POHOŽAEV IDENTITY FOR THE FRACTIONAL $p-$LAPLACIAN ON $\mathbb{R}^N$

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Abstract. By virtue of a suitable approximation argument, we prove a Pohožaev identity for nonlinear nonlocal problems on $\mathbb{R}^N$ involving the fractional $p-$Laplacian operator. Furthermore we provide an application of the identity to show that some relevant levels of the energy functional associated with the problem coincide.

Contents

1. Introduction

1.1. Overview. In the seminal paper [17] Pohožaev discovered the celebrated identity

$$N\int_\Omega F(u)\,dx=\frac{N-2}{2}\int_\Omega |\nabla u|^2\,dx+\frac{1}{2}\int_{\partial\Omega} x\cdot\nu\,\left|\frac{\partial u}{\partial \nu}\right|^2\,d\mathcal{H}^{N-1},$$

which is valid for weak energy solutions of the semilinear elliptic problem

$$-\Delta u=F'(u),\quad \text{in }\Omega,\qquad u=0,\quad \text{on }\partial\Omega. \tag{1.1}$$

As a major application of such an identity, he was able to get nonexistence of nontrivial solutions of (1.1), when $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain, star-shaped with respect to the origin and F satisfies suitable assumptions.

The result in [17] stimulated further developments and extensions and a general variational identity for Euler-Lagrange equations of functionals of the Calculus of Variations was eventually formulated in 1986 by Pucci-Serrin [18]. Their results also included identities for systems, as well as for higher order equations, such those involving the polyharmonic operator $(-\Delta)^m$. They

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also covered both the case of bounded and unbounded domains. In the last case some results in the half-space were previously obtained by Esteban-Lions in [10] in 1982.

The only drawback of the general formula obtained in [18] is that it was stated for solutions of class C^2, which is of course quite a serious restriction if one thinks about degenerate/singular quasilinear problems such as

$$-\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\,\nabla u) = f(u). \tag{1.2}$$

Indeed, in this case the optimal regularity of solutions is known to be $C^{1,\alpha}$, see for example [8]. For this reason, the subsequent improvements were also focused on weakening the regularity assumption of the solutions for the validity of Pohožaev-Pucci-Serrin type identities. In the one-dimensional case this was done in [19], while for the particular case of equation (1.2) in a smooth bounded domain, Guedda-Veron in 1989 proved the identity

$$N\int_\Omega F(u)\,dx = \frac{N-p}{p}\int_\Omega |\nabla u|^p\,dx + \frac{p-1}{p}\int_{\partial\Omega} x\cdot\nu\,\left|\frac{\partial u}{\partial \nu}\right|^p\,d\mathcal{H}^{N-1},$$

for solutions $u\in W^{1,p}_0(\Omega)\cap L^\infty(\Omega)$. The technique of [13] is essentially based upon a suitable approximation of the problem with smoother problems on which the formula holds and then on providing some a priori estimates in order to pass to the limit and get the desired identity.

Finally, in 2003 Degiovanni-Musesti-Squassina derived [9] a general Pohožaev identity for C^1 solutions to variationl equations

$$-\mathrm{div}\nabla_\xi\mathscr{L}(x,u,\nabla u) + D_s\mathscr{L}(x,u,\nabla u) = f, \qquad \text{in } \Omega,$$

where the function $\xi\mapsto\mathscr{L}(x,s,\xi)$ is strictly convex for each $(x,s)\in\overline{\Omega}\times\mathbb{R}$. This generalized the identity of [18] by removing the C^2 assumption on u and the C^1 assumption on $\nabla_\xi\mathscr{L}$ (excluding e.g. the case of the $p-$Laplacian when $1<p<2$) by imposing, instead, the natural assumption of strict convexity of $\mathscr{L}(x,s,\cdot)$.

The great interest arisen in the recent years in the study of nonlocal problem, led researchers to investigate some kind of *nonlocal counterparts* of the Pohožaev identities. Clearly, in this framework, the main problem is the lack of (sufficiently) high regularity for the solutions and, for bounded domains, the lack of regularity up to the boundary. In 2014, Ros-Oton-Serra proved a boundary regularity result [23] for the solutions of $(-\Delta)^s u = f$, where $(-\Delta)^s$ is the fractional Laplacian defined by

$$(-\Delta)^s u = \mathscr{F}^{-1}\Big(|\xi|^{2\,s}\mathscr{F}(u)\Big).$$

Here $\mathscr{F}$ denotes the Fourier transform on $\mathbb{R}^N$. Precisely, they proved that, if $\Omega\subset\mathbb{R}^N$ is a $C^{1,1}$ domain and $f:\Omega\to\mathbb{R}$ is bounded, then the ratio

$$\frac{u}{\mathrm{d}_\Omega^s}, \qquad \text{where } \mathrm{d}_\Omega(x) := \mathrm{dist}(x,\partial\Omega),$$

admits a continuous extension to $\overline{\Omega}$ which is of class C^α, for some $\alpha\in(0,1)$. By virtue of this regularity result, in [20, Theorem 1.1] the same authors were able to prove that if $f:\mathbb{R}\to\mathbb{R}$ is a locally Lipschitz function and $u\in H^s_0(\Omega)\cap L^\infty(\Omega)$ is a solution to $(-\Delta)^s u = f(u)$, then the following identity holds

$$N\int_\Omega F(u)\,dx = \frac{N-2\,s}{2}\int_\Omega \left|(-\Delta)^{s/2}u\right|^2dx + \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega} x\cdot\nu\,\left(\frac{u}{\mathrm{d}_\Omega^s}\right)^2 d\mathcal{H}^{N-1}. \tag{1.3}$$

This can be used to get non-existence results in star-shaped domains for critical or supercritical nonlinearities. In the critical, it should be noted that only positive solutions can be ruled out, due to the current lack of unique continuation results up to the boundary.

In passing from semilinear to quasilinear nonlocal problems, the regularity issue becomes much harder and for the *fractional $p-$Laplacian operator* $(-\Delta_p)^s$, formally defined by

$$(-\Delta_p)^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))}{|x-y|^{N+s\,p}}\,dy, \qquad x \in \mathbb{R}^N.$$

the boundary regularity for the solution of the problem $(-\Delta_p)^s u = f$ has been recently studied by Iannizzotto-Mosconi-Squassina in [14], where it was proved that, if $f \in L^\infty(\Omega)$, then $u \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, s]$.

The further expected improvement, which would eventually open the doors to the possibility of getting a quasilinear counterpart of (1.3) is to prove that, as in the linear case $p = 2$, u/d_Ω^s admits a continuous extension to $\overline{\Omega}$ which is of class C^α. This regularity result is currently missing, for a more detailed discussion we refer the reader to the recent survey paper [16]. We mention that, without a Pohožaev identity, in [21] a nonexistence result for $(-\Delta_p)^s u = f(u)$ *for supercritical nonlinearities* was derived for bounded solutions which belong to $W^{1,r}(\Omega)$ for some $r > 1$. The latter is a rather severe restriction (especially for very low values of s), although some progresses in this direction were recently obtained in [4].

1.2. **Main result.** Recently, the interest in the study of spatial decay for the optimizers of the fractional Sobolev embedding [5] has lead the authors to study the problem

$$(-\Delta_p)^s u = f(u) \quad \text{ in } \mathbb{R}^N. \tag{1.4}$$

It is thus natural to wonder wether bounded energy solutions of (1.4) satisfy a suitable Pohožaev identity. This is indeed the case. In order to state the main result of the paper, we let

$$\mathcal{D}^{s,p}(\mathbb{R}^N) := \left\{ u \in L^{\frac{N\,p}{N-s\,p}}(\mathbb{R}^N) \;:\; \|u\|^p_{\mathcal{D}^{s,p}} := \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy < +\infty \right\}.$$

Theorem 1.1 (Pohožaev identity on $\mathbb{R}^N$)**.** *Let $1 < p < \infty$, $s \in (0,1)$ be such that $s\,p < N$, $f \in C^0(\mathbb{R},\mathbb{R})$ and set $F(t) = \int_0^t f(\tau)\,d\tau$. If $u \in \mathcal{D}^{s,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a weak solution to (1.4), namely*

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+p\,s}}\,dx\,dy = \int_{\mathbb{R}^N} f(u)\,\varphi\,dx, \quad \forall\ \varphi \in C_0^\infty(\mathbb{R}^N),$$

with the property $F(u) \in L^1(\mathbb{R}^N)$, it holds

$$\frac{N-s\,p}{p}\,\|u\|^p_{\mathcal{D}^{s,p}} = N \int_{\mathbb{R}^N} F(u)\,dx. \tag{1.5}$$

A slight variant of the arguments proving the assertion of Theorem 1.1 yields a nonautonomous version of the Pohožaev identity (1.5). See Remark 5.1 for more details.
We stress that the assertion of Theorem 1.1 is new also in the semi-linear case $p = 2$. In this case, the formula was previously known but only for bounded weak solutions $u \in H^s(\mathbb{R}^n)$ (in particular, $u \in L^2(\mathbb{R}^N)$) to equation (1.4), see [11, Proposition 4.1]. Finally, the condition $F(u) \in L^1(\mathbb{R}^N)$ is often naturally verified for any weak energy solution, see Corollary 2.5 for some simple criteria.

We now give an heuristic derivation of formula (1.5). Let $\mathcal{E}$ be the energy functional associated with equation (1.4), that is

$$\mathcal{E}(v) := \frac{1}{p}\,\|v\|^p_{\mathcal{D}^{s,p}} - \int_{\mathbb{R}^N} F(v)\,dx, \qquad v \in \mathcal{D}^{s,p}(\mathbb{R}^N).$$

If u is a solution of (1.4), then we have

$$\langle d\mathcal{E}(u), \varphi\rangle = 0, \qquad \text{for every } \varphi \in \mathcal{D}^{s,p}(\mathbb{R}^N). \tag{1.6}$$

We now consider the family of rescaled functions $u_t(x) = u(x/t)$, then we formally get

$$\frac{d}{dt}\mathcal{E}\left(u_t\right)\Big|_{t=1} = \langle d\mathcal{E}(u), \nabla u \cdot x\rangle.$$

If we suppose that $\nabla u \cdot x$ is an admissible test function, from (1.6) one would obtain

$$\langle d\mathcal{E}(u), \nabla u \cdot x\rangle = 0.$$

On the other hand, by scaling we also have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(u_t)\Big|_{t=1} &= \frac{N - s\,p}{p}\,t^{N-sp-1}\,\|u\|^p_{\mathcal{D}^{s,p}} - N\,t^{N-1}\int_{\mathbb{R}^N} F(u)\,dx\Big|_{t=1}\\ &= \frac{N - s\,p}{p}\,\|u\|^p_{\mathcal{D}^{s,p}} - N\int_{\mathbb{R}^N} F(u)\,dx.\end{aligned}$$

By joining the last three displays, one would eventually get the desired Pohozaev identity. Of course, this argument assumes that $\nabla u \cdot x \in \mathcal{D}^{s,p}(\mathbb{R}^N)$, which seems hardly true, both from the regularity and from the summability point of view.

1.3. **Sketch of proof.** Let us spend some words on the proof of Theorem 1.1. The first step is to freeze the term $f(u) =: f$ and define regularizations f_ε in a sufficiently strong sense. Then we approximate the equation $d\mathcal{J}(u) = f$ (where $\mathcal{J}(v) = \frac{1}{p}\|v\|^p_{\mathcal{D}^{s,p}}$) with entire elliptic problems of the form $d\mathcal{J}_\varepsilon(u_\varepsilon) = f_\varepsilon$, where

$$\mathcal{J}_\varepsilon(v) = \frac{1}{p}\,\|v\|^p_{\mathcal{D}^{s,p}} + \varepsilon\frac{1}{2}\,\|v\|^2_{\mathcal{D}^{s,2}}, \tag{1.7}$$

and prove that $u_\varepsilon \to u$ in a sufficiently strong sense. Since there is no local integration by parts formula which allow to reduce the global Pohǒzaev identity from a local one, this approximation must be done globally in $\mathbb{R}^N$, forcing some decay properties for the test functions allowed in the weak form of the equation.

We then prove a series of regularity statements on u_ε, ensuring that for any $\eta \in C^\infty_c(\mathbb{R}^N)$, the function $\eta\,x \cdot \nabla u_\varepsilon$ is a viable test function for the weak form of $d\mathcal{J}_\varepsilon(u_\varepsilon) = f_\varepsilon$.

Then we derive an approximate local form of the Pohǒzaev identity arising from the domain perturbation $\Phi_t(x) = x + t\eta(x)\,x$. More precisely, we prove the validity of

$$\frac{d}{dt}\mathcal{J}_\varepsilon(u_\varepsilon \circ \Phi_t)\Big|_{t=0} = \langle d\mathcal{J}_\varepsilon(u_\varepsilon), \eta\,x \cdot \nabla u_\varepsilon\rangle = \int_{\mathbb{R}^N} f_\varepsilon \nabla u_\varepsilon \cdot \eta\,x\,dx. \tag{1.8}$$

On the other hand, the energy $\mathcal{J}_\varepsilon$ is a double integral of the form

$$\mathcal{J}_\varepsilon(v) = \int_{\mathbb{R}^{2N}} e_\varepsilon(x, y, v(x) - v(y))\,dx\,dy$$

so that changing variable it holds

$$\mathcal{J}_\varepsilon(u_\varepsilon \circ \Phi_t) = \int_{\mathbb{R}^{2N}} e_\varepsilon(\Phi_t^{-1}(x), \Phi_t^{-1}(y), u_\varepsilon(x) - u_\varepsilon(y))\,d\Phi_t^{-1}(x)\,d\Phi_t^{-1}(y).$$

We then prove that the left hand side in (1.8) can be computed changing variables first, and then differentiating in t, providing

$$\frac{d}{dt}\mathcal{J}_\varepsilon(u_\varepsilon\circ\Phi_t)\Big|_{t=0}=\mathcal{P}_\varepsilon(u_\varepsilon,\eta)$$

for some expression $\mathcal{P}_\varepsilon(u_\varepsilon,\eta)$ which represent a localized (through η) approximation (through ε) of the left hand side of (1.5).

The corresponding identity is at this stage

$$\mathcal{P}_\varepsilon(u_\varepsilon,\eta)=\int_{\mathbb{R}^N} f_\varepsilon\nabla u_\varepsilon\cdot\eta\, x\, dx,$$

and we proceed in letting $\varepsilon\to 0$. Through some a-priori estimates it is readily seen that $\mathcal{P}_\varepsilon(u_\varepsilon,\eta)\to\mathcal{P}_0(u,\eta)$. The difficulty is now to take the limit in the right hand side of the identity, since the gradient term is going to blow-up as $\varepsilon\to 0$ (unless one *assumes* $u\in W^{1,r}_{\rm loc}(\mathbb{R}^N)$ as in [21]). To deal with it, we integrate by parts *before* passing to the limit, introducing an error term. More precisely, we write

$$\int_{\mathbb{R}^N} f_\varepsilon\nabla u_\varepsilon\cdot\eta\, x\, dx=\int_{\mathbb{R}^N} f(u_\varepsilon)\nabla u_\varepsilon\cdot\eta\, x\, dx+{\rm err}_\varepsilon=-\int_{\mathbb{R}^N} F(u_\varepsilon){\rm div}(\eta\, x)\, dx+{\rm err}_\varepsilon,$$

where the error term can be estimated through

$$|{\rm err}_\varepsilon|=\left|\int_{\mathbb{R}^N}(f_\varepsilon-f(u_\varepsilon))\nabla u_\varepsilon\cdot\eta\, x\, dx\right|\le\|(f_\varepsilon-f(u_\varepsilon))\nabla u_\varepsilon\|_{H^{-1,t}}\|\eta\, x\|_{H^{1,t'}},$$

where $H^{1,t'}$ denotes the Bessel potentials space. To conclude, we prove in the appendix the following bilinear estimate, much in the spirit of the commutator estimate of Kato-Ponce:

$$\|h\nabla v\|_{H^{-1,t}}\le C\|h\|_{L^q}\|v\|_{L^r},\qquad \frac{1}{q}+\frac{1}{r}=\frac{1}{t}<1.$$

This implies, for suitable choices of q and r, that ${\rm err}_\varepsilon\to 0$ as $\varepsilon\to 0$, and a final passage to the limit $\eta\to 1$ provides the asserted identity, as long as $F(u)\in L^1(\mathbb{R}^N)$.

1.4. **Structure of the paper.** In section 2.1 we describe our functional analytic setting and prove some auxiliary results. In section 2.2 we deal with some basic properties of energy solutions with particular emphasis on the reaction term $f(u)$. Section 2.3 collects some inequalities involving the integrand of the approximating functional (1.7). In section 3 we construct the approximation f_ε and study the convergence of the corresponding family $\{u_\varepsilon\}$. Section 4 is devoted to the regularity theory for u_ε, culminating in the viability of the test function $\eta\, x\cdot\nabla u_\varepsilon$. In section 5 we prove Theorem 1.1 in a series of steps. Finally, we give some applications of the Pohǒzaev identity in the last section. Namely, in section 6.1 we study various energy level of the ground states for subcritical entire equations driven by the fractional p-Laplacian, while section 6.2 points out some non-existence results.

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2. Preliminaries

2.1. **Notations and functional analytic setting.** Given $A \subseteq \mathbb{R}^N$ we set $A^c = \mathbb{R}^N \setminus A$ and let χ_A be the characteristic function of A. For $1 \le q \le +\infty$ and a measurable function $u : \mathbb{R}^N \to \mathbb{R}$, we set

$$\|u\|_q = \|u\|_{L^q(\mathbb{R}^N)}.$$

If $h \in \mathbb{R}^N \setminus \{0\}$ and $\psi \in \mathbb{R}^N \to \mathbb{R}$ is a measurable function, we set

$$\psi_h(x) = \psi(x+h) \qquad \text{and} \qquad \delta_h \psi = \psi(x+h) - \psi(x).$$

In the case $\psi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, we use the same notation for

$$\psi_h(x,y) = \psi(x+h, y+h) \qquad \text{and} \qquad \delta_h \psi = \psi(x+h, y+h) - \psi(x,y).$$

On $\mathbb{R}^{2N}$ we will consider the measure

$$d\mu = \frac{dx\, dy}{|x-y|^N}.$$

Remark 2.1. It may be worth noting that the measure μ is not $\sigma-$finite on $\mathbb{R}^{2N}$. This can cause some troubles in applying Fubini's Theorem with respect to this measure, so that all the integrals involving μ should be defined on $\mathbb{R}^{2N} \setminus \{(x,x) : x \in \mathbb{R}^N\}$, where μ is indeed $\sigma-$finite. However, to ease the notation we will omit this technicality.

