following [DLS11], the map $\boldsymbol{\eta}: \mathcal{A}_Q(\mathbb{R}^M) \rightarrow \mathbb{R}^M$ which takes each measure $T = \sum_l \llbracket P_l \rrbracket$ to its center of mass:

$$\boldsymbol{\eta}(T) := \frac{1}{Q} \sum_{l=1}^Q P_l. \quad (6.94)$$

Lemma 6.17. *Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be Jac-minimizing. Then:*

- (i) *the map $\boldsymbol{\eta} \circ N: \Omega \rightarrow \mathbb{R}^M$ is a classical Jacobi field;*
- (ii) *if $\zeta: \Omega \rightarrow \mathbb{R}^M$ is a classical Jacobi field, then the Q -valued map $u := \sum_l \llbracket N^l + \zeta \rrbracket$ is Jac-minimizing.*

Proof. Recall (cf. Section 3 and the notation therein) that a normal vector field $\zeta \in \Gamma^{1,2}(\mathcal{N}\Omega) := \Gamma_1^{1,2}(\mathcal{N}\Sigma)$ is a Jacobi field if it is a weak solution of the linear elliptic PDE on the normal bundle $\mathcal{N}\Sigma$

$$\left(-\Delta_\Sigma^\perp - \mathcal{A} - \mathcal{R}\right) \zeta = 0,$$

that is, if the identity

$$\int_\Omega \left(\langle \nabla^\perp \zeta : \nabla^\perp \phi \rangle - \langle A \cdot \zeta : A \cdot \phi \rangle - \mathcal{R}(\zeta, \phi) \right) d\mathcal{H}^n = 0 \quad (6.95)$$

holds for all test functions $\phi \in C^1(\Omega; \mathbb{R}^M)$ with $\text{supp}(\phi) \subset \Omega' \Subset \Omega$ and $\phi(x) \in T_x^\perp \Sigma \subset T_x \mathcal{M}$ for every $x \in \Omega$.

In order to prove (i), first observe that the map $\boldsymbol{\eta}$ preserves the fibers of the normal bundle, so that $\boldsymbol{\eta} \circ N(x) \in T_x^\perp \Sigma$ for a.e. $x \in \Omega$ and thus $\boldsymbol{\eta} \circ N \in \Gamma^{1,2}(\mathcal{N}\Omega) = \Gamma_1^{1,2}(\mathcal{N}\Omega)$. Now, fix any vector field ϕ as above. It is immediate to see that we can test the outer variation formula (6.3) with $\psi(x, u) := \phi(x)$, and that the resulting equation is precisely

$$\int_\Omega \left(\langle \nabla^\perp(\boldsymbol{\eta} \circ N) : \nabla^\perp \phi \rangle - \langle A \cdot (\boldsymbol{\eta} \circ N) : A \cdot \phi \rangle - \mathcal{R}(\boldsymbol{\eta} \circ N, \phi) \right) d\mathcal{H}^n = 0,$$

that is $\boldsymbol{\eta} \circ N$ solves (6.95) and the proof of (i) is complete.

In order to prove (ii), we take any $h \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ such that $h|_{\partial\Omega} = N|_{\partial\Omega}$ and we show that

$$\text{Jac}(u, \Omega) \leq \text{Jac}(\tilde{h}, \Omega),$$

with $\tilde{h} = \sum_l \llbracket h + \zeta \rrbracket$. We compute:

$$\begin{aligned} \text{Jac}(u, \Omega) &= \int_{\Omega} \sum_{l=1}^Q \left(|\nabla^{\perp}(N^l + \zeta)|^2 - |A \cdot (N^l + \zeta)|^2 - \mathcal{R}(N^l + \zeta, N^l + \zeta) \right) d\mathcal{H}^n \\ &= \text{Jac}(N, \Omega) + Q \left(\int_{\Omega} (|\nabla^{\perp}\zeta|^2 - |A \cdot \zeta|^2 - \mathcal{R}(\zeta, \zeta)) d\mathcal{H}^n \right) \\ &\quad + 2Q \left(\int_{\Omega} (\langle \nabla^{\perp}(\boldsymbol{\eta} \circ N) : \nabla^{\perp}\zeta \rangle - \langle A \cdot (\boldsymbol{\eta} \circ N) : A \cdot \zeta \rangle - \mathcal{R}(\boldsymbol{\eta} \circ N, \zeta)) d\mathcal{H}^n \right). \end{aligned}$$

Using that $\text{Jac}(N, \Omega) \leq \text{Jac}(h, \Omega)$ and recalling the definition of \tilde{h} , we see that

$$\begin{aligned} \text{Jac}(u, \Omega) - \text{Jac}(\tilde{h}, \Omega) &\leq 2Q \int_{\Omega} \langle \nabla^{\perp}(\boldsymbol{\eta} \circ N - \boldsymbol{\eta} \circ h) : \nabla^{\perp}\zeta \rangle \\ &\quad - 2Q \int_{\Omega} \langle A \cdot (\boldsymbol{\eta} \circ N - \boldsymbol{\eta} \circ h) : A \cdot \zeta \rangle \\ &\quad - 2Q \int_{\Omega} \mathcal{R}(\boldsymbol{\eta} \circ N - \boldsymbol{\eta} \circ h, \zeta) = 0, \end{aligned}$$

because ζ is a Jacobi field and the function $\phi = \boldsymbol{\eta} \circ N - \boldsymbol{\eta} \circ h$ is a $W^{1,2}$ section of the normal bundle vanishing at $\partial\Omega$. This completes the proof. \square

Remark 6.18. As a simple corollary of Lemma 6.17, if N is Jac-minimizing then the Q -valued map $\tilde{N} = \sum_l \llbracket N^l - \boldsymbol{\eta} \circ N \rrbracket$ is a Jac-minimizer with $\boldsymbol{\eta} \circ \tilde{N} \equiv 0$ and the same singular set as N . Therefore, there is no loss of generality in assuming that $\boldsymbol{\eta} \circ N \equiv 0$, and, thus, that every Q -point p satisfies $N(p) = Q[0]$. In particular, all the results of the previous paragraph hold for $p \in D_Q(N)$. Furthermore, the content of Proposition 6.12 becomes more apparent in this context. Indeed, the dichotomy stated in there discriminates perfectly between regular and singular top-multiplicity points: $p \in \text{reg}_Q(N)$ if and only if the condition (i) is observed; on the other hand, $p \in \text{sing}_Q(N)$ if and only if the frequency function is well defined and bounded in a neighborhood of p .

