

IMPROVED LIPSCHITZ APPROXIMATION OF H -PERIMETER MINIMIZING BOUNDARIES

ROBERTO MONTI AND GIORGIO STEFANI

ABSTRACT. We prove two new approximation results of H -perimeter minimizing boundaries by means of intrinsic Lipschitz functions in the setting of the Heisenberg group \mathbb{H}^n with $n \geq 2$. The first one is an improvement of [19] and is the natural reformulation in \mathbb{H}^n of the classical Lipschitz approximation in \mathbb{R}^n . The second one is an adaptation of the approximation via maximal function developed by De Lellis and Spadaro, [11].

1. INTRODUCTION

The study of Geometric Measure Theory in the Heisenberg group \mathbb{H}^n started from the pioneering work [12] and the regularity of sets that are minimizers for the horizontal perimeter is one of the most important open problems in the field. The known regularity results assume some strong *a priori* regularity and/or some restrictive geometric structure of the minimizer, see [5–7, 20, 25]. On the other hand, examples of minimal surfaces in the first Heisenberg group \mathbb{H}^1 that are only Lipschitz continuous in the Euclidean sense have been constructed, see, e.g., [22, 23], but no similar examples of non-smooth minimizers are known in \mathbb{H}^n with $n \geq 2$.

The most natural approach to a regularity theory for H -perimeter minimizing sets in the Heisenberg group \mathbb{H}^n is to adapt the classical De Giorgi's regularity theory for perimeter minimizers in \mathbb{R}^n . His ideas have been recently improved and generalized by several authors, see the recent monograph [17]. In particular, one of the most important achievements is Almgren's regularity theory of area minimizing integral currents in \mathbb{R}^n of general codimension, [1]. For a survey on Almgren's theory and on the long term program undertaken by De Lellis and Spadaro to make Almgren's work more readable and exploitable for a larger community of specialists, we refer to [2] and to the references therein. For the recent extension of the theory to infinite dimensional spaces, see [3].

This paper deals with the first step of the regularity theory, namely, the Lipschitz approximation. In fact, in De Giorgi's original approach the approximation is made by convolution and the estimates are based on a monotonicity formula. In the Heisenberg group, however, the validity of a monotonicity formula is not clear, see [10]. A more flexible approach is the approximation of minimizing boundaries by means of Lipschitz graphs, see [24]. Although the boundary of sets with finite H -perimeter is not rectifiable in the standard sense and, in fact, may have fractional Hausdorff dimension, [16], the notion of *intrinsic Lipschitz graph* in the sense of [13] turns out to be effective in the approximation, as shown in [19].

Date: November 28, 2016.

2010 Mathematics Subject Classification. 49Q05, 53C17, 28A75.

Key words and phrases. Heisenberg group, regularity of H -minimal surfaces, Lipschitz approximation.

Here, we prove two new intrinsic Lipschitz approximation theorems for H -perimeter minimizers in the setting of the Heisenberg group \mathbb{H}^n with $n \geq 2$.

The first result is an improvement of [19] and is the natural reformulation in \mathbb{H}^n of the classical Lipschitz approximation in \mathbb{R}^n , see [17, Theorem 23.7]. Let $\mathbb{W} = \mathbb{R} \times \mathbb{H}^{n-1}$ be the hyperplane passing through the origin and orthogonal to the direction $\nu = -X_1$. The disk $D_r \subset \mathbb{W}$ centered at the origin is defined using the natural box norm of \mathbb{H}^n and the cylinder $C_r(p)$, $p \in \mathbb{H}^n$, is defined as $C_r(p) = p * C_r$, where $C_r = D_r * (-r, r)$. We denote by $\mathbf{e}(E, C_r(p), \nu)$ the excess of E in $C_r(p)$ with respect to the fixed direction ν , that is, the L^2 -averaged oscillation of ν_E , the inner horizontal unit normal to E , from the direction ν in the cylinder. The $2n + 1$ -dimensional spherical Hausdorff measure \mathcal{S}^{2n+1} is defined by the natural distance of \mathbb{H}^n . Finally, $\nabla^\varphi \varphi$ is the intrinsic gradient of φ . We refer the reader to Section 2 for precise definitions.

Theorem 1.1. *Let $n \geq 2$. There exist positive dimensional constants $C_1(n)$, $\varepsilon_1(n)$ and $\delta_1(n)$ with the following property. If $E \subset \mathbb{H}^n$ is an H -perimeter minimizer in the cylinder C_{5124} with $0 \in \partial E$ and $\mathbf{e}(E, C_{5124}, \nu) \leq \varepsilon_1(n)$ then, letting*

$$M = C_1 \cap \partial E, \quad M_0 = \left\{ q \in M : \sup_{0 < s < 256} \mathbf{e}(E, C_s(q), \nu) \leq \delta_1(n) \right\},$$

there exists an intrinsic Lipschitz function $\varphi: \mathbb{W} \rightarrow \mathbb{R}$ such that

$$\sup_{\mathbb{W}} |\varphi| \leq C_1(n) \mathbf{e}(E, C_{5124}, \nu)^{\frac{1}{2(2n+1)}}, \quad \text{Lip}_H(\varphi) \leq 1,$$

$$M_0 \subset M \cap \Gamma, \quad \Gamma = \text{gr}(\varphi|_{D_1}),$$

$$\mathcal{S}^{2n+1}(M \Delta \Gamma) \leq C_1(n) \mathbf{e}(E, C_{5124}, \nu),$$

$$\int_{D_1} |\nabla^\varphi \varphi|^2 d\mathcal{L}^{2n} \leq C_1(n) \mathbf{e}(E, C_{5124}, \nu).$$

The Lipschitz approximation proved in [19] is limited to the estimate $\mathcal{S}^{2n+1}(M \Delta \Gamma) \leq C_1(n) \mathbf{e}(E, C_{5124}, \nu)$. Here, we give a more elementary proof of a more general result following the scheme outlined in [17, Section 23.3]. The fundamental tool used in the proof is the height estimate recently established in [21, Theorem 1.3]. Theorem 1.1 holds also for (Λ, r_0) -minimizers of H -perimeter, see the more general formulation given in Theorem 3.1 of Section 3.

Theorem 1.1 is the starting point for the proof of our second result, which is obtained using an adaptation to the setting of H -perimeter minimizers in \mathbb{H}^n of the ideas developed in [11] by De Lellis and Spadaro for area minimizing integral currents.

Theorem 1.2. *Let $n \geq 2$ and $\alpha \in (0, \frac{1}{2})$. There exist positive constants $C_2(n)$, $\varepsilon_2(\alpha, n)$ and $k_2 = k_2(n)$ with the following property. For any set $E \subset \mathbb{H}^n$ that is an H -perimeter minimizer in the cylinder C_{k_2} with $0 \in \partial E$ and $\mathbf{e}(E, C_{k_2}, \nu) \leq \varepsilon_2(\alpha, n)$, there exist a set $K \subset D_1$ and an intrinsic Lipschitz function $\varphi: \mathbb{W} \rightarrow \mathbb{R}$ such that:*

$$\mathcal{L}^{2n}(D_1 \setminus K) \leq C_2(n) \mathbf{e}(E, C_{k_2}, \nu)^{1-2\alpha}$$

$$\text{gr}(\varphi|_K) = \partial E \cap (K * (-1, 1)),$$

$$\text{Lip}_H(\varphi) \leq C_2(n) \mathbf{e}(E, C_{k_2}, \nu)^\alpha,$$

$$\mathcal{S}^{2n+1}((\partial E \Delta \text{gr}(\varphi)) \cap C_1) \leq C_2(n) \mathbf{e}(E, C_{k_2}, \nu)^{1-2\alpha},$$

$$\int_{D_1} |\nabla^\varphi \varphi|^2 d\mathcal{L}^{2n} \leq C_2(n) \mathbf{e}(E, C_{k_2}, \nu).$$

Theorem 1.2 holds also for (Λ, r_0) -minimizers of H -perimeter, see the more general formulation of this result given in Corollary 5.5 of Section 5.

The first step in [11] is to establish a so-called *BV estimate* on the vertical slices of the area minimizing integral current, see [11, Lemma A.1]. The proof of this estimate uses several fundamental results of the theory of integral currents in \mathbb{R}^n . Thus far, a theory for integral currents in \mathbb{H}^n is not yet well established, see [14], and a similar estimate for the slices of the boundary of an H -perimeter minimizer is not clear. However, when the minimizer is the intrinsic epigraph of an intrinsic Lipschitz function, the *BV estimate* is an easy consequence of the Cauchy–Schwarz inequality and of the area formula. Therefore, when E is an H -perimeter minimizer, we can overcome the problem with the following trick: first, by Theorem 1.1, we approximate the boundary of E with the intrinsic graph of a suitable intrinsic Lipschitz function; second, up to an error which is comparable to the excess, we replace the *BV estimate* on the slices of the boundary of E with the *BV estimate* on the slices of the approximating graph. A fundamental tool used in our argument is the Poincaré inequality recently established in [9].

In the case of minimizing integral currents in \mathbb{R}^n , the Lipschitz approximation in the spirit of Theorem 1.2 is the starting point of the so-called harmonic approximation, that gives the decay estimates for excess. In the setting of \mathbb{H}^n , deriving the harmonic approximation from Theorem 1.2 is still an open problem, see [20].

2. PRELIMINARIES

In this section, we fix the notation and recall some basic facts on intrinsic Lipschitz functions, on the area formula, and on the height bound for H -perimeter minimizers. The reader familiar with these results can skip this section.

2.1. The Heisenberg group. The n -th Heisenberg group is the manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ endowed with the group law $(z, t) * (w, s) = (z + w, t + s + P(z, w))$ for $(z, t), (w, s) \in \mathbb{H}^n$, where $z, w \in \mathbb{C}^n$, $t, s \in \mathbb{R}$ and $P: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{R}$ is the bilinear form

$$P(z, w) = 2 \operatorname{Im} \left(\sum_{j=1}^n z_j \bar{w}_j \right), \quad z, w \in \mathbb{C}^n.$$

The *left translations* $\tau_q: \mathbb{H}^n \rightarrow \mathbb{H}^n$ are defined by $\tau_q(p) = q * p$. The automorphisms $\delta_\lambda: \mathbb{H}^n \rightarrow \mathbb{H}^n$, $\lambda > 0$, of the form

$$\delta_\lambda(z, t) = (\lambda z, \lambda^2 t), \quad (z, t) \in \mathbb{H}^n,$$

are called *dilations*. We use the abbreviations $\lambda p = \delta_\lambda(p)$ and $\lambda E = \delta_\lambda(E)$ for $p \in \mathbb{H}^n$ and $E \subset \mathbb{H}^n$.

