Thin elastic plates supported over small areas
II. Variational-asymptotic models.

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Abstract

An asymptotic analysis is performed for thin anisotropic elastic plate clamped along its lateral side and also supported at a small area θ_h of one base with diameter of the same order as the plate thickness $h \ll 1$. A three-dimensional boundary layer in the vicinity of the support θ_h is involved into the asymptotic form which is justified by means of the previously derived weighted inequality of Korn's type provides an error estimate with the bound $ch^{1/2}\left|\ln h\right|$. Ignoring this boundary layer effect reduces the precision order down to $\left|\ln h\right|^{-1/2}$. A two-dimensional variational-asymptotic model of the plate is proposed within the theory of self-adjoint extensions of differential operators. The only characteristics of the boundary layer, namely the elastic logarithmic potential matrix of size 4×4, is involved into the model which however keeps the precision order $h^{1/2}\left|\ln h\right|$ in certain norms. Several formulations and applications of the model are discussed.

Keywords: Kirchhoff plate, small support zone, asymptotic analysis, self-adjoint extensions, variational model.

MSC: 74K20, 74B05.

1 Introduction

1.1 Motivation

We consider a thin elastic anisotropic plate

$$\Omega_h = \left\{ x = (y, z) \in \mathbb{R}^2 \times \mathbb{R} : y = (y_1, y_2) \in \omega, \ |z| < h/2 \right\} \tag{1.1}$$

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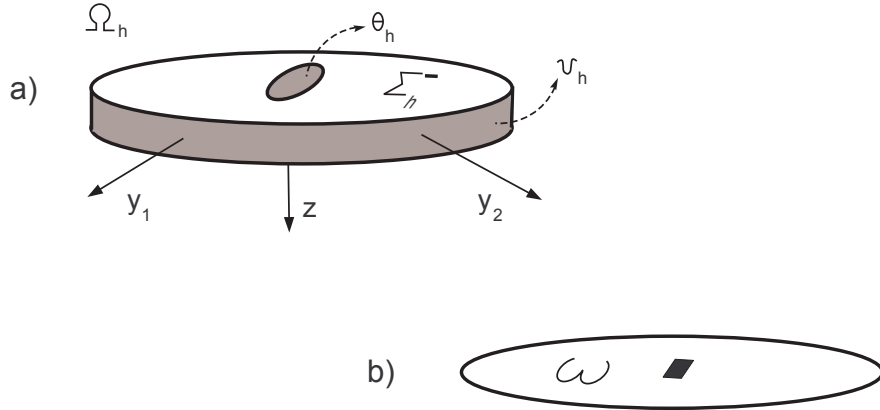
Figure 1: A plate with small clamped area (a) and its two-dimensional model where the black square indicates an asymptotic point condition (b).

of the relative thickness $h \ll 1$ with one or several small, of diameter $O(h)$, support areas at its lower base Σ_h^- (see Figure 1,a). In this paper we assume that the plate is clamped along the lateral side υ_h while the bases $\Sigma_h^\pm$ are traction-free, except at the only small support zone

$$\theta_h = \left\{ x : \eta := h^{-1} y \in \theta, \ z = -h/2 \right\}. \tag{1.2}$$

Here, ω and θ are domains in the plane $\mathbb{R}^2$ with smooth boundaries and compact closures (see Figure 1,b), the y-coordinate origin $\mathcal{O}$ is put inside θ,

$$\Sigma_h^\pm = \{ x : y \in \omega, \ z = \pm h/2 \}, \quad \upsilon_h = \{ x : y \in \partial\omega, \ |z| < h/2 \}. \tag{1.3}$$

This paper is a direct sequel of [4] where a convenient form of Korn's inequality in Ω_h was derived as well as a boundary layer effect near θ_h was investigated scrupulously. These results, combined with the classical procedure of dimension reduction, see [7, 8, 13, 23, 32] etc., and an asymptotic analysis of singular perturbation of boundaries, see [10, 15], allow us here to construct and justify asymptotic expansions of elastic fields in Ω_h, i.e., displacements, strains and stresses.

A small Dirichlet perturbation of boundary conditions may lead to neglectable changes in a solution but in our case, unfortunately, such a change is fair and significant in two aspects. First of all, fixing the plate (1.1) with the sharp supports at $\theta_h^1, ..., \theta_h^J$, cf. (1.2), results in the Sobolev point conditions, see, e.g., [5], namely the average deflexion w_3 vanishes at the points $\mathcal{O}^1, ..., \mathcal{O}^J$, the centers of the small support areas. Due to the Sobolev embedding theorem in $\mathbb{R}^2$ these additional restrictions are natural in the variational formulation of the traditional two-dimensional model of Kirchhoff's plate while the only implication defect turns into a certain lack of smoothness properties of the solution at the points $\mathcal{O}^1, ..., \mathcal{O}^J$. Much more disagreeable input of the small support areas becomes a crucial reduction of the convergence rate: instead of the the conventional rate $O(\sqrt{h})$ for the Kirchhoff model, we detect $O(|\ln h|^{-1/2})$, cf. [6], that is unacceptable for any application purpose.

The three-dimensional boundary layers, described and investigated in [4], see also [18, 21], help to construct specific corerctions in the vicinity of θ_h, to take into account their influence on

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the far-field asymptotics and, as a result, to achieve the appropriate error estimate $O(\sqrt{h}|\ln h|)$. At the same time, the global asymptotic approximation of the elastic fields becomes quite complicated so that its utility in application is rather doubtful. In this way, it is worth to restrict our analysis to the fair-field asymptotics whose dependence on the small parameter h remains complicated but can be expressed through rational functions in $\ln h$ and an integral characteristics of the support area, namely a 4×4-matrix of elastic capacity [4, 18].

To provide a correct and mathematically rigorous two-dimensional model of the supported plate, we employ two approaches. First, we turn to the technique of self-adjoint extensions of differential operators (see the pioneering paper [1], the review [30] and, e.g., the papers [11, 17, 25, 29], especially [20] connecting this approach with the method of matched asymptotic expansions). To simulate the influence of localized special boundary layer, we provide a proper choice of singular solutions by selecting a special self-adjoint extension in the Lebesgue space $L^2(\omega)$ of the matrix $\mathcal{L}(\nabla_y)$ of differential operators in the Kirchhoff plate model, see (2.30), (2.29). Parameters of this extension depend on the quantity $|\ln h|$ and the elastic logarithmic capacity matrix, see [4, Sect. 3.2], which is similar to the logarithmic capacity in harmonic analysis, cf. [12, 31].

The introduced model is two-fold and can be formulated as either an abstract equation with the obtained self-adjoint operator, or a problem on a stationary point of the natural energy functional in a weighted function space with detached asymptotics. Advantages and deficiencies of these approaches are discussed in Section 3. In any case the compelled removal of a major part of the three-dimensional layer from the asymptotic form surely reduces capabilities of the model which, in particular, does not give the global approximation of stresses and strains, but does outside a neighborhood of the sets $\overline{\theta_h}$ and $\overline{\upsilon_h}$ where the Dirichlet conditions are imposed. Thereupon, we mention that, first, the Kirchhoff model itself does not approximate correctly stresses and strains near the clamped edge, too, and, second, quite many applications demand to determine asymptotics of displacements only. In Corollaries 11 and 13 we will give some error estimates for the model, in particular, an asymptotic formula for the elastic energy of the three-dimensional thin-plate. Results [26] predict that the model also can furnish asymptotics of eigenfrequencies of the plate but this will be a topic of an oncoming paper.

1.2   Formulation of the problem

Similarly to [4], in the domain (1.1) we consider the mixed boundary-value problem of the elasticity theory

$$L(\nabla)u(h, x) := D\left(-\nabla\right)^\top AD\left(\nabla\right)u\left(h, x\right) = f\left(h, x\right), \quad x \in \Omega_h, \tag{1.4}$$

$$N^+(\nabla)u(h, x) := D\left(e_3\right)^\top AD\left(\nabla\right)u\left(h, x\right) = 0, \quad x \in \Sigma_h^+, \tag{1.5}$$

$$N^-(\nabla)u(h, x) := D\left(-e_3\right)^\top AD\left(\nabla\right)u\left(h, x\right) = 0, \quad x \in \Sigma_h^\bullet = \Sigma_h^- \setminus \theta_h,$$

$$u\left(h, x\right) = 0, \quad x \in \theta_h, \tag{1.6}$$

$$u\left(h, x\right) = 0, \quad x \in \upsilon_h. \tag{1.7}$$

Let us explain the Voigt-Mandel notation we use throughout the paper. The displacement vector $(u_1(x), u_2(x), u_3(x))^\top$ is regarded as a column in $\mathbb{R}^3$, $\top$ stands for transposition and the Hooke's law

$$\sigma\left(u\right) = A\varepsilon\left(u\right) = AD\left(\nabla\right)u \tag{1.8}$$

relates the strain column

$$\varepsilon = \left(\varepsilon_{11}, \varepsilon_{22}, 2^{1/2}\varepsilon_{12}, 2^{1/2}\varepsilon_{13}, 2^{1/2}\varepsilon_{23}, \varepsilon_{33}\right)^{\top}, \quad \varepsilon_{jk}\left(u\right) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right),$$

with the stress column $\sigma\left(u\right)$ of the same structure as $\varepsilon\left(u\right)$. Moreover, the stiffness matrix A of size 6×6, symmetric and positive definite, figures in (1.8) as well as the differential operators

$$D\left(\nabla\right)^{\top} = \begin{pmatrix} \partial_1 & 0 & 2^{-1/2}\partial_2 & 2^{-1/2}\partial_3 & 0 & 0 \\ 0 & \partial_2 & 2^{-1/2}\partial_1 & 0 & 2^{-1/2}\partial_3 & 0 \\ 0 & 0 & 0 & 2^{-1/2}\partial_1 & 2^{-1/2}\partial_2 & \partial_3 \end{pmatrix}, \quad \partial_j = \frac{\partial}{\partial x_j}, \quad \nabla = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}. \quad (1.9)$$

The variational formulation of problem (1.4)-(1.7) implies the integral identity

$$\left(AD\left(\nabla\right)u, D\left(\nabla\right)v\right)_{\Omega_h} = \left(f, v\right)_{\Omega_h}, \quad \forall v \in H_0^1\left(\Omega_h; \Gamma_h\right)^3, \quad (1.10)$$

where $(\ ,\ )_{\Omega_h}$ is the scalar product in $L^2\left(\Omega_h\right)$, $H_0^1\left(\Omega_h; \Gamma_h\right)$ is a subspace of functions in the Sobolev space $H^1\left(\Omega_h\right)$ which vanish on $\Gamma_h = \theta_h \cup \upsilon_h$, see (1.6) and (1.7), and the last superscript 3 in (1.10) indicates the number of components in the test (vector) function $v = \left(v_1, v_2, v_3\right)^{\top}$.

We here have listed only notation of further use while more explanation and an example of the isotropic material can be found in Section 1.2 of [4].

1.3   Architecture of the paper

In §2 we present the standard asymptotic procedure of dimension reduction to provide explicit formulas in the Kirchhoff plate's model. Based on the results obtained in [4], we construct in §3 asymptotic expansions of a solution $u(h, x)$ to problem (1.4)-(1.7) including the boundary layer near the support area (1.2) and derive the error estimates in various versions.

The technique of self-adjoint extensions of differential operators is outlined in §3 for the Kirchhoff plate supported at one point, i.e. with the Sobolev condition, together with the choice of the extension parameters on the base of the previous asymptotic analysis. Finally, the asymptotic-variational model mentioned in Section 1.1 is discussed at length in §4.

2   The limit two-dimensional problem

2.1   The dimension reduction

We suppose that the right-hand side of system (1.4) takes the form

$$f\left(h, x\right) = h^{-1/2}f^0\left(y, \zeta\right) + h^{1/2}f^1\left(y\right) + \widetilde{f}\left(h, x\right) \quad (2.1)$$

where $\zeta = h^{-1}z \in \left(-1/2, 1/2\right)$ is the stretched transverse coordinate and f^0, f^1 are vector functions subject to the conditions

$$\int_{-1/2}^{1/2} f_3^0\left(y, \zeta\right) d\zeta = 0, \quad f_1^1\left(y\right) = f_2^1\left(y\right) = 0, \quad y \in \omega. \quad (2.2)$$

Now we assume f^0 and f^1 smooth and postpone a detailed description of properties of f^0, f^1 and $\widetilde{f}$ to Section 3.2. We emphasize that, owing to linearity of the problem, any reasonable

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right-hand side can be represented as in (2.1), (2.2). After appropriate rescaling our choice of the factors $h^{-1/2}$ and $h^{1/2}$ in (2.1) as well as the orthogonality condition in (2.2) ensures that the potential energy stored by the plate Ω_h under the volume force (2.1) gets order $1 = h^0$, see Section 4.4.

As shown in [22], [23, Ch.4], the asymptotic ansatz

$$u(h, x) = h^{-3/2}U^0(y) + h^{-1/2}U^1(y, \zeta) + h^{1/2}U^2(y, \zeta) + \ldots \tag{2.3}$$

of the solution to problem (1.4)-(1.6) is perfectly adjusted with decomposition (2.1). The coordinate change $x \mapsto (y, \zeta)$ splits the differential operators $L(\nabla)$ and $N^{\pm}(\nabla)$ in (1.4) and (1.5), respectively, as follows:

$$L(\nabla) := D(-\nabla)^{\top} AD(\nabla) = h^{-2}L^0(\partial_\zeta) + h^{-1}L^1(\nabla_y, \partial_\zeta) + h^0 L^2(\nabla_y), \tag{2.4}$$
$$N^{\pm}(\nabla) := D(0, 0, \pm 1)^{\top} AD(\nabla) = h^{-1}N^{0\pm}(\partial_\zeta) + h^0 N^{1\pm}(\nabla_y),$$

where

$$L^0 = -D_\zeta^{\top} AD_\zeta, \quad L^1 = -D_\zeta^{\top} AD_y - D_y^{\top} AD_\zeta, \quad L^2 = -D_y^{\top} AD_y,$$
$$N^{0\pm} = \pm D_3^{\top} AD_\zeta, \quad N^{1\pm} = \pm D_3^{\top} AD_y$$
$$D_\zeta = D(0, 0, \partial_\zeta), \quad D_y = D(\nabla_y, 0), \quad D_3 = (0, 0, 1), \tag{2.5}$$
$$\partial_\zeta = \partial/\partial\zeta, \quad \nabla_y = (\partial/\partial y_1, \partial/\partial y_2).$$

We now insert formulas (2.4) and (2.3), (2.1) into (1.4) and (1.5) and collect coefficients of h^{p-2} and h^{p-1}, respectively. Equalizing their sums yields a recursive sequence of the Neumann problems on the interval $(-1/2, 1/2) \ni \zeta$ with the parameter $y \in \omega$. For $p = 0, 1, 2$, we have

$$L^0 U^p = F^p := -L^1 U^{p-1} - L^2 U^{p-2} \text{ for } \zeta \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right), \tag{2.6}$$
$$N^{0\pm}U^p = G^{p\pm} := -N^{1\pm}U^{p-1} \text{ at } \zeta = \pm\tfrac{1}{2},$$

where $U^{-2} = U^{-1} = 0$. Since $F^0 = 0$, $G^{0\pm} = 0$ and $F^1 = 0$, $G^{1\pm} = \mp D_3^{\top} AD_y U^0$, we readily take

$$U^0(y) = w_3(y) e_3, \quad U^1(y) = \sum_{i=1}^{2} e_i \left(w_i(y) - \zeta \frac{\partial w_3}{\partial y_i}(y)\right) \tag{2.7}$$

as solutions to problem (2.6) with $p = 0$ and $p = 1$. Here, e_j is the unit vector of the x_j-axis and

$$w = (w_1, w_2, w_3)^{\top} \tag{2.8}$$

is a vector function to be determined. Notice that $h^{-3/2}w_3(y)$ and $h^{-1/2}w_i(y)$ will be the averaged deflection and longitudinal displacements in the plate.