In view of the above remark, we assume from this point onward that N is Jac-minimizing and such that $\boldsymbol{\eta} \circ N = 0$. We fix a point $p \in \text{sing}_Q(N)$, and an orthonormal basis $(e_1, \dots, e_n, e_{n+1}, \dots, e_{n+k}, e_{n+k+1}, \dots, e_M)$ of the euclidean space \mathbb{R}^M with the property that $T_p\Sigma = \text{span}(e_1, \dots, e_n)$ and $T_p^{\perp}\Sigma = \text{span}(e_{n+1}, \dots, e_{n+k})$. Choose local orthonormal frames $(\xi_i)_{i=1}^n$ and $(\nu_{\alpha})_{\alpha=1}^k$ of $\mathcal{T}\Sigma$ and $\mathcal{N}\Sigma$ respectively which extend the basis at p , that is, such that $\xi_i(p) = e_i$ for $i = 1, \dots, n$ and $\nu_{\alpha}(p) = e_{n+\alpha}$ for $\alpha = 1, \dots, k$. With a slight abuse of notation, we will sometimes denote the linear subspace $\mathbb{R}^n \times \{0\} \times \{0\}$ by \mathbb{R}^n and $\{0\} \times \mathbb{R}^k \times \{0\}$ by \mathbb{R}^k .

Let $r_0 > 0$ be such that all the conclusions from the previous paragraphs hold. For every $r \in (0, r_0]$, translate and rescale the manifolds \mathcal{M} and Σ , setting

$$\mathcal{M}_r := \frac{\mathcal{M} - p}{r}, \quad \Sigma_r := \frac{\Sigma - p}{r},$$

that is $\mathcal{M}_r = \iota_{p,r}(\mathcal{M})$ and $\Sigma_r = \iota_{p,r}(\Sigma)$, where

$$\iota_{p,r}(x) := \frac{x - p}{r}.$$

The manifolds \mathcal{M}_r and Σ_r will be regarded as Riemannian submanifolds of \mathbb{R}^M with the induced euclidean metric. We will let

$$\mathbf{ex}_r: B_1 \subset T_0\Sigma_r \simeq \mathbb{R}^n \rightarrow \Sigma_r$$

be the exponential map, and we will use the symbol $\psi_{p,r}$ to denote the map

$$\psi_{p,r} := \iota_{p,r}^{-1} \circ \mathbf{ex}_r.$$

Observe that $\psi_{p,r}$ maps the euclidean ball $B_1(0) \subset \mathbb{R}^n$ diffeomorphically onto the geodesic ball $\mathbf{B}_r(p) \subset \Sigma$.

Remark 6.19. Observe that, since $T_0\mathcal{M}_r = T_p\mathcal{M} = \text{span}(e_1, \dots, e_{n+k})$ for every r , the ambient manifolds \mathcal{M}_r converge, as $r \downarrow 0$, to $\mathbb{R}^{n+k} \times \{0\}$ in $C^{3,\beta}$. For the same reason, the Σ_r 's converge to $\mathbb{R}^n \times \{0\} \times \{0\}$ in $C^{3,\beta}$ and the exponential maps \mathbf{ex}_r converge in $C^{2,\beta}$ to the identity map of the ball $B_1 \subset \mathbb{R}^n$ (cf. [DLS16b, Proposition A.4]).

Definition 6.20. We define the *blow-ups* of N at p as the one-parameter family of maps $N_{p,r}: B_1 \subset T_0\Sigma_r \rightarrow \mathcal{A}_Q(\mathbb{R}^M)$ indexed by $r \in (0, r_0]$ and given by

$$N_{p,r}(y) := \frac{r^{\frac{n}{2}} N(\psi_{p,r}(y))}{\|N\|_{L^2(\mathbf{B}_r(p))}} = \frac{r^{\frac{n}{2}} N(p + r\mathbf{ex}_r(y))}{\|N\|_{L^2(\mathbf{B}_r(p))}}. \quad (6.96)$$

Observe that the maps $N_{p,r}$ are well defined because N is not vanishing in any ball $\mathbf{B}_r(p)$ with $0 < r \leq r_0$.

The next theorem is the anticipated convergence result for the blow-up maps.

Theorem 6.21. *Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be Jac-minimizing with $\boldsymbol{\eta} \circ N = 0$. Assume $p \in \text{sing}_Q(N)$. Then, for any sequence N_{p,r_j} with $r_j \downarrow 0$, a subsequence, not relabeled, converges weakly in $W^{1,2}$, strongly in L^2 and locally uniformly to a continuous Q -valued function $\mathcal{N}_p: B_1 \subset \mathbb{R}^n \rightarrow \mathcal{A}_Q(\mathbb{R}^k)$ such that:*

- (a) $\mathcal{N}_p(0) = Q[[0]]$ and $\boldsymbol{\eta} \circ \mathcal{N}_p \equiv 0$, but $\|\mathcal{N}_p\|_{L^2(B_1)} = 1$, and thus, in particular, \mathcal{N}_p is non-trivial;
- (b) \mathcal{N}_p is locally Dir-minimizing in B_1 ;
- (c) \mathcal{N}_p is μ -homogeneous, with $\mu = I_0(p)$, the frequency of N at p defined in (6.75).

Remark 6.22. Any map \mathcal{N}_p which is the limit of a blow-up sequence N_{p,r_j} in the sense specified above will be called a *tangent map* to N at p .

Proof. Let N and p be as in the statement. For any sequence $r_j \downarrow 0$, let us denote $\mathcal{M}_j := \mathcal{M}_{r_j}$, $\Sigma_j := \Sigma_{r_j}$, $\mathbf{ex}_j := \mathbf{ex}_{r_j}$, $\psi_j := \psi_{p,r_j}$ and $N_j := N_{p,r_j}$ in order to simplify the notation. We will divide the proof into steps.

Step 1: boundedness in $W^{1,2}$. Assume for the moment that $j \in \mathbb{N}$ is fixed. We start estimating $\|N_j\|_{L^2(B_1)}$. Changing coordinates $x = \psi_j(y)$ in the integral, we compute explicitly

$$\begin{aligned} \|N\|_{L^2(\mathbf{B}_{r_j}(p))}^2 &= \int_{\mathbf{B}_{r_j}(p)} |N(x)|^2 d\mathcal{H}^n(x) \\ &= \|N\|_{L^2(\mathbf{B}_{r_j}(p))}^2 \int_{B_1} |N_j(y)|^2 \mathbf{Jex}_j(y) dy, \end{aligned} \quad (6.97)$$

and thus

$$\int_{B_1} |N_j(y)|^2 \mathbf{Jex}_j(y) dy = 1 \quad (6.98)$$

for every j . By the considerations in Remark 6.19, we can deduce that necessarily

$$\frac{1}{2} \leq \|N_j\|_{L^2(B_1)}^2 \leq 2 \quad (6.99)$$

when j is large enough.

Next, we bound the Dirichlet energy of the blow-up maps in B_1 . For any $y \in B_1 \subset T_0\Sigma_j$, and for all $i = 1, \dots, n$, let $\varepsilon_i = \varepsilon_i(y)$ be the vector in $T_0\Sigma_j$ defined by

$$d(\mathbf{ex}_j)|_y \cdot \varepsilon_i(y) = \xi_i(p + r_j \mathbf{ex}_j(y)).$$

We notice that, when $j \uparrow \infty$, ε_i converges to $e_i = \xi_i(p)$ uniformly in B_1 .

