Short-time behavior of the heat kernel and Weyl's law on $\mathrm{RCD}^*(K,N)$ spaces

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Abstract

In this paper, we prove pointwise convergence of heat kernels for mGH-convergent sequences of $\mathrm{RCD}^*(K,N)$-spaces. We obtain as a corollary results on the short-time behavior of the heat kernel in $\mathrm{RCD}^*(K,N)$-spaces. We use then these results to initiate the study of Weyl's law in the RCD setting.

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1 Introduction

More than a century ago, H. Weyl gave in [We11] a nice description of the asymptotic behavior of the eigenvalues of the Laplacian on bounded domains of $\mathbb{R}^n$ for $n = 2, 3$ (his result was later on extended for any integer $n \geq 2$). More precisely, if $\Omega \subset \mathbb{R}^n$ is a bounded domain, it is well-known that the spectrum of (minus) the Dirichlet Laplacian on Ω is a sequence of positive numbers $(\lambda_i)_{i\in\mathbb{N}^*}$ such that $\lambda_i \to +\infty$ as $i \to +\infty$. Weyl proved that

$$\lim_{\lambda\to+\infty} \frac{N(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n}\mathcal{L}^n(\Omega)$$

where $N(\lambda) = \sharp\{i \in \mathbb{N} : \ \lambda_i \leq \lambda\}$ (the eigenvalues being counted with multiplicity), ω_n is the volume of the n-dimensional euclidean unit ball, and $\mathcal{L}^n(\Omega)$ is the n-dimensional Lebesgue measure of Ω. This result is known as Weyl's law. It has been widely used to tackle some physical problems, and several refinements were found after Weyl's first

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article. For a complete overview of the history of Weyl's law and its refinements, we refer to [ANPS09].

Among the possible generalizations of Weyl's law, one can replace the bounded domain $\Omega \subset \mathbb{R}^n$ by a n-dimensional compact closed manifold. The Laplacian is then replaced by the Laplace-Beltrami operator of the manifold, and the term $\mathcal{L}^n(\Omega)$ is replaced by $\mathcal{H}^n(M)$, where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure. It has been proved by B. Levitan in [Le52] that Weyl's law is still true in that case.

Another generalization concerns compact Riemannian manifolds (M, g) equipped with the distance d induced by the metric g and a measure with positive smooth density e^{-f} with respect to the volume measure $\mathcal{H}^n$. For such spaces $(M, \mathsf{d}, e^{-f}\mathcal{H}^n)$, called weighted Riemannian manifolds, one has

$$\lim_{\lambda \to +\infty} \frac{N_{(M,\mathsf{d},e^{-f}\mathcal{H}^n)}(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n}\mathcal{H}^n(M), \tag{1.1}$$

holds, where $N_{(M,\mathsf{d},e^{-f}\mathcal{H}^n)}(\lambda)$ denotes the counting function of the (weighted) Laplacian $\Delta^f := \Delta - \langle \nabla f, \nabla \cdot \rangle$ of $(M, \mathsf{d}, e^{-f}\mathcal{H}^n)$. This result is a consequence of [H68]. We stress that in the asymptotic behavior (1.1) the information of the weight, e^{-f}, disappears (as we obtain by different means in Example 4.9). This sounds surprising: the Hausdorff dimension is a purely metric notion, whereas the Laplace-Beltrami operator on weighted Riemannian manifolds and more generally on $\mathrm{RCD}^*(K, N)$-spaces does depend on the reference measure.

In this paper we focus on infinitesimally Hilbertian metric measure spaces with Ricci curvature bounded from below, the so-called RCD-spaces. The curvature-dimension condition $\mathrm{CD}(K, N)$ was independently formulated in terms of optimal transport by Sturm in [St06] and Lott-Villani in [LV09]. The CD condition extends to a non-smooth setting the Riemannian notion of Ricci curvature bounded below. Indeed, for given $K \in \mathbb{R}$ and $N \in [1, +\infty]$, a Riemannian manifold satisfies the $\mathrm{CD}(K, N)$ condition if and only if it has Ricci curvature bounded below by K and dimension bounded above by N. The CD condition is also stable under Gromov-Hausdorff convergence: any metric measure space obtained as a measured Gromov-Hausdorff limit of a sequence of Riemannian manifolds with Ricci curvature bounded below by K and dimension bounded above by N satisfies the $\mathrm{CD}(K, N)$ condition. Such limit spaces are called Ricci limit spaces in the sequel.

In more recent times, two main requirements were added to that theory, namely, the CD^* condition introduced in [BS10] and the infinitesimal Hilbertianity introduced in [AGS14b], giving rise to the study of the so-called RCD (resp. RCD^*) spaces which are by definition infinitesimally Hilbertian spaces satisfying the CD (resp. CD^*) condition. All these notions are stable under Gromov-Hausdorff convergence. See also the papers [AGS15], [EKS15], [AMS15], where the $\mathrm{RCD}(K, \infty)/\,\mathrm{RCD}^*(K, N)$ theories have been proved to be essentially equivalent to the Bakry-Emery theory. The latter, based on diffusion operators and Bochner's inequality, for weighted Riemannian manifolds $(M, \mathsf{d}, e^{-f}\mathcal{H}^n)$ reads as follows:

$$\mathrm{Ric}_M + \mathrm{Hess}_f - \frac{\nabla f \otimes \nabla f}{N - n} \geq K g_M. \tag{1.2}$$

Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $\mathrm{RCD}^*(K, N)$-space. The main result of this paper is a sharp criterion (see (1.3) below) for the validity of Weyl's law on $(X, \mathsf{d}, \mathfrak{m})$. The authors do not know whether there exist $\mathrm{RCD}^*(K, N)$-spaces which do not satisfy this criterion, since all known examples satisfy it.

As observed in (1.1) it is expected that the asymptotic behavior of the counting function $N_{(X,\mathsf{d},\mathfrak{m})}(\lambda)$ is not related to the reference measure $\mathfrak{m}$, but rather to the Hausdorff measure $\mathcal{H}^\ell$, where ℓ is the Hausdorff dimension of (X, d).

In order to introduce the precise statement of our criterion let us recall Mondino-Naber's result [MN14]:

$$\mathfrak{m}(X \setminus \bigcup_{i=1}^{[N]} \mathcal{R}_i) = 0,$$

where $[N]$ is the integer part of N and $\mathcal{R}_i$ is the i-dimensional regular set of $(X, \mathsf{d}, \mathfrak{m})$. Recall that the i-dimensional regular set of (X, d) (Definition 3.4) is the set of points of X admitting a unique tangent cone isometric to $(\mathbb{R}^i, d_{eucl}, c_i\mathcal{H}^i)$. More recently, building on [DePhR16], more than one group of authors have shown that $\mathfrak{m}$-almost all of $\mathcal{R}_i$ can be covered by bi-Lipschitz charts defined in subsets of $\mathbb{R}^i$, and that the restriction of the reference measure to each $\mathcal{R}_i$ is absolutely continuous w.r.t. $\mathcal{H}^i$ (Proposition 3.7 and [KM16], [DePhMR16], [GP16]).

Let us define the maximal regular dimension $\dim_{\mathsf{d},\mathfrak{m}}(X)$ of $(X, \mathsf{d}, \mathfrak{m})$ as the largest integer k such that $\mathfrak{m}(\mathcal{R}_k) > 0$. We are now in a position to introduce our main result. We prove first in Theorem 4.1 that, setting $k = \dim_{\mathsf{d},\mathfrak{m}}(X)$, the limit

$$\lim_{r\to 0^+} \frac{r^k}{\mathfrak{m}(B_r(x))}$$

exists and is finite for $\mathfrak{m}$-a.e. $x \in X$. Then our criterion (Theorem 4.3) can be stated as follows:

$$\lim_{r\to 0^+} \int_X \frac{r^k}{\mathfrak{m}(B_r(x))} \mathrm{d}\mathfrak{m} = \int_X \lim_{r\to 0^+} \frac{r^k}{\mathfrak{m}(B_r(x))} \mathrm{d}\mathfrak{m} < \infty \tag{1.3}$$

if and only if

$$\lim_{\lambda\to+\infty} \frac{N_{(X,\mathsf{d},\mathfrak{m})}(\lambda)}{\lambda^{k/2}} = \frac{\omega_k}{(2\pi)^k}\mathcal{H}^k(\mathcal{R}_k^*) < \infty, \tag{1.4}$$

where $\mathcal{R}_k^* \subset \mathcal{R}_k$ denotes a suitable reduced regular set (defined in Theorem 4.1) such that $\mathfrak{m}(\mathcal{R}_k \setminus \mathcal{R}_k^*) = 0$ and $\mathfrak{m}\llcorner\mathcal{R}_k^*$ and $\mathcal{H}^k\llcorner\mathcal{R}_k^*$ are mutually absolutely continuous (in particular $\mathcal{H}^k(\mathcal{R}_k^*) > 0$).

Note that with this criterion, the asymptotic behavior of $N_{(X,\mathsf{d},\mathfrak{m})}(\lambda)$ (including the growth order) is determined by the sole top-dimensional reduced regular set. As typical examples, thanks to the dominated convergence theorem, the criterion is automatically satisfied when $k = N$, or when the metric measure structure is Ahlfors regular. As a consequence, we obtain a new result, namely Weyl's law for finite dimensional compact Alexandrov spaces (Corollaries 4.4 and 4.8). We can also provide examples (see (1.5) below) such that $k < N$ and Ahlfors regularity fails.

On the other hand, it is worth pointing out that from the viewpoint of RCD-theory the least number $N_{\min}$ such that $(X, \mathsf{d}, \mathfrak{m}) \in \mathrm{RCD}^*(K, N_{\min})$ for some $K \in \mathbb{R}$ might be naturally regarded as another dimension of $(X, \mathsf{d}, \mathfrak{m})$ (indeed, $N_{\min} = n$ for weighted Riemannian manifolds $(M, \mathsf{d}, e^{-f}\mathcal{H}^n)$). However, in general $N_{\min}$ is not equal to the Hausdorff dimension of (X, d) and need not be related to the asymptotic behaviour of $N_{(X,\mathsf{d},\mathfrak{m})}(\lambda)$, as the following example shows: for $N \in (1, +\infty)$, let us consider the metric measure space

$$(X, \mathsf{d}, \mathfrak{m}) := ([0, \pi], \mathsf{d}_{[0,\pi]}, \sin^{N-1} t\mathrm{d}t). \tag{1.5}$$

It is known that $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathrm{RCD}^*(N-1, N)$-space (see for instance [CM15a]). Moreover, since $\mathfrak{m}(B_r(x)) \sim r$ for $x \in (0, \pi)$ and $\mathfrak{m}(B_r(x)) \sim r^N$ for $x \in \{0, \pi\}$ as $r \to 0^+$, for this metric measure structure one has $N_{\min} = N$, because Bishop-Gromov inequality for $\mathrm{RCD}^*(\hat{K}, \hat{N})$-spaces implies a positive lower bound on $\mathfrak{m}(B_r(x))/r^{\hat{N}} > 0$.