For any $p = (z, t) \in \mathbb{H}^n$, let $\|p\|_\infty = \max\{|z|, |t|^{1/2}\}$ be the *box norm*. It satisfies the triangle inequality

$$\|p * q\|_\infty \leq \|p\|_\infty + \|q\|_\infty, \quad p, q \in \mathbb{H}^n.$$

The function $d_\infty: \mathbb{H}^n \times \mathbb{H}^n \rightarrow [0, \infty)$, $d(p, q) = \|p^{-1} * q\|$ for $p, q \in \mathbb{H}^n$, is a left invariant distance on \mathbb{H}^n equivalent to the Carnot–Carathéodory distance. We define the open ball centered at $p \in \mathbb{H}^n$ and with radius $r > 0$ as

$$(2.1) \quad B_r(p) = \{q \in \mathbb{H}^n : d_\infty(q, p) < r\} = p * \{q \in \mathbb{H}^n : \|q\|_\infty < r\}.$$

In the case $p = 0$, we let $B_r = B_r(0)$.

For any $s \geq 0$, we denote by \mathcal{S}^s the spherical Hausdorff measure in \mathbb{H}^n constructed with the left invariant metric d_∞ . Namely, for any $E \subset \mathbb{H}^n$ we let

$$\mathcal{S}^s(E) = \lim_{\delta \rightarrow 0} \mathcal{S}_\delta^s(E),$$

where

$$\mathcal{S}_\delta^s(E) = \inf \left\{ \sum_{n \in \mathbb{N}} (\text{diam } B_i)^s : E \subset \bigcup_{n \in \mathbb{N}} B_i, B_i \text{ balls as in (2.1), } \text{diam } B_i < \delta \right\}$$

and diam is the diameter in the distance d_∞ . The correct dimension to measure hyper-surfaces is $s = 2n + 1$.

We identify an element $z = x + iy \in \mathbb{C}^n$ with $(x, y) \in \mathbb{R}^{2n}$. The Lie algebra of left invariant vector fields in \mathbb{H}^n is spanned by the vector fields

$$(2.2) \quad X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}, \quad j = 1, \dots, n.$$

We denote by H the *horizontal sub-bundle* of $T\mathbb{H}^n$. Namely, for any $p = (z, t) \in \mathbb{H}^n$, we let

$$H_p = \text{span} \{X_1(p), \dots, X_n(p), Y_1(p), \dots, Y_n(p)\}.$$

Let g be the left invariant Riemannian metric on \mathbb{H}^n that makes orthonormal the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n, T$. The metric g induces a volume form on \mathbb{H}^n that is left invariant and coincides with the Lebesgue measure \mathcal{L}^{2n+1} . For tangent vectors $V, W \in T\mathbb{H}^n$, we let

$$\langle V, W \rangle_g = g(V, W) \quad \text{and} \quad |V|_g = g(V, V)^{1/2}.$$

Let $\Omega \subset \mathbb{H}^n$ be an open set. A *horizontal section* $V \in C_c^1(\Omega; H)$ is a vector field of the form

$$V = \sum_{j=1}^n V_j X_j + V_{j+n} Y_j,$$

where $V_j \in C_c^1(\Omega)$ for any $j = 1, \dots, 2n$. The sup-norm with respect to g of a horizontal section $V \in C_c^1(\Omega; H)$ is

$$\|V\|_g = \max_{p \in \Omega} |V(p)|_g.$$

The *horizontal divergence* of V is

$$\text{div}_H V = \sum_{j=1}^n X_j V_j + Y_j V_{j+n}.$$

2.2. Locally finite perimeter sets. A \mathcal{L}^{2n+1} -measurable set $E \subset \mathbb{H}^n$ has *finite H -perimeter* in an open set $\Omega \subset \mathbb{H}^n$ if

$$P_H(E; \Omega) = \sup \left\{ \int_E \text{div}_H V \, d\mathcal{L}^{2n+1} : V \in C_c^1(\Omega; H), \|V\|_g \leq 1 \right\} < \infty.$$

If $P_H(E; A) < \infty$ for any open set $A \subset\subset \Omega$, we say that E has *locally finite H -perimeter* in Ω . In this case, the mapping $A \mapsto P_H(E; A) = \mu_E(A)$ extends from open sets to a

Radon measure μ_E on Ω . By the Radon-Nykodim Theorem, there exists a μ_E -measurable function $\nu_E: \Omega \rightarrow H$ such that $|\nu_E|_g = 1$ μ_E -a.e., and the *Gauss–Green formula*

$$\int_E \operatorname{div}_H V \, d\mathcal{L}^{2n+1} = - \int_\Omega \langle V, \nu_E \rangle_g \, d\mu_E$$

holds for any $V \in C_c^1(\Omega; H)$. We call ν_E the *horizontal inner normal* of E in Ω . The *measure theoretic boundary* of a \mathcal{L}^{2n+1} -measurable set $E \subset \mathbb{H}^n$ is the set

$$\partial E = \left\{ p \in \mathbb{H}^n : \mathcal{L}^{2n+1}(E \cap B_r(p)) > 0 \text{ and } \mathcal{L}^{2n+1}(B_r(p) \setminus E) > 0 \text{ for all } r > 0 \right\}.$$

Let E be a set with locally finite H -perimeter in \mathbb{H}^n . Then the measure μ_E is concentrated on ∂E and, actually, on a subset $\partial^* E \subset \partial E$ called the *reduced boundary* of E . This follows from the structure theorem for sets with locally finite H -perimeter, see [12]. Moreover, up to modifying E on a Lebesgue negligible set, one can always assume that ∂E coincides with the topological boundary of E , see [25, Proposition 2.5].

2.3. Perimeter minimizers. Let $\Omega \subset \mathbb{H}^n$ be an open set and let E be a set with locally finite H -perimeter in \mathbb{H}^n . We say that the set E is a (Λ, r_0) -*minimizer of H -perimeter* in Ω if there exist two constants $\Lambda \in [0, \infty)$ and $r_0 \in (0, \infty]$ such that

$$P(E; B_r(p)) \leq P(F; B_r(p)) + \Lambda \mathcal{L}^{2n+1}(E \Delta F)$$

for any measurable set $F \subset \mathbb{H}^n$, $p \in \Omega$ and $r < r_0$ such that $E \Delta F \subset\subset B_r(p) \subset\subset \Omega$.

When $\Lambda = 0$ and $r_0 = \infty$, we say that the set E is a *locally H -perimeter minimizer* in Ω , that is, we have

$$P(E; B_r(p)) \leq P(F; B_r(p))$$

for any measurable set $F \subset \mathbb{H}^n$, $p \in \Omega$ and $r > 0$ such that $E \Delta F \subset\subset B_r(p) \subset\subset \Omega$.

If E is a (Λ, r_0) -minimizer of H -perimeter in Ω , then the difference $\partial E \setminus \partial^* E$ is \mathcal{L}^{2n+1} -negligible in Ω , see [21, Corollary 4.2]. Thus, in the following, up to modifying E on a Lebesgue negligible set, we will tacitly assume that the reduced boundary and the topological boundary of E coincide.

Remark 2.1 (Scaling of (Λ, r_0) -minimizer). If the set E is a (Λ, r_0) -minimizer of H -perimeter in the open set $\Omega \subset \mathbb{H}^n$ then, for every $p \in \mathbb{H}^n$ and $r > 0$, the set $E_{p,r} = \delta_{\frac{1}{r}}(\tau_{p^{-1}}(E))$ is a (Λ', r'_0) -minimizer of H -perimeter in $\Omega_{p,r}$, where $\Lambda' = \Lambda r$ and $r'_0 = r_0/r$. In particular, the product Λr_0 is invariant and thus it is convenient to assume that $\Lambda r_0 \leq 1$, as we shall always do in the following.

2.4. Cylindrical excess. The *height function* $\mathfrak{h}: \mathbb{H}^n \rightarrow \mathbb{R}$ is the group homomorphism $\mathfrak{h}(p) = x_1$, for $p = (x, y, t) \in \mathbb{H}^n$. Let \mathbb{W} be the (normal) subgroup of \mathbb{H}^n given by the kernel of \mathfrak{h} ,

$$\mathbb{W} := \ker \mathfrak{h} = \left\{ p \in \mathbb{H}^n : \mathfrak{h}(p) = 0 \right\}.$$

The *open disk* in \mathbb{W} of radius $r > 0$ centered at the origin is the set $D_r = \{w \in \mathbb{W} : \|w\|_\infty < r\}$. For any $p \in \mathbb{W}$, we let $D_r(p) = p * D_r \subset \mathbb{W}$. Note that, for all $p \in \mathbb{W}$ and $r > 0$,

$$(2.3) \quad \mathcal{L}^{2n}(D_r(p)) = \mathcal{L}^{2n}(D_r) = \kappa_n r^{2n+1},$$

with $\kappa_n = \mathcal{L}^{2n}(D_1)$. The *open cylinder* with central section D_r and height $2r$ is the set

$$C_r = D_r * (-r, r) := \{w * s e_1 \in \mathbb{H}^n : w \in D_r, s \in (-r, r)\},$$

where $se_1 = (s, 0, \dots, 0) \in \mathbb{H}^n$. For any $p \in \mathbb{H}^n$, we let $C_r(p) = p * C_r$.