If $p = 2$, one may easily verify that the right-hand sides in (2.6) become

$$F^2(y, \zeta) = D_\zeta^{\top} AD_y U^1 + D_y^{\top} A(D_\zeta U^1 + D_y U^0) = D_\zeta^{\top} A\mathcal{Y}(\zeta)\mathcal{D}(\nabla_y)w(y), \tag{2.9}$$
$$G^{2\pm}(y) = \mp D_3^{\top} A\mathcal{Y}(1/2)\mathcal{D}(\nabla_y)w(y)$$

where $D_\zeta U^1 + D_y U^0 = 0$ in view of (2.7) and

$$\mathcal{Y}(\zeta) = \begin{pmatrix} \mathbb{I}_3 & -2^{1/2}\mathbb{I}_3\zeta \\ \mathbb{O}_3 & \mathbb{O}_3 \end{pmatrix}, \quad \mathcal{D}(\nabla_y)^{\top} = \begin{pmatrix} \partial_1 & 0 & 2^{-1/2}\partial_2 & 0 & 0 & 0 \\ 0 & \partial_2 & 2^{-1/2}\partial_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2^{-1/2}\partial_1^2 & 2^{-1/2}\partial_2^2 & \partial_1\partial_2 \end{pmatrix},$$
$$\tag{2.10}$$

cf. (1.9), $\mathbb{I}_3$ and $\mathbb{O}_3$ are the unit and null matrices of size 3×3.

Hence,

$$U^2 (y, \zeta) = \mathcal{X} (\zeta) \mathcal{D} (\nabla_y) w (y) \tag{2.11}$$

and $\mathcal{X}$ is a matrix solution (of size 3×6) to the problem

$$-D_\zeta^\top A D_\zeta \mathcal{X} (\zeta) = D_\zeta^\top A \mathcal{Y} (\zeta), \ \ \zeta \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right), \ \ \pm D_3^\top A D_\zeta \mathcal{X} \left(\pm\tfrac{1}{2}\right) = \mp D_3^\top A \mathcal{Y} \left(\pm\tfrac{1}{2}\right). \tag{2.12}$$

Compatibility conditions in problem (2.12) are evidently satisfied.

The problem with $p = 3$

$$L^0 U^p = F^p \text{ for } \zeta \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right), \quad N^{0\pm} U^p = G^{p\pm} \text{ at } \zeta = \pm\tfrac{1}{2} \tag{2.13}$$

gets the right-hand sides

$$F^3 = -L^1 U^2 - L^0 U^1 + f^0 = D_\zeta^\top A D_y U^2 + D_y^\top A \left(D_\zeta U^2 + D_y U^1\right) + f^0, \tag{2.14}$$

$$G^{3\pm} = \mp N^{1\pm} U^2 = \mp D_3^\top A D_y U^2.$$

Let us consider the compatibility conditions

$$I_j^p (y) := \int_{-1/2}^{1/2} F_j^p (y, \zeta) \, d\zeta + G_j^{p+} (y) + G_j^{p-} (y) = 0, \quad j = 1, 2, 3, \tag{2.15}$$

with $p = 3$. By (2.14) and (2.7), (2.11), we have

$$D_\zeta U^2 (y, \zeta) + D_y U^1 (y, \zeta) = (D_\zeta \mathcal{X} (\zeta) + \mathcal{Y} (\zeta)) \mathcal{D} (\nabla_y) w (y) \tag{2.16}$$

and

$$I_i^3 (y) = e_i^\top D_y^\top \int_{-1/2}^{1/2} A (D_\zeta \mathcal{X} (\zeta) + \mathcal{Y} (\zeta)) \, d\zeta \mathcal{D} (\nabla_y) w (y) \, d\zeta + \int_{-1/2}^{1/2} f_i^0 (y, \zeta) \, d\zeta.$$

Since $e_i^\top D_y^\top = e_i^\top \mathcal{D} (\nabla_y)$ according to (1.9), (2.5), (2.10), the equalities $I_i^3 (y) = 0$ with $i = 1, 2$ are nothing but two upper lines in the system of differential equations

$$\mathcal{D} (-\nabla_y)^\top A \mathcal{D} (\nabla_y) w (y) = g (y), \quad y \in \omega, \tag{2.17}$$

where

$$\mathcal{A} = \int_{-1/2}^{1/2} \mathcal{Y} (\zeta)^\top A (D_\zeta \mathcal{X} (\zeta) + \mathcal{Y} (\zeta)) \, d\zeta \tag{2.18}$$

and

$$g_i (y) = \int_{-1/2}^{1/2} f_i^0 (y, \zeta) \, d\zeta, \quad i = 1, 2. \tag{2.19}$$

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The compatibility condition (2.15) with $j = 3$ is fulfilled automatically. Indeed, we take into account the orthogonality condition in (2.2) and write

$$I_3^3(y) = \int_{-1/2}^{1/2} e_3^\top D_y^\top A \left(D_\zeta \mathcal{X}(\zeta) + \mathcal{Y}(\zeta) \right) d\zeta \; \mathcal{D}(\nabla_y) w(y)$$

$$= \sum_{i=1}^{2} \frac{\partial}{\partial y_i} \int_{-1/2}^{1/2} (D_\zeta \zeta e_i)^\top A \left(D_\zeta \mathcal{X}(\zeta) + \mathcal{Y}(\zeta) \right) d\zeta \; \mathcal{D}(\nabla_y) w(y)$$

$$= \sum_{i=1}^{2} \frac{\partial}{\partial y_i} \left(- \int_{-1/2}^{1/2} e_i^\top \zeta D_\zeta^\top A \left(D_\zeta \mathcal{X}(\zeta) + \mathcal{Y}(\zeta) \right) d\zeta \right.$$

$$\left. + e_i^\top \zeta D_3^\top A \left(D_\zeta \mathcal{X}(\zeta) + \mathcal{Y}(\zeta) \right) \Big|_{\zeta=-1/2}^{1/2} \right) \mathcal{D}(\nabla_y) w(y) = 0.$$

Here, we again applied representation (2.16) together with the identity

$$D_y e_3 = \sum_{i=1}^{2} D_\zeta \zeta e_i \frac{\partial}{\partial y_i} \tag{2.20}$$

inherited from (1.9), (2.5), and finally recalled problem (2.12) for the 3×6-matrix $\mathcal{X}$.

Let us demonstrate that the third line of system (2.17) is equivalent to the equality

$$I_3^4(y) = 0, \tag{2.21}$$

that is the third compatibility condition (2.15) in problem (2.13) with the right-hand sides

$$F^3 = -L^1 U^3 - L^0 U^2 + f^1 = D_\zeta^\top A D_y U^3 + D_y^\top A \left(D_\zeta U^3 + D_y U^2 \right) + f_3^1,$$
$$G^{4\pm} = \mp N^{1\pm} U^3 = \mp D_3^\top A D_y U^3.$$

We have

$$I_3^4(y) = \int_{-1/2}^{1/2} e_3^\top D_y^\top A \left(D_\zeta U^3 + D_y U^2 \right) d\zeta + f_3^1 \tag{2.22}$$

$$= \sum_{i=1}^{2} \frac{\partial}{\partial y_i} \int_{-1/2}^{1/2} (D_\zeta \zeta e_i)^\top A \left(D_\zeta U^3 + D_y U^2 \right) d\zeta + f_3^1$$

$$= f_3^1 + \sum_{i=1}^{2} \frac{\partial}{\partial y_i} e_i^\top \left(- \int_{-1/2}^{1/2} \zeta D_\zeta^\top A \left(D_\zeta U^3 + D_y U^2 \right) d\zeta + \left(\zeta D_3^\top A \left(D_\zeta U^3 + D_y U^2 \right) \right) |_{\zeta=-1/2}^{1/2} \right)$$

$$= f_3^1 + \sum_{i=1}^{2} \frac{\partial}{\partial y_i} e_i^\top \int_{-1/2}^{1/2} \zeta \left(D_y^\top A \left(D_\zeta U^2 + D_y U^1 \right) + f^0 \right) d\zeta$$

$$= g_3 + \sum_{i=1}^{2} \frac{\partial}{\partial y_i} e_i^\top D_y^\top \int_{-1/2}^{1/2} \zeta A \left(D_\zeta \mathcal{X} + \mathcal{Y} \right) d\zeta \mathcal{D}(\nabla_y) w$$

where

$$g_3(y) = f_3^1(y) + \sum_{i=1}^{2} \int_{-1/2}^{1/2} \zeta \frac{\partial f_i^0}{\partial y_i}(y, \zeta) \, d\zeta. \tag{2.23}$$

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Let us clarify calculation (2.22). First, we used identity (2.20) and integrate by parts in the interval $(-1/2, 1/2)$. Then we took into account problem (2.6) with $p = 3$ and its right-hand sides (2.14). Finally, formula (2.16) was applied.

In view of (1.9) and (2.10) one observes that

$$\sum_{i=1}^{2} \frac{\partial}{\partial y_i} e_i^\top D_y^\top \zeta = e_3^\top \mathcal{D} \left(\nabla_y \right) \mathcal{Y} \left(\zeta \right)$$

and, hence, equality (2.21) indeed implies the third line of system (2.17) due to definition (2.18).

2.2 The plate equations

Owing to the Dirichlet condition (1.7) on the lateral side υ_h of the plate Ω_h, we supply system (2.17) with the boundary conditions

$$w_i \left(y \right) = 0, \ i = 1, 2, \quad w_3 \left(y \right) = 0, \quad \partial_n w_3 \left(y \right) = 0, \ y \in \partial \omega, \tag{2.24}$$

where $\partial_n = n^\top \nabla_y$ and $n = (n_1, n_2)^\top$ is the unit vector of the outward normal at the boundary of the domain $\omega \subset \mathbb{R}^2$. Notice that conditions (2.24) make the main asymptotic terms (2.7) in ansatz (2.3) vanish at the lateral boundary υ_h.

It follows from the weighted Korn's inequality (2.10) in [4] and will be shown by other means later that the Dirichlet conditions (1.6) at the small support zones (1.2) lead to the Sobolev (point) condition at the y-coordinates origin $\mathcal{O}$, namely

$$w_3 \left(\mathcal{O} \right) = 0. \tag{2.25}$$

To prove the unique solvability of the limit two-dimensional problem (2.17), (2.24), (2.25) needs a piece of information on the matrix $\mathcal{A}$ of the system.

Lemma 1 *Matrix (2.18) is block-diagonal and takes the form*

$$\mathcal{A} = \left(\begin{array}{cc} \mathcal{A}' & \mathbb{O}_3 \\ \mathbb{O}_3 & \mathcal{A}_{(3)} \end{array} \right) = \left(\begin{array}{cc} A^0 & \mathbb{O}_3 \\ \mathbb{O}_3 & \frac{1}{6} A^0 \end{array} \right) \tag{2.26}$$

where $A^0 = A_{(yy)} - A_{(yz)} A_{(zz)}^{-1} A_{(zy)}$ is a symmetric and positive definite 3×3-matrix constructed from submatrices of size 3×3 in the representation of the stiffness matrix

$$A = \left(\begin{array}{cc} A_{(yy)} & A_{(yz)} \\ A_{(zy)} & A_{(zz)} \end{array} \right). \tag{2.27}$$

Proof. We introduce the diagonal matrix $\mathbb{J} = \mathrm{diag} \left(2^{-1/2}, 2^{-1/2}, 1 \right)$ and, by (2.27) and (1.9), rewrite problem (2.12) as follows:

$$-A_{(zz)} \mathbb{J} \partial_\zeta^2 \mathcal{X} \left(\zeta \right) = A_{(zy)} (\mathbb{O}_3, -2^{1/2} \mathbb{I}_3), \ \zeta \in (-1/2, 1/2),$$

$$\pm A_{(zz)} \mathbb{J} \partial_\zeta \mathcal{X} \left(\pm 1/2 \right) = \mp A_{(zy)} (\mathbb{I}_3, \mp 2^{1/2} \mathbb{I}_3).$$

Its solution thus reads

$$\mathcal{X} \left(\zeta \right) = \mathbb{J}^{-1} A_{(zz)}^{-1} A_{(zy)} \left(-\zeta \mathbb{I}_3, 2^{1/2} \left(\frac{\zeta^2}{2} - \frac{1}{24} \right) \mathbb{I}_3 \right). \tag{2.28}$$

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Inserting this formula into (2.18) assures representation (2.26). Since matrix (2.27) is symmetric and positive definite, these properties are evidently passed to A^0. ∎

Representation (2.26) indicates a remarkable fact: for any homogeneous anisotropic plate the limit two-dimensional system (2.17) divides into the fourth-order differential operator

$$\mathcal{L}_3\left(\nabla_y\right) = \tfrac{1}{6}\mathcal{D}_3\left(\nabla_y\right)^\top A^0\mathcal{D}_3\left(\nabla_y\right) \tag{2.29}$$

for the deflection w_3 and the 2×2-matrix second-order of differential operators

$$\mathcal{L}'\left(\nabla_y\right) = \mathcal{D}'\left(-\nabla_y\right)^\top A^0\mathcal{D}'\left(\nabla_y\right) \tag{2.30}$$

for the longitudinal displacement vector $w' = \left(w_1, w_2\right)^\top$.