It turns out that our criterion can be applied to $(X, \mathsf{d}, \mathfrak{m})$, as (1.3) holds by the dominated convergence theorem. Thus

$$\lim_{\lambda \to +\infty} \frac{N_{(X,\mathsf{d},\mathfrak{m})}(\lambda)}{\lambda^{1/2}} = \frac{\omega_1}{2\pi}\mathcal{H}^1((0,\pi)) = 1 \tag{1.6}$$

and the number N does not appear in (1.6). Note that (1.6) is also new and that the same asymptotic behavior in (1.6) for $N = 1$ (i.e. the metric measure space is $([0,\pi], \mathsf{d}_{[0,\pi]}, \mathcal{H}^1)$) is well-known as Weyl's law on $[0, \pi]$ associated with homogeneous Neumann boundary conditions. See Example 4.5 for more details.

Let us conclude this introduction by pointing out that our technique is based on a pointwise convergence of heat kernels for Gromov-Hausdorff converging sequences of $\mathrm{RCD}^*(K,N)$ spaces (Theorem 3.3) which is a generalization of Ding's Riemannian results [D02]. A main advantage of our approach is the use of regularity theory of heat flows that avoids some technical difficulties of spectral theory. As a corollary, we obtain a precise short-time diagonal estimate of the heat kernel $p(x, x, t)$ on the regular sets of a compact $\mathrm{RCD}^*(K,N)$ space $(X, \mathsf{d}, \mathfrak{m})$. In fact, (1.3) allows to turn this estimate into a trace formula (see the proof of Theorem 4.3)

$$\lim_{t \to 0^+} (4\pi t)^{k/2} \int_X p(x,x,t) \mathrm{d}\mathfrak{m}(x) = \mathcal{H}^k(\mathcal{R}_k^*),$$

where $k = \dim_{\mathsf{d},\mathfrak{m}}(X)$, leading naturally to Weyl's law.

The paper is organized as follows. In Section 2 we recall the notions on $\mathrm{RCD}^*(K,N)$ spaces that shall be used in the sequel and give some useful lemmas. Section 3 begins with the treatment of weak/strong L^2-convergence for sequences of functions defined on converging $\mathrm{RCD}^*(K,N)$ spaces. Then, using the Gaussian estimates (3.2) of the heat kernel on $\mathrm{RCD}^*(K,N)$-spaces established in [JLZ16], we prove the pointwise convergence of heat kernels defined on a converging sequence of $\mathrm{RCD}^*(K,N)$ spaces, and we deduce from this fact the short-time estimate of the heat kernel on regular sets of $\mathrm{RCD}^*(K,N)$ spaces. Section 4 is devoted to the study of Weyl's law, first in a weak form (Theorem 4.2) and then strengthening the assumptions in a stronger and more classical form (Theorem 4.3). The rest of this section is dedicated to examples and applications (especially to compact Alexandrov spaces).

After completing our paper we learned of an independent work [ZZ17] by H-C. Zhang and X-P. Zhu on Weyl's law in the setting of $\mathrm{RCD}^*(K,N)$ spaces. The paper is based on a local analysis, along the same lines of [D02], and provides sufficient conditions for the validity of Weyl's law, different from our sharp criterion of Theorem 4.3.

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2 Notation and preliminaries about $\mathrm{RCD}^*(K,N)$ spaces

Let us recall basic facts about Sobolev spaces and heat flow in metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$, see [AGS14a] and [G15a] for a more systematic treatment of this topic. The so-called Cheeger energy $\mathsf{Ch} : L^2(X,\mathfrak{m}) \to [0,+\infty]$ is the convex and $L^2(X,\mathfrak{m})$-lower semicontinuous functional defined as follows:

$$\mathsf{Ch}(f) := \inf\left\{\liminf_{n\to\infty} \frac{1}{2}\int_X \mathrm{Lip}_a^2(f_n)\,\mathrm{d}\mathfrak{m} :\ f_n \in \mathrm{Lip_b}(X)\cap L^2(X,\mathfrak{m}),\ \|f_n - f\|_2 \to 0\right\}. \tag{2.1}$$

The original definition in [Ch99] involves generalized upper gradients of f_n in place of their asymptotic Lipschitz constant

$$\mathrm{Lip}_a(f) := \lim_{r\to 0^+} \mathrm{Lip}(f, B_r(x)),$$

but many other pseudo gradients (upper gradients, or the slope $\mathrm{lip}(f) \le \mathrm{Lip}_a(f)$, which is a particular upper gradient) can be used and all of them lead to the same definition, see [ACDM15] and the discussion in [AGS14a, Remark 5.12]).

The Sobolev space $H^{1,2}(X,\mathsf{d},\mathfrak{m})$ is simply defined as the finiteness domain of Ch. When endowed with the norm

$$\|f\|_{H^{1,2}} := \left(\|f\|^2_{L^2(X,\mathfrak{m})} + 2\mathsf{Ch}(f)\right)^{1/2}$$

this space is Banach, and reflexive if (X,d) is doubling (see [ACDM15]). The Sobolev space is Hilbert if Ch is a quadratic form. We say that a metric measure space $(X,\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian if Ch is a quadratic form.

By looking at minimal relaxed slopes and by a polarization procedure, one can then define a *Carré du champ*

$$\Gamma : H^{1,2}(X,\mathsf{d},\mathfrak{m}) \times H^{1,2}(X,\mathsf{d},\mathfrak{m}) \to L^1(X,\mathfrak{m})$$

playing in this abstract theory the role of the scalar product between gradients. In infinitesimally Hilbertian metric measure spaces the Γ operator satisfies all natural symmetry, bilinearity, locality and chain rule properties, and provides integral representation to Ch: $2\mathsf{Ch}(f) = \int_X \Gamma(f,f)\,\mathrm{d}\mathfrak{m}$ for all $f \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$. We can also define a densely defined operator $\Delta : D(\Delta) \to L^2(X,\mathfrak{m})$ by

$$f \in D(\Delta) \iff \exists h := \Delta f \in L^2(X,\mathfrak{m}) \text{ s.t. } \int_X hg\mathrm{d}\mathfrak{m} = -\int_X \Gamma(f,g)\mathrm{d}\mathfrak{m}\ \forall g \in H^{1,2}(X,\mathsf{d},\mathfrak{m}).$$

Another object canonically associated to Ch and then to the metric measure structure is the heat flow h_t, defined as the $L^2(X,\mathfrak{m})$ gradient flow of Ch; even in general metric measure structures one can use the Brezis-Komura theory of gradient flows of lower semicontinuous functionals in Hilbert spaces to provide existence and uniqueness of this gradient flow. In the special case of infinitesimally Hilbertian metric measure spaces, this provides a linear, continuous and self-adjoint contraction semigroup h_t in $L^2(X,\mathfrak{m})$ with the Markov property, characterized by: $t \mapsto h_t f$ is locally absolutely continuous in $(0,+\infty)$ with values in $L^2(X,\mathfrak{m})$ and

$$\frac{d}{dt}h_t f = \Delta h_t f \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0,+\infty)$$

for all $f \in L^2(X,\mathfrak{m})$. Thanks to the Markov property, this semigroup has a unique L^p continuous extension from $L^2 \cap L^p$ to L^p, $1 \le p < +\infty$, and by duality one defines also the w^*-continuous extension to $L^\infty(X,\mathfrak{m})$.

In order to introduce the class of $\mathrm{RCD}(K,\infty)$ and $\mathrm{RCD}^*(K,N)$ metric measure spaces we follow the Γ-calculus point of view, based on Bochner's inequality, because this is the point of view more relevant in the proof of heat kernel estimates, Li-Yau inequalities, etc. The equivalence with the Lagrangian point of view, based on the theory of optimal transport is discussed in [AGS15] (in the case $N = \infty$) and in [EKS15], [AMS15] (in the case $N < \infty$). The latter point of view does not play a role in this paper, but it plays indeed a key role in the proof of the results we need, mainly taken from [GMS13] and [MN14].

Definition 2.1 (RCD spaces)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space, with (X, d) complete, satisfying

$$\mathfrak{m}(B_r(\bar{x})) \le c_1 e^{c_2 r^2} \qquad \forall r > 0 \tag{2.2}$$

for some c_1, $c_2 > 0$ and $\bar{x} \in X$ and the so-called Sobolev to Lipschitz property: any $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})\mathcal{L}^\infty(\mathcal{X}, \mathfrak{m})$ with $\Gamma(f) \le 1$ $\mathfrak{m}$-a.e. in X has a representative in $\tilde{f} \in \mathrm{Lip_b}(X)$, with $\mathrm{Lip}(\tilde{f}) \le 1$.

For $K \in \mathbb{R}$, we say that $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathrm{RCD}(K, \infty)$ metric measure space if, for all $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \cap D(\Delta)$ with $\Delta f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$, Bochner's inequality

$$\frac{1}{2}\Delta\Gamma(f) \ge \Gamma(f, \Delta f) + K\Gamma(f)$$

holds in the weak form

$$\frac{1}{2}\int \Gamma(f)\Delta\varphi \mathrm{d}\mathfrak{m} \ge \int \varphi(\Gamma(f, \Delta f) + K\Gamma(f))\mathrm{d}\mathfrak{m} \quad \forall \varphi \in D(\Delta) \text{ with } \varphi \ge 0,\ \Delta\varphi \in L^\infty(X, \mathfrak{m}).$$

Analogously, for $K \in \mathbb{R}$ and $N > 0$, we say that $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathrm{RCD}^*(K, N)$ metric measure space if, for all $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \cap D(\Delta)$ with $\Delta f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$, Bochner's inequality

$$\frac{1}{2}\Delta\Gamma(f) \ge \Gamma(f, \Delta f) + \frac{1}{N}(\Delta f)^2 + K\Gamma(f)$$

holds in the weak form

$$\frac{1}{2}\int \Gamma(f)\Delta\varphi \mathrm{d}\mathfrak{m} \ge \int \varphi(\Gamma(f, \Delta f) + \frac{1}{N}(\Delta f)^2 + K\Gamma(f))\mathrm{d}\mathfrak{m}$$

for all $\varphi \in D(\Delta)$ with $\varphi \ge 0$ and $\Delta\varphi \in L^\infty(X, \mathfrak{m})$.

The assumption (2.2) is needed to ensure stochastic completeness, namely the property $h_t 1 = 1$. For our purposes it will be convenient not to add the assumption that $X = \operatorname{supp}\mathfrak{m}$, made in some other papers on this subject. Neverthless, it is obvious that $(X, \mathsf{d}, \mathfrak{m})$ is $\mathrm{RCD}(K, \infty)$ (resp. $\mathrm{RCD}^*(K, N)$) if and only if $(X, \mathsf{d}, \operatorname{supp}\mathfrak{m})$ is $\mathrm{RCD}(K, \infty)$ (resp. $\mathrm{RCD}^*(K, N)$).