Let $\pi: \mathbb{H}^n \rightarrow \mathbb{W}$ be the projection on \mathbb{W} defined, for any $p \in \mathbb{H}^n$, by the formula

$$(2.4) \quad p = \pi(p) * \mathfrak{h}(p)e_1.$$

By (2.4), for any $p \in \mathbb{H}^n$ and $r > 0$, we have

$$p \in C_r \iff \pi(p) \in D_r, \quad \mathfrak{h}(p) \in (-r, r) \iff \|\pi(p)\|_\infty < r, \quad |\mathfrak{h}(p)| < r.$$

We thus let $\|\cdot\|_C: \mathbb{H}^n \rightarrow [0, \infty)$ be the map

$$(2.5) \quad \|p\|_C := \max\{\|\pi(p)\|_\infty, |\mathfrak{h}(p)|\}$$

for any $p \in \mathbb{H}^n$, so that $C_r = \{p \in \mathbb{H}^n : \|p\|_C < r\}$. The map $\|\cdot\|_C$ is a quasi-norm and, by (2.4), we have

$$(2.6) \quad \|p\|_C \leq 2\|p\|_\infty, \quad \|p\|_\infty \leq 2\|p\|_C \quad p \in \mathbb{H}^n.$$

Let $d_C: \mathbb{H}^n \times \mathbb{H}^n \rightarrow [0, \infty)$ be the quasi-distance induced by $\|\cdot\|_C$. By (2.6), the cylinder $C_r(p)$ is comparable with the ball $B_r(p)$ induced by the box norm for any $p \in \mathbb{H}^n$. Namely, we have

$$(2.7) \quad B_{r/2}(p) \subset C_r(p) \subset B_{2r}(p) \quad \text{for all } p \in \mathbb{H}^n, \quad r > 0.$$

A concept which plays a key role in the regularity theory of (Λ, r_0) -minimizers of H -perimeter is the notion of excess.

Definition 2.2 (Cylindrical excess). Let E be a set with locally finite H -perimeter in \mathbb{H}^n . The *cylindrical excess* of E at the point $p \in \partial E$, at the scale $r > 0$, and with respect to the direction $\nu = -X_1$, is defined as

$$\mathbf{e}(E, p, r, \nu) := \frac{1}{r^{2n+1}} \int_{C_r(p)} \frac{|\nu_E - \nu|_g^2}{2} d\mu_E = \frac{\delta(n)}{r^{2n+1}} \int_{C_r(p) \cap \partial^* E} \left(1 - \langle \nu_E, \nu \rangle_g\right) d\mathcal{L}^{2n+1}$$

where μ_E is the Gauss-Green measure of E , ν_E is the horizontal inner normal and the multiplicative constant is $\delta(n) = \frac{2\omega_{2n-1}}{\omega_{2n+1}}$.

We refer the reader to [18] for the problem of the coincidence of perimeter measure and spherical Hausdorff measures.

For the sake of brevity, we will set $\mathbf{e}(p, r) = \mathbf{e}(E, p, r, \nu)$ and, in the case $p = 0$, $\mathbf{e}(r) = \mathbf{e}(0, r)$. For the elementary properties of the excess, see [21, Section 3.2].

2.5. Height bound. The following result is a fundamental estimate relating the height of the boundary of a (Λ, r_0) -minimizer of H -perimeter with the cylindrical excess, see [21, Theorem 1.3].

Theorem 2.3 (Height bound). *Let $n \geq 2$. There exist positive dimensional constants $\varepsilon_0(n)$ and $C_0(n)$ with the following property. If E is a (Λ, r_0) -minimizer of H -perimeter in the cylinder C_{16r_0} with*

$$\Lambda r_0 \leq 1, \quad 0 \in \partial E, \quad \mathbf{e}(16r_0) \leq \varepsilon_0(n),$$

then

$$(2.8) \quad \sup \left\{ \frac{|\mathfrak{h}(p)|}{r_0} : p \in C_{r_0} \cap \partial E \right\} \leq C_0(n) \mathbf{e}(16r_0)^{\frac{1}{2(2n+1)}}.$$

Remark 2.4. The estimate (2.8) does not hold when $n = 1$. In fact, there are sets $E \subset \mathbb{H}^1$ such that $\mathbf{e}(E, 0, r, \nu) = 0$ but ∂E is not flat in $C_{\varepsilon r}$ for any $\varepsilon > 0$, see the conclusion of [19, Proposition 3.7].

2.6. Intrinsic Lipschitz functions. We identify the vertical hyperplane

$$\mathbb{W} = \mathbb{H}^{n-1} \times \mathbb{R} = \{(z, t) \in \mathbb{H}^n : x_1 = 0\}$$

with \mathbb{R}^{2n} via the coordinates $w = (x_2, \dots, x_n, y_1, \dots, y_n, t)$. The line flow of the vector field X_1 starting from the point $(z, t) \in \mathbb{W}$ is the curve

$$(2.9) \quad \gamma(s) = \exp(sX_1)(z, t) = (z + se_1, t + 2y_1s), \quad s \in \mathbb{R},$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{H}^n$ and $z = (x, y) \in \mathbb{C}^n \cong \mathbb{R}^{2n}$.

Let $W \subset \mathbb{W}$ be a set and let $\varphi: W \rightarrow \mathbb{R}$ be a function. The set

$$(2.10) \quad E_\varphi = \left\{ \exp(sX_1)(w) \in \mathbb{H}^n : s > \varphi(w), w \in W \right\}$$

is called *intrinsic epigraph of φ* along X_1 , while the set

$$\text{gr}(\varphi) = \left\{ \exp(\varphi(w)X_1)(w) \in \mathbb{H}^n : w \in W \right\}$$

is called *intrinsic graph of φ* along X_1 . By (2.9), we easily find the identity

$$\exp(\varphi(w)X_1)(w) = w * \varphi(w)e_1 \quad \text{for any } w \in W,$$

thus the intrinsic graph of φ is the set $\text{gr}(\varphi) = \{w * \varphi(w)e_1 \in \mathbb{H}^n : w \in W\}$. The *graph map* of the function $\varphi: W \rightarrow \mathbb{R}$, $W \subset \mathbb{W}$, is the map $\Phi: W \rightarrow \mathbb{H}^n$, $\Phi(w) = w * \varphi(w)e_1$, $w \in W$. For any $A \subset W$, we let $\text{gr}(\varphi|_A) = \Phi(A)$.

The notion of intrinsic Lipschitz function was introduced in [13, Definition 3.1].

Definition 2.5 (Intrinsic Lipschitz function). Let $W \subset \mathbb{W}$. A function $\varphi: W \rightarrow \mathbb{R}$ is *L-intrinsic Lipschitz*, with $L \in [0, \infty)$, if for all $p, q \in \text{gr}(\varphi)$ we have

$$(2.11) \quad |\varphi(\pi(p)) - \varphi(\pi(q))| \leq L \|\pi(q^{-1} * p)\|_\infty.$$

The definition can be equivalently given in terms of intrinsic cones. We denote by $\text{Lip}_H(W)$ the set of intrinsic Lipschitz functions on the set $W \subset \mathbb{W}$. If $\varphi \in \text{Lip}_H(W)$, we denote by $\text{Lip}_H(\varphi)$ the intrinsic Lipschitz constant of φ , with no reference to the set if no confusion arises.

An extension theorem for intrinsic Lipschitz functions was proved for the first time in [15, Theorem 4.25]. The following result gives an explicit estimate of the Lipschitz constant of the extension. The first part is proved in [19, Proposition 4.8], while the second part follows from an easy modification of the proof of the first one.

Proposition 2.6. *Let $W \subset \mathbb{W}$ and let $\varphi: W \rightarrow \mathbb{R}$ be an L-intrinsic Lipschitz function. There exists an M-intrinsic Lipschitz function $\psi: \mathbb{W} \rightarrow \mathbb{R}$ with*

$$(2.12) \quad M = \left(\sqrt{1 + \frac{1}{L + 2L^2}} - 1 \right)^{-2}$$

such that $\psi(w) = \varphi(w)$ for all $w \in W$. If φ is bounded then there exists an extension that also satisfies $\|\psi\|_{L^\infty(\mathbb{W})} = \|\varphi\|_{L^\infty(W)}$.

Note that, in (2.12), we have $M \leq 2L$ for all $L \leq 0,07$.

We now introduce a non-linear gradient for functions $\varphi: W \rightarrow \mathbb{R}$ with $W \subset \mathbb{W}$ an open set. Let $\mathcal{B}: \text{Lip}_{loc}(W) \rightarrow L_{loc}^\infty(W)$ be the Burgers' operator defined by

$$\mathcal{B}\varphi = \frac{\partial\varphi}{\partial y_1} - 4\varphi \frac{\partial\varphi}{\partial t}.$$

When $\varphi \in C(W)$ is only continuous, we say that $\mathcal{B}\varphi$ exists in the sense of distributions and is represented by a locally bounded function if there exists a function $\vartheta \in L_{loc}^\infty(W)$ such that

$$\int_W \vartheta\psi \, dw = - \int_W \left\{ \varphi \frac{\partial\psi}{\partial y_1} - 2\varphi^2 \frac{\partial\psi}{\partial t} \right\} dw$$

for any $\psi \in C_c^1(W)$. In this case, we let $\mathcal{B}\varphi = \vartheta$.

Note that the vector fields $X_2, \dots, X_n, Y_2, \dots, Y_n$ can be naturally restricted to \mathbb{W} and that they are self-adjoint.

Let $\varphi: W \rightarrow \mathbb{R}$ be a continuous function on the open set $W \subset \mathbb{W}$. We say that the intrinsic gradient $\nabla^\varphi\varphi \in L_{loc}^\infty(W; \mathbb{R}^{2n-1})$ exists in the sense of distributions if the distributional derivatives $X_i\varphi$, $\mathcal{B}\varphi$ and $Y_i\varphi$, with $i = 2, \dots, n$, are represented by locally bounded functions in W . In this case, we let

$$(2.13) \quad \nabla^\varphi\varphi = (X_2\varphi, \dots, X_n\varphi, \mathcal{B}\varphi, Y_2\varphi, \dots, Y_n\varphi),$$

and we call $\nabla^\varphi\varphi$ the *intrinsic gradient* of φ . When $n = 1$, the intrinsic gradient reduces to $\nabla^\varphi\varphi = \mathcal{B}\varphi$.

The intrinsic gradient (2.13) has a strong non-linear character. This partially motivates the fact that $\text{Lip}_H(W)$ is not a vector space.