We moreover let

$$\mathfrak{D}^s u(x,y) = \frac{u(x) - u(y)}{|x-y|^s}, \qquad x \ne y.$$

It is easy to see that for every pair of measurable functions u, v we have the Leibniz-type rule

$$\mathfrak{D}^s(u\, v)(x,y) = \mathfrak{D}^s u(x,y)\, \frac{v(x)+v(y)}{2} + \mathfrak{D}^s v(x,y)\, \frac{u(x)+u(y)}{2}. \tag{2.1}$$

For any $1 < p < +\infty$ and measurable $u : \mathbb{R}^N \to \mathbb{R}$, the quantity

$$\|u\|_{\mathcal{D}^{s,p}} = \left(\int_{\mathbb{R}^{2N}} |\mathfrak{D}^s u|^p \, d\mu \right)^{\frac{1}{p}},$$

defines a uniformly convex norm on the reflexive Banach space

$$\mathcal{D}^{s,p}(\mathbb{R}^N) = \left\{ u \in L^{p^*}(\mathbb{R}^N) \,:\, \|u\|_{\mathcal{D}^{s,p}} < +\infty \right\}, \qquad \text{with} \quad p^* = \frac{N\, p}{N - s\, p}.$$

The topological dual of $\mathcal{D}^{s,p}(\mathbb{R}^N)$ will be denoted by $\mathcal{D}^{-s,p'}(\mathbb{R}^N)$ with, as usual, $p' = p/(p-1)$. We define

$$\mathcal{S}_{p,s} = \min_{u \in \mathcal{D}^{s,p}(\mathbb{R}^N)} \{ \|u\|^p_{\mathcal{D}^{s,p}} \,:\, \|u\|_{p^*} = 1 \},$$

i.e. the sharp Sobolev constant for the continuous embedding $\mathcal{D}^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$. We will also use the following space

$$\mathcal{D}^{s,p}_0(\Omega) := \left\{ u \in \mathcal{D}^{s,p}(\mathbb{R}^N) \,:\, u \equiv 0 \text{ in } \Omega^c \right\},$$

for a general open set $\Omega \subset \mathbb{R}^N$.

In Section 6 we will also need the Sobolev space

$$W^{s,p}(\mathbb{R}^N) := \mathcal{D}^{s,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N),$$

equipped with the norm

$$\|u\|^p_{W^{s,p}} = \|u\|^p_p + \|u\|^p_{\mathcal{D}^{s,p}}.$$

Notice that, as $\mathcal{D}^{s,p}(\mathbb{R}^N)$ is reflexive and $C_0^\infty(\mathbb{R}^N)$ is strongly dense in $\mathcal{D}^{s,p}(\mathbb{R}^N)$, the functionals

$$\mathcal{D}^{s,p}(\mathbb{R}^N) \ni u \mapsto \int_{\mathbb{R}^N} \varphi\, u\, dx,$$

form a dense subset of $\mathcal{D}^{-s,p'}(\mathbb{R}^N)$, which we will still denote by $C_0^\infty(\mathbb{R}^N)$.

The following result will be needed in the following.

Lemma 2.2. *Let $g \in C_0^\infty(\mathbb{R}^N)$, then for every $s \in (0,1)$ and $1 < p < \infty$ we have*

$$\sup_{|h|>0} \left\| \frac{\delta_{h\mathbf{e}_k} g}{h} \right\|_{\mathcal{D}^{-s,p'}} \leq \|g_{x_k}\|_{\mathcal{D}^{-s,p'}}, \qquad k = 1, \dots, N,$$

for some $C = C(N,s,p) > 0$.

Proof. Let us fix $k \in \{1,\dots,N\}$ and prove the first inequality. We have

$$\frac{\delta_{h\mathbf{e}_k} g(x)}{h} = \int_0^1 g_{x_k}(x + t\, h\mathbf{e}_k)\, dt,$$

thus by linearity we obtain for every $\varphi \in \mathcal{D}^{s,p}(\mathbb{R}^N)$

$$\begin{aligned}\left\langle \frac{\delta_{h\mathbf{e}_k} g(x)}{h}, \varphi \right\rangle &= \int_0^1 \langle g_{x_k}(x + t\, h\mathbf{e}_k), \varphi\rangle\, dt \\ &\leq \left(\int_0^1 \|g_{x_k}(\cdot + t\, h\mathbf{e}_k)\|_{\mathcal{D}^{-s,p'}}\, dt \right) \|\varphi\|_{\mathcal{D}^{s,p}}.\end{aligned}$$

By using the translation invariance of the dual norm and taking the supremum over $\varphi \neq 0$, we get the conclusion. ☐

Finally, for a set $\Omega \subset \mathbb{R}^N$ we will also make use of the following spaces:

$$L^p_\Omega(d\mu) := \Big\{ \mathcal{K} \in L^p(\mathbb{R}^{2N}, d\mu) : \mathrm{supp}(\mathcal{K}) \subseteq (\Omega \times \mathbb{R}^N) \cup (\mathbb{R}^N \times \Omega) \Big\}.$$

It is readily checked that

$$u \in \mathcal{D}^{s,p}_0(\Omega) \qquad \Longrightarrow \qquad \mathfrak{D}^s u \in L^p_\Omega(d\mu).$$

Moreover, $L^p_\Omega(d\mu)$ is an $L^\infty(\mathbb{R}^{2N}, d\mu)$ module, in the sense that $\mathcal{K}\,\mathcal{H} \in L^p_\Omega(d\mu)$ for any $\mathcal{K} \in L^p_\Omega(d\mu)$ and $\mathcal{H} \in L^\infty(\mathbb{R}^{2N}, d\mu)$.

A simple criterion to consider products with more general $\mathcal{H}$ is given in the following lemma.

Lemma 2.3. *Let $\Omega \subseteq \mathbb{R}^N$ be bounded, $\mathcal{K} \in L^{q'}_\Omega(d\mu)$ and $\mathcal{H} \in L^q_{\mathrm{loc}}(\mathbb{R}^{2N}, d\mu)$ be such that*

$$\mathcal{H}(x,y) \leq \frac{K}{|x-y|^\delta}, \quad \text{for some } \delta > 0 \text{ and } |x-y| \geq M \tag{2.2}$$

Then $\mathcal{K}\,\mathcal{H} \in L^1(\mathbb{R}^{2N}, d\mu)$ with

$$\|\mathcal{K}\,\mathcal{H}\|_{L^1(\mathbb{R}^{2N}, d\mu)} \leq C \|\mathcal{K}\|_{L^{q'}(\mathbb{R}^{2N}, d\mu)},$$

for some finite C depending only on q, Ω, δ, M, K and $\mathcal{H}$.

Proof. Let $R > 0$ be such that $\Omega \subseteq B_R$ and $A = B_{R+M}$. Then the inequality in (2.2) holds for any $x \in \Omega$ and $y \in A$. Using $\mathrm{supp}(\mathcal{K}\,\mathcal{H}) \subseteq (\Omega \times \mathbb{R}^N) \cup (\mathbb{R}^N \times \Omega)$, we split the integral over $\mathbb{R}^{2N}$ as follows

$$\int_{\mathbb{R}^{2N}} \mathcal{K}\,\mathcal{H}\, d\mu = \int_{A\times A} \mathcal{K}\,\mathcal{H}\, d\mu + \int_{\Omega \times A^c} \mathcal{K}\,\mathcal{H}\, d\mu + \int_{A^c \times \Omega} \mathcal{K}\,\mathcal{H}\, d\mu,$$

and apply Hölder's inequality. This provides

$$\|\mathcal{K}\,\mathcal{H}\|_{L^1(\mathbb{R}^{2N},d\mu)} \leq \|\mathcal{K}\|_{L^{q'}(\mathbb{R}^{2N},d\mu)} \Big(\|\mathcal{H}\|_{L^q(A\times A,d\mu)} + \|\mathcal{H}\|_{L^q(\Omega\times A^c,d\mu)} + \|\mathcal{H}\|_{L^q(A^c\times \Omega,d\mu)} \Big).$$

The first norm involving $\mathcal{H}$ is finite by assumption, and since assuption (2.2) is symmetric in x and y, it suffices to estimate

$$\int_{\Omega\times A^c} |\mathcal{H}|^q \, d\mu \leq K^q \int_{B_R} \int_{B^c_{R+M}} \frac{dx\, dy}{|x-y|^{N+q\,\delta}} \leq C(N,q,\delta)\, K^q\, R^N\, M^{-q\,\delta},$$

which concludes the proof. □

2.2. **Basic properties of energy solutions.** As mentioned in the introduction, we are concerned with bounded weak solutions to (1.4) in the sense that $u \in \mathcal{D}^{s,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^{2N}} |\mathfrak{D}^s u|^{p-2} \mathfrak{D}^s u\, \mathfrak{D}^s \varphi \, d\mu = \int_{\mathbb{R}^N} f(u)\, \varphi \, dx, \qquad \forall \varphi \in C^\infty_0(\mathbb{R}^N). \tag{2.3}$$

Clearly then $f(u) \in L^1_{\rm loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and Hölder's inequality ensures

$$\left| \int_{\mathbb{R}^N} f(u)\, \varphi \, dx \right| \leq \|u\|^{p-1}_{\mathcal{D}^{s,p}} \|\varphi\|_{\mathcal{D}^{s,p}}, \qquad \forall \varphi \in C^\infty_0(\mathbb{R}^N),$$

so that $f(u)$ actually extends to a unique element of $\mathcal{D}^{-s,p'}(\mathbb{R}^N)$, which we will still denote by $f(u)$. It is an old question (addressed for classical Sobolev spaces in [7]) whether $f(u)$, as an element of $\mathcal{D}^{s,p}(\mathbb{R}^N)$, can be represented as the L^2 multiplication by the function $f(u)$, i.e. to find conditions which ensures that for a suitable $v \in \mathcal{D}^{s,p}(\mathbb{R}^N)$ it holds

$$f(u)\, v \in L^1(\mathbb{R}^N), \qquad \langle f(u), v\rangle = \int_{\mathbb{R}^N} f(u)\, v\, d\mu,$$

where by $\langle \cdot,\cdot\rangle$ we denote the duality pairing between $\mathcal{D}^{-s,p'}(\mathbb{R}^N)$ and $\mathcal{D}^{s,p}(\mathbb{R}^N)$.

Lemma 2.4. *Let $p > 1$, $s \in (0,1)$, $f \in L^1_{\rm loc}(\mathbb{R}^N) \cap \mathcal{D}^{-s,p'}(\mathbb{R}^N)$. If $v \in \mathcal{D}^{s,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is such that $f\, v \geq h$ for some $h \in L^1(\mathbb{R}^N)$, then*

$$f\, v \in L^1(\mathbb{R}^N), \qquad \langle f, v\rangle = \int_{\mathbb{R}^N} f\, v\, dx.$$

Proof. We first observe that

$$\langle f, \varphi\rangle = \int_{\mathbb{R}^N} f\,\varphi\, dx \qquad \text{for every } \varphi \in \mathcal{D}^{s,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \text{ with bounded support.}$$

This follows by the density of $C^\infty_0(B_R)$ in $\mathcal{D}^{s,p}_0(B_R) \cap L^\infty(B_R)$ and dominated convergence, for a ball B_R such that $B_R \supseteq \Omega$.

For the second statement notice that it suffices to prove it separately for v_+ and v_- since they both satisfy $f\, v_\pm \geq -|h| \in L^1(\mathbb{R}^N)$. Consider $v_n \in C^\infty_0(\mathbb{R}^N)$ positive and such that

$$\lim_{n\to\infty} \|v_n - v_+\|_{\mathcal{D}^{s,p}} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|v_n - v_+\|_{L^p(B_R)} = 0, \quad \text{for every } B_R \subset \mathbb{R}^N.$$

Finally, we define $w_n = \min\{v_n,\, v_+\}$. Since the map $(a,b) \mapsto \min\{a,b\}$ is Lipschitz, we get that $w_n \in \mathcal{D}^{s,p}(\mathbb{R}^N)$ and

$$|\mathfrak{D}^s w_n| \leq |\mathfrak{D}^s v_n| + |\mathfrak{D}^s v_+|.$$

Clearly $w_n \to v_+$ a.e. and $0 \le w_n \le v_+$ for any n. Up to subsequence we can thus suppose that $w_n \rightharpoonup v_+$ in $\mathcal{D}^{s,p}(\mathbb{R}^N)$, which implies

$$\langle f, v_+\rangle = \lim_{n\to\infty} \langle f, w_n\rangle = \lim_{n\to\infty} \int_{\mathbb{R}^N} f\, w_n\, dx. \tag{2.4}$$

Since

$$f(x)\, w_n(x) \ge \begin{cases} 0 & \text{if } f(x) \ge 0, \\ f(x)\, v_+(x) & \text{if } f(x) < 0, \end{cases}$$

it holds $f\, w_n \ge -|h|$, so that Fatou's lemma ensures

$$-\int_{\mathbb{R}^N} |h|\, dx \le \int_{\mathbb{R}^N} f\, v_+\, dx \le \liminf_n \int_{\mathbb{R}^N} f\, w_n\, dx = \langle f, v_+\rangle,$$

and thus $f\, v_+ \in L^1(\mathbb{R}^N)$. Finally, using $|f\, w_n| \le |f\, v_+|$ and applying Dominated Convergence in (2.4), we get the conclusion. The proof for v_- is entirely analogous. □

Here we list some simple criteria ensuring that some of the assumptions in Theorem 1.1 can be removed for non-oscillating nonlinearities.

Corollary 2.5. *Let $p > 1$, $s \in (0, 1)$, $f \in C^0(\mathbb{R}, \mathbb{R})$ and $u \in \mathcal{D}^{s,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ solve problem* (2.3).

(1) If $f(t)\, t$ does not change sign in a neighborhood of 0 then $f(u)\, u \in L^1(\mathbb{R}^N)$.

(2) If f is monotone in a neighborhood of 0, then $F(u) \in L^1(\mathbb{R}^N)$.

Proof. We prove the two statement separately.

(1) Suppose without loss of generality that $f(t)\, t \ge 0$ for $|t| \le \delta$, where $\delta > 0$. Then $(u - \delta)_+$ has support of finite measure by Chebyshev inequality. Since $f(u)$ is bounded, it holds $v_1 := f(u)\, (u - \delta)_+ \in L^1(\mathbb{R}^N)$. Moreover

$$v_2 := f(u)\, \min\{\delta, u_+\} \ge -|f(u)|\, u\, \chi_{\{u>\delta\}},$$

so that by the previous Lemma, $v_2 \in L^1(\mathbb{R}^N)$. By observing that $f(u)\, u_+ = v_1 + v_2$, the latter belongs to $L^1(\mathbb{R}^N)$. A similar argument shows that $f(u)\, u_- \in L^1(\mathbb{R}^N)$.

(2) Suppose without loss of generality that f is non-decreasing for $|t| \le \delta$. We have two distinguish two cases: either $f(0) = 0$ or $f(0) \ne 0$. In the first case, then for some $0 < \delta' \le \delta$, $t \mapsto f(t)\, t$ does not change sign and the previous point gives $f(u)\, u \in L^1(\mathbb{R}^N)$. Clearly $F(u) \in L^1(\{|u| \ge \delta'\})$ by Chebyshev inequality and the boundedness of $F(u)$. Finally on $\{|u| \le \delta'\}$ it holds $0 \le F(u) \le f(u)\, u \in L^1(\mathbb{R}^N)$, which proves the claim.

If $f(0) \ne 0$, then f does not change in a neighborhood of the origin. For simplicity, we can assume that there exsist $\delta > 0$ such that $f(t) > 0$ for $|t| \le \delta$. By the same argument as before, we get that $f(u)\, u_+$ and $f(u)\, u_-$ both belong to $L^1(\mathbb{R}^N)$. This in turn implies that $u \in L^1(\mathbb{R}^N)$, indeed

$$\begin{aligned} \int_{\mathbb{R}^N} |u|\, dx &= \int_{\{|u|>\delta\}} |u|\, dx + \int_{\{|u|\le\delta\}} |u|\, dx \\ &\le \int_{\{|u|>\delta\}} |u|\, dx + \frac{1}{f(0)} \int_{\{|u|\le\delta\}} f(u)\, |u|\, dx < +\infty, \end{aligned}$$

where we used the monotonicity of f. By observing that $|F(u)| \le \|f(u)\|_\infty\, |u|$, we get the desired conclusion. □

Remark 2.6. It is natural to expect that, as in the local case, there are no global bounded finite energy solutions to (2.3) if f satisfies $f(0)\neq 0$. A proof of this fact, however, is still missing.

2.3. **Approximating functional.** The main tool to prove Theorem 1.1 will be a natural approximation procedure for the corresponding energy functional

$$\mathcal{E}(u)=\frac{1}{p}\,\|u\|^p_{\mathcal{D}^{s,p}}-\int_{\mathbb{R}^N}F(u)\,dx.$$

We will employ the following uniformly elliptic regularization on the kinetic part, given by

$$u\mapsto\int_{\mathbb{R}^{2N}}J_\varepsilon(\mathfrak{D}^s u)\,d\mu,$$

where for $\varepsilon\geq 0$

$$J_\varepsilon(t)=\frac{|t|^p}{p}+\varepsilon\,\frac{|t|^2}{2}.$$

We now prove some basic inequalities for the approximating integrand J_ε which will ensure regularity and well-posedness of the approximating procedure. The proof of the following result is straightforward.