For $\mathrm{RCD}(K, \infty)$ spaces it is proved in [AGS14b] that the dual semigroup $\tilde{h}_t$, acting on the space $\mathscr{P}_2(X)$ of probability measures with finite quadratic moments, is K-contractive and maps for all $t > 0$ $\mathscr{P}_2(X)$ into measures absolutely continuous w.r.t. $\mathfrak{m}$, with finite logarithmic entropy. Setting then

$$\tilde{h}_t \delta_x = p(x, \cdot, t)\mathfrak{m} \qquad x \in X,\ t > 0$$

this provides a version of the heat kernel $p(x, y, t)$ in this class of spaces (defined for any x in $\operatorname{supp}\mathfrak{m}$, up to a $\mathfrak{m}$-negligible set of points y), so that

$$h_t f(x) = \int_X p(x, y, t) f(y)\mathrm{d}\mathfrak{m} \qquad \forall f \in L^2(X, \mathfrak{m}).$$

In $\mathrm{RCD}^*(K, N)$ spaces with $N < \infty$, thanks to additional properties satisfied by the metric measure structure, one can find a version of p continuous in $\operatorname{supp}\mathfrak{m} \times \operatorname{supp}\mathfrak{m} \times (0, +\infty)$, as illustrated in the next section.

Definition 2.2 (Rectifiable sets)**.** Let (X, d) be a metric space and let $k \geq 1$ be an integer. We say that $S \subset X$ is countably k-rectifiable if there exist at most countably many sets $B_i \subset \mathbb{R}^k$ and Lipschitz maps $f_i : B_i \to X$ such that $S \subset \cup_i f_i(B_i)$.
For a nonnegative Borel measure μ in X (not necessarily σ-finite), we say that S is (μ, k)-rectifiable if there exists a countably k-rectifiable set $S' \subset S$ such that $\mu^*(S \setminus S') = 0$, namely $S \setminus S'$ is contained in a μ-negligible Borel set.

In the next proposition we recall some basic differentiation properties of measures.

Proposition 2.3. *If μ is a locally finite and nonnegative Borel measure in X and $S \subset X$ is a Borel set, one has*

$$\mu(S) = 0 \quad \Longrightarrow \quad \mu(B_r(x)) = o(r^k) \ \textit{for } \mathcal{H}^k\textit{-a.e. } x \in S. \tag{2.3}$$

In addition,

$$\mu(S) = 0, \ S \subset \{x : \limsup_{r \to 0^+} \frac{\mu(B_r(x))}{r^k} > 0\} \quad \Longrightarrow \quad \mathcal{H}^k(S) = 0. \tag{2.4}$$

Finally, if $\mu = f\mathcal{H}^k \llcorner S$ with S countably k-rectifiable, one has

$$\lim_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_k r^k} = f(x) \qquad \textit{for } \mathcal{H}^k\textit{-a.e. } x \in S. \tag{2.5}$$

Proof. The proof of (2.3) and (2.4) can be found for instance in [F69, 2.10.19] in a much more general context. See also [AT04, Theorem 2.4.3] for more specific statements and proofs. The proof of (2.5) is given in [K94] when $\mu = \mathcal{H}^k \llcorner S$, with S countably k-rectifiable and having locally finite $\mathcal{H}^k$-measure (the proof uses the fact that for any $\varepsilon > 0$ we can cover $\mathcal{H}^k$-almost all of S by sets S_i which are biLipschitz deformations, with biLipschitz constants smaller than $1 + \varepsilon$, of $(\mathbb{R}^i, \| \cdot \|_i)$, for suitable norms $\| \cdot \|_i$). In the general case a simple comparison argument gives the result. □

We conclude this section with two auxiliary results.

Lemma 2.4. *Let f_i, g_i, f, $g \in L^1(X, \mathfrak{m})$. Assume that f_i, $g_i \to f$, g $\mathfrak{m}$-a.e. respectively, that $|f_i| \leq g_i$ $\mathfrak{m}$-a.e., and that $\lim_{i \to \infty} \|g_i\|_{L^1} = \|g\|_{L^1}$. Then $f_i \to f$ in $L^1(X, \mathfrak{m})$.*

Proof. Obviously $|f| \leq g$ $\mathfrak{m}$-a.e. Applying Fatou's lemma for $h_i := g_i + g - |f_i - f| \geq 0$ yields

$$\int_X \liminf_{i \to \infty} h_i \mathrm{d}\mathfrak{m} \leq \liminf_{i \to \infty} \int_X h_i \mathrm{d}\mathfrak{m}.$$

Then by assumption the left hand side is equal to $2\|g\|_{L^1}$, and the right hand side is equal to $2\|g\|_{L^1} - \limsup_i \|f_i - f\|_{L^1}$. It follows that $\limsup_i \|f_i - f\|_{L^1} = 0$, which completes the proof. □

The proof of the next classical result can be found, for instance, in [F71, Sec. XIII.5, Theorem 2].

Theorem 2.5 (Karamata's Tauberian theorem)**.** *Let ν be a nonnegative and locally finite measure in $[0, +\infty)$ and set*

$$\hat{\nu}(t) := \int_{[0,+\infty)} e^{-\lambda t} \mathrm{d}\nu(\lambda) \qquad t > 0.$$

Then, for all $\gamma > 0$ *and* $a \in [0, +\infty)$ *one has*

$$\lim_{t\to 0^+} t^\gamma \hat\nu(t) = a \qquad \Longleftrightarrow \qquad \lim_{\lambda\to+\infty} \frac{\nu([0,\lambda])}{\lambda^\gamma} = \frac{a}{\Gamma(\gamma+1)}.$$

In particular, if $\gamma = k/2$ *with* k *integer, the limit in the right hand side can be written as* $a\omega_k/\pi^{k/2}$.

Remark 2.6 (One-sided versions). More generally we shall prove in the last section of the paper the so-called Abelian one-sided implications and inequalities:

$$\liminf_{t\to 0^+} t^\gamma \hat\nu(t) \geq \Gamma(\gamma+1) \liminf_{\lambda\to+\infty} \frac{\nu([0,\lambda])}{\lambda^\gamma}, \tag{2.6}$$

$$\limsup_{\lambda\to+\infty} \frac{\nu([0,\lambda])}{\lambda^\gamma} < +\infty \quad \Longrightarrow \quad \limsup_{t\to 0^+} t^\gamma \hat\nu(t) \leq \Gamma(\gamma+1) \limsup_{\lambda\to+\infty} \frac{\nu([0,\lambda])}{\lambda^\gamma} \tag{2.7}$$

as well as the so-called Tauberian one-sided implications and inequalities

$$\limsup_{\lambda\to+\infty} \frac{\nu([0,\lambda])}{\lambda^\gamma} \leq e \limsup_{t\to 0^+} t^\gamma \hat\nu(t), \tag{2.8}$$

$$\liminf_{t\to 0^+} t^\gamma \hat\nu(t) > 0, \; \limsup_{t\to 0^+} t^\gamma \hat\nu(t) < +\infty \qquad \Longrightarrow \qquad \liminf_{\lambda\to+\infty} \frac{\nu([0,\lambda])}{\lambda^\gamma} > 0. \tag{2.9}$$

Notice that (2.9) is not quantitative, and requires both bounds on the lim inf and the lim sup, see Remark 5.4 for an additional discussion.

3 Pointwise convergence of heat kernels

From now on, $K \in \mathbb{R}$ and $N \in [1, +\infty)$. Let us fix a pointed measured Gromov-Hausdorff (mGH for short in the sequel) convergent sequence $(X_i, \mathsf{d}_i, x_i, \mathfrak{m}_i) \overset{mGH}{\to} (X, \mathsf{d}, x, \mathfrak{m})$ of $\mathrm{RCD}^*(K,N)$-spaces. This means that there exist sequences of positive numbers $\epsilon_i \to 0$, $R_i \uparrow \infty$, and of Borel maps $\varphi_i : B_{R_i}(x_i) \to X$ such that:

(a) $|\mathsf{d}_i(x,y) - \mathsf{d}(\varphi_i(x), \varphi_i(y))| < \epsilon_i$ for any i and all x, $y \in B_{R_i}(x_i)$, so that $B_{R_i-\epsilon_i}(\varphi_i(x_i)) \subset B_{\epsilon_i}(\varphi_i(B_{R_i}(x_i)))$;

(b) $\varphi_i(x_i) \to x$ in X as $i \to \infty$ (we denote it by $x_i \overset{GH}{\to} x$ for short);

(c) $(\varphi_i)_\sharp \mathfrak{m}_i \overset{C_{\mathrm{bs}}(X)}{\rightharpoonup} \mathfrak{m}$.

In statement (c) we have denoted by $C_{\mathrm{bs}}(X)$ the space of continuous functions with bounded support, and by $f_\sharp$ the push forward operator between measures induced by a Borel map f. We shall use this notation also in the sequel and we call weak convergence the convergence in duality with $C_{\mathrm{bs}}(X)$.

Since $\mathfrak{m}_i$ are uniformly doubling (it follows directly from Bishop-Gromov inequality, known to be true even in the $\mathrm{CD}^*(K,N)$ case), the mGH-convergence is equivalent to the pointed measured Gromov (pmG for short) convergence introduced in [GMS13]. Recall that "$(X_i, \mathsf{d}_i, x_i, \mathfrak{m}_i)$ pmG-converges to $(X, \mathsf{d}, x, \mathfrak{m})$" means that there exist a doubling and complete metric space $\mathbb{X}$ and isometric embeddings $\psi_i : X_i \hookrightarrow \mathbb{X}$, $\psi : X \hookrightarrow \mathbb{X}$ such that $\psi_i(x_i) \to \psi(x)$ in $\mathbb{X}$ as $i \to \infty$ (we also write $x_i \overset{GH}{\to} x$ for short) and such that $(\psi_i)_\sharp \mathfrak{m}_i \overset{C_{\mathrm{bs}}(\mathbb{X})}{\rightharpoonup} (\psi)_\sharp \mathfrak{m}$. See [GMS13, Theorem 3.15] for the proof of the equivalence.

Since all objects we are dealing with are invariant under isometric and measure-preserving embeddings, we identify in the sequel $(X_i, \mathsf{d}_i, x_i, \mathfrak{m}_i)$ with its image by ψ_i, i.e. $(X_i, \mathsf{d}_i, x_i, \mathfrak{m}_i) = (\psi_i(X_i), \mathsf{d}, \psi_i(x_i), (\psi_i)_\sharp \mathfrak{m}_i)$. So, in the sequel the complete and doubling space (X, d) will be fixed (playing the role of $\mathbb{X}$), and we denote by $X_i \subset X$ the supports of the measures $\mathfrak{m}_i$, weakly convergent in X to $\mathfrak{m}$. Because of this, we also use the simpler notation $y_i \to y$ for $y_i \stackrel{GH}{\to} y$. We recall that complete and doubling spaces are proper (i.e. bounded closed sets are compact), hence separable.

Under this notation let us recall the definition of L^2-strong/weak convergence of functions with respect to the mGH-convergence. The following formulation is due to [GMS13] and [AST16], which fits the pmG-convergence well. Other good formulations of L^2-convergence, in connection with mGH-convergence, can be found in [H15, KS03]. However in our setting these formulations are equivalent by the volume doubling condition (e.g. [H16, Proposition 3.3]).