In the isotropic case (see formula (1.10) in [4]) we have

$$A^0 = \begin{pmatrix} \lambda'+2\mu & \lambda' & 0 \\ \lambda' & \lambda'+2\mu & 0 \\ 0 & 0 & 2\mu \end{pmatrix}, \quad \lambda' = \frac{2\lambda\mu}{\lambda+2\mu}$$

while $\mathcal{L}_3\left(\nabla_y\right)$ and $\mathcal{L}'\left(\nabla_y\right)$ are the bi-harmonic operator and the plane Lamé operator

$$\mathcal{L}_3'\left(\nabla_y\right) = \frac{\mu}{3}\frac{\lambda+\mu}{\lambda+2\mu}\Delta_y^2, \quad \mathcal{L}'\left(\nabla_y\right) = -\mu\Delta_y\mathbb{I}_2 - \left(\lambda'+\mu\right)\nabla_y\nabla_y^\top.$$

Remark 2 *Throughout the paper it is convenient to write the constructed solutions of problems (2.6) and (2.13) with f^0 in the form*

$$U^p\left(y,\zeta\right) = W^p\left(\zeta,\nabla_y\right)w\left(y\right), \quad p = 0,1,2,3, \tag{2.31}$$

where, according to (2.7), (2.11) and (2.28),

$$W^0\left(\zeta,\nabla_y\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad W^1\left(\zeta,\nabla_y\right) = \begin{pmatrix} 1 & 0 & -\zeta\partial_1 \\ 0 & 1 & -\zeta\partial_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \partial_i = \frac{\partial}{\partial y_i}, \tag{2.32}$$

$$W^2\left(\zeta,\nabla_y\right) = \mathbb{J}^{-1}A_{(zz)}^{-1}A_{(zy)}\left(-\zeta\mathbb{I}_3, 2^{1/2}\left(\frac{\zeta^2}{2} - \frac{1}{24}\right)\mathbb{I}_3\right)\mathcal{D}\left(\nabla_y\right).$$

A concrete formula for $W^3\left(\zeta,\nabla_y\right)$ will not be applied. We emphasize that (2.31) is a linear combination of derivatives of $w' = \left(w_1, w_2\right)$ of order $p-1$ and w_3 of order p. ∎

Proposition 3 *Let*

$$g_3 = g_3^0 - \sum_{i=1}^2 \frac{\partial}{\partial y_i}g_3^i, \quad g_3^0, \; g_3^i, \; g_i \in L^2\left(\omega\right). \tag{2.33}$$

Problem (2.17), (2.24), (2.25) admits a unique solution in the energy space $\mathcal{H} = H_0^1\left(\omega;\partial\omega\right)^2 \times H_0^2\left(\omega;\partial\omega\cup\mathcal{O}\right)$. Moreover, this solution falls into $H^2\left(\omega\right)^2 \times H^3\left(\omega\right)$ and meets estimate

$$\|w; H^2\left(\omega\right)^2 \times H^3\left(\omega\right)\| \leq c\mathcal{N} \tag{2.34}$$

where $\mathcal{N}$ is the sum of norms of functions indicated in (2.33). The space $\mathcal{H}$ consists of vector functions (2.8) in the direct product $H^1\left(\omega\right)^2 \times H^2\left(\omega\right)$ of Sobolev spaces such that the Dirichlet (2.24) and Sobolev (2.25) conditions are satisfied.

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Proof. The variational formulation of problem (2.17), (2.24), (2.25) reads: to find $w \in \mathcal{H}$ fulfilling the integral identity

$$\left(\mathcal{A}\mathcal{D}(\nabla_y)w, \mathcal{D}(\nabla_y)v\right)_\omega = \left(g_3^0, v_3\right)_\omega + \sum_{i=1}^2 \left((g_i, v_i)_\omega + \left(g_3^i, \frac{\partial v_3}{\partial y_i}\right)_\omega\right), \qquad \forall v \in \mathcal{H}. \qquad (2.35)$$

The latter, as usual, is obtained by multiplying system (2.17) scalarly with a test function $v \in \mathcal{H}$ and integrating by parts in ω with the help of conditions (2.24), (2.25) for v and the formula for g_3 in (2.33). Clearly, the right-hand side of (2.35) is a continuous linear functional in $\mathcal{H} \ni v$. The left-hand side of (2.35) may be chosen as a scalar product in $\mathcal{H}$. Indeed, the necessary properties of this bi-linear form are inherited from the positive definiteness of the matrix $\mathcal{A}^0$ together with the following Friedrichs and Korn inequalities:

$$||u; L^2(\omega)|| \le c_\omega ||\nabla_y u; L^2(\omega)||, \quad \forall u \in H_0^1(\omega; \partial\omega), \qquad (2.36)$$

$$||\nabla_y w'; L^2(\omega)|| \le c_\omega ||\mathcal{D}'(\nabla_y) w'; L^2(\omega)||, \quad \forall w' \in H_0^1(\omega; \partial\omega)^2.$$

Equivalently, we may write equation (2.35) as the minimization problem

$$\min\left\{\frac{1}{2}\left(\mathcal{A}\mathcal{D}(\nabla_y)w, \mathcal{D}(\nabla_y)w\right)_\omega - (g, w)_\omega \ : \ w \in \mathcal{H}\right\}$$

where $g = (g_1, g_2, g_3)$ is given by (2.33). Notice that the Sobolev embedding theorem $H^2 \subset C$ in $\mathbb{R}^2$ assures that $\mathcal{H}$ is a closed subspace in $H^1(\omega)^2 \times H^2(\omega)$ and the second inequality in (2.36) can be easily derived by integration by parts because w' vanishes at $\partial\omega$.

Now the Riesz representation theorem guarantees the existence of a unique weak solution to problem (2.35) and the estimate

$$||w; H^1(\omega)^2 \times H^2(\omega)|| \le c\mathcal{N}.$$

Finally, a result in [14, Ch.2] on lifting smoothness of solutions to elliptic problems, see also [5] for details about the fourth-order equation, concludes with the inclusion $w \in H^2(\omega)^2 \times H^3(\omega)$ and inequality (2.34). ∎

A classical assertion, cf. [7, 8, 13, 32] and others, was outlined in Theorem 7 of [4], namely the rescaled displacements $h^{3/2} u_3(h, y, h\zeta)$ and $h^{1/2} u_i(h, y, h\zeta)$, $i = 1, 2$, taken from the three-dimensional problem (1.4)-(1.7), converge in $L^2(\omega \times (-1/2, 1/2))$ as $h \to +0$ to the functions $w_3(y)$ and $w_i(y) - \zeta \frac{\partial w_3}{\partial y_i}(y)$, respectively, where $w = (w_1, w_2, w_3)^\top$ is the solution of the two-dimensional Dirichlet-Sobolev problem given in Proposition 3. Moreover, the convergence rate $O(|\ln h|^{-1/2})$, see Theorem 8 and Remark 6, is rather slow and in the next section we will improve the asymptotic result by constructing three-dimensional boundary layers studied in [4, 18, 21].

3 Constructing and justifying the asymptotics

3.1 Matching the outer and inner expansions

Intending to apply the method of matched asymptotic expansions, cf. [10, 33] and [15, Ch.2], we regard (2.3) as the *outer* expansion suitable at a distance from the small support area (1.2).

In the vicinity of the set θ_h we, as mentioned in Section 3 of [4], use the stretched coordinates

$$\xi = (\eta, \zeta) = (h^{-1}y, h^{-1}z) \tag{3.1}$$

and engage the *inner* expansion in the form

$$u(h, x) = h^{1/2}\mathcal{P}(\xi)\, a + \dots \tag{3.2}$$

where $\mathcal{P}$ is the elastic logarithmic potential, see Section 3.4 in [4], and $a \in \mathbb{R}^4$ is a column to be determined. The decompositions for this potential derived in [4] show that, when $\rho \to +\infty$, we have

$$h^{1/2}\mathcal{P}(\xi)\, a = h^{1/2}\Big(\sum_{p=0}^{3} h^p W^p(\zeta, \nabla_\eta)\, \Phi^\sharp(\eta) + d^\sharp(\eta, \zeta)\, C^\sharp \Big) a + \dots = \tag{3.3}$$

$$= \sum_{p=0}^{3} h^{p-3/2} W^p(\zeta, \nabla_y) \Big(\Phi^\sharp(y) - \Psi \ln h + d^\sharp(\eta, 0)\, C^\sharp \Big) a + \dots$$

Here, the operators $W^p(\zeta, \nabla_\eta)$ were introduced in Remark 2 and dots stand for lower-order asymptotic terms and we have taken into account formulas

$$W^p(\zeta, \nabla_\eta) = h^{p-1} W^p(\zeta, \nabla_y)\, H, \quad H = \mathrm{diag}\{1, 1, h\}, \tag{3.4}$$

$$d^\sharp(\eta, \zeta) = H^{-1} d^\sharp(y, \zeta), \quad \Phi^\sharp(\eta) = H^{-1}\Big(\Phi^\sharp(y) - d^\sharp(y, 0)\, \Psi \ln h \Big)$$

where $d^\sharp$ and $\Phi^\sharp$ are the following 3×4-matrices

$$d^\sharp(\eta, \zeta) = \begin{pmatrix} 1 & 0 & 0 & \zeta \\ 0 & 1 & -\zeta & 0 \\ 0 & 0 & \eta_2 & -\eta_1 \end{pmatrix} \tag{3.5}$$

$$\Phi^\sharp(y) = \Big(d^\sharp(-\nabla_y, 0)^\top \Phi(y)^\top \Big)^\top = \begin{pmatrix} \Phi'(y) & 0 & 0 \\ & 0 & 0 \\ 0 & 0 & \Phi_3^2(y) & \Phi_3^1(y) \end{pmatrix}, \quad \Phi_3^j = \frac{\partial \Phi_3}{\partial y_j},$$

which are composed, respectively, from rigid motions and the nondegenerate block-diagonal 4×4-matrix

$$\Psi = \mathrm{diag}\{\Psi', \Psi_3, \Psi_3\} \tag{3.6}$$

where Ψ' is a non-degenerate numeral 2×2-matrix and Ψ_3 is a scalar in the representations

$$\Phi'(y) = \Psi' \ln r + \psi'(\varphi), \quad \Phi_3(y) = r^2 \Big(-\frac{1}{2}\Psi_3 \ln r + \psi_3(\varphi) \Big) \tag{3.7}$$

of the fundamental matrix of the differential operator (2.30) of size 2×2 and the fundamental solution of the scalar differential operator (2.29). These were described in detail in Section 3.4 of [4] on the base of general results in [9]. Moreover, (r, φ) is the polar coordinate system and ψ', ψ_3 are smooth on the unit circle $\mathbb{S}^1 \ni \varphi$. Furthermore, we have used the diagonal matrix $H = \mathrm{diag}\{1, 1, h\}$ and the obvious formulas

$$d^\sharp(\eta, \zeta) = \big(W^0 + W^1(\zeta, \nabla_\eta) \big) d^\sharp(\eta, 0), \quad W^2(\zeta, \nabla_\eta) d^\sharp(\eta, 0) = W^3(\zeta, \nabla_\eta) d^\sharp(\eta, 0) = 0. \tag{3.8}$$

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Note that $\ln h$ comes to (3.4) from the relation $\ln |\eta| = \ln |y| - \ln h$ and H is caused by different orders of differentiation in $W^p(\zeta, \nabla_\eta)$ and different degrees of $\rho = |\eta|$ in matrix functions but H disappears from the final expression in (3.3).

We also will need the following expansion of the elastic logarithmic potential, see Section 3.6 in [4],

$$\mathcal{P}(\xi) = (1 - \chi_\theta(\eta)) \left(\sum_{p=0}^{3} W^p(\zeta, \nabla_\eta) \Phi^\sharp(\eta) + d^\sharp(\eta, 0) C^\sharp + \Upsilon^\sharp(\eta) \right) + \widetilde{\widetilde{\mathcal{P}}}(\xi) \qquad (3.9)$$

which had been used in (3.3). Ingredients of (3.9) were defined in (2.32) and (3.5)-(3.7) while $C^\sharp$ stands for the elastic capacity matrix, χ_θ a smooth compactly supported function which equals 1 in the vicinity of the clamped area θ, the vector function $\Upsilon^\sharp$ was described in Section 3.6 of [4] but it plays an auxiliary role and an explicit form is not used below, and the remainders $\widetilde{\widetilde{\mathcal{P}}}$ admits the estimates for a big $\rho = |\eta|$

$$|\nabla_\eta^p \partial_\zeta^q \widetilde{\widetilde{\mathcal{P}}}(\xi)| \leq c_{pq} \rho^{-1+\varepsilon}, \ \ |\nabla_\eta^p \partial_\zeta^q \widetilde{\widetilde{\mathcal{P}}}_3(\xi)| \leq c_{pq} \rho^\varepsilon, \ \ p, q = 0, 1, 2, ..., \ \ \varepsilon > 0. \qquad (3.10)$$

The presence of $\Phi^\sharp(y)$ drives all detached terms in (3.4) from the space $H^1(\omega)^2 \times H^2(\omega)$ where the solution w of the limit problem (2.17), (2.24), (2.25) was found in Proposition 3. On the other hand, the Neumann boundary condition (1.5) on the lower base Σ^- does not hold on the small set $\overline{\theta_h} = \Sigma^- \setminus \Sigma^\bullet$ which shrinks to the coordinate origin $\mathcal{O}$. Hence, the dimension reduction procedure in Section 2 was not able to deduce the differential equations at the point $\mathcal{O}$ and we did not have a regular reason to cast out solutions with reasonable singularities. Let us introduce such solutions.

We denote by G' the Green 2×2-matrix of the Dirichlet problem in ω for the matrix operator (2.30). Its particular value $G'(\cdot, \mathcal{O})$ is a distributional solution of the problem

$$\mathcal{L}'(\nabla_y) G'(y, \mathcal{O}) = \delta(y) \mathbb{I}_2, \ y \in \omega, \quad G'(y, \mathcal{O}) = 0, \ y \in \partial \omega, \qquad (3.11)$$

where δ is the Dirac mass and $\mathbb{I}_2$ is the unit 2×2-matrix.

As known, the Green function G_3 of the Dirichlet problem for the fourth-order scalar differential operator (2.29) belongs to $H^2(\omega) \subset C(\omega)$ and its value

$$G_3(\mathcal{O}, \mathcal{O}) = (\mathcal{A}_3 \mathcal{D}_3(\nabla_y) G_3(\cdot, \mathcal{O}), \mathcal{D}_3(\nabla_y) G_3(\cdot, \mathcal{O}))_\omega$$

is positive, see, e.g., [5]. The derivative in the second argument

$$G_{3,i}(y, \mathcal{O}) = \frac{\partial}{\partial \mathbf{y}_i} G_3(y, \mathbf{y})|_{\mathbf{y}=0} \qquad (3.12)$$

leaves the space $H^2(\omega)$ but remains Hölder-continuous in ω and satisfies the equation

$$\mathcal{L}_3(\nabla_y) G_{3.i}(y, \mathcal{O}) = -\frac{\partial}{\partial y_i} \delta(y), \ y \in \omega, \qquad (3.13)$$

and the Dirichlet conditions (2.24). However, the Sobolev condition (2.25) is not achieved yet and we set

$$G_3^i(y, \mathcal{O}) = G_{3,i}(y, \mathcal{O}) - G_{3,i}(\mathcal{O}, \mathcal{O}) G_3(\mathcal{O}, \mathcal{O})^{-1} G_3(y, \mathcal{O}), \qquad (3.14)$$

cf. [5], so that $G_3^i(\mathcal{O}, \mathcal{O}) = 0$ and $G_3^i(y, \mathcal{O}) = \partial_n G_3^i(y, \mathcal{O}) = 0$, $y \in \partial\omega$. Finally we introduce the matrix function of size 3×4

$$G^\sharp(y) = \begin{pmatrix} G'(y, \mathcal{O}) & 0 & 0 \\ & 0 & 0 \\ 0 & 0 & -G_3^2(y, \mathcal{O}) & G_3^1(y, \mathcal{O}) \end{pmatrix}$$

which, by virtue of (3.11) and (3.13), admits the representation

$$G^\sharp(y) = \Phi^\sharp(y) + \widehat{G}^\sharp(y) \tag{3.15}$$

where $\Phi^\sharp$ is the singular matrix function in (3.5) and $\widehat{G}^\sharp$ is the regular part. According to general properties of the Green matrices of second-order systems, see [9], first two lines $\widehat{G}_i^\sharp$, $i = 1, 2$, in $\widehat{G}^\sharp$ involve smooth functions in $\overline{\omega}$, however the singularity $O(r^2 |\ln r|)$ of the subtrahend in (3.14) moves some entries of $\widehat{G}_3^\sharp$ from $C^2(\omega)$ but keep them still in the Hölder space $C^{1,\alpha}(\omega)$ with any $\alpha \in (0, 1)$. Hence, the remainder in (3.15) verifies the relations

$$\widehat{G}^\sharp(y) = d^\sharp(y, 0) \mathcal{G}^\sharp + \widetilde{G}^\sharp(y), \quad \widetilde{G}_i^\sharp(y) = O(r), \quad i = 1, 2, \quad \widetilde{G}_3^\sharp(y) = O(r^2(1 + |\ln r|)), \tag{3.16}$$

where $\mathcal{G}^\sharp = \mathcal{G}^\sharp(A, \omega)$ is a numeric matrix of size 4×4. Properties of this matrix are just the same as ones of the elastic logarithmic capacity and verification of its symmetry is a sufficient simplification of the proof of Theorem 13 in [4].

