

A priori estimates for Donaldson's equation over compact Hermitian manifolds

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Received: 15 February 2013 / Accepted: 15 July 2013 / Published online: 26 July 2013
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Abstract In this paper we prove a priori estimates for Donaldson equation's

$$\omega \wedge (\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^{n-1} = e^F (\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^n,$$

over a compact complex manifold X of complex dimension n , where ω and χ are arbitrary Hermitian metrics. Our estimates answer a question of Tosatti-Weinkove (Asian J. Math. 14:19–40, 2010).

Mathematics Subject Classification (2000) Primary 53C55; Secondary 32W20 · 35J60 · 58J05

1 Introduction

1.1 Donaldson's equation over compact Kähler manifolds

Let (X, ω) be a compact Kähler manifold of the complex dimension n , and χ another Kähler metric on X . In [3], Donaldson considered the following interesting equation

$$\omega \wedge \eta^{n-1} = c\eta^n, \quad [\eta] = [\chi], \quad (1.1)$$

where c is a constant, depending only on the Kähler classes of $[\chi]$ and $[\omega]$, given by

$$c = \frac{\int_X \omega \wedge \chi^{n-1}}{\int_X \chi^n}. \quad (1.2)$$

Communicated by L. Ambrosio.

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He noted that a necessary condition for Eq. (1.1) is

$$nc\chi - \omega > 0, \tag{1.3}$$

and then conjectured that the condition (1.3) is also sufficient. For $n = 2$, Chen [1] observed that in this case the Eq. (1.1) reduces to a complex Monge-Ampère equation completely solved by Yau on his celebrated work on Calabi’s conjecture [24].

1.2 J -flow and Donaldson’s equation

To better understand the Eq. (1.1), Donaldson [3] and Chen [1] independently discovered the J -flow whose critical point gives the Eq. (1.1), and Chen showed that such flow always exists for all time. Using the J -flow, Chen [2] proved that if $n = 2$ and the holomorphic bisectional curvature of ω is nonnegative then the J -flow converges to a critical metric. Later, the curvature assumption was removed by Weinkove [22] and hence gave an alternative proof of Donaldson’s conjecture on Kähler surfaces. For higher dimensional case, Weinkove [23] solved Donaldson’s conjecture on a slightly strong condition

$$nc\chi - (n - 1)\omega > 0 \tag{1.4}$$

using the J -flow. For more detailed discussions and related works, we refer to [4–7, 15, 16].

1.3 Donaldson’s equation over compact Hermitian manifolds

Recently, the complex Monge-Ampère equation over compact Hermitian manifolds was solved Tosatti and Weinkove [17, 18]. Other interesting estimates can be found in [19, 25, 26]. A parabolic proof was late given by Gill [8] by considering a parabolic complex Monge-Ampère equation. Other parabolic flows over compact Hermitian manifolds were considered in [14, 19–21], where they obtained lots of interesting results parallel to those in Kähler case. By Tosatti-Weinkove’s work, the author considers Donaldson’s equation over compact Hermitian manifolds.

Let (X, ω) be a compact Hermitian manifold of the complex dimension n and χ another Hermitian metric on X . We denote by \mathcal{H}_χ the set of all real-valued smooth functions φ on X such that $\chi_\varphi := \chi + \sqrt{-1}\partial\bar{\partial}\varphi > 0$. Locally we have

$$\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}, \quad \chi = \sqrt{-1}\chi_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}. \tag{1.5}$$

For any real positive $(1, 1)$ -form $\alpha := \sqrt{-1}\alpha_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}$ and real $(1, 1)$ -form $\beta := \sqrt{-1}\beta_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}$ we set

$$\text{tr}_\alpha \beta := \alpha^{i\bar{j}}\beta_{i\bar{j}}. \tag{1.6}$$

We consider Donaldson’s equation

$$\omega \wedge \chi_\varphi^{n-1} = e^F \cdot \chi_\varphi^n, \quad \varphi \in \mathcal{H}_\varphi \tag{1.7}$$

on X , where F is a given smooth function on X .

The main result of this paper is the following a priori estimates.

Theorem 1.1 *Let (X, ω) be a compact Hermitian manifold of the complex dimension n and χ another Hermitian metric. Let φ be a smooth solution of Donaldson’s equation (1.7). Assume that*

$$\chi - \frac{n - 1}{ne^F}\omega > 0. \tag{1.8}$$

Then

- (1) *there exist uniform constant $A > 0$ and $C > 0$, depending only on X, ω, χ , and F , such that*

$$\text{tr}_\omega \chi_\varphi \leq C \cdot e^{A(\varphi - \inf_X \varphi)}; \tag{1.9}$$

- (2) *there exists a uniform constant $C > 0$, depending only on X, ω, χ , and F , such that*

$$\|\varphi\|_{C^0} \leq C; \tag{1.10}$$

- (3) *there are uniform C^∞ a priori estimates on φ depending only on X, ω, χ , and F .*

Meanwhile, Guan, Li and Sun [9, 11–13] considered a priori estimates for Donaldson’s equation over compact Hermitian manifolds under very general structure conditions rather than the condition (1.8).

Remark 1.2 As remarked in [17] (see page 22, line 27–28), to prove the zeroth estimate in Theorem 1.1 it suffices to show the second order estimate on φ . Our result gives an affirmative answer to the question in [17] (see page 22, line 28–30). Using the same argument in [17] (page 33), we can get a C^α estimate on φ for some $\alpha \in (0, 1)$. Differentiating (1.7) and applying the standard local elliptic estimates imply uniform C^∞ estimates on φ .

There are some natural questions about the Eq. (1.7). Is condition (1.8) sufficient to product a solution to (1.7)? When ω and χ both are Kähler, it has been proved in [2, 22, 23] that this condition is sufficient. The second question is to consider a parabolic flow over compact Hermitian manifolds like the J -flow. Can we prove the long time existence and convergence of such a flow? Song and Weinkove [16] gave a necessary and sufficient condition for existence of solutions to the Donaldson’s equation over compact Kähler manifolds (and also for convergence of the J -flow over compact Kähler manifolds). The last question then is whether we can find an analogous of above Song-Weinkove’s condition. Those questions will be answered later.

Remark 1.3 Here and henceforth, when we say a “uniform constant” it should be understood to be a constant that depends only on X, ω, χ , and F . We will often write C or C' for such a constant, where the value of C or C' may differ from line to line. For the relation $P \leq CQ$ for a uniform constant C in the above sense, we write it as $P \lesssim Q$. $\text{Re}(P)$ means the real part of P .

