QUASI-STATIC CRACK PROPAGATION BY GRIFFITH'S CRITERION

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Abstract: We consider the propagation of a crack in a brittle material along a prescribed crack path and define a quasi-static evolution by means of local minimizers of the free energy. We show that this evolution satisfies Griffith's criterion in a suitable form which takes into account both stable and unstable propagation, as well as an energy balance formula which accounts for dissipation in the unstable regime. If the load is monotonically increasing this solution is explicit and almost everywhere unique. For more general loads we construct a solution via time discretization, finding a local minimizer for every incremental problem. Finally, we consider a finite element discretization of the problem and prove convergence of the discrete solutions.

Keywords: brittle materials; Griffith criterion; quasi-static propagation
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1 Introduction

In most models of brittle fracture, since the seminal work of Griffith[13], the quasistatic evolution of a crack is governed by the energy release rate along the crack path, and thus is closely related to the meta-stability of the crack. By contrast, a new class of mathematical models for fracture in brittle materials, based primarily on global stability, was proposed much later by Francfort & Marigo[10]. While the latter model has some undeniable mathematical advantages, and indeed a solid mathematical theory has subsequently been developed (cf. Francfort & Larsen[11] or Dal Maso *et al.*[8]), there are some differences to Griffith's original model. In this article, we give a thorough mathematical and numerical analysis of Griffith's model in a simple situation with pre-defined crack path and discuss the differences to the model of Francfort & Marigo. In particular, in this simple situation it is possible to rely entirely on meta-stability for defining the crack evolution and we are thus able to extend Griffith's model to non-smooth crack evolutions. For brevity, we shall use (G) to refer to Griffith's original model and its subsequent developments, including the weak form which we define, and (FM) to refer to models based on global stability.

Before we introduce our model problem, we would like to mention the recent work of Toader

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& Zanini[18] where a meta-stable quasi-static evolution is defined by means of a vanishing viscosity approach. It follows from our uniqueness result (Theorem 2.4) that their solution coincides with the left-continuous version of our solution (cf. Theorem 2.4) in the case of monotonically increasing loads (cf. Section 3). We have not analyzed, however, whether this is also true in the general case.

We consider the case of antiplane deformation of an axisymmetric domain. To this end, let $\Omega = (-L, L) \times (-1, 1)$, where $L > 0$ is fixed, denote the domain occupied by (a cross-section of) an elastic body in its undamaged state. The boundary $\partial\Omega$ is decomposed into $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$, where $\partial_D\Omega = (-L, L) \times \{-1, 1\}$ and $\partial_N\Omega = \partial\Omega \setminus \partial_D\Omega$. As admissible cracks, we allow all symmetric, closed line segments $K(l) = \{(x, 0) : -l \le x \le l\}$, where $l \in (0, L]$. For each $l \in (0, L]$ and $u \in \mathrm{H}^1(\Omega \setminus K(l))$ the functional

$$E(l; u) = \frac{\mu}{2} \int_{\Omega \setminus K(l)} |\nabla u|^2 \, \mathrm{d}x,$$

where μ is the second Lamé constant, defines the elastic energy of the displacement field u (which corresponds to the third component of the elastic displacement in antiplane strain). At time $t \in [0, +\infty)$ a displacement $g(t) \in \mathrm{H}^{3/2}(\partial_D\Omega)$ is prescribed at $\partial_D\Omega$. Thus, in equilibrium, the elastic energy of the body with crack length l at time t is given by

$$\mathcal{E}(t, l) = \inf_{\substack{u \in \mathrm{H}^1(\Omega \setminus K(l)) \\ u|_{\partial_D\Omega} = g(t)}} E(l; u). \tag{1}$$

It is furthermore customary in the engineering literature to define the *energy release rate* $G(t, l)$ by

$$G(t, l) = -\frac{\partial}{\partial l} \mathcal{E}(t, l);$$

cf. Appendix 6.3 where we show that this is well-defined. According to Griffith's theory, a crack $K(l)$ is said to be in equilibrium at time t if $G(t, l) \le G_c$ where $G_c > 0$ is a material parameter called the fracture toughness. Since the largest admissible fracture, $K(L)$, is trivially in equilibrium, we define $G(t, L) = 0$ for all $t \in [0, +\infty)$.

Griffith's condition corresponds to meta-stability of the *free energy* of the elastic body at time t with crack $K(l)$, which is defined as

$$H(t, l) = \mathcal{E}(t, l) + G_c(l - l_0),$$

where $l_0 > 0$ and $K(l_0)$ is a pre-existing crack. In terms of H Griffith's stability criterion reads

$$\frac{\partial}{\partial l} H(t, l) = -G(t, l) + G_c \ge 0. \tag{2}$$

Thus, informally speaking, may say that a crack $K(l)$ is in equilibrium if it is a meta-stable state of the free energy H. (For the time being, we simply take (2) as our definition of meta-stability. However, we shall discuss this later and discard several solutions in favor of a stronger notion.) By contrast, in the variational theory of Francfort & Marigo [10] a crack $K(l)$ is said to be in equilibrium only if it is globally stable, i.e., if

$$\mathcal{E}(t, l) \le \mathcal{E}(t, l') \qquad \forall l' \in [l, L],$$

and thus, the stability condition in (FM) is much stronger than that in (G).

The second component of Griffith's theory is an activation criterion, which is not explicitly considered in (FM). By interpreting the stability criterion as a meta-stability principle, it is intuitive to allow a crack to advance if, and only if $G \geq G_c$. Together with the stability condition $G \leq G_c$ we may write this new condition as $(G(t,\ell) - G_c)\dot{\ell}(t) = 0$, where $\ell(t)$ is the crack length at time t. This is the form usually advocated in the engineering literature.

Despite the differences in the formulation of (G) and (FM), it is not immediately clear whether the two models are too different. In fact, it was demonstrated that, if a crack evolution generated by (FM) is smooth, then it satisfies Griffith's law[10]. The reverse is not true however. It is intuitively clear (and we shall provide a computational example) that a smooth solution of (G) is not necessarily globally stable.

To make this discussion possible, we first need to state precisely how Griffith's model should be interpreted mathematically. This is not entirely trivial as minimal changes can give very different notions of solution, many of which are clearly unphysical. We believe that Definition 2.2 gives a convenient notion of crack evolution, which generalizes the classical version of (G) and captures the concept of meta-stability which motivates (G). In Section 4 we give a general existence result and prove a generalized energy balance formula. In particular, we shall see that crack evolutions can have a quite general structure and not in all circumstances the physical prerequisites for Griffith's theory are met. To understand the situation better, we first consider the case of monotone increasing loads, i.e. $g(t) = tg(1)$, in Section 3. There, we completely characterize the set of solutions to our formulation of Griffith's model and even give a uniqueness result. In this setting we are able to clearly demonstrate the differences between (G) and (FM).

Finally, in Section 5, we use the techniques developed to prove some results concerning the numerical approximation of the weak form of (G). As opposed to the discretizations of (FM) (e.g. Bourdin *et al.*[4], Negri[17] or Giacomini & Ponsiglione[12]) which are based on global minimization, the main advantage from a computational point of view of our (i.e. of Griffith's) approach is the algorithmically computability of solutions. Since we assume a predefined crack path this argument is not entirely valid for our situation, however, it does carry over to more general situations while our analysis does not.

We conclude this introduction with a final remark. We have naively assumed that for the symmetric model problem which we have posed the crack should propagate in a straight line. While we are aware that this is generally false, we believe that our analysis is still interesting. First, we are only interested in stability along a crack path rather than the stability of the crack path, and second, our analysis applies with few changes to arbitrary smooth crack paths as long as they are known *a priori*.

2 Notation and main results

We require that the boundary values $g(t)$ on $\partial_D\Omega$ are the traces of H^2-functions, i.e., that they belong to $\mathrm{H}^{3/2}(\partial_D\Omega)$, and that map $t \mapsto g(t)$, $t \in [0,+\infty)$, satisfies

$$g \in C^1([0,+\infty); \mathrm{H}^{3/2}(\partial_D\Omega)) \quad \text{with} \quad g(0) = 0. \tag{3}$$

We assume that (3) holds throughout this article. Under this condition it can be shown (cf. §6.3) that $G \in \mathrm{C}((0,L) \times [0,+\infty))$.

We begin by reviewing a precise statement of Griffith's model in a classical sense. Suppose that, at time $t = 0$, the body contains a pre-existing crack $K(l_0)$, where $l_0 > 0$. We assume that the crack is irreversible, symmetric (thus $g(t,x)$ has to be symmetric) and propagates according to Griffith's law. By classical (or strong) solutions, we shall mean solutions which are absolutely continuous. A function $\ell : [0,+\infty) \to [l_0, L]$ is said to be absolutely continuous if $\ell(t) = \ell(0) + \int_0^t \dot{\ell}(t)\,\mathrm{d}t$, where $\dot{\ell}(t) \in \mathrm{L}^1(0,+\infty)$.

Definition 2.1 *A strong solution of (G) is an absolutely continuous, non-decreasing function $\ell(t)\colon [0,+\infty) \to [l_0, L]$ such that $\ell(0) = l_0$ and such that*

$$G(t,\ell(t)) \le G_c \qquad \forall t \in [0,+\infty), \quad \text{and} \tag{4}$$

$$\big(G(t,\ell(t)) - G_c\big)\,\dot{\ell}(t) = 0 \qquad \text{for a.e. } t \in [0,+\infty). \tag{5}$$

Let us say a few words on (5), which represents the evolution law in the so-called Kuhn-Tucker form (see for instance Maugin[16]). Roughly speaking, Griffith's criterion[13] asserts that the crack is stable when $G(t,\ell(t)) < G_c$ and that it propagates quasi-statically when $G(t,\ell(t)) = G_c$, i.e. when the energy release rate reaches the critical value G_c. The remaining case $G(t,\ell(t)) > G_c$ is usually related to unstable propagation and thus should be ruled out from the quasi-static framework. These conditions are represented by (4) and (5) which are indeed equivalent to

$$G(t,\ell(t)) \le G_c \quad \text{and} \quad \dot{\ell}(t) > 0 \Rightarrow G(t,\ell(t)) = G_c\,.$$

Moreover, for absolutely continuous functions $\ell(t)$, (5) is directly related to the conservation of energy (cf. Section 4).

We will see in §3 that it is not always possible to find a crack evolution ℓ which fulfills all the requirements of Definition 2.1. We shall demonstrate that under quite general conditions, solutions of Griffith's model are discontinuous and thus require a more careful definition. While this may seem just a technical point, it has instead a strong physical consequence. For example, we shall derive a generalized energy conservation principle which shows precisely under which conditions the assumption of rate-independence is invalid as well as the reason why.

We are therefore looking for a new formulation of (G) which generalizes Definition 2.1 but makes sense even if ℓ is a general monotone function. Solely based on the principle of meta-stability, we modify (5) to obtain the following definition.

Definition 2.2 *We say that a non-decreasing function $\ell(t)\colon [0,T] \to [l_0, L]$ is a weak solution of (G) if $\ell(0) = l_0$, if it satisfies the local stability condition* (4) *and the weak activation criterion*

$$\begin{aligned} \big|\ell(t+\Delta t) - \ell(t-\Delta t)\big| > 0 \quad \forall \Delta t < \eta \qquad &\Rightarrow \\ G(t,l) \ge G_c \quad \forall l \in [\ell(t-), \ell(t+)] \setminus \{L\}. \end{aligned} \tag{6}$$

The weak form of (G), given in Definition 2.2, is a mathematically correct and convenient restatement of the physical motivation behind (5). It means that a crack can advance at time t only if G has reached G_c before. Indeed, since G is continuous, by (6) we get $\lim_{s\to t^-} G(s,\ell(s)) = G(t,\ell(t-)) = G_c$. We remark that the two formulations are equivalent at every point t where ℓ is differentiable. On the contrary, if ℓ jumps at time t then (4) does not give any information on the behaviour of G for $l \in [\ell(t-), \ell(t+)]$. In this case, criterion (6) is stronger and prevents jumps over local energy minima.

For future reference, we denote the set of discontinuities (only jump discontinuities are possible) of a monotone map ℓ by $S(\ell)$. We are now in a position to state our main results: the first establishes the existence of weak solutions to Griffith's model and the generalized energy balance.