Again by changing variable $x = \psi_j(y)$ in the integral, we compute:

$$\begin{aligned} \text{Dir}(N, \mathbf{B}_{r_j}(p)) &= \int_{\mathbf{B}_{r_j}(p)} \sum_{i=1}^n |D_{\varepsilon_i} N(x)|^2 d\mathcal{H}^n(x) \\ &= \frac{\|N\|_{L^2(\mathbf{B}_{r_j}(p))}^2}{r_j^2} \int_{B_1} \sum_{i=1}^n |D_{\varepsilon_i} N_j(y)|^2 \mathbf{Jex}_j(y) dy. \end{aligned} \quad (6.100)$$

On the other hand, using that N takes values in the normal bundle, we have the usual estimate

$$\text{Dir}(N, \mathbf{B}_{r_j}(p)) \leq \text{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_{r_j}(p)) + C_0 \|N\|_{L^2(\mathbf{B}_{r_j}(p))}^2, \quad (6.101)$$

for some positive geometric constant $C_0 = C_0(\mathbf{A}, \overline{\mathbf{A}})$. From (6.100) and (6.101) we conclude that for any j

$$\int_{B_1} \sum_{i=1}^n |D_{\varepsilon_i} N_j(y)|^2 \mathbf{Jex}_j(y) dy \leq \frac{r_j^2 \text{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_{r_j}(p))}{\|N\|_{L^2(\mathbf{B}_{r_j}(p))}^2} + C_0 r_j^2 \leq C(1 + r_j^2), \quad (6.102)$$

because of the reverse Poincaré inequality (6.87). Thus, we conclude that the Dirichlet energy of the blow-up maps in B_1 is bounded:

$$\text{Dir}(N_j, B_1) := \int_{B_1} \sum_{i=1}^n |D_{e_i} N_j|^2 dy \leq C. \quad (6.103)$$

Step 2: convergence. The $W^{1,2}$ bounds given by estimates (6.99) and (6.103), together with Proposition 1.19, clearly imply the $W^{1,2}$ -weak and L^2 -strong convergence of a subsequence in B_1 . We claim now that the N_j 's are locally Hölder equicontinuous. This is an easy consequence of the Hölder estimate in (5.1) and of the reverse Poincaré inequality. Indeed, for any $0 < \theta < 1$ and for any points $y_1, y_2 \in \overline{B}_\theta$ one has the following:

$$\begin{aligned} \mathcal{G}(N_j(y_1), N_j(y_2)) &= \frac{r_j^{\frac{n}{2}}}{\|N\|_{L^2(\mathbf{B}_{r_j}(p))}} \mathcal{G}(N(p + r_j \mathbf{ex}_j(y_1)), N(p + r_j \mathbf{ex}_j(y_2))) \\ &\leq \frac{r_j^{\frac{n}{2}}}{\|N\|_{L^2(\mathbf{B}_{r_j}(p))}} [N]_{C^{0,\alpha}(\overline{\mathbf{B}}_{\theta r_j}(p))} \mathbf{d}(p + r_j \mathbf{ex}_j(y_1), p + r_j \mathbf{ex}_j(y_2))^\alpha \\ &\stackrel{(5.1)}{\leq} C \frac{r_j}{\|N\|_{L^2(\mathbf{B}_{r_j}(p))}} \left(\text{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_{r_j}(p)) + (\Lambda + C_0) \|N\|_{L^2(\mathbf{B}_{r_j}(p))}^2 \right)^{\frac{1}{2}} |y_1 - y_2|^\alpha \\ &\stackrel{(6.87)}{\leq} C(1 + r_j) |y_1 - y_2|^\alpha. \end{aligned}$$

Hence, for every $0 < \theta < 1$ there exists $C = C(\theta) > 0$ such that

$$[N_j]_{C^{0,\alpha}(\overline{B_\theta})} := \sup_{y_1, y_2 \in \overline{B_\theta}} \frac{\mathcal{G}(N_j(y_1), N_j(y_2))}{|y_1 - y_2|^\alpha} \leq C \quad (6.104)$$

for all j . Since $N_j(0) = Q[[0]]$, the N_j 's are also locally uniformly bounded, hence the Ascoli-Arzelà theorem implies that, up to extracting another subsequence if necessary, the convergence is locally uniform, and the limit is a continuous Q -valued function $\mathcal{N}_p: B_1 \subset \mathbb{R}^n \rightarrow \mathcal{A}_Q(\mathbb{R}^M)$.

Step 3: properties of the limit: proof of (a). It is immediate to see that $\boldsymbol{\eta} \circ \mathcal{N}_p \equiv 0$. Indeed, from the assumption that $\boldsymbol{\eta} \circ N \equiv 0$ and the definition of the blow-up maps, we deduce that $\boldsymbol{\eta} \circ N_j \equiv 0$ for every j . Now, the N_j 's converge to \mathcal{N}_p locally uniformly, and thus

$$\boldsymbol{\eta} \circ \mathcal{N}_p(y) = \lim_{j \rightarrow \infty} \boldsymbol{\eta} \circ N_j(y) = 0$$

for all $y \in B_1$. With the same argument, using the pointwise convergence of N_j to \mathcal{N}_p and the fact that $N_j(0) = Q[[0]]$ for every j we conclude that $\mathcal{N}_p(0) = Q[[0]]$.

Nonetheless, the map \mathcal{N}_p is non-trivial. Indeed, since $N_j \rightarrow \mathcal{N}_p$ strongly in L^2 , estimate (6.98) guarantees that:

$$\|\mathcal{N}_p\|_{L^2(B_1)}^2 = \lim_{j \rightarrow \infty} \int_{B_1} |N_j(y)|^2 \mathbf{J} \mathbf{ex}_j(y) dy = 1. \quad (6.105)$$

Next, we see that $\mathcal{N}_p(y) \in \mathcal{A}_Q(\mathbb{R}^k)$ for every $y \in B_1$. Indeed, considering the projection $\mathcal{N}_p^{(1)}$ of \mathcal{N}_p onto the subspace $\mathbb{R}^n \times \{0\} \times \{0\}$ we easily infer that

$$\begin{aligned} \int_{B_1} |\mathcal{N}_p^{(1)}(y)|^2 dy &= \int_{B_1} \sum_{l=1}^Q \sum_{i=1}^n |\langle \mathcal{N}_p^l(y), e_i \rangle|^2 dy \\ &= \lim_{j \rightarrow \infty} \int_{B_1} \sum_{l=1}^Q \sum_{i=1}^n |\langle N_j^l(y), \xi_i(p + r_j \mathbf{ex}_j(y)) \rangle|^2 dy = 0, \end{aligned}$$

because of the definition of N_j . Analogously, the projection $\mathcal{N}_p^{(3)}$ onto $\{0\} \times \{0\} \times \mathbb{R}^K$ satisfies

$$\begin{aligned} \int_{B_1} |\mathcal{N}_p^{(3)}(y)|^2 dy &= \int_{B_1} \sum_{l=1}^Q \sum_{\beta=1}^K |\langle \mathcal{N}_p^l(y), e_{n+k+\beta} \rangle|^2 dy \\ &= \lim_{j \rightarrow \infty} \int_{B_1} \sum_{l=1}^Q \sum_{\beta=1}^K |\langle N_j^l(y), \eta_\beta(p + r_j \mathbf{ex}_j(y)) \rangle|^2 dy = 0, \end{aligned}$$

where the η_β 's are a local orthonormal frame of the normal bundle of \mathcal{M} in \mathbb{R}^M extending the $e_{n+k+\beta}$'s in a neighborhood of p .

