Γ-CONVERGENCE OF VARIATIONAL FUNCTIONALS WITH BOUNDARY TERMS IN STEIN MANIFOLDS

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Abstract. Let Ω be an open subset of a Stein manifold Σ and let M be its boundary. It is well knwon that M inherits a natural contact structure. In this paper we consider a family of non-coercive variational functionals F_ε defined by the sum of two terms: a Dirichlet-type energy associated with a subriemannian structure in Ω and a potential term on the boundary M. We prove that the functionals F_ε Γ-converge to the intrinsic perimeter in M associated with its contact structure.

Similar results have been obtained in the Euclidean space by Alberti, Bouchitté, Seppecher. We stress that already in the Euclidean setting the situation is not covered by the classical Modica-Mortola Theorem because of the precense of the boundary term.

We recall also that Modica-Mortola type results (without a boundary term) have been proved in the Euclidean space for subriemannian energies by Monti and Serra Cassano.

Contents

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1. Introduction and statement of the results

It is well known that, roughly speaking, a contact manifold (M,θ) can be viewed as "the boundary" of a symplectic manifold (Ω,ω). We refer for instance to [13], Section 6.8. In particular, the Heisenberg group $\mathbb{H}^n$ can be seen as the boundary of the upper half-space $\mathcal{U}^n \subset \mathbb{C}^n$ (see, e.g. [50], Chapter XII).

The aim of this note is to show that – in the same spirit – the notion of perimeter associated with the contact structure of (M,θ) (see [8]) can be seen as a variational limit of "solid functionals" defined in the symplectic manifold (Ω,ω) that has M as boundary (notice that similar approximation "from within M" are already known, at least in the model case $\mathbb{H}^n$: see [38].)

More precisely, inspired by [5], we show that the perimeter in (M,θ) is the Γ-limit of a family of "phase transition" functionals with "low dimensional tension effect" in Ω.

Let us start by introducing the setting of our results. Let Ω be a bounded open set in a Stein manifold of complex dimension $N = n+1$, with symplectic form ω. A complex manifold Σ, endowed with a complex structure J, is said a Stein manifold if it admits an exhausting J-convex function ϕ. We recall that Σ is endowed with a Riemannian metric g associated with ω and J. We assume that $\Omega = \{\phi < c\}$ is a sublevel set of ϕ. Then its boundary $M = \partial\Omega$, of real dimension $2n+1$, inherits a natural contact structure (M,θ), where θ is (roughly speaking) the restriction to M of the 1-form ξ, the contraction of the symplectic form ω along the so-called Liouville vector field X_0, that plays the role of the normal vector to M.

All precise definitions will be given in Section 2.1, but the idea is that bounded open sets in Stein manifolds are the natural generalization of domains of holomorphy in $\mathbb{C}^n$, having a contact manifold as boundary. We denote also by dy the volume element in Ω with respect to the metric compatible with the symplectic form ω, and $dv_\theta := \theta \wedge (d\theta)^{N-1}$ the volume element in M with respect to the contact form θ.

Let V be a double well potential, i.e, a function $V : \mathbb{R} \to \mathbb{R}$ satisfying

$$V(0) = V(1) = 0, \quad V > 0 \ \text{in} \ \mathbb{R}\backslash\{0,1\}.$$

Given $\varepsilon > 0$ and $\lambda_\varepsilon > 0$, we define the energy functional

$$F_\varepsilon(u) := \varepsilon \int_\Omega f(y, Du(y))\, dy + \lambda_\varepsilon \int_M V(\operatorname{Tr} u)\, dv_\theta, \tag{1.1}$$

where Du denotes the Riemannian gradient of u. The first term in the functional $F_\varepsilon(u)$ is essentially the Dirichlet energy on Ω inherited from the sub-Riemannian structure of $(\Omega, \ker\xi)$, and it will be precisely written in Section 2.1 after we have introduced all the necessary notations. Thus F_ε will be well defined if $u \in W^{1,2}(\Omega)$; we assign it the value infinity otherwise. Note that functions in this space have a well defined trace $\operatorname{Tr} u$ on M with respect to the normal X_0.

The second term in the functional, coming from a double well potential (on the boundary), creates a phase transition on the boundary M as $\varepsilon \to 0$. Here the sub-Riemannian geometry of M plays an essential role in the understanding of the Γ-limit of the functional as $\varepsilon \to 0$, and this is the main innovation of the present paper.

The model we have in mind is M the n-Heisenberg group $\mathbb{H}^n$ and $\Omega = \mathbb{H}^n \times \mathbb{R}^+$, which is the flat model in this geometry; Indeed, by the Darboux Theorem, any $(2n+1)$-dimensional contact manifold is locally contact-diffeomorphic to the n-Heisenberg group (see e.g. Theorem 5.1.5, [1]). In this model case the functional reduces to (1.2).

For a general review on Heisenberg groups and their properties, we refer to [11], [28], [50], and [52]. We limit ourselves to fix some notations, following [23]. The Heisenberg group $\mathbb{H}^n$ is identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted by $p = (\eta, t)$, with $\eta \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}$. If p and $p' \in \mathbb{H}^n$, the group operation is defined as

$$p \cdot p' = \big(\eta + \eta', t + t' + \frac{1}{2}\sum_{j=1}^{n}(\eta_j \eta'_{j+n} - \eta_{j+n}\eta'_j)\big).$$

For fixed $q \in \mathbb{H}^n$ and for $r > 0$, left translations $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ and not isotropic dilations $\delta_r : \mathbb{H}^n \to \mathbb{H}^n$ are defined as

$$\tau_q(p) := q \cdot p \qquad \text{and as} \qquad \delta_r(p) := (r\eta, r^2 t).$$

We denote by $\mathfrak{h}$ the Lie algebra of the left invariant vector fields of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}$ is given, for $i = 1, \dots, n$, by

$$W_i^{\mathbb{H}} := \partial_{\eta_i} - \frac{1}{2}\eta_{i+n}\partial_t, \quad W_{i+n}^{\mathbb{H}} := \partial_{\eta_{i+n}} + \frac{1}{2}\eta_i \partial_t, \quad T := \partial_t.$$

The only non-trivial commutation relations are $[W_j^{\mathbb{H}}, W_{j+n}^{\mathbb{H}}] = T$, for $j = 1, \dots, n$.

The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $W_1^{\mathbb{H}}, \dots, W_{2n}^{\mathbb{H}}$. Coherently, from now on, we refer to $W_1^{\mathbb{H}}, \dots, W_{2n}^{\mathbb{H}}$ (identified with first order differential operators) as to the *horizontal derivatives*, and we write

$$\mathbf{W}^{\mathbb{H}} := \{W_1^{\mathbb{H}}, \dots, W_{2n}^{\mathbb{H}}\}.$$

Let $g_{\mathbb{H}} = g_{\mathbb{H}}(\cdot, \cdot)$ be the Riemannian metric on $\mathbb{H}^n$ making $W_1^{\mathbb{H}}, \dots, W_{2n}^{\mathbb{H}}, T$ orthonormal. We shall denote it by $\langle \cdot, \cdot \rangle_{\mathbb{H}}$. We denote by $\nabla_{\mathbb{H}}$ the *horizontal gradient*

$$\nabla_{\mathbb{H}} := (W_1^{\mathbb{H}}, \dots, W_{2n}^{\mathbb{H}}).$$

Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1 \mathfrak{h}$. The basis of $\bigwedge^1 \mathfrak{h}$, dual to the basis $\{W_1^{\mathbb{H}}, \dots, W_{2n}^{\mathbb{H}}, T\}$ is the family of covectors $\{d\eta_1, \dots, d\eta_{2n}, \theta_0\}$ where

$$\theta_0 := dt - \frac{1}{2}\sum_{j=1}^{n}(\eta_j d\eta_{j+n} - \eta_{j+n} d\eta_j)$$

is called the *contact form* in $\mathbb{H}^n$.

In this particular case, the functional (1.1) is written as

(1.2)
$$E_\varepsilon(u) := \varepsilon \int_{\mathbb{H}^n\times[0,\infty)} \Big(\sum_{j=1}^{2n} (W_j^{\mathbb{H}} u)^2 + (\partial_z u)^2 \Big)\, dv_{\theta_0} dz + \lambda_\varepsilon \int_{\mathbb{H}^n} V(\operatorname{Tr} u)\, dv_{\theta_0}.$$

where $dv_{\theta_0} = d\eta\, dt$. Here we realize that our functional corresponds to a hypoelliptic Dirichlet energy functional with a boundary phase transition on a contact manifold.

In general throughout this paper, if u is a real function defined on a smooth manifold and X is a smooth tangent vector field, we shall write

$$Xu := \mathcal{L}_X u,$$

to denote the Lie derivative of u along X.

Let us now state our main theorem, a *boundary* Γ-convergence result. For the rest of the paper, we will assume that

$$\lim_{\varepsilon\to 0} \varepsilon \log \lambda_\varepsilon = \kappa \quad \text{for some constant } \kappa \in (0,\infty). \tag{1.3}$$

We also define the limit functional on M as

$$F(v) = \begin{cases} \mathbf{c}\, \|S_v\|_\theta & \text{if } v \in BV_\theta(M,\{0,1\}), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.4}$$

where $\mathbf{c} = \kappa/\pi$. Here $\|\partial A\|_\theta$ denotes the intrinsic perimeter measure of the set $A \subset M$ associated with the contact form θ, and $S_v = \partial\{v \equiv 1\}$ the singular set of $v \in BV_\theta(M,\{0,1\})$. Precise definitions will be given in Section 3.

Theorem 1.1. *For $\varepsilon > 0$, consider the functional $F_\varepsilon : W^{1,2}(\Omega) \to [0,+\infty]$, Under scaling* (1.3) *we have that:*

i) Given a sequence $\{u_\varepsilon\}$ such that $F_\varepsilon(u_\varepsilon)$ is bounded when $\varepsilon \to 0$, then $\{\operatorname{Tr} u_\varepsilon\}$ is pre-compact in $L^1(M)$ and every cluster point belongs to $BV_\theta(M,\{0,1\})$.

ii) Lower bound inequality: for every $v \in BV_\theta(M,\{0,1\})$ and every sequence $\{u_\varepsilon\} \subset W^{1,2}(\Omega)$ such that $\operatorname{Tr} u_\varepsilon \to v$ in $L^1(M)$, there holds
$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq F(v).$$

iii) Upper bound inequality: for every $v \in BV_\theta(M,\{0,1\})$ there exists a sequence $\{u_\varepsilon\} \subset W^{1,2}(\Omega)$ such that $\operatorname{Tr} u_\varepsilon \to v$ in $L^1(M)$ and
$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = F(v).$$

The inspiration for this theorem comes from the Riemannian case. The classical theorem for phase transitions of Modica-Mortola states that a Dirichlet energy functional with a double well potential (in the interior) Γ-converges to the area functional, and thus, phase transitions happen at a minimal surface (see the survey paper [3] or [34], for instance). Later, Alberti, Bouchitté and Seppecher [5] considered an energy functional on domain $\Omega \subset \mathbb{R}^3$ with a double well potential defined on the boundary of Ω, which is a closed surface M. In this case the Γ-limit leads to a phase transition problem on the boundary surface M. This problem comes in relation to a model in capillarity with line tension effect.

Here we consider the sub-Riemannian version of [5], in which the phase transition occurs at the boundary of a complex domain Ω, which is a sub-Riemannian (contact) manifold M. Although the structure of the proof is similar to the Riemannian case, the main difficulties, detailed below, come precisely from the fact that the sub-Laplacian is a hypoelliptic, but not elliptic, operator, and from the intrinsic geometry of a contact manifold.

The first Γ-convergence result in the sub-Riemannian setting is by Monti and Serra Cassano [38], where they show the analog of the Modica-Mortola theorem for interior phase transitions in a subdomain Ω in the framework of Carnot-Carathéodory spaces. As a particular case, their result holds in the case of the Heisenberg group, which is the flat model in contact geometry.

In contrast, looking at boundary phase transitions on complex domains presents several difficulties that one needs to deal with. Therefore, we give now an overview of the paper, stressing the points at which we cannot plainly traslate Euclidean techniques to our geometric setting, but we have to use new approaches or new technical arguments.

First, in order to follow the methods in [5] for the Riemannian setting, one needs to compare our domain Ω to a product $M \times [0, \sigma)$ while still preserving the complex structure. However, in the process of flattening one needs to control the error in this procedure only by means of the derivatives appearing in the functional (1.2) and not of the whole gradient. This is the content of Section 2.2.

Second, while there is an extensive literature on sub-Riemannian geometry for the Heisenberg group, the Carnot-Carathéodory theory on a general contact manifold has just recently been developed in [8]. In [8], the authors developed the theory of perimeter and BV functions, but several results needed in our proofs were not available. One of the missing concepts was the Eikonal equation for the Carnot-Caratheodory (CC) distance, which we address in Section 3.2. Of course, the Eikonal equation holds in the viscosity sense in the CC setting (see Corollary 2.36 and Remark 2.37. in [14]), but we need a pointwise identity.

Section 4 deals with the proof of the compactness and the lower bound inequality for the model functional (1.2). This part essentially follows, as in the Riemannian case, using a slicing theorem by [35] to reduce the problem to a one dimensional one.

In Section 5, we prove point i) and ii) of Theorem 1.1. To do that, we need to pass from the corresponding results for the flat model, established in Section 4, to the ones for the original functional. In doing that, a crucial issue is to compare our boundary contact manifold to the Heisenberg group near a given point, in the spirit of the blow up theorems by [8]. Of course, the starting point is Darboux theorem. Let us give now a list of the difficulties we have subsequently to deal with. Precise technical features are described in Remark 5.7. In the Euclidean setting, for a smooth hypersurface S basically all reasonable notions of surface measure agree: De Giorgi perimeter, spherical Hausdorff measure with respect to Euclidean balls, as well as Minkowski content. Because of this, in [5] the authors use systematically the spherical Hausdorff measure. In a contact manifold the situation is different: indeed it is natural to formulate our results in terms of perimeter and Minkowski content, and we are forced to use the Carnot-Carathéodory distance on the contact manifold, since it satisfies the Eikonal equation. On the other hand, the proof of the liminf inequality (with exact constants) is reached in [5] by means of the estimate of the density of a suitable measure associated with the functional, yielding a comparison with the Carnot-Carathéodory spherical Hausdorff measure. Unfortunately, an explicit representation formula for the perimeter in terms of the Carnot-Carathéodory spherical Hausdorff measure is not known, and we have to use an indirect comparison argument, that is stated in Theorem 5.6.

Many of the results that are needed are summarized later in Section 7, as an appendix for the paper (see also [24]). Finally, Section 6, mostly analytical, concludes the proof of the main theorem, establishing the upper bound inequality (point iii) in Theorem 1.1).

2. Reduction to a model problem

2.1. **Geometric setting.** We refer to [13] for an introduction to the results in this section.

Definition 2.1. A complex manifold Σ is said a *Stein manifold* if admits an *exhausting J-convex function ϕ*. To be a complex manifold means that:

i) Σ is a smooth manifold of real dimension $2N$, endowed with an endomorphism (the *complex structure*) $J : T\Sigma \to T\Sigma$ satisfying $J^2 = -I$ on each fiber;

ii) *J is integrable*, i.e. J is induced by complex coordinates on Σ.

Let now $\phi : \Sigma \to \mathbb{R}$ be a smooth function. We say that ϕ is an exhausting function if:

iii) $\inf \phi > -\infty$;

iv) $\phi^{-1}(K)$ is compact for any compact set $K \subset \mathbb{R}$.

We denote by $d^{\mathbb{C}}$ the operator defined by

$$\langle d^{\mathbb{C}}\phi | X\rangle := \langle d\phi | JX\rangle \quad \text{for all smooth tangent vector fields } X.$$

We can associate with ϕ the 2-form

$$\omega = \omega_\phi := d\xi_\phi, \quad \text{where} \quad \xi = \xi_\phi := -d^{\mathbb{C}}\phi.$$

Then the function ϕ is said *J-convex* if

$$\omega_\phi(X, JX) > 0 \quad \text{for all smooth tangent vector fields } X.\text{[1]} \tag{2.1}$$

Proposition 2.2 ([13])**.** *Suppose Σ is a Stein manifold with respect to the complex structure J and the exhausting J-convex function ϕ. Then:*

i) *ω_ϕ is a symplectic form;*

ii) *ω_ϕ is J-invariant, i.e. $\omega_\phi(JX, JY) = \omega_\phi(X, Y)$ for all smooth tangent vector fields X, Y;*

[1]Through this paper, we denote by $\langle\cdot|\cdot\rangle$ the duality between cotangent h-vectors and tangent h-vectors. Moreover, for sake of simplicity we write sometimes $\omega_\phi(X, Y)$ for $\langle\omega_\phi | X \wedge Y\rangle$ and $\xi_\phi(X)$ for $\langle\xi_\phi | X\rangle$).

iii) *the bilinear form on $T\Sigma$ given by $g_\phi(X,Y) = g(X,Y) := \omega_\phi(X,JY)$ is a Riemannian scalar product and hence a Kähler metric. In particular the Riemannian volume form dy coincides with the symplectic volume form ω_ϕ^N ;*

iv) *J is a g-isometry;*

v) *if we denote by $\nabla_\phi = \nabla_g$ the gradient associated with the Riemannian scalar product g_ϕ, then the vector field $X_\phi := \nabla_\phi \phi$ satisfies*

$$\mathcal{L}_{X_\phi}\omega_\phi = \omega_\phi \quad \text{or, equivalently,} \quad \xi_\phi = \imath_{X_\phi}\omega_\phi,$$

i.e. X_ϕ is a Liouville vector field for the symplectic form ω_ϕ. Here $\imath_X$ denotes the contraction along the vector field X.

vi) *$g_\phi(X_\phi, Z) = 0$ in M for all $Z \in TM$.*

Proof. Assertions *i)* and *ii)* are proved in [13], Sections 2.1 and 2.2; assertions *iii)* and *v)* are contained in [13], Lemma 2.20. As for *iv)*, if $X, Y \in T\Sigma$

$$g_\phi(JX,JY) = \omega_\phi(JY, J^2X) = -\omega_\phi(JY,X) = \omega_\phi(X,JY) = g_\phi(X,Y).$$

Finally, vi) follows from the identity $g_\phi(X_\phi, Z) = \langle d\phi | Z\rangle$. □

The symplectic structure induced by ϕ is independent of ϕ in the following sense:

Theorem 2.3 ([13], Theorem 1.4.A)**.** *Let $\psi : \Sigma \to \mathbb{R}$ be another smooth function satisfying iii), iv) in Definition 2.1, and* (2.1)*. Then (Σ, ω_ϕ) and (Σ, ω_ψ) are symplectomorphic.*

Let now Σ be a Stein manifold, and let ϕ be the associated exhausting function. If $c \in \mathbb{R}$ is a regular value of ϕ, we set $\Omega_{\phi,c} = \phi^{-1}(]-\infty, c[)$. Clearly $\Omega_{\phi,c}$ is a bounded open set in Σ with smooth compact boundary M_ϕ. We assume here, for sake of simplicity, that M_ϕ has only one connected component.

From now on, the exhausting function ϕ and the regular level c will be fixed, and we drop the corresponding indices in our notations and thus we write $\Omega := \Omega_{\phi,c}$ and $M = \partial\Omega$. In addition, we shall write X_0 for the Liouville vector field $\nabla_\phi \phi$. We notice that $X_0 \neq 0$ in a neighborhood $\mathcal{M}$ of M since c is a regular value of ϕ and M is compact.

We denote by $T\Omega := (\Omega, T\Omega, \pi)$ the tangent bundle of Ω, and by $T_y\Omega$ the fiber of $T\Omega$ over $y \in \Omega$. Coherently, we denote by g_y the Riemannian metric g on $T_y\Omega$, and by ξ_y and ω_y the forms ξ and ω at the point y. However, as customary in differential geometry, we drop the index y whenever this does not lead to misunderstandings. An analogous notation will be used for TM, the tangent bundle of M.

Finally, we denote by d the Riemannian distance on $\overline{\Omega}$ with respect to the metric g.

The next step consists in proving that there is a natural $(2N-1)$-distribution associated with the the Liouville form ξ in a neighborhood $\mathcal{M}$ of M.

Proposition 2.4. *Set $\mathcal{H} := \ker \xi = \{X \in T\Omega;\ \imath_X \xi = 0\} \subset T\Omega$. Then*

i) *$X_0 \in \mathcal{H}$;*

ii) $\dim \mathcal{H} = 2N - 1$ *in* $\mathcal{M}$*;*

iii) $\mathcal{H}$ *has a orthonormal basis of the form*

$$\mathcal{B} := \{X_0, Z_1, JZ_1, Z_2, JZ_2, \dots, Z_{N-1}, JZ_{N-1}\}$$

(in particular, $Z_1, JZ_1, \dots, Z_{N-1}, JZ_{N-1} \in TM$ *on* M*);*

iv) $\omega(Z_i, Z_j) = 0$ *for all* $i, j = 1, \dots, N-1$*,* $\omega(JZ_i, JZ_j) = 0$ *for all* $i, j = 1, \dots, N-1$*,* $\omega(Z_i, JZ_j) = 0$ *for all* $i, j = 1, \dots, N-1$*,* $i \neq j$*, and* $\omega(Z_i, JZ_i) = 1$ *for all* $i = 1, \dots, N-1$*;*

v) $\xi([JZ_i, Z_i]) = 1$ *for* $i = 1, \dots, N$*;*

vi) $\mathcal{H} + [\mathcal{H}, \mathcal{H}] = T\Omega$*, so that* $(\mathcal{H}, g)$ *is a regular sub-Riemannian structure on* Ω*.*

Proof. To prove *i)* we write

$$\langle \xi | X_0 \rangle = \imath_{X_0}\omega(X_0) = \omega(X_0, X_0) = 0.$$

Next, obviously $\dim \ker \xi \geq 2N - 1$. Suppose *ii)* fails to be true. Then for some $y \in \mathcal{M}$ and for any $Y \in T_y\Omega$ in $\mathcal{M}$

$$0 = \langle \xi_y | Y \rangle_y = \omega_y(X_0, Y),$$

which contradicts $X_0 \neq 0$ since ω is symplectic.

To prove *iii)*, we prove first that, if $g(X, X_0) = 0$, then $\langle \xi | JX \rangle = 0$. Indeed

$$\langle \xi | JX \rangle = \omega(X_0, JX) = g(X_0, X) = 0. \tag{2.2}$$

Consider now $X_0^\perp \cap \ker \xi$, the g-orthogonal complement of X_0 in $\ker \xi$, that has dimension $2N - 2$, and take an unit vector $Z_1 \in X_0^\perp \cap \ker \xi$. Take now JZ_1, that is a unit vector by Theorem 2.2, part *iv)*. By (2.2) $JZ_1 \in \ker \xi$. We have also

$$\begin{aligned} g(X_0, JZ_1) &= \omega(X_0, J^2 Z_1) = -\omega(X_0, Z_1) \\ &= -\langle \imath_{X_0}\omega | Z_1 \rangle = \langle \xi | Z_1 \rangle = 0. \end{aligned}$$

Thus $JZ_1 \in X_0^\perp \cap \ker \xi$. Finally

$$g(JZ_1, Z_1) = \omega(JZ_1, JZ_1) = 0.$$

Summing up, Z_1 and JZ_1 are two orthonormal vectors in $X_0^\perp \cap \ker \xi$. We can take now an unitary vector $Z_2 \in \operatorname{span}\{X_0, Z_1, JZ_1\}^\perp \cap \ker \xi$. Arguing as above, Z_2 and JZ_2 are two orthonormal vectors in $\operatorname{span}\{X_0, Z_1, JZ_1\}^\perp \cap \ker \xi$. Repeating the argument, we achieve the proof of *iii)*.