Theorem 2.3 *Suppose that* (3) *holds. Then, there exists at least one evolution* $\ell : [0, +\infty) \to [l_0, L]$ *satisfying Definition 2.2. Furthermore, If* $0 \le t_1 < t_2$ *are points of continuity for* ℓ *and if* $\ell(t_2) < L$ *then*

$$H(t_2, \ell(t_2)) - H(t_1, \ell(t_1)) = \int_{t_1}^{t_2} \mathscr{P}(t)\,\mathrm{d}t + \sum_{t \in S(\ell) \cap (t_1, t_2)} \mathscr{D}(t),$$

where $S(\ell)$ *is the set of jump points of* ℓ, $\mathscr{P}$ *is the* power of the external forces *and* $\mathscr{D}$ *is the* dissipated energy *(respectively given by* (13) *and* (14)*).*

Note that the formulation of the energy balance allowing only intervals where the endpoints are points of continuity of ℓ is not a true restriction. By taking upper or lower limits, it can be immediately extended to arbitrary intervals; however, the formulation would have been extremely awkward since it requires the distinction of many different cases.

The proof of Theorem 2.3 is given in §4. Note in particular that we make no mention of uniqueness of solutions. This is generally false as our results in §3, which culminate in Theorem 2.4 below, show. Here, we consider the case of monotone increasing loads, i.e., we assume that $g(t) = tg(1)$ where $g(1) \in \mathrm{H}^{3/2}(\partial_D\Omega)$. In this case, the crack tip evolution can almost be written as a closed formula involving only the function $G(1, l)$, more precisely or its *monotone envelope* which is defined by

$$G_m(1, l) = \sup\{\varphi(l) : \varphi : [l_0, L] \to [0, +\infty]\,,\ \varphi \le G \quad \text{and} \quad \varphi \text{ is decreasing }\}.$$

Theorem 2.4 *Suppose that* $g(t) = tg(1)$ *where* $g(1) \in \mathrm{H}^{3/2}(\partial_D\Omega)$. *Then, for any weak solution* $\hat{\ell}(t)$ *of (G) (whose existence is guaranteed by Proposition 2.3) the entire set of weak solutions is characterized by*

$$\mathcal{L} = \{\ell \in \mathrm{BV}(0, +\infty) : \hat{\ell}(t-) \le \ell(t) \le \hat{\ell}(t+) \ \text{ and } G(t, \ell(t)) = G_m(t, \ell(t))\ \forall t\}.$$

Moreover, the right-continuous map $\ell^+(t) = \hat{\ell}(t+)$ *is always a solution and is the unique right-continuous inverse (cf. §3.3) of the map* $\tau : [l_0, L] \to [0, +\infty]$*, given by*

$$\tau(l) = \sqrt{G_c/G_m(1, l)}.$$

3 Monotonically increasing loads

Rather than with the general case, we begin our analysis with the case of monotone increasing loads. We believe that in this setting our results are in fact the most interesting as they provide the best intuition for the crack propagation problem.

We assume throughout this section that the load is monotonically increasing, i.e., $g(t) = tg(1)$. In this important case, the geometry of the energy landscape simplifies considerably. We have, in particular,

$$\mathscr{E}(t, l) = t^2 \mathscr{E}(1, l), \quad \text{and therefore} \quad G(t, l) = t^2 G(1, l).$$

The fracture criterion, at least in its naive form, becomes

$$\ell(0) = l_0, \quad G(1, \ell(t)) \le G_c/t^2 \ \text{ and } \ (G(1, \ell(t)) - G_c/t^2)\dot{\ell}(t) = 0.$$

Thus, we only need to know $G(1, l)$ in order to describe the fracture evolution. We begin by describing the construction heuristically at an illustrative example. Before making the ideas rigorous we then briefly discuss in Section 3.2 where (G) differs fundamentally from (FM) and possible consequences of these differences. Only then, in Section 3.3, do we turn to a rigorous analysis of the construction given in Section 3.1.

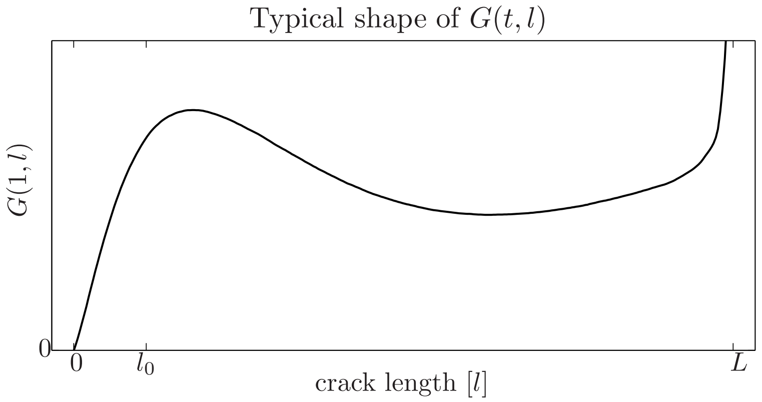

Figure 1: A typical shape of the energy release rate $G(1, l)$ in the case of monotone increasing loads.

3.1 Heuristic construction of the evolution

We describe the construction at the example,

$$g(1, x, y) = y\big[\tfrac{1}{2}(1 + \mathrm{e}^{-x^2}) + \tfrac{1}{27}x^2\big]$$

which gives the shape for $G(1, l)$, shown in Figure 1. We would like to stress that this shape for the map $l \mapsto G(1, l)$ is actually representative for many applied boundary displacements and we have only chosen our specific load in order to obtain a visually more attractive picture.

It is possible to construct the crack evolution from the plot in Figure 1 alone. Recall that the activation criterion, in its primitive form, reads $G(1, \ell(t)) = G_c/t^2$. Thus, we can draw a horizontal line in the picture at G_c/t^2 and push it down (as t increases) in order to determine the position of the crack tip.

In Figure 2 (a), the horizontal line at G_c/t^2 is just above the level $G(1, l_0)$ and therefore the crack has not started to move, i.e., $\ell(t) = l_0$. In step (b) it reaches precisely the value $G(1, l_0)$. At this point we can either decide to remain in l_0, or propagate the crack to the point shown in the figure. Since $G(1, l) > G_c/t^2$ in-between, only those two are possible equilibria for the crack position. We believe that the physically more appealing equilibrium, which is a strict local minimum, is the one we have chosen in the figure. Mathematically, however, the choice $\ell(t) = l_0$ is equally possible. In Figure 2 (c) we can see that the crack position moves continuously with the line G_c/t^2 and we have a unique value for $\ell(t)$. Finally, in steps (d) and (e) the line G_c/t^2 reaches, and then moves past the lowest point of $G(1, l)$ in the interval $[l_0, L]$. Since $G(t, l) > G_c$ for $l \geq \ell(t)$, it follows that the crack propagates all the way through to the boundary.

This last 'snap' is a point which requires some further discussion. In order to have $G(t, \ell(t)) = G(t, L) \leq G_c$ for all $t > 0$, it is necessary to define $G(1, L) = 0$. This also makes sense physically as the energy does not decrease of course when the crack length is increased past the boundary of the domain. However, in this case, the condition $(G - G_c)\dot{\ell} = 0$ is not satisfied (by whatever means $\dot{\ell}$ is interpreted, it should always be non-zero) at the time t when the crack tip leaves the domain — if ℓ is chosen right-continuous. We do not want to discard this solution, however. Our weak activation criterion (6), takes care of this problem as well.

The solid line in Figure 2 (e) traces the monotone envelope of $G(1, l)$ in the interval $[l_0, L]$ and it represents, up to a scaling for t, the inverse of the crack tip evolution $\ell(t)$. Thus,

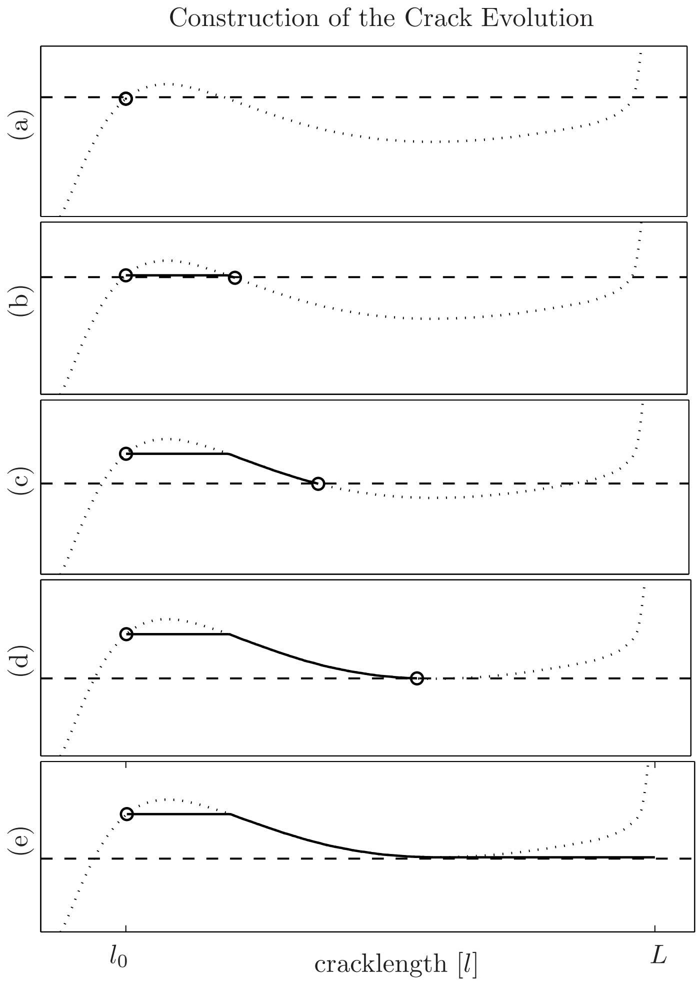

Figure 2: Construction of a weak solution to (G), for monotone increasing loads. The dashed line represents time in the form G_c/t^2, the dotted line represents $G(1,l)$ and the full line which traces the monotone envelope of $G(1,l)$ in $[l_0, L]$ represents the crack evolution.

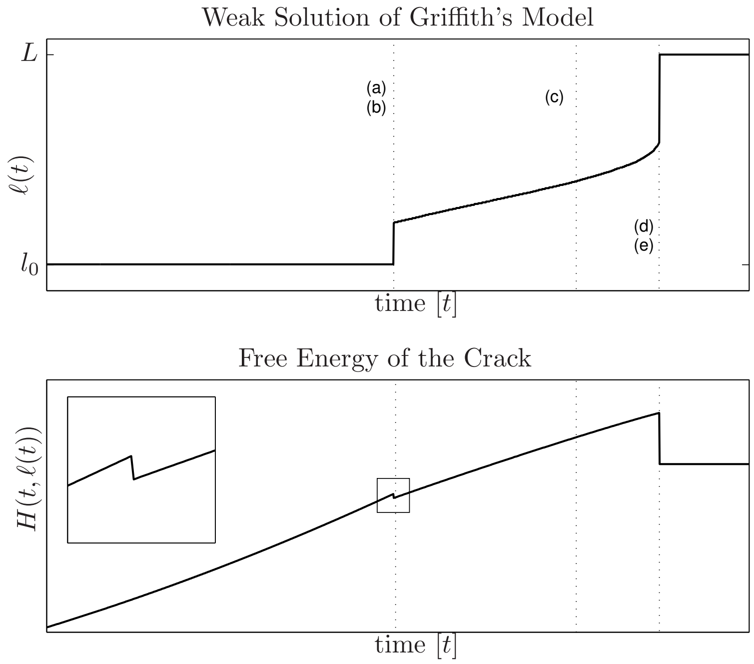

Figure 3: Crack tip evolution $\ell(t)$, constructed in Figure 2, which is a weak solution of (G) in the sense of Definition 2.2 and the corresponding free energy $H(t, \ell(t))$.

by interchanging and correcting the axis, we obtain Figure 3. We note, in particular, that $t \mapsto \ell(t)$ is discontinuous in two points and thus, the classical theory does not apply.