Definition 3.1 (L^2-convergence of functions with respect to variable measures)**.**

1. (L^2-strong/weak convergence) We say that $f_i \in L^2(X_i, \mathfrak{m}_i)$ L^2-*weakly converge to* $f \in L^2(X, \mathfrak{m})$ if $\sup_i \|f_i\|_{L^2} < \infty$ and $f_i \mathfrak{m}_i \stackrel{C_{\rm bs}(X)}{\rightharpoonup} f\mathfrak{m}$. Moreover, we say that $f_i \in L^2(X_i, \mathfrak{m}_i)$ L^2-*strongly converge to* $f \in L^2(X, \mathfrak{m})$ if f_i L^2-weakly converge to f with $\limsup_{i\to\infty} \|f_i\|_{L^2} \le \|f\|_{L^2}$.
2. ($L^2_{\rm loc}$-strong/weak convergence) We say that $f_i \in L^2_{\rm loc}(X_i, \mathfrak{m}_i)$ $L^2_{\rm loc}$-*weakly (or strongly, respectively) converge to* $f \in L^2_{\rm loc}(X, \mathfrak{m})$ if ζf_i $L^2_{\rm loc}$-weakly (or strongly, respectively) converge to ζf for any $\zeta \in C_{\rm bs}(X)$.

Proposition 3.2. *Let $f_i \in C^0(X_i)$ and $f \in C^0(X)$, with X proper and*

$$\sup_i \sup_{X_i \cap B_R(x_i)} |f_i| < +\infty \qquad \forall R > 0.$$

Assume that $\{f_i\}_i$ is locally equi-continuous, i.e. for any $\epsilon > 0$ and any $R > 0$ there exists $\delta > 0$ independent of i such that

$$(y, z) \in (X_i \cap B_R(x_i))^2 \;\; \mathsf{d}(y, z) < \delta \quad \Longrightarrow \quad |f_i(y) - f_i(z)| < \epsilon. \tag{3.1}$$

Then the following are equivalent:

(1) $\lim_{k\to\infty} f_{i(k)}(y_{i(k)}) = f(y)$ *whenever* $y \in \operatorname{supp} \mathfrak{m}$*,* $i(k) \to \infty$ *and* $y_{i(k)} \in X_{i(k)} \to y$*,*

(2) f_i $L^2_{\rm loc}$*-weakly converge to* f*,*

(3) f_i $L^2_{\rm loc}$*-strongly converge to* f*.*

Proof. We prove the implication from (1) to (3) and from (2) to (1), since the implication from (3) to (2) is trivial.

Assume that (2) holds, let $\epsilon > 0$ and let $y_i \to y$. Take ζ nonnegative, with support contained in $B_\delta(y)$ and with $\int \zeta \mathsf{d}\mathfrak{m} = 1$. Thanks to (3.1) and the continuity of f, for δ sufficiently small we have

$$(f_i(y_i) - \epsilon) \int \zeta \mathsf{d}\mathfrak{m}_i \le \int \zeta f_i \mathsf{d}\mathfrak{m}_i \le (f_i(y_i) + \varepsilon) \int \zeta \mathsf{d}\mathfrak{m}_i \qquad f(y) - \epsilon \le \int \zeta f \mathsf{d}\mathfrak{m} \le f(y) + \varepsilon$$

Since $\int \zeta f_i \mathsf{d}\mathfrak{m}_i \to \int \zeta f \mathsf{d}\mathfrak{m}$ and $\int \zeta \mathsf{d}\mathfrak{m}_i \to \int \zeta \mathsf{d}\mathfrak{m} = 1$, from the arbitrariness of ε we obtain that $f_i(y_i) \to f(y)$. A similar argument, for arbitrary subsequences, gives (1).

In order to prove the implication from (1) to (3) we prove the implication from (1) to (2). Assuming with no loss of generality that f_i and f are nonnegative, for any $\zeta \in C_{\rm bs}(X)$ nonnegative, (1) and the compactness of the support of ζ give that for any $\varepsilon > 0$ and any $s > 0$ the set $X_i \cap \{f_i\zeta > s\}$ is contained in the ϵ-neighbourhood of $\{f\zeta > s\}$ for i large enough, so that

$$\limsup_{i\to\infty} \mathfrak{m}_i(\{f_i\zeta > s\}) \le \mathfrak{m}(\{f\zeta \ge s\}).$$

Analogously, any open set $A \Subset \{f\zeta > s\}$ is contained for i large enough in the set $\{f_i\zeta > s\} \cup (X \setminus X_i)$, so that

$$\liminf_{i\to\infty} \mathfrak{m}_i(\{f_i\zeta > s\}) \ge \mathfrak{m}(\{f\zeta > s\}).$$

Combining these two informations, Cavalieri's formula and the dominated convergence theorem provide $\int_X f_i\zeta \mathsf{d}\mathfrak{m}_i \to \int_X f\zeta \mathsf{d}\mathfrak{m}$ and then, since ζ is arbitrary, (2).

Now we can prove the implication from (1) to (3). Thanks to the equiboundedness assumption, the sequence $g_i := f_i^2$ is locally equi-continuous as well and g_i pointwise converge to $g := |f|^2$ in the sense of (1), applying the implication from (1) to (2) for g_i gives

$$\lim_{i\to\infty} \int_{X_i} \zeta^2 f_i^2 \mathsf{d}\mathfrak{m}_i = \int_X \zeta^2 f^2 \mathsf{d}\mathfrak{m} \qquad \forall \zeta \in C_{\rm bs}(X),$$

which yields (3). □

Let us recall the regularity of the heat kernel $p_{(X,\mathsf{d},\mathfrak{m})} := p(x,y,t)$ of a $\mathrm{RCD}^*(K,N)$-space $(X,\mathsf{d},\mathfrak{m})$ we need, where $N \in [1,\infty)$ and $K \le 0$. The general theory of Dirichlet forms [St96], together with the doubling and Poincaré properties ensure that we can find a locally Hölder continuous representative of p in $X \times X \times (0,+\infty)$, which satisfies Gaussian bounds. See [St94, Theorem 4], [St95, Proposition 2.3], [St96, Sections 3 and 4].

On $\mathrm{RCD}^*(K,N)$-spaces, finer properties of the heat kernel are known, as follows. It was proven in [JLZ16] that for any $\epsilon \in (0,1)$ there exist $C_i := C_i(\epsilon,K,N) > 1$ $(i = 1,\, 2)$ (depending only on ϵ, K, N) such that

$$\frac{1}{C_1\mathfrak{m}(B_{t^{1/2}}(x))}\exp\left(-\frac{\mathsf{d}^2(x,y)}{4(1-\epsilon)t} - C_2t\right) \le p(x,y,t) \le \frac{C_1}{\mathfrak{m}(B_{t^{1/2}}(x))}\exp\left(-\frac{\mathsf{d}^2(x,y)}{4(1+\epsilon)t} + C_2t\right) \tag{3.2}$$

for all x, $y \in \operatorname{supp}\mathfrak{m}$ and any $t > 0$. This, combined with the Li-Yau inequality [GM14, J16] gives a gradient estimate:

$$|\nabla_x p(x,y,t)| \le \frac{C_3}{t^{1/2}\mathfrak{m}(B_{t^{1/2}}(x))}\exp\left(-\frac{\mathsf{d}^2(x,y)}{(4+\epsilon)t} + C_4t\right) \tag{3.3}$$

for any $t > 0$, $y \in \operatorname{supp}\mathfrak{m}$ and $\mathfrak{m}$-a.e. $x \in X$, where $C_i := C_i(\epsilon,K,N) > 1$ $(i = 3,\, 4)$. In particular one obtains a quantitative local Lipschitz bound on p, i.e., for any $z \in X$, any $R > 0$ and any $0 < t_0 \le t_1 < \infty$ there exists $C := C(K,N,R,t_0,t_1) > 0$ such that

$$|p(x,y,t) - p(\hat{x},\hat{y},t)| \le \frac{C}{\mathfrak{m}(B_{t_0^{1/2}}(z))}\mathsf{d}((x,y),(\hat{x},\hat{y})) \tag{3.4}$$

for all x, y, $\hat{x}$, $\hat{y} \in B_R(z) \cap \operatorname{supp}\mathfrak{m}$ and any $t \in [t_0,t_1]$. See [JLZ16, Theorem 1.2, Corollary 1.2] (see also [GM14, MN14]).

The following is a generalization/refinement of Ding's result [D02, Theorems 2.6, 5.54 and 5.58] from the Ricci limit setting to our setting, via a different approach.

Theorem 3.3 (Pointwise convergence of heat kernels)**.** *The heat kernels* p_i *of* $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ *satisfy*

$$\lim_{i\to\infty} p_i(x_i, y_i, t_i) = p(x, y, t)$$

whenever $(x_i, y_i, t_i) \in X_i \times X_i \times (0, +\infty) \to (x, y, t) \in \operatorname{supp}\mathfrak{m} \times \operatorname{supp}\mathfrak{m} \times (0, +\infty)$.

Proof. By rescaling $\mathsf{d} \to (t/t_i)^{1/2}\mathsf{d}$, without any loss of generality we can assume that $t_i \equiv t$. Let $f \in C_{\mathrm{bs}}(X)$ and recall that, viewing f as an element of $L^2 \cap L^\infty(X_i, \mathfrak{m}_i)$, $h_t^i f$ L^2-strongly converge to $h_t f$ [GMS13, Theorem 6.11]. By the Bakry-Emery estimate (see for instance [AGS14b, Theorem 6.5], here $I_0(t) = t$ and $I_S(t) := (e^{St} - 1)/S$ for $S \neq 0$)

$$\sqrt{2I_{2K}(t)}\operatorname{Lip}(h_t f, \operatorname{supp}\mathfrak{m}) \leq \|f\|_{L^\infty(X,\mathfrak{m})},$$

valid in all $\mathrm{RCD}(K, \infty)$ spaces, we see that $h_t^i f$ are equi-Lipschitz on X_i. Thus, applying Proposition 3.2 yields $h_t^i f(y_i) \to h_t f(y)$.

On the other hand, the Gaussian estimate (3.2) shows that $\sup_i \|p_i(\cdot, y_i, t)\|_{L^\infty} < \infty$. By definition, since

$$h_t f(y_i) = \int_{X_i} p_i(z, y_i, t) f(z) \mathrm{d}\mathfrak{m}_i(z), \qquad h_t f(y) = \int_X p(z, y, t) f(z) \mathrm{d}\mathfrak{m}(z),$$

we see that $p_i(\cdot, y_i, t)$ L^2_{loc}-weakly converge to $p(\cdot, y, t)$. Moreover, since thanks to (3.4) the functions $p_i(\cdot, y_i, t)$ are locally equi-Lipschitz continuous, choosing any continuous extension of $p(\cdot, y, t)$ to the whole of X and applying Proposition 3.2 once more to $p_i(\cdot, y_i, t)$ we obtain $p_i(x_i, y_i, t)$ converge to $p(x, y, t)$, which completes the proof. □

Definition 3.4 (k-dimensional regular sets $\mathcal{R}_k$ and maximal dimension $\dim_{\mathsf{d},\mathfrak{m}}(X)$)**.** Recall that the k-dimensional regular set $\mathcal{R}_k$ of a $\mathrm{RCD}^*(K, N)$-space $(X, \mathsf{d}, \mathfrak{m})$ in the sense of Mondino-Naber [MN14] is, by definition, the set of points $x \in \operatorname{supp}\mathfrak{m}$ such that

$$(X, r^{-1}\mathsf{d}, \mathfrak{m}_r^x, x) \overset{mGH}{\to} (\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, c_k\mathcal{H}^k, 0_k)$$

as $r \to 0^+$, where c_k is the normalization constant such that $\int_{B_1(0_k)}(1 - |x|)\mathrm{d}(c_k\mathcal{H}^k) = 1$, and

$$\mathfrak{m}_r^x := \left(\int_{B_r(x)} \left(1 - \frac{\mathsf{d}(x, \cdot)}{r}\right) \mathrm{d}\mathfrak{m}\right)^{-1} \mathfrak{m}.$$

We denote by $\dim_{\mathsf{d},\mathfrak{m}}(X)$ the largest integer k such that $\mathcal{R}_k$ has positive $\mathfrak{m}$-measure.