2 The second order estimates

2.1 Basic facts and notions

Let (X, ω) be a complex Hermitian manifold of the complex dimension n and χ another Hermitian metric on X . For a solution φ of Donaldson’s equation (1.7), we denote by

$$\chi' := \chi + \sqrt{-1} \partial \bar{\partial} \varphi = \sqrt{-1} (\chi_{i\bar{j}} + \varphi_{i\bar{j}}) dz^i \wedge d\bar{z}^{\bar{j}}. \tag{2.1}$$

Also, we set $\chi'_{i\bar{j}} := \chi_{i\bar{j}} + \varphi_{i\bar{j}}$. Then we observe that

$$\text{tr}_{\chi'} \omega = n \frac{\omega \wedge (\chi')^{n-1}}{(\chi')^n} = n e^F. \tag{2.2}$$

Consequently, $\text{tr}_{\chi'}\omega$ is uniformly bounded away from zero and infinity. Let Δ_ω denote the Laplacian operator of the Chern connection associated to the Hermitian metric ω , and similarly for Δ_χ . Note that

$$\text{tr}_\omega \chi' = g^{i\bar{j}}(\chi_{i\bar{j}} + \varphi_{i\bar{j}}) = \text{tr}_\omega \chi + \Delta_\omega \varphi. \tag{2.3}$$

Remark 2.1 $\text{tr}_\omega \chi'$ and $\text{tr}_{\chi'}\omega$ are uniformly bounded from below away from zero. More precisely,

$$\text{tr}_\omega \chi' \geq \frac{n}{e^F}, \quad \text{tr}_{\chi'}\omega = ne^F. \tag{2.4}$$

The second assertion follows from (2.2), while the first inequality is obtained as follows. We choose a normal coordinate system so that

$$g_{i\bar{j}} = \delta_{ij}, \quad \chi'_{i\bar{j}} = \lambda'_i \delta_{ij}$$

for some $\lambda'_1, \dots, \lambda'_n > 0$. Donaldson’s equation then yields

$$ne^F = \sum_{1 \leq i \leq n} \frac{1}{\lambda'_i}.$$

An elementary inequality shows that

$$\text{tr}_\omega \chi' = \sum_{1 \leq i \leq n} \lambda'_i \geq \frac{n^2}{\sum_{1 \leq i \leq n} \frac{1}{\lambda'_i}} = \frac{n^2}{ne^F} = \frac{n}{e^F}.$$

We will frequently use the following

Lemma 2.2 (Guan-Li [10]) *At any point $p \in X$ there exists a holomorphic coordinates system centered at p such that, at p ,*

$$g_{i\bar{j}} = \delta_{ij}, \quad \partial_j g_{i\bar{i}} = 0 \tag{2.5}$$

for all i and j . Furthermore, we can assume that $\chi'_{i\bar{j}}$ is diagonal.

Let $\tilde{\Delta}$ denote the Laplacian operator associated to the Hermitian metric $h_{i\bar{j}}$ whose inverse matrix is given by

$$h^{i\bar{j}} := \chi'^{i\bar{\ell}} \chi'^{k\bar{j}} g_{k\bar{\ell}}; \tag{2.6}$$

and $\tilde{\nabla}$ the associated covariant derivatives.

The basic idea to obtain the second order estimate, following from the method of Yau [24], is to consider the quantity

$$Q := \log(\text{tr}_\omega \chi') - A\varphi \tag{2.7}$$

for some suitable constant A . Our first step is to estimate the term $\tilde{\Delta} \log(\text{tr}_\omega \chi')$.

Definition 2.3 For convenience, we say that a term E is of **type I** if

$$|E|_\omega \lesssim 1, \tag{2.8}$$

and is of **type II** if

$$|E|_\omega \lesssim \text{tr}_\omega \chi'. \tag{2.9}$$

It is easy to see that any uniform constant is of type I and any type I term is of type II. We will use E_1 and E_2 to denote a type I and type II term, respectively.

2.2 The estimate for $\tilde{\Delta} \log(\text{tr}_\omega \chi')$

Direct computation shows

$$\tilde{\Delta} \log(\text{tr}_\omega \chi') = \frac{\tilde{\Delta} \text{tr}_\omega \chi'}{\text{tr}_\omega \chi'} - \frac{|\tilde{\nabla} \text{tr}_\omega \chi'|_h^2}{(\text{tr}_\omega \chi')^2}. \tag{2.10}$$

By the definition, we have

$$\begin{aligned} \tilde{\Delta} \text{tr}_\omega \chi' &= h^{i\bar{j}} \partial_i \partial_{\bar{j}} (g^{k\bar{\ell}} \chi'_{k\bar{\ell}}) \\ &= h^{i\bar{j}} \partial_i \left(-g^{k\bar{b}} g^{a\bar{\ell}} \partial_{\bar{j}} g_{a\bar{b}} \cdot \chi'_{k\bar{\ell}} + g^{k\bar{\ell}} \partial_{\bar{j}} \chi'_{k\bar{\ell}} \right) \\ &= h^{i\bar{j}} \left[g^{k\bar{\ell}} \partial_i \partial_{\bar{j}} \chi'_{k\bar{\ell}} - g^{k\bar{b}} g^{a\bar{\ell}} \partial_i g_{a\bar{b}} \cdot \partial_{\bar{j}} \chi'_{k\bar{\ell}} - g^{k\bar{b}} g^{a\bar{\ell}} \partial_{\bar{j}} g_{a\bar{b}} \cdot \partial_i \chi'_{k\bar{\ell}} \right. \\ &\quad \left. - \left(-g^{k\bar{q}} g^{p\bar{b}} \partial_i g_{p\bar{q}} \cdot g^{a\bar{\ell}} \partial_{\bar{j}} g_{a\bar{b}} - g^{k\bar{b}} g^{a\bar{q}} g^{p\bar{\ell}} \partial_i g_{p\bar{q}} \cdot \partial_{\bar{j}} g_{a\bar{b}} \right. \right. \\ &\quad \left. \left. + g^{k\bar{b}} g^{a\bar{\ell}} \partial_i \partial_{\bar{j}} g_{a\bar{b}} \right) \chi'_{k\bar{\ell}} \right]. \end{aligned}$$

Using the local coordinates in Lemma 2.2, we deduce that

$$\begin{aligned} \tilde{\Delta} \text{tr}_\omega \chi' &= \sum_{1 \leq i, j \leq n} h^{i\bar{i}} \partial_i \partial_{\bar{i}} \chi'_{k\bar{k}} - \sum_{1 \leq i, k, \ell \leq n} h^{i\bar{i}} \partial_i g_{\ell\bar{k}} \cdot \partial_{\bar{i}} \chi'_{k\bar{\ell}} \\ &\quad - \sum_{1 \leq i, k, \ell \leq n} h^{i\bar{i}} \partial_{\bar{i}} g_{\ell\bar{k}} \cdot \partial_i \chi'_{k\bar{\ell}} + \sum_{1 \leq i, k, p \leq n} h^{i\bar{i}} \partial_i g_{p\bar{k}} \cdot \partial_{\bar{i}} g_{k\bar{p}} \cdot \chi'_{k\bar{k}} \\ &\quad + \sum_{1 \leq i, k, q \leq n} h^{i\bar{i}} \partial_i g_{k\bar{q}} \cdot \partial_{\bar{i}} g_{q\bar{k}} \cdot \chi'_{k\bar{k}} - \sum_{1 \leq i, k \leq n} h^{i\bar{i}} \partial_i \partial_{\bar{i}} g_{k\bar{k}} \cdot \chi'_{k\bar{k}} \\ &= \sum_{1 \leq i, k \leq n} h^{i\bar{i}} \partial_i \partial_{\bar{i}} \chi'_{k\bar{k}} - 2 \cdot \text{Re} \left(\sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \partial_{\bar{i}} g_{j\bar{k}} \cdot \partial_i \chi'_{k\bar{j}} \right) + E_1, \tag{2.11} \end{aligned}$$

where

$$\begin{aligned} E_1 &= \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \partial_i g_{j\bar{k}} \cdot \partial_{\bar{i}} g_{k\bar{j}} \cdot \chi'_{k\bar{k}} + \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \partial_i g_{k\bar{j}} \cdot \partial_{\bar{i}} g_{j\bar{k}} \cdot \chi'_{k\bar{k}} \\ &\quad - \sum_{1 \leq i, k \leq n} h^{i\bar{i}} \partial_i \partial_{\bar{i}} g_{k\bar{k}} \cdot \chi'_{k\bar{k}}. \end{aligned}$$