Step 4: harmonicity of the limit: proof of (b). We show now that \mathcal{N}_p is locally Dir-minimizing in B_1 and, moreover, that for every $0 < \rho < 1$ the following identity holds true:

$$\text{Dir}(\mathcal{N}_p, B_\rho) = \liminf_{j \rightarrow \infty} \text{Dir}(N_j, B_\rho). \quad (6.106)$$

In order to obtain the proof of the above claim, we need to exploit the minimizing property of the Jacobi Q -valued field N in order to deduce some crucial information on the blow-up

sequence N_j . Fix $j \in \mathbb{N}$, and let $u: B_1 \subset T_0\Sigma_j \simeq \mathbb{R}^n \rightarrow \mathcal{A}_Q(\mathbb{R}^M)$ be any $W^{1,2}$ map such that $u|_{\partial B_1} = N_j|_{\partial B_1}$. Then, the map $\tilde{u} \in W^{1,2}(\mathbf{B}_{r_j}(p); \mathcal{A}_Q(\mathbb{R}^M))$ defined by

$$\begin{aligned} \tilde{u}(x) &:= r_j^{-\frac{n}{2}} \|N\|_{L^2(\mathbf{B}_{r_j}(p))} \left(u \circ \psi_j^{-1} \right)^\perp(x) \\ &= r_j^{-\frac{n}{2}} \|N\|_{L^2(\mathbf{B}_{r_j}(p))} \sum_{l=1}^Q \left[\sum_{\beta=1}^k \langle u^l \circ \psi_j^{-1}(x), \nu_\beta(x) \rangle \nu_\beta(x) \right] \end{aligned}$$

is a section of $\mathcal{N}\Sigma$ in $\mathbf{B}_{r_j}(p)$ such that $\tilde{u}|_{\partial \mathbf{B}_{r_j}(p)} = N|_{\partial \mathbf{B}_{r_j}(p)}$. By minimality, it follows then that

$$\text{Jac}(N, \mathbf{B}_{r_j}(p)) \leq \text{Jac}(\tilde{u}, \mathbf{B}_{r_j}(p)). \quad (6.107)$$

Standard computations show that (6.107) is equivalent to the condition

$$\mathcal{F}_j(N_j) \leq \mathcal{F}_j(u), \quad (6.108)$$

where $\mathcal{F}_j(u)$ is the functional defined by

$$\begin{aligned} \mathcal{F}_j(u) &:= \int_{B_1} \sum_{l=1}^Q \sum_{i=1}^n \sum_{\alpha=1}^k \left| \langle D_{\varepsilon_i} u^l(y), \nu_\alpha \circ \psi_j(y) \rangle + r_j \langle u^l(y), (D_{\xi_i} \nu_\alpha - \nabla_{\xi_i}^\perp \nu_\alpha) \circ \psi_j(y) \rangle \right|^2 \mathbf{J} \mathbf{e} \mathbf{x}_j(y) dy \\ &\quad - r_j^2 \int_{B_1} \sum_{l=1}^Q \left(\left| A \circ \psi_j(y) \cdot (u^l)^\perp{}_{\psi_j(y)} \right|^2 + \mathcal{R} \circ \psi_j((u^l)^\perp{}_{\psi_j(y)}, (u^l)^\perp{}_{\psi_j(y)}) \right) \mathbf{J} \mathbf{e} \mathbf{x}_j(y) dy \\ &= \mathcal{F}_j^{(1)}(u) + \mathcal{F}_j^{(2)}(u) \end{aligned} \quad (6.109)$$

on the space of $u \in W^{1,2}(B_1; \mathcal{A}_Q(\mathbb{R}^M))$ such that $u|_{\partial B_1} = N_j|_{\partial B_1}$. Note that the following notation has been adopted in formula (6.99): $(u^l)^\perp{}_{\psi_j(y)}$ is the orthogonal projection of the vector $u^l(y)$ onto $T_{\psi_j(y)}^\perp \Sigma$, given by

$$(u^l)^\perp{}_{\psi_j(y)} = \sum_{\beta=1}^k \langle u^l(y), \nu_\beta(\psi_j(y)) \rangle \nu_\beta(\psi_j(y)).$$

Hence, one has

$$\left| A \circ \psi_j(y) \cdot (u^l)^\perp{}_{\psi_j(y)} \right|^2 = \sum_{i,h=1}^n \left| \sum_{\beta=1}^k A_{ih}^\beta(\psi_j(y)) \langle u^l(y), \nu_\beta(\psi_j(y)) \rangle \right|^2,$$

with $A_{ih}^\beta := \langle A(\xi_i, \xi_h), \nu_\beta \rangle$, and

$$\mathcal{R} \circ \psi_j((u^l)^\perp{}_{\psi_j(y)}, (u^l)^\perp{}_{\psi_j(y)}) = \sum_{i=1}^n \sum_{\beta, \gamma=1}^k R_{i\beta\gamma}^i(\psi_j(x)) \langle u^l(y), \nu_\beta(\psi_j(y)) \rangle \langle u^l(y), \nu_\gamma(\psi_j(y)) \rangle,$$

with $R_{i\beta\gamma}^i := \langle R(\xi_i, \nu_\beta) \nu_\gamma, \xi_i \rangle$.