Let us prove *iv)*. Let $i \neq j$ be given. Thanks to the anti-commutativity of ω, we can assume $i < j$. Then $\omega(Z_i, Z_j) = \omega(JZ_i, JZ_j) = g(JZ_i, Z_j) = 0$, by construction. In addition, if $i \neq j$, then $\omega(Z_i, JZ_j) = g(Z_i, Z_j) = 0$, whereas $\omega(Z_i, JZ_i) = g(Z_i, Z_i) = 1$ for $i = 1, \dots, N-1$. This achieves the proof of *iv)*.

To prove *v)*, we have only to recall that, by classical Cartan's formula

$$\begin{aligned} 1 = \omega(Z_i, JZ_i) = d\xi(Z_i, JZ_i) &= JZ_i \langle \xi | Z_i \rangle - Z_i \langle \xi | JZ_i \rangle - \langle \xi | [Z_i, JZ_i] \rangle \\ &= -\langle \xi | [Z_i, JZ_i] \rangle. \end{aligned}$$

Finally, *vi)* follows from *ii)* and *v)*. □

Remark 2.5. We can always take Z_j and JZ_j, $j = 1, \dots, N-1$, that commute with X_0.

Let us remind now the following well-known definition.

Definition 2.6. Let M be a smooth $(2n+1)$-manifold. A 1-form θ is said a *contact form* if $\theta \wedge (d\theta)^{2n} \neq 0$ on M. The set $\ker \theta \subset TM$ is called a *contact distribution.* Let M_1 and M_2 be two contact $(2n+1)$-manifolds endowed with the contact forms θ_1 and θ_2. A smooth diffeomorphism $f : M_1 \to M_2$ is said a *contact map* if $\theta_1 = f^*\theta_2$ and hence $f_* \ker \theta_1 = \ker \theta_2$.

The following result is well known:

Proposition 2.7. *Denote by $i : M \to \overline{\Omega}$ the natural embedding. Then the 1-form $\theta := i^*(\imath_{X_0}\omega)$ is a contact form on M, and therefore $\ker \theta$ defines a contact distribution on M.*

Remark 2.8. By the previous proposition, we can choose $dv_\theta := \theta \wedge (d\theta)^{N-1}$ as the volume form in M. For sake of simplicity, if $A \subset M$ we shall write $v_\theta(A)$ for $\int_A dv_\theta$.

Moreover (see e.g. [10]) there exists a global vector field T on M satisfying $\langle \theta | T \rangle = 1$ and orthogonal to $\ker \theta$ with respect to the Riemannian metric induced by g on TM (still denoted by g), that is called the characteristic vector field or Reeb vector field of the contact structure.

Proposition 2.9. *The contact distribution $\ker \theta$ carries a natural symplectic structure*

$$d\theta = di^*(\xi) = i^*(d\xi) = i^*\omega.$$

Proof. We have only to prove that $i^*\omega$ is non-degenerate on $\ker \theta$. To this end, let $X \in \ker \theta$ be such that $i^*\omega(X, Y) = 0$ for all $Y \in \ker \theta$. If $x \in M$, then, keeping in mind that $i(x) = x$, we have

$$0 = i^*\omega_x(X, Y) = \omega_{i(x)}(di(X), di(Y)).$$

We remark now that any tangent vector Z to Ω at a point of M can be written in the form $Z = di(Y) + \lambda X_0$ with $\lambda \in \mathbb{R}$ and $Y \in TM$, since X_0 is normal to TM. On the other hand

$$\omega_{i(x)}(di(X), X_0) = -\xi_{i(x)}(di(X)) = -\theta_x(X) = 0,$$

and hence $\omega_{i(x)}(di(X), Z) = 0$ for all $Z \in T_{i(x)}\Omega$, achieving the proof of the proposition since di is injective. $\square$

Proposition 2.10. *The vector fields Z_j and JZ_j, $j = 1, \dots, N-1$ (that belong to $T\Omega$), being tangent to M at the points of M, can be identified with vectors in $\ker \theta \subset TM$ and are a symplectic basis of $\ker \theta$. Moreover, $\ker \theta$ inherits the Riemannian metric from the ambient space (denoted by the same letter g) and Z_j and JZ_j, $j = 1, \dots, N-1$ give an orthonormal basis of $\ker \theta$.*

Proof. It is enough to apply Theorem 2.4, *iv).* $\square$

We are ready now to introduce our main object of study. We write $N =: n + 1$. If p is a tangent smooth vector field to Ω, we denote

$$\Lambda(y, p) := \sum_{j=1}^{n} g_y(Z_j(y), p)^2 + \sum_{j=1}^{n} g_y(JZ_j(y), p)^2 + g_y(X_0(y), p)^2.$$

Let now $f : T\Omega \to \mathbb{R}$ be a smooth function such that:

H1. $0 \le f(y,p) \le C\, g_y(p,p)$ for all $y \in \Omega$ and $p \in T_y\Omega$.;

H2. for any $\sigma > 0$ small enough there exists a neighborhood U_σ of M in Ω, $U_\sigma \subset \mathcal{M}$, such that

$$(1-\sigma)\Lambda(y,p) \le f(y,p) \le (1+\sigma)\Lambda(y,p)$$

for all $y \in U_\sigma$ and $p \in T_y\Omega$.

If there is no way to misunderstanding, we denote by $\nabla = \nabla_g$ the Riemannian gradient in Ω. We notice that, if X is any vector field on Ω, then $g_y(X, \nabla_g u)^2 = |Xu|_g^2$. Keeping in mind that

$$g_y(X, \nabla_g u) = \langle du|X\rangle = \mathcal{L}_X u = Xu,$$

we can write

$$\int_{U_\sigma} \Lambda(y, \nabla u(y))\, dy = \int_{U_\sigma} \Big(\sum_{j=1}^n (Z_j u)^2 + \sum_{j=1}^n (JZ_j u)^2 + (X_0 u)^2 \Big)\, dy. \tag{2.3}$$

2.2. **Straightening the domain and freezing the functional.** It is well known that, straightening the integral curve of X_0, we can transform the neighborhood U_σ of M into the cylinder $M \times [0,\sigma)$. More precisely, we consider the map

$$\Phi = \Phi(x,z) : M \times [0,\sigma) \to \Omega$$

defined by

$$\frac{\partial \Phi}{\partial z} = -X_0(\Phi) \quad \text{and } \Phi(x,0) = i(x). \tag{2.4}$$

If $\sigma > 0$ is small enough, then Φ is a smooth diffeomorphism. We set now

$$\tilde{Z}_j := (\Phi^{-1})_* Z_j, \quad \widetilde{JZ_j} := (\Phi^{-1})_* JZ_j, \quad j = 1, \dots, n,$$

and

$$\tilde{\xi} := \Phi^*(\xi), \quad \tilde{\omega} := \Phi^*(\omega).$$

In addition, we define the projection

$$\pi : M \times [0,\sigma) \to M$$

given by $\pi(x,z) = x$. We notice that, if α is a differential form on M, then $\pi^*\alpha$ is its "natural" extension on $M \times [0,\sigma)$.

The following result follows straightforwardly by algebraic arguments.

Lemma 2.11. *We remind that we have set $\theta := i^*\xi$. Then we have:*

i) $\tilde{\xi} = e^{-z}\, \pi^*\theta$;
ii) $\tilde{\omega} = d(e^{-z}\, \pi^*\theta)$;
iii) $\ker \tilde{\xi} = \ker \theta \times \mathbb{R}$.

Moreover, we have the following Lemma:

Lemma 2.12. *We have:*

i) $(\Phi^{-1})_* X_0 = (0,-1) = -\partial_z$;
ii) $\Phi^*(\omega^N) = e^{-Nz}\, \pi^*(dv_\theta) \wedge dz$.

Proof. Point i) comes by the way we have defined Φ in (2.4). To prove ii), we notice that, by Lemma 2.11,

$$\begin{aligned}\Phi^*(\omega^N) = \tilde{\omega}^N &= (d(e^{-z}\,\pi^*\theta))^N = e^{-Nz}(-dz\wedge\pi^*\theta + \pi^*(d\theta))^N \\ &= -e^{-Nz}\,dz\wedge\pi^*\theta\wedge(\pi^*(d\theta))^{N-1} \\ &= e^{-Nz}\,\pi^*\theta\wedge(\pi^*(d\theta))^{N-1}\wedge dz \\ &= e^{-Nz}\,\pi^*\big(\theta\wedge(d\theta)^{N-1}\big)\wedge dz.\end{aligned}$$

□

Remark 2.13. For sake of simplicity, from now on we shall write $dv_\theta\wedge dz$ for $\pi^*(dv_\theta)\wedge dz$.

If we perform the change of variables $y=\Phi(x,z)$, keeping in mind that $X_0u = \partial_z(u\circ\Phi)$ and $Z_ju = \tilde{Z}_j(u\circ\Phi)$, and setting $\tilde{u} := u\circ\Phi$, the functional (2.3) becomes

$$\begin{aligned}&\int_{U_\sigma}\Lambda(y,Du(y))\,dy \\ &= \int_{M\times[0,\sigma)}\Big(\sum_{j=1}^n(\tilde{Z}_j\tilde{u})^2 + \sum_{j=1}^n(\widetilde{JZ}_j\tilde{u})^2 + (\partial_z\tilde{u})^2\Big)\,e^{-Nz}dv_\theta\wedge dz.\end{aligned}\tag{2.5}$$

We recall now that the vector fields $Z_1,\dots,Z_n$ and $JZ_1,\dots,JZ_n$ in $\overline{\Omega}$ are tangent to M in M, and hence can be identified with vector fields tangent to M at the points of the form $(x,0)\in M\times[0,\sigma)$. Thus in $M\times[0,\sigma)$ we set:

$$\tilde{Z}_j^0(x,z) := \tilde{Z}_j(x,0) = Z_j(i(x))$$

and

$$\widetilde{JZ}_j^0(x,z) := \widetilde{JZ}_j(x,0) = JZ_j(i(x)).$$

The core of this Section is the following Proposition, that states basically that our functional near the boundary M of Ω is equivalent – in a suitable way – to a variational functional $\tilde{F}_{\varepsilon,\sigma}$ satisfying the following properties:

- $\tilde{F}_{\varepsilon,\sigma}$ is defined in a cylindric region $M\times[0,\sigma)$;
- $\tilde{F}_{\varepsilon,\sigma}$ is associated with the vector fields $\widetilde{Z}_j^0$ and $\widetilde{JZ}_j^0$ (that are tangent to M and are independent of the "vertical" variable) and to a purely vertical vector field ∂_z.

More precisely, we write

$$\tilde{F}_{\varepsilon,\sigma}(\tilde{u}) := \int_{M\times[0,\sigma)}\Big(\sum_{j=1}^n(Z_j^0\tilde{u})^2 + \sum_{j=1}^n(JZ_j^0\tilde{u})^2 + (\partial_z\tilde{u})^2\Big)\,dv_\theta\wedge dz.$$

Proposition 2.14. *Using the above notations, we have*

$$(1+O(\sigma))\int_{U_\sigma}\Lambda(y,\nabla u(y))\,dy = \tilde{F}_{\varepsilon,\sigma}(\tilde{u})$$

provided we take σ small enough.

Obviously, the exponential e^{-Nz} in (2.5) gives no trouble. The remaining part of the proof of Proposition 2.14 is more delicate: in $M \times [0, \sigma)$ we have to replace (e.g.) the vector fields $\tilde{Z}_j$ by their value frozen at $z = 0$ and to control the error. However, a straightforward application of the mean value theorem does not fit our purposes, because this estimate of the error would involve *all* derivatives of $\tilde{u}$, that in turn are not controlled by the original functional, where only derivatives along a particular distribution appear. Thus, we have to show that we can control the error only by means of the derivatives appearing in the functional. This is the aim of the following technical lemma.

Lemma 2.15. *If $j = 1, \dots, n$ and $0 < s < z \leq 1$, then*

$$\begin{aligned} \partial_z \tilde{Z}_j(x,s) &= \sum_{\ell=1}^{n} \lambda_{\ell,j}(x,s,z)\tilde{Z}_\ell(x,z) \\ &\quad + \sum_{\ell=1}^{n} \lambda_{\ell+n,j}(x,s,z)\widetilde{JZ}_\ell(x,z) + \lambda_{0,j}(x,s,z)\partial_z. \end{aligned} \tag{2.6}$$

Similarly,

$$\begin{aligned} \partial_z \widetilde{JZ}_j(x,s) &= \sum_{\ell=1}^{n} \lambda'_{\ell,j}(x,s,z)\tilde{Z}_\ell(x,z) \\ &\quad + \sum_{\ell=1}^{n} \lambda'_{\ell+n,j}(x,s,z)\widetilde{JZ}_\ell(x,z) + \lambda'_{0,j}(x,s,z)\partial_z, \end{aligned} \tag{2.7}$$

Moreover, there exists a geometric constant $C > 0$ such that $|\lambda_{0,j}| + \cdots + |\lambda_{2n,j}| \leq C$ and $|\lambda'_{0,j}| + \cdots + |\lambda'_{2n,j}| \leq C$ for any $j = 1, \dots, n$.

Proof. We prove (2.6); the proof of (2.7) is analogue. First, we prove that for any $j = 1, \dots, n$, the vector fields $\partial_z \tilde{Z}_j(x,s)$, $\partial_z \widetilde{JZ}_j(x,s)$ belong to $\ker \tilde{\xi}(x,s)$. Then the assertion follows since $\ker \tilde{\xi}(x,s) = \ker \tilde{\xi}(x,z)$ for any $0 < s \leq z$, by Lemma 2.11, iii).

We show that for any $j = 1, \dots, n$

$$\begin{aligned} \partial_z \tilde{Z}_j &= \sum_{\ell=1}^{n} \big\{ g([Z_j, X_0], Z_\ell) \circ \Phi \big\} \tilde{Z}_\ell \\ &\quad + \sum_{\ell=1}^{n} \big\{ g([Z_j, X_0], JZ_\ell) \circ \Phi \big\} \widetilde{JZ}_\ell + \big\{ g([Z_j, X_0], X_0) \circ \Phi \big\} \partial_z. \end{aligned} \tag{2.8}$$

In order to prove (2.8), we notice preliminarily that

$$\partial_z \tilde{Z}_j = [(\Phi^{-1})_* Z_j, \partial_z] = [(\Phi^{-1})_* Z_j, (\Phi^{-1})_* X_0] = (\Phi^{-1})_*[Z_j, X_0],$$

where the last equality comes from [1], Proposition 4.2.23.

Let us prove now that $[Z_j, X_0] \in \ker \xi$. Using Proposition 7.4.11 in [1], we have

$$\begin{aligned} \omega(Z_j, X_0) &= d\xi(Z_j, X_0) \\ &= Z_j \langle \xi | X_0 \rangle - X_0 \langle \xi | Z_j \rangle - \langle \xi | [Z_j, X_0] \rangle = -\langle \xi | [Z_j, X_0] \rangle. \end{aligned}$$

On the other hand

$$\omega(Z_j, X_0) = \omega(X_0, J^2 Z_j) = g(X_0, JZ_j) = 0,$$

since the basis $\{X_0, Z_1, \dots, Z_n, JZ_1, \dots, JZ_n\}$ is orthonormal, hence $[Z_j, X_0] \in \ker \xi$. Thus,

$$[Z_j, X_0] = \sum_{\ell=1}^{n} g([Z_j, X_0], Z_\ell) Z_\ell + \sum_{\ell=1}^{n} g([Z_j, X_0], JZ_\ell) JZ_\ell + g([Z_j, X_0], X_0) X_0,$$

and hence

$$\begin{aligned}(\Phi^{-1})_*([Z_j, X_0]) = &\sum_{\ell=1}^{n} \big\{g([Z_j, X_0], Z_\ell) \circ \Phi\big\} \tilde{Z}_\ell + \sum_{\ell=1}^{n} \big\{g([Z_j, X_0], JZ_\ell) \circ \Phi\big\} \widetilde{JZ}_\ell \\ &+ \big\{g([Z_j, X_0], X_0) \circ \Phi\big\} \partial_z.\end{aligned}$$

This proves (2.8) and concludes the proof of Lemma 2.15. $\square$

For the sake of simplicity, sometimes we denote the vector fields

$$\tilde{Z}_1, \dots, \tilde{Z}_n, \widetilde{JZ}_1, \dots, \widetilde{JZ}_n \quad \text{by} \quad \widetilde{W}_1, \dots, \widetilde{W}_{2n},$$

and we set

$$\widetilde{\mathbf{W}} = \{\widetilde{W}_1, \dots, \widetilde{W}_{2n}\}.$$

Analogously we define the $\widetilde{W}_j^0$'s by freezing the $\widetilde{W}_j$ at $z = 0$ and we set

$$\widetilde{\mathbf{W}}^0 = \{\widetilde{W}_1^0, \dots, \widetilde{W}_{2n}^0\}.$$

With these notations, Lemma 2.15 reads as follows: for any $j = 1, \dots, 2n$, and $0 < s < z \le 1$, there exists $2n$ coefficients $\lambda_{0,j}, \lambda_{1,j}, \dots, \lambda_{2n,j}$ such that $|\lambda_{1,j}| + \cdots + |\lambda_{2n,j}| \le C$, and

$$\partial_z \widetilde{W}_j(x, s) = \sum_{\ell=1}^{2n} \lambda_{\ell,j}(x, s, z) \widetilde{W}_\ell(x, z) + \lambda_{0,j}(x, s, z) \partial_z. \tag{2.9}$$

We can give now the proof of Proposition 2.14.

Proof of Proposition 2.14. By (2.9), we have that for any $j = 1, \dots, 2n$, the following holds:

$$\begin{aligned}\widetilde{W}_j(x, z) &= \widetilde{W}_j(x, 0) + \int_0^z \partial_z \widetilde{W}(x, s) ds \\ &= \widetilde{W}_j(x, 0) + \sum_{\ell=1}^{2n} \left(\int_0^z \lambda_{\ell,j}(x, s, z) ds\right) \widetilde{W}_\ell(x, z) + z\lambda_{0,j}(x, z)\partial_z;\end{aligned}$$

so that

$$\widetilde{W}_j(x, z) = \widetilde{W}_j(x, 0) + \sum_{\ell=1}^{2n} \hat{\lambda}_{\ell,j}(x, z) \widetilde{W}_\ell(x, z) + \hat{\lambda}_{0,j}(x, z) \partial_z,$$

where $\hat{\lambda}_{0,j}, \dots, \hat{\lambda}_{2n,j} = O(z)$ as $z \to 0$ for $j = 1, \dots, 2n$. Setting, for any $j = 1, \dots, 2n$:

$$\widetilde{W}_j^0(x, z) := \widetilde{W}_j(x, 0),$$

we have

$$
\begin{aligned}
(2.10)\qquad (\widetilde{W}_j\tilde{u})(x,z) = (\widetilde{W}_j^0\tilde{u})(x,z) + \sum_{\ell=1}^{2n}\hat{\lambda}_{\ell,\,j}(x,z)(\widetilde{W}_\ell\tilde{u})(x,z) \\
+ \hat{\lambda}_{0,\,j}(x,z)\partial_z\tilde{u}(x,z).
\end{aligned}
$$

To conclude the proof we have to show that

$$
\begin{aligned}
&\sum_{j=1}^{2n}(\widetilde{W}_j\tilde{u})^2 + (\partial_z\tilde{u})^2 - \big(\sum_{j=1}^{2n}(\widetilde{W}_j^0\tilde{u})^2 + (\partial_z\tilde{u})^2\big) \\
(2.11)\qquad &= \sum_{j=1}^{2n}(\widetilde{W}_j\tilde{u})^2 - \sum_{j=1}^{2n}(\widetilde{W}_j^0\tilde{u})^2 \\
&= O(\sigma)\big(\sum_{j=1}^{2n}(\widetilde{W}_j\tilde{u})^2 + (\partial_z\tilde{u})^2\big).
\end{aligned}
$$

For any $j = 1,\dots,2n$, we set:

$$
a_j := \widetilde{W}_j\tilde{u}, \quad b_j := \widetilde{W}_j^0\tilde{u}, \quad c_0 = \partial_z\tilde{u},
$$

so that (2.11) becomes

(2.12)

$$
\sum_{j=1}^{2n}a_j^2 - \sum_{j=1}^{2n}b_j^2 = \big(\sum_{j=1}^{2n}a_j^2 + c_0^2\big) - \big(\sum_{j=1}^{2n}b_j^2 + c_0^2\big) = O(z)\big(\sum_{j=1}^{2n}a_j^2 + c_0^2\big).
$$

By (2.10), we have that

$$
a_j = b_j + \sum_{\ell=1}^{2n}\hat{\lambda}_{\ell,\,j}a_\ell + \hat{\lambda}_{0,\,j}c_0,
$$

and hence

$$
\sum_{j=1}^{2n}\big(a_j - \sum_{\ell=1}^{2n}\hat{\lambda}_{\ell,\,j}a_\ell - \hat{\lambda}_{0,\,j}c_0\big)^2 = \sum_{j=1}^{2n}b_j^2.
$$

We compute:

$$
\begin{aligned}
\sum_{j=1}^{2n}\big(a_j - \sum_{\ell=1}^{2n}\hat{\lambda}_{\ell,\,j}a_\ell - \hat{\lambda}_{0,\,j}c_0\big)^2 &= \sum_{j=1}^{2n}a_j^2 + c_0^2\sum_{j=1}^{2n}\hat{\lambda}_{0,\,j}^2 + \sum_{j=1}^{2n}\big(\sum_{\ell=1}^{2n}\hat{\lambda}_{\ell,\,j}a_\ell\big)^2 \\
&- 2\sum_{j=1}^{2n}a_j\sum_{\ell=1}^{2n}\hat{\lambda}_{\ell,\,j}a_\ell - 2c_0\sum_{j=1}^{2n}a_j\hat{\lambda}_{0,\,j} - 2c_0\sum_{j,\ell=1}^{2n}\hat{\lambda}_{\ell,\,j}a_\ell\hat{\lambda}_{0,\,j} \\
&= \sum_{j=1}^{2n}a_j^2 + I_0 + I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

It remains to estimate I_i for $i = 0, \dots, 4$:

$$I_0 = c_0^2 \sum_{j=1}^{2n} \lambda_{0,j}^2 \le O(\sigma) c_0^2;$$

$$I_1 \le \sum_{j=1}^{2n} \big(\sum_{\ell=1}^{2n} \hat\lambda_{\ell,j}^2 \big) \sum_{\ell=1}^{2n} a_\ell^2 \le O(\sigma) \sum_{\ell=1}^{2n} a_\ell^2;$$

$$|I_2| \le 2 \big(\sum_{j=1}^{2n} a_j^2 \big)^{1/2} \big(\sum_{j=1}^{2n} \big(\sum_{\ell=1}^{2n} \hat\lambda_{\ell,j} a_\ell \big)^2 \big)^{1/2} \le 2 \big(\sum_{j=1}^{2n} a_j^2 \big)^{1/2} \big(\sum_{j,\ell=1}^{2n} \hat\lambda_{\ell,j}^2 a_\ell^2 \big)^{1/2}$$
$$\le O(\sigma) \sum_{\ell=1}^{2n} a_\ell^2.$$

$$|I_3| \le 2|c_0| \sum_{j=1}^{2n} |\hat\lambda_{0,j} a_j| \le O(\sigma)|c_0| \big(\sum_{j=1}^{2n} a_j^2 \big)^{1/2} = O(\sigma) \big(\sum_{j=1}^{2n} a_j^2 + c_0^2 \big).$$

$$|I_4| \le 2|c_0| \sum_{j,\ell=1}^{2n} |\hat\lambda_{\ell,j} a_\ell \hat\lambda_{0,j}| \le O(\sigma)|c_0| \big(\sum_{j=1}^{2n} a_j^2 \big)^{1/2} = O(\sigma) \big(\sum_{j=1}^{2n} a_j^2 + c_0^2 \big).$$

This yields (2.12) and then achieves the proof of the proposition. □

For $\varepsilon > 0$, the functional $\tilde F_{\varepsilon,\sigma} : W^{1,2}(M \times [0,\sigma)) \to [0, +\infty]$ reads as

$$\tilde F_{\varepsilon,\sigma}(\tilde u) := \varepsilon \int_{M \times [0,\sigma)} \Big(\sum_{j=1}^{2n} (\tilde W_j^0 \tilde u)^2 + (\partial_z \tilde u)^2 \Big) \, dv_\theta \wedge dz + \lambda_\varepsilon \int_M V(\operatorname{Tr} \tilde u) \, dv_\theta, \tag{2.13}$$

that, according to Proposition 2.14, is nothing but an approximation of the original functional F_ε in a neighborhood of M, written in the new "straightened" coordinates.