By the Bishop-Gromov inequality, it is easily seen that $\mathcal{R}_k = \emptyset$ if $k > [N]$. It is conjectured that $\mathrm{RCD}^*(K, N)$ spaces cannot be made of pieces of different dimensions, i.e. there exists only one integer k such that $\mathfrak{m}(\mathcal{R}_k) > 0$. This property is known to be true for Ricci limit spaces, see [CN12].

Remark 3.5. By the L^2_{loc}-strong convergence of $\mathsf{d}(y_i, \cdot)$ to $\mathsf{d}(y, \cdot)$ for any mGH-convergent sequence $(Y_i, \mathsf{d}_i, \nu_i, y_i) \overset{GH}{\to} (Y, \mathsf{d}, \nu, y)$ of $\mathrm{RCD}^*(K, N)$-spaces, it is easy to check that $x \in X$ is a k-dimensional regular point if and only if

$$\left(X, r^{-1}\mathsf{d}, \frac{\mathfrak{m}}{\mathfrak{m}(B_r(x))}, x\right) \overset{mGH}{\to} \left(\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, \frac{\mathcal{H}^k}{\omega_k}, 0_k\right),$$

where recall that ω_k denotes the volume of a unit ball in the k-dimensional Euclidean space.

Corollary 3.6 (Short time diagonal behavior of heat kernel on the regular set)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a* $\mathrm{RCD}^*(K, N)$*-space with* $K \in \mathbb{R}$ *and* $N \in (1, +\infty)$*. Then*

$$\lim_{t \to 0^+} \mathfrak{m}(B_{t^{1/2}}(x)) p(x, x, t) = \frac{\omega_k}{(4\pi)^{k/2}} \tag{3.5}$$

for any k*-dimensional regular point* x *of* $(X, \mathsf{d}, \mathfrak{m})$*.*

Proof. Let us recall that for any $r > 0$ and any $C > 0$ the heat kernel $\hat{p}(x, y, t)$ of the rescaled $\mathrm{RCD}^*(r^2K, N)$-space $(X, r^{-1}\mathsf{d}, C\mathfrak{m})$ is given by $\hat{p}(x, y, t) = C^{-1}p(x, y, r^2 t)$. Applying this for $r := t^{1/2}$, $C := \frac{1}{\mathfrak{m}(B_t(x))}$ with Theorem 3.3 and Remark 3.5 shows

$$\lim_{t \to 0^+} \mathfrak{m}(B_{t^{1/2}}(x)) p(x, x, t) = \lim_{t \to 0^+} p^t(x, x, 1) = p_{\mathbb{R}^k}(0_k, 0_k, 1) = \frac{\omega_k}{(4\pi)^{k/2}},$$

where p^t, $p_{\mathbb{R}^k}$ denote the heat kernels of $(X, t^{-1/2}\mathsf{d}, \frac{\mathfrak{m}}{\mathfrak{m}(B_{t^{1/2}}(x))})$, $\left(\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, \frac{\mathcal{H}^k}{\omega_k}\right)$, respectively. □

In the proof of Weyl's law, in the next section, the following finer properties of $\mathcal{R}_k$ will be needed.

Theorem 3.7. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a* $\mathrm{RCD}^*(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in (1, +\infty)$*. For all* k *the set* $\mathcal{R}_k$ *is* $(\mathfrak{m}, k)$*-rectifiable and*

$$\mathfrak{m}(X \setminus \bigcup_{k=1}^{[N]} \mathcal{R}_k) = 0.$$

In addition, $\mathfrak{m} \llcorner \mathcal{R}_k \ll \mathcal{H}^k$*.*

Proof. See [MN14] for the proof of the first two statements (more precisely, it has been proved the stronger property that $\mathfrak{m}$-almost all of $\mathcal{R}_k$ can be covered by bi-Lipschitz charts with biLipschitz constant arbitrarily close to 1, defined in subsets of the k-dimensional Euclidean space). See [KM16], [DePhMR16] and [GP16] for the proof of the absolute continuity statement. □

4 Weyl's law

In a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ the sequence of eigenvalues can be defined appealing to Courant's min-max procedure:

$$\lambda_i := \min \left\{ \max_{f \in S, \, \|f\|_{L^2} = 1} \mathsf{Ch}(f) : \; S \subset H^{1,2}(X, \mathsf{d}, \mathfrak{m}), \;\; \dim(S) = i \right\} \qquad i \geq 1. \tag{4.1}$$

We then define

$$N_{(X,\mathsf{d},\mathfrak{m})}(\lambda) := \#\{i \geq 1 : \; \lambda_i \leq \lambda\}$$

as the "inverse" function of $i \mapsto \lambda_i$. Notice that the formula makes sense even though Ch is not quadratic, and that the formula shows that the growth rate of $N_{(X,\mathsf{d},\mathfrak{m})}$ does not change if we replace the distance d by a biLipschitz equivalent distance, or perturb the measure $\mathfrak{m}$ by a factor uniformly bounded away from 0 and $+\infty$. Notice also that if (X, d) is doubling we can always find a Dirichlet form $\mathcal{E}$ with $C^{-1}\mathcal{E} \leq \mathsf{Ch} \leq C\mathcal{E}$, with C depending only on the metric doubling constant, see [ACDM15] (a result previously proved in [Ch99] for metric measure spaces whose measure is doubling and satisfies Poincaré inequality). Thus, the replacement of Ch with $\mathcal{E}$ makes the standard tools of Linear Algebra applicable.

Let us come now to Weyl's law on RCD-spaces. It is not known yet to what extent the restriction of the measure of a RCD space (or even of a Ricci limit space) to a regular set is quantitatively comparable to the Hausdorff measure of the corresponding dimension. As the behavior of the Hausdorff measure on the regular sets turns out to be related to the asymptotic behavior of the eigenvalues of the Laplacian, this lack of knowledge seems to be a significant difficulty to establish Weyl's law in the RCD context in full generality. However, we can bypass this difficulty for a significant class of spaces (including the class of compact Alexandrov spaces, see Corollary 4.8), by noticing that these spaces satisfy a suitable criterion which implies Weyl's law (we provide the implication in Theorem 4.3). Let us point out that all known examples of compact RCD-spaces satisfy this criterion.

Let us start this section by giving the following, which is to some extent a generalization of [CC00, Theorem 4.6] to the RCD-setting:

Theorem 4.1 (Weak Ahlfors regularity)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a $\mathrm{RCD}^*(K,N)$-space with $K \in \mathbb{R}$ and $N \in (1,+\infty)$ and set*

$$\mathcal{R}_k^* := \left\{ x \in \mathcal{R}_k : \ \exists \lim_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{\omega_k r^k} \in (0,+\infty) \right\}. \tag{4.2}$$

Then $\mathfrak{m}(\mathcal{R}_k \setminus \mathcal{R}_k^*) = 0$, $\mathcal{H}^N(\mathcal{R}_N \setminus \mathcal{R}_N^*) = 0$ if N is an integer, $\mathfrak{m} \llcorner \mathcal{R}_k^*$ and $\mathcal{H}^k \llcorner \mathcal{R}_k^*$ are mutually absolutely continuous and

$$\lim_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{\omega_k r^k} = \frac{\mathsf{d}\mathfrak{m} \llcorner \mathcal{R}_k^*}{\mathsf{d}\mathcal{H}^k \llcorner \mathcal{R}_k^*}(x) \qquad \textit{for } \mathfrak{m}\textit{-a.e. } x \in \mathcal{R}_k^*. \tag{4.3}$$

Finally, if $k_0 = \dim_{\mathsf{d},\mathfrak{m}}(X)$, one has

$$\lim_{r \to 0^+} \frac{\omega_{k_0} r^{k_0}}{\mathfrak{m}(B_r(x))} = \chi_{\mathcal{R}_{k_0}^*}(x) \frac{\mathsf{d}\mathcal{H}^k \llcorner \mathcal{R}_{k_0}^*}{\mathsf{d}\mathfrak{m} \llcorner \mathcal{R}_{k_0}^*}(x) \qquad \textit{for } \mathfrak{m}\textit{-a.e. } x \in X. \tag{4.4}$$

Proof. Let S_k be a countably k-rectifiable subset of $\mathcal{R}_k$ with $\mathfrak{m}(\mathcal{R}_k \setminus S_k) = 0$. From (2.4) we obtain that the set $\mathcal{R}_k^* \setminus S_k$ is $\mathcal{H}^k$-negligible, hence $\mathcal{R}_k^*$ is $(\mathcal{H}^k, k)$-rectifiable. We denote $\mathfrak{m}_k = \mathfrak{m} \llcorner \mathcal{R}_k$ and recall that, thanks to Theorem 3.7, $\mathfrak{m}_k \ll \mathcal{H}^k$ and $\mathfrak{m} = \sum_k \mathfrak{m}_k$. We denote by $f : X \to [0,+\infty)$ a Borel function such that $\mathfrak{m}_k = f\mathcal{H}^k \llcorner \mathcal{R}_k^*$ (whose existence is ensured by the Radon-Nikodym theorem, being $\mathcal{R}_k^*$ σ-finite w.r.t. $\mathcal{H}^k$) and recall that (2.5) gives

$$\exists \lim_{r \to 0} \frac{\mathfrak{m}_k(B_r(x))}{\omega_k r^k} = f(x) \qquad \text{for } \mathcal{H}^k\text{-a.e. } x \in \mathcal{R}_k^*. \tag{4.5}$$

Now, in (4.5) we can replace $\mathfrak{m}_k$ by $\mathfrak{m}$ for $\mathcal{H}^k$-a.e. $x \in \mathcal{R}_k^*$; this is a direct consequence of (2.3) with $\mu = \mathfrak{m} - \mathfrak{m}_k$ and $S = \mathcal{R}_k^*$.

Calling then N_k the $\mathcal{H}^k$-negligible (and then $\mathfrak{m}_k$-negligible) subset of $\mathcal{R}_k^*$ where the equality

$$\lim_{r \to 0} \frac{\mathfrak{m}(B_r(x))}{\omega_k r^k} = f(x)$$

fails, we obtain existence and finiteness of the limit on $\mathcal{R}_k^* \setminus N_k$; since f is a density, it is also obvious that the limit is positive $\mathfrak{m}_k$-a.e., and that $\mathcal{H}^k \llcorner \mathcal{R}_k^* \cap \{f > 0\}$ is absolutely continuous w.r.t. $\mathfrak{m}_k$.