Now, for every $0 < \rho < 1$, set

$$D_\rho := \liminf_{j \rightarrow \infty} \text{Dir}(N_j, B_\rho) = \liminf_{j \rightarrow \infty} \int_{B_\rho} \sum_{i=1}^n |D_{e_i} N_j|^2 dy,$$

and suppose by contradiction that either \mathcal{N}_ρ is not Dir-minimizing in B_ρ or $\text{Dir}(\mathcal{N}_\rho, B_\rho) < D_\rho$ ⁶ for some ρ . In any of the two situations, there exists a $\rho_0 > 0$ such that for any $\rho \geq \rho_0$ there exists a multiple valued map $g \in W^{1,2}(B_\rho, \mathcal{A}_Q(\mathbb{R}^k))$ with

$$g|_{\partial B_\rho} = \mathcal{N}_\rho|_{\partial B_\rho} \quad \text{and} \quad \gamma_\rho := D_\rho - \text{Dir}(g, B_\rho) > 0. \quad (6.110)$$

A simple application of Fatou's lemma shows that for almost every $\rho \in (0, 1)$ both the quantities $\liminf_j \text{Dir}(N_j, \partial B_\rho)$ and $\liminf_j \|N_j\|_{L^2(\partial B_\rho)}^2$ are finite:

$$\begin{aligned} \int_0^1 \liminf_{j \rightarrow \infty} \text{Dir}(N_j, \partial B_\rho) d\rho &\leq \liminf_{j \rightarrow \infty} \int_0^1 \text{Dir}(N_j, \partial B_\rho) d\rho = \liminf_{j \rightarrow \infty} \text{Dir}(N_j, B_1) \leq M < \infty, \\ \int_0^1 \liminf_{j \rightarrow \infty} \|N_j\|_{L^2(\partial B_\rho)}^2 d\rho &\leq \liminf_{j \rightarrow \infty} \int_0^1 \|N_j\|_{L^2(\partial B_\rho)}^2 d\rho = \lim_{j \rightarrow \infty} \|N_j\|_{L^2(B_\rho)}^2 = \|\mathcal{N}_\rho\|_{L^2(B_\rho)}^2 \leq 1. \end{aligned}$$

Therefore, passing if necessary to a subsequence, we can fix a radius $\rho \geq \rho_0$ such that

$$\text{Dir}(\mathcal{N}_\rho, \partial B_\rho) \leq \lim_{j \rightarrow \infty} \text{Dir}(N_j, \partial B_\rho) \leq M < \infty \quad (6.111)$$

and

$$\|\mathcal{N}_\rho\|_{L^2(\partial B_\rho)}^2 \leq \lim_{j \rightarrow \infty} \|N_j\|_{L^2(\partial B_\rho)}^2 \leq 1. \quad (6.112)$$

This allows us also to fix the corresponding map g satisfying the conditions in (6.110). The strategy to complete the proof is now the following: we will use the map g to construct, for every j , a competitor u_j for the functional \mathcal{F}_j , that is a map $u_j \in W^{1,2}(B_1; \mathcal{A}_Q(\mathbb{R}^M))$ with $u_j|_{\partial B_1} = N_j|_{\partial B_1}$. Then, we will show that if j is chosen sufficiently large then $\mathcal{F}_j(u_j) < \mathcal{F}_j(N_j)$, thus contradicting (6.108) and, in turn, the minimality of N in $\mathbf{B}_{r_j}(p)$.

The construction of the maps u_j is analogous to the one presented in [DLS11, Proposition 3.20]: we fix a number $0 < \delta < \frac{\rho}{2}$ to be suitably chosen later, and for every $j \in \mathbb{N}$ we define u_j on B_1 as follows:

$$u_j(y) := \begin{cases} g\left(\frac{\rho y}{\rho - \delta}\right) & \text{for } y \in B_{\rho - \delta}, \\ h_j(y) & \text{for } y \in B_\rho \setminus B_{\rho - \delta}, \\ N_j(y) & \text{for } y \in B_1 \setminus B_\rho, \end{cases}$$

where the maps h_j interpolate between $g\left(\frac{\rho y}{\rho - \delta}\right) = \mathcal{N}_\rho\left(\frac{\rho y}{\rho - \delta}\right) \in W^{1,2}(\partial B_{\rho - \delta}, \mathcal{A}_Q)$ and $N_j \in W^{1,2}(\partial B_\rho, \mathcal{A}_Q)$. Observe that the existence of the h_j 's is guaranteed by Proposition 4.1 (also cf. [DLS11, Lemma 2.15]).

As anticipated, the goal is now to show that this map u_j has less \mathcal{F}_j energy than N_j when j is big enough (and thus r_j is suitably close to 0). We first notice that u_j differs from N_j only on B_ρ , therefore our analysis will be carried on this smaller ball only. Then, fix a small number $\theta > 0$, and recall that, in the limit as $j \uparrow \infty$, the exponential maps \mathbf{ex}_j converge uniformly to the identity map of the unit ball in \mathbb{R}^n , whereas the maps ψ_j converge uniformly to the constant map identically equal to p . Hence, the first line in the definition of $\mathcal{F}_j(u_j)$ can be estimated by

$$\mathcal{F}_j^{(1)}(u_j)|_{B_\rho} \leq (1 + \theta)\text{Dir}(u_j, B_\rho) + \theta\|u_j\|_{L^2(B_\rho)}^2 \quad (6.113)$$

⁶Observe that the inequality

$$\text{Dir}(\mathcal{N}_\rho, B_\rho) \leq D_\rho$$

is guaranteed for every ρ because the Dirichlet functional is lower semi-continuous with respect to weak convergence in $W^{1,2}$.

for all $j \geq j_0(\theta)$. On the other hand, the definition of u_j together with the estimate (4.3) imply that

$$\begin{aligned} \text{Dir}(u_j, B_\rho) &\leq \text{Dir}(u_j, B_{\rho-\delta}) + C\delta(\text{Dir}(u_j, \partial B_{\rho-\delta}) + \text{Dir}(N_j, \partial B_\rho)) + \frac{C}{\delta} \int_{\partial B_\rho} \mathcal{G}(u_j, N_j)^2 \\ &\leq \text{Dir}(g, B_\rho) + C\delta \text{Dir}(\mathcal{N}_p, \partial B_\rho) + C\delta \text{Dir}(N_j, \partial B_\rho) + \frac{C}{\delta} \int_{\partial B_\rho} \mathcal{G}(\mathcal{N}_p, N_j)^2 \end{aligned} \quad (6.114)$$

whereas

$$\begin{aligned} \|u_j\|_{L^2(B_\rho)}^2 &= \|u_j\|_{L^2(B_{\rho-\delta})}^2 + \|u_j\|_{L^2(B_\rho \setminus B_{\rho-\delta})}^2 \\ &\leq \|g\|_{L^2(B_\rho)}^2 + \|h_j\|_{L^2(B_\rho \setminus B_{\rho-\delta})}^2 \\ &\stackrel{(4.2)}{\leq} \|g\|_{L^2(B_\rho)}^2 + C\delta \left(\|\mathcal{N}_p\|_{L^2(\partial B_\rho)}^2 + \|N_j\|_{L^2(\partial B_\rho)}^2 \right) \\ &\stackrel{(6.112)}{\leq} \|g\|_{L^2(B_\rho)}^2 + 3C\delta. \end{aligned} \quad (6.115)$$

Concerning the second term in the functional \mathcal{F}_j , it is easy to compute that

$$\begin{aligned} \mathcal{F}_j^{(2)}(u_j)|_{B_\rho} &\leq \mathcal{F}_j^{(2)}(N_j)|_{B_\rho} + Cr_j^2 \left(\int_{B_\rho} |N_j|^2 dy \right)^{\frac{1}{2}} \left(\int_{B_\rho} \mathcal{G}(u_j, N_j)^2 dy \right)^{\frac{1}{2}} + Cr_j^2 \int_{B_\rho} \mathcal{G}(u_j, N_j)^2 dy \\ &\leq \mathcal{F}_j^{(2)}(N_j)|_{B_\rho} + Cr_j^2, \end{aligned} \quad (6.116)$$

because the L^2 norms of both the maps u_j and N_j are uniformly bounded in j .