Remark 2.16. From now on we shall work only on the straight cylinder $M \times [0,\sigma)$, and hence, to avoid cumbersome notations, we shall drop everywhere the tilde if there is no way of misunderstanding.

In addition, since the vector fields $W_1^0, \dots W_{2n}^0$ are independent of $z \in [0,\sigma)$, we can identify them with vector fields in TM.

The proof of our Γ-convergence Theorem 1.1, at least parts *i)* and *ii)*, will follow from the following analogue result for the approximate functional (2.13) using Proposition 2.14.

Theorem 2.17. *Assume that the scaling* (1.3) *holds. Then, for all $\sigma > 0$ small enough, we have:*

- *i*) Given a sequence $\{u_\varepsilon\}$ such that $\tilde F_{\varepsilon,\sigma}(u_\varepsilon)$ is bounded when $\varepsilon \to 0$, then $\{\operatorname{Tr} u_\varepsilon\}$ is pre-compact in $L^1(M)$ and every cluster point belongs to $BV_\theta(M, \{0,1\})$.*
- *ii*) For every $v \in BV_\theta(M, \{0,1\})$ and every sequence $\{u_\varepsilon\} \subset W^{1,2}(M \times [0,\sigma))$ such that $\operatorname{Tr} u_\varepsilon \to v$ in $L^1(M)$, there holds*

$$\liminf_{\varepsilon \to 0} \tilde F_{\varepsilon,\sigma}(u_\varepsilon) \ge F(v).$$

The scheme of this paper is the following: in Section 5 we shall prove *i*)* and *ii*)* of Theorem 2.17. Finally, in Section 6 we shall prove *iii)* of Theorem 1.1, thus completing the proof of of Theorem 1.1.

3. Sub-Riemannian structures

Although there is a wide literature on Carnot-Carathéodory spaces over $\mathbb{R}^n$, here we are looking at manifolds [8, 29], for which some of the theory needs to be developed. We will briefly recall all the necessary ingredients. Though several of the following results hold for general geometric structures, for reader's convenience we state them in our setting, i.e. in the contact manifold (M,θ) endowed with the metric g. According to Remark 2.16, we denote by

$$\mathbf{W}^0 = \{W_1^0,\dots,W_{2n}^0\}$$

our fixed orthonormal basis of $\ker\theta$, and by T the Reeb vector field.

We next define the distance d_c on M. Recall that an absolutely continuous curve $\gamma:[0,T]\to M$ is a *subunit curve* with respect to $W_1^0,\dots,W_{2n}^0$ if there are real measurable functions $c_1,\dots,c_{2n}$, defined in $[0,T]$, such that

$$\sum_{j=1}^{2n} c_j^2(s)\le 1 \quad\text{and}\quad \dot\gamma(s)=\sum_{j=1}^{2n} c_j(s)W_j^0(\gamma(s)),\quad \text{for a.e. } s\in[0,T].$$

Then, if $p,q\in M$, the cc-distance (Carnot-Carathéodory distance) $d_c(p,q)$ is

$$d_c(p,q)\stackrel{\text{def}}{=}\inf\{T>0:\ \gamma \text{ is subunit},\ \gamma(0)=p,\, \gamma(T)=q\}.$$

The set of subunit curves joining p and q is not empty, by Chow's theorem, since the rank of the Lie algebra generated by $W_1^0,\dots,W_{2n}^0$ is $2n+1$. Moreover, d_c is a distance on M inducing the same topology as the standard distance on M as a differentiable manifold (cf. [8, 2]). (M,d_c) is called a Carnot-Carathéodory space.

We recall that, because the topologies induced by d_c and the usual one coincide, the topological dimension of M is $2n+1$. On the contrary the *homogeneous dimension* of M is the integer $Q:=2n+2$.

In the particular case that M is the Heisenberg group, we write the Carnot-Carathéodory distance by $d_c^{\mathbb{H}}$.

Throughout the paper we will denote by $B_r(p)=B(r,p)$ the open ball (centered at p of radius r) in M associated with the distance d_c and by $B_r^{\mathbb{H}}(p)=B^{\mathbb{H}}(p,r)$ the open ball in $\mathbb{H}^n$ associated with the distance $d_c^{\mathbb{H}}$.

3.1. **Functions of bounded variation.** The aim of this section is to recall some basic facts about BV-functions on a contact manifold M and, in particular, the coarea formula, following [8] and [33]. Since the volume form dv_θ has been chosen once for all, if $X\in\Gamma(M,\ker\theta)$ is a continuously differentiable section of $\ker\theta$, we can define the function $\operatorname{div}X$ by the identity

$$(\operatorname{div}X)dv_\theta := \mathcal{L}_X(dv_\theta)=d(i_X(dv_\theta)).$$

Using properties of exterior derivatives and differential forms, we see that $\operatorname{div}X$ satisfies

$$-\int_M \phi\operatorname{div}X\,dv_\theta=\int_M (X\phi)dv_\theta \quad \text{for any }\ \phi\in C_c^1(M). \tag{3.1}$$

Applying (3.1) to the product $h\phi$, with $h \in C^1(M)$ and $\phi \in C^1_c(M)$, using Leibnitz rule and the identity

$$\mathrm{div}(\phi X) = \phi\, \mathrm{div}\, X + X\phi,$$

we deduce that

$$-\int_M h\, \mathrm{div}(\phi X) dv_\theta = \int_M \phi(Xh) dv_\theta.$$

We use this identity to define now the derivative of h along X in the sense of distributions. We say that a measure with finite total variation, that we will denote by $D_X h$, represents in an open set $U \subset M$ the derivative of h along X in the sense of distributions, if

$$-\int_U h\ \mathrm{div}(\phi X)\, dv_\theta = \int_U \phi\, dD_X h, \quad \forall \phi \in \mathcal{C}^\infty_0(U).$$

In [8], Proposition 2.1, it is proved that for $h \in L^1_{\mathrm{loc}}(M, dv_\theta)$, $D_X h$ is a signed measure with finite total variation in U if and only if

$$\sup\left\{\int_U h\, \mathrm{div}(\phi X) dv_\theta,\ \phi \in \mathcal{D}(U), |\phi| \leq 1\right\} < \infty, \tag{3.2}$$

and if this happens the supremum above equals $|D_X h|$. We can now define the space BV_θ.

Definition 3.1. Let $U \subset M$ be an open set. We say that $h \in L^1_{\mathrm{loc}}(M, dv_\theta)$ belongs to $BV_\theta(U)$ if

$$\sup\{|D_X h|(U) : X \in \Gamma(M, \ker\theta),\ g(X, X) \leq 1\} < \infty.$$

If $\mathbf{W}^0 := \{W^0_1, \dots W^0_{2n}\}$ is the orthonormal basis of $\ker\theta$ and $f \in L^1_{\mathrm{loc}}(M, dv_\theta)$, we define a vector-valued measure

$$\mathbf{W}^0 h := (W^0_1 h, \dots, W^0_{2n} h).$$

Proposition 3.2 (see [8], Theorem 3.1)**.** *If $h \in BV_\theta(U)$, then*

i) *the total variation of $\mathbf{W}^0 h$ in U is finite. We denote it by $|\mathbf{W}^0 h|(U)$;*
ii) *h belongs to $BV(U, d_c, dv_\theta)$, the BV-space in metric measure space (M, d_c, dv_θ) in the sense of* [33]*. We notice that (M, d_c, dv_θ) is a "good" metric space in the sense of* [33]*, as pointed out also in* [8]*;*
iii) $|\mathbf{W}^0 h|(U) = \sup\{|D_X h|(U) : X \in \Gamma(M, \ker\theta)\ g(X, X) \leq 1\}$*;*
iv) $|\mathbf{W}^0 h|(U) = \|Dh\|(U)$*, where $\|Dh\|(U)$ is the total variation of h in the sense of* [33]*.*

Definition 3.3. If $E \subset M$ is a Borel set, we say that E has (locally) *finite perimeter* in U if $\chi_E \in BV_\theta(U)$. Moreover we denote

$$\|\partial E\|_\theta(U) := |\mathbf{W}^0 \chi_E|(U).$$

For $h \in BV_\theta(U, \{0, 1\})$, i.e., $h = \chi_E$, we denote by S_h the set of points where the upper and lower approximate limits of h differ. In this case we write $S_h = \partial E \cap U$, the jump set of h in U.

Next, from (3.2) we know that if $\chi_E \in BV_\theta(U)$, then for $\|\partial E\|_\theta$-a.e. $x \in U$,

(3.3)
$$\liminf_{r\downarrow 0} \frac{\min\{v_\theta(B_r(p)\cap E), v_\theta(B_r(p)\setminus E)\}}{v_\theta(B_r(p))} > 0, \quad \limsup_{r\downarrow 0} \frac{\|\partial E\|_\theta(B_r)}{v_\theta(B_r(p))/r} < \infty.$$

Definition 3.4 (see [8], Definition 3.2)**.** (Dual normal and reduced boundary). We write in polar decomposition:

$$\mathbf{W}^0\chi_E = \nu_E^*|\mathbf{W}^0\chi_E|,$$

where $\nu_E^* = (\nu_{E,1}^*, \dots, \nu_{E,2n}^*) : M \to \mathbb{R}^{2n}$ is a Borel vector field with unit norm. We call ν_E^* the *dual normal* to E.

We denote by $\partial^* E$ the *reduced boundary* of E, i.e. the set of all points p in the support of $|\mathbf{W}^0\chi_E|$ satisfying (3.3) and

$$\lim_{r\downarrow 0} \frac{1}{|\mathbf{W}^0\chi_E|(B_r(p))} \int_{B_r(p)} |\nu_E^*(q) - \nu_E^*(p)|^2 d|\mathbf{W}^0\chi_E|(q) = 0.$$

We know that if E has locally finite perimeter in U, then $|\mathbf{W}^0\chi_E|$-almost every point in U belongs to $\partial^* E$. Moreover,

Theorem 3.5 (Riesz Theorem: see [8], Theorem 3.3)**.** *Let h be a function in $BV_\theta(M)$. Then, there exists a Borel vector field ν_h, satisfying $g(\nu_h, \nu_h) = 1$ $|\mathbf{W}^0 h| - a.e.$ in M and*

$$D_X h = g(X, \nu_u)|\mathbf{W}^0 h|, \quad \textit{for any } X \in \Gamma(M, \ker\theta).$$

If E is a set of finite perimeter and $u = \chi_E$, we call geometric normal *the vector field:*

$$\nu_E := \nu_{\chi_E}. \tag{3.4}$$

In addition $\nu_E = \sum_i \nu_{E,i}^* W_i$.

Finally, combining Proposition 3.2 above and Remark 4.3 in [33], we obtain

Proposition 3.6 (Coarea formula in M)**.** *If $h \in BV_\theta(M)$ and $f : M \to \mathbb{R}$ is a Borel-measurable function, $f \geq 0$, for any Borel set $U \subset M$ we have:*

$$\int_U f\, d|\mathbf{W}^0 h| = \int_{-\infty}^{+\infty} \left(\int_U f\, d\|\partial E_t\|_\theta(x) \right) dt,$$

where $E_t = \{h < t\}$.

3.2. **Carnot-Carathéodory distance and the Eikonal equation.** The aim of this subsection is to prove the Eikonal equation for the Carnot-Carathéodory distance.

First we recall the following regularity result about geodesics (see the survey [37], Theorem 4):

Theorem 3.7 (Theorem 4 in [37])**.** *In contact manifolds any length minimizing curve is smooth.*

A function $h : (M, d_c) \to \mathbb{R}$ is L-Lipschitz if

$$|h(p) - h(q)| \leq L d_c(p, q)$$

for all $p, q \in M$. The infimum of such constants L is denoted by $\mathrm{Lip}(h)$. Lipschitz functions are differentiable a.e. along the vector fields W_j, $j = 1, \dots, 2n$, as we see from the lemma below.

Lemma 3.8. *If $h : M \to \mathbb{R}$ is L-Lipschitz continuous with respect to d_c, then*

$$h \in BV_\theta(M),$$

$$|\mathbf{W}^0 h|(U) \le L v_\theta(U) \quad \text{for all open sets } U \subset M \tag{3.5}$$

and the Lie derivative

$$\mathcal{L}_X h(x^0) := \lim_{t \to 0} \frac{1}{t} \big(h(\exp(tX)x^0) - h(x^0) \big) \tag{3.6}$$

exists for all $X \in \ker \theta$ and for almost every $x^0 \in M$.

In addition $\mathcal{L}_X h$ is a distributional derivative, i.e. (with the notation of [8] *as in* (3.2)*)*

$$(\mathcal{L}_X h)\, dv_\theta = D_X h.$$

Proof. The first two assertions follows straightforwardly from [33], keeping in mind Theorem 3.1 of [8]. Let now $\bar{x} \in M$ be a fixed point. Then, by Darboux theorem there exists a neighborhood U of $\bar{x}$ and a contact diffeomorphism $\Psi : U \to \mathbb{H}^n$. The map Ψ is bi-Lipschitz continuous with respect to the Carnot-Carathéodory distance d_c in U and the canonical Carnot-Carathéodory distance $d_c^{\mathbb{H}}$ in $\mathbb{H}^n$. In particular, $h \circ \Psi^{-1}$ is $d_c^{\mathbb{H}}$-Lipschitz continuous. By Pansu-Rademacher theorem (see [43]), for a.e. $x^0 \in U$ there exist real numbers $\lambda_1(x^0), \dots, \lambda_{2n}(x^0)$ such that, if we set $\Psi(x^0) := p^0$ for $p^0 = (p_1^0, \dots, p_{2n+1}^0)$ and $p = (p_1, \dots, p_{2n+1})$,

$$h \circ \Psi^{-1}(p) - h \circ \Psi^{-1}(p^0) = \sum_{j=0}^{2n} \lambda_j(x^0)(p_j - p_j^0) + o(d_c^{\mathbb{H}}(p, p^0))$$

as $p \to p^0$ and hence, if $\Psi = (\Psi_1, \dots, \Psi_{2n+1})$,

$$h(x) - h(x^0) = \sum_{j=0}^{2n} \lambda_j(x^0)(\Psi_j(x) - \Psi_j(x^0)) + o(d_c(x, x^0)),$$

as $x \to x^0$. Thus, keeping in mind that $d_c(\exp(tX)x^0, x^0) = O(t)$ as $t \to 0$, we have:

$$\begin{aligned}
&\lim_{t \to 0} \frac{1}{t} \big(h(\exp(tX)x^0) - h(x^0) \big) \\
&\quad = \lim_{t \to 0} \frac{1}{t} \sum_{j=0}^{2n} \lambda_j(x^0)(\Psi_j(\exp(tX)x^0) - \Psi_j(x^0)) \\
&\quad + \lim_{t \to 0} \frac{1}{t} o(d_c(\exp(tX)x^0, x^0)) \\
&\quad = \sum_{j=0}^{2n} \mu_j^X(x_0),
\end{aligned} \tag{3.7}$$

where

$$\mu_j^X(x_0) = \lambda_j(x^0) \frac{d}{dt} \Psi_j(\exp(tX)x^0) \quad \text{at } t = 0, \quad j = 1, \dots, 2n.$$

Finally, the last statement follows from (3.6) and (3.5) by standard arguments.

□

Remark 3.9. We notice that, if $\gamma : [0,1] \to M$ is a continuously differentiable horizontal curve with $\gamma(0) = x^0$ and $\dot\gamma(0) = X$, then, arguing as in (3.7),

$$\lim_{t\to 0} \frac{1}{t}\big(h(\gamma(t)) - h(x^0)\big) = \mathcal{L}_X h(x^0).$$

Lemma 3.10. *Let $K \subset M$ be a compact set and let $x \in M$. We denote by $d_{c,K}(x)$ the Carnot-Carathéodory distance of x from K. Then*

i) *$d_{c,K}(x)$ is 1-Lipschitz continuous with respect to the d_c-distance;*

ii) *for a.e. $x^0 \in M$ and for all $X \in \ker\theta$, with $g(X,X) \le 1$*

$$\big|X d_{c,K}(x^0)\big| \le 1,$$

and there exists $X^0 = X(x_0) \in \ker\theta$, with $g(X^0, X^0) = 1$ such that

$$X^0 d_{c,K}(x^0) = 1.$$

Proof. The first assertion is trivial. Moreover, it is well known that for any $x \in M$, there exists $\bar x \in K$ such that $d_{c,K}(x) = d_c(\bar x, x)$. Let now x^0 be a point where all horizontal Lie derivatives exist, and let $\gamma : [0, d_c(\bar x, x^0)] \to M$ be a minimizing geodesic with $\gamma(d_c(\bar x, x^0)) = \bar x$ and $\gamma(0) = x^0$. By Theorem 3.7, γ is smooth. Without loss of generality, we may assume that $d_c(\gamma(t), x^0) = t$. Keeping in mind Remark 3.9, if we take $X^0 := X(x^0) = \dot\gamma(0)$, we have

$$X^0 d_{c,K}(x^0) = \lim_{t\to 0} \frac{1}{t}\big(d_c(\gamma(t), x^0)\big) = 1.$$

This concludes the proof of ii). □

We can finally state the Eikonal equation for the distance d_c:

Theorem 3.11 (The Eikonal equation). *Let $K \subset M$ be a closed set and let $d_{c,K}$ be the distance from K. Then*

$$|\mathbf{W}^0 d_{c,K}| = dv_\theta. \tag{3.8}$$

Proof. Let x^0 and $X^0 = X(x^0)$ be as in Lemma 3.10. We can write $X^0 = \sum_{j=1}^{2n} \lambda_j W_j^0$. Since $g(X^0, X^0) = 1$ we have

$$\sum_j \lambda_j^2 = 1.$$

Then

$$\left(\sum_{j=1}^{2n} (W_j^0 d_{c,K})^2\right)^{1/2} \ge \sum_{j=1}^{2n} \lambda_j (W_j^0 d_{c,K}) = X^0 d_{c,K} = 1.$$

The reverse estimate follows from (3.5) and Theorem 3.10, part i).

Finally, as in [8], page 20, we have that $|\mathbf{W}^0 d_{c,K}| = \left(\sum_{j=1}^{2n} (W_j^0 d_{c,K})^2\right)^{1/2}$, which concludes the proof of the Theorem. □

3.3. **Minkowski content and perimeter.** Let E be an open set in M and let $d_{c,\partial E}(x)$ denote the Carnot-Carathéodory distance of the point $x \in M$ from the boundary of E. We define the tubular neighborhood of ∂E in M:

$$\mathcal{U}_r(\partial E) := \{p \in M \ : \ d_{c,\partial E}(p) < r\}.$$

The *upper* and *lower Minkowski content* of ∂E in M are defined, respectively, as follows:

$$\mathcal{M}^+(\partial E) := \limsup_{r \downarrow 0} \frac{v_\theta(\mathcal{U}_r(\partial E))}{2r},$$

$$\mathcal{M}^-(\partial E) := \liminf_{r \downarrow 0} \frac{v_\theta(\mathcal{U}_r(\partial E))}{2r}.$$

When $\mathcal{M}^+(\partial E) = \mathcal{M}^-(\partial E)$, we call the common value the *Minkowski content* of E and we denote it by $\mathcal{M}(\partial E)$. The following theorem is the analogue of Theorem 5.1 in [38].

Theorem 3.12. *Let $E \subset\subset M$ be a bounded open set with C^∞ boundary. Then $\mathcal{M}^+(\partial E) = \mathcal{M}^-(\partial E)$ and we have*

$$\mathcal{M}(\partial E) = \|\partial E\|_\theta.$$

Proof. We follow the proof of Theorem 5.1 in [38]. We prove separately the two following inequalities:

$$\mathcal{M}^-(\partial E) \geq \|\partial E\|_\theta, \tag{3.9}$$

$$\mathcal{M}^+(\partial E) \leq \|\partial E\|_\theta. \tag{3.10}$$

We start by proving (3.9). Let us introduce the signed distance from ∂E:

$$\rho_c(x) = \begin{cases} d_{c,\partial E}(p) & \text{if } p \in E, \\ -d_{c,\partial E}(p) & \text{if } p \in M \setminus E. \end{cases} \tag{3.11}$$

For $\varepsilon > 0$ we define the function:

$$\varphi_\varepsilon(p) = \begin{cases} \frac{1}{2\varepsilon}\rho_c(p) + \frac{1}{2} & \text{if } |\rho_c(p)| < \varepsilon, \\ 1 & \text{if } \rho_c(p) \geq \varepsilon, \\ 0 & \text{if } \rho_c(p) \leq -\varepsilon. \end{cases}$$

Using that Theorem 3.11 on the Eikonal equation, we have

$$|\mathbf{W}^0\varphi_\varepsilon| = \frac{1}{2\varepsilon} \int_{\{|\rho_c(p)| < \varepsilon\}} |\mathbf{W}^0\varphi_\varepsilon(p)| \, dv_\theta(p) \leq \frac{1}{2\varepsilon} \, v_\theta(\mathcal{U}_\varepsilon(\partial E)).$$

By the lower semicontinuity of the total variation and since $\varphi_\varepsilon \to \chi_E$ in $L^1(M)$, we deduce that

$$\|\partial E\|_\theta \leq \liminf_{\varepsilon \to 0} |\mathbf{W}^0\varphi_\varepsilon| \leq \mathcal{M}^-(\partial E),$$

which concludes the proof of (3.9).