Since under the above mentioned local coordinates $\chi'_{i\bar{i}} = \lambda'_i \delta_{ij}$, it follows that $h^{i\bar{i}} = (\chi^{i\bar{i}})^2 = 1/\lambda_i'^2$; hence $h^{i\bar{i}} \leq e^{2F}$ using Remark 2.1. Therefore we see that E_1 is of type II, i.e.,

$$|E_1|_\omega \lesssim \text{tr}_\omega \chi'. \tag{2.12}$$

The first term on the right hand side of (2.11) can be computed as follows: From Donaldson’s equation (1.7), we obtain

$$ne^F = \text{tr}_{\chi'} \omega = \chi^{i\bar{j}} g_{i\bar{j}}$$

and, after taking the derivative with respect to $z^{\bar{\ell}}$,

$$n \partial_{\bar{\ell}} F \cdot e^F = -\chi^{i\bar{b}} \chi'^{a\bar{j}} \partial_{\bar{\ell}} \chi'_{a\bar{b}} \cdot g_{i\bar{j}} + \chi^{i\bar{j}} \partial_{\bar{\ell}} g_{i\bar{j}}.$$

Differentiating above equation again with respect to z^k yields

$$\begin{aligned} n\partial_k\partial_{\bar{\ell}}F \cdot e^F + n\partial_{\bar{\ell}}F\partial_kF \cdot e^F &= -\chi^{i\bar{i}}\chi^{a\bar{j}}g_{i\bar{j}}\partial_k\partial_{\bar{\ell}}\chi'_{a\bar{b}} - \chi^{i\bar{i}}\chi^{a\bar{j}}\partial_{\bar{\ell}}\chi'_{a\bar{b}}\partial_kg_{i\bar{j}} \\ &\quad - \left(-\chi^{i\bar{i}}\chi^{p\bar{q}}\partial_k\chi'_{p\bar{q}} \cdot \chi^{a\bar{j}}g_{i\bar{j}} - \chi^{i\bar{i}}\chi^{a\bar{q}}\chi^{p\bar{j}}\partial_k\chi'_{p\bar{q}} \cdot g_{i\bar{j}}\right)\partial_{\bar{\ell}}\chi'_{a\bar{b}} \\ &\quad - \chi^{i\bar{i}}\chi^{a\bar{j}}\partial_k\chi'_{a\bar{b}} \cdot \partial_{\bar{\ell}}g_{i\bar{j}} + \chi^{i\bar{j}}\partial_k\partial_{\bar{\ell}}g_{i\bar{j}} \\ &= -\chi^{i\bar{i}}\chi^{a\bar{j}}g_{i\bar{j}}\partial_k\partial_{\bar{\ell}}\chi'_{a\bar{b}} - \chi^{i\bar{i}}\chi^{a\bar{j}}\partial_k\chi'_{a\bar{b}} \cdot \partial_{\bar{\ell}}g_{i\bar{j}} + \chi^{i\bar{j}}\partial_k\partial_{\bar{\ell}}g_{i\bar{j}} \\ &\quad - \left(-\chi^{i\bar{i}}\chi^{p\bar{q}}\partial_k\chi'_{p\bar{q}} \cdot \chi^{a\bar{j}}g_{i\bar{j}} - \chi^{i\bar{i}}\chi^{a\bar{q}}\partial_k\chi'_{p\bar{q}} \cdot g_{i\bar{j}} + \chi^{i\bar{i}}\chi^{a\bar{j}}\partial_kg_{i\bar{j}}\right)\partial_{\bar{\ell}}\chi'_{a\bar{b}}. \end{aligned}$$

Multiplying above by $g^{k\bar{\ell}}$ on both sides implies

$$\begin{aligned} (\Delta_\omega F + |\nabla F|_\omega^2)ne^F &= -\sum_{1\leq i,j,k,\ell\leq n} \left(h^{i\bar{j}}g^{k\bar{\ell}}\partial_k\partial_{\bar{\ell}}\chi'_{i\bar{j}} - \chi^{i\bar{j}}g^{k\bar{\ell}}\partial_k\partial_{\bar{\ell}}g_{i\bar{j}}\right) \\ &\quad - \sum_{1\leq i,j,k,\ell,a,b\leq n} \chi^{i\bar{i}}\chi^{a\bar{j}}g^{k\bar{\ell}}\partial_k\chi'_{a\bar{b}} \cdot \partial_{\bar{\ell}}g_{i\bar{j}} + \sum_{1\leq i,j,k,\ell,p,q\leq n} h^{i\bar{q}}\chi^{p\bar{j}}g^{k\bar{\ell}}\partial_k\chi'_{p\bar{q}} \cdot \partial_{\bar{\ell}}\chi'_{i\bar{j}} \\ &\quad + \sum_{1\leq i,j,k,\ell,p,q\leq n} h^{p\bar{j}}\chi^{i\bar{q}}g^{k\bar{\ell}}\partial_k\chi'_{p\bar{q}} \cdot \partial_{\bar{\ell}}\chi'_{i\bar{j}} - \sum_{1\leq i,j,k,\ell,a,b\leq n} \chi^{i\bar{i}}\chi^{a\bar{j}}g^{k\bar{\ell}}\partial_kg_{i\bar{j}} \cdot \partial_{\bar{\ell}}\chi'_{a\bar{b}}. \end{aligned}$$

Using the local coordinates (2.5) we arrive at

$$\begin{aligned} (\Delta_\omega F + |\nabla F|_\omega^2)ne^F &= -\sum_{1\leq i,k\leq n} h^{i\bar{i}}\partial_k\partial_{\bar{k}}\chi'_{i\bar{i}} + \sum_{1\leq i,k\leq n} \chi^{i\bar{i}}\partial_k\partial_{\bar{k}}g_{i\bar{i}} + \sum_{1\leq i,j,k\leq n} h^{i\bar{i}}\chi^{j\bar{j}}\partial_k\chi'_{j\bar{j}} \cdot \partial_{\bar{k}}\chi'_{i\bar{j}} \\ &\quad + \sum_{1\leq i,j,k\leq n} h^{i\bar{i}}\chi^{j\bar{j}}\partial_k\chi'_{i\bar{j}} \cdot \partial_{\bar{k}}\chi'_{j\bar{i}} - 2 \cdot \operatorname{Re} \left(\sum_{1\leq i,j,k\leq n} \chi^{i\bar{i}}\chi^{j\bar{j}}\partial_kg_{i\bar{j}} \cdot \partial_{\bar{k}}\chi'_{j\bar{i}} \right). \end{aligned}$$