This proves that $\mathfrak{m}(\mathcal{R}_k \setminus \mathcal{R}_k^*) = 0$ and that $\mathfrak{m} \llcorner \mathcal{R}_k^*$ and $\mathcal{H}^k \llcorner \mathcal{R}_k^*$ are mutually absolutely continuous. In the special case $k = N$ a suitable density lower bound $(\mathfrak{m}(B_r(x))/r^N \geq$

$\mathfrak{m}(X)/(\mathrm{diam}(X))^N$ in the case $K \geq 0$, a more complex lower bound involving the comparison spaces also holds when $K \leq 0$) coming from the Bishop-Gromov inequality gives that $\mathcal{H}^N \ll \mathfrak{m}$, hence $\mathcal{R}_N \setminus \mathcal{R}_N^*$ is also $\mathcal{H}^N$-negligible. The last statement (4.4) follows by the fact that $r^{k_0} = o(\mathfrak{m}(B_r(x)))$ for $\mathfrak{m}$-a.e. $x \in \mathcal{R}_k$, $k < k_0$, since this property holds on the sets $\mathcal{R}_k^*$. □

Recall that, as direct consequence of standard arguments from spectral theory and elliptic regularity theory, for a compact $\mathrm{RCD}^*(K, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ the heat kernel p can be expressed by eigenfunctions:

$$p(x, y, t) = \sum_i e^{-\lambda_i t} \varphi_i(x) \varphi_i(y) \tag{4.6}$$

for any x, $y \in \operatorname{supp} \mathfrak{m}$ and any $t > 0$, where λ_i is the i-th eigenvalue of the Laplacian (counting with multiplicities) and φ_i is a corresponding eigenfunction, with $\|\varphi_i\|_{L^2} = 1$. More precisely, in (4.6) one choose the Hölder continuous representative of φ_i, whose Hölder norm grows linearly w.r.t. λ, so that the series in (4.6) is locally Hölder continuous in $X \times X \times (0, +\infty)$.

We are now in a position to introduce our first criterion. We always have $\mathcal{H}^k(\mathcal{R}_k^*) > 0$ and, if an assumption slightly stronger than the finiteness of k-dimensional Hausdorff measure holds, we obtain Weyl's law in the weak asymptotic form. For simplicity we use the notation; $f(\lambda) \sim g(\lambda)$ for the existence of $C > 1$ satisfying $C^{-1} f(\lambda) \leq g(\lambda) \leq C f(\lambda)$ for sufficiently large λ.

Theorem 4.2. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $\mathrm{RCD}^*(K, N)$-space with $K \in \mathbb{R}$ and $N \in (1, +\infty)$, let $k = \dim_{\mathsf{d},\mathfrak{m}}(X)$ and let $\mathcal{R}_k^*$ be as in (4.2) of Theorem 4.1. Then we have*

$$\liminf_{t \to 0^+} \left(t^{k/2} \sum_i e^{-\lambda_i t} \right) \geq \frac{1}{(4\pi)^{k/2}} \mathcal{H}^k(\mathcal{R}_k^*) > 0. \tag{4.7}$$

In particular, if $N_{(X,\mathsf{d},\mathfrak{m})}(\lambda) \sim \lambda^i$ as $\lambda \to +\infty$ for some i, then Remark 2.6 gives $i \geq k/2$. In addition

$$\limsup_{s \to 0^+} \int_X \frac{s^k}{\mathfrak{m}(B_s(x))} \mathsf{d}\mathfrak{m}(x) < +\infty \quad \Longleftrightarrow \quad N_{(X,\mathsf{d},\mathfrak{m})}(\lambda) \sim \lambda^{k/2} \ (\lambda \to +\infty). \tag{4.8}$$

Proof. In order to prove (4.7) we first notice that the combination of (3.5) and (4.4) gives

$$\lim_{t \to 0^+} t^{k/2} p(x, x, t) = \frac{1}{(4\pi)^{k/2}} \chi_{\mathcal{R}_k^*}(x) \frac{\mathsf{d}\mathcal{H}^k \llcorner \mathcal{R}_k^*}{\mathsf{d}\mathfrak{m} \llcorner \mathcal{R}_k^*}(x) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X.$$

Using the identity $t^{k/2} \sum_i e^{-\lambda_i t} = \int_X t^{k/2} p(x, x, t) \mathsf{d}\mathfrak{m}(x)$ and Fatou's lemma we obtain

$$\liminf_{t \to 0} \left(t^{k/2} \sum_i e^{-\lambda_i t} \right) \geq \frac{1}{(4\pi)^{k/2}} \int_{\mathcal{R}_k^*} \frac{\mathsf{d}\mathcal{H}^k \llcorner \mathcal{R}_k^*}{\mathsf{d}\mathfrak{m} \llcorner \mathcal{R}_k^*} \mathsf{d}\mathfrak{m} = \frac{1}{(4\pi)^{k/2}} \mathcal{H}^k(\mathcal{R}_k^*).$$

The heat kernel estimate (3.2) shows

$$C^{-1} \frac{t^{k/2}}{\mathfrak{m}(B_{t^{1/2}}(x))} \leq t^{k/2} p(x, x, t) \leq C \frac{t^{k/2}}{\mathfrak{m}(B_{t^{1/2}}(x))} \tag{4.9}$$

for some $C > 1$, which is independent of t and x. Thus the upper bound on p gives

$$\limsup_{t \to 0^+} t^{k/2} \int_X p(x, x, t) \mathsf{d}\mathfrak{m}(x) \leq C \limsup_{s \to 0^+} \int_X \frac{s^k}{\mathfrak{m}(B_s(x))} \mathsf{d}\mathfrak{m}(x) < +\infty.$$

We can now invoke Remark 2.6 to obtain the implication $\Rightarrow$ in (4.8). The proof of the converse implication is similar and uses the lower bound in (4.9). □

Under the stronger assumption (4.10) (notice that both the finiteness of the limit and the equality of the integrals are part of the assumption) we can recover Weyl's law in the stronger form.

Theorem 4.3. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a compact* $\mathrm{RCD}^*(K, N)$*-space with* $K \in \mathbb{R}$ *and* $N \in (1, +\infty)$*, and let* $k = \dim_{\mathsf{d},\mathfrak{m}}(X)$*. Then*

$$\lim_{s\to 0^+} \int_X \frac{s^k}{\mathfrak{m}(B_s(x))} \mathsf{d}\mathfrak{m}(x) = \int_X \lim_{s\to 0^+} \frac{s^k}{\mathfrak{m}(B_s(x))} \mathsf{d}\mathfrak{m}(x) < +\infty \tag{4.10}$$

if and only if

$$\lim_{\lambda\to+\infty} \frac{N_{(X,\mathsf{d},\mathfrak{m})}(\lambda)}{\lambda^{k/2}} = \frac{\omega_k}{(2\pi)^k} \mathcal{H}^k(\mathcal{R}_k^*) < +\infty. \tag{4.11}$$

Proof. We first assume that (4.10) holds. Taking (4.4) and (4.9) into account, we can apply Lemma 2.4 with $f_t(x) = t^{k/2} p(x, x, t)$ and $g_t(x) = Ct^{k/2}/\mathfrak{m}(B_{t^{1/2}}(x))$ to get

$$\begin{aligned}
\lim_{t\to 0^+} t^{k/2} \int_X p(x, x, t) \mathsf{d}\mathfrak{m}(x) &= \int_X \lim_{t\to 0^+} t^{k/2} p(x, x, t) \mathsf{d}\mathfrak{m}(x) \\
&= \int_{\mathcal{R}_k^*} \frac{1}{(4\pi)^{k/2}} \frac{\mathsf{d}\mathcal{H}^k \llcorner \mathcal{R}_k^*}{\mathsf{d}\mathfrak{m} \llcorner \mathcal{R}_k^*} \mathsf{d}\mathfrak{m} \\
&= \frac{1}{(4\pi)^{k/2}} \mathcal{H}^k(\mathcal{R}_k^*)
\end{aligned}$$

which shows (4.11) by Karamata's Tauberian theorem.

Next we assume that (4.11) holds. Then by (4.4) and Karamata's Tauberian theorem again, (4.11) is equivalent to

$$\lim_{t\to 0^+} t^{k/2} \int_X p(x, x, t) \mathsf{d}\mathfrak{m}(x) = \int_X \lim_{t\to 0^+} t^{k/2} p(x, x, t) \mathsf{d}\mathfrak{m}(x) < +\infty. \tag{4.12}$$

Let $f_t(x) := t^{k/2}/\mathfrak{m}(B_{t^{1/2}}(x))$. Then the heat kernel estimate (4.9) shows that we can apply Lemma 2.4 with $g_t(x) = Ct^{k/2} p(x, x, t)$ to get (4.10). □

By the stability of RCD-conditions with respect to mGH-convergence and [CC97, Theorem 5.1], noncollapsed Ricci limit spaces give typical examples of $\mathrm{RCD}^*(K, N)$-spaces $(X, \mathsf{d}, \mathfrak{m})$ with $\dim_{\mathsf{d},\mathfrak{m}} X = N$. For such metric measure spaces Weyl's law was proven in [D02] by Ding. Thus the following corollary also recovers his result.

Corollary 4.4. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a compact* $\mathrm{RCD}^*(K, N)$*-space with* $K \in \mathbb{R}$ *and* $N \in (1, +\infty)$*, and let* $k = \dim_{\mathsf{d},\mathfrak{m}} X$*. Assume that either* $k = N$*, or that for any integer* i *such that* $\mathfrak{m}(\mathcal{R}_i) > 0$ *there exists* $g_i \in L^1(\mathcal{R}_i^*, \mathcal{H}^i)$ *such that*

$$g_i(x, t) := \frac{t^k}{\mathfrak{m}(B_t(x))} \frac{\mathsf{d}\mathfrak{m} \llcorner \mathcal{R}_i^*}{\mathsf{d}\mathcal{H}^i \llcorner \mathcal{R}_i^*}(x) \le g_i(x) \qquad \forall t \in (0, 1)$$

for $\mathcal{H}^i$*-a.e.* $x \in \mathcal{R}_i^*$*. Then* (4.11) *holds.*

Proof. If the functions g_i exist, the proof follows by the dominated convergence theorem in conjunction with Theorem 4.3. When $k = N$ the existence of the functions g_i follows directly from the Bishop-Gromov inequality, since $\mathfrak{m}(B_r(x))/r^k$ is bounded from below by a positive constant. □