It remains to prove (3.10). Here we use a Riemannian approximation for Carnot-Carathéodory spaces (see e.g. [17] and [38]). We consider the Carnot-Carathéodory distance d_ε in M associated with the vector fields $\mathbf{W}^0_\varepsilon = \{W^0_1, \dots, W^0_{2n}, \varepsilon T\}$. Notice that $\mathbf{W}^0_\varepsilon$ is an orthonormal basis of TM with respect to the Riemannian metric g_ε defined as follows: if $X, Y \in TM$, we write $X = X' + X''$, $Y = Y' + Y''$, with $X', Y' \in \ker\theta$ and $X'', Y'' \in \operatorname{span}\{T\}$, and we set

$$g_\varepsilon(X, Y) := g(X', Y') + \frac{1}{\varepsilon^2} g(X'', Y'').$$

Obviously d_ε is a Riemannian distance.

Define also $d_{\varepsilon,\partial E}(p) = \min_{q\in\partial E} d_\varepsilon(x, y)$. We have that

$$d_\varepsilon(p, q) \le d_{c,\partial E}(p, q) \quad \text{for all } p, q. \tag{3.12}$$

In fact, $d_{c,\partial E}(p, q) = \sup_{\varepsilon>0} d_\varepsilon(p, q)$.

Define also ρ_ε to be the signed ε-distance to ∂E as in (3.11). Then ρ_ε is C^∞ near ∂E and it satisfies the Eikonal equation $|\mathbf{W}^0_\varepsilon(\rho_\varepsilon)| = 1$.

We consider the usual upper and lower Minkowski content for ρ_ε

$$\mathcal{M}^+_\varepsilon(\partial E) := \limsup_{r\downarrow 0} \frac{v_{g_\varepsilon}(\{|\rho_\varepsilon| < r\})}{2r}, \quad \mathcal{M}^-_\varepsilon(\partial E) := \liminf_{r\downarrow 0} \frac{v_{g_\varepsilon}(\{|\rho_\varepsilon| < r\})}{2r}.$$

From (3.12), $|\rho_\varepsilon| \le |\rho|$, from which we immediately have

$$\mathcal{M}^+(\partial E) \le \mathcal{M}^+_\varepsilon(\partial E). \tag{3.13}$$

To achieve the proof of Theorem 3.12, we need the following technical result.

Lemma 3.13. *If $E \subset M$ is an open set with smooth boundary ∂E, that is a compact $2n$-dimensional submanifold without boundary, we have*

$$|\mathbf{W}^0_\varepsilon \chi_E|(M) \to |\mathbf{W}^0 \chi_E|(M) \quad \textit{as } \varepsilon \to 0. \tag{3.14}$$

Proof. Without loss of generality, in (3.14) we can replace M by an open set U that is contained in the domain of a Darboux map $\Psi : U \to \mathbb{H}^n \equiv \mathbb{R}^{2n+1}$. We denote by $\mu \to \Psi_\# \mu$ the push-forward of a Borel measure μ, i.e.

$$\Psi_\#\mu(\mathcal{B}) = \mu(\Psi^{-1}(\mathcal{B})) \quad \text{for any } \mathcal{B} \subset \mathbb{R}^{2n+1} \text{ Borel}.$$

Moreover, we denote by $\Psi^* g$ the pull-back metric on $\mathbb{R}^{2n+1}$. By [8], Proposition 2.2, if $X \in \Gamma(M, TM)$, then

$$(D_X\chi_E)(\mathcal{B}) = \Psi_\#(D_X\chi_E)(\Psi(\mathcal{B})).$$

Thus

$$\begin{aligned}
|\mathbf{W}^0_\varepsilon \chi_E|(\mathcal{B}) &= \sup_{g_\varepsilon(X,X)\le 1} |D_X\chi_E|(\mathcal{B}) \\
&= \sup_{g_\varepsilon(X,X)\le 1} |\Psi_\#(D_X\chi_E)|(\Psi(\mathcal{B})) \\
&= \sup_{g_\varepsilon(X,X)\le 1} |\Psi_\#(D_{\Psi_* X}\chi_{\Psi(E)}|(\Psi(\mathcal{B})) \\
&= \sup_{g^*_\varepsilon(\Psi_* X, \Psi_* X)\le 1} |\Psi_\#(D_{\Psi_* X}\chi_{\Psi(E)}|(\Psi(\mathcal{B})) \\
&= |\Psi_*(\mathbf{W}^0_\varepsilon)\chi_{\Psi(E)}|(\Psi(\mathcal{B})),
\end{aligned} \tag{3.15}$$

where

$$\Psi_*(\mathbf{W}^0_\varepsilon) = \{\Psi_* W^0_1, \dots, \Psi_* W^0_{2n}\}.$$

As in [38], formula (5.5),

$$|\Psi_*(\mathbf{W}^0_\varepsilon)\chi_{\Psi(E)}|(\Psi(\mathcal{B})) \to |\Psi_*(\mathbf{W}^0)\chi_{\Psi(E)}|(\Psi(\mathcal{B})).$$

Thus, repeating backward the arguments of (3.15), we conclude the proof of the Lemma.

□

Let us go back to the proof of Theorem 3.12. We will prove soon that

$$\mathcal{M}^+_\varepsilon(\partial E) = \mathcal{M}^-_\varepsilon(\partial E) = |\mathbf{W}^0_\varepsilon \chi_E|(M). \tag{3.16}$$

Suppose for the moment that this is true. Then, by (3.13), (3.16), and (3.14), we have:

$$\mathcal{M}^+(\partial E) \leq \lim_{\varepsilon\to 0} \mathcal{M}^+_\varepsilon(\partial E) = \lim_{\varepsilon\to 0} |\mathbf{W}^0_\varepsilon \chi_\varepsilon|(M) = |\mathbf{W}^0 \chi_\varepsilon|(M),$$

which concludes the proof of the theorem. Therefore, it remains just to show (3.16).

Let $E_s = \{p \in M \,:\, \rho_\varepsilon(p) > s\}$. Using the coarea formula (3.6) and the Riemannian Eikonal equation, we have that

$$\begin{aligned} v_\theta(\{|\rho_\varepsilon| < t\}) &= \int_{\{|\rho_\varepsilon < t|\}} dv_\theta = \int_{-t}^{t} \frac{1}{|\mathbf{W}^0_\varepsilon \rho_\varepsilon|}\, d|\mathbf{W}^0_\varepsilon \chi_{E_s}|\, ds \\ &= \int_{-t}^{t} |\mathbf{W}^0_\varepsilon \chi_{E_s}|(M)\, ds. \end{aligned}$$

Thus, (3.16) will follow if we prove that

$$\text{the map} \quad s \to |\mathbf{W}^0_\varepsilon \chi_{E_s}|(M) \quad \text{is continuous at } s = 0. \tag{3.17}$$

This can be done using again the arguments of (3.15) to reduce ourselves to the "flat" case of $\mathbb{R}^{2n+1}$, where (3.17) has been already established in [38] (see the proof of Theorem 5.1 therein).

□

4. Compactness and liminf inequality in Heisenberg groups

The aim of this Section is to prove a liminf inequality for the "model case" where $M \times [0, \sigma)$ is replaced by $\mathbb{H}^n \times [0, \sigma)$. To this end, for a subset A of $\mathbb{H}^n \times \mathbb{R}^+$, and $A' = \partial A \cap \{z = 0\}$, and for a function $u : A \to \mathbb{R}$, we consider the localized functional:

$$E_\varepsilon(u, A, A') := \varepsilon \int_A \left(|\mathbf{W}^{\mathbb{H}} u|^2 + |\partial_z u|^2\right) d\eta dt dz + \lambda_\varepsilon \int_{A'} V(\operatorname{Tr} u)\, d\eta dt. \tag{4.1}$$

The following theorem is the analogue of Proposition 4.7 of [5]. It establishes a compactness result and a liminf inequality for the functional $E_\varepsilon(u_\varepsilon, C_R, B^{\mathbb{H}}_R)$, where $B^{\mathbb{H}}_R = B^{\mathbb{H}}(0, R)$ is the Carnot-Caratheodory ball in $\mathbb{H}^n$ of radius R centered at 0, $C_R := B^{\mathbb{H}}_R \times (0, R) \subset \mathbb{H}^n \times \mathbb{R}^+$ and for simplicity of notation we write $B^{\mathbb{H}}_R$ in place of $B^{\mathbb{H}}_R \times \{0\}$.

Theorem 4.1. *Let $\{u_\varepsilon\} \subset W^{1,2}(C_R)$ be a countable sequence with uniformly bounded energies $E_\varepsilon(u_\varepsilon, C_R, B^{\mathbb{H}}_R)$. Then the traces $\operatorname{Tr} u_\varepsilon$ are pre-compact in $L^1(B^{\mathbb{H}}_R)$ and every cluster point v belongs to $BV_{\theta_0}(B^{\mathbb{H}}_R, \{0, 1\})$. Moreover, if $\operatorname{Tr} u_\varepsilon \to v$ in $L^1(B^{\mathbb{H}}_R)$, then*

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, C_R, B^{\mathbb{H}}_R) \geq \mathbf{c} \left| \int_{B^{\mathbb{H}}_R} \nu_v \, d\|\partial\{v = 1\}\|_{\theta_0} \right|, \tag{4.2}$$

where ν_v *is the geometrical normal to the set* $\{v \equiv 1\}$ *and* $\mathbf{c} = k/\pi$ *with* k *given in* (1.3)*.*

The proof of Theorem 4.1 is articulated in several steps and requires a good amount of preliminary results.

4.1. **Slicing theorems.** We recall a Fubini type Theorem in Carnot groups, which is proven in [35]. Here, we state it for the case of the Heisenberg group, but it holds in general Carnot groups. Let $S \subset \mathbb{H}^n$ be a $\mathcal{C}^1$ smooth hypersurface. By the classical Implicit Function Theorem, we may assume that $S = \partial E$, where $E \subset \mathbb{H}^n$ is an open set with finite $\mathbb{H}$-perimeter. Suppose that there exists an horizontal left invariant vector field $W^{\mathbb{H}}$ which is globally transverse to S, i.e.

$$\left\langle W^{\mathbb{H}}(p), \nu(p) \right\rangle \neq 0 \quad \forall \ p \in S,$$

where ν is the Euclidean unit inward normal along S. The Cauchy problem

$$\begin{cases} \dot\gamma(t) & = W^{\mathbb{H}}(\gamma(t)) \\ \gamma(0) & = p \in S, \end{cases}$$

has a unique smooth solution defined on all $\mathbb{R}$, which we denote by $\gamma_p(t) = \exp(tW^{\mathbb{H}})(p)$ for $t \in \mathbb{R}$ and $p \in S$. We call this trajectory a horizontal line. Now we consider the family of horizontal $W^{\mathbb{H}}$-lines starting from S and we denote by R_S the subset of $\mathbb{H}^n$ reachable from S moving along horizontal $W^{\mathbb{H}}$-lines, that is

$$R_S := \{q \in \mathbb{H}^n : \exists\, p \in S,\ \exists\, t \in \mathbb{R} \ \text{ s.t. } \ q = \gamma_p(t) \text{ for some } \gamma_p\}\,. \tag{4.3}$$

Assume moreover that $\gamma_p(\mathbb{R}) \cap S = p$ for every $p \in S$. Since $W^{\mathbb{H}}$ is transverse to S, by the uniqueness of the solution of the Cauchy problem and by (4.3), any subset D of R_S has a natural projection on S along $W^{\mathbb{H}}$. We define the map $pr_S : D \subset R_S \to S$ in the following way: for $q \in D$ and $p \in S$, we set $p = pr_S(q)$ if and only if there exists $t \in \mathbb{R}$ such that $q = \gamma_p(t)$. Using this projection, every subset D of R_S can be foliated with one-dimensional leaves that are horizontal $W^{\mathbb{H}}$-lines. We define now the partial perimeter along a horizontal direction.

Definition 4.2. Let U be an open set in $\mathbb{H}^n$. Let E be a measurable subset of $\mathbb{H}^n$. We say that E has finite $W^{\mathbb{H}}$-perimeter in U if

$$\|\partial_{W^{\mathbb{H}}} E\|_\theta(U) := \sup\left\{ \int_U \chi_E \ W^{\mathbb{H}}\varphi \ d\eta dt \ : \ \ \varphi \in C^1_0(U),\ |\varphi| \le 1 \right\} < \infty.$$

With this notions, we can now state the Fubini type result, which will be used in the proof of the liminf inequality.

Theorem 4.3 (see Corollary 2.3 in [35])**.** *Let* $S \subset \mathbb{H}^n$ *be a* $\mathbb{H}$*-regular hypersurface and assume* $S = \partial E$ *globally, where* $E \subset \mathbb{H}^n$ *is a suitable open* $\mathbb{H}$*-Caccioppoli set. Let as before,* γ_p *be the horizontal* $W^{\mathbb{H}}$*-line starting from* $p \in S$ *and assume that* $\gamma_p(\mathbb{R}) \cap S = p$ *for every* $p \in S$*. Finally let* $D \subset R_S$ *be a Lebesgue measurable subset of* $\mathbb{H}^n$ *that is reachable from* S *by means of* $W^{\mathbb{H}}$*-lines. Then, for every function* $\psi \in L^1(D)$*, the following statement holds:*

(i) *let $\psi_{|D_p}$ denote the restriction of ψ to $D_p := D \cap \gamma_p(\mathbb{R})$ and let us define the mapping*

$$\psi_p : \gamma_p^{-1}(D_p) \subset \mathbb{R} \to \mathbb{R}, \quad \psi_p(s) = (\psi \circ \gamma_p)(s).$$

Then ψ_p is $\mathcal{L}^1$-measurable for $\|\partial E\|_{\theta_0}$-a.e. $p \in S$ or, equivalently, the restriction $\psi_{|D_p}$ is $\mathcal{H}_c^1$-measurable for $\|\partial E\|_{\theta_0}$-a.e. $p \in S$;

(ii) *the mapping defined by*

$$S \ni p \mapsto \int_{D_p} \psi \, d\mathcal{H}_c^1 = \int_{\gamma_p^{-1}(D_p)} \psi_p(s)\, ds$$

is $\|\partial E\|_{\theta_0}$-measurable on S and the following formula holds

(4.4)

$$\begin{aligned}\int_D \psi \, dv_{\theta_0} &= \int_{pr_S(D)} \left[\int_{D_p} \psi \, d\mathcal{H}_c^1 \right] d\|\partial_{W^{\mathbb{H}}} E\|_{\theta_0}(p) \\ &= \int_{pr_S(D)} \left[\int_{\gamma_p^{-1}(D_p)} \psi_p(s)\, ds \right] \cdot \left| \left\langle W^{\mathbb{H}}, \nu_E \right\rangle_{H\mathbb{H}_p} \right| \, d\|\partial E\|_{\theta_0}(p).\end{aligned}$$

Later we will apply this result to the case in which S is a vertical hyperplane. We stress that the $\mathbb{H}$-perimeter on any vertical hyperplane coincides with the Lebesgue measure ([12]).

The following result, which is contained in [35], allows to reduce the study of BV functions on Carnot groups to the study of their one-dimensional restrictions. First we introduce the following notation, concerning the one-dimensional total variation along an horizontal vector field $W^{\mathbb{H}}$ of a function. Let $W^{\mathbb{H}}$ be a horizontal vector field, such that $|W^{\mathbb{H}}|_{\mathcal{H}\mathbb{H}^n} = 1$ and let γ_p be a horizontal $W^{\mathbb{H}}$-line starting from $p \in \mathbb{H}^n$. We set

$$\begin{aligned} var^1_{W^{\mathbb{H}}}[f](\mathcal{U}) := \sup \Big\{ \int_{\mathcal{U}} f W^{\mathbb{H}} \varphi \, d\mathcal{H}_c^1 \ :\ & \varphi \in C_0^1(\mathcal{B}), |\varphi| \le 1, \\ & \text{where } \mathcal{B} \subset \mathbb{H}^n,\ \mathcal{B} \text{ open s.t. } \gamma_p \cap \mathcal{B} = \mathcal{U} \Big\}. \end{aligned}$$

We give the statement for the specific case of the Heisenberg group.

Theorem 4.4 (Theorem 3.7 in [35]). *Let $S \subset \mathbb{H}^n$ be a $\mathbb{H}$-regular hypersurfaces and assume that $S = \partial E$ globally, where $E \subset \mathbb{H}^n$ is a suitable open $\mathbb{H}^n$-Caccioppoli set. Let $W^{\mathbb{H}} \in \mathcal{H}\mathbb{H}^n$, $|W^{\mathbb{H}}|_{\mathcal{H}\mathbb{H}^n} = 1$, be a unit horizontal left invariant vector field which is transverse to S, and denote by $t \to \gamma_p(t) := p \cdot \exp(tW^{\mathbb{H}})$ the horizontal $W^{\mathbb{H}}$-line starting from $p \in S$. Let $D \subset R_S$ be a Lebesgue measurable subset of $\mathbb{H}^n$ that is reachable from S by means of $W^{\mathbb{H}}$-lines.*

Then

$$|W^{\mathbb{H}} f|(D) = \int_{pr_S(D)} var^1_{W^{\mathbb{H}}}[f_p](D_p) d\|\partial_{W^{\mathbb{H}}} E\|_{\theta_0}(p), \tag{4.5}$$

where $f_p := f \circ \gamma_p$ and $D_p := \gamma_p \cap D$.

Our next step will be to prove a compactness result in L^1 for a family of functions satisfying some kind of equicontinuity along 1-dimensional horizontal lines (see Theorem 4.6). To this end, we must factorize an arbitrary displacement through a finite number of *horizontal* displacements of controlled length. This is the content of the following Theorem 4.5.

Theorem 4.5 ([39], §3). *There exist $m \in \mathbb{N}$ and three multi-indexes I, J and ω of length m*

$$\begin{aligned} I &= (i_1, \dots, i_m), \quad i_n \in \{1, \dots, 2n\} \\ J &= (j_1, \dots, j_m), \quad j_n \in \{1, \dots, 2n+1\} \\ \omega &= (\omega_1, \dots, \omega_M) \quad \omega_n \in \{-1, 1\} \end{aligned}$$

and two geometric constants $0 < b < a < 1$ such that, if we set

$$\mathcal{E}_{I,J,\omega} : \mathbb{R}^{2n+1} \to \mathbb{H}^n$$

$$\mathcal{E}_{I,J,\omega}(t_1, \dots, t_{2n+1}) := \exp(\omega_1 t_{j_1} W_{i_1}^{\mathbb{H}}) \cdots \exp(\omega_m t_{j_m} W_{i_m}^{\mathbb{H}}),$$

then for all $R > 0$

$$B_c(0, bR) \subset \mathcal{E}_{I,J,\omega}(Q(0, aR)) \subset B_c(0, R),$$

where

$$Q(0, r) = \{(t_1, \dots, t_{2n+1}) \in \mathbb{R}^{2n+1}, \ \max_\ell \{|t_\ell|\} < r\}.$$

In particular, if $h \in \mathbb{H}^n$, then there exist $t_\ell = t_\ell(h)$, $\ell = 1, \dots, 2n+1$, $\max_\ell\{|t_\ell|\} < a\, d_c(0, h)/b$ such that

$$\mathcal{E}_{I,J,\omega}(t_1, \dots, t_{2n+1}) = h.$$

The main idea of Theorem 4.5 is that each point in $\mathbb{H}^n$ can be reached by integral curves of horizontal vector fields, and when a commutator of two vector fields is needed, it can be approximated by a finite length "square path" along the two fields, taken successively with opposite sign. This is an important difference between this result and the classical result due to Nagel, Stein and Wainger [40], Theorem 7, where instead the authors work directly with integral curves of commutators.

The following result is the analogue of Theorem 6.6 in [5], and will be used to deduce compactness of the $\mathrm{Tr}\, u_\varepsilon$ from the compactness of their restrictions to the horizontal slices. We first fix some notations. Let $e_1, ..., e_{2n}$ be the first $2n$ unit vectors of the canonical basis of $\mathbb{H}^n$. Let $D \subset \mathbb{H}^n$ and let Π_i be the vertical hyperplane orthogonal to e_i. Obviously we have that $W_i^{\mathbb{H}}$ is globally transverse to Π_i, and therefore we can consider the projection D_i of D on Π_i along $W_i^{\mathbb{H}}$. We denote by $\gamma_i^p(s)$ the horizontal $W_i^{\mathbb{H}}$-line starting from a point $p \in \Pi_i$. For a function v defined on D, we consider the function $v_i^p(s) := v(\gamma_i^p(s))$ defined on the set $D_i^p := \{s \in \mathbb{R} | \gamma_i^p(s) \in D\}$. Accordingly, for every family $\mathcal{F}$ of functions on D, we define the family $\mathcal{F}_i^p := \{v_i^p | v \in \mathcal{F}\}$.

We say that a family $\mathcal{F}'$ is δ-dense in $\mathcal{F}$ if $\mathcal{F}$ lies in a δ-neighborhood of $\mathcal{F}'$ with respect to the L^1 topology. We have the following theorem:

Theorem 4.6. *Let $\mathcal{F}$ be a family of functions $v : D \to [-L, L]$ and assume that for every $\delta > 0$ there exists a family $\mathcal{F}_\delta$ δ-dense in $\mathcal{F}$ such that $(\mathcal{F}_\delta)_i^p$ is pre-compact in $L^1(D_i^p)$ for $|\Pi_i|_{\mathcal{H}}$- a.e. $p \in D_i$ for every $i = 1, ..., 2n$. Then $\mathcal{F}$ is pre-compact in $L^1(D)$.*

Proof. We can assume $L = 1$ and $|D_i^p| \leq 1$ for every $p \in \Pi_i$. Every function defined on D is extended to be zero outside D, and accordingly every function defined D_i^p is extended to be zero outside D_i^p. Arguing as in [5], Theorem 6.6, we have but to show that for any $\delta > 0$

$$\int_{\mathbb{H}^n} |v(q \cdot h) - v(q)| \, dq \to 0 \tag{4.6}$$

as $d_c^{\mathbb{H}}(h, 0) \to 0$, uniformly for $v \in \mathcal{F}_\delta$.