Equivalently,

$$\begin{aligned} \sum_{1\leq i,k\leq n} h^{i\bar{i}}\partial_k\partial_{\bar{k}}\chi'_{i\bar{i}} &= \sum_{1\leq i,j,k\leq n} h^{i\bar{i}}\chi^{j\bar{j}}\partial_k\chi'_{j\bar{i}}\partial_{\bar{k}}\chi'_{i\bar{j}} + \sum_{1\leq i,j,k\leq n} h^{i\bar{i}}\chi^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}} \\ &\quad - 2 \cdot \operatorname{Re} \left(\sum_{1\leq i,j,k\leq n} \chi^{i\bar{i}}\chi^{j\bar{j}}\partial_kg_{i\bar{j}} \cdot \partial_{\bar{k}}\chi'_{j\bar{i}} \right) \\ &\quad + \sum_{1\leq i,k\leq n} \chi^{i\bar{i}}\partial_k\partial_{\bar{k}}g_{i\bar{i}} - (\Delta_\omega F + |\nabla F|_\omega^2)ne^F. \end{aligned} \tag{2.13}$$

Since

$$\begin{aligned} \partial_k\partial_{\bar{k}}\chi'_{i\bar{i}} &= \partial_k\partial_{\bar{k}}(X_{i\bar{i}} + \varphi_{i\bar{i}}) \\ &= \partial_k\partial_{\bar{k}}X_{i\bar{i}} + \partial_k\partial_{\bar{k}}\varphi_{i\bar{i}} \\ &= \partial_k\partial_{\bar{k}}X_{i\bar{i}} + \partial_i\partial_{\bar{i}}\varphi_{k\bar{k}} \\ &= \partial_k\partial_{\bar{k}}X_{i\bar{i}} + \partial_i\partial_{\bar{i}}(X'_{k\bar{k}} - X_{k\bar{k}}) \\ &= \partial_i\partial_{\bar{i}}X'_{k\bar{k}} + (\partial_k\partial_{\bar{k}}X_{i\bar{i}} - \partial_i\partial_{\bar{i}}X_{k\bar{k}}), \end{aligned}$$

we conclude that

$$\sum_{1 \leq i, k \leq n} h^{i\bar{i}} \partial_i \partial_{\bar{i}} \chi'_{k\bar{k}} = \sum_{1 \leq i, k \leq n} h^{i\bar{i}} \partial_k \partial_{\bar{k}} \chi'_{i\bar{i}} + \sum_{1 \leq i, k \leq n} h^{i\bar{i}} (\partial_i \partial_{\bar{i}} \chi_{k\bar{k}} - \partial_k \partial_{\bar{k}} \chi_{i\bar{i}}). \tag{2.14}$$

Combining (2.13) and (2.14) yields

$$\begin{aligned} \sum_{1 \leq i, k \leq n} h^{i\bar{i}} \partial_i \partial_{\bar{i}} \chi'_{k\bar{k}} &= \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{j\bar{j}} \cdot \partial_{\bar{k}} \chi'_{i\bar{j}} \\ &+ \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{i\bar{j}} \partial_{\bar{k}} \chi'_{j\bar{j}} \\ &- 2 \cdot \operatorname{Re} \left(\sum_{1 \leq i, j, k \leq n} \chi'^{i\bar{i}} \chi'^{j\bar{j}} \partial_k g_{j\bar{i}} \cdot \partial_{\bar{k}} \chi'_{i\bar{j}} \right) + E_2, \end{aligned} \tag{2.15}$$

where

$$E_2 = \sum_{1 \leq i, k \leq n} \chi'^{i\bar{i}} \partial_k \partial_{\bar{k}} g_{i\bar{i}} + \sum_{1 \leq i, k \leq n} h^{i\bar{i}} (\partial_i \partial_{\bar{i}} \chi_{k\bar{k}} - \partial_k \partial_{\bar{k}} \chi_{i\bar{i}}) - (\Delta_\omega F + |\nabla F|_\omega^2) ne^F.$$

By the same reason that $\chi'^{i\bar{i}} \leq e^F$ and $h^{i\bar{i}} \leq e^{2F}$, we observe that E_2 is of type I and

$$|E_2|_\omega \lesssim 1. \tag{2.16}$$

From (2.11) and (2.15), we get

$$\begin{aligned} \tilde{\Delta} \operatorname{tr}_\omega \chi' &= \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{j\bar{j}} \partial_{\bar{k}} \chi'_{i\bar{j}} + \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{i\bar{j}} \partial_{\bar{k}} \chi'_{j\bar{j}} \\ &- 2 \cdot \operatorname{Re} \left(\sum_{1 \leq i, j, k \leq n} \chi'^{i\bar{i}} \chi'^{j\bar{j}} \partial_k g_{j\bar{i}} \partial_{\bar{k}} \chi'_{i\bar{j}} \right) \\ &- 2 \cdot \operatorname{Re} \left(\sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \partial_i g_{j\bar{k}} \partial_i \chi'_{k\bar{j}} \right) + E_1 + E_2 \\ &= \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{j\bar{j}} \partial_{\bar{k}} \chi'_{i\bar{j}} + \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{i\bar{j}} \partial_{\bar{k}} \chi'_{j\bar{j}} \\ &- 2 \cdot \operatorname{Re} \left(\sum_{1 \leq i, j, k \leq n} \chi'^{i\bar{i}} \chi'^{j\bar{j}} \partial_k g_{j\bar{i}} \partial_{\bar{k}} \chi'_{i\bar{j}} \right) \\ &- 2 \cdot \operatorname{Re} \left(\sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \partial_i g_{j\bar{k}} \partial_i \chi'_{k\bar{j}} \right) + E_2, \end{aligned}$$

since any type I term is also of type II.

2.3 The estimate for $\tilde{\Delta} \log(\operatorname{tr}_\omega \chi')$, continued: ω is Kähler

In the case that ω is Kähler, we in addition have $\partial_k g_{ij} = 0$ for any i, j, k in Lemma 2.2, and we deduce from the above equation that

$$\tilde{\Delta} \operatorname{tr}_\omega \chi' = \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{j\bar{j}} \partial_{\bar{k}} \chi'_{i\bar{j}} + \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{i\bar{j}} \partial_{\bar{k}} \chi'_{j\bar{j}} + E_2. \tag{2.17}$$

It remains to control the term $|\widetilde{\nabla} \text{tr}_\omega \chi'|_h^2 / (\text{tr}_\omega \chi')^2$. Notice that

$$\partial_i (\text{tr}_\omega \chi') = \partial_i (g^{k\bar{\ell}} \chi'_{k\bar{\ell}}) = g^{k\bar{\ell}} \partial_i \chi'_{k\bar{\ell}} = \sum_{1 \leq k \leq n} \partial_i \chi'_{k\bar{k}}$$