Theorem 2.7 (Area formula). *Let $W \subset \mathbb{W}$ be an open set and let $\varphi: W \rightarrow \mathbb{R}$ be a locally intrinsic Lipschitz function. Then the intrinsic epigraph $E_\varphi \subset \mathbb{H}^n$ has locally finite H -perimeter in the cylinder*

$$W * \mathbb{R} = \{w * se_1 \in \mathbb{H}^n : w \in W, s \in \mathbb{R}\},$$

and for \mathcal{L}^{2n} -a.e. $w \in W$ the inner horizontal normal to ∂E_φ is given by

$$(2.14) \quad \nu_{E_\varphi}(\Phi(w)) = \left(\frac{1}{\sqrt{1 + |\nabla^\varphi\varphi(w)|^2}}, \frac{-\nabla^\varphi\varphi(w)}{\sqrt{1 + |\nabla^\varphi\varphi(w)|^2}} \right).$$

Moreover, for any $W' \subset\subset W$, the following area formula holds:

$$(2.15) \quad P_H(E_\varphi; W' * \mathbb{R}) = \int_{W'} \sqrt{1 + |\nabla^\varphi\varphi(w)|^2} \, d\mathcal{L}^{2n}.$$

Formula (2.14) for the inner horizontal normal to ∂E_φ and the area formula (2.15) are proved in [8], respectively in Corollary 4.2 and in Theorem 1.6. The area formula (2.15) can be improved in the following way

$$(2.16) \quad \int_{\partial E_\varphi \cap W' * \mathbb{R}} g(p) \, d\mu_{E_\varphi} = \int_{W'} g(\Phi(w)) \sqrt{1 + |\nabla^\varphi\varphi(w)|^2} \, d\mathcal{L}^{2n},$$

where $g: \partial E_\varphi \rightarrow \mathbb{R}$ is a Borel function. To avoid long equations, in the following we often omit the variables and the flow map Φ when we apply the area formula (2.15) and its general version (2.16).

3. INTRINSIC LIPSCHITZ APPROXIMATION

In this section, we prove the following result, which contains Theorem 1.1 in the Introduction as a particular case.

Theorem 3.1. *Let $n \geq 2$. There exist positive dimensional constants $C_1(n)$, $\varepsilon_1(n)$ and $\delta_1(n)$ with the following property. If $E \subset \mathbb{H}^n$ is a (Λ, r_0) -minimizer of H -perimeter in C_{5124} with $\mathbf{e}(5124) \leq \varepsilon_1(n)$, $\Lambda r_0 \leq 1$, $r_0 > 5124$, and $0 \in \partial E$, then, letting*

$$M = C_1 \cap \partial E, \quad M_0 = \left\{ q \in M : \sup_{0 < s < 256} \mathbf{e}(q, s) \leq \delta_1(n) \right\},$$

there exists an intrinsic Lipschitz function $\varphi: \mathbb{W} \rightarrow \mathbb{R}$ such that

$$(3.1) \quad \sup_{\mathbb{W}} |\varphi| \leq C_1(n) \mathbf{e}(5124)^{\frac{1}{2(2n+1)}}, \quad \text{Lip}_H(\varphi) \leq 1,$$

$$(3.2) \quad M_0 \subset M \cap \Gamma, \quad \Gamma = \text{gr}(\varphi|_{D_1}),$$

$$(3.3) \quad \mathcal{J}^{2n+1}(M \triangle \Gamma) \leq C_1(n) \mathbf{e}(5124),$$

$$(3.4) \quad \int_{D_1} |\nabla^\varphi \varphi|^2 d\mathcal{L}^{2n} \leq C_1(n) \mathbf{e}(5124).$$

Proof. The proof is divided in three steps.

Step 1: construction of φ . Let $\varepsilon_0(n)$ and $C_0(n)$ be the constants given in Theorem 2.3. Then we have

$$(3.5) \quad \sup \left\{ |\mathcal{H}_2(p)| : p \in C_1 \cap \partial E \right\} \leq C_0(n) \mathbf{e}(16)^{\frac{1}{2(2n+1)}},$$

provided that $\mathbf{e}(16) \leq \varepsilon_0(n)$; this follows from the elementary properties of the excess with $\varepsilon_1(n) \leq \varepsilon_0(n)$ suitably small.

Let $q \in M_0$ and $p \in M$ be fixed. Then $p, q \in C_1$, so $d_C(p, q) < 8$ by (2.7), where d_C is the quasi-distance induced by the quasi norm $\|\cdot\|_C$ defined in (2.5). We consider the blow-up of E at scale $d_C(p, q)$ centered in q , that is, $F = E_{q, d_C(p, q)} = \delta_{1/r}(\tau_{q^{-1}} E)$ with $r = d_C(p, q)$. By Remark 2.1, F is a (Λ', r'_0) -perimeter minimizer in $(C_{5124})_{q, d_C(p, q)}$, with

$$\Lambda' = \Lambda d_C(p, q), \quad r'_0 = \frac{r_0}{d_C(p, q)} > 1.$$

Since

$$C_{16} \subset (C_{5124})_{q, d_C(p, q)}, \quad \Lambda' r'_0 \leq 1, \quad 0 \in \partial F$$

and, by the scaling property of the excess and by definition of M_0 ,

$$\mathbf{e}(F, 0, 16, \nu) = \mathbf{e}(E, q, 16d_C(p, q), \nu) \leq \delta_1(n),$$

then, provided that $\delta_1(n) \leq \varepsilon_0(n)$, by Theorem 2.3 we have

$$\sup \left\{ |\mathcal{H}_2(w)| : w \in C_1 \cap \partial F \right\} \leq C_0(n) \delta_1(n)^{\frac{1}{2(2n+1)}}.$$

In particular, choosing

$$w = \frac{1}{d_C(p, q)} q^{-1} * p \in C_1 \cap \partial F,$$

we get

$$(3.6) \quad |\mathcal{H}_2(q^{-1} * p)| \leq C_0(n) \delta_1(n)^{\frac{1}{2(2n+1)}} d_C(p, q).$$

We now set

$$(3.7) \quad L(n) := C_0(n) \delta_1(n)^{\frac{1}{2(2n+1)}}$$

and we choose $\delta_1(n)$ so small that $L(n) < 1$. Then, by (3.6), we conclude that $d_C(p, q) = \|\pi(q^{-1} * p)\|_\infty$ and we get

$$(3.8) \quad |\mathfrak{f}(q^{-1} * p)| \leq L(n) \|\pi(q^{-1} * p)\|_\infty \quad \text{for all } p \in M, q \in M_0.$$

In particular, (3.8) proves that the projection π is invertible on M_0 . Therefore, we can define a function $\varphi: \pi(M_0) \rightarrow \mathbb{R}$ setting $\varphi(\pi(p)) = \mathfrak{f}(p)$ for all $p \in M_0$. From (3.8), we deduce that

$$|\varphi(\pi(p)) - \varphi(\pi(q))| \leq L(n) \|\pi(q^{-1} * p)\|_\infty \quad \text{for all } p, q \in M_0,$$

so that φ is an intrinsic Lipschitz function on $\pi(M_0)$ with $\text{Lip}_H(\varphi, \pi(M_0)) \leq L(n) < 1$ by (2.11). Since $M_0 \subset M$, by (3.5) we also have

$$|\varphi(\pi(p))| \leq C_0(n) \mathbf{e}(16)^{\frac{1}{2(2n+1)}} \quad \text{for all } p \in M_0.$$

Therefore, by Proposition 2.6, possibly choosing $\delta_1(n)$ smaller accordingly to (2.12), we can extend φ from $\pi(M_0)$ to the whole \mathbb{W} with $\text{Lip}_H(\varphi, \mathbb{W}) \leq L(n) < 1$ in such a way that

$$M_0 \subset M \cap \Gamma, \quad \Gamma = \text{gr}(\varphi|_{D_1}), \quad \text{and} \quad |\varphi(w)| \leq C_0(n) \mathbf{e}(16)^{\frac{1}{2(2n+1)}} \quad \text{for all } w \in \mathbb{W}.$$

We thus proved (3.1) and (3.2) for a suitable $C_1(n) \geq C_0(n)$.

Step 2: covering argument. We now prove (3.3) via a covering argument. By definition of M_0 , for every $q \in M \setminus M_0$ there exists $s = s(q) \in (0, 256)$ such that

$$(3.9) \quad \int_{C_s(q) \cap \partial E} \frac{|\nu_E - \nu|_g^2}{2} d\mathcal{S}^{2n+1} > \frac{\delta_1(n)}{\delta(n)} s^{2n+1},$$

with $\delta(n) = \frac{2\omega_{2n-1}}{\omega_{2n+1}}$ and $\nu = -X_1$ as in Definition 2.2. The family of balls

$$\{B_{2s}(q) : q \in M \setminus M_0, s = s(q)\}$$

is a covering of $M \setminus M_0$. By the 5r-covering Lemma, there exist a sequence of points $q_h \in M \setminus M_0$ and a sequence of radii $s_h = s(q_h)$, $h \in \mathbb{N}$, with q_h and s_h satisfying (3.9), such that the balls $B_{2s_h}(q_h)$ are pairwise disjoint and

$$\{B_{10s_h}(q_h) : h \in \mathbb{N}\}$$

is still a covering of $M \setminus M_0$. Note that $B_{10s_h}(q_h) \subset C_{5124}$, because if $p \in B_{10s_h}(q_h)$ then, by (2.6),

$$\|p\|_C \leq 2\|p\|_\infty \leq 2d_\infty(p, q_h) + 2\|q_h\|_\infty < 20s_h + 4\|q_h\|_C < 5124.$$

Therefore, by the density estimates in [21, Theorem 4.1], we get

$$\begin{aligned} \mathcal{S}^{2n+1}(M \setminus M_0) &\leq \sum_{h \in \mathbb{N}} \mathcal{S}^{2n+1}((M \setminus M_0) \cap B_{10s_h}(q_h)) \\ &\leq \sum_{h \in \mathbb{N}} \mathcal{S}^{2n+1}(M \cap B_{10s_h}(q_h)) \\ &\leq C(n) \sum_{h \in \mathbb{N}} s_h^{2n+1}, \end{aligned}$$

where $C(n)$ is a positive dimensional constant. Since $C_{s_h}(q_h) \subset B_{2s_h}(q_h)$ by (2.7), the cylinders $C_{s_h}(q_h)$ are pairwise disjoint and contained in C_{5124} , so we have

$$(3.10) \quad \mathcal{S}^{2n+1}(M \setminus M_0) \leq C(n) \sum_{h \in \mathbb{N}} \int_{C_{s_h}(q_h) \cap \partial E} \frac{|\nu_E - \nu|_g^2}{2} d\mathcal{S}^{2n+1} \leq C(n) \mathbf{e}(5124),$$

where $C(n)$ is a new positive dimensional constant. Therefore, since $M \setminus \Gamma \subset M \setminus M_0$, by (3.10) it follows that

$$(3.11) \quad \mathcal{S}^{2n+1}(M \setminus \Gamma) \leq C(n) \mathbf{e}(5124),$$

which is the first half of (3.3).