Our cartoon in Figure 2 provides us with all the necessary intuition for the following analysis. We shall give an explicit construction of all solutions $\ell(t)$ of the fracture problem and show that they are unique in a sense which we make precise. We then show that energy is conserved if, and only if (apart from some degenerate limiting cases), ℓ is continuous. Our interpretation of this phenomenon is that the quasistatic model is invalid if ℓ jumps, i.e., if the crack propagation is 'brutal'. Indeed, if we consider Figure 2 (b) again, we see that past the position l_0, the crack tip would propagate without increasing the load. We believe that this is an instance where it becomes necessary to model fracture dynamically in order to correctly describe the physics of the evolution.

3.2 Comparison of (G) and (FM)

Clearly, the formulation of (G) and of (FM) are fundamentally different. While (G) relies on local stability (in fact even equilibrium is sufficient), (FM) requires global stability, i.e., that its solutions are global minimizers of the free energy. It would therefore be truly surprising if, even in the presence of regularity, (FM) and (G) would coincide. Nevertheless, it was shown by Francfort & Marigo[10] that, if $\ell(t)$ is a solution to (FM) and if ℓ is absolutely continuous then it is also a strong solution to (G). Thus, we may be tempted to conclude that (FM) is a genuine weak theory for (G). However, one would still have to prove that any strong solution to (G) is also a solution to (FM) which is not true.

We demonstrate this computationally by returning to the example given in the previous section. The solution to (G) shown in Figure 4 is computed using Algorithm 2 of Section 5.2. The energy release rate is computed using the method described in Section 5.3. The solution to (FM) is obtained by using the same discretization of H and minimizing the energy "globally", i.e., over the finitely many points of a discretized crack path. In Figure 4 we show the computed solutions to (G) and to (FM), while in Figure 5 we plot the evolution of the total energy.

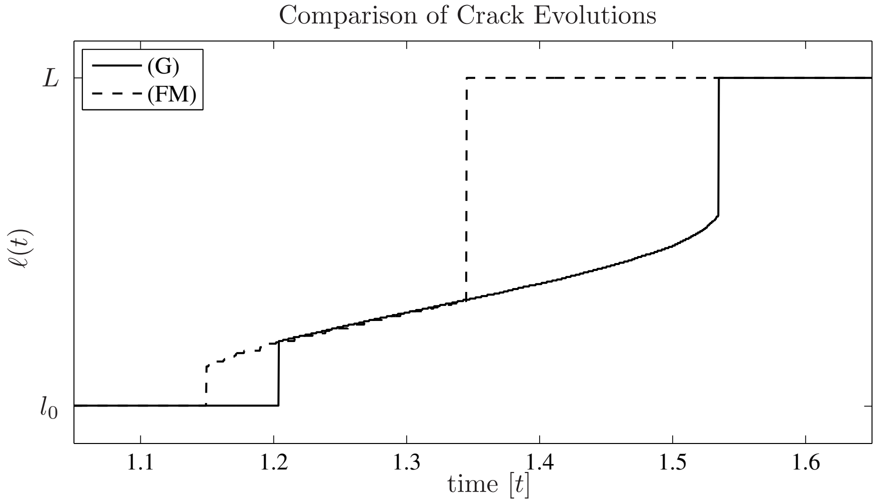

Figure 4: Comparison of the crack tip evolution $\ell(t)$ for the (G) and the (FM) model.

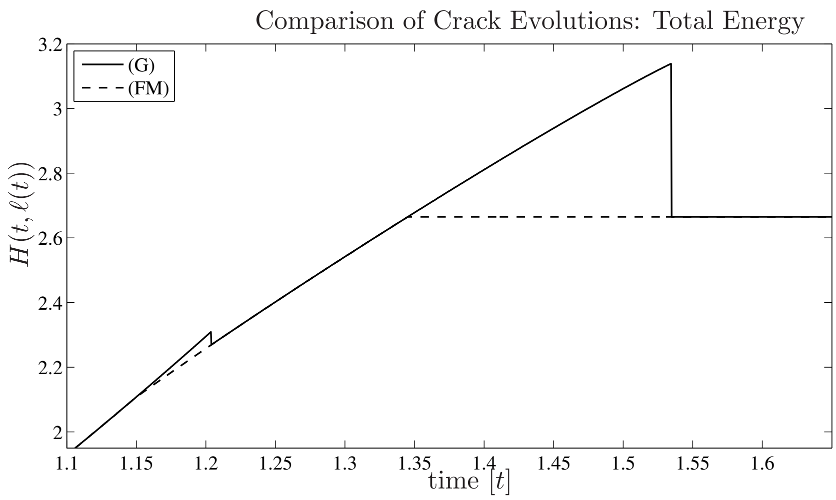

Figure 5: Comparison of the total energy $H(t, \ell(t))$ of the crack tip evolutions constructed from the (G) and the (FM) models.

Qualitatively the two evolutions seem quite similar at first glance. Both exhibit a jump followed by a smooth evolution followed by a final "snap". The jump in (G) occurs whenever the energy release rate $G(1, l)$ is increasing and a moment of reflection should reveal that this is precisely the situation where also (FM) jumps, however, it will always jump ahead of time — this behaviour is generic.

Figure 5 shows that the evolution of total energy of the (FM) solution is "smooth" whereas that of (G) jumps. This reflects the fact that (FM) conserves energy whereas (G) dissipates energy in the jump points of its solution. The generalized energy conservation formula valid for our weak Griffith model is proved in §4. Energy conservation is a crucial requirement of continuum mechanics and particularly of rate independent evolutions. From this point of view, we may actually prefer (FM) over (G). However, in the (FM) model, energy conservation is attained at the price of an evolution which is able to "look into the future" and jump to a new configuration (the global energy minimum) which is completely separated from the current configuration by arbitrarily large energy barriers.

Nevertheless, the fact that (G) dissipates energy is somewhat worrying. Our interpretation of this phenomenon, which is consistent with the view commonly taken in the engineering literature, is that in this situation the assumption of rate-independence becomes invalid and that a dynamic model should be used instead. It will be highly interesting to investigate in which ways a dynamic model would dissipate the energy which is stored in the system at this point. An understanding of this process might lead to a more advanced Griffith model where the unstable situation $G > G_c$ may be modelled correctly in a quasistatic setting.

Finally, it is worth commenting on the connection between the convexity of H and the monotonicity of G. Convexity of H is renders the minimization problem for (FM) well-posed while the monotonicity of G would render (G) well-posed in the sense that there exist unique solutions. Neither should be true (except in trivial cases) as boundary effects usually imply that $G(1, l) \to +\infty$ as $l \to L$. However, for (G) it seems to be fairly clear that the model, in the weak form which we have presented, is only valid in the domain where G is monotone. In this regime, however the strong form of (G) and the weak form of (G) are almost equivalent.

3.3 Explicit construction of crack evolutions

The above discussion has shown us a possible path for the construction of crack evolutions. We define the monotone envelope $G_m(1, l)$ of $G(1, l)$ by

$$G_m(1, l) = \sup\{\varphi(l) : \varphi\colon [l_0, L] \to \mathbb{R}\,, \varphi \text{ is decreasing}\,, \varphi \leq G\}.$$

Note that by this definition, we have $G_m(1, l_0) = G(1, l_0)$ while $G_m(1, L) = 0$. As for G, we define $G_m(t, l) = t^2 G_m(1, l)$.

Next, we define the map $\tau\colon [l_0, L] \to [0, +\infty]$ by

$$\tau(l) = \sqrt{G_c/G_m(1, l)}.$$

It should be clear from the discussion in Section 3.1, that the value $\tau(l)$ represents the time at which the crack tip reaches or moves past the crack length l. Informally for the moment, we may even write $\tau(l) = \inf\{t \geq 0 : \ell(t) < l\}$, where $\ell(t)$ may be any crack evolution. We notice therefore that $\tau(l)$ is an inverse for ℓ and therefore any crack evolution must be an inverse of τ. The purpose of the following discussion is to make this idea precise. We begin by reviewing some facts about the inverse(s) of a monotone function.

We denote the left- and right-continuous inverses of τ respectively by ℓ^- and ℓ^+. The functions $\ell^\pm : [0, +\infty) \to [l_0, L]$ are defined by

$$\ell^-(t) = \sup\{l \in [l_0, L] : \tau(l) < t\} \cup \{l_0\}, \qquad \forall t \in [0, +\infty), \quad \text{and} \tag{7}$$

$$\ell^+(t) = \inf\{l \in [l_0, L] : \tau(l) > t\} \cup \{L\}, \qquad \forall t \in [0, +\infty). \tag{8}$$

Since $\tau(l)$ is non-decreasing the previous definitions are equivalent to

$$\ell^-(t) = \inf\{l \in [l_0, L] : \tau(l) \geq t\} \cup \{L\}, \qquad \forall t \in [0, +\infty), \quad \text{and} \tag{9}$$

$$\ell^+(t) = \sup\{l \in [l_0, L] : \tau(l) \leq t\} \cup \{l_0\}, \qquad \forall t \in [0, \infty). \tag{10}$$

Since τ may in principle attain the value $+\infty$, we shall always use the topology of the compactification of $[0, +\infty]$. For example, with respect to this topology, τ is always continuous in $[l_0, L)$ (but not in $[l_0, L]$).

By definition, ℓ^+ and ℓ^- are monotone, ℓ^- is left-continuous, ℓ^+ is right-continuous and $\ell^- \leq \ell^+$.

Proposition 3.1 *(i) Let $\ell\colon [0, +\infty) \to [l_0, L]$ satisfy $\ell^- \leq \ell \leq \ell^+$; then, for each $\hat{t} \in [0, +\infty)$ we have $\hat{t} \in [\tau(\ell(\hat{t})-), \tau(\ell(\hat{t})+)]$. In particular, $\ell^\pm(t) = \ell(t\pm)$ for all $t \in [0, +\infty)$.*

(ii) If $t_1 < t_2 \in [0, +\infty)$ then $\ell^+(t_1) \leq \ell^-(t_2)$.

Proof. For the proof of item (i), let $\hat{l} = \ell(\hat{t})$ for $\hat{t} \in [0, +\infty)$ and assume by contradiction that $\hat{t} < \tau(\hat{l}-)$. If $\hat{l} = l_0$ by definition $\tau(\hat{l}-) = l_0 \geq \hat{t}$. Otherwise, there exists a $l' < \hat{l}$ such that $\tau(l') > \hat{t}$. This implies that $\ell^+(\hat{t}) < \hat{l}$ which contradicts the assumption that $\ell(\hat{t}) \leq \ell^+(\hat{t})$ and thus shows that $\hat{t} \geq \tau(\hat{l}-)$. By a similar argument, we also obtain $\hat{t} \leq \tau(\hat{l}+)$.

To prove item (ii) it is sufficient to write $\ell^+(t_1) \leq \ell^-(t_2)$ as

$$\inf\{l \in [l_0, L] : \tau(l) > t_1\} \cup \{L\} \leq \inf\{l \in [l_0, L] : \tau(l) \geq t_2\} \cup \{L\}$$

which is true since $t_1 < t_2$. ∎

Proposition 3.1 motivates the definition of the set of inverse functions of τ,

$$\{\tau^{-1}\} = \{\ell\colon [0, +\infty) \to [l_0, L] : \ell^- \leq \ell \leq \ell^+\}.$$

Corollary 3.2 *If $\ell \in \{\tau^{-1}\}$ then ℓ is non-decreasing. Furthermore, for every $t \in [0, +\infty)$, for which $l_0 < \ell(t) < L$, it holds that $\tau(\ell(t)) = t$.*

Proof. Both facts follow trivially from Proposition 3.1 and the fact that τ is continuous at $\ell(t)$. ∎

After establishing some facts about monotone functions and their inverses, we are in a position to define the set

$$\mathcal{L} = \big\{\ell \in \{\tau^{-1}\} : G(t, \ell(t)) = G_m(t, \ell(t)) \quad \forall t \in [0, +\infty]\big\},$$

which we claim to be the set of all weak solutions of (G) with $\ell(0) = l_0$. First, we show that $\mathcal{L}$ is non-empty. This would follow in fact from the results in the next section and the abstract existence result, Theorem 4.1. However, we give a slightly stronger result which will also be useful for the following proofs.