As in [17], we first give an inequality for $|\widetilde{\nabla} \text{tr}_\omega \chi'|_h^2 / \text{tr}_\omega \chi'$ and then we control the term $\text{Re} \left(\sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_i \chi'_{j\bar{j}} (\partial_i \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}}) \right)$. From

$$\begin{aligned} \frac{|\widetilde{\nabla} \text{tr}_\omega \chi'|_h^2}{\text{tr}_\omega \chi'} &= \sum_{1 \leq i, j, k \leq n} \frac{h^{i\bar{i}} \partial_i \chi'_{j\bar{j}} \partial_{\bar{i}} \chi'_{k\bar{k}}}{\text{tr}_\omega \chi'} = \sum_{1 \leq j, k, i \leq n} \frac{\sqrt{h^{i\bar{i}}} \partial_i \chi'_{j\bar{j}} \sqrt{h^{i\bar{i}}} \partial_{\bar{i}} \chi'_{k\bar{k}}}{\text{tr}_\omega \chi'} \\ &\leq \frac{1}{\text{tr}_\omega \chi'} \sum_{1 \leq j, k \leq n} \left(\sum_{1 \leq i \leq n} h^{i\bar{i}} |\partial_i \chi'_{j\bar{j}}|^2 \right)^{1/2} \left(\sum_{1 \leq i \leq n} h^{i\bar{i}} |\partial_i \chi'_{k\bar{k}}|^2 \right)^{1/2} \\ &= \frac{1}{\text{tr}_\omega \chi'} \left[\sum_{1 \leq j \leq n} \left(\sum_{1 \leq i \leq n} h^{i\bar{i}} |\partial_i \chi'_{j\bar{j}}|^2 \right)^{1/2} \right]^2 \\ &= \frac{1}{\text{tr}_\omega \chi'} \left[\sum_{1 \leq j \leq n} \sqrt{\chi'_{j\bar{j}}} \left(\sum_{1 \leq i \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} |\partial_i \chi'_{j\bar{j}}|^2 \right)^{1/2} \right]^2 \\ &\leq \sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} |\partial_i \chi'_{j\bar{j}}|^2 = \sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_i \chi'_{j\bar{j}} \partial_{\bar{i}} \chi'_{j\bar{j}}. \end{aligned}$$

From

$$\begin{aligned} \partial_i \chi'_{j\bar{j}} &= \partial_i (\chi_{j\bar{j}} + \varphi_{j\bar{j}}) = \partial_i \chi_{j\bar{j}} + \partial_j \varphi_{i\bar{j}} = \partial_i \chi_{j\bar{j}} - \partial_j \chi_{i\bar{j}} + \partial_j \chi'_{i\bar{j}}, \\ \partial_{\bar{i}} \chi'_{j\bar{j}} &= \partial_{\bar{i}} (\chi_{j\bar{j}} + \varphi_{j\bar{j}}) = \partial_{\bar{i}} \chi_{j\bar{j}} + \partial_{\bar{j}} \varphi_{j\bar{i}} = \partial_{\bar{i}} \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}} + \partial_{\bar{j}} \chi'_{j\bar{i}}, \end{aligned}$$

it follows that

$$\begin{aligned} \frac{|\widetilde{\nabla} \text{tr}_\omega \chi'|_h^2}{\text{tr}_\omega \chi'} &\leq \sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} (\partial_j \chi'_{i\bar{j}} + \partial_i \chi_{j\bar{j}} - \partial_j \chi_{i\bar{j}}) (\partial_{\bar{j}} \chi'_{j\bar{i}} + \partial_{\bar{i}} \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}}) \\ &= \sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_j \chi'_{i\bar{j}} \partial_{\bar{j}} \chi'_{j\bar{i}} + \sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} |\partial_i \chi_{j\bar{j}} - \partial_j \chi_{i\bar{j}}|^2 \\ &\quad + 2 \cdot \text{Re} \left[\sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_j \chi'_{i\bar{j}} (\partial_{\bar{i}} \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}}) \right]. \end{aligned} \tag{2.18}$$

Note that

$$\partial_j \chi'_{i\bar{j}} = \partial_j (\chi_{i\bar{j}} + \varphi_{i\bar{j}}) = \partial_j \chi_{i\bar{j}} + \partial_i \varphi_{j\bar{j}} = \partial_j \chi_{i\bar{j}} - \partial_i \chi_{j\bar{j}} + \partial_i \chi'_{j\bar{j}}. \tag{2.19}$$

Substituting (2.19) into (2.18) we obtain

$$\begin{aligned} \frac{|\tilde{\nabla} \operatorname{tr}_\omega \chi'|_h^2}{\operatorname{tr}_\omega \chi'} &\leq \sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_j \chi'_{i\bar{j}} \partial_{\bar{j}} \chi'_{j\bar{i}} - \sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \left| \partial_i \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{i\bar{j}} \right|^2 \\ &\quad + 2 \cdot \operatorname{Re} \left[\sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_i \chi'_{j\bar{j}} \left(\partial_{\bar{i}} \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}} \right) \right] \\ &\leq \sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_j \chi'_{i\bar{j}} \partial_{\bar{j}} \chi'_{j\bar{i}} + 2 \cdot \operatorname{Re} \left[\sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_i \chi'_{j\bar{j}} \left(\partial_{\bar{i}} \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}} \right) \right]. \end{aligned} \tag{2.20}$$

Lemma 2.4 *If ω is Kähler, then $\tilde{\Delta} \log(\operatorname{tr}_\omega \chi') \gtrsim -1$.*

Proof Calculate, since $h^{j\bar{j}} = \chi'^{j\bar{j}} \chi'^{j\bar{j}}$,

$$\begin{aligned} &\left| 2 \cdot \operatorname{Re} \left[\sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_i \chi'_{j\bar{j}} \left(\partial_{\bar{i}} \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}} \right) \right] \right| \\ &= \left| 2 \cdot \operatorname{Re} \left[\sum_{1 \leq i, j \leq n} \sqrt{h^{j\bar{j}}} \sqrt{\chi'^{j\bar{j}}} \partial_i \chi'_{j\bar{j}} \cdot \sqrt{\chi'_{j\bar{j}}} h^{i\bar{i}} \left(\partial_{\bar{i}} \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}} \right) \right] \right| \\ &\leq \sum_{1 \leq i, j \leq n} h^{j\bar{j}} \chi'^{j\bar{j}} \partial_i \chi'_{j\bar{j}} \partial_{\bar{i}} \chi'_{j\bar{j}} + \sum_{1 \leq i, j \leq n} \chi'_{j\bar{j}} (h^{i\bar{i}})^2 |\partial_{\bar{i}} \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}}|^2 \\ &\leq \sum_{1 \leq i, j, k \leq n} h^{k\bar{k}} \chi'^{j\bar{j}} \partial_i \chi'_{j\bar{k}} \partial_{\bar{i}} \chi'_{k\bar{j}} + E_2 \\ &= \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{j\bar{i}} \partial_{\bar{k}} \chi'_{i\bar{j}} + E_2, \end{aligned} \tag{2.21}$$

where E_2 is a term of type II:

$$E_2 = \sum_{1 \leq i, j \leq n} \chi'_{j\bar{j}} (h^{i\bar{i}})^2 |\partial_{\bar{i}} \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}}|^2.$$