We now bound the second half of (3.3). We choose $\varepsilon_1(n)$ so small that

$$\mathbf{e}(2) \leq \omega(n, \frac{1}{2}, \frac{1}{5124}, 5124),$$

where $\omega(n, t, \Lambda, r_0)$, with $t \in (0, 1)$, is the constant given in [21, Lemma 3.3]. This is possible by the scaling property of the excess. Then, by (3.57) in [21, Lemma 3.4], we have

$$\mathcal{L}^{2n}(G) \leq \mathcal{S}^{2n+1}(M \cap \pi^{-1}(G))$$

for any Borel set $G \subset D_1$. Therefore, by the area formula (2.15) in Theorem 2.7, we can estimate

$$(3.12) \quad \begin{aligned} \delta(n) \mathcal{S}^{2n+1}(\Gamma \setminus M) &= \int_{\pi(\Gamma \setminus M)} \sqrt{1 + |\nabla^\varphi \varphi(w)|^2} d\mathcal{L}^{2n} \\ &\leq \sqrt{1 + \|\nabla^\varphi \varphi\|_{L^\infty(D_1)}^2} \mathcal{L}^{2n}(\pi(\Gamma \setminus M)) \\ &\leq \sqrt{1 + \|\nabla^\varphi \varphi\|_{L^\infty(D_1)}^2} \mathcal{S}^{2n+1}(M \cap \pi^{-1}(\pi(\Gamma \setminus M))). \end{aligned}$$

Since φ is intrinsic Lipschitz on D_1 with $\text{Lip}_H(\varphi) < 1$ by construction, by [8, Proposition 4.4] there exists a positive dimensional constant $C(n)$ such that

$$(3.13) \quad \|\nabla^\varphi \varphi\|_{L^\infty(D_1)} \leq C(n) \text{Lip}_H(\varphi) (\text{Lip}_H(\varphi) + 1) < 2C(n).$$

Thus, by (3.12) and (3.13), there exists a positive dimensional constant $C(n)$ such that

$$(3.14) \quad \mathcal{S}^{2n+1}(\Gamma \setminus M) \leq C(n) \mathcal{S}^{2n+1}(M \cap \pi^{-1}(\pi(\Gamma \setminus M))).$$

Since we have

$$M \cap \pi^{-1}(\pi(\Gamma \setminus M)) \subset M \setminus \Gamma,$$

by (3.11) and (3.14) we conclude that, for some positive dimensional constant $C'(n)$,

$$(3.15) \quad \mathcal{S}^{2n+1}(\Gamma \setminus M) \leq C(n) \mathcal{S}^{2n+1}(M \setminus \Gamma) \leq C'(n) \mathbf{e}(5124),$$

which is the second half of (3.3). Combining (3.11) and (3.15), we prove (3.3).

Step 3: L^2 -estimate. Finally, we prove (3.4). We first notice that, by Theorem 2.7 and [4, Corollary 2.6], for \mathcal{S}^{2n+1} -a.e. $p \in M \cap \Gamma$ there exists $\lambda(p) \in \{-1, 1\}$ such that

$$(3.16) \quad \nu_E(p) = \lambda(p) \frac{(1, -\nabla^\varphi \varphi(\pi(p)))}{\sqrt{1 + |\nabla^\varphi \varphi(\pi(p))|^2}}.$$

Taking into account that, for \mathcal{S}^{2n+1} -a.e. $p \in M \cap \Gamma$,

$$(3.17) \quad \frac{|\nu_E(p) - \nu(p)|_g^2}{2} = 1 - \langle \nu_E(p), \nu(p) \rangle_g \geq \frac{1 - \langle \nu_E(p), \nu(p) \rangle_g^2}{2},$$

by (3.16) and by the area formula (2.16) we find that

$$\begin{aligned} \mathbf{e}(1) &\geq \int_{M \cap \Gamma} \frac{1 - \langle \nu_E(p), \nu(p) \rangle_g^2}{2} d\mu_E \\ &= \frac{1}{2} \int_{M \cap \Gamma} \frac{|\nabla^\varphi \varphi(\pi(p))|^2}{1 + |\nabla^\varphi \varphi(\pi(p))|^2} d\mu_E \\ &= \frac{1}{2} \int_{\pi(M \cap \Gamma)} \frac{|\nabla^\varphi \varphi(w)|^2}{\sqrt{1 + |\nabla^\varphi \varphi(w)|^2}} d\mathcal{L}^{2n}. \end{aligned}$$

Recalling (3.13) and the scaling property of the excess, we conclude that there exists a positive dimensional constant $C(n)$ such that

$$(3.18) \quad \int_{\pi(M \cap \Gamma)} |\nabla^\varphi \varphi(w)|^2 dw \leq C(n) \mathbf{e}(5124).$$

Moreover, again by the area formula (2.16), there exists a positive dimensional constant $C(n)$ such that

$$\begin{aligned} \int_{\pi(M \Delta \Gamma)} |\nabla^\varphi \varphi(w)|^2 d\mathcal{L}^{2n} &= \int_{M \Delta \Gamma} \frac{|\nabla^\varphi \varphi(\pi(p))|^2}{\sqrt{1 + |\nabla^\varphi \varphi(\pi(p))|^2}} d\mu_E \\ &\leq C(n) \|\nabla^\varphi \varphi\|_{L^\infty(D_1)}^2 \mathcal{S}^{2n+1}(M \Delta \Gamma). \end{aligned}$$

By (3.13) and (3.3), we find a positive dimensional constant $C(n)$ such that

$$(3.19) \quad \int_{\pi(M \Delta \Gamma)} \|\nabla^\varphi \varphi(w)\|^2 dw \leq C(n) \mathbf{e}(5124).$$

Combining (3.18) and (3.19), we prove (3.4). \square

Remark 3.2 (σ -representative). Let $0 < \sigma \leq 1$ and $I = (-1, 1)$. We let $\mathcal{A}(\sigma)$ be the family of sets $A \subseteq D_\sigma$ such that

$$|\mathfrak{h}(q^{-1} * p)| \leq L(n) \|\pi(q^{-1} * p)\|_\infty \quad \text{for all } p \in M \cap D_\sigma * I, q \in M \cap A * I,$$

where $L(n)$ is the dimensional constant in (3.7). The family $\mathcal{A}(\sigma)$ is partially ordered by inclusion and is closed under union. Thus $\mathcal{A}(\sigma)$ has a unique maximal element A_σ^* . Then, by (3.8), we have that

$$|\mathfrak{h}(q^{-1} * p)| \leq L(n) \|\pi(q^{-1} * p)\|_\infty \quad \text{for all } p, q \in M_0 \cup (M \cap A_\sigma^* * I).$$

Therefore, in Step 1 of the proof of Theorem 3.1, it is not restrictive to assume that the intrinsic Lipschitz approximation $\varphi: \mathbb{W} \rightarrow \mathbb{R}$ is defined in such a way that

$$\varphi(\pi(p)) = \mathfrak{h}(p) \quad \text{for all } p \in M_0 \cup (M \cap A_\sigma^* * I).$$

We define such an intrinsic Lipschitz function a σ -representative of Theorem 3.1. Moreover, if Theorem 3.1 is applied with a scaling factor $\lambda > 0$, then we have $0 < \sigma \leq \lambda$, $I = (-\lambda, \lambda)$ and we can define in the same way the family $\mathcal{A}(\sigma, \lambda)$, its maximal element $A_{\sigma, \lambda}^*$ and a (σ, λ) -representative of Theorem 3.1.

4. LOCAL MAXIMAL FUNCTIONS

In this section, we prove some lemmas on maximal functions of measures that are used in the proof of Theorem 1.2.

4.1. Maximal function on disks. Given $s > 0$ and a non-negative measure μ on $D_{4s} \subset \mathbb{W}$, the *local maximal function* of μ is defined as

$$(4.1) \quad M\mu(x) := \sup_{0 < r < 4s - \|x\|_\infty} \frac{\mu(D_r(x))}{\kappa_n r^{2n+1}} \quad \text{for } x \in D_{4s},$$

where $\kappa_n = \mathcal{L}^{2n}(D_1)$ as in (2.3).

Lemma 4.1. *Let $s > 0$ and let $\mu: D_{4s} \rightarrow [0, \infty)$ be as above. Assume that $\theta > 0$ is such that*

$$(4.2) \quad \mu(D_{4s}) \leq \frac{\theta}{5^{2n+1}} \kappa_n s^{2n+1}$$

and define

$$J_\theta = \{x \in D_{4s} : M\mu(x) > \theta\}.$$

Then for all $r \leq 3s$ we have

$$(4.3) \quad \mathcal{L}^{2n}(J_\theta \cap D_r) \leq \frac{5^{2n+1}}{\theta} \mu(J_{\theta/2^{2n+1}} \cap D_{r+\frac{s}{5}}).$$

Proof. Let $r \leq 3s$ be fixed. If $x \in J_\theta \cap D_r$, then there exists $r_x > 0$ such that

$$\mu(D_{r_x}(x)) > \theta \kappa_n r_x^{2n+1}.$$

By the 5r-covering Lemma applied to the family $\{D_{r_x}(x) : x \in J_\theta \cap D_r\}$, we find a sequence of pairwise disjoint balls $\{D_{r_i}(x_i)\}_{i \in \mathbb{N}}$, with $x_i \in J_\theta \cap D_r$ and $r_i > 0$, such that

$$J_\theta \cap D_r \subset \bigcup_{x \in J_\theta \cap D_r} D_{r_x}(x) \subset \bigcup_{i \in \mathbb{N}} D_{5r_i}(x_i), \quad \mu(D_{r_i}(x_i)) > \theta \kappa_n r_i^{2n+1}.$$