We can finally close the argument. Choose δ such that $4C\delta(M+1) \leq \gamma_\rho$, where M and γ_ρ are the constants in (6.111) and (6.110) respectively. Invoking (6.111) and using the uniform convergence of N_j to \mathcal{N}_p , from (6.114) follows

$$\begin{aligned} \text{Dir}(u_j, B_\rho) &\leq D_\rho - \gamma_\rho + C\delta M + C\delta(M+1) + \frac{C}{\delta} \int_{\partial B_\rho} \mathcal{G}(\mathcal{N}_p, N_j)^2 \\ &\leq D_\rho - \frac{\gamma_\rho}{2} + \frac{C}{\delta} \int_{\partial B_\rho} \mathcal{G}(\mathcal{N}_p, N_j)^2 \leq D_\rho - \frac{\gamma_\rho}{4} \end{aligned} \quad (6.117)$$

whenever j is sufficiently large. A suitable choice of the parameter θ in (6.113) depending on γ_ρ , D_ρ and $\|g\|_{L^2(B_\rho)}$ allows to conclude that for j big enough (depending on the same quantities):

$$\begin{aligned} \mathcal{F}_j^{(1)}(u_j)|_{B_\rho} &\leq D_\rho - \frac{\gamma_\rho}{8} \\ &\leq \text{Dir}(N_j, B_\rho) - \frac{\gamma_\rho}{16} \\ &\leq \mathcal{F}_j^{(1)}(N_j)|_{B_\rho} - \frac{\gamma_\rho}{32}. \end{aligned} \quad (6.118)$$

Observe that in the last inequality we have used again that the manifolds Σ_j are becoming more and more flat in the limit $j \uparrow \infty$, and also that the projection of the N_j 's on the orthogonal

complement to \mathbb{R}^k is vanishing in an L^2 sense in the same limit. Now, summing (6.116) and (6.118), we conclude:

$$\mathcal{F}_{r_j}(u_j) \leq \mathcal{F}_{r_j}(N_j) - \frac{\gamma\rho}{32} + Cr_j^2. \quad (6.119)$$

The desired contradiction is immediately obtained by choosing j so big that $Cr_j^2 \leq \frac{\gamma\rho}{64}$.

Step 5: homogeneity of the limit: proof of (c). We conclude the proof of the theorem showing that the limit map \mathcal{N}_p admits a homogeneous extension to the whole \mathbb{R}^n . In other words, the goal is to show that

$$\mathcal{N}_p\left(\frac{\rho y}{|y|}\right) = \left(\frac{\rho}{|y|}\right)^\mu \mathcal{N}_p(y)$$

for all $y \in B_1 \setminus \{0\}$, for all $0 < \rho < 1$, and with $\mu = I_0(p)$.

The strategy is to take advantage of [DLS11, Corollary 3.16]: since \mathcal{N}_p is Dir-minimizing, in order to prove that it is homogeneous it suffices to show that its frequency function at the origin $y = 0$ is constant. Hence, we set for $0 < \rho < 1$:

$$\mathcal{I}(\rho) := \frac{\rho \mathcal{D}(\rho)}{\mathcal{H}(\rho)}, \quad (6.120)$$

where

$$\mathcal{D}(\rho) := \text{Dir}(\mathcal{N}_p, B_\rho) = \int_{B_\rho} |D\mathcal{N}_p|^2 dy = \int_{B_\rho} \sum_{l=1}^Q \sum_{i=1}^n \sum_{\alpha=1}^k |\langle D_{e_i} \mathcal{N}_p^l, e_{n+\alpha} \rangle|^2 dy, \quad (6.121)$$

and

$$\mathcal{H}(\rho) := \int_{\partial B_\rho} |\mathcal{N}_p|^2 dy. \quad (6.122)$$

We first observe that $\mathcal{I}(\rho)$ is well defined for all $\rho \in (0, 1)$. Indeed, if there is ρ_0 such that $\mathcal{H}(\rho_0) = 0$, then by minimality it must be $\mathcal{N}_p \equiv Q[0]$ in B_{ρ_0} . On the other hand, the unique continuation property of Dir-minimizers (cf. [DLS16a, Lemma 7.1]) would then imply that $\mathcal{N}_p \equiv Q[0]$ in the whole B_1 , which in turn contradicts the fact that $\|\mathcal{N}_p\|_{L^2(B_1)} = 1$. In other words, this shows that the origin is singular for \mathcal{N}_p .

Extract, if necessary, a subsequence such that the lim inf in (6.106) can be replaced by a lim, and compute:

$$\begin{aligned} \mathcal{I}(\rho) &= \frac{\rho \int_{B_\rho} \sum_{l=1}^Q \sum_{i=1}^n \sum_{\alpha=1}^k |\langle D_{e_i} \mathcal{N}_p^l, e_{n+\alpha} \rangle|^2 dy}{\int_{\partial B_\rho} |\mathcal{N}_p|^2 dy} \\ &= \lim_{j \rightarrow \infty} \frac{\rho \int_{B_\rho} \sum_{l=1}^Q \sum_{i=1}^n \sum_{\alpha=1}^k |\langle D_{e_i} N_j^l, \nu_\alpha \circ \psi_j \rangle|^2 \mathbf{J} \mathbf{e} \mathbf{x}_j(y) dy}{\int_{\partial B_\rho} |N_j|^2 \mathbf{J} \mathbf{e} \mathbf{x}_j(y) dy} \\ &= \lim_{j \rightarrow \infty} \frac{\rho r_j \text{Dir}^{\mathcal{N}^\Sigma}(N, \mathbf{B}_{\rho r_j}(p))}{\int_{\partial \mathbf{B}_{\rho r_j}(p)} |N|^2 d\mathcal{H}^{n-1}} \\ &= \lim_{j \rightarrow \infty} \frac{\rho r_j \mathbf{D}(\rho r_j)}{\mathbf{H}(\rho r_j)} = \lim_{j \rightarrow \infty} \mathbf{I}(\rho r_j) = I_0, \end{aligned} \quad (6.123)$$

where we have used (modifications of) formulas (6.97), (6.100) and finally (6.82).