Lemma 4 *The 4×4-matrix $\mathcal{G}^\sharp = \mathcal{G}^\sharp(A, \omega)$ in (3.16) is symmetric.*

Let us put into the outer expansion (2.3) the singular solution

$$w(y) = \widehat{w}(y) + G^\sharp(y) a \tag{3.17}$$

of problem (2.17), (2.24), (2.25) with the same coefficient column $a \in \mathbb{R}^4$ as in (3.2) and the smooth solution $\widehat{w} \in H^2(\omega)^2 \times H^3(\omega)$ given in Proposition 3. With a reference to the Sobolev embedding theorem $H^2 \subset C$ and the standard Hardy inequalities (see, e.g., Section 2.1 in [4]), we write

$$\widehat{w}(y) = d^\sharp(y, 0) F + \widetilde{w}(y) \tag{3.18}$$

where the coefficient column $F \in \mathbb{R}^4$ and the remainder $\widetilde{w}$ satisfy the estimates

$$|F| \leq c\mathcal{N}, \tag{3.19}$$

$$\int_\omega r^{-2}(1 + |\ln r|)^{-2} \left(|\nabla_y \widetilde{w}'(y)|^2 + r^{-2} |\widetilde{w}'(y)|^2 + |\nabla_y^2 \widetilde{w}_3(y)|^2 \right. \tag{3.20}$$

$$\left. + r^{-2} |\nabla_y \widetilde{w}_3(y)|^2 + r^{-4} |\widetilde{w}_3(y)|^2 \right) dy \leq c\mathcal{N},$$

while $\mathcal{N}$ is the sum of norms of functions in (2.33).

Notice that, according to definition of the Green matrix, we have

$$F = \int_\omega G^\sharp(y) g(y) dy.$$

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Using formulas (3.17), (3.18) and notation (2.31), we obtain for the outer expansion (2.3) that

$$h^{-3/2} \sum_{p=0}^{3} h^p U^p (y, \zeta) = h^{-3/2} \sum_{p=0}^{3} h^p W^p (\zeta, \nabla_y) \, w (y) \tag{3.21}$$

$$= \sum_{p=0}^{3} h^{p-3/2} W^p (\zeta, \nabla_y) \left(d^\sharp (y, 0) F + \left(\Phi^\sharp (y) + d^\sharp (y, 0) \right) a \right) + \ldots$$

where dots again stand for lower-order asymptotic terms. The method of matched asymptotic expansions requires that the expressions detached in (3.21) and (3.3), coincide with each other. This coincidence implies the system of linear algebraic equations

$$F + \mathcal{G}^\sharp a = -\ln h \; \Psi a + C^\sharp a \tag{3.22}$$

and, thus,

$$a (\ln h) = \left(|\ln h| \, \Psi + C^\sharp (A, \theta) - \mathcal{G}^\sharp (A, \omega) \right)^{-1} F. \tag{3.23}$$

We here display the dependence of $C^\sharp$ and $\mathcal{G}^\sharp$ on the stiffness matrix A and the domains θ and ω. Since the matrix (3.6) is not degenerate, the matrix

$$M^\sharp (\ln h) = |\ln h| \, \Psi + C^\sharp (A, \theta) - \mathcal{G}^\sharp (A, \omega) \tag{3.24}$$

is invertible for a small $h \in (0, 1)$, too. Moreover, column (3.23) is a rational vector function in $|\ln h| = -\ln h$ while clearly

$$a (\ln h) = |\ln h|^{-1} \Psi^{-1} F + O(|\ln h|^{-2}). \tag{3.25}$$

Formula (3.23) gives concrete expressions to all terms in (2.3), (3.17) and (3.2) so that our formal construction of main terms in the outer and inner expansions is completed.

3.2 The final assumptions on the right-hand sides

To provide the necessary properties of the regular solution (3.18) of problem (2.17), (2.24), (2.25), we put the following requirement on terms in representation (2.1):

$$f^0 \in L^2 \left(\omega \times \left(-\tfrac{1}{2}, \tfrac{1}{2} \right) \right)^3, \quad f_3^1 \in H^{-1} (\omega). \tag{3.26}$$

The latter inclusion, as usual, cf. [14], means that

$$f_3^1 (y) = \nabla_y^\top \overrightarrow{f}_3^1 (y) + f_{30}^1 (y), \quad \overrightarrow{f}_3^1 = \left(f_{31}^1, f_{32}^1 \right)^\top \in L^2 (\omega)^2, \quad f_{30}^1 \in L^2 (\omega). \tag{3.27}$$

More precisely, the right-hand side (2.23) of the third equation in system (2.17) gives rise to the continuous functional

$$(g_3, \mathsf{v}_3)_\omega := \left(f_{30}^1, \mathsf{v}_3 \right)_\omega - \sum_{i=1}^{2} \left(f_{3i}^1 + \int_{-1/2}^{1/2} \zeta f_i^0 (\cdot, \zeta) \, d\zeta, \frac{\partial \mathsf{v}_3}{\partial y_i} \right)_\omega, \quad \mathsf{v}_3 \in H_0^1 (\omega). \tag{3.28}$$

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In view of (2.19), (3.26) and (3.28), (3.27) condition (2.33) in Proposition 3 is fulfilled so that the solution $\widehat{w} \in H^2 (\omega)^2 \times H^3 (\omega)$ meets estimate (2.34) where, as well as in (3.19), (3.20), $\mathcal{N}$ can be now fixed as

$$\mathcal{N} = \left\| f^0; L^2 \left(\omega \times \left(-\tfrac{1}{2}, \tfrac{1}{2} \right) \right) \right\| + \sum_{p=0}^{2} \left\| f^1_{3p}; L^2 (\omega) \right\|. \tag{3.29}$$

For the remainder $\widetilde{f}$ in (2.1), we suppose that the expression

$$\widetilde{\mathcal{N}} = h^{-1/2} |\ln h|^{-1} \left(h^{-1} \| s_h^2 S_{h2}^{-1} \widetilde{f}_3; L^2 (\Omega_h) \| + \sum_{i=1}^{2} \| s_h S_{h1}^{-1} \widetilde{f}_i; L^2 (\Omega_h) \| \right) \tag{3.30}$$

gets the same order as the expression (3.29). Notice that weights

$$s_h(y) = h + \operatorname{dist}(y, \partial\omega), \ \ S_{hq}(y) = (h^2 + |y|^2)^{-q/2}(1 + |\ln(h^2 + |y|^2)|)^{-1}, \ \ q = 0, 1, \tag{3.31}$$

come from the anisotropic Korn inequality (2.10) in [4] and make the norms on the right-hand side of (3.30) much weaker that the common Lebesgue norms. The factor $h^{-1/2} |\ln h|^{-1}$ is consistent with the final error estimate in Theorem 7. In other words, our way to signify the smallness of this remainder does not spoil the precision of the two-dimensional model.

3.3 The global approximation solution

We proceed with constructing a vector function $\mathbf{u} \in H^1_0 (\Omega_h; \upsilon_h \cup \theta_h)^3$ which is a bit cumbersome but very convenient for an estimation of the difference $u - \mathbf{u}$ between the true and approximate solutions of problem (1.4)-(1.7). In the next section we will get rid of unnecessary terms in $\mathbf{u}$ to conclude a fair asymptotics of u.

Although in the previous section we have used the method of matched expansions, we now apply an asymptotic structure attributed to the method of compound expansions, see a comparison of the methods in [15, Ch.2]. We give priority to expansion (3.2) and insert into (2.3) the decaying near $\mathcal{O}$ component $\widetilde{w}$, see (3.18) and (3.20), instead of the whole singular solution (3.17). We also introduce the cut-off functions

$$X_h^\omega (y) = 1 - \chi \left(h^{-1} |n| \right), \quad X_h^\theta (y) = 1 - \chi \left(2h^{-1} r/R_\theta \right) \tag{3.32}$$
$$\chi \in C^\infty (R), \ \chi(r) = 1 \ \text{ for } r < \tfrac{1}{2}, \ \chi(r) = 0 \ \text{ for } r \geq 1, \ 0 \leq \chi \leq 1,$$

which help to fulfill the Dirichlet conditions (1.7) and (1.6) because X_h^ω and X_h^θ vanish near the clamped sets υ_h and θ_h respectively. To this end, R_θ is fixed such that $\overline{\theta}$ belongs to the disk $\mathbb{B}^2_R$ of radius $R = R_\theta$.

We set

$$\mathbf{u} (h, x) = h^{-1/2} X_h^\omega (y) P \left(h^{-1} x \right) a (\ln h) + \tag{3.33}$$

$$+ h^{-3/2} X_h^\omega (y) \sum_{p=0}^{2} h^p W^p (\zeta, \nabla_y) \left(X_h^\theta (y) \widetilde{w} (y) - h\mathbf{w} (h, y) \right)$$

where the following correction term is introduced

$$\mathbf{w} (h, y) = \left(1 - \chi \left(R_\omega^{-1} r \right) \right) \Upsilon^\sharp \left(h^{-1} y \right) \tag{3.34}$$

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with $\Upsilon^\sharp$ taken from (3.9) and $R_\omega > 0$ is such that $\overline{\mathbb{B}_{R_\omega}} \subset \omega$. We emphasize that, according to definition of the differential operators (2.32) in (2.31), the inclusion $\widetilde{w} \in H^2(\omega)^2 \times H^3(\omega)$ assures that

$$W^p(\zeta, \nabla_y)\widetilde{w} \in H^{3-p}(\omega)^3 \text{ for any } \zeta \in (-1/2, 1/2) \tag{3.35}$$

and, therefore, the term $W^3(\zeta, \nabla_y)\widetilde{w}(y)$ does not belong to the energy space $H^1(\Omega_h)^3$ and is excluded from the global approximation solution, i.e., $p = 0, 1, 2$ but $p \neq 3$ in (3.33).

Thanks to the cut-off functions (3.32) and the Dirichlet conditions for $\mathcal{P}$, the displacement field satisfies conditions (1.6), (1.7). Hence, the difference $\mathbf{v} = u - \mathbf{u} \in \mathring{H}^1(\Omega_h; v_h \cup \theta_h)^3$ can be taken as a test function in the integral identity (1.10). Subtracting from both sides the scalar product $(AD(\nabla)\mathbf{u}, D(\nabla)\mathbf{v})_{\Omega_h}$, we arrive at the formula

$$(AD(\nabla)\mathbf{v}, D(\nabla)\mathbf{v})_{\Omega_h} = (f, \mathbf{v})_{\Omega_h} - (AD(\nabla)\mathbf{u}, \mathbf{v})_{\Omega_h}. \tag{3.36}$$

By the Korn inequality (see (2.10) and Theorem 2 in [4]), the left-hand side of (3.36) exceed the product $c_A |||v; \Omega_h|||^2_\bullet$ with the weighted anisotropic Sobolev norm

$$|||u; \Omega_h|||^2_\bullet = \int_{\Omega_h} \left(\sum_{i=1}^2 \left(|\nabla_y u_i|^2 + \frac{h^2}{s_h^2} S_{h1}^2 \left(\left|\frac{\partial u_i}{\partial z}\right|^2 + \left|\frac{\partial u_3}{\partial y_i}\right|^2 \right) + \frac{1}{s_h^2} S_{h1}^2 |u_i|^2 \right) \right. \tag{3.37}$$
$$\left. + \left|\frac{\partial u_3}{\partial z}\right|^2 + \frac{h^2}{s_h^4} S_{h2}^2 |u_3|^2 \right) dx.$$

and our immediate objective becomes to examine the right-hand side. The complication arises from the cut-off functions (3.32) which are included into (3.33) and meet the relations

$$|\nabla_y X_h^\omega(y)| \leq ch^{-1}, \quad \text{supp} |\nabla_y X_h^\omega| \subset \omega(h) := \{y \in \omega : 0 > n > -h\}, \tag{3.38}$$
$$|\nabla_y X_h^\theta(y)| \leq ch^{-1}, \quad \text{supp} |\nabla_y X_h^\theta| \subset \overline{\mathbb{B}_{2R_\omega h}}.$$

Denoting $\mathbf{v}^\omega = X_h^\omega \mathbf{v}$ and $\widetilde{w}^\theta = X_h^\theta \widetilde{w}$, we obtain

$$||D(\nabla)\mathbf{v}^\omega; L^2(\Omega_h)|| \leq c \left(||D(\nabla)\mathbf{v}; L^2(\Omega_h)|| + h^{-1}||\mathbf{v}; L^2(\omega(h) \times (-h/2, h/2))|| \right)$$
$$\leq C||D(\nabla)\mathbf{v}; L^2(\Omega_h)|| =: C\mathbf{n},$$
$$||\widetilde{w}^\theta; H^2(\omega)^2 \times H^3(\omega)|| \leq c \left(||\widetilde{w}; H^2(\omega)^2 \times H^3(\omega)||^2 + J \right)$$
$$\leq c||\widetilde{w}; H^2(\omega)^2 \times H^3(\omega)||^2 \leq C\mathcal{N},$$

where J is the weighted integral on the left in (3.20). In both inequalities we have taken into account that the weights $s_h(y)$ and $r(1 + |\ln r|)$ are small in the boundary strip $\omega(h)$ and the disk $\mathbb{B}_{2R_\theta h}$.