Lemma 3.3 *The left- and right- continuous inverses $\ell^\pm$ of τ are elements of $\mathcal{L}$.*

Proof. Let $\ell \in \{\ell^+, \ell^-\}$. Since, by its definition, $\ell \in \{\tau^{-1}\}$, we only have to prove that $G(1, \ell(t)) = G_m(1, \ell(t))$. For $l = l_0$ and $l = L$ we always have (by definition) $G(1, l) = G_m(1, l)$, hence we only need to consider $l_0 < \ell(t) < L$. In those points G and G_m are continuous. Assume that $G(1, \ell(t)) > G_m(1, \ell(t))$. Then, there exists a neighbourhood of $l = \ell(t)$ where this still holds and hence G_m and also τ must be constant in that neighbourhood. This, however, contradicts the definition of ℓ^- and ℓ^+, as $\ell^-(t)$ would be strictly less than l and $\ell^+(t)$ would be strictly greater than l. ∎

Lemma 3.4 *For every $t \in [0, +\infty]$ such that $\ell^+(t) < L$ we have*

$$G_m(t, \ell^+(t)) = G_m(t, \ell^-(t)).$$

If $\ell^+(t) = L$ we still have $G_m(t, \ell^-(t)) = G_m(t, l)$ for all $l \in [\ell^-(t), \ell^+(t))$.

Proof. If $\ell^+(t) = \ell^-(t)$ then the equality is trivial. If $\ell^+(t) > \ell^-(t)$ then (by the definitions of $\ell^\pm$) it follows that τ is constant in the interval $(\ell^-(t), \ell^+(t))$ and hence so is G_m. If $\ell^+(t) < L$, by continuity $G_m(t, \ell^-(t)) = G_m(t, l) = G_m(t, \ell^+(t))$ for every $l \in (l_0, L)$, hence G_m is constant in $[l_0, L]$. If $\ell^+(t) = L$ the same argument gives $G_m(t, \ell^-(t)) = G_m(t, l)$ for all $l \in [\ell^-(t), \ell^+(t))$. ■

We have now assembled all the results required to prove that any member of $\mathcal{L}$ is a solution of the fracture problem.

Lemma 3.5 *If $\ell \in \mathcal{L}$ then ℓ is a weak solution of (G), i.e., it satisfies Definition 2.2.*

Proof. Clearly, any $\ell \in \mathcal{L}$ is monotone and, since $0 \leq G_m(1, l_0) < +\infty$ it follows that $\tau(l_0) \in (0, +\infty]$ and hence

$$\ell^+(0) = \inf\{l \in [l_0, L] : \tau(l) > 0\} = l_0.$$

Thus we only have to prove local stability (4) and the weak activation criterion (6) for all $\ell \in \mathcal{L}$. To this end, fix $\ell \in \mathcal{L}$ and define

$$\begin{aligned} T_0 &= \sup\{t \in [0, +\infty) : \ell^+(t) = l_0\}, \quad \text{and} \\ T_1 &= \inf\{t \in [0, +\infty] : \ell^-(t) = L\}. \end{aligned}$$

Hence, for $0 \leq t < T_0$ we have $\ell(t) = l_0$ and for $T_1 < t < +\infty$, we have $\ell(t) = L$.

Since $\ell^-(T_0) = \ell^+(T_0-) = l_0$, we can use definition (7) of $\ell^-(T_0)$ to deduce that $\tau(l) \geq T_0$ for all $l \geq l_0$ and hence $\tau(l_0) \geq T_0$. The reverse inequality is trivial; thus we have $\tau(l_0) = T_0$.

Hence, for $t < T_0$, we obtain

$$G(t, \ell(t)) = G(t, l_0) = G_m(t, l_0) \leq G_m(\tau(l_0), l_0) = G_c,$$

from which is follows that $\dot{\ell}(t) = 0$ since ℓ is constant in a neighbourhood of t.

If $T_0 < t < T_1$ then $l_0 < \ell^-(t) \leq \ell(t) \leq \ell^+(t) < L$. Thus, τ is continuous at $\ell(t)$ and from Corollary 3.2 it follows that

$$G(t, \ell(t)) = G_m(t, \ell(t)) = G_m(\tau(\ell(t)), \ell(t)) = G_c.$$

Since ℓ was arbitrary, the result holds in particular for $\ell = \ell^-$ and therefore, by Lemma 3.4,

$$G(t, l) \geq G_m(t, l) = G_m(t, \ell(t-)) = G_c \qquad \forall l \in [\ell(t-), \ell(t+)] \setminus \{L\}.$$

This implies (6) in the case $t \in (T_0, T_1)$.

If $t > T_1$ then $G(t, \ell(t)) = G(t, L) = 0 \leq G_c$ and $\dot{\ell}(t) = 0$ since ℓ is constant in a neighbourhood of t.

We are left to deal only with the points T_0 and T_1. For this we need to distinguish two cases. Assume first that $T_0 < T_1$. If $t = T_0$ then, using right-continuity and Lemma 3.4,

$$G(t, \ell^-(t)) = G(t, \ell(t)) = G(t, \ell^+(t)) = G_c$$

If $t = T_1$ then either $\ell(t) = \ell^+(t)$ in which case we have $G(t, \ell(t)) = 0$, or, using left-continuity,

$$G(t, \ell^-(t)) = G(t, \ell(t)) = G_c.$$

In either case, we can apply the above argument to show that the weak activation criterion (6) holds.

Second, assume that $t = T_0 = T_1$. As $\ell^-(s) = l_0$ for $s < t$ the left continuity of ℓ^- implies that $\ell^-(t) = l_0$ and similarly $\ell^+(t) = L$. As before, we also have $\tau(l_0) = t$ and $\tau(L) = +\infty$. Thus, we have $G(t, \ell(t+)) = 0 \leq G_c$ and

$$G(t, \ell(t-)) = G(t, l_0) = G(\tau(l_0), l_0) = G_c,$$

which proves the local stability. Since $G(t, \ell(t-)) = G_m(t, \ell(t-))$, it follows that

$$G(t, l) \geq G_m(t, l) = G_m(t, l_0) = G(t, l_0) = G_c \quad \forall l \in [\ell(t-), \ell(t+)),$$

which proves the activation criterion. ■

Finally, we show the *uniqueness* of the crack evolution. This is done in two steps. First we prove that any solution $\ell(t)$ must satisfy $G(t, \ell(t)) = G_m(t, \ell(t))$ and then, that $\ell \in \{\tau^{-1}\}$.

Lemma 3.6 *If ℓ is a weak solution of (G) then $G(t, \ell(t)) = G_m(t, \ell(t))$ for all $t \in [0, +\infty)$.*

Proof. If this were not true then there exists $t_0 \in [0, +\infty)$ such that $G(1, \ell(t_0)) > G_m(1, \ell(t_0))$. This implies that there exists $l_1 < \ell(t_0)$ such that $G(1, l_1) < G(1, \ell(t_0))$. Let $t_1 \leq t_0$ be a time for which $l_1 \in [\ell(t_1-), \ell(t_1+)]$. Hence, by condition (6), it holds that

$$G(t_0, \ell(t_0)) \geq G(t_1, \ell(t_0)) > G(t_1, l_1) \geq G_c,$$

which contradicts the stability condition (4). ■

Lemma 3.7 *If ℓ is a weak solution of (G) then $\ell \in \{\tau^{-1}\}$.*

Proof. Let $\hat{\tau} \colon [l_0, L] \to [0, +\infty]$ be the right-continuous inverse of ℓ. Recall that $\hat{\tau}(l)$ is defined by

$$\begin{aligned} \hat{\tau}(l) &= \inf\{t \in [0, +\infty] : \ell(t) > l\} \cup \{+\infty\} \\ &= \sup\{t \in [0, +\infty] : \ell(t) \leq l\} \cup \{0\}. \end{aligned}$$

Our aim is to show that $\hat{\tau} = \tau$. By definition, $\hat{\tau}(L) = +\infty$ which automatically gives $\tau(L) = \hat{\tau}(L)$.

First we show that $\hat{\tau}(l) \leq \tau(l)$, or equivalently, if $t > \tau(l)$ then $\ell(t) > l$. If it were true that $\ell(t) \leq l$ then

$$G(t, \ell(t)) \geq G_m(t, \ell(t)) \geq G_m(t, l) > G_m(\tau(l), l) = G_c.$$

This contradicts the stability condition (4). We have therefore established that $\hat{\tau}(l) \leq \tau(l)$.

On the other hand, to show $\hat{\tau}(l) \geq \tau(l)$, we note that

$$G_m(t', l') < G_c \qquad \forall t' < \tau(l), \, \forall l' \geq l,$$

which implies that $\ell(t') \leq l$ forall $t' < \tau(l)$. Indeed, if we had $\ell(t') > l$ then, by Lemma 3.6, $G(t', \ell(t')) = G_m(t', \ell(t')) < G_c$ which contradict the activation criterion (6). ■

The proof of Theorem 2.4 follows immediately from Lemmas 3.3–3.7.

4 A general existence result

This section is devoted to the proof of Theorem 2.3 in the case of general loads, i.e., the proof of existence of weak solutions to (G) and of the generalized energy balance.

Proposition 4.1 *If* (3) *is satisfied then there exists a non-decreasing map* $\ell\colon [0,+\infty) \to [l_0, L]$ *satisfying Definition 2.2.*

Proof. From Appendix 6.3 it follows that G is continuous in $[0,+\infty) \times [l_0, L)$. Furthermore, since $g(0) = 0$, it follows that $G(0, l) = 0$ for all $l \in [l_0, L]$.

For each $h > 0$ set $t^h_j = hj$ for $j \in \mathbb{N}$, $\ell_h(0) = l_0$, and define by induction

$$\ell_h(t^h_j) = \inf\{l \geq \ell_h(t^h_{j-1}) : G(t^h_j, l) < G_c\}, \qquad j \geq 1\,.$$

Note that, since $G(t, L) = 0$ by definition, $L \in \{l \geq \ell_h(t^h_{j-1}) : G(t^h_j, l) < G_c\}$ for every t^h_j and hence the $\ell_h(t^h_j)$ is well defined. This discretely constructed approximate solution is extended to $[0,+\infty)$ by setting

$$\ell_h(t) = \ell_h(t^h_j), \quad \text{if } t^h_{j-1} < t \leq t^h_j.$$

Clearly, the functions ℓ_h are non-decreasing and $l_0 \leq \ell_h(t) \leq L$, and hence there exists a sequence $h_n \downarrow 0$ and a non-decreasing function $\ell(t)\colon [0,+\infty) \to [l_0, L]$ such that

$$\begin{aligned} \ell_{h_n}(t) \to \ell(t) \qquad & \forall t \in [0, T], \quad \text{and} \\ \ell_{h_n} \overset{\star}{\rightharpoonup} \ell \qquad & \text{in BV}(0,+\infty), \quad \text{as } n \to \infty. \end{aligned}$$

The pointwise convergence follows from Helly's selection theorem while the weak-$*$ convergence follows from standard compactness results in BV (cf. for example Ambrosio *et al.* [1]). For simplicity of notation, we shall drop the subscript n whenever possible.

Since $\ell_h(0) \to \ell(0)$ it follows that $\ell(0) = l_0$.

Now, we check that $G(t, \ell(t)) \leq G_c$ for every $t \in [0,+\infty)$. If $\ell(t) = L$ then, by definition, $G(t, L) = 0 \leq G_c$. If $\ell(t) < L$ then G is continuous at $(t, \ell(t))$. Furthermore, there exists a sequence $\hat{t}_n = t^{h_n}_j$ such that $|t - \hat{t}_n| \leq h_n$ and $\ell_{h_n}(\hat{t}_n) = \ell_{h_n}(t) \to \ell(t)$ and hence $G(t, \ell(t)) = \lim_{n\to\infty} G(\hat{t}_n, \ell_{h_n}(\hat{t}_n)) \leq G_c$.

The proof of the weak activation criterion (6) will be split into three parts: $l = \ell(t-)$, $l \in (\ell(t-), \ell(t+))$ and finally $l = \ell(t+)$. Recall that we only have to prove (6) for $l < L$.