As already anticipated, [DLS11, Corollary 3.16] implies now that \mathcal{N}_p is a μ -homogeneous Q -valued function, with $\mu = I_0(p) > 0$. \square

Remark 6.23. Note that from the proof of Theorem 6.21 it follows that the convergence of (a subsequence of) the N_{p,r_j} to \mathcal{N}_p is actually *strong* in $W^{1,2}$ in any ball $B_\rho \subset B_1$ (cf. formula (6.106)). This stronger convergence has been in fact tacitly used in deriving (6.123).

6.5. The closing argument: proof of Theorem 0.7.

Proposition 6.24. *Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be Jac-minimizing. Assume $\Omega \subset \Sigma^n$ is connected. Then:*

- (i) *either $N = Q[\zeta]$ with $\zeta: \Omega \rightarrow \mathbb{R}^M$ a classical Jacobi field,*
- (ii) *or the set $D_Q(N)$ of multiplicity Q points is a relatively closed proper subset of Ω consisting of isolated points if $n = 2$ and with $\dim_{\mathcal{H}}(D_Q(N)) \leq n - 2$ if $n \geq 3$.*

Proof. Assume without loss of generality that $\boldsymbol{\eta} \circ N = 0$, so that $p \in D_Q(N)$ if and only if $N(p) = Q[0]$. We first observe the following fact: the set $D_Q(N)$ is relatively closed in Ω . This can be rapidly seen writing $D_Q(N) = \sigma^{-1}(\{Q\})$, where $\sigma: \Omega \rightarrow \mathbb{N}$ is the function given by

$$\sigma(x) := \text{card}(\text{spt}(N(x))), \quad (6.124)$$

and noticing that, since N is continuous, σ is lower semi-continuous.

We will now treat the two cases $n = 2$ and $n \geq 3$ separately.

Case 1: dimension $n = 2$. In this case, we claim that the points $p \in \text{sing}_Q(N)$ are isolated in $D_Q(N)$. Assume by contradiction that this is not the case, and let $p \in \text{sing}_Q(N)$ be the limit of a sequence $\{x_j\}_{j=1}^\infty$ of points in $D_Q(N)$. Set $r_j := \mathbf{d}(x_j, p)$. Since $r_j \downarrow 0$, by Theorem 6.21 the corresponding blow-up family N_{p,r_j} converges uniformly, up to a subsequence, to a Dir-minimizing, μ -homogenous tangent map $\mathcal{N}_p: B_1 \subset \mathbb{R}^2 \rightarrow \mathcal{A}_Q(\mathbb{R}^k)$ with $\|\mathcal{N}_p\|_{L^2(B_1)} = 1$ and $\boldsymbol{\eta} \circ \mathcal{N}_p \equiv 0$. Moreover, since each $x_j \in D_Q(N)$, the points $y_j := \psi_{p,r_j}^{-1}(x_j)$ are a sequence of multiplicity Q points for the corresponding N_{p,r_j} in $\mathbb{S}^1 = \partial B_1$: from this, we conclude that there exists $w \in \mathbb{S}^1$ such that $\mathcal{N}_p(w) = Q[0]$. Up to rotations, we can assume that $w = e_1$. Denote $z := \frac{1}{2}e_1$, and observe that, since \mathcal{N}_p is homogeneous, necessarily $\mathcal{N}_p(z) = Q[0]$. Consider now the blow-up of \mathcal{N}_p at z : by [DLS11, Lemma 3.24], any tangent map h to \mathcal{N}_p at z is a non-trivial β -homogeneous Dir-minimizer, with β equal to the frequency of \mathcal{N}_p at z , and such that $h(x_1, x_2) = \hat{h}(x_2)$, for some function $\hat{h}: \mathbb{R} \rightarrow \mathcal{A}_Q(\mathbb{R}^k)$ which is Dir-minimizing on every interval. Moreover, since $\text{Dir}(h, B_1) > 0$, it must also be $\text{Dir}(\hat{h}, I) > 0$, where $I := [-1, 1]$. On the other hand, every 1-dimensional Dir-minimizer \hat{h} is affine, that is it has the form $\hat{h}(x) = \sum_{i=1}^Q \llbracket L_i(x) \rrbracket$, where the L_i 's are affine functions such that either $L_i \equiv L_j$ or $L_i(x) \neq L_j(x)$ for every $x \in \mathbb{R}$, for any i, j . Now, since $\hat{h}(0) = Q[0]$, we deduce that $\hat{h} = Q\llbracket L \rrbracket$; on the other hand, $\boldsymbol{\eta} \circ h \equiv 0$, and thus necessarily $L = 0$. This contradicts $\text{Dir}(\hat{h}, I) > 0$.

Hence, if $p \in D_Q(N)$ then either p is isolated or, in case p is a regular multiplicity Q point, there exists an open neighborhood V of p such that $V \subset D_Q(N)$. From this we deduce that $\text{reg}_Q(N)$ is both open and closed in Ω . Since Ω is connected, then either $\text{reg}_Q(N) = \Omega$, and $N \equiv Q[0]$, or $\text{reg}_Q(N) = \emptyset$, and $D_Q(N) = \text{sing}_Q(N)$ consists of isolated points. This completes the proof in the dimension $n = 2$ case.

Case 2: dimension $n \geq 3$. In this case, the goal is to show that $\mathcal{H}^s(D_Q(N)) = 0$ for every $s > n - 2$, unless $N \equiv Q[0]$. Consider the set $\text{sing}_Q(N)$. We first claim that $\mathcal{H}^s(\text{sing}_Q(N)) = 0$ for every $s > n - 2$. Suppose by contradiction that this is not the case. Then, by [Sim83b, Theorem 3.6 (2)], there exist $s > n - 2$ and a subset $F \subset \text{sing}_Q(N)$ with $\mathcal{H}^s(F) > 0$ such that

every point $p \in F$ is a point of positive upper s -density for the measure \mathcal{H}_∞^s , that is

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}_\infty^s(\text{sing}_Q(N) \cap \mathbf{B}_r(p))}{r^s} > 0 \quad \text{for every } p \in F. \quad (6.125)$$

Here, the symbol \mathcal{H}_∞^s denotes as usual the s -dimensional Hausdorff pre-measure, defined by

$$\mathcal{H}_\infty^s(A) := \inf \left\{ \sum_{h=1}^{\infty} \omega_s \left(\frac{\text{diam}(E_h)}{2} \right)^s : A \subset \bigcup_{h=1}^{\infty} E_h \right\},$$

with $\omega_s := \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}$, where $\Gamma(s)$ is the usual Gamma function. Among the properties of \mathcal{H}_∞^s , it is worth recalling now that it is upper semi-continuous with respect to Hausdorff convergence of compact sets: in other words, if K_j is a sequence of compact sets converging to K in the sense of Hausdorff, then

$$\limsup_{j \rightarrow \infty} \mathcal{H}_\infty^s(K_j) \leq \mathcal{H}_\infty^s(K). \quad (6.126)$$