If $i = 1, \dots, 2n$ is fixed, $p \in D_i$, $r > 0$, we set

$$\omega_\delta^p(r) = \sup \Big\{ \int_{\mathbb{R}} |v_i^p(s + \sigma) - v_i^p(s)| \, ds \ : \ v \in \mathcal{F}_\delta, \ |\sigma| \leq r \Big\}.$$

By our assumptions, $\omega_\delta^p(r) \leq 2$ for all $r > 0$ and, as in [5], by Fréchet-Kolmogorov compactness theorem, $\omega_\delta^p(r) \searrow 0$ as $r \searrow 0$.

By Theorem 4.5 we can write

$$h = \mathcal{E}_{I,J,\omega}(t_1, \dots, t_{2n+1}),$$

with $t_\ell = t_\ell(h)$, $\ell = 1, \dots, 2n + 1$, $\max_\ell\{|t_\ell|\} < ad_c(0, h)/b$. For sake of brevity we write $t_h = (t_1, \dots, t_{2n+1})$. With the notations of Theorem 4.5, for $1 \leq k \leq m$ we set

$$I_k = (i_1, \dots, i_k) \quad , \quad J_k = (j_1, \dots, j_k) \quad \text{and} \quad \omega_k = (\omega_1, \dots, \omega_k).$$

If we set $\mathcal{E}(I_0, J_0, \omega_0) = e$, we have

$$\begin{aligned}
v(x \cdot h) - v(x) &= \sum_{k=1}^{m} \Big(v(x \cdot \mathcal{E}_{I_k, J_k, \omega_k}(t_h)) - v(x \cdot \mathcal{E}_{I_{k-1}, J_{k-1}, \omega_{k-1}}(t_h)) \Big) \\
&= \sum_{k=1}^{m} \Big(v(x \cdot \mathcal{E}_{I_{k-1}, J_{k-1}, \omega_{k-1}}(t_h) \cdot \exp(\omega_k t_{j_k} W_{i_k}^{\mathbb{H}})) - v(x \cdot \mathcal{E}_{I_{k-1}, J_{k-1}, \omega_{k-1}}(t_h)) \Big).
\end{aligned}$$

Thus, keeping in mind that Lebesgue measure in $\mathbb{H}^n$ (that is unimodular) is the group Haar measure and therefore is right invariant, we have

$$\begin{aligned}
&\int_{\mathbb{H}^n} |v(q \cdot h) - v(q)| \, dq \\
&\qquad \leq \sum_{k=1}^{m} \int_{\mathbb{H}^n} |v(q \cdot \exp(\omega_k t_{j_k} W_{i_k}^{\mathbb{H}})) - v(q)| \, dq.
\end{aligned}$$

Take now $i = i_k$ for a generic $k = 1, \dots, m$, and set $t := t_{j_k}$ and, for example, $\omega_k = 1$. By (4.4), we have

$$\begin{aligned}
\int_{\mathbb{H}^n} |v(q \cdot \exp(tW_i^{\mathbb{H}})) - v(q)| \, dq &= \int_{D_i} \Big(\int_{\mathbb{R}} |v_i^p(s + t) - v_i^p(s)| \, ds \Big) \, dp \\
&\leq \int_{D_i} \omega_\delta^p(t) \, dp \leq \int_{D_i} \omega_\delta^p(ad_c(h, 0)/b) \, dp,
\end{aligned}$$

and (4.6) follows as in [5].

□

4.2. **Fractional energy in $\mathbb{R}$.** In this Subsection we recall a liminf inequality for a one-dimensional fractional energy. We follow [4]. Let $A \subset \mathbb{R}$ be an interval, $v \in L^1(A)$, we define

$$G_\varepsilon(v, A) := \frac{\varepsilon}{2\pi} \int_{A^2} \left| \frac{v(s) - v(s')}{s - s'} \right|^2 ds\, ds' + \lambda_\varepsilon \int_A V(v(s))\, ds. \tag{4.7}$$

We recall two results that we will use in the proof of the liminf inequality, and that are contained in [27] and [5]. The first one is a trace inequality in rectangles with optimal constant.

Theorem 4.7 ([27], Theorem 19)**.** *Let $u \in W^{1,2}((0,1) \times (0,1))$. Then, the trace of u on $(0,1) \times \{0\}$, call it v, is a well defined function $v \in H^{1/2}(0,1)$, and we have*

$$\iint_{(0,1)^2} \left| \frac{v(s) - v(s')}{s - s'} \right|^2 ds\, ds' \le 2\pi \int_0^1 \int_0^1 |\nabla u|^2\, ds\, dz. \tag{4.8}$$

The following theorem is a liminf inequality for the energy functional G_ε.

Theorem 4.8 (Lemma 1 in [4] and Theorem 4.4 in [5])**.** *We have:*

(i) *Every countable sequence $\{v_\varepsilon\} \subset L^1(A)$ with uniformly bounded energies $G_\varepsilon(v_\varepsilon, A)$ is pre-compact in $L^1(A)$ and every cluster point belongs to $BV(A, \{0,1\})$;*

(ii) *For every $v \in BV(A, \{0,1\})$ and every sequence $\{v_\varepsilon\}$ such that $v_\varepsilon \to v$ in $L^1(A)$,*

$$\liminf_{\varepsilon \to 0} G_\varepsilon(v_\varepsilon, A) \ge \mathbf{c}\#(S_v),$$

where $\#(S_v)$ denotes the number of points of discontinuity of v and $\mathbf{c} = \kappa/\pi$ with k given in (1.3).

4.3. **Proof of Theorem 4.1.** With these preliminaries in hand, we can give now the proof of our Theorem 4.1. By a standard truncation argument, we can assume that $0 \le u_\varepsilon \le 1$ for every $\varepsilon > 0$. We follow the proof of Proposition 4.7 in [5], which is based on a slicing argument. Let $\boldsymbol{e}$ be an horizontal vector at the origin with $|\boldsymbol{e}| = 1$, and let $W^{\mathbb{H}}$ be a left invariant horizontal vector field such that $W^{\mathbb{H}}(0) = \boldsymbol{e}$. We denote by Π the $(2n)$-dimensional vertical hyperplane orthogonal to $W^{\mathbb{H}}(0) = \boldsymbol{e}$. We apply Theorem 4.3 above with $S = \Pi \cap B_R^{\mathbb{H}}$, $D = B_R^{\mathbb{H}}$, $D_p = D \cap \gamma^p$, where as before γ^p is the integral curve of $W^{\mathbb{H}}$ starting from $p \in S$. Observe that if $u \in H^1(C_R)$, where as before $C_R = B_R^{\mathbb{H}} \times (0, R)$, then for a.e. $p \in S = \Pi \cap B_R^{\mathbb{H}}$ its restriction to D_p, denoted by u_p, belongs to $H^1(D_p)$ (see Proposition 6.8 in [5]). Moreover, using that $|\boldsymbol{e}| = 1$ and $W^{\mathbb{H}}$ is left invariant, a simple computation show that

$$\sum_{i=1}^{2n} |W_i^{\mathbb{H}} u_\varepsilon|^2 \ge |W^{\mathbb{H}} u_\varepsilon|^2.$$

Indeed, if we write $\boldsymbol{e} = \sum_{i=1}^{2n} c_j W_i^{\mathbb{H}}(0)$ with $\sum_{i=1}^{2n} c_i^2 = 1$, by the left invariance of $W^{\mathbb{H}}$ we have

$$|W^{\mathbb{H}} u_\varepsilon|^2 = |\sum_{i=1}^{2n} c_i W_i^{\mathbb{H}} u_\varepsilon|^2 \le \left(\sum_{i=1}^{2n} c_i^2\right)\left(\sum_{i=1}^{2n} |W_i^{\mathbb{H}} u_\varepsilon|^2\right) \le \sum_{i=1}^{2n} |W_i^{\mathbb{H}} u_\varepsilon|^2.$$

Hence we have:

$$\begin{aligned}
E_\varepsilon(u_\varepsilon, C_R, B_R^{\mathbb{H}}) = \varepsilon \int_{C_R} &\left(\sum_{i=1}^{2n} |W_i^{\mathbb{H}} u_\varepsilon(\eta,t,z)|^2 + (\partial_z u_\varepsilon(\eta,t,z))^2 \right) d\eta dt dz \\
&+ \lambda_\varepsilon \int_{B_R^{\mathbb{H}}} V(\mathrm{Tr}\, u_\varepsilon(\eta,t,0)) d\eta dt, \\
&\geq \varepsilon \int_0^R \int_{B_R^{\mathbb{H}}} \left(|W^{\mathbb{H}} u_\varepsilon(\eta,t,z)|^2 + (\partial_z u_\varepsilon(\eta,t,z))^2 \right) d\eta dt dz \\
&+ \lambda_\varepsilon \int_{B_R^{\mathbb{H}}} V(\mathrm{Tr}\, u_\varepsilon(\eta,t,0)) d\eta dt.
\end{aligned}$$

Set $D^p = (\gamma^p)^{-1}(\gamma^p(\mathbb{R}) \cap B_R^{\mathbb{H}}) = \{ s \in \mathbb{R} \mid \gamma^p(s) \in B_R^{\mathbb{H}} \}$, and $d\mathcal{L}_\Pi$ the Lebesgue measure on Π. Using (4.4), we obtain

$$\begin{aligned}
&E_\varepsilon(u_\varepsilon, C_R, B_R^{\mathbb{H}}) \\
&\geq \varepsilon \int_{\Pi \cap B_R^{\mathbb{H}}} d\mathcal{L}_\Pi(p) \left(\int_0^R dz \int_{D^p} \left(|W^{\mathbb{H}} u_\varepsilon(\gamma_p(s), z)|^2 + |\partial_z u_\varepsilon(\gamma_p(s), z)|^2 \right) ds \right. \\
&\quad \left. + \lambda_\varepsilon \int_{D^p} V(\mathrm{Tr}\, u_\varepsilon(\gamma_p(s), 0)) ds \right).
\end{aligned}$$

Since γ^p is the integral curve of $W^{\mathbb{H}}$, setting

$$\widetilde{u}_\varepsilon^p(s,z) = u_\varepsilon(\gamma^p(s), z),$$

we deduce that

$$W^{\mathbb{H}} u_\varepsilon(\gamma^p(s), z) = \partial_s \widetilde{u}_\varepsilon^p(s,z) \quad \text{and} \quad \partial_z u_\varepsilon(\gamma_p(s), z) = \partial_z \widetilde{u}_\varepsilon^p(s,z).$$

Therefore, we get

$$\begin{aligned}
&E_\varepsilon(u_\varepsilon, C_R, B_R^{\mathbb{H}}) \geq \\
&\varepsilon \int_{\Pi \cap B_R^{\mathbb{H}}} d\mathcal{L}_\Pi(p) \left(\int_0^R dz \int_{D^p} \left(|\partial_s \widetilde{u}_\varepsilon^p(s,z)|^2 + |\partial_z \widetilde{u}_\varepsilon^p(s,z)|^2 \right) ds \right. \\
&\quad \left. + \lambda_\varepsilon \int_{D^p} V(\mathrm{Tr}\, \widetilde{u}_\varepsilon^p(s,0)) ds \right).
\end{aligned}$$

We apply now the trace inequality (4.8) to get

(4.9)

$$\begin{aligned}
E_\varepsilon(u_\varepsilon, C_R, B_R^{\mathbb{H}}) \geq \int_{\Pi \cap B_R^{\mathbb{H}}} &d\mathcal{L}_\Pi(p) \left[\frac{\varepsilon}{2\pi} \int_{(D^p)^2} \left| \frac{\mathrm{Tr}\, \widetilde{u}_\varepsilon^p(s',0) - \mathrm{Tr}\, \widetilde{u}_\varepsilon^p(s,0)}{s'-s} \right|^2 ds ds' \right. \\
&\left. + \lambda_\varepsilon \int_{D^p} V(\mathrm{Tr}\, \widetilde{u}_\varepsilon^p(s,0)) \right] ds \\
&= \int_{\Pi \cap B_R^{\mathbb{H}}} d\mathcal{L}_\Pi(p)\, G_\varepsilon(\mathrm{Tr}\, \widetilde{u}_\varepsilon^p, D^p),
\end{aligned}$$

where G_ε is defined as in (4.7). The proof of Theorem 4.1 follows from the following two steps:

Step 1. Compactness: We first show that the sequence $\operatorname{Tr} u_\varepsilon$ is pre-compact in $L^1(B_R^{\mathbb{H}})$. In order to prove this, it is enough to show that the family $\mathcal{F} := \{\operatorname{Tr} u_\varepsilon\}$ satisfies the assumptions of Theorem 4.6. We choose a constant C such that

$$E_\varepsilon(u_\varepsilon, C_R, B_R^{\mathbb{H}}) \le C. \tag{4.10}$$

Fix now $\delta > 0$ and consider the sequence $v_\varepsilon : B_R^{\mathbb{H}} \to [0,1]$ defined as follows: $v_\varepsilon(\gamma^p(s)) := v_\varepsilon^p(s)$, where

$$v_\varepsilon^p := \begin{cases} \operatorname{Tr} \widetilde{u}_\varepsilon^p & \text{for all } p \in \Pi \cap B_R^{\mathbb{H}} \text{ such that } G_\varepsilon(\operatorname{Tr} \widetilde{u}_\varepsilon^p, E_p) \le |\Pi \cap B_R^{\mathbb{H}}| C/\delta, \\ 1 & \text{otherwise.} \end{cases} \tag{4.11}$$

Observe that v_ε is well-defined by the uniqueness of integral curves of horizontal vector fields starting from a given point. Using (4.9), (4.10), and (4.11) we deduce that $v_\varepsilon^p = \operatorname{Tr} \widetilde{u}_\varepsilon^p$ for all $p \in \Pi \cap B_R^{\mathbb{H}}$ apart from a subset of measure smaller that $\delta/|\Pi \cap B_R^{\mathbb{H}}|$. Therefore $v_\varepsilon = \operatorname{Tr} \widetilde{u}_\varepsilon$ in $B_R^{\mathbb{H}}$ minus a set of measure smaller than δ and, since $0 \le \operatorname{Tr} u_\varepsilon \le 1$, we deduce that $\|v_\varepsilon - \operatorname{Tr} u_\varepsilon\|_{L^1(B_R^{\mathbb{H}})} \le \delta$. This implies that the family $\mathcal{F}_\delta$ is δ-dense in $\mathcal{F}$. By (4.11) we have that $G_\varepsilon(v_\varepsilon^p, D^p) \le |\Pi \cap B_R^{\mathbb{H}}| C/\delta$ for every $p \in \Pi \cap B_R^{\mathbb{H}}$ and every ε, and hence we can apply statement (i) of Theorem 4.8 to deduce that the sequence (v_ε^p) is pre-compact in $L^1(D^p)$. Thus the family $\mathcal{F}$ satisfies the assumption of Theorem 4.6 for any horizontal tangent vector $\boldsymbol{e}$ at the origin, and thus in particular for $e_1, \dots, e_{2n}$, and we conclude that the sequence $(\operatorname{Tr} u_\varepsilon)$ is pre-compact in $B_R^{\mathbb{H}}$.

Step 2. Liminf inequality: It remains to prove that if $\operatorname{Tr} u_\varepsilon \to v$ in $L^1(B_R^{\mathbb{H}})$, then $v \in BV_{\theta_0}(B_R^{\mathbb{H}}, \{0,1\})$ and inequality (4.2) holds. Using (4.9) and passing to the limit as $\varepsilon \to 0$, by Fatou's Lemma we deduce that

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, C_R, B_R^{\mathbb{H}}) \ge \int_{\Pi \cap B_R^{\mathbb{H}}} \liminf_{\varepsilon \to 0} G_\varepsilon(\operatorname{Tr} \widetilde{u}_\varepsilon^p, D^p)\, d\mathcal{L}_\Pi(p),$$

and then $\liminf_{\varepsilon\to 0} G_\varepsilon(\operatorname{Tr} \widetilde{u}_\varepsilon^p, D^p)$ is finite for a.e. $p \in \Pi \cap B_R^{\mathbb{H}}$. Since $\operatorname{Tr} u_\varepsilon \to v$ in $L^1(B_R^{\mathbb{H}})$, possibly passing to a subsequence, we have that $\operatorname{Tr} \widetilde{u}_\varepsilon^p \to v^p$ in $L^1(D^p)$ for a.e. $p \in \Pi \cap B_R^{\mathbb{H}}$ (see Remark 6.7 in [5]). Then, using Theorem 4.8 we deduce that $v^p \in BV(D^p, \{0,1\})$ and

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, C_R, B_R^{\mathbb{H}}) \ge \int_{\Pi \cap B_R^{\mathbb{H}}} \mathbf{c}\#(S_{v^p})\, d\mathcal{L}_\Pi(p). \tag{4.12}$$

Finally, applying Theorem 4.4 we deduce that $v \in BV_{\theta_0}(B_R^{\mathbb{H}}, \{0,1\})$, that S_{v^p} agrees with $S_v \cap D^p$ for a.e. $p \in \Pi \cap B_R^{\mathbb{H}}$, and that

$$\begin{aligned} \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, C_R, B_R^{\mathbb{H}}) &\ge \mathbf{c} \int_{B_R^{\mathbb{H}} \cap S_v} \langle \nu_v, \boldsymbol{e} \rangle \; d\|\partial\{v=1\}\|_{\theta_0} \\ &= \Big\langle \int_{B_R^{\mathbb{H}} \cap S_v} \nu_v \, d\|\partial\{v=1\}\|_{\theta_0}, \boldsymbol{e} \Big\rangle. \end{aligned}$$

We conclude the proof of Theorem 4.1 by choosing a suitable vector $\boldsymbol{e}$.

5. Proof of the liminf inequality near the boundary M

In this Section we prove Theorem 2.17. To this aim, we need to pass from the "flat case" $\mathbb{H}^n \times [0,\sigma)$ to $M \times [0,\sigma)$. This will be the content of the following Sections 5.1, 5.2 and 7.

Given $A \subset M \times [0,\sigma)$, and $A' \subset M$, we define the localized energy

$$\tilde{F}_\varepsilon(u, A, A') := \varepsilon \int_A \Big(\sum_{j=1}^{2n} (W_j^0 u)^2 + (\partial_z u)^2 \Big)\, dv_\theta \wedge dz + \lambda_\varepsilon \int_{A'} V(\operatorname{Tr} u)\, dv_\theta$$

(compare with (2.13) and keep in mind Remark 2.16).

5.1. Flattening.

Following [5] we give the definition of *contact isometry defect.*

Definition 5.1. Let M_1 and M_2 be two contact $(2n+1)$-manifolds endowed with the contact forms θ_1 and θ_2, and let g_{θ_1} and g_{θ_2} be fixed Riemannian metrics on $\ker\theta_1$ and $\ker\theta_2$, respectively. If $p_i \in M_i$, $i = 1,2$, we denote by $HO(T_{p_1}M_1, T_{p_2}M_2)$ the space of linear maps from $T_{p_1}M_1$ to $T_{p_2}M_2$ that are isometries on $\ker\theta_1(p_1)$ and are induced by contact maps.

Definition 5.2. Let M_1 and M_2 be two contact $(2n+1)$-manifolds endowed with the contact forms θ_1 and θ_2, respectively, and let $U_1 \subset M_1$ and $U_2 \subset M_2$ be open sets. Let $\Psi : U_1 \to U_2$ be a diffeomorphism. We call *contact isometry defect* $\delta(\Psi)$ the smallest $\delta > 0$ such that

$$\operatorname{dist}(d\Psi(p), HO(T_pM_1, T_{\Psi(p)}M_2)) \le \delta \quad \text{for a.e. } p \in U_1.$$

Theorem 5.3. *Let (M,θ) be the $(2n+1)$-dimensional contact manifold endowed with the Riemannian metric g, as in Propositions 2.7 and 2.10. Let $\bar p \in M$ be any fixed point. Let $(W_1^0, \dots, W_{2n}^0)$ be the orthonormal symplectic basis of $\ker\theta(\bar p)$ (see Remark 2.16), and let $(W_1^{\mathbb{H}}, \dots, W_{2n}^{\mathbb{H}})$ be the orthonormal symplectic basis of $\ker\theta_0$ at the origin in $\mathbb{H}^n$. Then there exist an open neighborhood $\mathcal{U}$ of $\bar p$ and a local diffeomorphism*

$$\Psi : \mathcal{U} \to \mathbb{H}^n,$$

such that

- *i) Ψ is a contact map (i.e. $\Psi^*\theta_0 = \theta$);*
- *ii) $\Psi(\bar p) = 0$ and $\mathcal{U}_0 := \Psi(\mathcal{U})$ is open;*
- *iii) $D\Psi(\bar p)W_j^0 = W_j^{\mathbb{H}}$, $j = 1,\dots,2n$. In particular, $D\Psi(\bar p) : \ker\theta(\bar p) \to \ker\theta_0$ is an isometry when the horizontal fiber of $\ker\theta_0$ at the origin is endowed with the canonical Riemannian metric $\langle\cdot,\cdot\rangle_{\mathbb{H}}$.*

Proof. Darboux Theorem implies that there exists a neighborhood $\mathcal{U}$ of $\bar p$ and a diffeomorphism $\Psi_0 : \mathcal{U} \to \mathbb{H}^n$ such that $\Psi_0^*\theta_0 = \theta$, and thus $\Psi_0^*(d\theta_0) = d\theta = i^*\omega$. Hence

$$(\hat W_1, \cdots, \hat W_{2n}) := ((\Psi_0)_* W_1^0, \dots, (\Psi_0)_* W_{2n}^0)$$

is a symplectic basis of $\ker\theta_0$. Then, in particular,

$$(\hat W_1(0), \dots, \hat W_{2n}(0))$$

can be identified with a symplectic basis of $\mathbb{R}^{2n}$, and therefore there exists $A \in Sp(n)$ such that

$$A\hat{W}_j(0) = e_j = W_j^{\mathbb{H}}(0) \qquad j = 1, \dots, 2n.$$

Put now

$$\Psi := \begin{pmatrix} A & 0_{2n\times 1} \\ 0_{1\times 2n} & 1 \end{pmatrix} \Psi_0.$$

Obviously, Ψ satisfies *i)* by Lemma 5.4 below and *ii)*. Moreover

$$D\Psi(\bar{p})(W_i^0(\bar{p})) = \begin{pmatrix} A & 0_{2n\times 1} \\ 0_{1\times 2n} & 1 \end{pmatrix} \hat{W}_i(0) = W_i^{\mathbb{H}}(0),$$

and the assertion follows. □

Lemma 5.4 (see [19, 44])**.** *If $a > 0$ and $\frac{1}{\sqrt{a}}A \in Sp(n)$, then the (Euclidean) linear map $T : \mathbb{H}^n \to \mathbb{H}^n$*

$$T := \begin{pmatrix} A & 0_{2n\times 1} \\ 0_{1\times 2n} & a \end{pmatrix}$$

belongs to $GL(\mathbb{R}^{2n+1}, \mathbb{R}^{2n+1})$ and is a contact map.