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3.4 Estimating the discrepancies

Let $h^{-3/2}X_h^\omega \mathcal{W} - h^{-1/2}X_h^\omega \mathbf{W}$ stand for the last sum in (3.33). We have

$$
\begin{aligned}
(AD\,(\nabla)\,\mathbf{u}, D\,(\nabla)\,\mathbf{v})_{\Omega_h} &= h^{-1/2}\,(AD\,(\nabla)\,\mathcal{P}a, D\,(\nabla)\,\mathbf{v}^\omega)_{\Omega_h} \\
&\quad + h^{-1/2}\,\left(A\,(D\,(\nabla)\,X_h^\omega)\,(\mathcal{P}a + h^{-1}\mathcal{W} - \mathbf{W}), D\,(\nabla)\,\mathbf{v}\right)_{\Omega_h} \\
&\quad + \left(AD\,(\nabla)\,(\mathcal{P}a + h^{-1}\mathcal{W} - \mathbf{W}), (D\,(\nabla)\,X_h^\omega)\,\mathbf{v}\right)_{\Omega_h} \\
&\quad + h^{-1/2}\,(AD\,(\nabla)\,\mathbf{W}, D\,(\nabla)\,\mathbf{v}^\omega)_{\Omega_h} + h^{-3/2}\,(AD\,(\nabla)\,\mathcal{W}, D\,(\nabla)\,\mathbf{v}^\omega)_{\Omega_h} \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}
\tag{3.39}
$$

First of all, we observe that $\mathbf{v} = 0$ at the lateral side υ_h and, therefore, after extending $\mathbf{v}^\omega$ as null over the layer $\mathbb{R}^2 \times (-h/2, h/2)$ we make the coordinate dilation (3.1) and integrate by parts to conclude that $I_1 = 0$ since $\mathcal{P}$ is a solution of the homogeneous elasticity problem in the infinite layer $\Lambda = \mathbb{R}^2 \times (-1/2, 1/2)$ clamped over the set $\overline{\theta} \times \{-1/2\}$ on its lower base.

Next, formulas (3.9) and (3.17), (3.18) together with the matching condition (3.22) yield

$$
\mathcal{P}\,(\xi)\,a + h^{-1}\mathcal{W}\,(h, x) - \mathbf{W}\,(h, x)
$$
$$
= \widetilde{\widetilde{\mathcal{P}}}\,(\xi)\,a + h^2\left(W^3\,(\zeta, \nabla_y)\,\Phi^\sharp\,(\eta) + d^\sharp\,(\eta, 0)\,C^\sharp + \Upsilon^\sharp\,(\eta)\right)a + h^{-1}\sum_{p=0}^{2}h^p W^p\,(\zeta, \nabla_y)\,w\,(y).
$$

The first term on the right is estimated by means of (3.10) noticing that $\rho^{-1} = hr^{-1} = O\,(h)$ in the thin boundary strip $\omega\,(h)$. The second term and its derivatives become $O\left(h^2\,(1 + |\ln h|)\right)$ near the plate edge due to formulas (3.5), (3.8) and (3.25). Furthermore,

$$
\begin{aligned}
&\int_{\omega(h)}\left(s_h^{-2}\,|w'|^2 + |\nabla_y w'|^2 + s_h^{-4}\,|w_3|^2 + s_h^{-2}\,|\nabla_y w_3|^2 + |\nabla_y^2 w_3|^2\right)dy \\
&\leq c\int_{\omega(h)}\left(|\nabla_y w'|^2 + |\nabla_y^2 w_3|^2\right)dy \\
&\leq ch\left(\left\|w'; H^2\,(\omega \setminus \mathbb{B}_{R_\omega/2})\right\|^2 + \left\|w_3; H^3\,(\omega \setminus \mathbb{B}_{R_\omega/2})\right\|^2\right) \\
&\leq ch\mathcal{N}
\end{aligned}
$$

follows from formulas (3.20), (3.25), the standard Hardy inequalities (cf. Section 2.1 in [4]) and another consequence of the Newton-Leibnitz formula with some fixed $T \geq R_\omega/2$, namely

$$
\int_0^h |U\,(t)|^2\,dt = \int_0^h \int_t^T \frac{\partial}{\partial \tau}\left(\chi\,(\tau/T)\,U\,(\tau)^2\right)d\tau dt \leq c\int_0^h \int_t^T\left(\left|\frac{dU}{d\tau}\,(\tau)\right|^2 + |U\,(\tau)|^2\right)d\tau dt
$$
$$
\leq ch\left\|U; H^1\,(0, T)\right\|^2.
$$

Estimating the matrix function $D\,(\nabla)\,X_h^\omega$ according to (3.38) and mentioning that, by virtue of weight (3.31) in norm (3.37),

$$
\left\|(D\,(\nabla)\,X_h^\omega)\,\mathbf{v}; L^2\,(\Omega_h)\right\| \leq c|||\mathbf{v}; \omega\,(h) \times (-h/2, h/2)\,|||_\bullet \leq c\mathbf{n},
$$

we deduce that

$$|I_2| \leq ch^{-1/2}h^{1/2}\big(h^{-1}h\,(1+|\ln h|)^2\,|a|\,(\text{mes}_2\,\omega\,(h))^{1/2} \tag{3.40}$$
$$+ \big\|w; H^1\,(\omega\,(h))^2 \times H^2\,(\omega\,(h))\,\big\|^2 + h^2\,(1+|\ln h|)^2\,|a|\,(\text{mes}_2\,\omega\,(h))^{1/2}\,\big)\mathbf{n}$$
$$\leq ch^{1/2}\,|\ln h|\,\mathbf{n}.$$

Note that, in the middle part of (3.40), $h^{-1/2}$ is the original common factor, $h^{1/2}$ comes from integration in z, $\text{mes}_2\,\omega\,(h) = O\,(h)$ and $(1+|\ln h|)^2$ reduces finally to $1+|\ln h|$ due to the inequality $|a\,(\ln h)| \leq c\,(1+|\ln h|)^{-1}\,\mathcal{N}$ inherited from (3.23), (3.19) and (3.25).

The third term I_3 is examined in the simplest way. Indeed, formulas (2.32) and (2.11), (2.9) assure that

$$h^{-1/2}D\,(\nabla)\,\mathbf{W}\,(h,x) \tag{3.41}$$
$$= h^{1/2}((D_\zeta\mathcal{X}\,(\zeta) + \mathcal{Y}\,(\zeta))\,\mathcal{D}\,(\nabla_y)\,\mathbf{w}\,(h,x) + hD_yW^2\,(\zeta,\nabla_y)\,\mathbf{w}\,(h,x))$$

and we readily obtain that

$$|I_3| \leq ch^{1/2}h^{1/2}\,|\ln h|\,\mathcal{N}\mathbf{n}.$$

Estimating the fourth term I_4 in (3.39) follows the standard scheme in the theory of thin plates with a modification caused by the cut-off function X_h^θ present in (3.33). The function $\widetilde{w}^\theta$ satisfies the system

$$\mathcal{L}\,(\nabla_y)\,\widetilde{w}^\theta = \widetilde{g}^\theta := X_h^\theta g - [\mathcal{L}(\nabla_y), X_h^\theta]$$

where $[\mathcal{L}, X_h^\theta]$ is the commutator of the differential operator with the cut-off function X_h^θ. By (3.20) and (3.38), we have

$$\int_\omega \Big(S_{1,h}^2|(1-X_h^\theta)g'|^2 + h^{-2}S_{2,h}^2|(1-X_h^\theta)g_3|^2\Big)\,dy \leq ch^2\,(1+|\ln h|)^2\,\mathcal{N}, \tag{3.42}$$

$$\int_\omega S_{1,h}^2|[L',X_h^\theta]\widetilde{w}'|^2dy \leq c\int_{\mathbb{B}_{2R_\theta h}} (r+h)^2\,\big(1+\big|\ln\big(r^2+h^2\big)\big|\big)^2\,\big(h^{-4}|\widetilde{w}'|^2 + h^{-2}|\nabla_y\widetilde{w}'|^2\big)dy$$

$$\leq ch^2\,|\ln h|^4\,\mathcal{N},$$

$$\int_\omega S_{2,h}^2|[L_3,X_h^\theta]\widetilde{w}_3|^2dy$$

$$\leq c\int_{\mathbb{B}_{2R_\theta h}} (r+h)^4\,\big(1+\big|\ln\big(r^2+h^2\big)\big|\big)^2\,\big(h^{-8}|\widetilde{w}_3|^2 + h^{-6}|\nabla_y\widetilde{w}_3|^2 + h^{-4}|\nabla_y^2\widetilde{w}_3|^2 + h^{-2}|\nabla_y^3\widetilde{w}_3|^2\big)dy$$

$$\leq ch^2\,|\ln h|^4\,\mathcal{N}.$$

In these calculations we used that weights are small on the disk $\mathbb{B}_{2R_\theta h}$ and coefficients of order k derivatives in the first- and third-order operators $[L', X_h^\theta]$ and $[L_3, X_h^\theta]$ are $O(h^{k-2})$ and $O(h^{k-4})$, respectively. Returning to the notation in Section 2.2, we set $\widetilde{U}^p\,(y,\zeta) = W^p\,(\zeta,\nabla_y)\,\widetilde{w}^\theta\,(y)$,

$p = 0, 1, 2$, and, similarly to (3.41), write

$$I_4 = h^{-5/2}\big(AD_\zeta\widetilde{U}^0, D\left(\nabla\right)\mathbf{v}^\omega\big)_{\Omega_h} + h^{-3/2}\big(A\big(D_\zeta\widetilde{U}^1 + D_y\widetilde{U}^0\big), D\left(\nabla\right)\mathbf{v}^\theta\big)_{\Omega_h} \tag{3.43}$$

$$+ h^{-1/2}\big(A\big(D_\zeta\widetilde{U}^2 + D_y\widetilde{U}^1\big), D\left(\nabla\right)\mathbf{v}^\theta\big)_{\Omega_h} + h^{1/2}\big(AD_y\widetilde{U}^2, D\left(\nabla\right)\mathbf{v}^\theta\big)_{\Omega_h}$$

$$= h^{-3/2}\Big(A\left(D_\zeta\mathcal{X} + \mathcal{Y}\right)\mathcal{D}\left(\nabla_y\right)w^\theta, D_\zeta\mathbf{v}^\omega\Big)_{\Omega_h} + h^{-1/2}\big(A\big(D_\zeta\widetilde{U}^2 + D_y\widetilde{U}^1\big), D_y\mathbf{v}^\omega\big)_{\Omega_h}$$

$$+ \big(AD_y\widetilde{U}^2, D_\zeta\mathbf{v}^\omega\big)_{\Omega_h} + h^{1/2}\big(AD_y\widetilde{U}^2, D_y\mathbf{v}^\omega\big)_{\Omega_h}$$

$$=: h^{-3/2}I_{40} + h^{-1/2}I_{41} + h^{1/2}I_{42}.$$

Here, we have taken into account that $D_\zeta\widetilde{U}^0 = 0$, $D_\zeta\widetilde{U}^1 + D_y\widetilde{U}^0 = 0$ due to (2.7) and, moreover, $I_{40} = 0$ because the 3×6-matrix $\mathcal{X}$ solves problem (2.12) and w^θ is independent of ζ. Recalling problem (2.13) with $p = 3$, we see that

$$I_{41} = -\big(D_y^\top A\big(D_\zeta\widetilde{U}^2 + D_y\widetilde{U}^1\big) + D_\zeta^\top AD_y\widetilde{U}^1, \mathbf{v}^\theta\big)_{\Omega_h} + \sum_\pm \pm\big(D_3^\top AD_y\widetilde{U}^2, \mathbf{v}^\omega\big)_{\Sigma_h^\pm} \tag{3.44}$$

$$= \big(D_\zeta^\top AD_\zeta\widetilde{U}^3 + f^0, \mathbf{v}^\omega\big)_{\Omega_h} - \sum_\pm \pm\big(D_3^\top AD_\zeta\widetilde{U}^3, \mathbf{v}^\omega\big)_{\Sigma_h^\pm}$$

$$= \big(f^0, \mathbf{v}^\omega\big)_{\Omega_h} - \big(AD_\zeta\widetilde{U}^3, D_\zeta\mathbf{v}^\omega\big)_{\Omega_h}$$

$$= \big(f^0, \mathbf{v}^\omega\big)_{\Omega_h} - h\big(AD_\zeta\widetilde{U}^3, D\left(\nabla\right)\mathbf{v}^\omega\big)_{\Omega_h} + h\big(AD_\zeta\widetilde{U}^3, D_y\mathbf{v}^\omega\big)_{\Omega_h}.$$

We emphasize that the differential properties (3.35) put all entries of scalar products into the Lebesgue space $L^2\left(\Omega_h\right)$. Furthermore, we have the estimates

$$h^{-1/2}\big|\big(f^0, \mathbf{v}^\omega\big)_{\Omega_h} - \big(f^0, \mathbf{v}\big)_{\Omega_h}\big| = h^{-1/2}\left|\sum_{i=1}^2 \big(f_i^0, (1 - X_h^\omega)\mathbf{v}_i\big)_{\Omega_h}\right| \tag{3.45}$$

$$\leq ch^{-1/2}h^{1/2}\left\|f^0; L^2\left(\Omega_1\right)\right\| h\,\||\mathbf{v}; \omega\left(h\right) \times \left(-h/2, h/2\right)\||_\bullet$$

$$\leq ch\mathcal{N}\mathbf{n},$$

$$h^{-1/2}\big|h\big(AD_\zeta\widetilde{U}^3, D\left(\nabla\right)\mathbf{v}^\omega\big)_{\Omega_h}\big| \leq ch^{1/2}h^{1/2}\mathcal{N}\left\|D\left(\nabla\right)\mathbf{v}^\omega; L^2\left(\Omega_h\right)\right\| \leq ch\mathcal{N}\mathbf{n}.$$

Notice that the factors $h^{1/2}$ came into (3.45) due to the change $z \mapsto \zeta = h^{-1}z$. It remains to consider the last terms in (3.43) and (3.44). We write

$$h^{1/2}\big(A\big(D_y\widetilde{U}^2 + D_\zeta\widetilde{U}^3\big), D_y\mathbf{v}^\omega\big)_{\Omega_h} = h^{1/2}\big(A\big(D_y\widetilde{U}^2 + D_\zeta\widetilde{U}^3\big), D_ye_3\overline{\mathbf{v}}_3^\omega\big)_{\Omega_h}$$

$$+ h^{1/2}\big(A\big(D_y\widetilde{U}^2 + D_\zeta\widetilde{U}^3\big), D_y\big(\mathbf{v}^\omega - e_3\overline{\mathbf{v}}_3^\omega\big)\big)_{\Omega_h} =: h^{1/2}I_{42}^1 + h^{1/2}I_{42}^2,$$

where $\overline{\mathbf{v}}_3^\omega\left(y\right) = h^{-1}\int_{-h/2}^{h/2}\mathbf{v}_3^\omega\left(y, z\right)dz$. The known inequality

$$\left\|D_y\big(\mathbf{v}^\omega - e_3\overline{\mathbf{v}}_3^\omega\big); L^2\left(\Omega_h\right)\right\| \leq \left\|D\left(\nabla\right)\mathbf{v}^\omega; L^2\left(\Omega_h\right)\right\|, \quad \forall\mathbf{v}^\omega \in H_0^1\left(\Omega_h, \upsilon_h\right)^3,$$

see, e.g., [22, Section 4.3] and [23, Proposition 3.3.13], yields:

$$h^{1/2}\big|I_{42}^2\big| \leq ch^{1/2}h^{1/2}\mathcal{N}\mathbf{n}.$$

Finally, calculation (2.22) together with formulas (2.18), (2.26) for the matrix $\mathcal{A}$ and the relation $\mathcal{D}_3\left(\nabla_y\right) = 2^{-1/2}\nabla_y\mathcal{D}'\left(\nabla_y\right)$ between blocks in the matrix $\mathcal{D}\left(\nabla_y\right)$, see (2.10), demonstrate

that

$$
h^{1/2}I_{42}^1 - h^{1/2}\left(f_3^1, \overline{\mathbf{v}}_3^\omega\right)_{\Omega_h} = h^{3/2}\left(2^{-1/2}\left(\mathcal{D}'\left(-\nabla_y\right)^\top \mathcal{A}_{(3)}\mathcal{D}_3\left(\nabla_y\right)\widetilde{w}_3^\theta, \nabla_y\overline{\mathbf{v}}_3^\omega\right)_\omega \right.
$$

$$
\left. - \left(f_{30}^1, \overline{\mathbf{v}}_3^\omega\right)_\omega + \sum_{i=1}^{2}\left(f_{3i}^1 + X_h^\theta \int_{-1/2}^{1/2}\zeta f_i^0\left(\cdot, \zeta\right)d\zeta, \frac{\partial \overline{\mathbf{v}}_3^\omega}{\partial y_i}\right)_\omega\right).
$$