First, we prove by contraposition that, if $|\ell(t + \Delta t) - \ell(t - \Delta t)| > 0$ for all $\Delta t < t$, then $G(t, \ell(t-)) \geq G_c$. We will show that, if $G(t, \ell(t-)) < G_c$ then ℓ is constant in a neighborhood of t, i.e. $|\ell(t + \Delta t) - \ell(t - \Delta t)| = 0$ for Δt sufficiently small.

Since G is continuous, there exists $\delta > 0$ such that, for all $|t' - t| \leq \delta$ and $|l' - \ell(t-)| \leq \delta$ we have $G(t', l') < G_c$. By continuity and pointwise convergence we can find $t_1 \in (t - \delta, t)$ such that $|\ell(t_1) - \ell(t-)| \leq \delta/2$. Hence for h sufficiently small $|\ell(t-) - \ell_h(t_1)| \leq \delta$ and therefore $G(t', \ell_h(t_1)) < G_c$ for $t_1 < t' \leq t + \delta$. It follows from the definition of ℓ_h that $\ell_h(t') = \ell_h(t_1)$ for $t' \in (t_1, t + \delta)$. Hence $\ell(t') = \ell(t_1)$ for $t' \in (t_1, t + \delta)$.

Next, assume that $l \in (\ell(t-), \ell(t+))$ and that $G(t, l) < G_c$. Again, there exists $\delta > 0$ such that $[l - \delta, l + \delta] \subset (\ell(t-), \ell(t+))$ and, for $|t - t'| \leq \delta$ and for $|l - l'| \leq \delta$, $G(t', l') < G_c$. Take, furthermore, $t_1 < t$ and $t_2 > t$ such that $|t_i - t| \leq \delta$, $i = 1, 2$. By continuity and pointwise convergence, for h sufficiently small, $\ell_h(t_1) \leq l - \delta$ and $\ell_h(t_2) \geq l + \delta$. By the definition of ℓ_h this is, however, impossible since for $t \in [t_1, t_2]$ we must necessarily have $\ell_h(t) \leq l$.

If $\ell(t+) > \ell(t-)$ then the third case, $l = \ell(t+) < L$, can be reduced to the case $l < \ell(t+)$. Namely, if $G(t, \ell(t+)) < G_c$ then, by continuity of G, there exists $l \in (\ell(t-), \ell(t+)$ such that $G(t, l) < G_c$ which again leads to a contradiction.

Finally, if $t = 0$ we have $G(0, l_0) = 0$ and by continuity, $G(t, l_0) < G_c$ for sufficiently small t. This implies that $\ell_h(t) = l_0$ and hence $\dot{\ell}(0) = 0$ for sufficiently small t. ∎

Lemma 4.2 *Let $T \in (0, +\infty)$ such that $T = \inf\{\tau : \ell(\tau) = L\}$. Then $G(t, \ell(t))$ is continuous in $[0, T)$.*

Proof. We show that $\lim_{s\to t} G(s, \ell(s)) = G(t, \ell(t))$. By Appendix 6.3 $G(t, l)$ is continuous in $(0, +\infty) \times (0, L)$ and thus the statement is trivial for any time t where $\ell(t)$ is continuous. Let us therefore consider $t \in S(l)$. By (4), $G(t, \ell(t)) \leq G_c$. Hence, by continuity, $\lim_{s\to t^-} G(s, \ell(s)) = G(t, \ell(t-)) \leq G_c$. By (6) we know that $G(t, \ell(t-)) \geq G_c$, thus $\lim_{s\to t^-} G(s, \ell(s)) = G(t, \ell(t-)) = G_c$. Similarly we get $\lim_{s\to t^+} G(s, \ell(s)) = G_c$. To conclude the proof, we note that, upon combining (4) and (6) again, it follows immediately that $G(t, \ell(t)) = G_c$. ∎

Any bounded monotone map ℓ is of bounded variation and hence its distributional derivative $\dot{\ell}$ can be decomposed as (cf. the monograph of Ambrosio *et al.* [1] and Appendix 6.1)

$$\dot{\ell} = D^a\ell + D^j\ell + D^c\ell = \tilde{D}\ell + D^j\ell.$$

The support of $D^j\ell$ is the set of jump discontinuities of ℓ which we have denoted by $S(\ell)$. We use this decomposition to prove a weak form of the activation criterion (5).

Lemma 4.3 *Let ℓ be a weak solution of (G); then*

$$(G(t, \ell(t)) - G_c)\, \tilde{D}\ell = 0 \qquad \textit{in the sense of measures,} \tag{11}$$

$$(G(t, \ell(t)) - G_c)\, D^j\ell(t) = 0 \quad \forall t \in S(\ell) \textit{ such that } \ell(t) \neq L\,. \tag{12}$$

Note that since the measures $D^a\ell$ and $D^c\ell$ are mutually singular, (11) implies that $(G(t, \ell(t)) - G_c)\, D^k\ell = 0$ for $k \in \{a, c\}$. Moreover, for every $t \in S(\ell)$ we have $(G(t, \ell(t)) - G_c))(\ell^+(t) - \ell^-(t)) = 0$, hence $G(t, \ell(t)) = G_c$ whenever $\ell(t) \neq L$. The special case $\ell(t) = L$ is not included since, by definition, $G(t, L) = 0$.

Proof. By (4) we know that $G(t, \ell(t)) \leq G_c$ for every $t \in (0, +\infty)$.

Let $t \in (0, +\infty)$ with $\ell(t) \neq L$ and such that $\left|\ell(t + \Delta t) - \ell(t - \Delta t)\right| > 0$ for $\Delta t < t$. By monotonicity $\ell(t-) \leq \ell(t) \leq \ell(t+)$ and by (6) $G(t, l) \geq G_c$ for $l \in [\ell(t-), \ell(t+)] \setminus L$. For $l = \ell(t)$ it follows that $G(t, \ell(t)) = G_c$.

Let $t \in (0, +\infty)$ such that $\left|\ell(t + \Delta t) - \ell(t - \Delta t)\right| = 0$ for some Δt. By monotonicity $\ell(s) = \ell(t - \Delta t)$ for $s \in [t - \Delta t, t + \Delta t]$. Hence $D^a\ell = \dot{\ell} = 0$ in $(t - \Delta t, t + \Delta t)$. In other terms, $(t - \Delta t, t + \Delta t)$ is not contained in the support of the measure $\dot{\ell}$.

Therefore, (11) and (12) are satisfied. ∎

Proposition 4.4 *Suppose that (3) is satisfied and that ℓ is a weak solution of (G). If $0 \leq t_1 < t_2$ are points of continuity for ℓ and if $\ell(t_2) < L$ then*

$$H(t_2, \ell(t_2)) - H(t_1, \ell(t_1)) = \int_{t_1}^{t_2} \mathscr{P}(t)\, \mathrm{d}t + \sum_{t \in S(\ell) \cap (t_1, t_2)} \mathscr{D}(t),$$

where $\mathscr{P}$ is the power of external forces and $\mathscr{D}$ the dissipated energy, respectively given by

$$\mathscr{P}(t) = \mu \int_{\partial_D \Omega} \dot{g}(t) \nabla u(t, \ell(t)) \cdot \hat{n} \, \mathrm{d}y \tag{13}$$

$$\mathscr{D}(t) = \int_{\ell(t-)}^{\ell(t+)} \big[G(t,l) - G_c\big] \, \mathrm{d}l. \tag{14}$$

Before we prove this result, let us note that the integrand in (14) is non-negative by condition (6). **Proof.** Since $H \in C^1((0,+\infty) \times (0,L))$, by Lemma 6.2 it follows that $H(t,\ell(t)) \in BV(t_1,t_2)$. Hence $D(H(t,\ell(t))) = \tilde{D}H + D^j H$. In particular, by Lemma 4.3

$$\begin{aligned}
\tilde{D}H(t,\ell(t)) &= \frac{\partial H}{\partial t}(t,\ell(t))dt + \frac{\partial H}{\partial l}(t,\ell(t))\tilde{D}\ell \\
&= \mathscr{P}(t)\, dt - (G(t,\ell(t)) - G_c)\tilde{D}\ell = \mathscr{P}(t) dt\,.
\end{aligned}$$

Moreover, setting $\ell^{\pm} = \ell(t\pm)$, we have

$$\begin{aligned}
D^j H &= \sum_{t \in S(\ell)} \Big(H(t,\ell^+) - H(t,\ell^-) \Big) \delta_t \\
&= \sum_{t \in S(\ell)} \Big(\mathscr{E}(t,\ell^+(t)) - \mathscr{E}(t,\ell^-(t)) + G_c(\ell^+ - \ell^-) \Big) \delta_t \\
&= \sum_{t \in S(\ell)} \left(- \int_{\ell^-}^{\ell^+} G(t,s)\, \mathrm{d}s + G_c(\ell^+ - \ell^-) \right),
\end{aligned}$$

and hence

$$D^j H = - \sum_{t \in S(\ell)} \int_{\ell^-(t)}^{\ell^+(t)} \big(G(t,s) - G_c \big)\, \mathrm{d}s = \sum_{t \in S(\ell)} \mathscr{D}(t)\,,$$

where $\mathscr{D}$ denotes the 'dissipated' energy.

It follows that,

$$\begin{aligned}
H(t_2,\ell(t_2)) - H(t_1,\ell(t_1)) &= \int_{t_1}^{t_2} DH(t,\ell(t)) \\
&= \int_{t_1}^{t_2} \mathscr{P}(t)\, dt + \sum_{t \in S(\ell) \cap (t_1,t_2)} \mathscr{D}(t).
\end{aligned}$$

■

5 Numerical approximation

In the previous sections, we have provided an abstract existence theory for a "weak" form of Griffith's model of fracture in brittle solids. One of our main motivations to pursue this line of work were questions of computability. While abstract convergence proofs for numerical discretizations of the model of Francfort & Marigo[10] exist, they could, so far, not be formulated algorithmically due to the non-convexity of the energy functional. By contrast, since (G) is a local theory, the solutions can be computed by local searches (rather than global searches as in (FM)) which renders the discretizations computable. We formulate two general algorithms for the computation of fracture if the crack path is known *a priori*, prove their convergence, and, in the case of monotone increasing loads, even give a convergence rate.

5.1 The general case

We begin by describing a general numerical algorithm for crack propagation if the crack path is known *a priori*. For each $h > 0$ let $(t_j^h)_{j=0}^{\infty}$ be a partition of $[0, +\infty)$, satisfying

$$0 = t_0^h < t_1^h < \cdots < t_j^h \to +\infty \quad \text{as } j \to \infty, \text{ and}$$
$$\sup_{j \geq 1}(t_j^h - t_{j-1}^h) \to 0 \quad \text{as } h \to 0.$$

Similarly, let $(l_m^h)_{m=0}^{M_h}$, $M_h \in \mathbb{N}$, be a partition of the crack path, i.e.,

$$l_0 = l_0^h < l_1^h < \cdots < l_{M_h}^h = L,$$

satisfying also $\max_{m=1,\ldots,M_h}(l_m^j - l_{m-1}^h) \to 0$ as $h \to 0$. Finally, we assume that for each t_j^h and l_m^h we can compute a *discrete energy release rate* $G_h(t_j^h, l_m^h)$ which should be an approximation to the exact energy resease rate $G(t_j^h, l_m^h)$.

Let us discuss these assumptions in the light of a finite element method. For each $h > 0$ let $\mathscr{T}_h$ be a regular finite element mesh with $\operatorname{diam}(\kappa) \leq h$ for all $\kappa \in \mathscr{T}_h$. Assume that the mesh is such that the crack path is a union of edges in $\mathscr{T}_h$. This gives a natural discretization of the crack path. For the time-discretization we could simply take $t_j^h = j\Delta t$, where $\Delta t \to 0$ as $h \to 0$. The energy release rate could be computed in several different ways. A naive approach would be to use finite differences, i.e., to take

$$G_h(t_j^h, l_m^h) = \frac{\mathcal{E}_h(t_j^h, l_{m+1}^h) - \mathcal{E}_h(t_j^h, l_m^h)}{l_{m+1}^h - l_m^h}.$$

This seems like a reasonable choice, however, for a standard finite element method we are unable to prove convergence of the evolution with this choice of G_h (we do observe it in practice though). If, however, we supplement the finite element space with the singularities of the domain then $\mathcal{E}_h$ converges fast enough so that we do obtain a condition on G_h to guarantee convergence of the method.