Then, for each $p \in M$ and any $r > 0$ (close to 0), there exists a neighborhood $U(p, r) \subset M$ and a diffeomorphism Ψ_p such that the image $\Psi_p(U(p, r))$ is the $d_c^{\mathbb{H}}$-ball of radius r centered at the origin in the Heisenberg group, denoted by $B_r^{\mathbb{H}}$, and

$$\|D(\Psi_p) - I_{2n+1}\| \le \delta(r),$$

for some $\delta(r) \to 0$ when $r \to 0$. Here I_n denotes the identity map in n-dimensions. We also point out that, by Lemma 7.1 (which will be proven later on in Section 7), we have that in M:

$$U(p, r) \subset B(p, r(1 + o(1))) \qquad \text{as } r \to 0. \tag{5.1}$$

Adding the normal variable $z > 0$, we may cover $M \times [0, r]$ by a finite number of neighborhoods $\{\tilde{U}(p_j, r)\}_{j=1}^K$, $p_j \in M$ such that for each j, there exists a diffeomorphism

$$\tilde{\Psi}_{p_j} : \{\tilde{U}(p_j, r)\}_{j=1}^K \to \mathbb{H}^n \times [0, r]$$

satisfying

$$\begin{aligned} &\tilde{\Psi}_{p_j}(\tilde{U}(p_j, r)) = C_r^{\mathbb{H}} \subset \mathbb{H}^n \times \mathbb{R}_+, \\ &\tilde{\Psi}_{p_j}(U(p_j, r)) = B_r^{\mathbb{H}} \subset \mathbb{H}^n, \\ &\tilde{\Psi}_{p_j}((p_j, 0)) = (0, 0), \end{aligned}$$

and

$$\|D\tilde{\Psi}_{p_j} - I_{2(n+1)}\| \le \tilde{\delta}(r),$$

for some $\tilde{\delta}(r) \to 0$ when $r \to 0$.

Since

$$|D(u \circ \tilde{\Psi}_{p_j}^{-1})| \le (1 + \delta)|Du \circ \tilde{\Psi}_{p_j}^{-1}|, \tag{5.2}$$

this in particular implies that the localized energy $\tilde{F}_\varepsilon(u_\varepsilon, \tilde{U}(p_j, r), U(p_j, r))$ can be replaced by the energy $E_\varepsilon(w_\varepsilon, C_r^{\mathbb{H}}, B_r^{\mathbb{H}})$, where $w_\varepsilon = u_\varepsilon \circ \tilde{\Psi}_{p_j}$. More precisely, arguing exactly as in [5], Proposition 4.9, we have that

$$\tilde{F}_\varepsilon(u_\varepsilon, \tilde{U}(p_j, r), U(p_j, r)) \geq (1 - \delta^5) E_\varepsilon(w_\varepsilon, C_r^{\mathbb{H}}, B_r^{\mathbb{H}}). \tag{5.3}$$

5.2. **Conclusion of the proof of Theorem 2.17.** Let $\{u_\varepsilon\} \subset W^{1,2}(\Omega)$ be a countable sequence such that $\tilde{F}_{\varepsilon,r}(u_\varepsilon)$ is bounded independently of ε. We have to prove that the sequence of the traces $\{\operatorname{Tr} u_\varepsilon\}$ is pre-compact in $L^1(M)$. But since we have just shown that we can cover $M \times [0, r]$ with finitely many neighborhoods $\{\tilde{U}(p_j, r)\}_{j=1}^K$, it is enough to show that $\{\operatorname{Tr} u_\varepsilon\}$ is is pre-compact in $L^1(U(p_j, r))$ for every $j = 1, \dots, K$.

For every fixed j, let $w_\varepsilon = u_\varepsilon \circ \tilde{\Psi}_{p_j}^{-1}$. In particular, (5.2) implies that $E_\varepsilon(w_\varepsilon, C_r^{\mathbb{H}}, B_r^{\mathbb{H}})$ is uniformly bounded in ε. Hence the pre-compactness follows from Theorem 4.6. This proves statement *i*)* of Theorem 2.17.

Next, we would like to prove statement *ii*)* in Theorem 2.17. Then things become more delicate.

Let us start by recalling some classical definitions. For $m > 0$, we denote

$$\alpha_m := \frac{\Gamma(\frac{1}{2})^m}{\Gamma(\frac{m}{2} + 1)},$$

being Γ the Euler function and

$$\beta_m := 2^{-m} \alpha_m. \tag{5.4}$$

According to Federer's notation [16], we define a *centered* density of an outer measure μ on X:

Definition 5.5. Let (X, d) be a separable metric space, and let μ be an outer measure on X. If $m > 0$, the *upper and lower centered m-densities* of μ at $p \in X$ are

$$\Theta^{*\,m}(\mu, p) := \limsup_{r \to 0} \frac{\mu(\overline{B}(p, r))}{\beta_m \,(\operatorname{diam} \overline{B}(p, r))^m}$$

and

$$\Theta_*^m(\mu, p) := \liminf_{r \to 0} \frac{\mu(\overline{B}(p, r))}{\beta_m \,(\operatorname{diam} \overline{B}(p, r))^m}.$$

If they agree their common value

$$\Theta^m(\mu, p) := \Theta^{*\,m}(\mu, p) = \Theta_*^m(\mu, p)$$

is called the *m-density* of μ at p.

The crucial step of the proof of the liminf inequality ii^*) is provided by the following theorem that allows us to pass from an inequality between densities to the corresponding inequality between measures. We point out that this theorem is well known in the Euclidean setting, but fails to be true in general Carnot-Carathéodory spaces, and its proof in our special setting is postponed to Section 7.

We have:

Theorem 5.6. *Let M be $(2n+1)$-dimensional contact manifold endowed with a contact form θ and a Riemannian metric g on the fibers of* $\ker\theta$. *Let* $\mathbf{W}^0 := (W_1^0, \dots, W_{2n}^0)$ *be an orthonormal basis of* $\ker\theta$, *and let $E \subset M$ be a set of locally finite sub-Riemannian perimeter associated with $\mathbf{W}^0$. We denote by $|\mathbf{W}^0\chi_E|$ the associated perimeter measure. If μ is a σ-finite Borel measure on X, then*

$$\Theta^{*,2n+1}(\mu, p) \geq \Theta^{*,2n+1}(|\mathbf{W}^0\chi_E|, p) \qquad \textit{for } \mathcal{H}_d^{2n+1}\textit{-a.e. } p \in \partial^* E \tag{5.5}$$

yields

$$\mu \llcorner \partial E(\mathcal{B}) \geq |\mathbf{W}^0\chi_E|(\mathcal{B}) \tag{5.6}$$

for any Borel set $\mathcal{B} \subset \partial E$.

Remark 5.7. Let us explain why we do need Theorem 5.6 precisely in that form, and then we have to go through all the arguments of Section 7. First of all, we recall the following definition: let μ be an outer measure on the metric space (X, d). Then the *m-Federer densities* of μ at $x \in X$ are

$$\Theta_F^{*m}(\mu, x) := \inf_{\varepsilon>0} \sup \left\{ \frac{\mu(B(y,r))}{\beta_m \operatorname{diam}(B(y,r))^m} : x \in B(y,r),\ \rho_0\, r \leq \varepsilon \right\}.$$

It is easy to see that

$$\Theta^{*\,m}(\mu, x) \leq \Theta_F^{*m}(\mu, x) \leq 2^m\, \Theta^{*\,m}(\mu, x) \quad \forall x \in X\,. \tag{5.7}$$

If X is separable and endowed with a Radon measure μ, absolutely continuous with respect to the m-dimensional spherical Hausdorff measure $\mathcal{S}^m$, by [30] (see also [24]), the area formula for μ with respect to $\mathcal{S}^m$ i.e.

$$\mu(B) = \int_B \Theta_F^{*m}(\mu, x)\, d\mathcal{S}^m(x) \tag{5.8}$$

for any Borel set B may fail to be true in general, if the m-dimensional density $\Theta_F^{*m}(\mu, \cdot)$ is replaced by the centered m-dimensional density $\Theta^{*\,m}(\mu, \cdot)$ (see Definition 5.5).

To be more precise, the representation formula (5.8) is known to hold in Heisenberg groups only for suitable left-invariant distances, as d_∞ (see [24], Remark 4.25). In particular, we do not know whether it holds for the spherical Hausdorff measure associated with the Carnot-Carathéodory distance, that we use throughout the present paper (keep in mind its connection with the Minkowski content).

In fact, Magnani provides a counterexample precisely in the first Heisenberg group.

On the other hand, following [5], a crucial step of the proof of the liminf inequality *ii**) is provided by the following estimate:

$$\Theta^{*,2n+1}(\mu, p) \geq \mathbf{c}\, \Theta^{*,2n+1}(|\mathbf{W}^0\chi_E|, p), \tag{5.9}$$

where μ is the limit measure of the energy distribution associated with $\tilde{F}_\varepsilon$ and $p \in S_v$.

Unfortunately, due to Magnani's result, if $\mathcal{B}$ is a Borel set, we cannot derive from (5.9) the corresponding inequality *with the explicit constant* $\mathbf{c}$ for the measures $\mu(\mathcal{B})$ and $|\mathbf{W}^0\chi_E|(\mathcal{B})$, that we would need in the sequel.

Assuming Theorem 5.6, we can complete the proof of Theorem 2.17 as follows.

Let now $\{u_\varepsilon\}$ be a sequence in $W^{1,2}(M \times [0,\sigma))$ such that $\{\mathrm{Tr}\, u_\varepsilon\}$ converges to $v \in BV_\theta(M, \{0,1\})$ in the $L^1(M)$ norm. We need to show that

$$\liminf_{\varepsilon \to 0} \tilde{F}_\varepsilon(u_\varepsilon) \geq F(v).$$

If we write $v = \chi_E$, then $F(v) = |\mathbf{W}^0 \chi_E|$.

Without loss of generality, assume that this liminf is finite.

For every $\varepsilon \in (0,1)$, let μ_ε be the energy distribution associated with $\tilde{F}_\varepsilon$ for u_ε, i.e., μ_ε is the positive measure given by

$$\mu_\varepsilon(\mathcal{B}) := \varepsilon \int_{\mathcal{B}} \left(\sum_{j=1}^{2n} (W_j^0 u_\varepsilon)^2 + (\partial_z u_\varepsilon)^2 \right) dv_\theta \wedge dz + \lambda_\varepsilon \int_{\mathcal{B}_0} V(\mathrm{Tr}\, u_\varepsilon)\, dv_\theta$$

for every Borel set $\mathcal{B} \subset M \times [0,\sigma)$, $\mathcal{B}_0 = \overline{\mathcal{B}} \cap M$. The total variation $\|\mu_\varepsilon\|$ of the measure μ_ε is equal to $\tilde{F}_\varepsilon(u_\varepsilon)$.

Without loss of generality, we can assume $0 \leq \tilde{F}_\varepsilon(u_\varepsilon) \leq C$ for every $0 < \varepsilon < 1$, and therefore the $\{\mu_\varepsilon\}$ is an equibounded family of Radon measures in Ω. By De La Vallée Poussin's Theorem ([7], Theorem 1.59), there exist a subsequence $(\varepsilon_h)_{h \in \mathbb{N}}$ and a Radon measure μ in Ω such that $\mu_{\varepsilon_h} \to \mu$ in the sense of the convergence of measures. Then, by the lower semicontinuity of the total variation we have

$$\liminf_{\varepsilon \to 0} \tilde{F}_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon \to 0} \|\mu_\varepsilon\| \geq \|\mu\|\,.$$

Similarly, we define

$$\mu_0(\mathcal{B}) := |\mathbf{W}^0 \chi_E|(\mathcal{B}).$$

We just need to show that

$$\mu \geq \mu_0. \tag{5.10}$$

Take now a point $p \in S_v$. For r small enough, we choose a map $\tilde{\Psi} := \tilde{\Psi}_p$ as in the discussion right after Theorem 5.3. Set $w_\varepsilon := u_\varepsilon \circ \tilde{\Psi}^{-1}$ and $\bar{v} := v \circ \Psi^{-1}$. Hence, $\mathrm{Tr} w_\varepsilon \to \bar{v}$ in $L^1(B_r^{\mathbb{H}})$ and $\bar{v} \in BV(B_r^{\mathbb{H}}, \{0,1\})$. Moreover, if $v = \chi_E$, then $\bar{v} = \chi_{\Psi(E)}$ and $\nu_v(\Psi(z)) = D\Psi^{-1}(z) \cdot \nu_{\bar{v}}^{\mathbb{H}}(z)$, for any $z \in S_{\bar{v}}$ (here $\nu_{\bar{v}}^{\mathbb{H}}$ denotes the geometric normal to $S_{\bar{v}}$ in $\mathbb{H}^n$) . Keeping in mind (5.1) and (5.3), we have

$$\begin{aligned} \mu(B(p,r(1+o(1))) &\geq \mu(U(p,r)) = \lim_{\varepsilon \to 0} \mu_\varepsilon(\tilde{U}(p,r)) \\ &= \lim_{\varepsilon \to 0} \tilde{F}_\varepsilon(u_\varepsilon, \tilde{U}(p,r), U(p,r)) \\ &\geq \liminf_{\varepsilon \to 0} (1 - \delta(\Psi))^5 E_\varepsilon(w_\varepsilon, C_r^{\mathbb{H}}, B_r^{\mathbb{H}}). \end{aligned}$$

Notice that $\delta(\Psi) \to 0$ as $r \to 0$. On the other hand, by Theorem 4.1, we have that

$$\liminf_{\varepsilon \to 0} E_\varepsilon(w_\varepsilon, C_r^{\mathbb{H}}, B_r^{\mathbb{H}}) \geq \mathbf{c} \left| \int_{B_r^{\mathbb{H}}} \nu_{\bar{v}}^{\mathbb{H}} d|\mathbf{W}^{\mathbb{H}} \chi_{\Psi(E)}| \right|.$$

We have now, by Lemma 7.2, ii), and [22], Lemma 3.8, iii),

$$
\begin{aligned}
\Theta^{*\,2n+1}(\mu,p) &:= \limsup_{r\to 0} \frac{\mu(\overline{B}(p,r))}{\beta_{2n+1}\,(\operatorname{diam}\overline{B}(p,r))^{2n+1}} \\
&= \limsup_{r\to 0} \frac{\mu(\overline{B}(p,r))}{\alpha_{2n+1}\,r^{2n+1}} \\
&\geq \mathbf{c}\,\liminf_{r\to 0} \frac{|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B_r^{\mathbb{H}})}{\alpha_{2n+1}\,r^{2n+1}} \left| \fint_{B_r^{\mathbb{H}}} \nu_{\bar v}^{\mathbb{H}}\, d|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}| \right|.
\end{aligned} \tag{5.11}
$$

Let us prove now the following approximation lemma.

Lemma 5.8.

$$
\lim_{r\to 0} \frac{|\mathbf{W}^0\chi_E|(\overline{B}(p,r))}{|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B_r^{\mathbb{H}})} = 1. \tag{5.12}
$$

Proof. In the notation from Section 3.1, the perimeter measure in M is defined as

(5.13)

$$
\begin{aligned}
&|\mathbf{W}^0\chi_E|(\overline{B}(p,r)) \\
&\qquad = \sup\{|D_X(\chi_E)|(\overline{B}(p,r))\ :\ X \in \Gamma(M,\ker\theta),\ g(X,X)\leq 1\}.
\end{aligned}
$$

Note that from the definition of D_X in (3.2), it is enough to restrict our attention to vector fields X supported on $\overline{B}(p,r)$.

On the other hand, by Lemma 7.1 and with the notations therein, if we put

$$
\rho = \rho(r) := r(1 + Cr^{1/2}), \qquad \text{then } B(p,r) \subset U(p,\rho).
$$

Then

$$
|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B_r^{\mathbb{H}}) = |\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(\delta_{r/\rho}^{\mathbb{H}}(B_\rho^{\mathbb{H}})) = (1+o(1))|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B_\rho^{\mathbb{H}}),
$$

where δ is the standard group dilation in the Heisenberg group. We recall now that

$$
\begin{aligned}
&|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B_\rho^{\mathbb{H}}) \\
&\qquad = \sup\{|D_Y(\chi_{\Psi(E)})|(B_\rho^{\mathbb{H}})\ :\ Y \in \Gamma(\mathbb{H}^n,\ker\theta_0),\ \langle Y,Y\rangle_{\mathbb{H}} \leq 1\},
\end{aligned} \tag{5.14}
$$

where again we can assume $\operatorname{supp} Y \subset B_\rho^{\mathbb{H}}$.

It remains to compare the metrics g on M and $\langle\ ,\ \rangle_{\mathbb{H}}$ on $\mathbb{H}^n$. Note that Ψ is a contact map, so we can always write $Y = \Psi_* X$ for $X \in \Gamma(M,\ker\theta)$. By the change of variables formula (14) in [8]

$$
\Psi_\#(D_X h) = D_{\Psi_* X}(h\circ\Psi^{-1}), \tag{5.15}
$$

we have

$$
|D_{\Psi_* X}(h\circ\Psi^{-1})| = |\Psi_\# D_X h|. \tag{5.16}
$$

Using also the definition of push forward of a measure,

$$
\begin{aligned}
|D_Y(\chi_{\Psi(E)})|(B_\rho^{\mathbb{H}}) &= \Psi_\#|D_X(\chi_E)|(B_\rho^{\mathbb{H}}) \\
&= |D_X(\chi_E)|(U(p,\rho)) \geq |D_X(\chi_E)|(B(p,r)).
\end{aligned}
$$

Finally, in order to compare the perimeter measures (5.13) and (5.14), we notice that, by Theorem 5.3, *iii)* if $\langle Y, Y\rangle_{\mathbb{H}} \le 1$, then $g(X,X) \le 1 + o(1)$ as $r \to 0$.

This proves that

$$\limsup_{r\to 0} \frac{|\mathbf{W}^0 \chi_E|(\overline{B}(p,r))}{|\mathbf{W}^{\mathbb{H}} \chi_{\Psi(E)}|(B_r^{\mathbb{H}})} \le 1.$$

The proof of the reverse inequality can be carried out in the same fashion. □

Before going back to the proof of the lower bound inequality, we need the following last lemma.

Lemma 5.9. *We have:*

$$\left| \fint_{B_r^{\mathbb{H}}} \nu_{\bar{v}}^{\mathbb{H}} \, d|\mathbf{W}^{\mathbb{H}} \chi_{\Psi(E)}| \right| = 1 + o(1) \qquad \text{as } r \to 0. \tag{5.17}$$

Proof. We use Lemma 7.1, with the notations therein, and we put $\phi(r) := (1 + C\sqrt{r})^{-1}$. We have

$$\phi(r)(1 + C\sqrt{r\phi(r)}) \le 1 \qquad \text{and} \qquad \phi(r) = 1 + o(1) \quad \text{as } r \to 0.$$

Let us prove first that

$$\begin{aligned} &\left| \fint_{B_{r\phi(r)}^{\mathbb{H}}} \nu_{\bar{v}}^{\mathbb{H}} \, d|\mathbf{W}^{\mathbb{H}} \chi_{\Psi(E)}| \right| \\ &\qquad = \frac{1}{|\mathbf{W}^{\mathbb{H}} \chi_{\Psi(E)}|(B_{r\phi(r)}^{\mathbb{H}})} \left| \int_{\Psi(B(p,r))} \nu_{\bar{v}}^{\mathbb{H}} \, d|\mathbf{W}^{\mathbb{H}} \chi_{\Psi(E)}| \right| + o(1). \end{aligned} \tag{5.18}$$

First of all, we notice that

$$B_{r\phi(r)}^{\mathbb{H}} \subset \Psi(B(p,r)), \qquad 0 < r < r_0.$$

Indeed, take $z \in B_{r\phi(r)}^{\mathbb{H}}$. Since Ψ is a diffeomorphism, we can assume that $z = \Psi(\zeta)$, with $\zeta \in M$, provided r is small enough. Therefore

$$d_c(p,\zeta) = d_c^{\Psi}(0,z) \le r\phi(r)(1 + C\sqrt{r\phi(r)}) \le r.$$

Analogously

$$\Psi(B(p,r)) \subset B_{r/\phi(r)}^{\mathbb{H}}, \qquad 0 < r < r_0.$$

Therefore, in order to prove (5.18), we have to show in the first place that

$$\frac{1}{|\mathbf{W}^{\mathbb{H}} \chi_{\Psi(E)}|(B_{r\phi(r)}^{\mathbb{H}})} \left| \int_{\Psi(B(p,r)) \setminus B_{r\phi(r)}^{\mathbb{H}}} \nu_{\bar{v}}^{\mathbb{H}} \, d|\mathbf{W}^{\mathbb{H}} \chi_{\Psi(E)}| \right| = o(1).$$

On the other hand, keeping in mind the homogeneity of $|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|$ with respect to group dilations $\delta^{\mathbb{H}}$, we have:

$$
\begin{aligned}
\frac{1}{|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B^{\mathbb{H}}_{r\phi(r)})} & \left| \int_{\Psi(B(p,r))\setminus B^{\mathbb{H}}_{r\phi(r)}} \nu^{\mathbb{H}}_{\bar{v}}\, d|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}| \right| \\
&\le \frac{|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(\Psi(B(p,r))) - |\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B^{\mathbb{H}}_{r\phi(r)})}{|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B^{\mathbb{H}}_{r\phi(r)})} \\
&\le \frac{|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B^{\mathbb{H}}_{r/\phi(r)}) - |\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B^{\mathbb{H}}_{r\phi(r)})}{|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B^{\mathbb{H}}_{r\phi(r)})} \\
&= \frac{|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B^{\mathbb{H}}_{1/\phi(r)}) - |\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B^{\mathbb{H}}_{\phi(r)})}{|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B^{\mathbb{H}}_{\phi(r)})} \\
&= o(1).
\end{aligned}
$$

This yields (5.18).

Take now $Y := \Psi_* X$, with $\langle Y, Y\rangle_{\mathbb{H}} = 1$. By the change of variable formula (5.15),

$$\Psi_{\#}(D_X(\chi_E)) = D_{\Psi_* X}(\chi_{\Psi(E)}),$$

and thus

$$
\begin{aligned}
\Big\langle Y, \int_{\Psi(B(p,r))} \nu_{\bar{v}}\, d|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}| \Big\rangle_{\mathbb{H}} &= \int_{\Psi(B(p,r))} \langle Y, \nu_{\bar{v}}\rangle\, d|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}| \\
&= D_Y\chi_{\Psi(E)}(\Psi(B(p,r))) = \Psi_{\#}(D_X\chi_E)(\Psi(B(p,r))) \\
&= D_X\chi_E(B(p,r)) = g\big(X, \int_{B(p,r)} \nu_v\, d|\mathbf{W}^0\chi_E|\big) \qquad (5.19) \\
&\le \|X\|_g \Big\| \int_{B(p,r)} \nu_v\, d|\mathbf{W}^0\chi_E| \Big\|_g .
\end{aligned}
$$

As in the proof of previous lemma, $\|X\|_g = 1 + o(1)$. On the other hand, keeping in mind that p belongs to the reduced boundary of E,

$$\lim_{\rho\to 0} \frac{1}{|\mathbf{W}^0\chi_E(B(p,r))|} \Big\| \int_{B(p,r)} \nu_v\, d|\mathbf{W}^0\chi_E| \Big\|_g = 1.$$

But by the previous formula (5.12), and the fact that

$$\lim_{r\to 0} \frac{|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B^{\mathbb{H}}_{r\phi(r)})}{|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B^{\mathbb{H}}_{r})} = 1$$

using a rescaling argument by dilations in the Heisenberg group, we conclude from (5.19) that

$$\lim_{\rho\to 0} \Big\langle Y, \frac{1}{|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}|(B^{\mathbb{H}}_{r\phi(r)})} \int_{\Psi(B(p,r))} \nu^{\mathbb{H}}_{\bar{v}}\, d|\mathbf{W}^{\mathbb{H}}\chi_{\Psi(E)}| \Big\rangle_{\mathbb{H}} \le 1.$$

A standard argument taking the sup among all Y (or equivalently, all X) with norm less than one, looking back at (5.18), completes the proof of the Lemma. $\square$

We can go back to the proof of (5.10). Replacing both (5.12) and (5.17) into (5.11) we conclude that

$$\Theta^{*,2n+1}(\mu, p) \geq \mathbf{c}\, \Theta^{*,2n+1}(|\mathbf{W}^0 \chi_E|, p).$$

The proof of the lower bound inequality is completed by Theorem 5.6.