We here took into account that in the approximate solution (3.33) the component $\widetilde{w}$ of (3.18) is multiplied with the cut-off function X_h^θ, see (3.32). Hence, to use the integral identity (2.35) with particular test functions $v_i = 0$ and $v_3 = X_h^\theta \overline{\mathbf{v}}_3^\omega$, which vanish in neighborhood of $\partial\omega$ and $\mathcal{O}$, we need to process several commutators with coefficient supports located in the small disk $\overline{\mathbb{B}_{2R_\theta h}}$ in the same way as we had done in (3.42). All of them receive bounds of similar order in h, always much better than in (3.40), and therefore we list only typical estimates

$$
h^{3/2}\left|\left(\left[\mathcal{D}'\left(\nabla_y\right)^\top \mathcal{A}_{(3)}\mathcal{D}_3\left(\nabla_y\right), X_h^\theta\right]\widetilde{w}_3, \nabla_y\overline{\mathbf{v}}_3^\omega\right)_\omega\right|
$$

$$
\leq ch^{3/2}\left(\int_{\mathbb{B}_{2R_\theta h}}\left(h^{-6}\left|\widetilde{w}_3\right|^2 + h^{-4}\left|\nabla_y\widetilde{w}_3\right|^2 + h^{-2}\left|\nabla_y^2\widetilde{w}_3\right|^2\right)dy\right)^{1/2}\left\|\nabla_y\overline{\mathbf{v}}_3^\omega; L^2\left(\mathbb{B}_{2R_\theta h}\right)\right\|
$$

$$
\leq ch^{3/2}\left|\ln h\right|\mathcal{N}h^{1/2}\left\|\nabla_y\mathbf{v}_3^\omega; L^2\left(\mathbb{B}_{2R_\theta h}\times\left(-h/2, h/2\right)\right)\right\| \leq ch\left|\ln h\right|^2\mathcal{N}\mathbf{n},
$$

$$
h^{3/2}\left|\left(f_{30}^1, \overline{\mathbf{v}}_3^\omega\right)_\omega - \left(f_{30}^1, X_h^\theta \overline{\mathbf{v}}_3^\omega\right)_\omega\right|
$$

$$
\leq ch^{3/2}\left\|f_{30}^1; L^2\left(\omega\right)\right\|h^{1/2}\left\|\mathbf{v}_3^\omega; L^2\left(\mathbb{B}_{2R_\theta h}\times\left(-h/2, h/2\right)\right)\right\| \leq ch^2\left|\ln h\right|^2\mathcal{N}\mathbf{n},
$$

$$
h^{3/2}\left|\left(f_{3i}^1, \overline{\mathbf{v}}_3^\omega \partial_i X_h^\theta\right)_\omega\right|
$$

$$
\leq ch^{3/2}\left\|f_{3i}^1; L^2\left(\omega\right)\right\|h^{-3/2}\left\|\mathbf{v}_3^\omega; L^2\left(\mathbb{B}_{2R_\theta h}\times\left(-h/2, h/2\right)\right)\right\| \leq ch\left|\ln h\right|^2\mathcal{N}\mathbf{n}.
$$

In this way we finally obtain:

$$
h^{1/2}\left|I_{41}^1 - \left(f_3^1, \overline{\mathbf{v}}_3^\omega\right)_{\Omega_h}\right| \leq ch\left|\ln h\right|^2\mathcal{N}\mathbf{n}.
$$

We are in position to reckon calculations performed. Comparing bounds in all estimates for terms on the right of (3.39) we detect the worst one in (3.40), namely $ch^{1/2}\left|\ln h\right|\mathcal{N}\mathbf{n}$. Recalling also supposition (3.30), we finally derive from (3.36) and (3.39), (2.1) the formula

$$
\left(AD\left(\nabla\right)\mathbf{v}, D\left(\nabla\right)\mathbf{v}\right)_{\Omega_h} \leq ch^{1/2}\left|\ln h\right|\left(\mathcal{N}+\widetilde{\mathcal{N}}\right)\left\|D\left(\nabla\right)\mathbf{v}; L^2\left(\Omega_h\right)\right\|
$$

which together with the weighted anisotropic Korn inequality of Theorem 1 in [4] lead to the following assertion.

Proposition 5 *Under assumptions (2.2) and (3.26), (3.27) on the right-hand side (2.1) in problem (1.4)-(1.7) (or (1.10) in the variational form) the difference $\mathbf{v}$ of the true u and approximate $\mathbf{u}$ solutions meets the estimate*

$$
\left|\left|\left|u - \mathbf{u}; \Omega_h\right|\right|\right|_\bullet \leq c\left\|D\left(\nabla\right)\left(u - \mathbf{u}\right); L^2\left(\Omega_h\right)\right\| \leq Ch^{1/2}\left|\ln h\right|\left(\mathcal{N}+\widetilde{\mathcal{N}}\right) \tag{3.46}
$$

where $\mathbf{u}$ is determined in (3.33), $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ in (3.29) and (3.30), $\left|\left|\left|\cdot\right|\right|\right|_\bullet$ stands for the norm (3.37) and C is a constant independent of both, the right-hand side f and the small parameter $h \in \left(0, h_0\right]$, $h_0 > 0$.

Remark 6 *As verified in [34], the error $O(h^{1/2})$ of the Kirchhoff plate model (that is without the small support (1.6)) cannot be improved because of the boundary layer phenomenon near the edge. We will see that the boundary layer near the support θ_h brings a perturbation of much bigger order $O(|\ln h|^{-1})$.*

3.5 Theorem on asymptotics

The complicated form (3.33) with various cut-off functions was technically needed to satisfy the stable boundary conditions (1.6) and (1.7) in the variational formulation (1.10) of the problem but after verifying estimate (3.46) we may get rid of some waste items while keeping the same order of proximity. We consider two versions of the simplified asymptotic structures, namely

$$\mathbf{u}^\theta\left(h, x\right) = h^{-1/2}\mathcal{P}\left(h^{-1}x\right) a\left(\ln h\right) + h^{-3/2}\sum_{p=0}^{2} h^p W^p\left(\zeta, \nabla_y\right)\widetilde{w}\left(y\right) \tag{3.47}$$

with the notation in (3.33) and

$$\mathbf{u}^\omega\left(h, x\right) = h^{-3/2}\sum_{p=0}^{2} h^p W^p\left(\zeta, \nabla_y\right)\left(\widehat{w}\left(y\right) + G^{\sharp\theta}\left(h, x\right) a\left(\ln h\right)\right) + h^{-1/2}\mathcal{P}\left(h^{-1}x\right) a\left(\ln h\right) \tag{3.48}$$

where, comparing formula (3.15) with the content of Section 3.5 in [4], we set

$$G^{\sharp\theta}\left(h, x\right) = (1 - \chi(r/2hR_\theta))\Phi^\sharp\left(y\right) + \widehat{G}^\sharp\left(y\right), \tag{3.49}$$

$$\widetilde{\mathcal{P}}\left(\xi\right) = \mathcal{P}\left(\xi\right) - \sum_{p=1}^{2} W^p\left(\zeta, \nabla_\eta\right)\left(1 - \chi\left(\rho/2R_\theta\right)\right)\left(\Phi^\sharp\left(\eta\right) + d^\sharp\left(\eta, 0\right) C^\sharp\right).$$

Theorem 7 *Under conditions of Proposition 5, the vector functions (3.47), (3.48) meet estimate (3.46).*

Proof. According to (3.49), the approximate asymptotic solutions (3.47) and (3.48) differ from each other only for the singular term, cf. (3.4),

$$\left(1 - \chi\left(2r/hR_\theta\right)\right)\Phi^\sharp\left(y\right) = (1 - \chi\left(2\rho/R_\theta\right))H(\Phi^\sharp\left(\eta\right) + d^\sharp\left(\eta, 0\right)\Psi\ln h),$$

and the concomitant rearranging of the rigid motion

$$h^{-1/2}H^{-1}d^\sharp(y, 0)(C^\sharp - \Psi\ln h)a\left(\ln h\right),$$

taking into account the matching condition (3.22).

Hence, it is sufficient to check up the estimate for $\mathbf{u}^\theta$. First of all, we observe that the discrepancy of (3.47) in the Dirichlet conditions (1.7) on the lateral side υ_h of the plate is equal to

$$h^{-3/2}\sum_{p=0}^{2} h^p W^p\left(\zeta, \nabla_y\right) H\Upsilon^\sharp\left(h^{-1}y\right) + h^{-1/2}\widetilde{\widetilde{\mathcal{P}}}\left(\varepsilon^{-1}x\right) + h^{1/2}W^2\left(\zeta, \nabla_y\right)\widetilde{w}\left(y\right). \tag{3.50}$$

Fast decay of the first two terms in (3.50) and the small coefficient $h^{1/2}$ in the third term allow us to withdraw from (3.33) the cut-off function X_h^ω as well as term (3.34) which gets the small

factor h in (3.33) and vanishes in the disk $\mathbb{B}_{R_\omega/2} \ni y$, that is at a distance from θ_h. This requires a standard and simple calculation based on weights in norm (3.37) and has been outlined in many publications (see, e.g., [22], [23, §4.3] etc.). To take off from (3.33) the other cut-off function X_h^θ which is null in the vicinity of θ_h, becomes much more delicate issue since the support of $\left|\nabla_y X_h^\theta\right|$ is located very near singularities of the asymptotic terms. We make use of the weighted estimate (3.20) for $\widetilde{w}$ and observe that the expressions $h^{p-3/2} W^p\left(\zeta, \nabla_y\right)\left(1 - X_h^\theta\left(y\right)\right)\widetilde{w}\left(y\right)$ vanish outside the cylinder $\mathbb{B}_{2R_\theta h} \times (-h/2, h/2)$ where all weights in (3.20) and (3.37) are big. In this way we write

$$h^{p-3/2}|||W^p(1 - X_h^\theta)\widetilde{w}; \Omega_h|||_\bullet$$

$$\leq ch^{p-3/2}h^{1/2}\left(\sum_{q=0}^{p-1} h^{-(p-1-q)}h^{-1}|\ln h|^{-1}||\nabla_y^q \widetilde{w}'; L^2(\mathbb{B}_{2R_\theta h})||\right.$$

$$\left. + \sum_{q=0}^p h^{-(p-q)}h^{-1}|\ln h|^{-1}||\nabla_y^q \widetilde{w}_3; L^2(\mathbb{B}_{2R_\theta h})||\right)$$

$$\leq ch^{p-1}\left(\sum_{q=0}^{p-1} h^{-(p-1-q)}h^{1-q}||r^{-2+q}(1 + |\ln r|)^{-1}\nabla_y^q \widetilde{w}'; L^2(\mathbb{B}_{2R_\theta h})||\right.$$

$$\left. + \sum_{q=0}^p h^{-(p-q)}h^{2-q}||r^{-3+q}(1 + |\ln r|)^{-1}\nabla_y^q \widetilde{w}_3; L^2(\mathbb{B}_{2R_\theta h})||\right).$$

Note that the factors $h^{1/2}$ and $h^{-1}|\ln h|^{-1}$ arrived due to integration in $z \in (-h/2, h/2)$ and taking the weights S_{h1} and hS_{h2} into account. Plugging into norms the weights $r^{-2-q}(1 + |\ln r|)^{-1}$ and $r^{-3-q}(1 + |\ln r|)^{-1}$ according to (3.20), we have gained back $h^{2-q}|\ln h|$ and $h^{3-q}|\ln h|$, respectively, that had led to the bound chN which is even better than in (3.46). ■

3.6   Further simplifications of asymptotic forms

By virtue of (3.20), (3.23) and the definition of the elastic capacity potential in [4], the elastic energy of the boundary layer term $h^{-1/2}\widetilde{\mathcal{P}}\left(h^{-1}x\right) a\left(\ln h\right)$ in (3.48) gets order

$$\left\|D\left(\nabla\right)h^{-1/2}\widetilde{\mathcal{P}}a; L^2(\Omega_h)\right\|^2 = O((h^{-1}h^3 h^{-2}|a\left(\ln h\right)|^2) = O(|\ln h|^{-2}) \qquad (3.51)$$

and therefore it cannot be omitted in the asymptotic expansion of the elastic fields in the plate Ω_h when one attends to achieve an error estimate with the power-law bound $c_\delta h^\delta$, $\delta > 0$. However, contenting ourselves with the logarithmic precision $O(|\ln h|^{-1})$, we may write much simpler asymptotics.

Theorem 8 *Under conditions of Proposition 5, the regular solution $\widehat{w} \in H^1\left(\omega\right)^2 \times H^2\left(\omega\right)$ of the two-dimensional Sobolev-Dirichlet problem (2.17),(2.24), (2.25) is related to the solution $u \in H_0^1\left(\Omega_h; \Gamma_h\right)^3$ of the three-dimensional problem (1.4)-(1.7) by the inequality*

$$|||u - h^{-3/2}\sum_{p=0}^2 h^p W^p \widehat{w}; \Omega_h|||_\bullet \leq c|\ln h|^{-1/2}(\mathcal{N} + \widetilde{\mathcal{N}}),$$

where $W^p\left(\zeta, \nabla_y\right)$ are differential operators in (2.32) and c is a constant independent of $h \in (0, h_0]$ and quantities $\mathcal{N}$, $\widetilde{\mathcal{N}}$ given in (3.29), (3.30).