Lemma 2.7. *Let $\varepsilon>0$ and $1<p<+\infty$, for every $a,b\in\mathbb{R}$ we have*

$$\Big(J'_\varepsilon(a)-J'_\varepsilon(b)\Big)\,(a-b)\geq\varepsilon\,(a-b)^2. \tag{2.5}$$

Lemma 2.8. *For every $a,b\in\mathbb{R}$ and every $g:\mathbb{R}\to\mathbb{R}$ Lipschitz non-decreasing function, we have*

$$J'_\varepsilon(a-b)\,(g(a)-g(b))\geq|G(a)-G(b)|^p, \tag{2.6}$$

where

$$G(t)=\int_0^t\big(g'(\tau)\big)^{\frac{1}{p}}\,d\tau.$$

Proof. We observe that

$$(a-b)\,(g(a)-g(b))\geq 0,$$

thanks to the monotonicity of g. Thus we obtain

$$\begin{aligned}J'_\varepsilon(a-b)\,(g(a)-g(b))&=\Big(|a-b|^{p-2}+\varepsilon\Big)\,(a-b)\,(g(a)-g(b))\\&\geq|a-b|^{p-2}\,(a-b)\,(g(a)-g(b)).\end{aligned}$$

We can now apply [3, Lemma A.2] and get the conclusion. □

Lemma 2.9. *Let $\varepsilon>0$, $1<p<+\infty$ and $g:\mathbb{R}\to\mathbb{R}$ be a Lipschitz non-decreasing function. For every $a,b,c,d\in\mathbb{R}$ and $\lambda>0$, $h\neq 0$ we have*

$$\frac{1}{h}\left(J'_\varepsilon\left(\frac{a-b}{\lambda}\right)-J'_\varepsilon\left(\frac{c-d}{\lambda}\right)\right)\frac{g\left(\dfrac{a-c}{h}\right)-g\left(\dfrac{b-d}{h}\right)}{\lambda}\geq\varepsilon\left|\frac{H\left(\dfrac{a-c}{h}\right)-H\left(\dfrac{b-d}{h}\right)}{\lambda}\right|^2, \tag{2.7}$$

where

$$H(t)=\int_0^t\sqrt{g'(\tau)}\,d\tau.$$

Proof. We first prove (2.7) in the case $h = 1$. We observe that if $a - b = c - d$, then we have $a - c = b - d$ as well and the inequality is trivially true. Let us assume $a - b \neq c - d$, without loss of generality we can assume $a - b > c - d$ (which is equivalent to $a - c > b - d$). We then have

$$
\begin{aligned}
&\left(J'_\varepsilon\left(\frac{a-b}{\lambda}\right) - J'_\varepsilon\left(\frac{c-d}{\lambda}\right)\right)\frac{g(a-c)-g(b-d)}{\lambda} \\
&\qquad = \frac{J'_\varepsilon\left(\frac{a-b}{\lambda}\right) - J'_\varepsilon\left(\frac{c-d}{\lambda}\right)}{\frac{a-b}{\lambda} - \frac{c-d}{\lambda}}\left(\frac{a-c}{\lambda} - \frac{b-d}{\lambda}\right)\frac{g(a-c)-g(b-d)}{\lambda} \\
&\qquad \geq \frac{\varepsilon}{\lambda^2}\,(a-c-(b-d))\,\Big(g(a-c)-g(b-d)\Big),
\end{aligned}
$$

where we used (2.5). We now call $X = a - c$ and $Y = b - d$, then it is sufficient to observe that by Jensen's inequality

$$
(X-Y)\,(g(X)-g(Y)) = (X-Y)^2 \fint_Y^X g'(\tau)\,d\tau \geq \left(\int_Y^X \sqrt{g'(\tau)}\,d\tau\right)^2 = (H(X)-H(Y))^2.
$$

This concludes the proof for $h = 1$.

For $h = -1$, we just need to observe that the left-hand side of (2.7) is

$$
\left(J'_\varepsilon\left(\frac{c-d}{\lambda}\right) - J'_\varepsilon\left(\frac{a-b}{\lambda}\right)\right)\frac{g\,(c-a)-g\,(d-b)}{\lambda},
$$

and thus we are back to the previous case.

Finally, for a general $h \neq 0$ we have

$$
\begin{aligned}
&\frac{1}{h}\left(J'_\varepsilon\left(\frac{a-b}{\lambda}\right) - J'_\varepsilon\left(\frac{c-d}{\lambda}\right)\right)\frac{g\left(\frac{a-c}{h}\right)-g\left(\frac{b-d}{h}\right)}{\lambda} \\
&\qquad = \frac{1}{h^2}\left(J'_\varepsilon\left(\frac{\frac{a-b}{h}}{\lambda/h}\right) - J'_\varepsilon\left(\frac{\frac{c-d}{h}}{\lambda/h}\right)\right)\frac{g\left(\frac{a-c}{h}\right)-g\left(\frac{b-d}{h}\right)}{\lambda/h},
\end{aligned}
$$

and we conclude by using the first part of the proof with

$$
\frac{a}{h},\quad \frac{b}{h},\quad \frac{c}{h},\quad \frac{d}{h}\quad \text{and}\quad \frac{\lambda}{h}.
$$

This gives the desired conclusion. □

3. Existence for the approximating problem

For $\varepsilon \in (0, 1]$ we now construct a family of functions $\{u_\varepsilon\}_\varepsilon$ which approximate the solution u of (1.4). In what follows, we set

$$
\varrho_\varepsilon(x) = \frac{1}{\varepsilon^N}\,\varrho\left(\frac{x}{\varepsilon}\right), \qquad x \in \mathbb{R}^N,
$$

where ϱ is a standard compactly supported positive mollifier. In order to approximate the term $f(u)$ in the desired norms, we need the following technical result.

Lemma 3.1. *Let* $\eta \in C_0^\infty(B_1)$ *be a function such that*

$$0 \le \eta \le 1, \qquad \eta \equiv 1 \text{ on } B_{\frac{1}{2}}, \qquad |\nabla \eta| \le C.$$

For every $\varepsilon \in (0,1]$, *we define* $\eta_\varepsilon(x) = \eta(\varepsilon\, x)$, *for* $x \in \mathbb{R}^N$. *We then introduce the linear smoothing operator*

$$\begin{array}{cccc} T_\varepsilon & : & \mathcal{D}^{-s,p'}(\mathbb{R}^N) & \to & \mathcal{D}^{-s,p'}(\mathbb{R}^N) \\ & & \Lambda & \mapsto & (\Lambda\, \eta_\varepsilon) * \varrho_\varepsilon, \end{array}$$

defined by

$$\langle T_\varepsilon(\Lambda), v\rangle := \langle \Lambda, \eta_\varepsilon\,(v * \varrho_\varepsilon)\rangle, \qquad \text{for every } v \in \mathcal{D}^{s,p}(\mathbb{R}^N).$$

Then $T_\varepsilon(\Lambda) \in C_0^\infty(\mathbb{R}^N)$ *for every* $\Lambda \in \mathcal{D}^{-s,p'}(\mathbb{R}^N)$ *and*

$$\lim_{\varepsilon \searrow 0} \|T_\varepsilon(\Lambda) - \Lambda\|_{\mathcal{D}^{-s,p'}} = 0. \tag{3.1}$$

Proof. The fact that $T_\varepsilon(\Lambda) \in C_0^\infty(\mathbb{R}^N)$ is a standard fact in the theory of distributions, let us prove (3.1).

We observe that $\{T_\varepsilon\}_\varepsilon$ is an equi-bounded family of linear operators, indeed for every $v \in \mathcal{D}^{s,p}(\mathbb{R}^N)$ with unit norm we get

$$\begin{aligned} |\langle T_\varepsilon(\Lambda), v\rangle| &\le \|\Lambda\|_{\mathcal{D}^{-s,p'}}\, \|\eta_\varepsilon\,(v * \varrho_\varepsilon)\|_{\mathcal{D}^{s,p}} \\ &= \varepsilon^{s-\frac{N}{p}}\, \|\Lambda\|_{\mathcal{D}^{-s,p'}}\, \|\eta\,(v_{\frac{1}{\varepsilon}} * \varrho_{\varepsilon^2})\|_{\mathcal{D}^{s,p}} \\ &\le C\, \varepsilon^{s-\frac{N}{p}}\, (\|\eta\|_\infty + \|\nabla \eta\|_\infty)\, \|\Lambda\|_{\mathcal{D}^{-s,p'}}\, \|v_{\frac{1}{\varepsilon}} * \varrho_{\varepsilon^2}\|_{\mathcal{D}^{s,p}} \\ &\le C'\, \varepsilon^{s-\frac{N}{p}}\, (\|\eta\|_\infty + \|\nabla \eta\|_\infty)\, \|\Lambda\|_{\mathcal{D}^{-s,p}}\, \|v_{\frac{1}{\varepsilon}}\|_{\mathcal{D}^{s,p}} \\ &\le C'\, (\|\eta\|_\infty + \|\nabla \eta\|_\infty)\, \|\Lambda\|_{\mathcal{D}^{-s,p}}. \end{aligned}$$

The first equality follows by scaling, the second inequality follows by [6, Lemma A.1], while the third one follows from Young's inequality for convolutions. In particular, from the previous argument it follows that

$$\|T_\varepsilon(\Lambda_1) - T_\varepsilon(\Lambda_2)\|_{\mathcal{D}^{-s,p'}} \le C\, \|\Lambda_1 - \Lambda_2\|_{\mathcal{D}^{-s,p'}}, \qquad \text{for every } \Lambda_1, \Lambda_2 \in \mathcal{D}^{-s,p'}(\mathbb{R}^N),$$

with $C > 0$ independent of $\varepsilon > 0$. By using this and recalling that $C_0^\infty(\mathbb{R}^N)$ is a dense subspace of $\mathcal{D}^{-s,p'}(\mathbb{R}^N)$, in order to prove (3.1) it is thus sufficient to prove that

$$\lim_{\varepsilon \to 0} \|T_\varepsilon\,(\Lambda) - \Lambda\|_{\mathcal{D}^{-s,p'}} = 0, \qquad \text{for every } \Lambda \in C_0^\infty(\Omega).$$

To this aim, let us take $\Lambda \in C_0^\infty(\mathbb{R}^N)$, then we observe that $T_\varepsilon(\Lambda)$ can be written as an integral, i.e.

$$\langle T_\varepsilon(\Lambda), v\rangle = \int_{\mathbb{R}^N} \Lambda\, \eta_\varepsilon\,(v * \varrho_\varepsilon)\, dx = \int_{\mathbb{R}^N} (\Lambda\, \eta_\varepsilon) * \varrho_\varepsilon\, v\, dx,$$

by Lemma 2.4, since the term $\Lambda\, \eta_\varepsilon\,(v * \varrho_\varepsilon)$ is in $L^1(\mathbb{R}^N)$. By using this writing and observing that $\Lambda\, \eta_\varepsilon = \Lambda$ for ε sufficiently small, we get

$$\begin{aligned} \|T_\varepsilon\,(\Lambda) - \Lambda\|_{\mathcal{D}^{-s,p'}} &= \sup_{\|v\|_{\mathcal{D}^{s,p}}=1} \langle T_\varepsilon(\Lambda) - \Lambda, v\rangle = \sup_{\|v\|_{\mathcal{D}^{s,p}}=1} \int_{\mathbb{R}^N} \Big[(\Lambda\, \eta_\varepsilon) * \varrho_\varepsilon - \Lambda\Big]\, v\, dx \\ &= \sup_{\|v\|_{\mathcal{D}^{s,p}}=1} \int_{\mathbb{R}^N} \Big[\Lambda * \varrho_\varepsilon - \Lambda\Big]\, v\, dx \le \sup_{\|v\|_{\mathcal{D}^{s,p}}=1} \|\Lambda * \varrho_\varepsilon - \Lambda\|_{(p^*)'}\, \|v\|_{p^*} \\ &\le \mathcal{S}_{p,s}^{-\frac{1}{p}}\, \|\Lambda * \varrho_\varepsilon - \Lambda\|_{(p^*)'} \to 0, \qquad \text{as } \varepsilon \to 0, \end{aligned}$$

where in the last inequality we used Sobolev inequality. This concludes the proof. □

Remark 3.2 (Approximation of $f(u)$)**.** By recalling that $f(u) \in L^1_{\rm loc}(\mathbb{R}^N) \cap \mathcal{D}^{-s,p'}(\mathbb{R}^N)$, in light of the previous result in what follows we will take, by using the notation above,

$$f_\varepsilon = T_\varepsilon(f(u)) = \Big((f(u))\,\eta_\varepsilon\Big) * \varrho_\varepsilon,$$

(notice that the last equality holds true just by Fubini). Such an approximation has the following properties

$$f_\varepsilon \to f(u) \quad \text{strongly in } \mathcal{D}^{-s,p'}(\mathbb{R}^N),$$

$$\sup_{\varepsilon\in(0,1]} \|f_\varepsilon\|_\infty \le C < +\infty, \qquad \text{and} \qquad f_\varepsilon \to f(u) \ \text{ in } L^q_{\rm loc}(\mathbb{R}^N), \text{ for every } q \ge 1.$$

The first property follows from Lemma 3.1, while the last two properties follows from the fact that $u \in L^\infty(\mathbb{R}^N)$ (thus the same is true for $f(u)$).

Proposition 3.3. *Let $\varepsilon \in (0,1]$ and $1 < p < +\infty$. Then the minimization problem*

$$\mathcal{I} := \inf_{v\in\mathcal{D}^{s,2}(\mathbb{R}^N)\cap\mathcal{D}^{s,p}(\mathbb{R}^N)} \left\{\int_{\mathbb{R}^{2N}} J_\varepsilon(\mathfrak{D}^s v)\,d\mu - \int_{\mathbb{R}^N} f_\varepsilon\, v\,dx\right\}, \tag{3.2}$$

admits a unique solution u_ε. Moreover, we have

$$\int_{\mathbb{R}^{2N}} J'_\varepsilon(\mathfrak{D}^s u_\varepsilon)\,\mathfrak{D}^s\varphi\,d\mu - \int_{\mathbb{R}^N} f_\varepsilon\,\varphi\,dx = 0, \tag{3.3}$$

for every $\varphi \in \mathcal{D}^{s,2}(\mathbb{R}^N)\cap\mathcal{D}^{s,p}(\mathbb{R}^N)$.

Proof. We first observe that by definition of J_ε, we immediately get

$$0 \le \int_{\mathbb{R}^{2N}} J_\varepsilon(\mathfrak{D}^s v)\,d\mu = \frac{\varepsilon}{2}\,\|v\|^2_{\mathcal{D}^{s,2}} + \frac{1}{p}\,\|v\|^p_{\mathcal{D}^{s,p}} < +\infty,$$

for every $v \in \mathcal{D}^{s,2}(\mathbb{R}^N)\cap\mathcal{D}^{s,p}(\mathbb{R}^N)$. Thus the functional is well-defined on this space. Moreover, the functional is weakly coercive on $\mathcal{D}^{s,2}(\mathbb{R}^N)\cap\mathcal{D}^{s,p}(\mathbb{R}^N)$. Indeed, we have

$$\int_{\mathbb{R}^{2N}} J_\varepsilon(\mathfrak{D}^s v)\,d\mu - \int_{\mathbb{R}^N} f_\varepsilon\,v\,dx \ge \frac{\varepsilon}{2}\,\|v\|^2_{\mathcal{D}^{s,2}} + \frac{1}{p}\,\|v\|^p_{\mathcal{D}^{s,p}} - \|f_\varepsilon\|_{\mathcal{D}^{-s,p'}}\,\|v\|_{\mathcal{D}^{s,p}}.$$

A standard use of Young's inequality leads to

$$\int_{\mathbb{R}^{2N}} J_\varepsilon(\mathfrak{D}^s v)\,d\mu - \int_{\mathbb{R}^N} f_\varepsilon\,v\,dx \ge \frac{\varepsilon}{2}\,\|v\|^2_{\mathcal{D}^{s,2}} + c\,\|v\|^p_{\mathcal{D}^{s,p}} - \frac{1}{c}\,\|f_\varepsilon\|^{p'}_{\mathcal{D}^{-s,p'}}, \tag{3.4}$$

for some $c = c(p) > 0$. The previous estimate implies that the infimum in (3.2) is finite and that any minimizing sequence $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{D}^{s,2}(\mathbb{R}^N)\cap\mathcal{D}^{s,p}(\mathbb{R}^N)$ is bounded. Thus we can infer weak convergence (up to a subsequence) to a function $u_\varepsilon \in \mathcal{D}^{s,2}(\mathbb{R}^N)\cap\mathcal{D}^{s,p}(\mathbb{R}^N)$. We also observe that for every ball B_R, we have

$$\sup_{n\in\mathbb{N}} \|v_n\|_{W^{s,p}(B_R)} \le C_R.$$

By exploiting the compactness of the embedding $W^{s,p}(B_R) \hookrightarrow L^p(B_R)$, we have in particular (up to a subsequence)

$$v_n \to u_\varepsilon, \qquad \text{for a. e. } x \in B_R.$$

In conclusion, by Fatou's Lemma we get

$$\begin{aligned}\int_{B_R\times B_R} J_\varepsilon(\mathfrak{D}^s u_\varepsilon)\,d\mu - \int_{\mathbb{R}^N} f_\varepsilon\, u_\varepsilon\,dx &\le \liminf_{n\to\infty}\int_{B_R\times B_R} J_\varepsilon(\mathfrak{D}^s v_n)\,d\mu - \int_{\mathbb{R}^N} f_\varepsilon\, v_n\,dx\\ &\le \liminf_{n\to\infty}\int_{\mathbb{R}^{2N}} J_\varepsilon(\mathfrak{D}^s v_n)\,d\mu - \int_{\mathbb{R}^N} f_\varepsilon\, v_n\,dx = \mathcal{I}.\end{aligned}$$

By taking the limit as R goes to $+\infty$ and using the Monotone Convergence Theorem, we finally get that u_ε is the desired minimizer. Uniqueness of the minimizer is now a plain consequence of the strict convexity of the functional. Finally, equation (3.3) is precisely the optimality condition. □

Proposition 3.4 (Convergence of minimizers)**.** *We have*

$$\lim_{\varepsilon\to 0}\|u_\varepsilon - u\|_{\mathcal{D}^{s,p}} = 0.$$

Proof. We first observe that $\{u_\varepsilon\}_\varepsilon$ is bounded in $\mathcal{D}^{s,p}(\mathbb{R}^N)$. Indeed, by testing the minimality of u_ε against $\varphi = 0$ and using (3.4), we obtain

$$\frac{\varepsilon}{2}\,\|u_\varepsilon\|^2_{\mathcal{D}^{s,2}} + c\,\|u_\varepsilon\|^p_{\mathcal{D}^{s,p}} - \frac{1}{c}\,\|f_\varepsilon\|^{p'}_{\mathcal{D}^{-s,p'}} \le 0.$$

By using that f_ε is uniformly bounded in $\mathcal{D}^{-s,p'}(\mathbb{R}^N)$ by construction, we get the claimed bound.