Example 4.5. Let us apply Theorem 4.3 to the following $\mathrm{RCD}^*(N-1,N)$-space:

$$(X,\mathsf{d},\mathfrak{m}) := \left([0,\pi],\mathsf{d}_{[0,\pi]},\sin^{N-1} t\mathsf{d}t\right)$$

for $N\in(1,\infty)$ (note that this is a Ricci limit space if N is an integer, see for instance [AH16]). Then we can apply Theorem 4.3 with $k=1$ and $\mathcal{R}_1^*=\mathcal{R}_1=(0,\pi)$, because of $\sup_{t<1}\|g_1(\cdot,t)\|_{L^\infty}<\infty$, where g_1 is as in Corollary 4.4. Thus we have Weyl's law:

$$\lim_{\lambda\to+\infty}\frac{N_{(X,\mathsf{d},\mathfrak{m})}(\lambda)}{\lambda^{1/2}} = \frac{\omega_1}{2\pi}\mathcal{H}^1((0,\pi)) = 1.$$

Example 4.6 (Iterated suspensions). Let us apply now Theorem 4.3 to iterated suspensions of $(X,\mathsf{d},\mathfrak{m})$ as in Example 4.5:

$$\begin{cases}(X_1,\mathsf{d}_1,\mathfrak{m}_1) := ([0,\pi],\mathsf{d}_{[0,\pi]},\sin^{N-1} t\mathsf{d}t),\\ (X_{n+1},\mathsf{d}_{n+1},\mathfrak{m}_{n+1}) := ([0,\pi],\mathsf{d}_{[0,\pi]},\sin t\mathsf{d}t)\times^1(X_n,\mathsf{d}_n,\mathfrak{m}_n).\end{cases}$$

Recall that the spherical suspension $([0,\pi],\mathsf{d}_{[0,\pi]},\sin t\mathsf{d}t)\times^1(X,\mathsf{d},\mathfrak{m})$ of a metric measure space $(X,\mathsf{d},\mathfrak{m})$ is the quotient of the product $[0,\pi]\times X$ by the identification of every point of $\{0\}\times X$ and $\{\pi\}\times X$ into two distinct points, equipped with the product measure $\mathsf{d}\mu:=\sin t\mathsf{d}t\times\mathfrak{m}$ and with the distance $\mathsf{d}_{\mathrm{susp}}$ defined by

$$\cos\mathsf{d}_{\mathrm{susp}}((t,x),(s,y)) = \cos t\cos s+\sin t\sin s\cos(\min\{\mathsf{d}(x,y),\pi\}).$$

Note that $(X_n,\mathsf{d}_n,\mathfrak{m}_n)$ is a $\mathrm{RCD}^*(N+n-2,N+n-1)$-space (see [K15a]) and that (X_n,d_n) are isometric to a hemisphere of the n-dimensional unit sphere $\mathbb{S}^n(1)$ as metric spaces.

Then we can apply Theorem 4.3 because an elementary calculation similar to the one of Example 4.5 shows that $\sup_{t<1}\|g_n(\cdot,t)\|_{L^\infty}<\infty$. Thus Weyl's law follows:

$$\lim_{\lambda\to+\infty}\frac{N_{(X_n\mathsf{d}_n,\mathfrak{m}_n)}(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n}\mathcal{H}^n(X_n) = \frac{\omega_n}{(2\pi)^n}\frac{\mathcal{H}^n(\mathbb{S}^n(1))}{2}.$$

Example 4.7 (Gaussian spaces). For noncompact $\mathrm{RCD}(K,\infty)$-spaces the behavior of the spectrum is different, and requires a more delicate analysis. For instance (see [Mil15, (2.2)]) the n-dimensional Gaussian space $(X,\mathsf{d},\mathfrak{m}):=(\mathbb{R}^n,\mathsf{d}_{\mathbb{R}^n},\gamma_n)$ satisfies

$$\lim_{\lambda\to+\infty}\frac{N_{(X,\mathsf{d},\mathfrak{m})}(\lambda)}{\lambda^n} = \frac{1}{\Gamma(n+1)}.$$

Corollary 4.8 (Weyl's law on compact Ahlfors regular $\mathrm{RCD}^*(K,N)$-spaces - especially Alexandrov spaces)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be a compact* $\mathrm{RCD}^*(K,N)$*-space with $K\in\mathbb{R}$ and $N\in(1,+\infty)$. Assume that $(X,\mathsf{d},\mathfrak{m})$ is Ahlfors n-regular for some $n\in\mathbb{N}$, i.e. there exists $C>1$ such that*

$$C^{-1}r^n\le\mathfrak{m}(B_r(x))\le Cr^n \qquad \forall x\in X,\ r\in(0,1).$$

Then we have Weyl's law:

$$\lim_{\lambda\to+\infty}\frac{N_{(X,\mathsf{d},\mathfrak{m})}(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n}\mathcal{H}^n(X). \tag{4.13}$$

In particular this holds if $(X,\mathsf{d},\mathfrak{m})$ is an n-dimensional compact Alexandrov space.

Proof. Note that by the Ahlfors n-regularity of $(X, \mathsf{d}, \mathfrak{m})$, any tangent cone at x also satisfies the Ahlfors n-regularity, which implies that $\mathcal{R}_i = \emptyset$ for any $i \neq n$. In particular since $\mathcal{H}^n \ll \mathfrak{m} \ll \mathcal{H}^n$, we have

$$\mathfrak{m}(X \setminus \mathcal{R}_n) = \mathcal{H}^n(X \setminus \mathcal{R}_n) = 0. \tag{4.14}$$

Then Theorem 4.3 can be applied with $g_k \equiv c$ for some $c > 0$, which proves (4.13) by (4.14). The final statement follows from the compatibility between Alexandrov spaces and RCD-spaces [Pet11, ZZ10]. □

Example 4.9. Let us discuss the simplest case we can apply Corollary 4.8; let M be a compact n-dimensional manifold and let $f \in C^2(M)$. Then, thanks to (1.2), for any $N \in (n, \infty)$ there exists $K \in \mathbb{R}$ such that $(M, \mathsf{d}, e^{-f}\mathcal{H}^n)$ is a $\mathrm{RCD}^*(K, N)$-space. Moreover since $\left(M, \mathsf{d}, e^{-f}\mathcal{H}^n\right)$ is Ahlfors n-regular, Corollary 4.8 yields Weyl's law:

$$\lim_{\lambda \to +\infty} \frac{N_{(M, \mathsf{d}, e^{-f}\mathcal{H}^n)}(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n}\mathcal{H}^n(M).$$

In order to give another application of Weyl's law on compact finite dimensional Alexandrov spaces, let us recall that two compact finite dimensional Alexandrov spaces are said to be *isospectral* if the spectrums of their Laplacians coincide. See for instance [S85, EW13] for constructions of isospectral manifolds and of isospectral Alexandrov spaces (see also [KMS01] for analysis on Alexandrov spaces).

It is also well-known as a direct consequence of Perelman's stability theorem [Per91] (see also [K07]) that for fixed $n \in \mathbb{N}$, $K \in \mathbb{R}$ and $d, v > 0$ the isometry class of n-dimensional compact Alexandrov spaces X of sectional curvature bounded below by K with $\operatorname{diam} X \le d$ and $\mathcal{H}^n(X) \ge v$ has only finitely many topological types. By using this and Weyl's law, we have the following which is a generalization of topological finiteness results for isospectral spaces proven in [BPP92, Har16, Stan05] to Alexandrov spaces.

Corollary 4.10 (Topological finiteness theorem for isospectral Alexandrov spaces)**.** *Let $\chi := \{(X_u, \mathsf{d}_u, \mathcal{H}^{n_u})\}_{u \in U}$ be a class of compact finite dimensional Alexandrov spaces with a uniform sectional curvature bound from below. Assume that there exists $C > 1$ such that*

$$\limsup_{\lambda \to +\infty} \frac{N_{(X_u, \mathsf{d}_u, \mathcal{H}^{n_u})}(\lambda)}{N_{(X_v, \mathsf{d}_v, \mathcal{H}^{n_v})}(\lambda)} \le C \tag{4.15}$$

for all u, $v \in U$. Then χ has only finitely many topological types.

In particular, any class of isospectral compact finite dimensional Alexandrov spaces with a uniform sectional curvature bound from below has only finitely many members up to homeomorphism.

Proof. By an argument similar to the proof of [BPP92, Corollary 1.2] (or [Stan05, Proposition 7.4]) with [VR04, Corollary 1] there exists $d > 0$ such that $\operatorname{diam} X_\lambda \le d$. Since Weyl's law (4.13) with (4.15) implies that there exist $n \in \mathbb{N}$ and $v > 0$ such that $\dim X_\lambda \equiv n$ and $\mathcal{H}^n(X_\lambda) \ge v$ for any $\lambda \in \Lambda$, the topological finiteness result stated above completes the proof. □

5 Appendix: refinements of Karamata's theorem

In this section we prove Theorem 2.5 and its one-sided versions mentioned in Remark 2.6. We follow the proofs in Theorems 10.2 and 10.3 of [S79], borrowing also the terminology "Abelian", "Tauberian" from there.

Throughout this section ν is a nonnegative and σ-finite Borel measure on $[0,+\infty)$. The results will then be applied to the case when $\nu := \sum_i \delta_{\lambda_i}$.

Lemma 5.1. *For all $t > 0$ one has*

$$\int_{[0,+\infty)} e^{-tx} \mathsf{d}\nu(x) = \int_0^\infty t\nu([0,y])e^{-ty} \mathsf{d}y. \tag{5.1}$$

Proof. By Cavalieri's formula and the change of variables $r = e^{-ty}$ we get

$$\int_{[0,+\infty)} e^{-tx} \mathsf{d}\nu(x) = \int_0^1 \nu(\{x : \ e^{-tx} \geq r\})\mathsf{d}r = \int_0^\infty te^{-ty}\nu(\{x : \ e^{-tx} \geq e^{-ty}\})\mathsf{d}y$$

and we conclude, since $\{x : \ e^{-tx} \geq e^{-ty}\} = [0,y]$. □

We start with the Abelian case, easier when compared to the Tauberian one.

Theorem 5.2 (Abelian theorem)**.** *Assume that there exist $\gamma \in [0,+\infty)$ and $C \in [0,+\infty)$ such that*

$$\lim_{a\to+\infty} \frac{\nu([0,a])}{a^\gamma} = C. \tag{5.2}$$

Then

$$\lim_{t\to 0^+} t^\gamma \int_{[0,+\infty)} e^{-tx} \mathsf{d}\nu(x) = C\Gamma(\gamma+1). \tag{5.3}$$

More generally,

$$\limsup_{a\to+\infty} \frac{\nu([0,a])}{a^\gamma} \leq C < +\infty \quad \Longrightarrow \quad \limsup_{t\to 0^+} t^\gamma \int_{[0,+\infty)} e^{-tx} \mathsf{d}\nu(x) \leq C\Gamma(\gamma+1) \tag{5.4}$$

and

$$\liminf_{a\to+\infty} \frac{\nu([0,a])}{a^\gamma} \geq c \quad \Longrightarrow \quad \liminf_{t\to 0^+} t^\gamma \int_{[0,+\infty)} e^{-tx} \mathsf{d}\nu(x) \geq c\Gamma(\gamma+1). \tag{5.5}$$

Proof. Let $F(a) := \nu([0,a])$ and $G(a) := (a+1)^{-\gamma}F(a)$. Then (5.2) yields

$$\lim_{a\to+\infty} G(a) = C. \tag{5.6}$$

In particular $\sup_a G(a) < \infty$. Then Lemma 5.1 gives

$$t^\gamma \int_{[0,+\infty)} e^{-tx} \mathsf{d}\nu(x) = t^{\gamma+1} \int_0^\infty e^{-tx}(x+1)^\gamma G(x)\mathsf{d}x = \int_0^\infty e^{-y}(y+t)^\gamma G(y/t)\, \mathsf{d}y. \tag{5.7}$$

Since for any $t \in (0,1]$

$$e^{-y}(y+t)^\gamma G(y/t) \leq e^{-y}(y+1)^\gamma \sup_a G(a) \in L^1([0,+\infty)), \tag{5.8}$$

applying the dominated convergence theorem to (5.7) as $t \downarrow 0$ shows (5.3) because $G(y/t) \to C$ as $t \downarrow 0$ by (5.6).