6. Proof of the main theorem - limsup

Now we show statement *iii)* of Theorem 1.1. Given $v \in BV_\theta(M, \{0,1\})$, we need to construct a sequence $\{u_\varepsilon\}$ in $W^{1,2}(\Omega)$ such that $\mathrm{Tr}\, u_\varepsilon \to v$ in $L^1(M)$ and

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leq F(v).$$

The proof of the limsup inequality will be divided into several steps:

Step 1: It is enough to assume that S_v is a smooth closed submanifold in M. This fact follows from the next two results. The first one is a reduction Lemma. It is valid for general metric spaces, and the proof is only a minor variant of the one given in [34], Lemma *IV* (see also [3]), hence we shall omit such a proof.

Lemma 6.1. *Let $(\mathcal{X}, \mathrm{d})$ be a metric space, let $F_k,\ F : \mathcal{X} \longrightarrow [-\infty, +\infty]$ with $k \in \mathbb{N}$; consider $\mathcal{D} \subset \mathcal{X}$ and $x \in \mathcal{X}$. Let us suppose that*

1) *for every $y \in \mathcal{D}$ there exists a sequence $(y_k)_{k \in \mathbb{N}} \subset \mathcal{X}$ such that $y_k \to y$ in $\mathcal{X}$ and*
$$\limsup_{k \to \infty} F_k(y_k) \leq F(y);$$
2) *there exists a sequence $(x_k)_{k \in \mathbb{N}} \subset \mathcal{D}$ such that $x_k \to x$ and*
$$\limsup_{k \to \infty} F(x_k) \leq F(x);$$

then there exists a sequence $(\overline{x}_k)_{k \in \mathbb{N}} \subset \mathcal{X}$ such that $\limsup_{k \to \infty} F_k(\overline{x}_k) \leq F(x)$.

The following approximation result is the analogue of Corollary 2.3.6 in [20] for the case of contact manifolds.

Lemma 6.2. *Each $v \in BV_\theta(M, \{0,1\})$ may be approximated in $L^1(M)$ by a sequence $\{v_k\}$ in $BV_\theta(M, \{0,1\})$ such that S_{v_k} is a smooth closed submanifold and*

$$\|S_{v_k}\|_\theta \to \|S_v\|_\theta.$$

Proof. The result follows by standard arguments from the Meyers-Serrin type result, Theorem 2.4 in [8], and the coarea formula (Proposition 3.6). □

Next, possibly modifying v on a negligible subset, we can assume that it is constant in each connected component of $M \setminus S_v$.

Step 2: (Preliminary calculations). Following the idea in [5], we take a function defined as follows: consider the half-plane $\mathbb{R}^2_+$ with coordinates $s \in \mathbb{R}$, $z > 0$. Let (ρ, ϑ), $\rho > 0$, $\vartheta \in [0, \pi]$ be the polar coordinates in $\mathbb{R}^2_+$.

We set

$$\bar{w}_\varepsilon(\rho,\vartheta) := \begin{cases} \rho\dfrac{\lambda_\varepsilon}{\varepsilon}(1-\frac{2}{\pi}\vartheta) & \text{if } 0\le\rho\le\dfrac{\varepsilon}{\lambda_\varepsilon}, \\ 1-\frac{1}{\pi}\vartheta & \text{if } \dfrac{\varepsilon}{\lambda_\varepsilon}\le\rho, \end{cases}$$

and $w_\varepsilon(s,z)=\bar{w}_\varepsilon(\rho,\vartheta)$. A straightforward calculation gives:

$$|\partial_s w_\varepsilon|, |\partial_z w_\varepsilon| \le \begin{cases} C\dfrac{\lambda_\varepsilon}{\varepsilon} & \text{if } 0\le\rho\le\dfrac{\varepsilon}{\lambda_\varepsilon}, \\ \dfrac{C}{\rho} & \text{if } \dfrac{\varepsilon}{\lambda_\varepsilon}\le\rho, \end{cases} \tag{6.1}$$

and

$$|\partial_{ss} w_\varepsilon|, |\partial_{zs} w_\varepsilon| \le \begin{cases} \dfrac{C}{\rho}\dfrac{\lambda_\varepsilon}{\varepsilon} & \text{if } 0\le\rho\le\dfrac{\varepsilon}{\lambda_\varepsilon}, \\ \dfrac{C}{\rho^2} & \text{if } \dfrac{\varepsilon}{\lambda_\varepsilon}\le\rho. \end{cases} \tag{6.2}$$

Moreover, the following estimates hold:

Lemma 6.3. *Let $t_\varepsilon\to 0$ as $\varepsilon\to 0$ and $\sigma>0$ in such a way that $\frac{\varepsilon}{\lambda_\varepsilon}\ll t_\varepsilon\ll\sigma$. Then, as $\varepsilon\to 0$,*

$$\varepsilon\int_{\{\rho<t_\varepsilon\}}|\nabla w_\varepsilon|^2\,dsdz = \frac{1}{\pi}\varepsilon\log\frac{\lambda_\varepsilon}{\varepsilon}(1+o(1)),$$

$$\varepsilon\int_{\{t_\varepsilon<\rho<\sigma\}}|\nabla w_\varepsilon|^2\,dsdz = \varepsilon\log t_\varepsilon(1+o(1)) = o\left(\varepsilon\log\frac{\lambda_\varepsilon}{\varepsilon}\right),$$

$$\lambda_\varepsilon\int_{\{z=0\}\cap\{\rho<t_\varepsilon\}}V(\operatorname{Tr}w_\varepsilon)\,ds = O(\varepsilon), \quad \lambda_\varepsilon\int_{\{z=0\}\cap\{\rho>t_\varepsilon\}}V(\operatorname{Tr}w_\varepsilon)\,ds = O(\varepsilon).$$

Proof. While the first two identities follow from straightforward calculation from the previous estimates, for the third one we use that $V\equiv 0$ unless $0\le\rho\le\frac{\varepsilon}{\lambda_\varepsilon}$. Also, from the proof it follows that these estimates are independent of the choice of σ. □

Step 3: (Set up). As we saw in Section 2.2, given $\sigma>0$ small enough, there exists a diffeomorphism Φ such that a tubular neighborhood of M in $\overline{\Omega}$ may be written as $M\times[0,\sigma)$, with coordinates $p\in M$ and $z\in[0,\sigma)$. In the product $M\times[0,\sigma)$ we shall define the distance

$$d((p',z'),(p'',z'')) = \sqrt{d_c(p',p'')^2+(z'-z'')^2}.$$

For each r small consider the following subset of $M\times[0,\sigma)$:

$$\tilde{A}_r = \{(p,z)\in M\times[0,\sigma) \,:\, d(p,S_v)<r\},$$

and set

$$\partial^0\tilde{A}_r = \overline{\tilde{A}_r}\cap M.$$

In coordinates $(p,z)\in\tilde{A}_\sigma$ where $p\in M$ and $z>0$, let

$$u_\varepsilon(p,z) := w_\varepsilon(d_c(p,S_v),z),$$

and transplant it back to Ω by

$$u_\varepsilon = \tilde{u}_\varepsilon\circ\Phi^{-1}, \quad A_r=\Phi(\tilde{A}_r),$$

for each $0 < r < \sigma$. Note that Φ can be defined independently of ε. Next, because of hypothesis *H2.* for f in Section 2.1, and Proposition 2.14, in the calculation of the energy functional F_ε in a neighborhood of M we have

$$F_\varepsilon(u_\varepsilon, A_\sigma, \partial^0 A_\sigma) \leq (1 + O(\sigma))\tilde{F}_{\varepsilon,\sigma}(\tilde{u}_\varepsilon, \tilde{A}_\sigma, \partial^0 \tilde{A}_\sigma), \tag{6.3}$$

so it is enough to estimate the integral in the right hand side.

Now, the phase transition should happen at scale ε. For this, let t_ε be as in Lemma 6.3, actually it is enough to take $t_\varepsilon = \varepsilon$. Then,

$$\tilde{F}_{\varepsilon,\sigma}(\tilde{u}_\varepsilon, \tilde{A}_\sigma, \partial^0 \tilde{A}_\sigma) = \tilde{F}_{\varepsilon,\sigma}(\tilde{u}_\varepsilon, \tilde{A}_\sigma \setminus \tilde{A}_{t_\varepsilon}, \partial^0(\tilde{A}_\sigma \setminus \tilde{A}_{t_\varepsilon})) + \tilde{F}_{\varepsilon,\sigma}(\tilde{u}_\varepsilon, \tilde{A}_{t_\varepsilon}, \partial^0 \tilde{A}_{t_\varepsilon}).$$

The last term in the right hand side above will be considered in Step 4, while the first one will be handled in Step 5.

On the other hand, it is not important how we define u_ε in the set $\Omega \backslash A_\sigma$, as long as $u_\varepsilon = v$ in $\Omega \backslash \partial^0 A_\sigma$ and its Lipschitz constant is bounded by $\frac{C}{\sigma}$. Recall that v is a function that only attains the values 0 or 1 on $M \setminus \partial^0 A_\sigma$, so that for the potential energy we have

$$\int_{M \backslash \partial^0 A_\sigma} V(\operatorname{Tr} u_\varepsilon) \, dv_\theta = 0.$$

Then we immediately have that

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega \backslash A_\sigma, M \backslash \partial^0 A_\sigma) = 0. \tag{6.4}$$

Step 4. (Construction near the singular set). We follow the ideas of [38] to estimate the value of $\tilde{F}_{\varepsilon,\sigma}(\tilde{u}_\varepsilon, \tilde{A}_{t_\varepsilon}, \partial^0 \tilde{A}_{t_\varepsilon})$. Let $s = d_c(p, S_v)$. Then, using Fubini's theorem,

$$\begin{aligned} &\tilde{F}_{\varepsilon,\sigma}(\tilde{u}_\varepsilon, \tilde{A}_{t_\varepsilon}, \partial^0 \tilde{A}_{t_\varepsilon}) \\ &\quad = \int_{\partial^0 \tilde{A}_{t_\varepsilon}} \left[\varepsilon \int_0^{\sqrt{t_\varepsilon^2 - s^2}} \sum_{j=1}^{2n} |\tilde{W}_j \tilde{u}_\varepsilon(p,z)|^2 \, dz + \lambda_\varepsilon V(\operatorname{Tr} \tilde{u}_\varepsilon(p)) \right] dv_\theta. \end{aligned} \tag{6.5}$$

Using the coarea formula from Theorem 3.6 and the Eikonal equation for d_c (3.8) we have

$$\tilde{F}_{\varepsilon,\sigma}(\tilde{u}_\varepsilon, \tilde{A}_{t_\varepsilon}, \partial^0 \tilde{A}_{t_\varepsilon}) = \int_{-t_\varepsilon}^{t_\varepsilon} h_\varepsilon(s) \, d\|\partial H_s\|_\theta \, ds,$$

where we have set

$$h_\varepsilon(s) := \varepsilon \int_0^{\sqrt{t_\varepsilon^2 - s^2}} \left[(\partial_s w_\varepsilon(s,z))^2 + (\partial_z w_\varepsilon(s,z))^2 \right] dz + \lambda_\varepsilon V(\operatorname{Tr} w_\varepsilon(s)) \tag{6.6}$$

and $H_s = \{p \in M : d_c(p, S_v) > s\}$. Next, notice that for all $s \in [-t_\varepsilon, t_\varepsilon]$, $h_\varepsilon(s) = h_\varepsilon(-s)$, so that

$$\tilde{F}_{\varepsilon,\sigma}(u_\varepsilon, \tilde{A}_{t_\varepsilon}, \partial^0 \tilde{A}_{t_\varepsilon}) \leq \int_0^{t_\varepsilon} h_\varepsilon(s) \; (d\|\partial H_s\|_\theta + d\|\partial H_{-s}\|_\theta) \, ds.$$

We can rewrite this expression as follows: let

$$Z(t) = \int_{-t}^{t} \|\partial H_s\|_\theta \, ds, \quad Z'(t) = \|\partial H_s\|_\theta + \|\partial H_{-s}\|_\theta,$$

so that

$$\tilde{F}_{\varepsilon,\sigma}(\tilde{u}_\varepsilon, \tilde{A}_{t_\varepsilon}, \partial^0 \tilde{A}_{t_\varepsilon}) \le \int_0^{t_\varepsilon} h_\varepsilon(s) Z'(s)\, ds = -\int_0^{t_\varepsilon} h'_\varepsilon(s) Z(s)\, ds \tag{6.7}$$

after integration by parts. Note that we have used that $h_\varepsilon(t_\varepsilon) = 0$.

Next, by Theorem 3.12 we have

$$\lim_{t\to 0^+} \frac{Z(t)}{2t} = L := \|\partial H\|_\theta,$$

and thus, there exists a function $\delta : [0,\infty) \to \mathbb{R}$ such that

$$Z(t) = 2Lt + \delta(t)t, \quad \text{with } \lim_{\varepsilon\to 0^+} \sup_{t\in[0,t_\varepsilon]} |\delta(t)| = 0. \tag{6.8}$$

Substituting the above into (6.7) we obtain that

$$\begin{aligned} \tilde{F}_{\varepsilon,\sigma}(\tilde{u}_\varepsilon, \tilde{A}_{t_\varepsilon}, \partial^0 \tilde{A}_{t_\varepsilon}) &\le -\int_0^{t_\varepsilon} s\delta(s) h'_\varepsilon(s)\, ds - 2L\int_0^{t_\varepsilon} s h'_\varepsilon(s)\, ds. \\ &=: I_\varepsilon + J_\varepsilon. \end{aligned} \tag{6.9}$$

In order to estimate the term J_ε above, we use again integration by parts

$$J_\varepsilon = 2L\int_0^{t_\varepsilon} h_\varepsilon(s)\, ds = L\int_{-t_\varepsilon}^{t_\varepsilon} h_\varepsilon(s)\, ds.$$

From the estimates in Lemma 6.3, using our initial hypothesis on λ_ε from (1.3), we may conclude

$$J_\varepsilon \longrightarrow \frac{\kappa}{\pi} L \quad \text{as } \varepsilon \to 0.$$

Finally, we need to show that the remaining term I_ε has limit zero when $\varepsilon \to 0$. But

$$|I_\varepsilon| \le \sup_{t\in[0,t_\varepsilon]} |\delta(t)| \int_0^{t_\varepsilon} s|h'_\varepsilon(s)|\, ds.$$

From the behavior of δ in (6.8), it is enough to show that the integral

$$\tilde{I}_\varepsilon := \int_0^{t_\varepsilon} s|h'_\varepsilon(s)|\, ds \tag{6.10}$$

is bounded independently of ε. Differentiating in (6.6), $h'_\varepsilon(s) = h^1_\varepsilon + h^2_\varepsilon + h^3_\varepsilon$ for

$$\begin{aligned} h^1_\varepsilon(s) &= \varepsilon\left[(\partial_s w_\varepsilon(s, \sqrt{t_\varepsilon^2 - s^2}))^2 + (\partial_z w_\varepsilon(s, \sqrt{t_\varepsilon^2 - s^2}))^2\right] \cdot \left(-\frac{s}{\sqrt{t_\varepsilon^2 - s^2}}\right), \\ h^2_\varepsilon(s) &= 2\varepsilon \int_0^{\sqrt{t_\varepsilon^2 - s^2}} \left[\partial_s w_\varepsilon \partial_{ss} w_\varepsilon + \partial_z w_\varepsilon \partial_{zs} w_\varepsilon\right] dz, \\ h^3_\varepsilon(s) &= \lambda_\varepsilon V'(\operatorname{Tr} w_\varepsilon(s)) \partial_s w_\varepsilon(s, 0). \end{aligned}$$

Since we know that $t_\varepsilon \gg \frac{\varepsilon}{\lambda_\varepsilon}$, using the estimates in (6.1), we deduce

$$|h^1_\varepsilon(s)| \le C\frac{\varepsilon}{t_\varepsilon^2} \frac{s}{\sqrt{t_\varepsilon^2 - s^2}},$$

so we may conclude

$$
\begin{aligned}
\int_0^{t_\varepsilon} s|h^1_\varepsilon(s)|\,ds &\le C\frac{\varepsilon}{t_\varepsilon^2}\int_0^{t_\varepsilon}\frac{s^2}{\sqrt{t_\varepsilon^2-s^2}}\,ds \\
&\le C\frac{\varepsilon t_\varepsilon}{t_\varepsilon^2}\int_0^{t_\varepsilon}\frac{s}{\sqrt{t_\varepsilon^2-s^2}}\,ds \\
&\le C\frac{\varepsilon t_\varepsilon}{t_\varepsilon^2}\left[\sqrt{t_\varepsilon^2-s^2}\right]_0^{t_\varepsilon} \le C
\end{aligned} \tag{6.11}
$$

independent of ε. For the second integral, note that the estimates in (6.1)-(6.2) give

$$
\begin{aligned}
\int_0^{t_\varepsilon} s|h^2_\varepsilon(s)|\,ds &\le C\varepsilon\left[\int_{\{0<\rho<\frac{\varepsilon}{\lambda_\varepsilon}\}} s\left(\frac{\lambda_\varepsilon}{\varepsilon}\right)^2 d\rho + \int_{\{\frac{\varepsilon}{\lambda_\varepsilon}<\rho<t_\varepsilon\}}\frac{s}{\rho^2}\,d\rho\right] \\
&\le C\varepsilon\log\lambda_\varepsilon < \infty
\end{aligned} \tag{6.12}
$$

by our initial hypothesis (1.3). Finally, looking again at the estimates (6.1) for $\partial_s w_\varepsilon$, we have

$$
\int_0^{t_\varepsilon} s|h^3_\varepsilon(s)|\,ds \le C\lambda_\varepsilon\int_0^{\frac{\varepsilon}{\lambda_\varepsilon}} s\frac{\lambda_\varepsilon}{\varepsilon}\,ds < \infty. \tag{6.13}
$$

Putting together (6.11), (6.12) and (6.13) we conclude that the integral $\tilde{I}_\varepsilon$ from (6.10) is uniformly bounded independently of ε. This shows that, looking at (6.9) and (6.3),

$$
\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, A_{t_\varepsilon}, \partial^0 A_{t_\varepsilon}) \le (1+O(\sigma))\frac{\kappa}{\pi}L, \tag{6.14}
$$

as desired.

Step 5: (Construction in $A_\sigma\setminus A_{t_\varepsilon}$). This argument is very close to that of [5].

First we set $u_\varepsilon \equiv v$ on $M\setminus\partial^0 A_{t_\varepsilon}$ (recall that v is a function that only attains the values 0 or 1 on $M\setminus\partial^0 A_{t_\varepsilon}$), so that

$$
\int_{M\setminus\partial^0 A_{t_\varepsilon}} V(\operatorname{Tr} u_\varepsilon)\,dv_\theta = 0.
$$

To conclude the proof we need the following extension lemma, which is a much simplified version of Lemma 4.11 in [5].

Lemma 6.4. *Let A be a domain in $\mathbb{R}^{2N}$ and $A'\subset\partial A$. Let $\varepsilon\in(0,1)$, and v a Lipschitz function $v: A'\to[0,1]$. Then v admits an extension $u: A\to[0,1]$ such that its Lipschitz constant satisfies*

$$
Lip(u) \le \frac{1}{\varepsilon} + \operatorname{Lip}(v)
$$

and

$$
\varepsilon\int_A |\nabla u|^2 \le (\varepsilon\operatorname{Lip}(v))^2(|\partial A| + o(1)),
$$

and $o(1)$ is a function of ε which does not depend on v.

From the previous steps we have constructed a function u_ε that has a smooth transition from 0 to 1 along $\partial A_{t_\varepsilon}$ and along A_σ, so at most its Lipschitz constant is $\frac{C}{t_\varepsilon}$ (recall that $t_\varepsilon \ll \sigma$). Thus, using the previous Lemma, we may extend u_ε to $A_\sigma \backslash A_{t_\varepsilon}$ in a Lipschitz fashion while

(6.15)
$$F_\varepsilon(u_\varepsilon, A_\sigma\backslash A_{t_\varepsilon}, \partial^0(A_\sigma\backslash A_{t_\varepsilon})) = \varepsilon \int_{A_\sigma\backslash A_{t_\varepsilon}} f(y, Du_\varepsilon(y))\, dy \le C(1+o(1))O(\sigma).$$

as $\varepsilon \to 0$ because of our hypothesis on f.

By construction, it is clear that $Tu_\varepsilon \to v$ in $L^1(M)$. Putting together (6.4), (6.14) and (6.15), the proof of the lim sup is completed by taking σ small enough.

7. Appendix: Densities and measures

In this Appendix we prove Theorem 5.6, which was a crucial ingredient in the proof of the liminf inequality. In order to do that, we need some preliminaries on densities and measures.

As in Theorem 5.3, let $(W_1^0, \dots, W_{2n}^0)$ be an orthonormal symplectic basis of $\ker\theta(\bar p)$, and let $(W_1^{\mathbb{H}}, \dots, W_{2n}^{\mathbb{H}})$ be the canonical orthonormal symplectic basis of $\ker\theta_0$ (θ_0 being the canonical contact form of $\mathbb{H}^n$). Let now $\mathcal{U} \subset M$ and, for $\bar p \in \mathcal{U}$, let $\Psi : \mathcal{U} \to \mathbb{H}^n$ be the *contact* diffeomorphism constructed in Theorem 5.3. In $\Psi(\mathcal{U})$, consider now the vector fields $\Psi_* W_i^0$, $i = 1, \dots, 2n$. Notice that

$$\operatorname{span}\{\Psi_* W_1^0, \dots, \Psi_* W_{2n}^0\} = \ker\theta_0 = \operatorname{span}\{W_1^{\mathbb{H}}, \dots, W_{2n}^{\mathbb{H}}\}.$$

Remember that $\Psi(\bar p) = 0$. By the same theorem, $\Psi_* W_i^0(0) = W_i^{\mathbb{H}}(0)$ for $i = 1, \dots, 2n$. We denote by d_c^Ψ the Carnot-Carathéodory distance in $\Psi(\mathcal{U})$ associated with the Riemannian metric $(\Psi^{-1})^* g$, and by $d_c^{\mathbb{H}}$ the standard Carnot-Carathéodory distance in $\mathbb{H}^n$. We denote also by $\overline{B}_\Psi$ and $\overline{B}_{\mathbb{H}}$ the closed balls associated with d_c^Ψ and $d_c^{\mathbb{H}}$, respectively.