From (2.10), (2.17), (2.20), and (2.21), we have

$$\begin{aligned} \tilde{\Delta} \log(\operatorname{tr}_\omega \chi') &\geq \frac{1}{\operatorname{tr}_\omega \chi'} \left[\sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{i\bar{j}} \partial_{\bar{k}} \chi'_{j\bar{i}} + E_2 - \sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_j \chi'_{i\bar{j}} \partial_{\bar{j}} \chi'_{j\bar{i}} \right] \\ &= \frac{1}{\operatorname{tr}_\omega \chi'} \left(\sum_{1 \leq i \leq n} \sum_{1 \leq j \neq k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{i\bar{j}} \partial_{\bar{k}} \chi'_{j\bar{i}} + E_2 \right) \\ &= \frac{1}{\operatorname{tr}_\omega \chi'} \left(\sum_{1 \leq i \leq n} \sum_{1 \leq j \neq k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \left| \partial_k \chi'_{i\bar{j}} \right|^2 + E_2 \right) \\ &\geq \frac{E_2}{\operatorname{tr}_\omega \chi'}. \end{aligned} \tag{2.22}$$

By the definition of type II terms, there exists a positive universal constant C satisfying $|E_2|_\omega \leq C \cdot \text{tr}_\omega \chi'$. Therefore

$$\tilde{\Delta} \log(\text{tr}_\omega \chi') \gtrsim -1.$$

Thus we complete the proof of the lemma. □

Theorem 2.5 *Let (X, ω) be a compact Kähler manifold of complex dimension n , and χ a Hermitian metric. Let φ be a smooth solution of Donaldson's equation*

$$\omega \wedge \chi_\varphi^{n-1} = e^F \chi_\varphi^n$$

where F is a smooth function on X . Assume that

$$\chi - \frac{n-1}{ne^F} \omega > 0.$$

Then there are uniform constants $A > 0$ and $C > 0$, depending only on X, ω, χ , and F , such that

$$\text{tr}_\omega \chi_\varphi \leq C \cdot e^{A(\varphi - \inf_X \varphi)}.$$

Proof Use the local coordinates in Lemma 2.2. The proof is similar to that in [22,23]. By the definition, one has

$$\tilde{\Delta} \varphi = h^{k\bar{k}} \varphi_{k\bar{k}} = (\chi'^{k\bar{k}})^2 (\chi'_{k\bar{k}} - \chi_{k\bar{k}}) = \sum_{1 \leq k \leq n} \chi'^{k\bar{k}} - \text{tr}_h \chi = \text{tr}_{\chi'} \omega - \text{tr}_h \chi.$$

Lemma 2.4 and (2.7) imply that

$$\begin{aligned} \tilde{\Delta} Q &= \tilde{\Delta} [\log(\text{tr}_\omega \chi') - A\varphi] \geq -C - A (\text{tr}_{\chi'} \omega - \text{tr}_h \chi) \\ &\geq -C - A \sum_{1 \leq i \leq n} \chi'^{i\bar{i}} + A \sum_{1 \leq i \leq n} \chi'^{i\bar{i}} \chi'^{i\bar{i}} \chi_{i\bar{i}}. \end{aligned}$$

Since φ is a solution of Donaldson's equation, it follows that $\text{tr}_{\chi'} \omega = ne^F$ by (2.7) and hence, for any given positive uniform constants A and B (we will chose those constants later),

$$\tilde{\Delta} Q \geq (Bne^F - C) - (A + B) \sum_{1 \leq i \leq n} \chi'^{i\bar{i}} + A \sum_{1 \leq i \leq n} \chi'^{i\bar{i}} \chi'^{i\bar{i}} \chi_{i\bar{i}}.$$

By the assumption we have $\chi \geq \frac{n-1}{ne^F} (1 + \epsilon) \omega$ for some suitable number ϵ such that $0 < \epsilon < \frac{1}{n-1}$. Let $p \in X$ be a point where Q achieves its maximum; so $\tilde{\Delta} Q \leq 0$. At this point, we conclude that

$$\begin{aligned} 0 &\geq (Bne^F - C) - (A + B) \sum_{1 \leq i \leq n} \chi'^{i\bar{i}} + A \sum_{1 \leq i \leq n} \chi'^{i\bar{i}} \chi'^{i\bar{i}} \chi_{i\bar{i}} \\ &\geq (Bne^F - C) - (A + B) \sum_{1 \leq i \leq n} \chi'^{i\bar{i}} + A \frac{n-1}{ne^F} (1 + \epsilon) \sum_{1 \leq i \leq n} \chi'^{i\bar{i}} \chi'^{i\bar{i}}. \end{aligned}$$

We denote by λ'_i the eigenvalues of χ' at point p such that $\lambda'_1 \leq \dots \leq \lambda'_n$. Hence

$$0 \geq (Bne^F - C) - (A + B) \sum_{1 \leq i \leq n} \frac{1}{\lambda'_i} + A \frac{n-1}{ne^F} (1 + \epsilon) \sum_{1 \leq i \leq n} \frac{1}{\lambda_i^2}.$$

In order to obtain the upper bound for λ'_i we need the following □

Lemma 2.6 *Let $\lambda_1, \dots, \lambda_n$ be a sequence of positive numbers. Suppose*

$$0 \geq 1 - \alpha \sum_{1 \leq i \leq n} \frac{1}{\lambda_i} + \beta \sum_{1 \leq i \leq n} \frac{1}{\lambda_i^2}$$

for some $\alpha, \beta > 0$ and $n \geq 2$. If

$$\frac{4}{n} \leq \frac{\alpha^2}{\beta} < \frac{4}{n-1} \tag{2.23}$$

holds, then

$$\lambda_i \leq \frac{2\beta}{\alpha - \sqrt{n\alpha^2 - 4\beta}} \tag{2.24}$$

for each i .

Proof Note that $\alpha - \sqrt{n\alpha^2 - 4\beta} > 0$ by (2.23). Since

$$1 + \sum_{1 \leq i \leq n} \left(\frac{\alpha}{2\sqrt{\beta}} - \frac{\sqrt{\beta}}{\lambda_i} \right)^2 \leq \frac{n\alpha^2}{4\beta}$$

it implies that

$$\sum_{1 \leq i \leq n} \left(\frac{\alpha}{2\sqrt{\beta}} - \frac{\sqrt{\beta}}{\lambda_i} \right)^2 \leq \frac{n\alpha^2 - 4\beta}{4\beta}.$$

The right hand side of the above inequality is nonnegative by (2.23). Consequently,

$$\frac{\alpha}{2\sqrt{\beta}} - \frac{\sqrt{\beta}}{\lambda_i} \leq \sqrt{\frac{n\alpha^2 - 4\beta}{4\beta}}$$

and then

$$\frac{\alpha - \sqrt{n\alpha^2 - 4\beta}}{2\sqrt{\beta}} \leq \frac{\sqrt{\beta}}{\lambda_i}.$$

Hence we obtain (2.24). □

To apply Lemma 2.6, we assume

$$Bne^F > C, \tag{2.25}$$

and set

$$\alpha \doteq \frac{A + B}{Bne^F - C}, \quad \beta \doteq \frac{A \frac{n-1}{ne^F} (1 + \epsilon)}{Bne^F - C}. \tag{2.26}$$

In the following we will find the explicit formulas for A and B in terms of C such that the assumption (2.25) and the condition (2.23) are both satisfied.