Now, (6.125) together with Theorem 6.21 imply the existence of a point $p \in \text{sing}_Q(N)$ and a sequence of radii $r_j \downarrow 0$ such that the blow-up maps $N_j = N_{p,2r_j}$ converge uniformly to a Dir-minimizing homogeneous Q -valued function $\mathcal{N}_p: B_1 \subset \mathbb{R}^n \rightarrow \mathcal{A}_Q(\mathbb{R}^k)$ with $\boldsymbol{\eta} \circ \mathcal{N}_p \equiv 0$ and $\|\mathcal{N}_p\|_{L^2(B_1)} = 1$, and furthermore such that

$$\limsup_{j \rightarrow \infty} \frac{\mathcal{H}_\infty^s(\text{sing}_Q(N) \cap \mathbf{B}_{r_j}(p))}{r_j^s} > 0, \quad (6.127)$$

or, equivalently,

$$\limsup_{j \rightarrow \infty} \mathcal{H}_\infty^s(\text{sing}_Q(N_j) \cap B_{\frac{1}{2}}) > 0, \quad (6.128)$$

where $B_{\frac{1}{2}} \subset T_0\Sigma_j \simeq \mathbb{R}^n$. Set $K_j := \text{sing}_Q(N_j) \cap \overline{B_{\frac{1}{2}}}$, and observe that, since N_j converge to \mathcal{N}_p uniformly, the compact sets K_j converge in the sense of Hausdorff to a compact set $K \subset D_Q(\mathcal{N}_p)$. From the semi-continuity property (6.126), we can therefore deduce that:

$$\begin{aligned} \mathcal{H}^s(D_Q(\mathcal{N}_p)) &\geq \mathcal{H}_\infty^s(D_Q(\mathcal{N}_p)) \geq \mathcal{H}_\infty^s(K) \\ &\geq \limsup_{j \rightarrow \infty} \mathcal{H}_\infty^s(K_j) \geq \limsup_{j \rightarrow \infty} \mathcal{H}_\infty^s(\text{sing}_Q(N_j) \cap B_{\frac{1}{2}}) > 0. \end{aligned} \quad (6.129)$$

Since $s > n - 2$, [DLS11, Proposition 3.22] implies that this can happen only if $\mathcal{N}_p \equiv Q[\zeta]$, where $\zeta: B_1 \rightarrow \mathbb{R}^k$ is a harmonic function. Since $\boldsymbol{\eta} \circ \mathcal{N}_p \equiv 0$, then it must be $\mathcal{N}_p \equiv Q[0]$, which in turns contradicts the fact that $\|\mathcal{N}_p\|_{L^2(B_1)} = 1$.

We can therefore conclude that necessarily $\mathcal{H}^s(\text{sing}_Q(N)) = 0$ for every $s > n - 2$. Since $\text{sing}_Q(N) = \partial D_Q(N) \cap \Omega$, either $D_Q(N) = \Omega$ and $N \equiv Q[0]$, or $D_Q(N) = \text{sing}_Q(N)$. The proof is complete. \square

Remark 6.25. As a corollary of Proposition 6.24, one easily deduces the following: if N is a Jac-minimizing Q -valued vector field in the open and connected subset $\Omega \subset \Sigma$ which is not of the form $N = Q[\zeta]$ for some classical Jacobi field ζ , then $D_Q(N) = \text{sing}_Q(N)$, that is all multiplicity Q points are singular.

We have now all the tools that are needed to prove Theorem 0.7.

Proof of Theorem 0.7. Since our manifolds are always assumed to be second-countable spaces, Ω can have at most countably many connected components, and these connected components

are open. Hence, there is no loss of generality in assuming that Ω itself is connected: in the general case, we would just work on each connected component separately.

Let σ be the function defined in (6.124). If $x \in \Omega$ is a regular point, then it is clear that σ is continuous at x . On the other hand, assume x is a point of continuity for σ , and write $N(x) = \sum_{j=1}^J k_j \llbracket P^j \rrbracket$, where the k_j 's are integers such that $\sum_{j=1}^J k_j = Q$, each $P^j \in T_x^\perp \Sigma$ and $P^i \neq P^j$ if $i \neq j$. Since the target of σ is discrete, the fact that σ is continuous at x implies that in fact $\sigma(z) = J$ for all z in a neighborhood U of x . Hence, since N is continuous, there exists a neighborhood $x \in V \subset U$ such that the map $N|_V$ admits a continuous decomposition $N(z) = \sum_{j=1}^J k_j \llbracket N^j(z) \rrbracket$, where each map $N^j : V \rightarrow \mathcal{N}\Sigma$ is a classical Jacobi field. Therefore, $x \in \text{reg}(N)$.

The above argument implies that $\text{sing}(N)$ coincides with the set of points where σ is discontinuous. The proof that its Hausdorff dimension cannot exceed $n - 2$ will be obtained via induction on Q . If $Q = 1$, there is nothing to prove, since single-valued Jac-minimizers are classical Jacobi fields. Assume now that the theorem holds for every Q^* -valued Jac-minimizer with $Q^* < Q$ and we prove it for N . If $N \equiv Q \llbracket \zeta \rrbracket$ with ζ a classical Jacobi field, then $\text{sing}(N)$ is empty, and the theorem follows. Assume, therefore, this is not the case. By Proposition 6.24, the set $D_Q(N) = \text{sing}_Q(N)$ is a closed subset of Ω which is at most countable if $n = 2$ and has Hausdorff dimension at most $n - 2$ if $n \geq 3$. Consider now the open set $\Omega' := \Omega \setminus D_Q(N)$. Since N is continuous, we can find countably many open geodesic balls \mathbf{B}_j such that $\Omega' = \bigcup_j \mathbf{B}_j$ and $N|_{\mathbf{B}_j}$ can be written as the superposition of two multiple-valued functions:

$$N|_{\mathbf{B}_j} = \llbracket N_{j,Q_1} \rrbracket + \llbracket N_{j,Q_2} \rrbracket \quad \text{with } Q_1 < Q, Q_2 < Q \tag{6.130}$$

and

$$\text{spt}(N_{j,Q_1}(x)) \cap \text{spt}(N_{j,Q_2}(x)) = \emptyset \quad \text{for every } x \in \mathbf{B}_j. \tag{6.131}$$

From this last condition, it follows that

$$\text{sing}(N) \cap \mathbf{B}_j = \text{sing}(N_{j,Q_1}) \cup \text{sing}(N_{j,Q_2}). \tag{6.132}$$

The maps N_{j,Q_1} and N_{j,Q_2} are both Jac-minimizing, and thus, by inductive hypothesis, their singular set has Hausdorff dimension at most $n - 2$, and is at most countable if $n = 2$. Finally:

$$\text{sing}(N) = \text{sing}_Q(N) \cup \bigcup_{j=1}^{\infty} (\text{sing}(N_{j,Q_1}) \cup \text{sing}(N_{j,Q_2})) \tag{6.133}$$

also has Hausdorff dimension at most $n - 2$ and is at most countable if $n = 2$. □

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