The one-sided versions (5.4), (5.5) follow by an analogous argument, using Fatou's lemma and noticing that in the lim sup case the functions in (5.8) are dominated as $t \to 0^+$ by an integrable function. □

Now we deal with the Tauberian case.

Theorem 5.3 (Tauberian theorem)**.** *Assume that there exist* $\gamma \in [0, +\infty)$ *and* $D \in [0, +\infty)$ *such that*

$$\lim_{t \to 0^+} t^\gamma \int_{[0,+\infty)} e^{-tx} \mathsf{d}\nu(x) = D. \tag{5.9}$$

Then

$$\lim_{a \to +\infty} \frac{\nu([0, a))}{a^\gamma} = \frac{D}{\Gamma(\gamma + 1)}. \tag{5.10}$$

Proof. If $\gamma = 0$, then applying the monotone convergence theorem to (5.9) shows (5.10), hence we can assume $\gamma > 0$. For any $t \in (0, 1]$ let ν_t, μ be Borel measures on $[0, +\infty)$ be respectively defined by

$$\nu_t(A) := t^\gamma \nu(t^{-1}A), \qquad \mu(A) := \int_A x^{\gamma-1} \mathsf{d}x \tag{5.11}$$

for any Borel subset A. Then (5.10) is equivalent to

$$\lim_{t \to 0^+} \nu_t([0, 1)) = \frac{D}{\Gamma(\gamma)} \mu([0, 1)) \tag{5.12}$$

because

$$\nu_t([0, 1)) = t^\gamma \nu([0, t^{-1})) \quad \text{and} \quad \mu([0, 1)) = \int_0^1 x^{\gamma-1} \mathsf{d}x = \frac{1}{\gamma} = \frac{\Gamma(\gamma)}{\Gamma(\gamma + 1)}. \tag{5.13}$$

In order to prove (5.12), we will show

$$\lim_{t \to 0^+} \int f(x) \mathsf{d}\nu_t(x) = \frac{D}{\Gamma(\gamma)} \int f(x) \mathsf{d}\mu(x) \tag{5.14}$$

for any $f \in C_c([0, +\infty))$ as follows.

Let $\hat{\nu}_t := e^{-x} \mathsf{d}\nu_t(x)$ and $\hat{\mu} := e^{-x} \mathsf{d}\mu(x)$ be the corresponding weighted measures on $[0, +\infty)$. Then (5.9) with Lemma 5.1 yields

$$\lim_{t \to 0^+} \hat{\nu}_t([0, +\infty)) = \lim_{t \to 0^+} \int e^{-x} \mathsf{d}\nu_t(x) = \lim_{t \to 0^+} \int e^{-tx} t^\gamma \mathsf{d}\nu(x) = \frac{D}{\Gamma(\gamma)} \hat{\mu}([0, +\infty)). \tag{5.15}$$

In particular

$$\sup_{t<1} \hat{\nu}_t([0, +\infty)) < +\infty. \tag{5.16}$$

More strongly, (5.9) yields

$$\lim_{t \to 0^+} \int g(x) \mathsf{d}\hat{\nu}_t(x) = \frac{D}{\Gamma(\gamma)} \int g(x) \mathsf{d}\hat{\mu}(x) \tag{5.17}$$

for any polynomial $g(x)$ in e^{-x} (i.e. $g(x) = \sum_{i=1}^N a_i e^{-ix}$). Because

$$\begin{aligned}
\lim_{t \to 0^+} \int e^{-kx} \mathsf{d}\hat{\nu}_t(x) &= \lim_{t \downarrow 0} \int e^{-(k+1)x} \mathsf{d}\nu_t(x) \\
&= \lim_{t \to 0^+} \int e^{-(k+1)tx} t^\gamma \mathsf{d}\nu(x) \\
&= \frac{D}{(k+1)^\gamma} = \frac{D}{\Gamma(\gamma)} \int e^{-kx} \mathsf{d}\hat{\mu}(x).
\end{aligned}$$

Let $C_0([0, +\infty))$ be the set of continuous functions f on $[0, +\infty)$ such that $f(x) \to 0$ as $x \to +\infty$. Then since the set of polynomials in e^{-x} is dense in $C_0([0, +\infty))$ with respect to the norm $\sup|f|$, applying the Stone-Weierstrass theorem to $(C_0([0,+\infty)), \sup|\cdot|)$ with (5.16) shows that (5.17) is satisfied for any $g \in C_0([0,+\infty))$, which implies (5.14).

We are now in a position to prove (5.12) by using (5.14). Indeed, it is well-known that the weak convergence implies $\nu_t(E) \to D\mu(E)/\Gamma(\gamma)$ for any compact set $E \subset [0,+\infty)$ with $\mu(\partial E) = 0$. Choosing $E = [0,1]$ we obtain (5.14). $\square$

Remark 5.4. The difficulty to obtain a one sided version out of the previous proof, as we did for the Abelian case, can also be explained as follows: if we consider the push forward σ_t of the measures $\hat{\nu}_t$ under the map $x \mapsto e^{-x}$, the argument above shows that all moments of all weak limit points of σ_t are uniquely determined. Hence, since a finite Borel measure in $[0,1]$ is uniquely determined by its moments, uniqueness follows. If we replace the assumption (5.9) by a bound on the lim inf or the lim sup, we find only an inequality between the moments of the measures, which does not seem to imply, in general, the corresponding inequality for the measures.

Proposition 5.5. *Assume that for some $\gamma \in [0,+\infty)$ one has*

$$\limsup_{t\to 0^+} t^\gamma \int_{[0,+\infty)} e^{-st} \mathrm{d}\nu(s) \le C_0 < +\infty. \tag{5.18}$$

Then

$$\limsup_{\lambda\to+\infty} \frac{\nu([0,\lambda])}{\lambda^\gamma} \le eC_0. \tag{5.19}$$

Proof. Note that for any $\lambda > 0$ and any $t > 0$

$$\nu([0,\lambda]) \le e^{\lambda t} \int_{[0,\lambda]} e^{-st} \mathrm{d}\nu(s) \le e^{\lambda t} \int_{[0,+\infty)} e^{-st} \mathrm{d}\nu(s). \tag{5.20}$$

By (5.18), for any $\epsilon > 0$ there exists $t_0 > 0$ such that $\int_{[0,+\infty)} e^{-st} \mathrm{d}\nu(s) \le (C_0+\epsilon)t^{-\gamma}$ for any $t < t_0$. Thus (5.20) yields $\nu([0,\lambda]) \le e^{\lambda t}(C_0+\epsilon)t^{-\gamma}$ for any $\lambda > 0$ and any $t < t_0$. Letting $\lambda := t^{-1}$ and then letting $t \downarrow 0$ shows (5.19). $\square$

Proposition 5.6. *Assume that for some $\gamma \in [0,+\infty)$ one has*

$$\liminf_{t\to 0^+} t^\gamma \int_{[0,+\infty)} e^{-st} \mathrm{d}\nu(s) > 0, \qquad \limsup_{t\to 0^+} t^\gamma \int_{[0,+\infty)} e^{-st} \mathrm{d}\nu(s) < +\infty. \tag{5.21}$$

Then

$$\liminf_{\lambda\to+\infty} \frac{\nu([0,\lambda])}{\lambda^\gamma} > 0. \tag{5.22}$$

Proof. Call $C_0 > 0$ the lim inf and $C_1 < +\infty$ the lim sup in (5.21). Note that for any $\lambda > 0$ and any $t > 0$

$$\begin{aligned}
\int_{[0,+\infty)} e^{-st} \mathrm{d}\nu(s) &= \int_{[0,\lambda]} e^{-st} \mathrm{d}\nu(s) + \sum_{\ell=1}^{\infty} \int_{(\ell\lambda,(\ell+1)\lambda]} e^{-st} \mathrm{d}\nu(s) \\
&\le \nu([0,\lambda]) + \sum_{\ell=1}^{\infty} e^{-\ell\lambda t} \nu([0,(\ell+1)\lambda]) \\
&= \sum_{\ell=0}^{\infty} e^{-\ell\lambda t} \nu([0,(\ell+1)\lambda]).
\end{aligned}$$

In particular, letting $\lambda := t^{-1}$ yields

$$t^\gamma \int_{[0,+\infty)} e^{-st} \mathsf{d}\nu(s) \leq t^\gamma \sum_{\ell=0}^{\infty} e^{-\ell} \nu([0, \frac{\ell+1}{t}]). \tag{5.23}$$

Thus there exists $t_0 > 0$ such that for any $t < t_0$

$$0 < \frac{C_0}{2} \leq t^\gamma \sum_{\ell=0}^{\infty} e^{-\ell} \nu([0, \frac{\ell+1}{t}]). \tag{5.24}$$

Next let us discuss the right hand side of (5.24). By (5.21) and Proposition 5.5 there exists $\hat{\lambda} > 0$ such that $\nu([0,\lambda]) \leq (eC_1 + 1)\lambda^\gamma$ for any $\lambda \geq \hat{\lambda}$. Thus for any $t > 0$ with $t^{-1} \geq \hat{\lambda}$ we get

$$\nu([0, \frac{\ell+1}{t}]) \leq (eC_1 + 1)\frac{(\ell+1)^\gamma}{t^\gamma}.$$

In particular

$$t^\gamma \sum_{\ell=k}^{\infty} e^{-\ell} \nu([0, \frac{\ell+1}{t}]) \leq (eC_1 + 1) \sum_{\ell=k}^{\infty} e^{-\ell} (\ell+1)^\gamma \tag{5.25}$$

for any $k \in \mathbb{N}$ and any $t > 0$ with $t^{-1} \geq \hat{\lambda}$.

For any $\delta > 0$ there exists $k_0 \in \mathbb{N}$ such that $\sum_{\ell=k_0+1}^{\infty} e^{-\ell}(\ell+1)^\gamma < \delta$. Then, combining (5.24) with (5.25) yields

$$0 < \frac{C_0}{2} < t^\gamma \sum_{\ell=0}^{k_0} e^{-\ell} \nu([0, \frac{\ell+1}{t}]) + (eC_1 + 1)\delta \tag{5.26}$$

for any $t > 0$ with $t < t_0$ and $t^{-1} \geq \hat{\lambda}$, which easily shows (5.22) choosing $\delta > 0$ so small that $(eC_1 + 1)\delta < C_0/2$. □

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