We choose a real number η satisfying

$$0 \leq \eta < 1. \tag{2.27}$$

Set

$$\frac{\alpha^2}{\beta} = \frac{4}{n - \eta}, \tag{2.28}$$

where α and β are given in (2.26). If (2.28) was valid, then the condition (2.23) is true. Equations (2.26) and (2.28) imply

$$(A + B)^2 = \frac{4}{n - \eta}(1 + \epsilon) \left(Bne^F - C \right) \frac{n - 1}{ne^F} A$$

so that

$$A^2 + B^2 + 2 \left[1 - \frac{2(1 + \epsilon)(n - 1)}{n - \eta} \right] AB + \frac{4(1 + \epsilon)(n - 1)C}{(n - \eta)ne^F} A = 0.$$

The above relation can be rewritten as

$$\left[A + \left(1 - \frac{2(1 + \epsilon)(n - 1)}{n - \eta} \right) B \right]^2 = \left[\left(1 - \frac{2(1 + \epsilon)(n - 1)}{n - \eta} \right)^2 - 1 \right] B^2 - \frac{4(1 + \epsilon)(n - 1)C}{(n - \eta)ne^F} A.$$

Taking

$$A = \left(-1 + \frac{2(1 + \epsilon)(n - 1)}{n - \eta} \right) B \tag{2.29}$$

we have $A > B$ and

$$B = \frac{\frac{4(1 + \epsilon)(n - 1)C}{(n - \eta)ne^F} \left(-1 + \frac{2(1 + \epsilon)(n - 1)}{n - \eta} \right)}{\left(1 - \frac{2(1 + \epsilon)(n - 1)}{n - \eta} \right)^2 - 1} = \frac{C}{ne^F} \cdot \frac{-(n - \eta) + 2(1 + \epsilon)(n - 1)}{-(n - \eta) + (1 + \epsilon)(n - 1)}, \tag{2.30}$$

assuming

$$(1 + \epsilon) > \frac{n - \eta}{n - 1}. \tag{2.31}$$

From (2.30) and (2.31) we see that

$$\frac{Bne^F}{C} = \frac{-(n - \eta) + 2(1 + \epsilon)(n - 1)}{-(n - \eta) + (1 + \epsilon)(n - 1)} > 1.$$

From the assumption $0 < \epsilon < \frac{1}{n - 1}$ we have $0 < n - (n - 1)(1 + \epsilon) < 1$ and then such a η always exists. Hence Lemma 2.6 yields

$$\lambda'_i \leq \frac{2\beta}{\alpha - \sqrt{n\alpha^2 - 4\beta}}$$

where α and β are determined by (2.26), (2.29), and (2.30). Since $\text{tr}_\omega \chi' = \sum_{i=1}^n \lambda'_i$, it follows that, at $p \in X$, $\text{tr}_\omega \chi' \leq C$ for some uniform constant C and, for any point $q \in X$,

$$Q(q) \leq Q(p) = \log(\text{tr}_\omega \chi')(p) - A\varphi(p) \leq C - A \cdot \inf_X \varphi.$$

Equivalently, $\log(\text{tr}_\omega \chi') \leq C + A(\varphi - \inf_X \varphi)$.

2.4 The estimate for $\tilde{\Delta}\log(\text{tr}_\omega\chi')$, continued: general case

Now we consider the general case that both ω and χ may not be Kähler. Using Lemma 2.2 we have

$$\begin{aligned} \tilde{\Delta}\text{tr}_\omega\chi' &= \sum_{1\leq i,j,k\leq n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{j\bar{j}}\partial_{\bar{k}}\chi'_{i\bar{j}} + \sum_{1\leq i,j,k\leq n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}} + E_2 \\ &\quad - 2 \cdot \text{Re} \left(\sum_{1\leq i,j,k\leq n} \chi'^{i\bar{i}}\chi'^{j\bar{j}}\partial_k g_{j\bar{i}}\partial_{\bar{k}}\chi'_{i\bar{j}} \right) \\ &\quad - 2 \cdot \text{Re} \left(\sum_{1\leq i,j,k\leq n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}\partial_i\chi'_{k\bar{j}} \right). \end{aligned} \tag{2.32}$$

As in [17] we deal with the last two terms by using the local coordinates in Lemma 2.2. Starting from the last term, we calculate

$$\begin{aligned} \sum_{1\leq i,j,k\leq n} h^{i\bar{i}}\partial_i\chi'_{k\bar{j}}\partial_{\bar{i}}g_{j\bar{k}} &= \sum_{1\leq i,j,k\leq n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}\partial_i(\chi_{k\bar{j}} + \varphi_{k\bar{j}}) \\ &= \sum_{1\leq i,j,k\leq n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}(\partial_i\chi_{k\bar{j}} + \partial_k\varphi_{i\bar{j}}) \\ &= \sum_{1\leq i,j,k\leq n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}(\partial_i\chi_{k\bar{j}} + \partial_k\chi'_{i\bar{j}} - \partial_k\chi_{i\bar{j}}) \\ &= \sum_{i=1}^n \sum_{1\leq j\neq k\leq n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}\partial_k\chi'_{i\bar{j}} + E_1, \end{aligned} \tag{2.33}$$

where E_1 is a term of type I and is given by

$$E_1 = \sum_{1\leq i,j,k\leq n} h^{i\bar{i}}\partial_{\bar{i}}g_{j\bar{k}}(\partial_i\chi_{k\bar{j}} - \partial_k\chi_{i\bar{j}}). \tag{2.34}$$

Taking the real part of (2.33) gives

$$\begin{aligned} &\left| 2 \cdot \text{Re} \left(\sum_{1\leq i,j,k\leq n} h^{i\bar{i}}\partial_i\chi'_{k\bar{j}}\partial_{\bar{i}}g_{j\bar{k}} \right) \right| \\ &= \left| 2 \cdot \text{Re} \left(\sum_{1\leq i\leq n} \sum_{1\leq j\neq k\leq n} \sqrt{h^{i\bar{i}}}\sqrt{\chi'^{j\bar{j}}}\partial_k\chi'_{i\bar{j}} \cdot \sqrt{h^{i\bar{i}}}\sqrt{\chi'_{j\bar{j}}}\partial_{\bar{i}}g_{j\bar{k}} \right) \right| + E_1 \\ &\leq \sum_{1\leq i\leq n} \sum_{1\leq j\neq k\leq n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}} + \sum_{i=1}^n \sum_{1\leq j\neq k\leq n} h^{i\bar{i}}\chi'_{j\bar{j}}\partial_{\bar{i}}g_{j\bar{k}}\partial_i g_{k\bar{j}} + E_1 \\ &\leq \sum_{1\leq i\leq n} \sum_{1\leq j\neq k\leq n} h^{i\bar{i}}\chi'^{j\bar{j}}\partial_k\chi'_{i\bar{j}}\partial_{\bar{k}}\chi'_{j\bar{i}} + E_2, \end{aligned} \tag{2.35}$$

since $\sum_{i=1}^n \sum_{1 \leq j \neq k \leq n} h^{i\bar{i}} \chi'_{j\bar{j}} \partial_{\bar{i}} g_{j\bar{k}} \partial_i g_{k\bar{j}}$ is of type II. Similarly we have