It is easy to see that for $p, q \in \mathcal{U}$

$$d_c(p,q) = d_c^\Psi(\Psi(p), \Psi(q)).$$

In the sequel, B^Ψ will be the open balls with respect to d_c^Ψ.

Lemma 7.1. *For z in a neighborhood of $0 \in \mathbb{H}^n$, the following estimates hold:*

$$d_{\mathbb{H}}(z,0) \le d_c^\Psi(z,0)(1 + C d_c^\Psi(z,0)^{1/2}); \tag{7.1}$$

$$d_c^\Psi(z,0) \le d_{\mathbb{H}}(z,0)(1 + C d_{\mathbb{H}}(z,0)^{1/2}). \tag{7.2}$$

Proof. We denote by $\mathcal{W}_\Psi$ and $\mathcal{W}_{\mathbb{H}}$ the $(2n \times 2n)$-matrices whose columns are $\Psi_* W_1^0, \dots, \Psi_* W_{2n}^0$ and $W_1^{\mathbb{H}}, \dots, W_{2n}^{\mathbb{H}}$, respectively. If we set

$$\mathcal{A} := (a_{ij})_{i,j=1,\dots,2n} := \mathcal{W}_{\mathbb{H}}^{-1} \mathcal{W}_\Psi,$$

we obtain that $\mathcal{A}$ transforms the coordinates with respect to $(\Psi_* W_1^0, \dots, \Psi_* W_{2n}^0)$ of a generic point in $\ker\theta_0$ into its coordinates with respect to $(W_1^{\mathbb{H}}, \dots, W_{2n}^{\mathbb{H}})$. If we denote by z a generic point of $\Psi(\mathcal{U})$, by Theorem 5.3,

$$\mathcal{A}(z) = \mathrm{Id} + O(|z|) \qquad \text{as } z \to 0.$$

Let now $z \in K \subset\subset \Psi(\mathcal{U})$ be fixed, and let $\gamma : [0,1] \to \mathbb{H}^n$ a (smooth) d_c^Ψ-geodesic connecting 0 and z. If $t \in [0,1]$, we can write

$$\gamma'(t) = \sum_i \gamma_i(t)(\Psi_* W_i^0)(\gamma(t)) \quad \text{and} \quad d_c^\Psi(z,0) = \int_0^1 \big(\sum_i \gamma_i^2(t)\big)^{1/2} dt.$$

Thus, if $t \in [0,1]$, we have

$$\gamma'(t) = \sum_i \big\{\sum_j a_{i,j}(\gamma(t))\gamma_j(t)\big\} W_i^{\mathbb{H}}(\gamma(t)),$$

and hence

$$\begin{aligned} d_{\mathbb{H}}(z,0) &\leq \int_0^1 \Big(\sum_i \big\{\sum_j a_{i,j}(\gamma(t))\gamma_j(t)\big\}^2\Big)^{1/2} dt \\ &= \int_0^1 \Big(\sum_i \big\{\sum_j (\delta_{i,j} + O(|\gamma(t)|))\gamma_j(t)\big\}^2\Big)^{1/2} dt \\ &= \int_0^1 \Big(\sum_i \big\{\gamma_i(t) + O(|\gamma(t)|^2)\big\}^2\Big)^{1/2} dt \\ &\leq \int_0^1 \Big(\sum_i \gamma_i(t)^2\Big)^{1/2} dt + \int_0^1 O(|\gamma(t)|^{3/2})\, dt \\ &= d_c^\Psi(z,0) + \int_0^1 O(|\gamma(t)|^{3/2})\, dt. \end{aligned}$$

On the other hand, since the Euclidean distance may be locally bounded by d_c^Ψ,

$$|\gamma(t)| \leq C_1 d_c^\Psi(\gamma(t),0) \leq C d_c^\Psi(z,0),$$

so that (7.1) follows. We can carry out the same argument interchanging the roles of $d_{\mathbb{H}}$ and d_c^Ψ, and we get (7.2).

$\square$

To keep our paper as self-contained as possible, we gather here few more or less known results about Hausdorff measures in metric spaces. This part is taken almost verbatim from [24].

We recall first the definition of a centered density for an outer measure μ on X from Definition 5.5. In Euclidean spaces (and more generally in Carnot groups) we can replace in this definition the diameter $\operatorname{diam} \overline{B}(x,r)$ by $2r$. This "elementary" statement fails to be true in general metric spaces, but still holds in contact manifolds endowed with their Carnot-Carathéodory distance. This will follow from the following results.

Lemma 7.2. *Let M be a $(2n+1)$-dimensional contact manifold endowed with the contact form θ, with the volume form $v_\theta := \theta \wedge (d\theta)^n$, and the Riemannian metric g on $\ker\theta$ as introduced in Propositions 2.7 and 2.10. We denote by d_c the associated Carnot-Carathéodory distance. Let $\bar{p} \in M$ be a fixed point. We have:*

i) *if c_0 is the volume of the unit ball in $\mathbb{H}^n$ for the Carnot-Carathéodory distance associated with the canonical basis $(W_1^{\mathbb{H}}, \dots, W_{2n}^{\mathbb{H}})$ of $\mathbb{H}^n$ (see Theorem 5.3), then*

$$\lim_{r\to 0} \frac{v_\theta(\overline{B}(x,r))}{r^{2n+2}} = c_0;$$

ii) *Moreover,*

$$\lim_{r\to 0} \frac{\operatorname{diam} \overline{B}(x,r)}{2r} = 1.$$

Proof. Take a ball $\overline{B}_r := \overline{B}(\bar{p}, r) \subset M$ with $r > 0$ sufficiently small. For sake of simplicity, in Lemma 7.1, put $\phi(t) := t(1 + C\sqrt{t})$. Obviously, $\phi(r) = r + o(r)$ and $\phi^{-1}(s) = s + o(s)$ as $s \to 0$.

By (7.1) and (7.2)

$$\overline{B}_{\mathbb{H}}(0, \phi^{-1}(r)) \subset \Psi(\overline{B}_r) = B^\Psi(0,r) \subset \overline{B}_{\mathbb{H}}(0, \phi(r)). \tag{7.3}$$

We recall now that for $\rho > 0$

$$c_0 \rho^{2n+2} = \mathcal{L}^{2n+1}(\overline{B}_{\mathbb{H}}(0,\rho)) = \int_{\overline{B}_{\mathbb{H}}} dv_{\theta_0},$$

and that

$$\begin{aligned} v_\theta(\overline{B}_r) &= \int_{\overline{B}_r} \theta \wedge (d\theta)^n = \int_{\Psi(\overline{B}_r)} (\Psi^{-1})^*(\theta \wedge (d\theta)^n) \\ &= \int_{\Psi(\overline{B}_r)} (\Psi^{-1})^*\theta \wedge (d(\Psi^{-1})^*(\theta)^n) = \int_{\Psi(\overline{B}_r)} \theta_0 \wedge (d\theta_0)^n \\ &= \int_{B^\Psi(0,r)} dv_{\theta_0} = v_{\theta_0}(B^\Psi(0,r)), \end{aligned}$$

so that

$$c_0(\phi^{-1}(r))^{2n+2} \le v_\theta(\overline{B}_r) \le c_0 \phi(r)^{2n+2}.$$

Then i) follows straightforwardly.

Let us prove ii). If $r > 0$ By [22], Proposition 2.4, there exist $z_r, \zeta_r \in \overline{B}_{\mathbb{H}}(0, \phi^{-1}(r))$ such that $d_{\mathbb{H}}(z_r, \zeta_r) = 2\phi^{-1}(r)$. Arguing as above, if $\gamma : [0,1] \to \mathbb{H}^n$ is a d_c^Ψ-geodesic connecting z_r and ζ_r, then

$$d_{\mathbb{H}}(z_r, \zeta_r) \le d_c^\Psi(z_r, \zeta_r) + \int_0^1 O(|\gamma(t)|^{3/2})\, dt.$$

On the other hand, $\gamma(t) \in \overline{B}_{\mathbb{H}}(0, 3\phi^{-1}(r))$, and hence, if $r > 0$ is sufficiently small,

$$O(|\gamma(t)|^{3/2}) \le C_1 |\gamma(t)|^{3/2} \le C_2 d_{\mathbb{H}}(0, \gamma(t))^{3/2} \le C(\phi^{-1}(r))^{3/2} = Cr^{3/2}(1+o(1)),$$

so that

$$2\phi^{-1}(r) = d_{\mathbb{H}}(z_r, \zeta_r) \le d_c^\Psi(z_r, \zeta_r) + Cr^{3/2}(1 + o(1)).$$

Therefore

$$d_c^\Psi(z_r, \zeta_r) \ge 2r(1 + o(1)).$$

By (7.3), $z_r, \zeta_r \in B_r^\Psi$, so that

$$\Psi(z_r), \Psi(\zeta_r) \in \overline{B}_r.$$

Hence

$$1 \geq \frac{\operatorname{diam}(\overline{B}_r)}{2r} \geq \frac{d_c(\Psi(z_r), \Phi(\zeta_r))}{2r} = \frac{d_c^{\Phi}(z_r, \zeta_r)}{2r} \geq 1 + o(1),$$

and ii) follows. □

Lemma 7.2 immediately yields the following equivalent definition of densities in contact manifolds:

Corollary 7.3. *Let M be $(2n+1)$-dimensional contact manifold endowed with a contact form θ and a Riemannian metric g on the fibers of θ as introduced in Propositions 2.7 and 2.10. We denote by d_c the associated Carnot-Carathéodory distance. Let μ be an outer measure on M. Then*

$$\Theta^{*m}(\mu, x) := \limsup_{r \to 0} \frac{\mu(\overline{B}(x,r))}{\alpha_m\, r^m}$$

and

$$\Theta_*^m(\mu, x) := \liminf_{r \to 0} \frac{\mu(\overline{B}(x,r))}{\alpha_m\, r^m}\,.$$

Remark 7.4. In Corollary 7.3 we can replace closed balls $\overline{B}(x,r)$ by open balls $B(x,r)$ (see [9], Remark 2.4.2).

Keeping in mind Corollary 7.3 and Remark 7.4, the following result can be proved by the same arguments used in the proof of Theorem 3.1 in [24].

Proposition 7.5. *Let M be $(2n+1)$-dimensional contact manifold endowed with a contact form θ and a Riemannian metric g on the fibers of θ as introduced in Propositions 2.7 and 2.10. We denote by d_c the associated Carnot-Carathéodory distance. Let μ be a σ-finite regular Borel measure on M. Then the map*

$$\Theta^{*m}(\mu, \cdot) : X \to [0, +\infty]$$

is Borel measurable.

We give now the following:

Definition 7.6. Let $A \subset X$, $m \in [0, \infty)$, $\delta \in (0, \infty)$, and let β_m be the constant (5.4).

(i) The *m-dimensional Hausdorff measure* $\mathcal{H}^m$ is defined as

$$\mathcal{H}^m(A) := \lim_{\delta \to 0} \mathcal{H}^m_\delta(A)$$

where

$$\mathcal{H}^m_\delta(A) = \inf \left\{ \sum_i \beta_m \operatorname{diam}(E_i)^m : \ A \subset \bigcup_i E_i, \quad \operatorname{diam}(E_i) \leq \delta \right\}.$$

(ii) The *m-dimensional spherical Hausdorff measure* $\mathcal{S}^m$ is defined as

$$\mathcal{S}^m(A) := \lim_{\delta \to 0} \mathcal{S}^m_\delta(A)$$

where

$$\mathcal{S}^m_\delta(A) = \inf \Big\{ \sum_i \beta_m \operatorname{diam}(B(x_i, r_i))^m : \ A \subset \bigcup_i B(x_i, r_i),$$
$$\operatorname{diam}(B(x_i, r_i)) \leq \delta \Big\}$$

(iii) The *m-dimensional centered Hausdorff measure* $\mathcal{C}^m$ is defined as

$$\mathcal{C}^m(A) := \sup_{E\subseteq A} \mathcal{C}_0^m(E)\,.$$

where $\mathcal{C}_0^m(E) := \lim_{\delta\to 0^+} \mathcal{C}_\delta^m(E)$, and, in turn, $\mathcal{C}_\delta^m(E) = 0$ if $E = \emptyset$ and for $E \neq \emptyset$,

$$\mathcal{C}_\delta^m(E) = \inf\Big\{ \sum_i \beta_m \mathrm{diam}\,(B(x_i,r_i))^m : E \subset \bigcup_i B(x_i,r_i), \\ x_i \in E, \quad \mathrm{diam}\,(B(x_i,r_i)) \leq \delta \Big\}.$$

Notice that the set function $\mathcal{C}_0^m$ is not necessarily monotone (see [46, Sect. 4]) while $\mathcal{C}^m$ is monotone.

For reader's convenience we collect a few results about the measures $\mathcal{C}^m$. Most of these results are taken from [15] and [24].
Let

$$\mathrm{dist}(E,F) := \inf\{d(x,y) : x \in E,\, y \in F\}$$

denote the *distance* between E and F. Recall that an outer measure μ on X is said to be *metric* if

$$\mu(A\cup B) = \mu(A) + \mu(B) \qquad \text{whenever } \mathrm{dist}(A,B) > 0\,.$$

Being obtained by Caratheödory's construction, $\mathcal{H}^m$ and $\mathcal{S}^m$ are metric (outer) measures (see [16, 2.10.1] or [31, Theorem 4.2]). Also the measures $\mathcal{C}^m$ are metric measures in any metric space, but this fact is not as immediate as for $\mathcal{H}^m$ and $\mathcal{S}^m$.

Lemma 7.7 ([15], Proposition 4.1)**.** *$\mathcal{C}^m$ is a Borel regular outer measure.*

Remark 7.8. The measures $\mathcal{H}^m$, $\mathcal{S}^m$ and $\mathcal{C}^m$ are all equivalent measures. Indeed, it is well known that (see, for instance, [16, 2.10.2])

$$\mathcal{H}^m \leq \mathcal{S}^m \leq 2^m\,\mathcal{H}^m$$

and, by definition,

$$\mathcal{H}^m \leq \mathcal{S}^m \leq \mathcal{C}^m\,.$$

The opposite inequality between $\mathcal{H}^m$ (or $\mathcal{S}^m$) and $\mathcal{C}^m$ is less immediate: it was proved in [46, Lemma 3.3] for the case $X = \mathbb{R}^n$. See also [49], but for a differently defined centered Hausdorff-type measure. The comparison in a general metric space is contained in [15].

Lemma 7.9 ([15], Proposition 4.2)**.** $\mathcal{H}^m \leq \mathcal{C}^m \leq 2^m\,\mathcal{H}^m\,.$

By Lemma 7.9, it follows in particular that the metric dimensions induced by $\mathcal{H}^m$ or $\mathcal{S}^m$ or $\mathcal{C}^m$ are the same.

The estimates needed to relate the m-dimensional density $\Theta^{*\,m}(\mu,\cdot)$ with the centered Hausdorff measure $\mathcal{C}^m$ are the following ones.

Theorem 7.10 ([15], Theorem 4.15)**.** *Let (X,d) be a separable metric space, let μ be a finite Borel outer measure in X and let $B \subset X$ be a Borel set. Then*

(i)

$$\mu(B) \leq \sup_{x\in B} \Theta^{*\,m}(\mu,x)\,\mathcal{C}^m(B),$$

except when the product is $\infty\cdot 0$;

(ii)

$$\inf_{x\in B} \Theta^{*\,m}(\mu, x)\,\mathcal{C}^m(B) \le \mu(B)\,.$$

By easy modifications of the proof of Theorem 7.10, one gets the following density estimates involving $\Theta^{*\,m}(\mu, x)$ and $\mathcal{C}^m$. These estimates are analogous to Federer's ones involving $\Theta^{*\,m}_F(\mu, x)$ and $\mathcal{S}^m$ (see [16]).

Theorem 7.11. *Let (X, d) be a separable metric space, let μ be an outer measure in X and $t > 0$.*

(i) *If μ is Borel regular and*

$$\Theta^{*\,m}(\mu \llcorner A, x) < t, \qquad \forall x \in A \subset X$$

then

$$\mu(A) \le t\,\mathcal{C}^m(A)\,.$$

(ii) *If $V \subset X$ is an open set and*

$$\Theta^{*\,m}(\mu, x) > t, \qquad \forall x \in B \subset V$$

then

$$\mu(V) \ge t\,\mathcal{C}^m(B)\,.$$

Remark 7.12. If μ is supposed to be a Radon measure, approximating from above by open sets, we can strengthen the conclusion in Theorem 7.11 (ii) getting the inequality $\mu(B) \ge t\,\mathcal{C}^m(B)$.

Using Lemma 7.2 (i.e. relying on the equivalence of the two notions of density) and Proposition 7.5, the following result can be proved following step by step the proof of Theorem 3.1 in [24].

Theorem 7.13. *Let M be $(2n + 1)$-dimensional contact manifold endowed with a contact form θ and a Riemannian metric g on the fibers of θ as introduced in Propositions 2.7 and 2.10. We denote by d_c the associated Carnot-Carathéodory distance. Let μ be a σ-finite regular Borel measure on M, and let $A \subset X$ be a Borel set. If $\mathcal{C}^m(A) < \infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{C}^m \llcorner A$, then for each Borel set $B \subset A$,*

$$\mu(B) = \int_B \Theta^{*\,m}(\mu, x)\, d\mathcal{C}^m(x).$$

Remark 7.14. Since $\mathcal{C}^m$ and $\mathcal{S}^m$ are equivalent, then $\mathcal{C}^m(A) < \infty$ if and only if $\mathcal{S}^m(A) < \infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{C}^m$ if and only if $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{S}^m$.

Now we can give the proof of Theorem 5.6.

Proof of Theorem 5.6. Since $|\mathbf{W}^0\chi_E|$ is supported on $\partial^* E$, without loss of generality we may assume that (5.5) holds for all $x \in \partial E$.

Suppose first

$$\mu \llcorner \partial E \ll \mathcal{H}^{2n+1} \llcorner \partial E, \tag{7.4}$$

and denote by $A \subset \partial E$ the set of points where (5.5) holds, so that $\mathcal{H}^{2n+1}(\partial E \setminus A) = 0$. We remind also that $|\mathbf{W}^0\chi_E| \ll \mathcal{H}^{2n+1} \llcorner \partial E$, by [6], Lemma 5.2.

Thus, if $B \subset \partial E$ is a Borel set, we can apply Theorem 7.13 to get

$$\begin{aligned}
\mu \llcorner \partial E(B) = \mu(\partial E \cap B) &= \int_{\partial E \cap B} \Theta^{*,2n+1}(\mu, x) d\mathcal{C}^{2n+1}(x) \\
&\geq \int_{\partial E \cap B} \Theta^{*,2n+1}(|\mathbf{W}^0 \chi_E|, x) d\mathcal{C}^{2n+1}(x) = |\mathbf{W}^0 \chi_E|(\partial E \cap B) \\
&= |\mathbf{W}^0 \chi_E|(B).
\end{aligned}$$

Let us drop now the assumption (7.4). We can write

$$\mu \llcorner \partial E = \mu_{ac} + \mu_s$$

with

$$\mu_{ac} \ll \mathcal{H}^{2n+1} \llcorner \partial E \quad \text{and} \quad \mu_s \perp \mathcal{H}^{2n+1} \llcorner \partial E$$

(see [45] Theorem 6.10), i.e. there exists $K \subset M$ such that

$$\mu_s = \mu_s \llcorner K \quad \text{and} \quad (\mathcal{H}^{2n+1} \llcorner \partial E)(K) = 0.$$

Set now

$$S_0 := \{x \in M \; ; \; \Theta^{*,2n+1}(\mu_s, x) = 0\}.$$

Notice that S_0 is a Borel set, since $\Theta^{*2n+1}(\mu_s, \cdot)$ is a Borel function.

If $x \in S_0$, then

$$\begin{aligned}
\Theta^{*,2n+1}(|\mathbf{W}^0 \chi_E|, x) &\leq \Theta^{*,2n+1}(\mu, x) \\
&\leq \Theta^{*,2n+1}(\mu_s, x) + \Theta^{*,2n+1}(\mu_{ac}, x) \\
&= \Theta^{*,2n+1}(\mu_{ac}, x).
\end{aligned}$$

Thus, as above, we can apply Theorem 7.13 to get for any Borel set B

$$|\mathbf{W}^0 \chi_E|(B \cap S_0) \leq \mu_{ac}(B \cap S_0) \leq \mu(B \cap S_0) \leq \mu(B).$$

To complete the proof of (5.6), we shall prove that

$$(\mathcal{H}^{2n+1} \llcorner \partial E)(S_0^c) = 0, \tag{7.5}$$

that yields

$$|\mathbf{W}^0 \chi_E|(S_0^c) = 0,$$

by [6], Lemma 5.2 (here S_0^c denotes the complement of S_0).

In order to prove (7.5), we can write

$$S_0^c = \cup_{n=1}^{\infty} \{x \in M \; ; \; \Theta^{*,2n+1}(\mu_s, x) > \tfrac{1}{n}\} := \cup_{n=1}^{\infty} T_n.$$

Then

(7.6)
$$\begin{aligned}
(\mathcal{H}^{2n+1} \llcorner \partial E)(S_0^c) &= (\mathcal{H}^{2n+1} \llcorner \partial E)(S_0^c \cap K) + (\mathcal{H}^{2n+1} \llcorner \partial E)(S_0^c \cap K^c) \\
&= (\mathcal{H}^{2n+1} \llcorner \partial E)(S_0^c \cap K^c),
\end{aligned}$$

since

$$(\mathcal{H}^{2n+1} \llcorner \partial E)(S_0^c \cap K) \leq (\mathcal{H}^{2n+1} \llcorner \partial E)(K) = 0.$$

On the other hand

$$(\mathcal{H}^{2n+1} \llcorner \partial E)(S_0^c \cap K^c) = \lim_{n \to \infty} (\mathcal{H}^{2n+1} \llcorner \partial E)(S_0^c \cap K^c \cap T_n). \tag{7.7}$$

The set $\partial E \cap S_0^c \cap K^c \cap T_n$ is a Borel set, so that, by Federer's differentiation theorem (see, e.g., [9] Theorem 2.4.3)

$$(\mathcal{H}^{2n+1} \llcorner \partial E)(S_0^c \cap K^c \cap T_n) \leq n\, \mu_s(S_0^c \cap K^c \cap T_n) \tag{7.8}$$
$$= n\, (\mu_s \llcorner K)(S_0^c \cap K^c \cap T_n) = 0.$$

Combining (7.6), (7.7) and (7.8) we obtain eventually (7.5). This completes the proof of the theorem.

□

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