We now need to identify the weak limit of $\{u_\varepsilon\}_\varepsilon$. Let us call $v \in \mathcal{D}^{s,p}(\mathbb{R}^N)$ such weak limit. Since $u_\varepsilon \rightharpoonup v$ in $\mathcal{D}^{s,p}(\mathbb{R}^N)$, $J_\varepsilon(u_\varepsilon) \ge \|u_\varepsilon\|_{\mathcal{D}^{s,p}}$ and $f_\varepsilon \to f(u)$ in $D^{-s,p'}(\mathbb{R}^N)$, it holds

$$\begin{aligned}\frac{1}{p}\,\|v\|^p_{\mathcal{D}^{s,p}} - \langle f(u), v\rangle &\le \liminf_{\varepsilon\to 0}\frac{1}{p}\,\|u_\varepsilon\|^p_{\mathcal{D}^{s,p}} - \langle f(u), u_\varepsilon\rangle\\ &\le \liminf_{\varepsilon\to 0}\int_{\mathbb{R}^{2N}} J_\varepsilon(\mathfrak{D}^s u_\varepsilon)\,d\mu - \int_{\mathbb{R}^N} f_\varepsilon\, u_\varepsilon\,dx.\end{aligned}$$

Let us fix $\varphi \in C_0^\infty(\mathbb{R}^N)$. By the minimality of u_ε and the fact that $f_\varepsilon \to f(u)$ in $L^1_{\rm loc}(\mathbb{R}^N)$ we obtain

$$\begin{aligned}\frac{1}{p}\,\|v\|^p_{\mathcal{D}^{s,p}} - \langle f(u), v\rangle &\le \liminf_{\varepsilon\to 0}\int_{\mathbb{R}^{2N}} J_\varepsilon(\mathfrak{D}^s \varphi)\,d\mu - \int_{\mathbb{R}^N} f_\varepsilon\, \varphi\,dx\\ &\le \frac{1}{p}\,\|\varphi\|^p_{\mathcal{D}^{s,p}} - \langle f(u), \varphi\rangle.\end{aligned}$$

By density of $C_0^\infty(\mathbb{R}^N)$ in $\mathcal{D}^{s,p}(\mathbb{R}^N)$, the previous estimate implies that v minimizes the strictly convex functional

$$\varphi \mapsto \frac{1}{p}\,\|\varphi\|^p_{\mathcal{D}^{s,p}} - \langle f(u), \varphi\rangle,$$

on $\mathcal{D}^{s,p}(\mathbb{R}^N)$. Since by assumption u is a critical point of the latter, the strict convexity forces $v = u$. By weak convergence, we also have

$$\|u\|^p_{\mathcal{D}^{s,p}} \le \liminf_{\varepsilon\to 0}\|u_\varepsilon\|^p_{\mathcal{D}^{s,p}}. \tag{3.5}$$

In order to improve the convergence, we observe that

$$\langle f(u), u\rangle = \|u\|^p_{\mathcal{D}^{s,p}},$$

and

$$\langle f_\varepsilon, u_\varepsilon\rangle = \int_{\mathbb{R}^{2N}} J'_\varepsilon(\mathfrak{D}^s u_\varepsilon)\,\mathfrak{D}^s u_\varepsilon\,d\mu \ge \|u_\varepsilon\|^p_{\mathcal{D}^{s,p}}.$$

By the first part of the proof and the strong convergence of f_ε to $f(u)$ in $\mathcal{D}^{-s,p'}(\mathbb{R}^N)$, we get

$$\|u\|^p_{\mathcal{D}^{s,p}} = \langle f(u), u\rangle = \lim_{\varepsilon\to 0}\langle f_\varepsilon, u_\varepsilon\rangle \geq \limsup_{\varepsilon\to 0} \|u_\varepsilon\|^p_{\mathcal{D}^{s,p}},$$

which, together with (3.5), ensures the conclusion by uniform convexity of the norm. □

4. Regularity for the approximating problem

In this section we will prove some regularity estimates for the solution u_ε of (3.2). For ease of notation, in the proofs we will drop the subscript ε and simply write u and f, in place of u_ε and f_ε.

Proposition 4.1 (Boundedness)**.** *Let $s \in (0,1)$ and $1 < p < +\infty$, then we have $u_\varepsilon \in L^\infty(\mathbb{R}^N)$. Moreover, it holds*

$$\|u_\varepsilon\|_\infty \leq C\,\|u_\varepsilon\|_{p^*}^{\frac{p^*-p}{p^*-1}},$$

for some $C = C(N,p,s,\|f(u)\|_\infty) > 0$.

Proof. For every $M > 0$, we set

$$g_M(t) = \begin{cases} \min\{t, M\}, & \text{if } t \geq 0,\\ 0, & \text{if } t < 0.\end{cases}$$

We then define

$$u_M = g_M \circ u \in \mathcal{D}^{s,p}(\mathbb{R}^N) \cap \mathcal{D}^{s,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

We observe that for every $\beta \geq 1$, the function u_M^β is an admissible test function in (3.3). Thus we obtain

$$\int_{\mathbb{R}^{2N}} J'_\varepsilon(\mathfrak{D}^s u)\,\mathfrak{D}^s u_M^\beta\, d\mu \leq \int_{\mathbb{R}^N} |f|\,u_M^\beta\, dx.$$

We define

$$G_{\beta,M}(t) = \beta^{\frac{1}{p}} \int_0^t g_M(\tau)^{\frac{\beta-1}{p}}\,(g'_M(\tau))^{\frac{1}{p}}\, d\tau = \beta^{\frac{1}{p}}\,\frac{p}{\beta+p-1}\, g_M(t)^{\frac{\beta+p-1}{p}},$$

then by (2.6) we get

$$\beta\left(\frac{p}{\beta+p-1}\right)^p \int_{\mathbb{R}^{2N}} \left|\mathfrak{D}^s u_M^{\frac{\beta+p-1}{p}}\right|^p d\mu \leq \int_{\mathbb{R}^N} |f|\,u_M^\beta\, dx.$$

By Sobolev inequality, this in turn implies

$$\left(\int_{\mathbb{R}^N} u_M^{p^*\frac{\beta+p-1}{p}}\, dx\right)^{\frac{p}{p^*}} \leq \frac{1}{\mathcal{S}_{p,s}\,\beta}\left(\frac{\beta+p-1}{p}\right)^p \int_{\mathbb{R}^N} |f|\,u_M^\beta\, dx < +\infty.$$

By using the uniform bound on $|f|$ and $\beta > 1$ we obtain

$$\left(\int_{\mathbb{R}^N} u_M^{p^*\frac{\beta+p-1}{p}}\, dx\right)^{\frac{p}{p^*}} \leq C^{\frac{p}{p^*}}\left(p^*\,\frac{\beta+p-1}{p}\right)^p \int_{\mathbb{R}^N} u_M^\beta\, dx,$$

for some $C = C(N,p,s,\|f(u)\|_\infty) \geq 1$. By setting

$$\beta_{n+1} = \frac{p^*}{p}\,(\beta_n + p - 1), \qquad \beta_0 = p^*, \qquad \sigma_n = \frac{\beta_n}{\beta_n+p-1} < 1,$$

we obtain $\beta_n \nearrow +\infty$ and

$$\|u_M\|_{\beta_{n+1}} \leq C^{\frac{1}{\beta_{n+1}}}\,\beta_{n+1}^{\frac{p^*}{\beta_{n+1}}}\,\|u_M\|_{\beta_n}^{\sigma_n}.$$

We can iterate this inequality and by using that $\sigma_n < 1$, we get for any $n \geq 1$

$$\|u_M\|_{\beta_{n+1}} \leq C^{\sum\limits_{i=1}^{n+1} \frac{1}{\beta_i}} \left(\prod_{i=1}^{n+1} \beta_i^{\frac{1}{\beta_i}} \right)^{p^*} \|u_M\|_{p^*}^{\prod_{i=0}^n \sigma_i}.$$

Now β_n can be determined explicitly: by setting $\gamma = p^*/p > 1$, this is

$$\beta_n = \gamma^n\, \beta_0 + (p-1)\, \frac{\gamma^{n+1} - \gamma}{\gamma - 1},$$

so it holds

$$\lim_{n\to\infty} \frac{\beta_n}{\gamma^n} = \beta_0 + (p-1)\, \frac{\gamma}{\gamma-1} = p^*\, \frac{p^*-1}{p^*-p}.$$

Therefore

$$\sum_{i=1}^{+\infty} \frac{1}{\beta_i} < +\infty \qquad \text{and} \qquad \prod_{i=1}^{+\infty} \beta_i^{\frac{1}{\beta_i}} < +\infty,$$

and also

$$\lim_{n\to\infty} \prod_{i=0}^{n} \sigma_i = \lim_{n\to\infty} \prod_{i=0}^{n} \gamma\, \frac{\beta_i}{\beta_{i+1}} = \lim_{n\to\infty} \gamma^{n+1}\, \frac{\beta_0}{\beta_{n+1}} = \frac{p^*-p}{p^*-1}.$$

This provides the estimate

$$\|u_M\|_\infty \leq C\, \|u_M\|_{p^*}^{\frac{p^*-p}{p^*-1}},$$

for some $C = C(N, p, s, \|f\|_\infty) \geq 1$. We now let M go to $+\infty$, which gives $u_+ \in L^\infty(\mathbb{R}^N)$. By repeating the argument above for[1] u_-, we can obtain $u \in L^\infty(\mathbb{R}^N)$. □

Proposition 4.2 (Higher differentiability). *Let $s \in (0,1)$ and $1 < p < +\infty$, then we have $\nabla u_\varepsilon \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ and*

$$\|(u_\varepsilon)_{x_k}\|_{\mathcal{D}^{s,2}} \leq \frac{1}{\varepsilon}\, \|(f_\varepsilon)_{x_k}\|_{\mathcal{D}^{-s,2}}, \qquad k = 1, \dots, N.$$

Proof. We need to differentiate the equation in a discrete sense. Let us fix $k \in \{1, \dots, N\}$ and take $h \neq 0$. We then consider equation (3.3) and plug-in the test function $\varphi_{-h\mathbf{e}_k}$, with $\varphi \in \mathcal{D}^{s,2}(\mathbb{R}^N) \cap \mathcal{D}^{s,p}(\mathbb{R}^N)$. By changing variables, we thus get

$$\int_{\mathbb{R}^{2N}} J'_\varepsilon(\mathfrak{D}^s u_{h\mathbf{e}_k})\, \mathfrak{D}^s \varphi\, d\mu - \int_{\mathbb{R}^N} f_{h\mathbf{e}_k}\, \varphi\, dx = 0.$$

By subtracting (3.3) from this, we get

$$\int_{\mathbb{R}^{2N}} \delta_{h\mathbf{e}_k} \Big(J'_\varepsilon(\mathfrak{D}^s u) \Big)\, \mathfrak{D}^s \varphi\, d\mu - \int_{\mathbb{R}^N} \delta_{h\mathbf{e}_k} f\, \varphi\, dx = 0, \tag{4.1}$$

which holds true for every $\varphi \in \mathcal{D}^{s,2}(\mathbb{R}^N) \cap \mathcal{D}^{s,p}(\mathbb{R}^N)$. We now test the previous identity with $\varphi = \delta_{h\mathbf{e}_k} u$, which is admissible. We observe at first that

$$\mathfrak{D}^s(\delta_{h\mathbf{e}_k} u) = \frac{u(x+h\mathbf{e}_k) - u(y+h\mathbf{e}_k) - (u(x) - u(y))}{|x-y|^s} = \delta_{h\mathbf{e}_k}\, \mathfrak{D}^s u,$$

thus we have

$$\int_{\mathbb{R}^{2N}} \delta_{h\mathbf{e}_k} \Big(J'_\varepsilon(\mathfrak{D}^s u) \Big)\, \delta_{h\mathbf{e}_k}\, \mathfrak{D}^s u\, d\mu - \int_{\mathbb{R}^N} \delta_{h\mathbf{e}_k} f\, \delta_{h\mathbf{e}_k} u\, dx = 0.$$

---

[1]It is sufficient to test (3.3) with $-g_M(-u)^\beta$. Observe that $t \mapsto -g_M(-t)$ is non-decreasing.

We now observe that

$$\delta_{h\mathbf{e}_k}\Big(J_\varepsilon'(\mathfrak{D}^s u)\Big)\,\delta_{h\mathbf{e}_k}\,\mathfrak{D}^s u=\Big(J_\varepsilon'(\mathfrak{D}^s u_{h\mathbf{e}_k})-J_\varepsilon'(\mathfrak{D}^s u)\Big)\,(\mathfrak{D}^s u_{h\mathbf{e}_k}-\mathfrak{D}^s u),$$

thus we can use Lemma 2.7 with the choices

$$a=\mathfrak{D}^s u_{h\mathbf{e}_k}\qquad\text{and}\qquad b=\mathfrak{D}^s u.$$

This yields for every $h\neq 0$

$$\begin{aligned}\varepsilon\int_{\mathbb{R}^{2N}}|\mathfrak{D}^s(\delta_{h\mathbf{e}_k}u)|^2\,d\mu&\le\int_{\mathbb{R}^N}|\delta_{h\mathbf{e}_k}f|\,|\delta_{h\mathbf{e}_k}u|\,dx\\&\le\|\delta_{h\mathbf{e}_k}f\|_{\mathcal{D}^{-s,2}}\,\|\delta_{h\mathbf{e}_k}u\|_{\mathcal{D}^{s,2}}.\end{aligned}$$

We can now absorb the last term, divide by $|h|^2$ and use Lemma 2.2 to estimate the term containing $\delta_{h\mathbf{e}_k}f$. This gives

$$\left\|\frac{\delta_{h\mathbf{e}_k}u}{h}\right\|_{\mathcal{D}^{s,2}}\le\frac{1}{\varepsilon}\,\|f_{x_k}\|_{\mathcal{D}^{-s,2}}.\tag{4.2}$$

By using Sobolev inequality, we get in particular that

$$\sup_{|h|>0}\left\|\frac{\delta_{h\mathbf{e}_k}u}{h}\right\|_{2^*}<+\infty.$$

By the finite differences characterization of integer order Sobolev spaces, we get $\nabla u\in L^{2^*}(\mathbb{R}^N)$. Moreover, there exists $\{h_n\}_{n\in\mathbb{N}}\subset\mathbb{R}\setminus\{0\}$ converging to 0 such that

$$\frac{\delta_{h_n\mathbf{e}_k}u}{h_n}\to u_{x_k}\qquad\text{strongly in }L^{2^*}(\mathbb{R}^N).$$

We can thus pass to the limit in (4.2) by using Fatou's Lemma and get the desired result. □

Proposition 4.3 (Lipschitz regularity). *Let $s\in(0,1)$ and $1<p<+\infty$, then we have*

$$\|\nabla u_\varepsilon\|_\infty\le C\,\|\nabla u_\varepsilon\|_{2^*}^{\frac{2^*-2}{2^*-1}},$$

for some $C=C(\varepsilon,N,p,s,\|\nabla f_\varepsilon\|_\infty)>0$ which blows-up as $\varepsilon\searrow 0$.