In particular, by (4.2), we get

$$r_i < \sqrt[2n+1]{\frac{\mu(D_{r_i}(x_i))}{\theta \kappa_n}} \leq \sqrt[2n+1]{\frac{\mu(D_{4s})}{\theta \kappa_n}} \leq \frac{s}{5},$$

and so, for any $i \in \mathbb{N}$, we have

$$D_{r_i}(x_i) \subset D_{\|x_i\|_\infty + r_i} \subset D_{r+\frac{s}{5}}.$$

We claim that

$$D_{r_i}(x_i) \subset J_{\theta/2^{2n+1}} \cap D_{r+\frac{s}{5}}$$

for any $i \in \mathbb{N}$. Indeed, by contradiction assume that there exists $y \in D_{r_i}(x_i)$ such that $M\mu(y) \leq \frac{\theta}{2^{2n+1}}$. Then $D_{r_i}(x_i) \subset D_{2r_i}(y)$ and

$$4s - \|y\|_\infty \geq 4s - r - \frac{s}{5} \geq 4s - 3s - \frac{s}{5} = \frac{4}{5}s > 2r_i.$$

Hence, we have

$$\begin{aligned} \frac{\theta}{2^{2n+1}} \geq M\mu(y) &= \sup_{0 < \delta < 4s - \|y\|_\infty} \frac{\mu(D_\delta(y))}{\kappa_n \delta^{2n+1}} \\ &\geq \sup_{2r_i < \delta < 4s - \|y\|_\infty} \frac{\mu(D_\delta(y))}{\kappa_n \delta^{2n+1}} \\ &\geq \sup_{2r_i < \delta < 4s - \|y\|_\infty} \frac{\mu(D_{r_i}(x_i))}{\kappa_n \delta^{2n+1}} = \frac{\mu(D_{r_i}(x_i))}{\kappa_n (2r_i)^{2n+1}} > \frac{\theta}{2^{2n+1}}, \end{aligned}$$

a contradiction.

We can finally estimate:

$$\begin{aligned} \mathcal{L}^{2n}(J_\theta \cap D_r) &\leq \sum_{i \in \mathbb{N}} \mathcal{L}^{2n}(D_{5r_i}(x_i)) = 5^{2n+1} \kappa_n \sum_{i \in \mathbb{N}} r_i^{2n+1} \leq \frac{5^{2n+1}}{\theta} \sum_{i \in \mathbb{N}} \mu(D_{r_i}(x_i)) \\ &= \frac{5^{2n+1}}{\theta} \mu\left(\bigcup_{i \in \mathbb{N}} D_{r_i}(x_i)\right) \leq \frac{5^{2n+1}}{\theta} \mu(J_{\theta/2^{2n+1}} \cap D_{r+\frac{s}{5}}), \end{aligned}$$

and (4.3) follows. \square

4.2. Maximal function on φ -balls. We recall the Poincaré inequality for intrinsic Lipschitz functions. The notion of intrinsic Lipschitz function can be equivalently restated on bounded open sets introducing a suitable notion of graph distance, see [8, Definition 1.1] or [9]. Let $W \subset \mathbb{W}$ be set and let $\varphi: W \rightarrow \mathbb{R}$ be a function. The map $d_\varphi: W \times W \rightarrow [0, \infty)$ given by

$$(4.4) \quad d_\varphi(w, w') = \frac{1}{2} \left(\left\| \pi(\Phi(w)^{-1} * \Phi(w')) \right\|_\infty + \left\| \pi(\Phi(w')^{-1} * \Phi(w)) \right\|_\infty \right)$$

for any $w, w' \in W$, where $\Phi(w) = w * \varphi(w) e_1$ for all $w \in W$, is the *graph distance* induced by φ .

Comparing (2.11) with (4.4), it is easy to see that, if $W \subset \mathbb{W}$ is a bounded open set and $\varphi: W \rightarrow \mathbb{R}$ is a continuous function, then φ is an intrinsic L -intrinsic Lipschitz function if and only if

$$|\varphi(w) - \varphi(w')| \leq L d_\varphi(w, w'), \quad w, w' \in W.$$

If φ is an intrinsic L -Lipschitz function on W , then d_φ turns out to be a quasi-distance on W , that is, $d_\varphi(x, y) = 0$ if and only if $x = y$ for all $x, y \in W$, d_φ is symmetric and, for all $x, y, z \in W$,

$$(4.5) \quad d_\varphi(x, y) \leq c_L (d_\varphi(x, z) + d_\varphi(z, y)),$$

where $c_L \geq 1$ depends only on L and

$$(4.6) \quad \lim_{L \rightarrow 0} c_L = 1,$$

see [8, Section 3].

The following Poincaré inequality is proved in [9], see Theorem 1.2 and also Corollary 1.3 therein for the case $p = 1$.

Theorem 4.2 (Poincaré inequality). *Let $W \subset \mathbb{W} \subset \mathbb{H}^n$, $n \geq 2$, be a bounded open set and let $1 \leq p < \infty$. Then there exist two constants $C_1^L, C_2^L > 0$ with $C_2^L > 1$, depending on $L > 0$, such that for any L -intrinsic Lipschitz function $\varphi: W \rightarrow \mathbb{R}$ we have*

$$(4.7) \quad \int_{U_\varphi(x,r)} |\varphi - (\varphi)_{x,r}|^p d\mathcal{L}^{2n} \leq C_1^L r^p \int_{U_\varphi(x, C_2^L r)} |\nabla^\varphi \varphi|^p d\mathcal{L}^{2n}$$

for every $U_\varphi(x, C_2^L r) \subset W$, where

$$(4.8) \quad U_\varphi(x, r) = \{y \in W : d_\varphi(x, y) < r\}$$

and

$$(\varphi)_{x,r} = \int_{U_\varphi(x,r)} \varphi d\mathcal{L}^{2n} = \frac{1}{\mathcal{L}^{2n}(U_\varphi(x,r))} \int_{U_\varphi(x,r)} \varphi d\mathcal{L}^{2n}.$$

For future convenience, we define

$$(4.9) \quad \gamma_2(n) = \lim_{L \rightarrow 0} C_2^L \geq 1.$$

The \mathcal{L}^{2n} -measure of the ball $U_\varphi(x, r)$ defined in (4.8) is comparable to r^{2n+1} . Namely, there exist two constants $c_1^L, c_2^L > 0$ depending on L such that, for all $U_\varphi(x, r) \subset W$, we have

$$(4.10) \quad c_1^L \leq \frac{\mathcal{L}^{2n}(U_\varphi(x, r))}{r^{2n+1}} \leq c_2^L,$$

see [9, Section 2.3] and the references therein.

We can now introduce the *local φ -maximal function*. Let $n \geq 2$, $s > 0$, and let $\varphi: \mathbb{W} \rightarrow \mathbb{R}$ be an L -intrinsic Lipschitz function. By (4.6) and by (4.9), there exists a dimensional constant $\ell(n) > 0$ such that

$$(4.11) \quad L \in [0, \ell(n)] \implies c_L \leq 2 \text{ and } C_2^L \leq 2\gamma_2(n),$$

where c_L is as in (4.5) and C_2^L is as in Theorem 4.2. For all $L \in [0, \ell(n)]$, we define the *local φ -maximal function* of μ_φ as

$$(4.12) \quad [\mu_\varphi](x) := \sup_{0 < r < r_\varphi(x,s)} \frac{\mu_\varphi(U_\varphi(x, r))}{\mathcal{L}^{2n}(U_\varphi(x, r))}, \quad x \in U_\varphi(0, s),$$

where we set

$$(4.13) \quad r_\varphi(x, s) = \frac{\rho(n)}{c_L} s - d_\varphi(x, 0), \quad x \in U_\varphi(0, s),$$

the dimensional constant is

$$(4.14) \quad \rho(n) = 64\gamma_2(n) + 2,$$

and the non-negative measure μ_φ on $U_\varphi(0, \rho(n)s)$ is given by

$$d\mu_\varphi = |\nabla^\varphi \varphi| d\mathcal{L}^{2n}.$$

The maximal function introduced in (4.12) is well-defined, since

$$x \in U_\varphi(0, s), r < r_\varphi(x, s) \implies U_\varphi(x, r) \subset U_\varphi(0, \rho(n)s),$$

by the quasi-triangular inequality (4.5).

We use the Poincaré inequality (4.7) to prove the following result on $[\mu_\varphi]$.

Lemma 4.3. *Let $n \geq 2$, $s > 0$, $\varphi: \mathbb{W} \rightarrow \mathbb{R}$, μ_φ , $[\mu_\varphi]$, $L \in [0, \ell(n)]$ be as above. Let $\theta > 0$ and define*

$$(4.15) \quad J_\theta^\varphi = \left\{ x \in U_\varphi(0, s) : [\mu_\varphi](x) > \theta \right\}.$$

Then there exists a constant $C = C(n, L)$ such that for all $x, y \in U_\varphi(0, s) \setminus J_\theta^\varphi$ we have

$$(4.16) \quad |\varphi(x) - \varphi(y)| \leq C\theta d_\varphi(x, y).$$

Proof. Let $x \in U_\varphi(0, s) \setminus J_\theta^\varphi$ and let $C_2^L r < r_\varphi(x, s)$. Then, by Theorem 4.2 with $p = 1$, we have

$$\int_{U_\varphi(x, r)} |\varphi - (\varphi)_{x, r}| d\mathcal{L}^{2n} \leq C_1^L r \int_{U_\varphi(x, C_2^L r)} |\nabla^\varphi \varphi| d\mathcal{L}^{2n} = C_1^L r \mu_\varphi(U_\varphi(x, C_2^L r)).$$

By (4.12) and by (4.15), we have

$$\mu_\varphi(U_\varphi(x, C_2^L r)) \leq \theta \mathcal{L}^{2n}(U_\varphi(x, C_2^L r)).$$

Therefore, by (4.10), we have

$$\int_{U_\varphi(x, r)} |\varphi - (\varphi)_{x, r}| d\mathcal{L}^{2n} \leq C_1^L r \theta c_2^L (C_2^L r)^{2n+1},$$

and so, again by (4.10), we get

$$\int_{U_\varphi(x, r)} |\varphi - (\varphi)_{x, r}| d\mathcal{L}^{2n} \leq \frac{c_2^L}{c_1^L} C_1^L (C_2^L)^{2n+1} \theta r,$$

for all $x \in U_\varphi(0, s) \setminus J_\theta^\varphi$ and $C_2^L r < r_\varphi(x, s)$.