A more sophisticated way to compute G_h, which is also dominant in the fracture mechanics literature is to use the relationship between G and the stress intensity factors. We shall discuss this option in more detail in §5.3 where we show that there exist discrete energy release rates G_h which approximate G is a sufficiently strong fashion.

In general, we shall require that G_h converges to G locally uniformly in $[0, +\infty) \times [l_0, L)$, i.e., that for all $T' \in (0, +\infty)$ and for all $L' \in (l_0, L)$, it holds that

$$\max_{\substack{0 \leq t_j^h \leq T' \\ l_0 \leq l_m^h \leq L'}} |G(t_j^h, l_m^h) - G_h(t_j^h, l_m^h)| \to 0 \quad \text{as } h \to 0. \tag{15}$$

After these preparations, we formulate the algorithm for crack propagation as follows:

Algorithm 1 **1.** *Set* $\ell_h(t_0^h) = l_0 = l_0^h$.

2. *For* $j = 1, 2, 3, \ldots,$

2.1. *Set* $\ell_h(t_j^h) = \min\{l_m^h \geq \ell_h(t_{j-1}^h) : G_h(t_j^h, l_m^h) < G_c\}$.

2.2. *If* $\ell(t_j^h) = L$ *go to (3).*

3. *Set $\ell(t_i^h) = L$ for $i > j$.*

Note that $\ell_h(t_j^h)$ can be computed recursively upon checking the stability inequality $G(t,l) < G_c$ for $l = l_m^h, l_{m+1}^h, ...,$ where $l_m^h = \ell_h(t_{j-1}^h)$. As in the case of the existence proof we extend ℓ_h by

$$\ell_h(t) = \ell_h(t_j^h) \qquad \text{if } t_{j-1}^h < t \leq t_j^h.$$

Theorem 5.1 *If the discretization G_h of the energy release rate satisfies* (15)*, then there exists a subsequence $h_n \downarrow 0$ and an increasing map $\ell\colon [0,+\infty) \to [l_0, L]$ which is a weak solution to (G) (in the sense of Definition 2.2) such that*

$$\begin{aligned} \ell_{h_n}(t) \to \ell(t) \qquad & \forall t \in [0,+\infty), \text{ and} \\ \ell_{h_n} \overset{*}{\rightharpoonup} \ell \qquad & \text{in } \mathrm{BV}(0,+\infty). \end{aligned}$$

Proof. As in the existence proof we can extract a subsequence for which we can find a monotone limit function so that the stated convergence holds. For simplicity of notation, we drop the subscript n, i.e., we assume that $\ell_h \to \ell$ pointwise and weakly-$*$ in $\mathrm{BV}(0,+\infty)$. We are thus left to prove that the limit curve ℓ is a weak solution to (G).

The pointwise convergence implies that $\ell(0) = l_0$.

To prove that $G(t,\ell(t)) \leq G_c$, suppose that $\ell(t) < L$ and let $t_j^h = \min\{t_k^h : k \in \mathbb{N}, t_k^h \geq t\}$ so that $(t_j^h, \ell(t_j^h)) \to (t, \ell(t))$ and hence $G_h(t_j^h, \ell(t_j^h)) \to G(t,\ell(t))$ as $h \to 0$. Thus, $G(t,\ell(t)) \leq G_c$.

For the proof of the activation criterion (6) we also follow closely the proof of Theorem 4.1. Suppose that $G(t,\ell(t-)) < G_c$; we will show that $\ell(t)$ is constant in a neighborhood of t. In this case there exist $\delta, \gamma > 0$ such that, for all $|t'-t| < \delta$ and $|l' - \ell(t-)| < \delta$, we have $G(t',l') \leq G_c - \gamma < G_c$. Since $G_h \to G$ locally uniformly, it follows that for n sufficiently small, we have $G_{h_n}(t_n, l_n) < G_c$ for all gridpoints t_n, l_n such that $|t - t_n| < \delta$ and $|\ell(t-) - l_n| < \delta$.

In the following we always assume implicitly that h is sufficiently small (viz. n sufficiently large). Let $t_1 \in (t-\delta, t)$ such that $|\ell(t_1) - \ell(t-)| \leq \delta/2$ and, for every h, let $t_{j_1}^h = \min\{t_k^h \geq t_1 : k \in \mathbb{N}\}$. In this case $t_{j_1}^h \to t_1$ as $h \downarrow 0$; then $\ell_h(t_{j_1}^h) \to \ell(t_1)$ and $|\ell(t_{j_1}^h) - \ell(t-)| \leq \delta$ (for h small). Hence by the continuity of G we get $G(t_{j_1}^h, \ell_h(t_{j_1}^h)) < G_c$. Furthermore, by the construction of ℓ_h, $\ell_h(t) = \ell_h(t_{j_1}^h)$ for $t \in (t_{j_1}^h, t+\delta)$. Since $\ell_h \to \ell$ pointwise, $\ell(t)$ is constant in $(t_1, t+\delta)$.

Next, assume that $l \in (\ell(t-), \ell(t+))$ and that $G(t,l) < G_c$. There exists $\delta > 0$ such that $[l-\delta, l+\delta] \subset (\ell(t-), \ell(t+))$ and, similarly as above, for h sufficiently small, for $|t_j^h - t| \leq \delta$ and for $|l_m^h - l| \leq \delta$, we have $G_h(t_j^h, l_m^h) < G_c$. Take $t_1 < t$ and $t_2 > t$ such that $|t_i - t| < \delta$, $i = 1,2$ and set

$$t_{j_1} = \min\{t_k^h \geq t_1 : k \in \mathbb{N}\} \quad \text{and} \quad t_{j_2}^h = \max\{t_k^h \leq t_2 : k \in \mathbb{N}\}.$$

For sufficiently small h, we have $\ell_h(t_{j_1}^h) \leq l - \delta$ and $\ell_h(t_{j_2}^h) \geq l + \delta$. By the construction of ℓ_h this is, however, impossible since for $t \leq t + \delta$ we must have $\ell_h(t) \leq l$.

The case $l = \ell(t+) < L$ and the case $t = 0$ can be treated as in the existence theorem. ∎

5.2 The case of monotone increasing loads

If the load $g(t)$ is monotone increasing ($g(t) = tg(1)$) then the numerical algorithm does not require a time-discretization but we still need a discretization $(l_m^h)_{m=0}^{M_h}$ of the crack path as in §5.1. A useful crack-propagation algorithm is given by the following formulation.

Algorithm 2 **1.** *Set* $t_0 = 0$.

2. *For* $m = 1, 2, \dots, M_h$ *do*

2.1. *Let* $t_m = \sup\{t \geq t_{m-1} : G_h(t, l^h_{m-1}) < G_c\}$.

2.2. *Set* $\ell_h(t) = l^h_{m-1}$ *for* $t \in [t_{m-1}, t_m)$.

Note, in particular, that t_m is computed explicitly by the formula

$$t_m = \max(t_{m-1}, \sqrt{G_c/G_h(1, l_{m-1})})\,.$$

For $t < t_m$ we have $G_h(t, l^h_{m-1}) < G_c$ and hence the crack must remain in l^h_{m-1}. For $t > t_m$, $G_h(t, l^h_{m-1}) > G_c$ and hence the crack is required to propagate. For this algorithm, we can see that ℓ_h is in fact constructed by the same procedure as in Section 3.3.

Proposition 5.2 *The function* ℓ_h *computed by Algorithm 2 is the right-continuous monotone inverse of* $\tau_h = \sqrt{G_c/G_{h,m}}$ *where* $G_{h,m}$ *is the monotone decreasing envelope of the function*

$$G_h(1, l) = G_h(1, l^h_{m-1}) \qquad \text{if } l \in [l^h_{m-1}, l^h_m).$$

Proof. For this proof we only need to show that

$$t_m = \tau_h(l^h_{m-1}) = \sqrt{G_c/G_{h,m}(1, l^h_{m-1})} \tag{16}$$

for $m = 1, 2, \dots, M_h - 1$. The result then follows immediately.

For $m = 1$, the statement is trivial since $t_0 = 0$ and $G_h(1, l_0) = G_{h,m}(1, l_0)$. Note that

$$G_{h,m}(1, l^h_m) = \min\{G_{h,m}(1, l^h_{m-1}), G_h(1, l^h_m)\}\,.$$

As $\tau_h(l^h_m) = \sqrt{G_c/G_{h,m}(1, l^h_m)}$

$$\tau_h(l^h_m) = \max\{\tau_h(l^h_{m-1}), \sqrt{G_c/G_h(1, l^h_m)}\}\,.$$

Since $\tau_h(l^h_1) = t_1$ by induction we get $\tau_h = t_m$ for $m \geq 1$. ∎

Recall that $\tau(l)$ can be interpreted as the time at which ℓ reaches or moves past the crack length l. Similarly, $\tau_h(l^h_m)$ is the time at which ℓ_h reaches or moves past the discrete crack length l^h_m. This is an alternative observable for which the error $\tau - \tau_h$ may be of interest. Therefore, we begin to estimate the error $G_m - G_{h,m}$.

Lemma 5.3 *Let* $G_h(1, \cdot)\colon [l_0, L] \to [0, +\infty)$ *with* $G_h(L) = 0$*; then*

$$|G_{h,m}(1, l) - G_m(1, l)| \leq \|G_h(1, \cdot) - G(1, \cdot)\|_{\mathrm{L}^\infty(l_0, l)} \qquad \forall l_0 < l < L.$$

Proof. First of all, we note that taking the monotone envelope taken on $[l_0, L]$ and restricting it to $[l_0, l]$, we obtain the monotone envelope on $[l_0, l]$.

Let us write $G_m(1, l) - \|G - G_h\|_{\mathrm{L}^\infty(l_0, l)} \leq G(1, l) - \|G - G_h\|_{\mathrm{L}^\infty(l_0, l)} \leq G_h(1, l)$. Since $G_m(1, l) - \|G - G_h\|_{\mathrm{L}^\infty(l_0, l)}$ is monotone, it follows (by the definition of monotone envelope) that $G_m(1, l) - \|G - G_h\|_{\mathrm{L}^\infty(l, l_0)} \leq G_{h,m}$.

For the opposite inequality, note that $G_{h,m}(1, l) \leq G_h(1, l) \leq G(1, l) + \|G - G_h\|_{\mathrm{L}^\infty(l_0, l)}$. As $G_{h,m}$ is monotone and $\|G - G_h\|_{\mathrm{L}^\infty(l_0, l)}$ is constant, we get $G_{h,m}(1, l) \leq G_m(1, l) + \|G - G_h\|_{\mathrm{L}^\infty(l_0, l)}$. ∎

We remark that, thanks to Lemma 5.3, from the convergence of G_h to G, which is studied in the next section, follows also the convergence of τ_h to τ. Hence ℓ_h converges to ℓ where $\ell \in \{\tau\}^{-1}$ is a weak solution of (G). Concrete error estimates can immediately be given at any point t such that τ is uniformly increasing in a neighbourhood of $\ell(t)$.

5.3 Discrete Energy Release Rates

For both convergence proofs of the two preceding sections we used the assumption that the "discrete energy release rate" G_h converges locally uniformly to the exact energy release rate G in the sense of (15). The purpose of this section is to briefly discuss this assumption and show that it can be quite easily satisfied in practice. To this end, we give a very simple method for which we prove that the condition of local uniform convergence is satisfied. This section is not intended as a detailed review of the practical computation of stress intensity factors. We only demonstrate at an example how error estimates which are available for several methods in the literature can be extended to satisfy the conditions required by our convergence analysis.

The difficulty for the computation of elastic solutions in the presence of cracks are the singularities which the exact solution exhibits at the crack tips and which cause a significant deterioration of the convergence rate for standard finite element methods. Several methods have been suggested to overcome this difficulty, usually based on local mesh refinement or on supplementing the finite element space in some way with the (known) singularity. The latter has probably led to the most reliable and efficient methods available.