Proof. For every $M>0$, we still define

$$g_M(t)=\begin{cases}\min\{t,M\}, & \text{if } t\ge 0,\\ 0, & \text{if } t<0.\end{cases}$$

Let us fix $k\in\{1,\dots,N\}$ and take $h\neq 0$. We insert in (4.1) the test function

$$\varphi=g_M\left(\frac{\delta_{h\mathbf{e}_k}u}{h}\right)^\beta,\qquad\beta\ge 1,$$

which gives

$$\int_{\mathbb{R}^{2N}}\frac{\delta_{h\mathbf{e}_k}\Big(J_\varepsilon'(\mathfrak{D}^s u)\Big)}{h}\,\mathfrak{D}^s g_M\left(\frac{\delta_{h\mathbf{e}_k}u}{h}\right)^\beta d\mu=\int_{\mathbb{R}^N}\frac{\delta_{h\mathbf{e}_k}f}{h}\,g_M\left(\frac{\delta_{h\mathbf{e}_k}u}{h}\right)^\beta dx.$$

We now observe that

$$\frac{\delta_{h\mathbf{e}_k}\Big(J'_\varepsilon(\mathfrak{D}^s u)\Big)}{h}\,\mathfrak{D}^s g_M\left(\frac{\delta_{h\mathbf{e}_k}u}{h}\right)^\beta=\frac{1}{h}\,\left(J'_\varepsilon\left(\frac{u_{h\mathbf{e}_k}(x)-u_{h\mathbf{e}_k}(y)}{|x-y|^s}\right)-J'_\varepsilon\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\right)$$
$$\times\,\frac{g_M\left(\dfrac{u_{h\mathbf{e}_k}(x)-u(x)}{h}\right)^\beta-g_M\left(\dfrac{u_{h\mathbf{e}_k}(y)-u(y)}{h}\right)^\beta}{|x-y|^s},$$

thus we can use Lemma 2.9 with the choices

$$a=u_{h\mathbf{e}_k}(x),\quad c=u(x),\quad b=u_{h\mathbf{e}_k}(y),\quad d=u(y)\qquad\text{and}\qquad\lambda=|x-y|^s.$$

We obtain

$$\varepsilon\int_{\mathbb{R}^{2N}}\left|\mathfrak{D}^s H_{\beta,M}\left(\frac{\delta_{h\mathbf{e}_k}u}{h}\right)\right|^2\,d\mu\le\int_{\mathbb{R}^N}|\delta_{h\mathbf{e}_k}f|\,\left|g_M\left(\frac{\delta_{h\mathbf{e}_k}u}{h}\right)^\beta\right|\,dx,$$

where

$$H_{\beta,M}(t)=\sqrt{\beta}\int_0^t g_M(\tau)^{\frac{\beta-1}{2}}\,(g'_M(\tau))^{\frac12}\,d\tau=\frac{2\sqrt{\beta}}{\beta+1}\,g_M(t)^{\frac{\beta+1}{2}}.$$

By Sobolev inequality in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, we obtain

$$\mathcal{S}_{2,s}\,\varepsilon\left(\int_{\mathbb{R}^N}\left|g_M\left(\frac{\delta_{h\mathbf{e}_k}u}{h}\right)^{\frac{\beta+1}{2}}\right|^{2^*}\,dx\right)^{\frac{2}{2^*}}\le\frac{(\beta+1)^2}{4\,\beta}\int_{\mathbb{R}^N}\left|\frac{\delta_{h\mathbf{e}_k}f}{h}\right|\,\left|g_M\left(\frac{\delta_{h\mathbf{e}_k}u}{h}\right)^\beta\right|\,dx.$$

By using that $\nabla f\in L^\infty(\mathbb{R}^N)$, we can proceed with a Moser's iteration as for the L^∞ estimate. □

Proposition 4.4. *For any $p>1$ and $\varepsilon>0$ it holds*

$$J'_\varepsilon(\mathfrak{D}^s u_\varepsilon)\in L^2_{\rm loc}(\mathbb{R}^{2N},d\mu)\tag{4.3}$$

Moreover, if $p\ge 2$ $J'_\varepsilon(\mathfrak{D}^s u_\varepsilon)\in L^2(\mathbb{R}^{2N},d\mu)$ while if $1<p<2$ it holds

$$|J'_\varepsilon(\mathfrak{D}^s u_\varepsilon)(x,y)|\le\frac{C(u_\varepsilon)}{|x-y|^{s\,(p-1)}},\qquad\textit{if } |x-y|\ge 1.\tag{4.4}$$

Proof. By Proposition 4.3 we have

$$|\mathfrak{D}^s u_\varepsilon(x,y)|\le\|\nabla u_\varepsilon\|_\infty|x-y|^{1-s}\tag{4.5}$$

which, being $|J'(t)|\le\varepsilon|t|+|t|^{p-1}$, yields

$$|J'_\varepsilon(\mathfrak{D}^s u_\varepsilon)(x,y)|\le\varepsilon\,\|\nabla u_\varepsilon\|_\infty|x-y|^{1-s}+\|\nabla u_\varepsilon\|_\infty^{p-1}|x-y|^{(p-1)(1-s)}$$

proving (4.3) by a direct computation. Moreover, Propositions 4.1 gives

$$|\mathfrak{D}^s u_\varepsilon(x,y)|\le\frac{|u_\varepsilon(x)|+|u_\varepsilon(y)|}{|x-y|^s}\le\frac{2\,\|u_\varepsilon\|_\infty}{|x-y|^s}.\tag{4.6}$$

For $p\in(1,2)$ and $|x-y|\ge 1$ it holds $|x-y|^{-s}\le|x-y|^{-s(p-1)}$, hence (4.6) proves (4.4) through

$$|J'_\varepsilon(\mathfrak{D}^s u_\varepsilon)(x,y)|\le\varepsilon\,\frac{2\,\|u_\varepsilon\|_\infty}{|x-y|^s}+\frac{2\,\|u_\varepsilon\|_\infty^{p-1}}{|x-y|^{s(p-1)}}\le 2\,\frac{\|u_\varepsilon\|_\infty+\|u_\varepsilon\|_\infty^{p-1}}{|x-y|^{s(p-1)}}.$$

Finally, using (4.5) for $|x-y|\le 1$ and (4.6) for $|x-y|>1$ we obtain $\mathfrak{D}^s u_\varepsilon\in L^\infty(\mathbb{R}^{2N})$. Therefore, for $p\ge 2$, it holds

$$|J'_\varepsilon(\mathfrak{D}^s u_\varepsilon)|\le\varepsilon\,|\mathfrak{D}^s u_\varepsilon|+\|\mathfrak{D}^s u_\varepsilon\|_\infty^{p-2}\,|\mathfrak{D}^s u_\varepsilon|,$$

which implies $J'_\varepsilon(\mathfrak{D}^s u_\varepsilon)\in L^2(\mathbb{R}^{2N},d\mu)$ by $\mathfrak{D}^s u_\varepsilon\in L^2(\mathbb{R}^{2N},d\mu)$. □

Corollary 4.5. *Let $\varepsilon > 0$, $s \in (0,1)$, $1 < p < +\infty$ and $\Omega \subset \mathbb{R}^N$ be open and bounded. For any $\mathcal{K} \in L^2_\Omega(d\mu)$ it holds*

$$J'_\varepsilon(\mathfrak{D}^s u_\varepsilon)\,\mathcal{K} \in L^1(\mathbb{R}^{2N}, d\mu). \tag{4.7}$$

Moreover, if $p \geq 2$ equation (3.3) holds for every $\varphi \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ and if $1 < p < 2$ it holds for any $\varphi \in \mathcal{D}^{s,2}_0(\Omega)$.

Proof. For $p \geq 2$, (4.7) directly follows from $J'_\varepsilon(\mathfrak{D}^s u_\varepsilon) \in L^2(\mathbb{R}^{2N}, d\mu)$ and $\mathcal{K} \in L^2(\mathbb{R}^{2N}, d\mu)$. For $1 < p < 2$ we use (4.3) and (4.4) in conjunction with Lemma 2.3 to get (4.7) again. The last statement for $p \geq 2$ follows immediately from $J'_\varepsilon(\mathfrak{D}^s u_\varepsilon) \in L^2(\mathbb{R}^{2N}, d\mu)$ and the density of $\mathcal{D}^{s,p}(\mathbb{R}^N) \cap \mathcal{D}^{s,2}(\mathbb{R}^N)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, while for $p \in (1,2)$, we simply observe that $\mathcal{D}^{s,2}_0(\Omega) \subseteq \mathcal{D}^{s,p}(\mathbb{R}^N) \cap \mathcal{D}^{s,2}(\mathbb{R}^N)$ for any open bounded set $\Omega \subset \mathbb{R}^N$.

□

5. Proof of the main result

In this section we proof Theorem 1.1, differentiating the energy functional under suitable compactly supported perturbations of the domain. In order to do so, we will need to use a particular version of the rule of derivation under the integral sign, whose statement is postponed at the end of the proof (see Theorem 5.2 below).

For ease of readability, we divide the proof into various intermediate steps.

Step 1: construction of the perturbation. Let $R > 1$ and $\eta \in C^\infty_0(\mathbb{R}^N)$ be a positive cut-off function, supported in B_R. For $|t| < \delta < 1$ (δ depending on η), the map defined by

$$x \mapsto x + t\,\eta(x)\,x =: x' = \Phi_t(x)$$

is a smooth diffeomorphism of $\mathbb{R}^N$ which is uniformly bilipschitz for $|t| < \delta$, i.e.

$$\sup_{|t|<\delta} \left(\|D\Phi_t\|_\infty + \|D\Phi_t^{-1}\|_\infty \right) < +\infty.$$

Since $\partial_t \Phi_t(x) = \eta(x)\,x$ and $\Phi_0(x) = x = x'$, it holds

$$\partial_t \Phi_t^{-1}(\Phi_t(x)) + D\Phi_t^{-1}(\Phi_t(x))\,\partial_t \Phi_t(x) = 0,$$

so that

$$\partial_t \Phi_t^{-1}(x') = -D\Phi_t^{-1}(x')\,\eta(\Phi_t^{-1}(x'))\,\Phi_t^{-1}(x'), \qquad \partial_t \Phi_t^{-1}(x')\Big|_{t=0} = -\eta(x')\,x'. \tag{5.1}$$

Moreover, for any fixed x', by the Jacobi formula we get

$$\partial_t \mathrm{det} D\Phi_t^{-1}(x') = \mathrm{tr}\left(D\partial_t \Phi_t^{-1}(x') \right) \qquad \text{and} \qquad \partial_t \mathrm{det} D\Phi_t^{-1}(x')\Big|_{t=0} = -\mathrm{div}(\eta(x')\,x'). \tag{5.2}$$

We set

$$u_{\varepsilon,t}(x) = u_\varepsilon \circ \Phi_t(x) = u_\varepsilon(x + t\,\eta(x)\,x),$$

and observe that changing variable $x' = \Phi_t(x)$ it holds

$$\int_{\mathbb{R}^{2N}} J_\varepsilon(\mathfrak{D}^s u_{\varepsilon,t})\,d\mu = \int_{\mathbb{R}^{2N}} J_\varepsilon\left(\frac{u_\varepsilon(x') - u_\varepsilon(y')}{|\Phi_t^{-1}(x') - \Phi_t^{-1}(y')|^s} \right) \frac{\det D\Phi_t^{-1}(x')\,\det D\Phi_t^{-1}(y')}{|\Phi_t^{-1}(x') - \Phi_t^{-1}(y')|^N}\,dx'\,dy'$$

which can be written as

$$\int_{\mathbb{R}^{2N}} J_\varepsilon(\mathfrak{D}^s u_{\varepsilon,t})\,d\mu = \int_{\mathbb{R}^{2N}} J_\varepsilon\left(\mathcal{K}^s_t\,\mathfrak{D}^s u_\varepsilon \right)\,\mathcal{H}_t \mathcal{K}^N_t\,d\mu \tag{5.3}$$

where we set for $x \neq y$

$$\mathcal{K}_t(x,y) = \frac{|x-y|}{|\Phi_t^{-1}(x) - \Phi_t^{-1}(y)|}, \qquad \mathcal{H}_t(x,y) = \det D\Phi_t^{-1}(x)\, \det D\Phi_t^{-1}(y).$$

Observe that, for $|t| < \delta$,

$$\sup_{|t|<\delta} \|\mathcal{K}_t\|_\infty + \sup_{|t|<\delta} \|\mathcal{H}_t\|_\infty < +\infty \tag{5.4}$$

and for any $(x,y) \in \mathbb{R}^{2N}$ the maps $t \mapsto \mathcal{K}_t(x,y)$ and $t \mapsto \mathcal{H}_t(x,y)$ are smooth. For future purposes we compute

$$\partial_t \mathcal{K}_t^\alpha = -\alpha\, \mathcal{K}_t^\alpha \frac{\Phi_t^{-1}(x) - \Phi_t^{-1}(y)}{|\Phi_t^{-1}(x) - \Phi_t^{-1}(y)|} \frac{\partial_t \Phi_t^{-1}(x) - \partial_t \Phi_t^{-1}(y)}{|\Phi_t^{-1}(x) - \Phi_t^{-1}(y)|}, \tag{5.5}$$

$$\partial_t \mathcal{H}_t = \mathrm{tr} D\partial_t \Phi_t^{-1}(x)\, \det D\Phi_t^{-1}(y) + \det D\Phi_t^{-1}(x)\, \mathrm{tr} D\partial_t \Phi_t^{-1}(y).$$

Hence, according to (5.1) and the bilipschitz character of Φ_t,

$$\frac{|\partial_t \Phi_t^{-1}(x) - \partial_t \Phi_t^{-1}(y)|}{|\Phi_t^{-1}(x) - \Phi_t^{-1}(y)|} \leq \|D\Phi_t\|_\infty \mathrm{Lip}\left(D\Phi_t^{-1}\eta(\Phi_t^{-1})\Phi_t^{-1}\right) \leq C(\|\eta\|_{C^2}) < +\infty.$$

By (5.2) and (5.4), we infer that $t \mapsto \partial_t \mathcal{K}_t$ and $t \mapsto \partial_t \mathcal{H}_t$ are continuous for any $(x,y) \in \mathbb{R}^{2N}$ and

$$\sup_{|t|<\delta} \|\partial_t \mathcal{K}_t\|_\infty + \|\partial_t \mathcal{H}_t\|_\infty < +\infty. \tag{5.6}$$

Step 2: differentiating under the integral sign. According to (5.3), our aim is to prove the following chain of equalities for $t = 0$

$$\begin{aligned} \int_{\mathbb{R}^{2N}} \frac{d}{dt} J_\varepsilon(\mathfrak{D}^s u_{\varepsilon,t})\, d\mu &= \frac{d}{dt} \int_{\mathbb{R}^{2N}} J_\varepsilon(\mathfrak{D}^s u_{\varepsilon,t})\, d\mu \\ &= \frac{d}{dt} \int_{\mathbb{R}^{2N}} J_\varepsilon(\mathcal{K}_t^s \mathfrak{D}^s u_\varepsilon) \mathcal{H}_t \mathcal{K}_t^N\, d\mu = \int_{\mathbb{R}^{2N}} \frac{d}{dt} \left(J_\varepsilon(\mathcal{K}_t^s \mathfrak{D}^s u_\varepsilon) \mathcal{H}_t \mathcal{K}_t^N \right) d\mu. \end{aligned} \tag{5.7}$$

In order to do so, notice that u_ε is Lipschitz, so that the integrands above are all well defined. In view of an application of Theorem 5.2 below, we claim that the maps

$$\begin{aligned} & t \mapsto \int_{\mathbb{R}^{2N}} J_\varepsilon(\mathfrak{D}^s u_{\varepsilon,t})\, d\mu = \int_{\mathbb{R}^{2N}} J_\varepsilon(\mathcal{K}_t^s \mathfrak{D}^s u_\varepsilon) \mathcal{H}_t \mathcal{K}_t^N\, d\mu, \\ & t \mapsto \int_{\mathbb{R}^{2N}} \frac{d}{dt} J_\varepsilon(\mathfrak{D}^s u_{\varepsilon,t})\, d\mu, \\ & t \mapsto \int_{\mathbb{R}^{2N}} \frac{d}{dt} \left(J_\varepsilon(\mathcal{K}_t^s \mathfrak{D}^s u_\varepsilon) \mathcal{H}_t \mathcal{K}_t^N \right) d\mu, \end{aligned}$$

are well defined and continuous in $|t| < \delta$. For the first map, changing variable $x' = \Phi_t(x)$ we get

$$\int_{\mathbb{R}^{2N}} J_\varepsilon(\mathfrak{D}^s u_{\varepsilon,t})\, d\mu = \int_{\mathbb{R}^{2N}} J_\varepsilon(\mathcal{K}_t^s\, \mathfrak{D}^s u_\varepsilon)\, \mathcal{H}_t\, \mathcal{K}_t^N\, d\mu,$$

and by the smoothness of $\mathcal{K}_t$ and $\mathcal{H}_t$ and the bound (5.4), we immediately infer its continuity through dominated convergence. For the second map, using the same change of variable as before we have

$$
\begin{aligned}
\int_{\mathbb{R}^{2N}} \frac{d}{dt} J_\varepsilon(\mathfrak{D}^s u_{\varepsilon,t})\, d\mu &= \int_{\mathbb{R}^{2N}} J_\varepsilon'(\mathfrak{D}^s u_{\varepsilon,t})\, \mathfrak{D}^s(\eta\, x\, \nabla u_\varepsilon \circ \Phi_t) d\mu \\
&= \int_{\mathbb{R}^{2N}} J'\left(\mathcal{K}_t^s\, \mathfrak{D}^s u_\varepsilon\right) \mathfrak{D}^s \left(\eta \circ \Phi_t^{-1}\, \Phi_t^{-1}\, \nabla u_\varepsilon\right)\, \mathcal{K}_t^{s+N}\, \mathcal{H}_t\, d\mu,
\end{aligned} \tag{5.8}
$$

and the integrand is pointwise continuous in t, therefore it suffices to dominate it uniformly in $|t| < \delta$. Notice that for any $\lambda \in \mathbb{R}$ it holds

$$
|J'(\lambda t)| = |\lambda t| + |\lambda t|^{p-1} \le (|\lambda| + |\lambda|^{p-1})|J'(t)|,
$$

therefore

$$
|J_\varepsilon'(\mathcal{K}_t^s \mathfrak{D}^s u_\varepsilon)| \le (\|\mathcal{K}_t^s\|_\infty + \|\mathcal{K}_t^{s(p-1)}\|_\infty)|J_\varepsilon'(\mathfrak{D}^s u_\varepsilon)|. \tag{5.9}
$$

and using (5.4) it holds

$$
|J'\left(\mathcal{K}_t^s\, \mathfrak{D}^s u_\varepsilon\right) \mathfrak{D}^s \left(\eta \circ \Phi_t^{-1}\, \Phi_t^{-1}\, \nabla u_\varepsilon\right)\, \mathcal{K}_t^{s+N}\, \mathcal{H}_t| \le C|J'(\mathfrak{D}^s u_\varepsilon)||\mathfrak{D}^s \left(\eta \circ \Phi_t^{-1}\, \Phi_t^{-1}\, \nabla u_\varepsilon\right)|
$$

Notice that $\mathrm{supp}(\eta \circ \Phi_t^{-1}) \subseteq B_{R+1}$, hence the last factor is supported in $A := (B_{R+1} \times \mathbb{R}^N) \cup (\mathbb{R}^N \times B_{R+1})$. Using (2.1) as

$$
|\mathfrak{D}^s \left(\eta \circ \Phi_t^{-1}\, \Phi_t^{-1}\, \nabla u_\varepsilon\right)| \le \|\nabla u_\varepsilon\|_\infty \left|\mathfrak{D}^s \left(\eta \circ \Phi_t^{-1}\, \Phi_t^{-1}\right)\right| + \left\|\eta(\Phi_t^{-1})\, \Phi_t^{-1}\right\|_\infty |\mathfrak{D}^s u_\varepsilon|,
$$

and using the bounds

$$
\sup_{|t|<\delta} \|\eta(\Phi_t^{-1})\, \Phi_t^{-1}\|_\infty = \|\eta\, x\|_\infty \le R
$$

and

$$
\sup_{|t|<\delta} \left|\mathfrak{D}^s \left(\eta \circ \Phi_t^{-1}\, \Phi_t^{-1}\right)(x, y)\right| \le C(\eta, \delta)\, \min\left\{|x-y|^{-s}, |x-y|^{1-s}\right\} \in L^2(\mathbb{R}^{2N}, d\mu),
$$

we thus obtained

$$
\begin{aligned}
&\left|J'\left(\mathcal{K}_t^s\, \mathfrak{D}^s u_\varepsilon\right) \mathfrak{D}^s \left(\eta \circ \Phi_t^{-1}\, \Phi_t^{-1}\, \nabla u_\varepsilon\right)\, \mathcal{K}_t^{s+N}\, \mathcal{H}_t\right| \\
&\qquad \le C\, |J'(\mathfrak{D} u_\varepsilon)|\, \chi_A\, \left(|\mathfrak{D}^s u_\varepsilon| + \min\left\{|x-y|^{-s}, |x-y|^{1-s}\right\}\right).
\end{aligned}
$$

Corollary 4.5 ensures that the right hand side is in $L^1(\mathbb{R}^{2N}, d\mu)$, which provides the claimed continuity through dominated convergence.