In particular, for all $j = 0, 1, 2, \dots$, we have

$$\begin{aligned} |(\varphi)_{x, \frac{r}{2^{j+1}}} - (\varphi)_{x, \frac{r}{2^j}}| &\leq \int_{U_\varphi(x, \frac{r}{2^{j+1}})} |\varphi(u) - (\varphi)_{x, \frac{r}{2^j}}| d\mathcal{L}^{2n}(u) \\ &\leq 2^{2n+1} \frac{c_2^L}{c_1^L} \int_{U_\varphi(x, \frac{r}{2^j})} |\varphi(u) - (\varphi)_{x, \frac{r}{2^j}}| d\mathcal{L}^{2n}(u) \\ &\leq \frac{2^{2n+1}}{2^j} \left(\frac{c_2^L}{c_1^L} \right)^2 C_1^L (C_2^L)^{2n+1} \theta r. \end{aligned}$$

Since φ is continuous, we get

$$|\varphi(x) - (\varphi)_{x, r}| \leq \sum_{j=0}^{\infty} |(\varphi)_{x, \frac{r}{2^{j+1}}} - (\varphi)_{x, \frac{r}{2^j}}| \leq 2^{2n+2} \left(\frac{c_2^L}{c_1^L} \right)^2 C_1^L (C_2^L)^{2n+1} \theta r,$$

for all $x \in U_\varphi(0, s) \setminus J_\theta^\varphi$ and $C_2^L r < r_\varphi(x, s)$.

Finally, let $x, y \in U_\varphi(0, s) \setminus J_\theta^\varphi$, $r = d_\varphi(x, y)$ and $c_3^L = 2c_L$. Then, by the quasi-triangular inequality (4.5), we have

$$U_\varphi(x, r) \cup U_\varphi(y, r) \subset U_\varphi(x, c_3^L r) \cap U_\varphi(y, c_3^L r).$$

Notice that, again by (4.5), we have

$$x, y \in U_\varphi(0, s), r = d_\varphi(x, y) \implies U_\varphi(x, c_3^L r) \cup U_\varphi(y, c_3^L r) \subset U_\varphi(0, \rho(n)s),$$

because, by (4.11) and (4.14),

$$c_L(2c_L c_3^L + 1) = c_L(4c_L^2 + 1) \leq \rho(n).$$

Therefore we obtain

$$\begin{aligned} |(\varphi)_{x,c_3^L r} - (\varphi)_{y,c_3^L r}| &\leq \int_{U_\varphi(x,c_3^L r) \cap U_\varphi(y,c_3^L r)} |\varphi(u) - (\varphi)_{x,c_3^L r}| + |\varphi(u) - (\varphi)_{y,c_3^L r}| d\mathcal{L}^{2n}(u) \\ &\leq \frac{c_2^L}{c_1^L} (c_3^L)^{2n+1} \left(\int_{U_\varphi(x,c_3^L r)} |\varphi(u) - (\varphi)_{x,c_3^L r}| d\mathcal{L}^{2n}(u) + \right. \\ &\quad \left. + \int_{U_\varphi(y,c_3^L r)} |\varphi(u) - (\varphi)_{y,c_3^L r}| d\mathcal{L}^{2n}(u) \right). \end{aligned}$$

Since $x, y \in U_\varphi(0, s) \setminus J_\theta^\varphi$, by (4.12) and by (4.15) we have

$$\mu_\varphi(U_\varphi(x, c_3^L C_2^L r)) \leq \theta \mathcal{L}^{2n}(U_\varphi(x, c_3^L C_2^L r))$$

and, analogously,

$$\mu_\varphi(U_\varphi(y, c_3^L C_2^L r)) \leq \theta \mathcal{L}^{2n}(U_\varphi(y, c_3^L C_2^L r)),$$

provided that

$$c_3^L C_2^L d_\varphi(x, y) < \min\{r_\varphi(x, s), r_\varphi(y, s)\}.$$

By (4.11), since $x, y \in U_\varphi(0, s)$, we have

$$\min\{r_\varphi(x, s), r_\varphi(y, s)\} > \frac{\rho(n)s}{c_L} - s \geq \left(\frac{\rho(n)}{2} - 1\right)s$$

and

$$c_3^L C_2^L d_\varphi(x, y) < 4c_L^2 C_2^L s \leq 32\gamma_2(n)s,$$

so it is enough to check that

$$32\gamma_2(n) \leq \frac{\rho(n)}{2} - 1,$$

but this is true thanks to the definition of $\rho(n)$ in (4.14).

We can now conclude the proof. Let $x, y \in U_\varphi(0, s) \setminus J_\theta^\varphi$ and $r = d_\varphi(x, y)$. Then we have

$$\begin{aligned} |\varphi(x) - \varphi(y)| &\leq |\varphi(x) - (\varphi)_{x,c_3^L r}| + |(\varphi)_{x,c_3^L r} - (\varphi)_{y,c_3^L r}| + |(\varphi)_{y,c_3^L r} - \varphi(y)| \\ &\leq \left(2(c_3^L)^{2n+2} + 2^{2n+3}c_3^L\right) \left(\frac{c_2^L}{c_1^L}\right)^2 C_1^L (C_2^L)^{2n+1} \theta r \\ &= C(n, L)\theta d_\varphi(x, y) \end{aligned}$$

and (4.16) follows. \square

5. APPROXIMATION VIA MAXIMAL FUNCTIONS

In this section, we develop the ideas contained in [11, Appendix A] to prove the following result. In the proof, we use Theorem 3.1 with a suitable scaling factor.

Theorem 5.1. *Let $n \geq 2$ and $\alpha \in (0, \frac{1}{2})$. There exist positive constants $C_2(n)$, $\varepsilon_2(\alpha, n)$ and $k_2 = k_2(n)$ with the following property. For any set $E \subset \mathbb{H}^n$ that is a (Λ, r_0) -minimizer of H -perimeter in C_{k_2} with $\mathbf{e}(k_2) \leq \varepsilon_2(\alpha, n)$, $\Lambda r_0 \leq 1$, $r_0 > k_2$ and $0 \in \partial E$, there exist a function $\varphi: \mathbb{W} \rightarrow \mathbb{R}$ and a set $K \subset D_1$ such that*

$$(5.1) \quad \mathcal{L}^{2n}(D_1 \setminus K) \leq C_2(n) \mathbf{e}(k_2)^{1-2\alpha},$$

$$(5.2) \quad \text{gr}(\varphi|_K) = \partial E \cap (K * (-1, 1)),$$

$$(5.3) \quad \text{Lip}_H(\varphi|_K) \leq C_2(n) \mathbf{e}(k_2)^\alpha.$$

We need some preliminaries. The following result is an easy consequence of Cauchy–Schwarz inequality.

Lemma 5.2. *Let $W \subset \mathbb{W}$ be an open set and let $\varphi: W \rightarrow \mathbb{R}$ be an L -intrinsic Lipschitz function. For any Borel set $A \subset\subset W$, we have*

$$(5.4) \quad \left(\int_A |\nabla^\varphi \varphi| \, d\mathcal{L}^{2n} \right)^2 \leq \sqrt{1 + \|\nabla^\varphi \varphi\|_{L^\infty(W)}^2} \mathcal{L}^{2n}(A) \int_{\text{gr}(\varphi|_A)} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} \, d\mu_{E_\varphi}.$$

Proof. Let $A \subset\subset W$ be fixed. Then, by the area formula (2.16),

$$\begin{aligned} \int_A |\nabla^\varphi \varphi| \, d\mathcal{L}^{2n} &= \int_{\text{gr}(\varphi|_A)} \frac{|\nabla^\varphi \varphi|}{\sqrt{1 + |\nabla^\varphi \varphi|^2}} \, d\mu_{E_\varphi} \\ &\leq \left(\int_{\text{gr}(\varphi|_A)} d\mu_{E_\varphi} \right)^{\frac{1}{2}} \left(\int_{\text{gr}(\varphi|_A)} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} \, d\mu_{E_\varphi} \right)^{\frac{1}{2}} \\ &= \left(\int_A \sqrt{1 + |\nabla^\varphi \varphi|^2} \, d\mathcal{L}^{2n} \right)^{\frac{1}{2}} \left(\int_{\text{gr}(\varphi|_A)} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} \, d\mu_{E_\varphi} \right)^{\frac{1}{2}} \\ &\leq \sqrt[4]{1 + \|\nabla^\varphi \varphi\|_{L^\infty(W)}^2} \mathcal{L}^{2n}(A)^{\frac{1}{2}} \left(\int_{\text{gr}(\varphi|_A)} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} \, d\mu_{E_\varphi} \right)^{\frac{1}{2}} \end{aligned}$$

and (5.4) follows squaring both sides. \square

The following lemma compares the distance d_φ with the distance of points of the graph of an intrinsic Lipschitz function φ .

Lemma 5.3. *Let $W \subset \mathbb{W}$ be an open set and let $\varphi: W \rightarrow \mathbb{R}$ be an intrinsic Lipschitz function. Then, for all $x \in W$, $r > 0$ and $0 < C < 1/(1 + \text{Lip}_H(\varphi))$, we have*

$$(5.5) \quad U_\varphi(x, Cr) \subset \pi(B_r(\Phi(x)) \cap \text{gr}(\varphi)) \subset U_\varphi(x, r),$$

where $U_\varphi(x, r)$ is as in (4.8) and $\Phi(x) = x * \varphi(x)\mathbf{e}_1$.

For the proof, see [8, Proposition 3.6].

Finally, the following result compares the distance d_φ with the distance d_∞ in W . Its proof easily follows from the definition of d_φ in (4.4) and is left to the reader.

Lemma 5.4. *Let $W \subset \mathbb{W}$ be an open set and let $\varphi: W \rightarrow \mathbb{R}$ be a bounded intrinsic Lipschitz function. Then, for all $x \in W$, and $r > 0$, we have*

$$U_\varphi(x, r) \subset D_R(x) \quad \text{and} \quad D_r(x) \subset U_\varphi(x, R),$$

where $R = r + 2\|\varphi\|_{L^\infty(W)}^{1/2} r^{1/2}$.

Proof of Theorem 5.1. The proof is divided in three steps.