To simplify the following discussion, we require H^2-regularity of the elastic solution (except at the singular corner) and we therefore need to assume that

$$\frac{\partial g(t,x,y)}{\partial x} = 0 \text{ for } x \in \{0, L\}, \quad \text{and} \quad g(t,x,y) = -g(t,x,-y) \text{ for } (x,y) \in \Omega. \tag{17}$$

It should be possible to use $\mathrm{W}^{2,p}$ regularity instead as we have done in Section 6.2, however, the corresponding finite element theory is far less developed. We use the symbol $g(t)$ also to denote the extension of $g(t)$ from the boundary $\partial_D\Omega^+$ to the domain given by the formula $g(t,x,y) = yg(t,x,1)$.

We begin by rewriting the elastic minimization problem as a weak Laplace problem in a polygonal domain. Using the symmetry of g about the y axis, it follows that u is symmetric and that, in the subdomain $\Omega^+ = (0,L)\times(-1,1)$ (we define $\partial_D\Omega^+$ and $\partial_N\Omega^+$ accordingly) it is the unique weak solution of

$$\begin{aligned} -\Delta u &= 0 \qquad \text{in } \Omega^+ \setminus K(l) \\ u &= g(t) \qquad \text{on } \partial_D\Omega^+ \\ \nabla u \cdot n &= 0 \qquad \text{on } \partial_N\Omega^+ \cup K(l). \end{aligned} \tag{18}$$

From a straightforward energy argument, it follows that

$$\|\nabla u\|_{\mathrm{L}^2(\Omega^+)} \le \|\nabla g(t)\|_{\mathrm{L}^2(\Omega^+)}. \tag{19}$$

The special structure of Ω^+ (a union of two rectangles) implies that the Poincaré constant is independent of l, i.e, we obtain

$$\|u\|_{\mathrm{L}^2} \le C_p \|\nabla g(t)\|_{\mathrm{L}^2}, \tag{20}$$

where C_p is independent of l.

After these preliminaries, we review the singular function representation of the solution. It can be shown that

$$u = \kappa S \eta + w, \tag{21}$$

where $\kappa \in \mathbb{R}$, $w \in \mathrm{H}^2(\Omega)$ and

$$S(\rho, \theta) = \rho^{1/2} \sin(\theta/2).$$

The cut-off function $\eta \in \mathrm{H}^2(\Omega)$ is radially symmetric about the crack tip, $\eta = 1$ in $B((l,0),R_1)$ and $\eta = 0$ in $\Omega^+ \setminus B((l,0),R_2)$, where $0 < R_1 < R_2$ such that $\mathrm{dist}((l,0),\partial\Omega^+) < R_2$. The pair (ρ,θ) are polar coordinates at the tip $(l,0)$.

This result can be obtained, for example from Theorem 2.4.3 and the remarks in Section 2.7 in the monograph of Grisvard [15]. Alternatively, one can apply Corollary 4.4.3.8 from Grisvard's earlier monograph[14] with $p = 2+\varepsilon$, letting $\varepsilon \to 0$ and using some simple H^2 *a priori* estimates on the solution.

The coefficient $\kappa = \kappa(t,l)$ of the singularity is called the *stress intensity factor*. Moreover, $G(t,l)$ is given by

$$G(t,l) = \frac{\mu\pi}{4}\kappa(t,l)^2. \tag{22}$$

For a proof of (22) see §6.3.

If the solution u to (18) is known then the stress intensity factor κ can be extracted via the formula

$$\kappa = \int_\Omega u(\Delta S^*)\,\mathrm{d}x, \tag{23}$$

where S^* is the dual singular function given by

$$S^*(\rho,\theta) = \rho^{-1/2}\sin(\theta/2).$$

The proof of (23) is a straightforward modification of Theorem 2.1 in Blum & Dobrowolski[3].

For each $h > 0$ let S_h be a P1 finite element space with a mesh $\mathscr{T}_h$ of Ω^+ of size h and uniform quality triangles (cf. [7]). Assume furthermore that the mesh is symmetric about the x-axis, so that the edges are aligned and give a natural crack path discretization $(l_m^h)_{m=0}^{M_h}$ where $(l_m^h, 0)$ are vertices of the finite element mesh and $l_0 = l_0^h$ for all h. Let $u_h \in S_h$ be the solution to the P1 finite element discretization of (18), i.e., $u_h|_{\partial_D\Omega^+} = g(t)$ (for simplicity we assume that $g \in S_h$) and

$$\int_\Omega \nabla u_h \nabla v_h\,\mathrm{d}x = 0 \qquad \forall v_h \in S_{0,h},$$

where $S_{0,h} = \{v_h \in S_h : v_h|_{\partial_D\Omega^+} = 0\}$.

For some $l = l_m^h$, we can thus define the discrete stress intensity factor κ_h by the extraction formula

$$\kappa_h(l_m^h) = \int_{\Omega\setminus K(l_m^h)} u_h(\Delta S^*)\,\mathrm{d}x. \tag{24}$$

Proposition 5.4 *Suppose that g satisfies (17). For fixed l and t we have*

$$|\kappa_h - \kappa| \le C\,\|\nabla g(t)\|_{\mathrm{L}^2}\,\max(e_h, h),$$

where $e_h \to 0$ as $h \to 0$ and C is a constant that depends only on $L-l$ but not on h or on t.

Proof. It follows from standard finite element theory, in particular Céa's Lemma[7], that

$$\|\nabla(u-u_h)\|_{\mathrm{L}^2} \le C_1 h|w|_{\mathrm{H}^2} + |\kappa| \inf_{v_h\in S_h} \|\nabla(v_h - S)\|_{\mathrm{L}^2},$$

where C_1 (resulting from the interpolation error estimate) depends only on the mesh quality and can be taken independent of t and l.

Upon splitting the domain into an upper and a lower part and noting that $u - g = 0$ at $\{y = 0\}$, we may use Theorem 2.2.1 in Grisvard[15] to estimate

$$\begin{aligned} |w|_{\mathrm{H}^2} &\le |w-g|_{\mathrm{H}^2} + |g|_{\mathrm{H}^2} \le 2\|\Delta(w-g)\|_{\mathrm{L}^2} + |g|_{\mathrm{H}^2} \\ &\le 2\|\Delta w\|_{\mathrm{L}^2} + 3|g|_{\mathrm{H}^2} = 2|\kappa|\|\Delta S\|_{\mathrm{L}^2} + 3|g|_{\mathrm{H}^2}. \end{aligned}$$

From the extraction formula (23) we deduce

$$|\kappa| \le \|u\|_{\mathrm{L}^2}\|\Delta S^*\|_{\mathrm{L}^2}.$$

The norms $\|\Delta S\|_{\mathrm{L}^2}$ and $\|\Delta S^*\|_{\mathrm{L}^2}$ depends only on R_1 and R_2 and can be bounded by a constant C_2, which depends only on $L - l$ and we therefore obtain

$$|w|_{\mathrm{H}^2} \le 2C_2^2\|u\|_{\mathrm{L}^2} + 3|g|_{\mathrm{H}^2}.$$

We note here that while S has a singularity at $(l, 0)$, $\Delta S = \Delta S^* = 0$ in a neighbourhood of this point and are therefore sufficiently smooth. Recalling that the Poincaré constant C_p of $\Omega^+ \setminus K(l)$ can be bounded independently of l, we obtain from (20)

$$|\kappa| + |w|_{\mathrm{H}^2} \le C_3\|\nabla g(t)\|_{\mathrm{L}^2} + 3|g|_{\mathrm{H}^2},$$

where $C_3 = 2C_2^2C_p + C_2C_p$.

Since $S \in \mathrm{H}^1$, we have

$$e_h := \inf_{v_h \in S_h} \|v_h - \hat{s}\|_{\mathrm{H}^1} \to 0$$

as $h \to 0$ (in fact $e_h \sim h^{1/2}$) and hence,

$$|\kappa - \kappa_h| \le C_2\|u - u_h\|_{\mathrm{L}^2} \le C\|g(t)\|_{\mathrm{H}^2} \max(e_h, h)$$

where $C = \max(C_pC_2C_3, 3)$. ∎

Since we assumed that $t \mapsto g(t) \in \mathrm{C}^1_{\mathrm{loc}}([0, +\infty); \mathrm{H}^2(\Omega))$ it follows immediately from Proposition 5.4 that condition (15) is satisfied if we define

$$G_h(t_j^h, l_m^h) = \frac{\pi\mu}{4}\kappa_h^2.$$

There exist many improvements in the literature over the simple idea which we used here. We would like to mention the recent works of Brenner and Sung[5] on multigrid methods and that of Cai *et al*[6] which presents a particularly sophisticated application of the extraction formula. Unfortunately, it is common in the numerical analysis literature to only consider the pure Dirichlet case which does not cover a true fracture problem. The latter reference is a notable exception in this respect.

6 Appendix

6.1 BV functions of a scalar variable

This section contains a brief presentation of some standard results on functions of bounded variation which are used throughout the paper. We refer to the monograph of Ambrosio et al.[1] for further details and proofs.

Let $I=(a,b)$ where $a,b\in[-\infty,+\infty]$. Let $\mathcal{M}(I)$ be the space of finite Radon measures on I, $\mathrm{C}_c(I)$ be the space of continuous functions with compact support and $\mathrm{C}_0(I)$ be the closure of $\mathrm{C}_c(I)$ with respect to the L^∞ norm. By the Riesz Representation Theorem $\mathcal{M}(I)$ is the dual of $\mathrm{C}_0(I)$ by means of the duality pairing

$$\langle \mu,\varphi\rangle=\int_I \varphi\,\mathrm{d}\mu\,.$$

This formula also makes sense also if φ is bounded and continuous.

A function $u\in \mathrm{L}^1(I)$ is said to be of bounded variation if its distributional derivative is a finite Radon measure, i.e. if there exists $\mu\in\mathcal{M}(I)$ such that

$$-\langle u,\varphi'\rangle=\int_I \varphi\,\mathrm{d}\mu\qquad \forall\varphi\in \mathrm{C}^\infty_c(I)\,.$$

If $u\in BV(I)$ we write Du instead of μ and we denote by $|Du|(I)$ its total variation in I. For $u\in BV(I)$, the measure Du can be decomposed in terms of mutually singular measures by

$$Du=D^lu+D^ju+D^cu=\tilde{D}u+D^ju\,,$$

where D^lu is absolutely continuous with respect to the Lebesgue measure, D^ju is concentrated on the jump set $S(u)$ and D^cu is the Cantor part ($\tilde{D}u$ stands for the sum of D^lu and D^cu). Denoting by $\mathrm{d}t$ the Lebesgue measure and by δ_t the Dirac measure concentrated in t, we write

$$D^lu=u'dt,\qquad \text{and}\quad D^ju=\sum_{t\in S(u)}(u^+-u^-)\,\delta_t\,.$$

Remark 6.1 *Note that thinking of t and u as time and space variables may give the wrong intuition that Du is a velocity. As a matter of fact, Du is dimensionally a space variable as are all the measures D^lu, D^ju and D^cu. On the other hand, the density u' can be interpreted as a velocity.*

Finally, we recall a chain rule in BV for the proof of which we refer to Theorem 3.93 of Ambrosio *et al.*[1].

Lemma 6.2 *For $l_0>0$ let $u:(0,+\infty)\to[l_0,L]$ be monotone. Let $f\in C^1((0,+\infty)\times(l_0,L))$. Then, $v=f\circ u\in \mathrm{BV}_{\mathrm{loc}}(0,+\infty)$ and*

$$Dv=\frac{\partial f}{\partial t}\big(t,u(t)\big)\,\mathrm{d}t+\frac{\partial f}{\partial l}(t,u(t))\tilde{D}u+\sum_{t\in S(u)}\big(f(t,u^+(t))-f(t,u^-(t))\big)\delta_t\,.$$

6.2 Regularity of the displacement field

In this section we investigate the regularity of the elastic solutions. The following technical Lemma prepares for a proof of the regularity of G in the next section.

Let $\Omega^+=(0,L)\times(-1,1)$ and let $\partial_D\Omega^+=[0,L]\times\{-1,1\}$, $\partial_N\Omega^+=\partial\Omega^+\setminus\partial_D\Omega^+$. For $l\in(0,L)$ let $K^+(l)=[0,l]\times\{0\}$.