For the third map, we compute

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{d}{dt} \left(J_\varepsilon(\mathcal{K}_t^s\, \mathfrak{D}^s u_\varepsilon)\, \mathcal{H}_t\, \mathcal{K}_t^N\right)\, d\mu \\
&\quad = \int_{\mathbb{R}^{2N}} J_\varepsilon'\left(\mathcal{K}_t^s\, \mathfrak{D}^s u_\varepsilon\right)\, \partial_t \mathcal{K}_t^s\, \mathfrak{D}^s u_\varepsilon\, \mathcal{H}_t \mathcal{K}_t^N + J_\varepsilon\left(\mathcal{K}_t^s\, \mathfrak{D}^s u_\varepsilon\right)\, (\partial_t \mathcal{H}_t\, \mathcal{K}_t^N + \mathcal{H}_t\, \partial_t \mathcal{K}_t^N)\, d\mu.
\end{aligned} \tag{5.10}
$$

The second term on the right-hand side is bounded through (5.4) and (5.6) by a multiple of $J_\varepsilon(\mathfrak{D}^s u_\varepsilon)$, uniformly in $|t| < \delta$. For the first term we proceed as before, noting first that by inspection of (5.5) shows that $\mathrm{supp}\,(\partial_t \mathcal{K}_t^s) \subseteq A$. Therefore (5.9), (5.4) and (5.6) imply the bound

$$
|J_\varepsilon'\left(\mathcal{K}_t^s\, \mathfrak{D}^s u_\varepsilon\right)\, \partial_t \mathcal{K}_t^s\, \mathfrak{D}^s u_\varepsilon\, \mathcal{H}_t\, \mathcal{K}_t^N| \le C\, |J_\varepsilon'(\mathfrak{D}^s u_\varepsilon)|\, \chi_A\, |\mathfrak{D}^s u_\varepsilon|,
$$

which is in $L^1(\mathbb{R}^{2N}, d\mu)$ again by Corollary 4.5. Thus we conclude as before.

Therefore (5.7) is proved for $|t| < \delta$ and the right-hand sides of (5.8) and (5.10) are equal.

Step 3: Pohožaev identity for the approximating problem. By computing them at $t = 0$ through (5.1) and (5.2), we thus have

$$\int_{\mathbb{R}^{2N}} J'_\varepsilon(\mathfrak{D}^s u_\varepsilon)\, \mathfrak{D}^s(\eta\, x\, \nabla u_\varepsilon) d\mu = s \int_{\mathbb{R}^{2N}} J'_\varepsilon\left(\mathfrak{D}^s u_\varepsilon\right)\, \mathfrak{D}^s u_\varepsilon\, \frac{x-y}{|x-y|} \cdot \frac{\eta(x)\, x - \eta(y)\, y}{|x-y|}\, d\mu$$
$$+ \int_{\mathbb{R}^{2N}} J_\varepsilon\left(\mathfrak{D}^s u_\varepsilon\right) \left[N \frac{x-y}{|x-y|} \cdot \frac{\eta(x)\, x - \eta(y)\, y}{|x-y|} - 2\mathrm{div}(\eta\, x)\right] d\mu.$$

Since $\eta\, x\, \nabla u_\varepsilon$ is a feasible test function for (3.3), we obtained the identity

$$\begin{aligned}\int_{\mathbb{R}^N} f_\varepsilon\, \eta\, x\, \nabla u_\varepsilon\, dx &= s \int_{\mathbb{R}^{2N}} J'_\varepsilon\left(\mathfrak{D}^s u_\varepsilon\right)\, \mathfrak{D}^s u_\varepsilon\, \frac{x-y}{|x-y|} \cdot \frac{\eta(x)\, x - \eta(y)\, y}{|x-y|}\, d\mu \\ &\quad + \int_{\mathbb{R}^{2N}} J_\varepsilon\left(\mathfrak{D}^s u_\varepsilon\right) \left[N \frac{x-y}{|x-y|} \cdot \frac{\eta(x)\, x - \eta(y)\, y}{|x-y|} - 2\, \mathrm{div}(\eta\, x)\right] d\mu.\end{aligned} \tag{5.11}$$

Step 4: taking the limit. We now let ε go to 0 in the previpus equality, by starting with the right-hand side. By Proposition 3.4, we have that $\mathfrak{D}^s u_\varepsilon \to \mathfrak{D}^s u$ strongly in $L^p(\mathbb{R}^{2N}, d\mu)$. Testing (3.4) with u_ε, one gets

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\mathbb{R}^{2N}} |\mathfrak{D}^s u_\varepsilon|^2\, d\mu = \lim_{\varepsilon\to 0} \left[\int_{\mathbb{R}^N} f_\varepsilon\, u_\varepsilon\, dx - \|u_\varepsilon\|^p_{\mathcal{D}^{s,p}}\right] = \langle f(u), u\rangle - \|u\|^p_{\mathcal{D}^{s,p}} = 0,$$

therefore

$$\sqrt{\varepsilon}\, u_\varepsilon \to 0, \qquad \text{in } \mathcal{D}^{s,2}(\mathbb{R}^N). \tag{5.12}$$

Since

$$J_\varepsilon(\mathfrak{D}^s u_\varepsilon) - J_0(\mathfrak{D}^s u) = \frac{\varepsilon}{2}\, |\mathfrak{D}^s u_\varepsilon|^2 + \frac{|\mathfrak{D}^s u_\varepsilon|^p}{p} - \frac{|\mathfrak{D}^s u|^p}{p},$$

we obtain that $J_\varepsilon(\mathfrak{D}^s u_\varepsilon) \to J_0(\mathfrak{D}^s u)$ strongly in $L^1(\mathbb{R}^{2N}, d\mu)$. Moreover the functions

$$\frac{x-y}{|x-y|}\, \frac{\eta(x)\, x - \eta(y)\, y}{|x-y|}, \qquad \mathrm{div}(\eta\, x),$$

are both bounded, thus we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{\mathbb{R}^{2N}} J_\varepsilon\left(\mathfrak{D}^s u_\varepsilon\right) &\left[N \frac{x-y}{|x-y|} \cdot \frac{\eta(x)\, x - \eta(y)\, y}{|x-y|} - 2\mathrm{div}(\eta\, x)\right] d\mu \\ &= \int_{\mathbb{R}^{2N}} \frac{|\mathfrak{D}^s u|^p}{p} \left[N \frac{x-y}{|x-y|} \cdot \frac{\eta(x)\, x - \eta(y)\, y}{|x-y|} - 2\mathrm{div}(\eta\, x)\right] d\mu.\end{aligned}$$

Similarly, using

$$J'_\varepsilon(\mathfrak{D}^s u_\varepsilon)\, \mathfrak{D}^s u_\varepsilon - |\mathfrak{D}^s u|^p = \varepsilon\, |\mathfrak{D}^s u_\varepsilon|^2 + |\mathfrak{D}^s u_\varepsilon|^p - |\mathfrak{D}^s u|^p,$$

we obtain through (5.12)

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^{2N}} J'_\varepsilon\left(\mathfrak{D}^s u_\varepsilon\right) \mathfrak{D}^s u_\varepsilon\, \frac{x-y}{|x-y|} \cdot \frac{\eta(x)\, x - \eta(y)\, y}{|x-y|}\, d\mu = \int_{\mathbb{R}^{2N}} |\mathfrak{D}^s u|^p\, \frac{x-y}{|x-y|} \cdot \frac{\eta(x)\, x - \eta(y)\, y}{|x-y|}\, d\mu.$$

Therefore the right-hand side in (5.11) converges to the corresponding quantity with $\varepsilon = 0$.

To compute the limit as $\varepsilon \searrow 0$ of the left-hand side of (5.11), we split it as

$$\int_{\mathbb{R}^N} f_\varepsilon\, \eta\, x\, \nabla u_\varepsilon\, dx = \int_{\mathbb{R}^N} (f_\varepsilon - f(u_\varepsilon))\, \eta\, x\, \nabla u_\varepsilon\, dx + \int_{\mathbb{R}^N} f(u_\varepsilon)\, \eta\, x\, \nabla u_\varepsilon\, dx.$$

Integrating by parts the last term we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} f(u_\varepsilon)\,\eta\,x\,\nabla u_\varepsilon\,dx &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \nabla F(u_\varepsilon)\,\eta\,x\,dx \\ &= -\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} F(u_\varepsilon)\,\mathrm{div}(\eta\,x)\,dx = -\int_{\mathbb{R}^N} F(u)\,\mathrm{div}(\eta\,x)\,dx.\end{aligned} \tag{5.13}$$

For the other term, observe that u_ε is uniformly bounded by Proposition 4.1 and f is continuous. Since by Proposition 3.4 we have $u_\varepsilon \to u$ in $L^{p^*}(\mathbb{R}^N)$, then $f_\varepsilon - f(u_\varepsilon) \to 0$ in $L^{p^*}_{\mathrm{loc}}(\mathbb{R}^N)$. Indeed, for every bounded set $\Omega \subset \mathbb{R}^N$ we have

$$\|f_\varepsilon - f(u_\varepsilon)\|_{L^{p^*}(\Omega)} \le \|f_\varepsilon - f(u)\|_{L^{p^*}(\Omega)} + \|f(u) - f(u_\varepsilon)\|_{L^{p^*}(\Omega)},$$

and the first term converges to 0 thanks to Lemma 3.1, while for the second one we can use the Dominated Convergence Theorem.

Since u_ε is bounded in $L^{p^*}(\mathbb{R}^N)$ and $u_\varepsilon \in W^{1,\infty}(\mathbb{R}^N) \subseteq W^{1,p^{*\prime}}_{\mathrm{loc}}(\mathbb{R}^N)$, Corollary A.2 applies with $r = q = p^*$, ensuring that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \Big(f_\varepsilon - f(u_\varepsilon)\Big)\,\eta\,x\,\nabla u_\varepsilon\,dx = 0.$$

Thus we proved that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} f_\varepsilon\,\eta\,x\,\nabla u_\varepsilon\,dx = -\int_{\mathbb{R}^N} F(u)\,\mathrm{div}(\eta\,x)\,dx.$$

By letting $\varepsilon \searrow 0$ in (5.11), we obtain

$$\begin{aligned}-\int_{\mathbb{R}^N} F(u)\,\mathrm{div}(\eta\,x)\,dx &= s\int_{\mathbb{R}^{2N}} |\mathfrak{D}^s u|^p\,\frac{x-y}{|x-y|}\,\frac{\eta(x)\,x - \eta(y)\,y}{|x-y|}\,d\mu \\ &+ \int_{\mathbb{R}^{2N}} \frac{|\mathfrak{D}^s u|^p}{p}\left[N\,\frac{x-y}{|x-y|}\cdot\frac{\eta(x)\,x - \eta(y)\,y}{|x-y|} - 2\,\mathrm{div}(\eta\,x)\right]d\mu.\end{aligned} \tag{5.14}$$

Step 5: conclusion. Finally, we take η of the form $\eta_R(x) = \varphi(x/R)$, with $\varphi \in C_0^\infty(B_1)$ positive, such that $\varphi \equiv 1$ in $B_{1/2}$. Clearly

$$\left|\frac{x-y}{|x-y|}\cdot\frac{\eta_R(x)\,x - \eta_R(y)\,y}{|x-y|}\right| \le \|\varphi\|_\infty + \|\nabla\varphi\|_\infty,$$

and

$$|\mathrm{div}(\eta_R\,x)| \le \left|\frac{1}{R}\,\nabla\varphi\left(\frac{x}{R}\right)\cdot x\right| + \mathrm{div}(x)\,\left|\varphi\left(\frac{x}{R}\right)\right| \le (1+N)\,\left(\|\varphi\|_\infty + \|\nabla\varphi\|_\infty\right).$$

Moreover, for any $(x,y) \in \mathbb{R}^{2N}$

$$\frac{x-y}{|x-y|}\cdot\frac{\eta_R(x)\,x - \eta_R(y)\,y}{|x-y|} \to 1, \qquad \mathrm{div}(\eta_R\,x) \to N,$$

as $R \to +\infty$. Hence, by the Dominated Convergence Theorem, we can let $R \to +\infty$ into (5.14) with $\eta = \eta_R$ to obtain the desired identity (1.5). □

Remark 5.1. In the non-autonomous case $f = f(x,u)$ the proof above provides a general version of the Pohožaev identity when $f(x,t)$ and $D_x F(x,t)$ are Carathéodory and both $F(u)$

and $x\cdot D_x F(x,u)$ belong to $L^1(\mathbb{R}^N)$. Indeed, it suffices to substitute (5.13) with

$$\int_{\mathbb{R}^N} f(x,u_\varepsilon)\,\eta\,x\,\nabla u_\varepsilon\,dx = \int_{\mathbb{R}^N} \nabla F(x,u_\varepsilon)\,\eta\,x\,dx - \int_{\mathbb{R}^N} D_x F(x,u_\varepsilon)\,\eta\,x\,dx$$
$$= -\int_{\mathbb{R}^N} F(x,u_\varepsilon)\,\mathrm{div}(\eta\,x)\,dx - \int_{\mathbb{R}^N} D_x F(x,u_\varepsilon)\,\eta\,x\,dx,$$

and proceed as before. Under the previous assumptions one therefore gets the Pohožaev identity

$$\frac{N-s\,p}{p}\,\|u\|^p_{\mathcal{D}^{s,p}} = N\int_{\mathbb{R}^N} F(x,u)\,dx + \int_{\mathbb{R}^N} x\cdot D_x F(x,u)\,dx. \tag{5.15}$$

In the proof above we needed the following result.

Theorem 5.2. *Let $I\subseteq\mathbb{R}$ be a closed bounded interval, (X,μ) be a σ-finite measure space and $G: I\times X\to\mathbb{R}$ e measurable function. Suppose that*

(1) For any $x\in X$ the function $t\mapsto G(t,x)$ is absolutely continuous with derivative $\partial_t G(t,x)\in L^1(I)$ and $\partial_t G: I\times X\to\mathbb{R}$ is measurable.

(2) The maps

$$I\ni t\mapsto \int_X |G(t,x)|\,d\mu, \qquad I\ni t\mapsto \int_X |\partial_t G(t,x)|\,d\mu$$

are continuous.

Then $t\mapsto \int_X G(t,x)\,d\mu$ is differentiable in I and

$$\frac{d}{dt}\int_X G(t,x)\,d\mu = \int_X \partial_t G(t,x)\,d\mu.$$

Proof. By the continuity hypthesis and Fubini's Theorem G and $\partial_t G$ belong to $L^1(I\times X)$. Applying again Fubini's Theorem, for any $\varphi\in C^\infty_0(\mathbb{R})$ it holds

$$\int_{\mathbb{R}}\left[\int_X G(t,x)\,d\mu\right]\varphi'(t)\,dt = \int_{X\times\mathbb{R}} G(t,x)\varphi'(t)\,d\mu\,dt = \int_X\int_{\mathbb{R}} G(t,x)\varphi'(t)\,dt\,d\mu$$
$$= -\int_X\int_{\mathbb{R}} \partial_t G(t,x)\varphi(t)\,dt\,d\mu = -\int_{\mathbb{R}}\left[\int_X \partial_t G(t,x)\,d\mu\right]\varphi(t)\,dt.$$

Therefore the distributional derivative of $t\mapsto\int_X G(t,x)\,d\mu$ coincides with the continuous function $t\mapsto\int_X \partial_t G(t,x)\,d\mu$. Applying [22, II.5, Theorem V] we obtain the claim. □

6. Applications

In this section, we consider some applications of our main result.

6.1. Least energy characterizations.

Let $f\in C^0(\mathbb{R},\mathbb{R})$ and let us set

$$F(t)=\int_0^t f(\tau)\,d\tau, \qquad t\in\mathbb{R}.$$

We assume that, for some $\ell<0$,

$$\lim_{t\to 0}\frac{f(t)}{|t|^{p-2}\,t}=\ell \qquad\text{and}\qquad \lim_{|t|\to+\infty}\frac{f(t)}{|t|^{p^*-1}}=0. \tag{6.1}$$

This implies in particular that for every $\delta>0$ we have

$$|f(t)|\le C_\delta\,|t|^{p-1}+\delta\,|t|^{p^*-1}, \qquad t\in\mathbb{R}, \tag{6.2}$$

for some $C_\delta = C_\delta(\ell, p, f) > 1$ which may blow-up as $\delta \searrow 0$. For later reference, we also record the following estimate

$$f(t) \leq (\ell + \delta)\,|t|^{p-1} + C_\delta\,|t|^{p^*-1}, \qquad t \in \mathbb{R},$$

which follows from the conditions on f. Correspondigly, we get

$$|F(t)| \leq C_\delta\,\frac{|t|^p}{p} + \delta\,\frac{|t|^{p^*}}{p^*}, \qquad t \in \mathbb{R}, \tag{6.3}$$

and

$$F(t) \leq (\ell + \delta)\,\frac{|t|^p}{p} + C_\delta\,\frac{|t|^{p^*}}{p^*}, \qquad t \in \mathbb{R}. \tag{6.4}$$

Thus from (6.3) we get $F(u) \in L^1(\mathbb{R}^N)$ for any $u \in W^{s,p}(\mathbb{R}^N)$. The functional $\mathcal{E} : W^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\mathcal{E}(u) := \frac{1}{p}\,\|u\|^p_{\mathcal{D}^{s,p}} - \int_{\mathbb{R}^N} F(u)\,dx,$$

is the energy functional associated with problem

$$(-\Delta_p)^s u = f(u) \quad \text{in } \mathbb{R}^N. \tag{6.5}$$

It is readily checked that under the previous growth assumptions on f, $\mathcal{E}$ is of class C^1.