Proof. We still have to estimate the ingredients $h^{p-3/2}W^p\left(\zeta,\nabla_y\right)G^{\sharp\theta}\left(h,x\right)a\left(\ln h\right)$, $p=0,1,2$, of the asymptotic solution (3.34). We take into account representation (3.15) of the Green matrix $G^\sharp$ which is smooth in the punctured domain $\omega\setminus\mathcal{O}$ and recall Remark 2 about $W^p\left(\zeta,\nabla_y\right)$ together with formulas (3.7), (3.5) for the fundamental matrix. Then we write

$$h^{-3/2}|||G_3^{\sharp\theta}a;\Omega_h|||_\bullet \leq ch^{-3/2}h^{1/2}h\left(1+\int_{\omega\setminus\mathbb{B}_{R_\theta h}}\left(\frac{r^2\left(1+|\ln r|\right)^2}{\left(r^2+h^2\right)^2\left(1+|\ln\left(r+h\right)|\right)^2}\right.\right. \tag{3.52}$$

$$\left.\left.+\frac{\left(1+|\ln r|\right)^2}{\left(r^2+h^2\right)\left(1+|\ln\left(r+h\right)|\right)^2}\right)dy\right)^{1/2}|a\left(\ln h\right)|$$

$$\leq c\left|\ln h\right|^{1/2}\left|\ln h\right|^{-1}=c\left|\ln h\right|^{-1/2},$$

$$h^{-1/2}|||W^1G^{\sharp\theta}a;\Omega_h|||_\bullet$$

$$\leq ch^{-1/2}h^{1/2}\left(1+\int_{\omega\setminus\mathbb{B}_{R_\theta h}}\frac{\left(1+|\ln r|\right)^2}{\left(r^2+h^2\right)\left(1+|\ln\left(r+h\right)|\right)^2}dy\right)^{1/2}|a\left(\ln h\right)|\leq c\left|\ln h\right|^{-1/2},$$

$$h^{1/2}|||W^2G^{\sharp\theta}a;\Omega_h|||_\bullet$$

$$\leq ch^{1/2}h^{1/2}\left(1+\int_{\omega\setminus\mathbb{B}_{R_\theta h}}\frac{\left(1+|\ln r|\right)^2}{r^2\left(r^2+h^2\right)\left(1+|\ln\left(r+h\right)|\right)^2}dy\right)^{1/2}|a\left(\ln h\right)|\leq c\left|\ln h\right|^{-1/2}.$$

For $p=0$, we considered a specific input of the third component u_3 into the weighted norm (3.37) while in the other cases $p=1,2$ the estimates were derived in the standard way: $h^{1/2}$ comes from integration in z and 1 is inserted to bound the regular part of $G^\sharp$. Note that the first two integrals in (3.52) are $O\left(|\ln h|\right)$ while the third one is $O\left(h^{-1}|\ln h|\right)$ so that the relation $|a\left(\ln h\right)|=O(|\ln h|^{-1})$ is very important. ∎

A similar effect of the catastrophic precision drop due to the Dirichlet condition at a small region was underlined in [6] for an homogenization problem in a perforated domain, too.

Our last simplification of asymptotic formulas is based on the following two observations. First,

$$||h^{-1/2}\widetilde{\mathcal{P}}a;L^2\left(\Omega_h\right)||^2=O((h^{-1}h^3|a\left(\ln h\right)|^2)=O(h^2|\ln h|^{-2})$$

because the factors h^{-2} and h^3 were introduced into (3.51) due to differentiation and integration in the fast variables $\xi=h^{-1}x$. Second, logarithmic singularities do not lead the Green matrix $G^\sharp$ out from the spaces $L^2\left(\omega\right)$ and $L^2\left(\Omega_h\right)$. In other words, the boundary layer may be excluded from the asymptotic form but $G^\sharp$ may be included without any cut-off function when dealing with the Lebesgue norms of displacements (for stresses and strains this does not hold true, of course).

Let us formulate a simple but substantial consequence of our previous results.

Theorem 9 *Under conditions of Proposition 5, the inequality*

$$\sum_{i=0}^2\left\|u_i-h^{-1/2}\left(w_i-h^{-1}z\frac{\partial w_3}{\partial y_i}\right);L^2\left(\Omega_h\right)\right\|+h\left\|u_3-h^{-3/2}w_3;L^2\left(\Omega_h\right)\right\|\leq ch^{1/2}\left|\ln h\right|(\mathcal{N}+\widetilde{\mathcal{N}})$$

is valid where $w\left(\ln h;x\right)$ is the singular solution (3.17) of the two-dimensional problem (2.17), (2.24), (2.25) with the column $a\left(\ln h\right)\in\mathbb{R}^4$ calculated in (3.23) and c is a constant independent of $h\in\left(0,h_0\right]$ and quantities $\mathcal{N}$, $\widetilde{\mathcal{N}}$ given in (3.29), (3.30).

4 A variational asymptotic model of a plate with small supports

4.1 A symmetric unbounded operator and its adjoint

We intend to apply a traditional scheme to model media with small defect in the framework of self-adjoint extension of differential operators, see the pioneering paper [1] together with the review paper [30] where a physical terminology refers to "potentials of zero radii". Thanks to the block-diagonal structure of the matrix $\mathcal{L}(\nabla_y)$, cf. Lemma 1, all preparatory results below are substantiated by the papers [11] and [20] where a scalar fourth-order differential operator and a second-order system are considered. At the same time, a straight way verification on them is an accessible task, too.

Let $\mathfrak{A}^0$ be an unbounded operator in the Hilbert space $L^2(\omega)^3$ with the differential expression $\mathcal{L}(\nabla_y)$ and the domain is

$$\mathfrak{D}(\mathfrak{A}^0) = \left\{ w \in C_c^\infty(\overline{\omega} \setminus \mathcal{O})^3 : (2.24) \text{ is fulfilled} \right\}. \tag{4.1}$$

Since the matrix $\mathcal{L}(\nabla_y)$ is elliptic and block-diagonal, the closure $\mathfrak{A} = \overline{\mathfrak{A}^0}$ gets the same differential expression but the domain

$$\mathfrak{D}(\mathfrak{A}) = \left\{ w \in H^2(\omega)^2 \times H^4(\omega) : w(y) = 0, \ \partial_n w_3(y) = 0, \ y \in \partial\omega, \right. \tag{4.2}$$
$$\left. w(\mathcal{O}) = 0 \in \mathbb{R}^3, \ \nabla_y w_3(\mathcal{O}) = 0 \in \mathbb{R}^2 \right\}.$$

Here, the Sobolev theorem on embedding $H^2 \subset C$ is taken into account as well as the fact that $H^1(\omega) \not\subset C(\omega)$. In this way the property of w in (4.1) to vanish in a neighborhood of the coordinate origin $\mathcal{O}$ were converted into five point conditions in (4.2) including the Sobolev one (2.25).

Both $\mathfrak{A}^0$ and $\mathfrak{A}$ are symmetric but not self-adjoint. Indeed, their adjoint operator $\mathfrak{A}^*$ keeps, as it is shown in [11, 20] and can be verified directly, the differential expression $\mathcal{L}(\nabla_y)$ but its domain becomes much bigger, namely

$$\mathfrak{D}(\mathfrak{A}^*) = \left\{ w = w_{(\text{sing})} + w_{(\text{reg})} : w_{(\text{reg})} \in H^2(\omega)^2 \times H^4(\omega), \ w_{(\text{reg})}(y) = 0, \right. \tag{4.3}$$
$$\partial_n w_{(\text{reg})3}(y) = 0, \ y \in \partial\omega, \ w'_{(\text{sing})}(y) = G'(y, \mathcal{O}) a', \ a' = (a_1, a_2)^\top,$$
$$w'_{(\text{sing})3}(y) = G_3(y, \mathcal{O}) a_0 + G_3^1(y, \mathcal{O}) a_3 + G_3^2(y, \mathcal{O}) a_4,$$
$$\left. a_0 \in \mathbb{R}, \ a = (a_1, a_2, a_3, a_4)^\top \in \mathbb{R}^4 \right\}.$$

Recall that the Green matrix G' of problem (3.11) and the Green function G_3 of the Dirichlet problem for the operator $\mathcal{L}_3(\nabla_y)$ together with its derivatives (3.12) live in the Lebesgue space but higher-order derivatives do not. In particular, $\mathfrak{D}(\mathfrak{A}^*) \subset L^2(\omega)^3$.

4.2 Self-adjoint extensions

One readily observes that codimension of the subspace $\mathfrak{D}(\mathfrak{A}) \subset \mathfrak{D}(\mathfrak{A}^*)$ is $10 = 5+5$, i.e. five free constants $a_0, ..., a_4$ in (4.3) and $b_0 = w_{(\text{reg})3}(\mathcal{O})$, $b = d^\sharp(\nabla_y, 0) w_{(\text{reg})}(\mathcal{O}) \in \mathbb{R}^4$. Moreover, the defect index of $\mathfrak{A}^*$ equals $(5 : 5)$, cf. [11, 20]. Hence, according to the von Neumann theorem, see [2, §4.4], the operator $\mathfrak{A}$ admits self-adjoint extensions.

The Friedrichs extension (or the hard one, [2, §10.3]) is obtained, for example, as a restriction of $\mathfrak{A}^*$ onto the subspace $\left\{w \in \mathfrak{D}(\mathfrak{A}^*) : a_0 = 0, \ a = 0 \in \mathbb{R}^4\right\}$ of codimension 5 and corresponds to the classical formulation in $H^2(\omega)^2 \times H^4(\omega)$ of problem (2.17),(2.24) with the right-hand side $g \in L^2(\omega)^3$. The Dirichlet-Sobolev problem (2.17),(2.24), (2.25), again with $g \in L^2(\omega)^3$, is associated with a self-adjoint extension possessing the domain

$$\left\{w \in \mathfrak{D}(\mathfrak{A}^*) : a = 0 \in \mathbb{R}^4, \ b_0 = w_3(\mathcal{O}) \text{ but } a_0 \text{ is arbitrary}\right\}.$$

However, to realize modeling of the three-dimensional elasticity problem (1.4)-(1.7) in Ω_h we ought to deal with a different self-adjoint extension.

Theorem 10 *Let M be a symmetric (real) 4×4-matrix. The restriction $\mathfrak{A}_M$ of $\mathfrak{A}^*$ onto the linear space*

$$\mathfrak{D}(\mathfrak{A}_M) = \left\{w \in \mathfrak{D}(\mathfrak{A}^*) : d^\sharp (\nabla_y, 0)^\top \widehat{w}(\mathcal{O}) = Ma, \ \widehat{w}_3(\mathcal{O}) = 0, \right. \tag{4.4}$$
$$\left. \widehat{w}(y) = w_{(reg)}(y) + a_0 G(y, \mathcal{O}) \right\}$$

is a self-adjoint extension of the operator $\mathfrak{A}$ with domain (4.2).

Proof. To verify that (4.4) constitutes the domain of a self-adjoint operator, we first of all mention that $\mathfrak{D}(\mathfrak{A}) \subset \mathfrak{D}(\mathfrak{A}_M)$ and $\dim(\mathfrak{D}(\mathfrak{A}^*)/\mathfrak{D}(\mathfrak{A}_M)) = 5$ because five linear restrictions are imposed on the free coefficients in (4.3), namely on a, $w_{(reg)3}(\mathcal{O})$ and $d^\sharp (\nabla_y, 0) \widehat{w}(\mathcal{O}) = (\widehat{w}_1(\mathcal{O}), \widehat{w}_2(\mathcal{O}), \partial_1\widehat{w}_3(\mathcal{O}), \partial_2\widehat{w}_3(\mathcal{O}))^\top$, see (3.5). Then the symplectic form

$$\mathfrak{q}(w, v) = (\mathfrak{A}^*w, v)_\omega + (w, \mathfrak{A}^*v)_\omega \tag{4.5}$$

is properly defined on the direct product. At the same time, with a clear reason, form (4.5) vanishes in the case both w and v belong to the domain of a self-adjoint extension of $\mathfrak{A}$. Thus, if we confirm that $\mathfrak{q}(w, v) = 0$, $\forall w, v \in \mathfrak{D}(\mathfrak{A}_M)$ the theorem is concluded.

We take functions from (4.3) with the attributes $a_{(w)}, a_{(v)} \in \mathbb{R}^4$ and write

$$\mathfrak{q}(w, v) = \lim_{t \to +0} \left((\mathcal{L}w, v)_{\omega \setminus \mathbb{B}_t} - (w, \mathcal{L}v)_{\omega \setminus \mathbb{B}_t}\right)$$
$$= \lim_{t \to +0} \left(\left(\mathcal{N}'w', v'\right)_{\mathbb{S}_t} + (\mathcal{N}_3 w_3, (1, \nabla_y) v_3)_{\mathbb{S}_t} - \left(w', \mathcal{N}'v'\right)_{\mathbb{S}_t} - ((1, \nabla_y) w_3, \mathcal{N}_3 v_3)_{\mathbb{S}_t}\right)$$

where $\mathbb{B}_t$ and $\mathbb{S}_t$ are the disk and circle with radius t and the center at $y = 0$ while the Neumann operators $\mathcal{N}'$ and $\mathcal{N}_3$ are taken from the Green formula for the differential operator matrix $\mathcal{L} = \text{diag}\{\mathcal{L}', \mathcal{L}_3\}$ in the domain $\omega \setminus \mathbb{B}_t$ with a small hole. Natural integration properties of the fundamental matrix (3.7), given, e.g., by formulas (3.36) and (3.42) in Section 3.4 in [4], provide a direct calculation of the last limit according to the representation in (4.3) and we finally obtain

$$\mathfrak{q}(w, v) = (\mathcal{L}w, v)_\omega - (w, \mathcal{L}v)_\omega \tag{4.6}$$
$$= (d^\sharp (\nabla_y, 0)^\top \widehat{v}(\mathcal{O}) + \mathcal{G}^\sharp a_{(v)})^\top a_{(w)} - a_{(w)}^\top (d^\sharp (\nabla_y, 0)^\top \widehat{w}(\mathcal{O}) + \mathcal{G}^\sharp a_{(w)}).$$

Since the matrices M and $\mathcal{G}^\sharp$ are symmetric, inserting relationship given in (4.4) shows that the right-hand side of (4.6) vanishes. ∎

We call (4.6) the generalized Green formula, cf. [20, 28], [27, Ch.6]; it holds for any $w, v \in \mathfrak{D}(\mathfrak{A}^*)$.

4.3 The operator model for the supported plate

We simplify representation (2.1) of the right-hand side in the three-dimensional problem (1.4)-(1.7) and set

$$f(h, x) = h^{-1/2} \left(f_1^0(y), f_2^0(y), h f_3^0(y) \right)^\top, \quad f_j^0 \in L^2(\omega). \tag{4.7}$$

The introduced restrictions (2.2) and (3.26), (3.27) are, of course, satisfied and, according to (2.19), (2.23), (3.29)

$$g(y) = f^0(y) = \left(f_1^0(y), f_2^0(y), f_3^0(y) \right)^\top, \quad \mathcal{N} = \left\| f^0; L^2(\omega) \right\|, \quad \widetilde{\mathcal{N}} = 0. \tag{4.8}$$

Let $\mathfrak{A}(\ln h)$ be the self-adjoint operator $\mathfrak{A}_{M^\sharp(\ln h)}$ given by Theorem 10 with the numeral matrix (3.24). We consider the abstract equation

$$\mathfrak{A}(\ln h) w = g \in L^2(\omega)^3. \tag{4.9}$$

A solution of this equation takes form (3.17) where $\widehat{w} \in H^1(\omega)^2 \times H^2(\omega)$ is the unique solution of the Sobolev-Dirichlet problem (2.17), (2.24), (2.25) and the column $a = a(\ln h) = M^\sharp(\ln h)^{-1} d^\sharp(\nabla_y, 0) \widehat{w}(\mathcal{O})$ is perfectly defined through formulas (3.23) and (3.18) for a small $h > 0$. In other words, the abstract equation (4.9) is uniquely solvable and, moreover, its solution $w \in \mathfrak{D}(\mathfrak{A}(\ln h))$ coincides with generator (3.17) of the outer asymptotic expansion (2.3) constructed in Sections 2.1 and 3.1.