Lemma 6.3 *Fix $p < 2$. For $f \in \mathrm{L}^p(\Omega^+)$, $g \in \mathrm{W}^{2-1/p,p}(\partial_D\Omega^+)$ and $l \in (0, L)$ there exists a unique solution $v \in \mathrm{H}^1(\Omega^+ \setminus K^+(l))$ of*

$$\begin{cases} -\Delta v = f & \Omega^+ \setminus K^+(l) \\ \nabla v \cdot n = 0 & \partial_N\Omega^+ \cup K^+(l) \\ v = g & \partial_D\Omega^+ \,. \end{cases} \tag{25}$$

For a unique real value κ the function v can be written in the form

$$v(x) = \kappa\, \hat{s}(\rho, \theta) + w(x)\,,$$

where $w \in \mathrm{W}^{2,p}(\Omega^+ \setminus K^+(l))$ and $\hat{s}(\rho, \theta) = \rho^{1/2} \sin(\theta/2)$, and (ρ, θ) are the polar coordinates with origin $(l, 0)$. Moreover, the mapping $\mathcal{V} : \mathrm{L}^p(\Omega^+) \times \mathrm{W}^{2-1/p,p}(\partial_D\Omega^+) \to \mathbb{R} \times \mathrm{W}^{2,p}(\Omega^+)$ defined by $(f, g) \mapsto (\kappa, w)$ is linear and continuous. Hence, there exists $C > 0$ (depending on l and p) such that $|\kappa| + \|w\|_{W^{2,p}} \leq C(\|f\|_{\mathrm{L}^p} + \|g\|_{\mathrm{W}^{2-1/p,p}})$.

Proof. Invoking Corollary 4.4.3.8 in Grisvard[14] it is easy to prove the first part of this Lemma, i.e uniqueness of v and its representation formula which imply that $\mathcal{V}$ is well-defined. The linearity of $\mathcal{V}$ is trivial. It remains to show that $\mathcal{V}$ is continuous. This however, follows from the inverse mapping theorem and the easily established fact that $\mathcal{V}$ is bijective. ■

We remark that the reason we obtain $\mathrm{W}^{2,p}$ regularity of w for $p < 2$ only is due to the non-homogeneous mixed boundary condition. As a matter of fact the theory of Grisvard, based on Fredholm operators, does not include the case $p = 2$. In our case, to have $w \in \mathrm{H}^2(\Omega^+)$ it is sufficient that the function

$$\psi = \begin{cases} \nabla g \cdot \hat{e}_1 & \text{in } \partial_D\Omega^+ \\ 0 & \text{in } \partial_N\Omega^+ \end{cases}$$

belongs to $\mathrm{H}^{1/2}(\partial\Omega^+)$. This condition can be found in an interesting paper of Banasiak and Roach[2]. In Section 5.3, where we required a somewhat more stringent regularity we achieved this through (17). Lemma 6.3, however, is aimed at proving the regularity of the energy release for which $\mathrm{W}^{2,p}$-regularity is sufficient.

6.3 Regularity of the free energy

In this section we will show that the free energy $H(t, l)$ belongs to the space $\mathrm{C}^1([0, +\infty) \times (0, L))$. Obviously it is enough to show that $\mathscr{E}(t, l) \in \mathrm{C}^1$.

Let us consider $\partial\mathscr{E}/\partial t$. By Lemma 6.3 the mapping $g \mapsto u$ (where u solves (25) with $f = 0$) is linear and continuous from $\mathrm{W}^{2-1/p,p}(\partial_D\Omega^+)$ to $\mathrm{H}^1(\Omega^+ \setminus K(l))$. Since $g(t)$ belongs to $\mathrm{C}^1([0, +\infty), \mathrm{H}^{3/2}(\partial_D\Omega))$ it follows easily that $t \mapsto g(t) \mapsto u(t, \cdot)$ is C^1 as well. As $\mathscr{E}$ is just the H^1 seminorm squared then $H(t, \cdot)$ is C^1 with respect to t and

$$\begin{aligned} \frac{\partial\mathscr{E}}{\partial t}(t, l) &= \mu \int_{\Omega\setminus K(l)} \nabla u(t, l) \nabla \dot{u}(t, l)\, \mathrm{d}x \\ &= \mu \int_{\partial_D\Omega} (\nabla u(t, l) \cdot n) \dot{u}(t, l)\, \mathrm{d}y \\ &= \mu \int_{\partial_D\Omega} (\nabla u(t, l) \cdot n) \dot{g}(t, l)\, \mathrm{d}y\,, \end{aligned}$$

where the last expression is the power of external forces.

The partial derivative $\partial\mathscr{E}/\partial l$ requires additional effort. Part of this result is already contained in Grisvard[15] Section 6.4.2 where it is shown that $l \mapsto \mathscr{E}(t,l)$ is differentiable in $(0,L)$ and that

$$G(t,l) = -\frac{\partial\mathscr{E}}{\partial l}(t,l) = \kappa^2\mu(\pi/4)\,,$$

where $\kappa = \kappa(t,l)$ is the stress intensity factor appearing in Lemma 6.3. It thus remains for us to prove that κ is continuous.[3]

Let us consider the pairs (t_1,l_1) and (t_2,l_2). For $K_i = K(l_i)$ let u_i be the solution of

$$\begin{cases} -\Delta u_i = 0 & \Omega\setminus K_i \\ \nabla u_i\cdot n = 0 & \partial_N\Omega\cup K_i \\ u_i = g(t_i) & \partial_D\Omega\,. \end{cases}$$

Let $\rho \in \mathrm{C}_c^\infty(\Omega,[0,1])$ be symmetric with respect to x_1 and x_2 and such that $\rho = 1$ in a (sufficiently small) neighborhood of K_2. Let us denote by ψ the mapping

$$\psi(x) = \rho(x)x(l_1/l_2) + \big(1-\rho(x)\big)x = x + \big((l_1/l_2)-1\big)\rho(x)x.$$

Since our arguments are local, we can assume that $l_2 - l_1$ is sufficiently small, in such a way that ψ is a smooth diffeomorphism of Ω to itself, mapping $\Omega\setminus K_2$ to $\Omega\setminus K_1$. Clearly, $\psi(x) = (l_1/l_2)x$ in a neighborhood of K_2 (where $\rho = 1$) and $\psi(x) = x$ in a neighborhood of $\partial\Omega$ (where $\rho = 0$) while

$$\nabla\psi(x) = I + \big((l_1/l_2)-1\big)\nabla(\rho(x)x)\,.$$

Let us define $\tilde{u}_2(x) = u_1(\psi(x))$ and let $f = -\Delta\tilde{u}_2$. Considering that $\nabla\psi = (l_1/l_2)I$ in a neighborhood of K_2 and that $\nabla\psi = I$ in a neighborhood of $\partial\Omega$ we get easily

$$\begin{cases} -\Delta\tilde{u}_2 = f & \Omega\setminus K_2 \\ \nabla\tilde{u}_2\cdot n = 0 & \partial_N\Omega\cup K_2 \\ \tilde{u}_2 = g(t_1) & \partial_D\Omega\,. \end{cases}$$

Next, we show that $f \in \mathrm{L}^2(\Omega)$ and that $\|f\|_{\mathrm{L}^2} \le C\,|l_1 - l_2|$. In a neighborhood of K_2 and in a neighborhood of $\partial_D\Omega$, where ψ is an homothety, we have $f = -\Delta\tilde{u}_2 = 0$. Thus, f has compact support in $\Omega\setminus K_2$. Moreover it is smooth in the interior, since u is harmonic and ψ is smooth. Thus, it is sufficient to control $\|f\|_{\mathrm{L}^2}$ in a set $\Omega' \subset\subset (\Omega\setminus K_2)$ containing the support of f. Denoting by id the identity function, by $\mathrm{tr}(\cdot)$ the trace operator and writing explicitly the term $\nabla\psi$ we get

$$\begin{aligned} f &= -\mathrm{div}\big(\nabla u_1(\psi)\nabla\psi\big) \\ &= -\mathrm{div}\big(\nabla u_1(\psi)\big) - \big((l_1/l_2)-1\big)\mathrm{div}\big(\nabla u_1(\psi)\nabla(\rho\mathrm{id})\big) \\ &= -\mathrm{tr}\big(\nabla^2 u_1(\psi)\nabla\psi\big) - \big((l_1/l_2)-1\big)\mathrm{div}\big(\nabla u_1(\psi)\nabla(\rho\mathrm{id})\big)\,. \\ &= -(l_1-l_2)/l_1\big[\mathrm{tr}\big(\nabla^2 u_1(\psi)\nabla(\rho\mathrm{id})\big) + \mathrm{div}\big(\nabla u_1(\psi)\nabla(\rho\mathrm{id})\big)\big]\,. \end{aligned}$$

Since u_1 belongs to $\mathrm{H}^2(\Omega')$ and ψ is smooth in Ω, it turns out that the term between square brackets is bounded in $L^2(\Omega')$ and, i.e., $\|f\|_{\mathrm{L}^2} \le C\,|l_1 - l_2|$, where C depends on $\|u_1\|_{\mathrm{H}^2(\Omega')}$. Note that we only aim to prove continuity of G and thus not require a uniform bound on this norm.

[3] As a matter of fact, Grisvard asserts that Destuynder & Djaoua[9] give a proof of the C^1 regularity of $\mathscr{E}$. Actually, we could not find a clear statement about the continuity of $\partial\mathscr{E}/\partial l$ suitable for our problem; we preferred then to give a complete proof.

By symmetry, the functions u_i are determined also by the corresponding boundary value problems in Ω_i^+. From Lemma 6.3 we know that $u_i = \kappa_i \hat{s}_i + w_i$ in Ω_i^+. Hence we can write $\tilde{u}_2(x) = \kappa_1 \hat{s}(\psi(x)) + w_1(\psi(x))$ and since ψ is an homothety close to K_2 we can write also $\tilde{u}_2(x) = \kappa_1 (l_1/l_2)^{1/2} \rho^{1/2} \sin(\theta/2) + \tilde{w}_2(x)$ for a suitable function $\tilde{w}_2 \in W^{2,p}(\Omega_2^+)$. Moreover, $\tilde{u}_2$ solves the boundary value problem

$$\begin{cases} -\Delta \tilde{u}_2 = f & \Omega_2^+ \\ \nabla \tilde{u}_2 \cdot n = 0 & \partial_N \Omega_2^+ \cup K_2^+ \\ \tilde{u}_2 = g(t_1) & \partial_D \Omega_2^+ \,. \end{cases}$$

Then $(\tilde{u}_2 - u_2)$ is of form $\big((l_2/l_1)\kappa_1 - \kappa_2\big)\hat{s} + w$ and solves

$$\begin{cases} -\Delta(\tilde{u}_2 - u_2) = f & \Omega_2^+ \\ \nabla(\tilde{u}_2 - u_2) \cdot n = 0 & \partial_N \Omega_2^+ \cup K_2^+ \\ (\tilde{u}_2 - u_2) = g(t_1) - g(t_2) & \partial_D \Omega_2^+ \,. \end{cases}$$

Invoking Lemma 6.3 we get $\big|\kappa_2 - (l_2/l_1)\kappa_1\big| \leq C(\|f\|_{\mathrm{L}^2} + \|g(t_1) - g(t_2)\|_{\mathrm{W}^{2-1/p,p}})$. Finally, since $\|f\|_{L^2} \leq C|l_1 - l_2|$, we get

$$\begin{aligned} |\kappa_2 - \kappa_1| &\leq |\kappa_2 - (l_1/l_2)\kappa_1| + |(l_1/l_2) - 1||\kappa_1| \\ &\leq C\|f\|_{L^2} + C\|g(t_1) - g(t_2)\|_{W^{2,p}} + |l_2 - l_1|\big(|\kappa_1|/l_2\big) \\ &\leq C'|l_2 - l_1| + C\|g(t_1) - g(t_2)\|_{W^{2,p}} + |l_2 - l_1|\big(|\kappa_1|/l_2\big) \,. \end{aligned}$$

Since $g \in \mathrm{C}^1([0,+\infty), \mathrm{H}^2(\Omega))$, it follows that κ (and therefore G) is continuous at the point (t, l_1).

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