Lemma 6.1 (Pohožaev identity)**.** *Let $u \in W^{s,p}(\mathbb{R}^N)$ be any nontrivial weak solution to (6.5) under assumption (6.2). Then*

$$u \in \mathcal{P} := \left\{ u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\} : \frac{N - s\,p}{p}\,\|u\|^p_{\mathcal{D}^{s,p}} - N \int_{\mathbb{R}^N} F(u)\,dx = 0 \right\}. \tag{6.6}$$

Proof. By Theorem 1.1, it suffices to show that any weak solution to (6.5) under the growth assumptions (6.2) is bounded. For $k > 0$ and $t \in \mathbb{R}$, we set

$$|t|_k = \min\{k, |t|\}.$$

For any $\alpha > 0$ we test the equation with $v = u\,|u|_k^\alpha \in W^{s,p}(\mathbb{R}^N)$. From the growth condition (6.2) we readily get $f(u)\,v \in L^1(\mathbb{R}^N)$, so that Lemma 2.4 is in force. We set $g(t) = t\,|t|_k^\alpha$ and observe that

$$G(t) = \int_0^t \left(g'(\tau)\right)^{\frac{1}{p}}\,d\tau \geq \frac{p}{p+\alpha}\,t\,|t|_k^{\frac{\alpha}{p}}.$$

By testing the equation with $\varphi = g(u)$ and then applying (2.6), we obtain

$$\|G(u)\|^p_{\mathcal{D}^{s,p}} \leq \int_{\mathbb{R}^{2N}} |\mathfrak{D}^s u|^{p-2}\,\mathfrak{D}^s u\,\mathfrak{D}^s v\,d\mu = \int_{\mathbb{R}^N} f(u)\,g(u)\,dx.$$

Using Sobolev inequality on the left-hand side and (6.2) on the right, we get

$$\left\| u\,|u|_k^{\frac{\alpha}{p}} \right\|^p_{p^*} \leq \frac{1}{\mathcal{S}_{p,s}} \left(\frac{p+\alpha}{p}\right)^p \int_{\mathbb{R}^N} \left[|u|^p\,|u|_k^\alpha + |u|^{p^*}\,|u|_k^\alpha \right] dx. \tag{6.7}$$

We now introduce the sequence of exponents

$$\alpha_0 = p^* - p, \qquad \alpha_{i+1} = \frac{p^*}{p}\,\alpha_i, \qquad i \in \mathbb{N}.$$

For every $i \in \mathbb{N}$, we also choose $M_i > 0$ such that

$$\frac{1}{\mathcal{S}_{p,s}} \left(\frac{p+\alpha_i}{p}\right)^p \left(\int_{\{|u| \geq M_i\}} |u|^{p^*}\,dx \right)^{1-\frac{p}{p^*}} < \frac{1}{2}.$$

By Hölder's inequality, for every $i\in\mathbb{N}$ we have

$$\int_{\mathbb{R}^N} |u|^{p^*}|u|_k^{\alpha_i}\,dx \le M_i^{\alpha_i}\int_{\{|u|<M_i\}}|u|^{p^*}\,dx + \left(\int_{\{|u|\ge M_i\}}|u|^{p^*}\,dx\right)^{1-\frac{p}{p^*}}\left(\int_{\{|u|\ge M\}}|u|^{p^*}|u|_k^{\frac{p^*}{p}\alpha_i}\,dx\right)^{\frac{p}{p^*}}.$$

By using this in (6.7) and absorbing the last term (this is possible thanks to the choice of M_i), we obtain

$$\frac{1}{2}\left\|u\,|u|_k^{\frac{\alpha_i}{p}}\right\|_{p^*}^p \le \frac{1}{\mathcal{S}_{p,s}}\left(\frac{p+\alpha_i}{p}\right)^p\left[\int_{\mathbb{R}^N}|u|^p\,|u|_k^{\alpha_i} + M_i^{\alpha_i}\int_{\mathbb{R}^N}|u|^{p^*}\,dx\right].$$

We can now use the estimate to prove that

$$u\in L^{p^*+\alpha_i}(\mathbb{R}^N)\quad\Longrightarrow\quad u\in L^{p^*+\alpha_{i+1}}(\mathbb{R}^N),\qquad \text{for every } i\in\mathbb{N}.$$

By starting from $i=0$ and iterating infinitely many times the previous scheme, we thus obtain at first $u\in L^q(\mathbb{R}^N)$, for every $p\le q<+\infty$.

We now want to enforce this information into $u\in L^\infty(\mathbb{R}^N)$. To this aim, we take $\alpha>0$ and define

$$\gamma=\sqrt{\frac{p^*}{p}},\qquad q_\alpha=\frac{p^*}{\alpha}+\gamma,$$

then by Hölder's inequality

$$\int_{\mathbb{R}^N}\left[|u|^{p+\alpha}+|u|^{p^*+\alpha}\right]dx \le \left\||u|^p+|u|^{p^*}\right\|_{q'_\alpha}\left\||u|^\alpha\right\|_{q_\alpha}.$$

Observe that the first term on the right-hand side is uniformly bounded: indeed, by observing that

$$1<q'_\alpha<\gamma',$$

by Lebesgue interpolation and Young's inequality

$$\begin{aligned}\left\||u|^p+|u|^{p^*}\right\|_{q'_\alpha} &\le \left\||u|^p+|u|^{p^*}\right\|_1^{\vartheta_\alpha}\left\||u|^p+|u|^{p^*}\right\|_{\gamma'}^{1-\vartheta_\alpha}\\ &\le \left\||u|^p+|u|^{p^*}\right\|_1+\left\||u|^p+|u|^{p^*}\right\|_{\gamma'}=:Y,\end{aligned}$$

for suitable $\vartheta_\alpha\in(0,1)$ determined by scale invariance. Observe that $Y<+\infty$ by the first part of the proof.

By taking the limit as k goes to $+\infty$ in (6.7) and using the previous estimate, we obtain

$$\left\||u|^{1+\frac{\alpha}{p}}\right\|_{p^*}^p \le \frac{Y}{\mathcal{S}_{p,s}}\left(\frac{p+\alpha}{p}\right)^p\left\||u|^\alpha\right\|_{q_\alpha}.$$

By recalling the definitions of γ and q_α, this estimate can be rewritten as

$$\left(\int_{\mathbb{R}^N}|u|^{p^*+\gamma^2\alpha}\,dx\right)^{\frac{1}{p^*+\gamma^2\alpha}} \le \left(\frac{Y}{\mathcal{S}_{p,s}}\right)^{\frac{1}{p+\alpha}}\left(\frac{p+\alpha}{p}\right)^{\frac{p}{p+\alpha}}\left(\int_{\mathbb{R}^N}|u|^{p^*+\gamma\alpha}\,dx\right)^{\frac{1}{p^*+\gamma\alpha}\,\frac{\alpha}{p+\alpha}}.$$

A standard iteration for $\alpha_i=\gamma^i$ now provides the desired L^∞ estimate. □

The following is a modification of [2]. Notice that in our setting we are not assuming symmetry conditions on f and a Radial Lemma of Strauss–type does not hold in general for $s\in(0,1)$.

Lemma 6.2 (Solvability)**.** *Let us suppose that* (6.1) *hold. If*

$$\mathcal{S} := \left\{ u \in W^{s,p}(\mathbb{R}^N) \, : \, \int_{\mathbb{R}^N} F(u)\, dx = 1 \right\} \neq \emptyset, \tag{6.8}$$

then there exists a nontrivial solution to (6.5)*.*

Proof. By proceeding as in [2], the solution will be a suitable rescaling of a minimizer of the constrained problem

$$E := \inf_{u\in\mathcal{S}} \mathcal{E}(u). \tag{6.9}$$

Indeed, observe that if w is such a minimizer, then it is a weak solution of

$$(-\Delta_p)^s w = \lambda\, f(w).$$

We first observe that $\lambda \neq 0$, indeed if one would have $\lambda = 0$, then from the equation we would get $w \equiv 0$ which does not verify the constraint (recall that $F(0) = 0$). Moreover, by using Lemma 6.1, it is not difficult to see that $\lambda > 0$. More precisely, we have

$$\frac{N - s\,p}{N\,p}\, \|w\|^p_{\mathcal{D}^{s,p}} = \lambda \int_{\mathbb{R}^N} F(w)\, dx = \lambda. \tag{6.10}$$

Then a solution of (6.5) is obtained by taking the rescaled function $x \mapsto w(\lambda^{-1/s\,p}\, x)$.

To solve the minimization problem (6.9), consider a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$. We will construct another minimizing sequence $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ which is radially monotone. For any $n \in \mathbb{N}$,

$$1 = \int_{\mathbb{R}^N} F(u_n^+)\, dx + \int_{\mathbb{R}^N} F(-u_n^-) =: I_n^+ + I_n^-.$$

Suppose that $I_n^- \leq 0$. Then $I_n^+ \geq 1$ and we let

$$v_n^+(x) := u_n^+\left((I_n^+)^{1/N} x\right). \tag{6.11}$$

By scaling it holds $v_n^+ \in \mathcal{S}$ and

$$\|v_n^+\|^p_{\mathcal{D}^{s,p}} = (I_n^+)^{-\frac{N-s\,p}{N}} \|u_n^+\|^p_{\mathcal{D}^{s,p}} \leq \|u_n\|^p_{\mathcal{D}^{s,p}}.$$

If $I_n^+ \leq 0$, we proceed similarly.

Suppose then that $0 < I_n^\pm < 1$ for all $n \geq n_0$, and let $v_n^\pm$ be defined as in (6.11). By [12, Remark 3.3], we know that

$$\|z^+\|^p_{\mathcal{D}^{s,p}} + \|z^-\|^p_{\mathcal{D}^{s,p}} \leq \|z\|^p_{\mathcal{D}^{s,p}}, \qquad \text{for all } z \in \mathcal{D}^{s,p}.$$

It follows by scaling that

$$\begin{aligned}
\|u_n\|^p_{\mathcal{D}^{s,p}} &\geq \|u_n^+\|^p_{\mathcal{D}^{s,p}} + \|u_n^-\|^p_{\mathcal{D}^{s,p}} \\
&\geq (I_n^+)^{\frac{N-s\,p}{N}} \|v_n^+\|^p_{\mathcal{D}^{s,p}} + (I_n^-)^{\frac{N-s\,p}{N}} \|v_n^-\|^p_{\mathcal{D}^{s,p}} \\
&\geq \left((I_n^+)^{\frac{N-s\,p}{N}} + (I_n^-)^{\frac{N-s\,p}{N}}\right) \min\left\{\|v_n^+\|^p_{\mathcal{D}^{s,p}}, \|v_n^-\|^p_{\mathcal{D}^{s,p}}\right\} \\
&\geq \left(I_n^+ + I_n^-\right)^{\frac{N-s\,p}{N}} \min\left\{\|v_n^+\|^p_{\mathcal{D}^{s,p}}, \|v_n^-\|^p_{\mathcal{D}^{s,p}}\right\}.
\end{aligned}$$

By recalling that $I_n^+ + I_n^- = 1$, up to a subsequence, either $\{v_n^+\}$ or $\{-v_n^-\}$ is minimizing. Suppose without loss of generality that $\{v_n^+\}$ is minimizing and let $w_n = (v_n^+)^*$ be its radially symmetric decreasing rearrangement. By Pólya-Szegő principle (see [1, Theorem 9.2]), $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ is still minimizing and we can suppose that w_n converges to $w \in \mathcal{D}^{s,p}(\mathbb{R}^N)$, weakly in $\mathcal{D}^{s,p}(\mathbb{R}^N)$ and almost everywhere. Observe that by lower semicontinuity, we have

$$\|w\|_{\mathcal{D}^{s,p}}^p \leq E+1. \tag{6.12}$$

Proceeding as in [2] provides a uniform bound on $\|w_n\|_p$, so that $\|w_n\|_q$ is bounded for any $p \leq q \leq p^*$. If $F = F^+ - F^-$, then using (6.1) we have

$$\lim_{t\to 0^+} \frac{F^+(t)}{t^p + t^{p^*}} = \lim_{t\to+\infty} \frac{F^+(t)}{t^p + t^{p^*}} = 0.$$

Moreover by [5, Lemma 2.9] with $\vartheta = p$, we have the decay estimate $0 \leq w_n(x) \leq C|x|^{-N/p}$ for some $C > 0$ independent on n. Therefore [2, Theorem A.I] applies, giving

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F^+(w_n)\,dx = \int_{\mathbb{R}^N} F^+(w)\,dx,$$

while by Fatou's Lemma

$$\begin{aligned}\int_{\mathbb{R}^N} F^-(w)\,dx &\leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} F^-(w_n)\,dx \\ &= \liminf_{n\to\infty} \int_{\mathbb{R}^N} F^+(w_n)\,dx - 1 = \int_{\mathbb{R}^N} F^+(w)\,dx - 1,\end{aligned}$$

that is

$$I := \int_{\mathbb{R}^N} F(w)\,dx \geq 1.$$

In order to conclude, we just need to show that $I = 1$. Let us assume that $I > 1$, then we define as above

$$\widetilde{w}(x) = w\Big(I^{\frac{1}{N}}\,x\Big).$$

We have $\widetilde{w} \in \mathcal{S}$ and

$$E+1 \leq \|\widetilde{w}\|_{\mathcal{D}^{s,p}}^p = I^{-\frac{N-s\,p}{N}}\,\|w\|_{\mathcal{D}^{s,p}}^p < E+1,$$

where in the second inequality we used (6.12) and $I > 1$. This gives a contradiction, thus $I = 1$ and w is the desired minimizer. □

We have the following result, which was first obtained in the semilinear local case in [15].

Theorem 6.3 (Energy characterization)**.** *Suppose* (6.2) *holds. We define the following energetic levels:*

- *Mountain-Pass value*
$$\mathfrak{c} := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathcal{E}(\gamma(t)),$$
where
$$\Gamma := \Big\{\gamma \in C([0,1], W^{s,p}(\mathbb{R}^N)) : \gamma(0) = 0,\, \mathcal{E}(\gamma(1)) < 0\Big\};$$
- *least energy of solutions*
$$\mathfrak{m} := \inf\left\{\mathcal{E}(u) : u \in W^{s,p}(\mathbb{R}^N)\setminus\{0\} \text{ is a weak solution to } (6.5)\right\};$$
- *Pohožaev value*
$$\mathfrak{p} := \inf_{u\in\mathcal{P}} \mathcal{E}(u),$$
where $\mathcal{P}$ *is defined in* (6.6)*;*

- *ground state value*

$$\mathfrak{s} := \frac{s}{N}\left(\frac{N-s\,p}{N\,p}\right)^{\frac{N-s\,p}{s\,p}} \inf_{u\in\mathcal{S}} \|u\|_{\mathcal{D}^{s,p}}^{N/s},$$

where $\mathcal{S}$ is defined in (6.8).

Then

$$\mathfrak{c} = \mathfrak{m} = \mathfrak{p} = \mathfrak{s}.$$

Proof. We shall divide the proof into five steps.

Step 1: $\mathfrak{c} \le \mathfrak{m}$. Let $u \in W^{s,p}(\mathbb{R}^N)$ be any nontrivial solution of (6.5) and consider the curve $\gamma \in C([0,\infty); W^{s,p}(\mathbb{R}^N))$ defined by

$$\gamma(t)(x) := \begin{cases} u\left(\dfrac{x}{t}\right) & \text{for } t > 0,\\ 0 & \text{for } t = 0.\end{cases}$$

Then, we have

$$\mathcal{E}(\gamma(t)) = \frac{1}{p}\|\gamma(t)\|_{\mathcal{D}^{s,p}}^p - \int_{\mathbb{R}^N} F(\gamma(t))\,dx = \frac{t^{N-s\,p}}{p}\|u\|_{\mathcal{D}^{s,p}}^p - t^N \int_{\mathbb{R}^N} F(u)\,dx. \tag{6.13}$$

Since $u \in \mathcal{P}$ by Lemma 6.1, we have

$$\int_{\mathbb{R}^N} F(u)\,dx > 0.$$

Notice that

$$\frac{d}{dt}\mathcal{E}(\gamma(t)) = \frac{N-s\,p}{p}\,t^{N-sp-1}\,\|u\|_{\mathcal{D}^{s,p}}^p - N\,t^{N-1}\int_{\mathbb{R}^N} F(u)\,dx,$$

thus $t \mapsto \mathcal{E}(\gamma(t))$ is increasing for

$$t \le \frac{N\displaystyle\int_{\mathbb{R}^N} F(u)\,dx}{\dfrac{N-s\,p}{p}\|u\|_{\mathcal{D}^{s,p}}^p} = 1,$$

and decreasing otherwise. Hence, we have

$$\max_{t\ge 0}\mathcal{E}(\gamma(t)) = \mathcal{E}(\gamma(1)) = \mathcal{E}(u).$$

Observe that $t \mapsto \mathcal{E}(\gamma(t))$ diverges to $-\infty$ as t goes to $+\infty$, thus there exists $\mu > 1$ such that

$$\mathcal{E}(\gamma(\mu)) = \frac{\mu^{N-s\,p}}{p}\|u\|_{\mathcal{D}^{s,p}}^p - \mu^N\int_{\mathbb{R}^N} F(u)dx < 0.$$

We consider the rescaled curve $\widetilde{\gamma} \in C([0,1]; W^{s,p}(\mathbb{R}^N))$ defined by

$$\widetilde{\gamma}(t)(x) := \gamma(\mu\,t)(x), \qquad t\in[0,1],\ x\in\mathbb{R}^N.$$

Then, we have $\widetilde{\gamma}(0) = 0$ and $\mathcal{E}(\widetilde{\gamma}(1)) < 0$, which yields $\widetilde{\gamma} \in \Gamma$. Therefore, we have

$$\mathfrak{c} \le \max_{t\in[0,1]}\mathcal{E}(\widetilde{\gamma}(t)) = \max_{t\in[0,\mu]}\mathcal{E}(\gamma(t)) = \mathcal{E}(\gamma(1)) = \mathcal{E}(u).$$

Taking the infimum over all the nontrivial solutions u to (6.1), we get $\mathfrak{c} \le \mathfrak{m}$.