Theorem 9 assures the following result.

Corollary 11 *A solution w of equation (4.9) with the chosen self-adjoint extension $\mathfrak{A}(\ln h)$ and a right-hand side as in (4.7) and (4.8) satisfies the estimate*

$$\int_{\Omega_h} \sum_{i=1}^2 \left| u_i(h, x) - h^{-1/2} w_i(\ln h, y) \right|^2 + h^2 \left| u_3(h, x) - h^{-3/2} w_3(\ln h, y) \right|^2 dx \leq c h^{1/2} \left| \ln h \right| \mathcal{N}$$

where $\mathcal{N}$ is given in (4.8) and c is a constant independent of data (4.7) and the small parameter $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$.

4.4 The variational model for the supported plate

The self-adjoint operator $\mathfrak{A}(\ln h)$ traditionally gives rise to the energy functional

$$\tfrac{1}{2} \left(\mathfrak{A}(\ln h) w, w \right)_\omega - (g, w)_\omega. \tag{4.10}$$

We also introduce the functional

$$\mathfrak{E}(w; g) = \tfrac{1}{2} \left(\mathfrak{A}^* w, w \right)_\omega - (g, w)_\omega + \tfrac{1}{2} a^\top \left(M^\sharp(\ln h) a - d^\sharp(\nabla_y, 0) \widehat{w}(\mathcal{O}) \right) \tag{4.11}$$

which according to (4.4) at $M = M^\sharp(\ln h)$, coincides with (4.10) for $v \in \mathfrak{D}(\mathfrak{A}(\ln h))$ but is defined on the whole space

$$\mathfrak{H} = \left\{ v \in \mathfrak{D}(\mathfrak{A}^*) : \widehat{w}_3(\mathcal{O}) = 0 \right\} \tag{4.12}$$

which becomes Hilbert with the norm

$$\| w; \mathfrak{H} \| = \left(\left\| w_{(\text{reg})}; H^2(\omega)^2 \times H^4(\omega) \right\|^2 + |a_0|^2 + |a|^2 \right)^{1/2}.$$

Theorem 12 *A solution* $w \in \mathfrak{D}\left(\mathfrak{A}\left(\ln h\right)\right)$ *of equation (4.9) is the only stationary point of functional (4.11) in* $\mathfrak{H}$.

Proof. Since, by definition, $\mathfrak{A}^* G^\sharp = 0 \in L^2\left(\omega\right)^4$, the generalized Green formula (4.6) and the symmetry of the matrices $M^\sharp\left(\ln h\right)$ and $\mathcal{G}^\sharp$ convert the variation of $\mathfrak{E}$ into

$$
\frac{1}{2}\left(\mathcal{L}w, v\right)_\omega + \frac{1}{2}\left(\mathcal{L}v, w\right)_\omega - \left(g, v\right)_\omega + \frac{1}{2}a_{(v)}^\top \left(M^\sharp\left(\ln h\right)a_{(w)} - d^\sharp\left(\nabla_y, 0\right)\widehat{w}\left(\mathcal{O}\right)\right) \qquad (4.13)
$$
$$
+ \frac{1}{2}a_{(w)}^\top \left(M^\sharp\left(\ln h\right)a_{(v)} - d^\sharp\left(\nabla_y, 0\right)\widehat{v}\left(\mathcal{O}\right)\right)
$$
$$
= \frac{1}{2}\left(\mathcal{L}w, v\right)_\omega + \frac{1}{2}\left(v, \mathcal{L}w\right)_\omega - \left(g, v\right)_\omega + \frac{1}{2}a_{(w)}^\top \left(d^\sharp\left(\nabla_y, 0\right)\widehat{v}\left(\mathcal{O}\right) + \mathcal{G}^\sharp a_{(v)}\right)
$$
$$
- \frac{1}{2}a_{(v)}^\top \left(d^\sharp\left(\nabla_y, 0\right)\widehat{w}\left(\mathcal{O}\right) + \mathcal{G}^\sharp a_{(w)}\right) + \frac{1}{2}a_{(v)}^\top \left(M^\sharp\left(\ln h\right)a_{(w)} - d^\sharp\left(\nabla_y, 0\right)\widehat{w}\left(\mathcal{O}\right)\right)
$$
$$
+ \frac{1}{2}a_{(w)}^\top \left(M^\sharp\left(\ln h\right)a_{(v)} - d^\sharp\left(\nabla_y, 0\right)\widehat{v}\left(\mathcal{O}\right)\right)
$$
$$
= \left(\mathcal{L}\widehat{w} - g, v\right)_\Omega + a_{(v)}^\top \left(M^\sharp\left(\ln h\right)a_{(w)} - d^\sharp\left(\nabla_y, 0\right)\widehat{w}\left(\mathcal{O}\right)\right).
$$

Equating (4.13) to null and taking a test function $v \in C_c^\infty\left(\omega \setminus \mathcal{O}\right)^3$ yields

$$
\left(\mathcal{L}\widehat{w} - g, v\right)_\Omega = 0 \Longrightarrow \mathcal{L}\left(\nabla_y\right)\widehat{w}\left(y\right) = g\left(y\right), \quad y \in \omega \setminus \mathcal{O},
$$

while conditions (2.24) and (2.25) are inherited from the inclusion $w \in \mathfrak{H}$. Since $a_{(v)} \in \mathbb{R}^4$ is arbitrary, we conclude that expression (4.13) vanishes provided $M^\sharp\left(\ln h\right)a_{(w)} = d^\sharp\left(\nabla_y, 0\right)\widehat{w}\left(\mathcal{O}\right)$. Thus, w falls into the domain $\mathfrak{D}\left(\mathfrak{A}\left(\ln h\right)\right)$.

In a similar way one verifies that a solution w of (4.9) annuls the variation of functional (4.11). ∎

From this theorem it follows that the abstract equation (4.9) is equivalent to the variational problem for the quadratic functional (4.11), cf. [16]. The latter is much more suitable for, e.g., numerical implementation because the space (4.12) does not involve any linear restriction on w except for the Sobolev condition (2.25).

The potential energy of the three-dimensional plate (1.1) under the volume force (4.7) is equal to

$$
E_h\left(u; f\right) = \tfrac{1}{2}\left(AD\left(\nabla_x\right)u, D\left(\nabla_x\right)u\right)_{\Omega_h} - \left(u, f\right)_{\Omega_h} = -\tfrac{1}{2}\left(u, f\right)_{\Omega_h};
$$

here the Green formula for a solution u of (1.4)-(1.7) was applied. The last expression demonstrates that, although the introduced model cannot provide "good" approximation of the strain and stress fields near the support area θ_h, Corollary 11 leads to a simple asymptotic formula for the energy.

Corollary 13 *For the right-hand sides (4.7) and (4.8), the solutions* $u\left(h, x\right)$ *of the three-dimensional problem (1.4)-(1.7) and the solution* $w\left(\ln h, y\right)$ *of its two-dimensional model provide the following relationship between the corresponding energy functionals*

$$
\left|E_h\left(u; f\right) - \mathfrak{E}\left(w; g\right)\right| \leq c\sqrt{\varepsilon}\mathcal{N}
$$

where $\mathcal{N}$ *is given in (4.8) and* c *is a constant independent of data (4.7) and the small parameter* $\varepsilon \in \left(0, \varepsilon_0\right), \varepsilon_0 > 0$.

Modifying calculation (4.13) and applying the standard Green formula in ω, we derive that

$$
\begin{aligned}
\mathfrak{E}(w;g) &= \tfrac{1}{2}(\mathcal{L}\widehat{w}, \widehat{w} + G^{\sharp}a)_{\omega} - (g, \widehat{w} + G^{\sharp}a)_{\omega} + 0 \\
&= \tfrac{1}{2}(\mathcal{L}\widehat{w}, \widehat{w})_{\omega} - (g, \widehat{w})_{\omega} - \tfrac{1}{2}a^{\top}d\left(\nabla_{y}, 0\right)\widehat{w}\left(\mathcal{O}\right) + a^{\top}d\left(\nabla_{y}, 0\right)\widehat{w}\left(\mathcal{O}\right) \\
&= \mathcal{E}(\widehat{w};g) + \tfrac{1}{2}a^{\top}M^{\sharp}\left(\ln h\right)a,
\end{aligned}
\tag{4.14}
$$

where

$$
\mathcal{E}(\widehat{w};g) = \frac{1}{2}(\mathcal{A}\mathcal{D}\left(\nabla_{y}\right)\widehat{w}, \mathcal{D}\left(\nabla_{y}\right)\widehat{w})_{\omega} - (g, \widehat{w})_{\omega}.
\tag{4.15}
$$

In other words, the energy functional (4.11) computed for the solution $w \in \mathfrak{D}\left(\mathfrak{A}\left(\ln h\right)\right)$ of the two-dimensional plate model (4.9) is equal to the sum of the potential energy (4.15), stored by the Kirchhoff plate with the Sobolev-Dirichlet conditions, and the correction term

$$
\frac{1}{2}a^{\top}M^{\sharp}\left(\ln h\right)a
\tag{4.16}
$$

which describes the energy concentrated in the vicinity of the small clamped zone θ_{h}. It should be mentioned that, in view of (3.25) and (3.24), value (4.16) gets order $O(|\ln h|^{-1})$ and is positive. The latter complies extension (1.2) of the Dirichlet area in the minimization problem

$$
E_{h}\left(u;f\right) = \min_{v \in H_{0}^{1}(\Omega_{h};\Gamma_{h})^{3}} E_{h}\left(v;f\right)
$$

compared with the traditional problem on the plate (1.1) clamped along the lateral side υ_{h}.

Although the model energy functional (4.14) gains the positive increment (4.16), the stationary point indicated in Theorem 12 does not constitute its minimum because of the calculation

$$
\begin{aligned}
\mathfrak{E}\left(w+v;g\right) - \mathfrak{E}\left(w;g\right) &= \frac{1}{2}\left(\mathcal{L}\widehat{v}, \widehat{v} + G^{\sharp}a_{(v)}\right)_{\omega} + \frac{1}{2}a_{(v)}^{\top}M^{\sharp}\left(\ln h\right)a_{(v)} \\
&= \frac{1}{2}\left(\mathcal{A}\mathcal{D}\left(\nabla_{y}\right)\widehat{v}, \mathcal{D}\left(\nabla_{y}\right)\widehat{v}\right)_{\omega} + \frac{1}{2}a_{(v)}^{\top}\left(M^{\sharp}\left(\ln h\right)a_{(v)} - d^{\sharp}\left(\nabla_{y}, 0\right)\widehat{v}\left(\mathcal{O}\right)\right)
\end{aligned}
$$

where the last terms can get any sign in $\mathfrak{H}$. This terms vanishes in $\mathfrak{D}\left(\mathfrak{A}\left(\ln h\right)\right)$ and, quite expected, the energy functional (4.11) admits the global minimum over the intrinsic linear space (4.4) at the solution w of equation (4.9).

4.5 Conclusive remarks

Asymptotics derived and justified in §3 indicates a distinguishing feature of the influence of the small support θ_{h} on the strain-stress state of the plate Ω_{h}, see (1.1) and (1.3). Namely, the three-dimensional boundary layer phenomenon in the vicinity of θ_{h} overrides the standard error estimate $O(h^{1/2})$ for the two-dimensional Kirchhoff model and the convergence rate $O(|\ln h|^{-1/2})$ in

$$
h^{1/2}u_{i}\left(h, y, h\zeta\right) \to \widehat{w}_{i}\left(y\right) - \zeta\frac{\partial\widehat{w}_{3}}{\partial y_{i}}, \ \ i = 1, 2,
\tag{4.17}
$$

$$
h^{3/2}u_{3}\left(h, y, h\zeta\right) \to \widehat{w}_{3}\left(y\right) \ \text{in} \ H^{1}\left(\omega \times \left(-1/2, 1/2\right)\right),
$$

see Theorem 8, becomes too sluggish so that cannot maintain an engineering application. An acceptable error estimate in Theorem 7, comparable with the usual one in the Kirchhoff theory,

requires to include into the asymptotic form the three-dimensional elastic field, that is the elastic logarithmical capacity potential $\mathcal{P}\left(\varepsilon^{-1}x\right)$ in Section 3.5 of [4] which is not described explicitly yet even for the disk θ_h of radius h and the isotropic material. However, replacing in (4.17) the regular solution $\widehat{w} \in H^1\left(\omega\right)^2 \times H^2\left(\omega\right)$ of the Dirichlet-Sobolev problem (2.17), (2.24), (2.25) for the singular solution (3.17) procures the convergence rate $O\left(h^{1/2}\left|\ln h\right|\right)$ in

$$h^{1/2}u_i\left(h,y,h\zeta\right) - w_i\left(\ln h, y\right) + \zeta\frac{\partial \widehat{w}_3}{\partial y_i}\left(\ln h, y\right) \to 0, \quad i = 1, 2, \tag{4.18}$$

$$h^{3/2}u_3\left(h,y,h\zeta\right) - w_3\left(\ln h, y\right) \to 0 \text{ in } L^2\left(\omega \times (-1/2, 1/2)\right),$$

however in the Lebesgue, not Sobolev norm.

The problem to determine the necessary singular solution $w \in L^2\left(\omega\right)^2 \times H^1\left(\omega\right)$ is well-posed with different formulations in Sections 4.3 and 4.4. Its structure (3.17), (3.23) of w, refers to the Green matrix (3.15), (3.16) and the integral characteristics $C^\sharp\left(A, \theta\right)$ in the representation of the logarithmic elastic potential $\mathcal{P}$ in the elastic layer, see Theorem 13 in [4]. The elastic logarithmic capacity matrix is a numeral 4×4-matrix and can be computed numerically.

Convergence (4.18) does not provide information about the stress and strain fields in the plate, however, using a technique of local estimates in [19, 23], it is possible to verify that, under certain restrictions on the volume forces (3.26), this convergence becomes pointwise outside a neighborhood of $\overline{\theta_h}$. We mention that, as discovered in [34], see also [24], pointwise convergence of stresses and strains is always broken near the clamped edge υ_h of the plate with or without small perturbation at θ_h.

Both, the asymptotic procedure and the models, can be easily adapted to the plate (1.1) with several small support areas $\theta_h^1, ..., \theta_h^J$ sparsely distributed on one or two bases $\Sigma_h^\pm$.

In view of block-diagonal structure (2.26) of the matrix $\mathcal{A}$ the Dirichlet-Sobolev problem (2.17), (2.24), (2.25) naturally decouples into independent problems for the vector $w' = (w_1, w_2)$ of longitudinal displacements and the deflection w_3. Since the elasticity problem for the infinite layer $\Lambda = \mathbb{R}^2 \times (-1/2, 1/2)$ clamped over the only area $\theta \times \{-1/2\}$ loses the symmetry with respect to the plane $\left\{\xi \in \mathbb{R}^3 : \xi_3 = \zeta = 0\right\}$, the elastic capacity matrix $C^\sharp$ does not get in general a block-diagonal structure and thus the relationship imposed in (4.4) on the regular and similar components of the solution $w = (w', w_3)$ of the model equation (4.9) couples the vector w' and the scalar w_3 at the level $O(|\ln h|^{-1})$.

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