$$\begin{aligned} & \left| 2 \cdot \operatorname{Re} \left(\sum_{1 \leq i, j, k \leq n} \chi^{i\bar{i}} \chi'^{j\bar{j}} \partial_{\bar{k}} \chi'_{j\bar{i}} \partial_k g_{i\bar{j}} \right) \right| \\ &= \left| 2 \cdot \operatorname{Re} \left(\sum_{1 \leq i, j, k \leq n} \sqrt{h^{j\bar{j}}} \sqrt{\chi^{i\bar{i}}} \partial_{\bar{k}} \chi'_{j\bar{i}} \cdot \sqrt{\chi^{i\bar{i}}} \partial_k g_{i\bar{j}} \right) \right| \\ &\leq \frac{1}{2} \sum_{1 \leq i, j, k \leq n} h^{j\bar{j}} \chi^{i\bar{i}} \partial_{\bar{k}} \chi'_{j\bar{i}} \partial_k \chi'_{i\bar{j}} + E_1 = \frac{1}{2} \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_{\bar{k}} \chi'_{i\bar{j}} \partial_k \chi'_{j\bar{i}} + E_1, \end{aligned} \tag{2.36}$$

where

$$E_1 = 2 \sum_{1 \leq i, j, k \leq n} \chi^{i\bar{i}} \partial_k g_{i\bar{j}} \partial_{\bar{k}} g_{j\bar{i}} \tag{2.37}$$

is a term of type I.

From (2.32), (2.35), and (2.36), we conclude that

$$\begin{aligned} \tilde{\Delta} \operatorname{tr}_\omega \chi' &\geq \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{j\bar{i}} \partial_{\bar{k}} \chi'_{i\bar{j}} + \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{i\bar{j}} \partial_{\bar{k}} \chi'_{j\bar{i}} \\ &\quad - \sum_{i=1}^n \sum_{1 \leq j \neq k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{i\bar{j}} \partial_{\bar{k}} \chi'_{j\bar{i}} - \frac{1}{2} \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{j\bar{i}} \partial_{\bar{k}} \chi'_{i\bar{j}} + E_2 \\ &= \frac{1}{2} \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{j\bar{i}} \partial_{\bar{k}} \chi'_{i\bar{j}} + \sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_j \chi'_{i\bar{j}} \partial_{\bar{j}} \chi'_{i\bar{j}} + E_2. \end{aligned} \tag{2.38}$$

It remains to control the term $|\tilde{\nabla} \operatorname{tr}_\omega \chi'|_h^2 / (\operatorname{tr}_\omega \chi')^2$. As in (2.20) one has

$$\begin{aligned} \frac{|\tilde{\nabla} \operatorname{tr}_\omega \chi'|_h^2}{\operatorname{tr}_\omega \chi'} &\leq \sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_j \chi'_{i\bar{j}} \partial_{\bar{j}} \chi'_{i\bar{j}} \\ &\quad + 2 \cdot \operatorname{Re} \left[\sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_i \chi'_{j\bar{j}} \left(\partial_{\bar{i}} \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}} \right) \right]. \end{aligned} \tag{2.39}$$

Lemma 2.7 *One has $\tilde{\Delta} \log(\operatorname{tr}_\omega \chi') \gtrsim -1$.*

Proof As in the proof of Lemma 2.4 we have

$$\begin{aligned} & \left| 2 \cdot \operatorname{Re} \left[\sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi^{i\bar{i}} \partial_i \chi'_{j\bar{j}} \left(\partial_{\bar{i}} \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}} \right) \right] \right| \\ &= \left| 2 \cdot \operatorname{Re} \left[\sum_{1 \leq i, j \leq n} \sqrt{h^{j\bar{j}}} \sqrt{\chi^{i\bar{i}}} \partial_i \chi'_{j\bar{j}} \cdot \sqrt{\chi'_{j\bar{j}}} h^{i\bar{i}} \left(\partial_i \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}} \right) \right] \right| \\ &\leq \frac{1}{2} \sum_{1 \leq i, j \leq n} h^{j\bar{j}} \chi'^{j\bar{j}} \partial_i \chi'_{j\bar{j}} \partial_{\bar{i}} \chi'_{j\bar{j}} + 2 \sum_{1 \leq i, j \leq n} \chi'_{j\bar{j}} (h^{i\bar{i}})^2 \left| \partial_{\bar{i}} \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}} \right|^2 \\ &\leq \frac{1}{2} \sum_{1 \leq i, j, k \leq n} h^{k\bar{k}} \chi'^{j\bar{j}} \partial_i \chi'_{j\bar{k}} \partial_{\bar{i}} \chi'_{k\bar{j}} + E_2 = \frac{1}{2} \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{j\bar{i}} \partial_{\bar{k}} \chi'_{i\bar{j}} + E_2, \end{aligned} \tag{2.40}$$

where E_2 is a term of type II and given by

$$E_2 = 2 \sum_{1 \leq i, j \leq n} \chi'_{j\bar{j}} (h^{i\bar{i}})^2 \left| \partial_{\bar{i}} \chi_{j\bar{j}} - \partial_{\bar{j}} \chi_{j\bar{i}} \right|^2.$$

Combining (2.40) with (2.10), (2.38), and (2.39), we arrive at

$$\begin{aligned} \tilde{\Delta} \log(\text{tr}_\omega \chi') &\geq \frac{1}{\text{tr}_\omega \chi'} \left[\frac{1}{2} \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{j\bar{i}} \partial_{\bar{k}} \chi'_{i\bar{j}} + \sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_j \chi'_{i\bar{j}} \partial_{\bar{j}} \chi'_{i\bar{j}} \right. \\ &\left. - \sum_{1 \leq i, j \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_j \chi'_{i\bar{j}} \partial_{\bar{j}} \chi'_{j\bar{i}} - \frac{1}{2} \sum_{1 \leq i, j, k \leq n} h^{i\bar{i}} \chi'^{j\bar{j}} \partial_k \chi'_{j\bar{i}} \partial_{\bar{k}} \chi'_{i\bar{j}} + E_2 \right] = \frac{E_2}{\text{tr}_\omega \chi'}. \end{aligned}$$

By the definition of type II terms, there exists a positive uniform constant C satisfying $|E_2|_\omega \leq C \cdot \text{tr}_\omega \chi'$. Therefore

$$\tilde{\Delta} \log(\text{tr}_\omega \chi') \geq -C.$$

This complete the proof. □

By using the similar method as in the proof of Theorem 2.5, we have

Theorem 2.8 *Let (X, ω) be a compact Hermitian manifold of the complex dimension n , and χ another Hermitian metric. Let φ be a smooth solution of Donaldson’s equation*

$$\omega \wedge \chi_\varphi^{n-1} = e^F \chi_\varphi^n,$$

where F is a smooth function on X . Assume that

$$\chi - \frac{n-1}{ne^F} \omega > 0.$$

Then there are uniform constants $A > 0$ and $C > 0$, depending only on X, ω, χ , and F , such that

$$\text{tr}_\omega \chi_\varphi \leq C \cdot e^{A(\varphi - \inf_X \varphi)}.$$

Acknowledgments The author would like to thank Kefeng Liu, Valentino Tosatti, Xiaokui Yang for useful discussions on Donaldson’s equation, the complex Monge-Ampère equation and geometric flows. The author thanks referees’s helpful suggestions.

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