Step 2: $\gamma([0,1])\cap\mathcal{P}\neq\emptyset$ for all $\gamma\in\Gamma$. Consider the Pohožaev functional

$$\mathscr{P}(u):=\frac{N-s\,p}{p}\,\|u\|^p_{\mathcal{D}^{s,p}}-N\int_{\mathbb{R}^N}F(u)\,dx.$$

We first prove that there exists $\rho>0$ such that, if $0<\|u\|_{W^{s,p}}\le\rho$, then $\mathcal{E}(u)>0$. Taking into account the growth conditions (6.4) we obtain

$$\begin{aligned}\mathscr{P}(u)&=\frac{N-s\,p}{p}\,\|u\|^p_{\mathcal{D}^{s,p}}-N\int_{\mathbb{R}^N}F(u)dx\\&\ge\frac{N-sp}{p}\|u\|^p_{\mathcal{D}^{s,p}}+\frac{N\,(-\ell-\delta)}{p}\int_{\mathbb{R}^N}|u|^pdx-C_\delta\int_{\mathbb{R}^N}|u|^{p^*}\,dx\\&\ge C\,\|u\|^p_{W^{s,p}}-C_\ell\,\|u\|^{p^*}_{W^{s,p}},\end{aligned}$$

by choosing $\delta=-\ell/2>0$. The previous computation show that we can choose $\rho>0$ small enough, so that $\mathscr{P}(u)>0$ if $0<\|u\|_{W^{s,p}}\le\rho$. Observe now that

$$\mathscr{P}(u)=N\,\mathcal{E}(u)-s\,\|u\|^p_{\mathcal{D}^{s,p}}.$$

If $\gamma\in\Gamma$, we have

$$\mathscr{P}(\gamma(0))=0\qquad\text{and}\qquad\mathscr{P}(\gamma(1))\le N\mathcal{E}(\gamma(1))<0.$$

From the previous discussion and the last property, we have $\|\gamma(1)\|_{W^{s,p}}>\rho$. We define

$$t_0=\sup\{t\in[0,1]\,:\,\|\gamma(t)\|_{W^{s,p}}\le\rho\}.$$

By continuity of $t\mapsto\|\gamma(t)\|_{W^{s,p}}$, we have $0<t_0<1$. Moreover, by the definition of t_0

$$\|\gamma(t_0)\|_{W^{s,p}}=\rho\qquad\text{and}\qquad\|\gamma(t)\|_{W^{s,p}}\ge\rho,\quad\text{for }t_0\le t\le 1.$$

Then by continuity of $t\mapsto\mathscr{P}(\gamma(t))$, there exists $t_0<t_1<1$ such that

$$\|\gamma(t_1)\|_{W^{s,p}}\ge\rho\qquad\text{and}\qquad\mathscr{P}(\gamma(t_1))=0.$$

This means $\gamma(t_1)\in\mathcal{P}$.

Step 3: $\mathfrak{p}=\mathfrak{s}$. The function $\Phi:\mathcal{S}\to\mathcal{P}$ defined by

$$\Phi(u)(x):=u\left(\frac{x}{t_u}\right),\qquad t_u:=\left(\frac{N-s\,p}{N\,p}\right)^{1/s\,p}\|u\|^{1/s}_{\mathcal{D}^{s,p}}.$$

establishes a bijective correspondence between $\mathcal{S}$ and $\mathcal{P}$. This implies

$$\mathfrak{p}=\min_{u\in\mathcal{P}}\mathcal{E}(u)=\min_{u\in\mathcal{S}}\mathcal{E}(\Phi(u)).$$

Moreover, by (6.13), for any $u\in\mathcal{S}$ it holds

$$\mathcal{E}(\Phi(u))=\frac{s}{N}\left(\frac{N-s\,p}{N\,p}\right)^{(N-s\,p)/sp}\|u\|^{N/s}_{\mathcal{D}^{s,p}}.$$

From the previous discussion we directly get $\mathfrak{p}=\mathfrak{s}$, by recalling the definition of $\mathfrak{s}$.

Step 4: $\mathfrak{m}=\mathfrak{p}=\mathfrak{s}$. We can suppose that $\mathcal{S}\neq\emptyset$ (otherwise the claim is trivial). Then Lemma 6.2 shows that (6.5) has at least a solution. Moreover, by Lemma 6.1 we get that any solution to (6.5) belongs to $\mathcal{P}$. This implies that

$$\mathfrak{m}\ge\mathfrak{p}.$$

We now take a minimizer $w\in\mathcal{S}$ for the problem

$$\min_{u\in\mathcal{S}}\mathcal{E}(u).$$

As observed in the proof of Lemma 6.2, there exists $\lambda > 0$ such that $w_\lambda(x) = w(\lambda^{-1/s\,p}\,x)$ is a solution of (6.5). We then have

$$\begin{aligned}\mathfrak{m} \le \mathcal{E}(w_\lambda) &= \frac{\lambda^{\frac{N}{s\,p}-1}}{p}\,\|w\|^p_{\mathcal{D}^{s,p}} - \lambda^{\frac{N}{s\,p}}\int_{\mathbb{R}^N} F(w)\,dx \\ &= \lambda^{\frac{N}{s\,p}}\left(\frac{\|w\|^p_{\mathcal{D}^{s,p}}}{p\,\lambda} - 1\right) = \left(\frac{N - s\,p}{N\,p}\right)^{\frac{N}{s\,p}}\,\|w\|^{N/s}_{\mathcal{D}^{s,p}}\,\frac{s\,p}{N - s\,p},\end{aligned}$$

where we used the relation (6.10) between the norm of w and λ. By recalling the definition of $\mathfrak{s}$ and using **Step 3**, we thus obtain

$$\mathfrak{m} \le \mathfrak{s} = \mathfrak{p}.$$

Step 5: conclusion. By **Step 1**, we know that $\mathfrak{c} \le \mathfrak{m}$. Now, given $\gamma \in \Gamma$, by virtue of **Step 2** there exists $t_1 \in (0,1)$ such that $\gamma(t_1) \in \mathcal{P}$. In turn, by exploiting **Step 4**, we get

$$\mathfrak{m} = \mathfrak{p} \le \mathcal{E}(\gamma(t_1)) \le \max_{t\in[0,1]} \mathcal{E}(\gamma(t)).$$

Hence, by the arbitrariness of γ, we conclude $\mathfrak{m} \le \mathfrak{c}$. This concludes the proof. □

6.2. **Nonexistence results.** From Theorem 1.1, we get the following

Corollary 6.4 (Nonexistence for power nonlinearities)**.** *Let $q > 1$ be such that $q \neq p^*$. Then*

$$(-\Delta_p)^s u = |u|^{q-2} u, \qquad \text{in } \mathbb{R}^N,$$

admits no nontrivial solution $u \in \mathcal{D}^{s,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$.

Proof. We have $f(t) = |t|^{q-2}\,t$ and by hypothesis $f(u)\,u \in L^1(\mathbb{R}^N)$. Thus by testing the equation with u itself we get

$$\|u\|^p_{\mathcal{D}^{s,p}} = \int_{\mathbb{R}^N} |u|^q\,dx.$$

On the other hand, by observing that $F(t) = |t|^q/q$, from Theorem 1.1 we get

$$\frac{N - s\,p}{N\,p}\,\|u\|^p_{\mathcal{D}^{s,p}} = \frac{1}{q}\int_{\mathbb{R}^N} |u|^q\,dx.$$

By comparing the last two displays, we eventually get the conclusion. □

More generally, under the assumption of the previous Section, we have the following

Proposition 6.5 (Nonexistence)**.** *Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that $f(\cdot, t)$ is of class C^1 for all $t \in \mathbb{R}$ and let $F(x,t) = \int_0^t f(x,\tau)\,d\tau$. Moreover, suppose that*

$$G(x,t) = N\,F(x,t) + x \cdot D_x F(x,t) - \frac{N - s\,p}{p} f(x,t)\,t \le 0,$$

and for any $x \in \mathbb{R}^N$ it vanishes at $t = 0$ only. Then the equation

$$(-\Delta_p)^s u = f(x,u), \qquad \text{in } \mathbb{R}^N, \tag{6.14}$$

admits no nontrivial weak solution $u \in \mathcal{D}^{s,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $F(x,u)$ and $x \cdot D_x F(x,u)$ are in $L^1(\mathbb{R}^N)$.

Proof. By hypothesis on G, the function $f(x,u)\,u$ is bounded from below by an $L^1(\mathbb{R}^N)$ function, which implies by Lemma 2.4 that $f(x,u)\,u \in L^1(\mathbb{R}^N)$. Testing equation (6.14) with u yields, again through Lemma 2.4

$$\|u\|^p_{\mathcal{D}^{s,p}} = \langle f(\cdot,u), u\rangle = \int_{\mathbb{R}^N} f(x,u)\,u\,dx.$$

This, combined with the Pohožaev identity (5.15), yields

$$\int_{\mathbb{R}^N} \mathcal{H}(x,u)\,dx = N\int_{\mathbb{R}^N} F(x,u)\,dx + \int_{\mathbb{R}^N} x\cdot D_x F(x,u)\,dx - \frac{N-s\,p}{p}\int_{\mathbb{R}^N} f(x,u)\,u\,dx = 0.$$

By assumption, this implies $u = 0$ almost everywhere, concluding the proof. □

Corollary 6.6. *Let us suppose that $f:\mathbb{R}\to\mathbb{R}$ is such that the function*

$$G(t) := N\,F(t) - \frac{N-s\,p}{p}\,f(t)\,t, \qquad t\in\mathbb{R},$$

is non positive and vanishes at $t = 0$ only. Then problem (6.5) has no nontrivial weak solution $u \in \mathcal{D}^{s,p}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ such that $F(u) = \int_0^u f(t)\,dt$ is in $L^1(\mathbb{R}^N)$.

Appendix A. A bilinear estimate

Let $\mathscr{F}$ denote the Fourier transform, $\mathcal{S}$ the Schwartz class and $\mathcal{S}'$ the space of tempered distributions. For every $s\in\mathbb{R}$ and $1<p<+\infty$, by $H^{s,p}(\mathbb{R}^N)$ we denote the *Bessel potential space*, defined as

$$\left\{u\in\mathcal{S}' \,:\, \mathscr{F}^{-1}\Big((1+|\xi|^2)^{s/2}\,\mathscr{F}(u)\Big)\in L^p(\mathbb{R}^N)\right\}.$$

This is endowed with the norm

$$\|u\|_{H^{s,p}} := \left\|\mathscr{F}^{-1}\Big((1+|\xi|^2)^{s/2}\,\mathscr{F}(u)\Big)\right\|_p.$$

We recall that for $s\in\mathbb{R}$ and $1<p<+\infty$, we have

$$\Big(H^{s,p}(\mathbb{R}^N)\Big)^* = H^{-s,p'}(\mathbb{R}^N), \tag{A.1}$$

see [24, Theorem 2.5.6 & Theorem 2.11.2].

Given a polynomial P, the differential operator $P(D)$ is a continuous operator $\mathcal{S}'\to\mathcal{S}'$. We are concerned with the product of two tempered distributions of the form $P(D)\,\Lambda_1$ and $Q(D)\,\Lambda_2$. In this case we are not only looking for the well-posedness issue, but also on stability with respect to approximation and (negative) Bessel norm estimates.

Proposition A.1. *Let*

$$\frac{1}{q}+\frac{1}{r}=\frac{1}{t}$$

for some $t,p,q>1$ and P, Q two polynomials with

$$\deg(P)+\deg(Q)=k.$$

Then there exists $C = C(P,Q,q,r,N)>0$ such that for any $u,v\in\mathcal{S}$ it holds

$$\|P(D)\,u\cdot Q(D)\,v\|_{H^{-k,t}} \le C\|u\|_q\,\|v\|_r. \tag{A.2}$$

In particular, the product $P(D)\,u\cdot Q(D)\,v$ extends to a bilinear continuous operator $\Pi_{P,Q}$: $L^r(\mathbb{R}^N)\times L^q(\mathbb{R}^N)\to H^{-k,t}(\mathbb{R}^N)$.

Proof. This is a direct consequence of the *Coifman-Meyer multilinear Theorem.* Since $P(D)\,u$, $Q(D)\,v \in \mathcal{S}$, taking the Fourier trasform gives

$$\mathscr{F}\Big(P(D)\,u \cdot Q(D)v\Big)(\xi) = \int_{\{\xi_1+\xi_2=\xi\}} P(i\,\xi_1)\,Q(i\,\xi_2)\,\mathscr{F}(u)(\xi_1)\,\mathscr{F}(v)(\xi_2)\,d\xi_1.$$

We define

$$\mathfrak{m}(\xi_1,\xi_2) = \frac{P(i\,\xi_1)\,Q(i\,\xi_2)}{(1+|\xi|^2)^{\frac{k}{2}}}.$$

By induction, we can verify that $\mathfrak{m}$ is a Coifman-Meyer multiplier, i.e.

$$\left|\partial^i_{\xi_1^i}\partial^j_{\xi_2^j}\mathfrak{m}(\xi_1,\xi_2)\right| \le \frac{C_{ij}}{(|\xi_1|+|\xi_2|)^{|i|+|j|}}, \qquad \text{for any multi-index } i, j.$$

Indeed, since $\mathfrak{m}$ is of the form $R(\xi_1,\xi_2)/(1+|\xi_1+\xi_2|^2)^\alpha$ where R is a polynomial function, any partial derivative is of the same form, and an inductive degree argument on $|i|+|j|$ shows that for any multi-indexes i, j it holds

$$\partial^i_{\xi_1^i}\partial^j_{\xi_2^j}\mathfrak{m}(\xi_1,\xi_2) = \frac{R_{i,j}(\xi_1,\xi_2)}{(1+|\xi_1+\xi_2|^2)^{\frac{k}{2}+|i|+|j|}}, \qquad \text{with } \deg(R_{i,j}) \le k+|i|+|j|.$$

Therefore the bilinear multiplier operator $T_{\mathfrak{m}}$ defined as

$$\mathscr{F}\left(T_{\mathfrak{m}}(u,v)\right)(\xi) = \int_{\{\xi_1+\xi_2=\xi\}} \mathfrak{m}(\xi_1,\xi_2)\,\mathscr{F}(u)(\xi_1)\,\mathscr{F}(v)(\xi_2)\,d\xi_1,$$

satisfies

$$\|T_{\mathfrak{m}}(u,v)\|_t \le C_{\mathfrak{m}}\,\|u\|_q\,\|v\|_r,$$

which is equivalent to (A.2) by the definition of $H^{-k,t}(\mathbb{R}^N)$. The last statement clearly follows. □

For practical applications, we will need the following consequence.

Corollary A.2. *We take*

$$\frac{1}{q}+\frac{1}{r}<1,$$

and consider $\{u_n\}_n \subset W^{1,r'}_{\rm loc}(\mathbb{R}^N)\cap L^q_{\rm loc}(\mathbb{R}^N)$ *and* $\{v_n\}_n \subset L^r_{\rm loc}(\mathbb{R}^N)$ *satisfying*

$$\|u_n\|_{L^q(\Omega)} \le C_\Omega < +\infty, \quad \textit{for any bounded } \Omega\subset\mathbb{R}^N, \qquad v_n \to 0 \quad \textit{in } L^r_{\rm loc}(\mathbb{R}^N).$$

Then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} v_n\,\nabla u_n\cdot F\,dx = 0, \qquad \textit{for every } F\in C^\infty_0(\mathbb{R}^N;\mathbb{R}^N). \tag{A.3}$$

Proof. We take $F\in C^\infty_0(\mathbb{R}^N)$ and fix $\eta\in C^\infty_0(\mathbb{R}^N)$ such that $\eta\equiv 1$ on a bounded neighbourhood of $\mathrm{supp}(F)$. Then we set

$$\widetilde{v}_n = \eta\,v_n \in L^r(\mathbb{R}^N) \qquad \text{and} \qquad \widetilde{u}_n = \eta\,u_n \in W^{1,r'}(\mathbb{R}^N)\cap L^q(\mathbb{R}^N).$$

By construction, the integral in (A.3) does not change with these substitutions and by our assumptions we have

$$\|\widetilde{u}_n\|_q \le C \qquad \text{and} \qquad \lim_{n\to\infty}\|\widetilde{v}_n\|_r = 0.$$

We may notice that, for any $k=1,\dots,N$, if $v\in L^r(\mathbb{R}^N)$ and $u\in W^{1,r'}(\mathbb{R}^N)$, then $v\,\partial_{x_k}u \in L^1(\mathbb{R}^N)\subseteq\mathcal{S}'$. It is readily checked that this function coincides with the tempered distribution

$\Lambda = v \cdot \partial_{x_k} u$ (uniquely) defined by density in Proposition A.1. Indeed, let $\tilde{u}_j \in C_0^\infty(\mathbb{R}^N)$ such that $\tilde{u}_j \to u$ strongly in $W^{1,r'}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$. For any $\varphi \in \mathcal{S}$, it holds

$$\langle \Lambda, \varphi \rangle = \lim_{j\to\infty} \int_{\mathbb{R}^N} v\, \partial_{x_k} \tilde{u}_j\, \varphi\, dx = \int_{\mathbb{R}^N} v\, \partial_{x_k} u\, \varphi\, dx + \lim_{j\to\infty} \int_{\mathbb{R}^N} v\, \partial_{x_k} (\tilde{u}_j - u)\, \varphi\, dx,$$

and the last limit vanishes by Hölder inequality. The claimed limit now immediately follows from (A.2), since by taking

$$\frac{1}{t} = \frac{1}{p} + \frac{1}{q},$$

we have

$$\begin{aligned}\lim_{n\to\infty} \left| \int_{\mathbb{R}^N} v_n\, \nabla u_n \cdot F\, dx \right| = \lim_{n\to\infty} \left| \int_{\mathbb{R}^N} \widetilde{v}_n\, \nabla \widetilde{u}_n \cdot F\, dx \right| &\le \lim_{n\to\infty} \|\widetilde{v}_n\, \nabla \widetilde{u}_n\|_{H^{-1,t}}\, \|F\|_{H^{1,t'}} \\ &\le C \lim_{n\to\infty} \|\widetilde{v}_n\|_r\, \|\widetilde{u}_n\|_q\, \|F\|_{H^{1,t'}} = 0,\end{aligned}$$

where we used (A.1). This concludes the proof. □

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