Step 1: construction of φ , K and proof of (5.2). Let $\alpha \in (0, \frac{1}{2})$ be fixed. We assume $\varepsilon_2(n, \alpha) \leq \varepsilon_1(n)$ and $k_2 > 5124$. Apply Theorem 3.1 with scaling factor $\frac{k_2}{5124}$ and let $\varphi: \mathbb{W} \rightarrow \mathbb{R}$ be the corresponding approximating function. Without loss of generality, we can assume that φ is a $(1, \frac{k_2}{5124})$ -representative in the sense of Remark 3.2. Moreover, choosing $\varepsilon_2(n, \alpha)$ sufficiently small, we can also assume that $\sup_{\mathbb{W}} |\varphi| < 1$.

Let $I = (-\frac{k_2}{5124}, \frac{k_2}{5124})$ and let $A \subset D_{\frac{k_2}{5124}}$ be a Borel set. By (3.16) and (3.17), we have

$$\begin{aligned} \int_{\text{gr}(\varphi|_A)} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} d\mu_{E_\varphi} &= \delta(n) \int_{\text{gr}(\varphi|_A)} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} d\mathcal{S}^{2n+1} = \\ &= \delta(n) \left(\int_{\text{gr}(\varphi|_A) \cap \partial E \cap A * I} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} d\mathcal{S}^{2n+1} + \int_{(\text{gr}(\varphi|_A) \setminus \partial E) \cap A * I} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} d\mathcal{S}^{2n+1} \right) \\ &\leq 2 \int_{\partial E \cap A * I} \frac{|\nu_E - \nu|_g^2}{2} d\mu_E + \int_{(\text{gr}(\varphi|_A) \setminus \partial E) \cap A * I} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} d\mu_{E_\varphi}, \end{aligned}$$

where $\delta(n) = \frac{2\omega_{2n-1}}{\omega_{2n+1}}$. Let μ be the non-negative measure on $D_{\frac{k_2}{5124}}$ defined as

$$(5.6) \quad \mu(A) = 2 \int_{\partial E \cap A * I} \frac{|\nu_E - \nu|_g^2}{2} d\mu_E + \int_{(\text{gr}(\varphi|_A) \setminus \partial E) \cap A * I} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} d\mu_{E_\varphi},$$

for any Borel set $A \subset D_{\frac{k_2}{5124}}$, where $\nu = -X_1$ as usual.

Let $0 < \eta < 1$ be a number that will be fixed later. We let

$$K_\eta = \left\{ x \in D_{\frac{k_2}{5124}} : M\mu(x) \leq \eta \right\},$$

where $M\mu$ is the local maximal function of μ defined in (4.1) with $s = \frac{k_2}{20496}$. We assume $k_2 > 20496$ and we define

$$K = K_\eta \cap D_1.$$

We now prove (5.2). Since φ is a 1-representative of Theorem 3.1 (with the scaling factor $\frac{k_2}{5124}$), by Remark 3.2 it is enough to prove that $K \in \mathcal{A}(1, \frac{k_2}{5124})$. To this end, let us fix $p \in M \cap D_1 * I$ and $q \in M \cap K * I$. We proceed as in Step 1 of the proof of Theorem 3.1. Indeed, by [21, Lemma 3.3], we have

$$(5.7) \quad |\mathfrak{h}(\xi)| < 1 \quad \text{for all } \xi \in C_{\frac{k_2}{5124}} \cap \partial E,$$

since E is a $(\frac{1}{k_2}, k_2)$ -minimizer of H -perimeter in $C_{\frac{k_2}{2562}}$ and, by the scaling property of the excess, we can estimate

$$\mathbf{e}(\frac{k_2}{2562}) \leq 2562^{2n+1} \mathbf{e}(k_2) \leq 2562^{2n+1} \varepsilon_2(n, \alpha) \leq \omega(n, \frac{1}{2}, \frac{1}{k_2}, k_2),$$

provided we assume

$$\varepsilon_2(n, \alpha) \leq 2562^{-2n-1} \omega(n, \frac{1}{2}, \frac{1}{k_2}, k_2).$$

Here, as in the proof of Theorem 3.1, $\omega(n, t, \Lambda, r_0)$, with $t \in (0, 1)$, is the constant given in [21, Lemma 3.3]. Thus we have $p, q \in C_1$ and $d_C(p, q) < 8$, where d_C is the quasi-distance given by the quasi-norm $\|\cdot\|_C$ defined in (2.5). Moreover, $q = \pi(q) * \mathfrak{h}(q)e_1$ with $\pi(q) \in K$ and $|\mathfrak{h}(q)| < 1$. Since

$$(5.8) \quad C_s(\xi) \subset \pi(C_s(\xi)) * (-s - \mathfrak{h}(\xi), \mathfrak{h}(\xi) + s) \subset D_{2s}(\pi(\xi)) * I$$

for any $\xi \in C_1$ and $0 < s < \frac{k_2}{5124} - 1$, we can estimate

$$\begin{aligned} \mathbf{e}(q, s) &= \frac{1}{s^{2n+1}} \int_{C_s(q) \cap \partial E} \frac{|\nu_E - \nu|_g^2}{2} d\mu_E \\ &\leq \frac{1}{s^{2n+1}} \int_{\partial E \cap D_{2s}(\pi(q)) * I} \frac{|\nu_E - \nu|_g^2}{2} d\mu_E \\ &\leq 2^{2n+1} \kappa_n \sup_{0 < \rho < \frac{k_2}{5124} - \|\pi(q)\|_\infty} \frac{1}{\kappa_n \rho^{2n+1}} \int_{\partial E \cap D_\rho(\pi(q)) * I} \frac{|\nu_E - \nu|_g^2}{2} d\mu_E \\ &\leq 2^{2n} \kappa_n M \mu(\pi(q)) \leq 2^{2n} \kappa_n \eta \end{aligned}$$

for any $0 < s < \frac{k_2}{10248}$, where $\kappa_n = \mathcal{L}^{2n}(D_1)$ as in (2.3).

We consider the blow-up of E at scale $d_C(p, q)$ centered at q , that is, $F = E_{q, d_C(p, q)}$. By Remark 2.1, F is a (Λ'', r_0'') -perimeter minimizer in $(C_{k_2})_{q, d_C(p, q)}$, with

$$\Lambda'' = \Lambda' d_C(p, q), \quad r_0'' = \frac{r_0'}{d_C(p, q)} > 1.$$

Now

$$C_{16} \subset (C_{k_2})_{q, d_C(p, q)}, \quad \Lambda'' r_0'' \leq 1, \quad 0 \in \partial F$$

and, by the scaling property of the excess and by definition of M_0 ,

$$\mathbf{e}(F, 0, 16, \nu) = \mathbf{e}(E, q, 16d_C(p, q), \nu) \leq 2^{2n} \kappa_n \eta,$$

since we can choose $k_2 > 1311744$. Therefore, provided we assume

$$2^{2n} \kappa_n \eta \leq \varepsilon_0(n),$$

by Theorem 2.3 we have

$$\sup \left\{ |\mathfrak{f}_2(\xi)| : \xi \in C_1 \cap \partial F \right\} \leq C(n) \eta^{\frac{1}{2(2n+1)}},$$

where $C(n)$ is a dimensional constant. In particular, choosing

$$\xi = \frac{1}{d_C(p, q)} q^{-1} * p \in C_1 \cap \partial F,$$

we get

$$(5.9) \quad |\mathfrak{f}_2(q^{-1} * p)| \leq C(n) \eta^{\frac{1}{2(2n+1)}} d_C(p, q).$$

We now set

$$L'(n, \eta) = C(n) \eta^{\frac{1}{2(2n+1)}}$$

and we choose η so small that $L'(n, \eta) \leq L(n)$, where $L(n) < 1$ is as in (3.7). Then, by (5.9), we conclude that $d_C(p, q) = \|\pi(q^{-1} * p)\|_\infty$ and we get

$$(5.10) \quad |\mathfrak{f}_2(q^{-1} * p)| \leq L(n) \|\pi(q^{-1} * p)\|_\infty \quad \text{for all } p \in M \cap D_1 * I, \quad q \in M \cap K * I,$$

so $K \in \mathcal{A}(1, \frac{k_2}{5124})$. Thus, by (5.7) and (5.10), equality (5.2) follows.

Step 2: proof of (5.1). We now apply Lemma 4.1 with $s = \frac{k_2}{20496}$ and measure μ as defined in (5.6). By Theorem 3.1, we have

$$(5.11) \quad \begin{aligned} \mu(D_{k_2/5124}) &= 2 \int_{\partial E \cap C_{k_2/5124}} \frac{|\nu_E - \nu|_g^2}{2} d\mu_E + \int_{(\text{gr}(\varphi) \setminus \partial E) \cap C_{k_2/5124}} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} d\mu_{E_\varphi} \\ &\leq 2 \left(\frac{k_2}{5124} \right)^{2n+1} \mathbf{e}\left(\frac{k_2}{5124}\right) + C(n) \mathcal{L}^{2n+1} \left((\partial E \triangle \text{gr}(\varphi)) \cap C_{k_2/5124} \right) \leq C'(n) \mathbf{e}(k_2), \end{aligned}$$

where $C(n)$ and $C'(n)$ are dimensional constants. We now choose $\eta = \mathbf{e}(k_2)^{2\alpha}$. In order to apply Lemma 4.1, we need to check that

$$\mu(D_{k_2/5124}) \leq \frac{\eta}{5^{2n+1}} \kappa_n \left(\frac{k_2}{20496} \right)^{2n+1}.$$

By (5.11), this follows if we assume that

$$\varepsilon_2(n, \alpha) \leq \left(\frac{\kappa_n}{C'(n)} \left(\frac{k_2}{102480} \right)^{2n+1} \right)^{\frac{1}{1-2\alpha}}.$$

This condition on $\varepsilon_2(n, \alpha)$ is the only one that depends also on the parameter α . Thus, by (4.3) in Lemma 4.1 and by (5.11), we conclude that

$$\begin{aligned} \mathcal{L}^{2n}(D_1 \setminus K) &= \mathcal{L}^{2n}(J_\eta \cap D_1) \leq \frac{5^{2n+1}}{\eta} \mu \left(J_{\eta/2^{2n+1}} \cap D_{1+\frac{k_2}{102480}} \right) \\ &\leq \frac{5^{2n+1}}{\mathbf{e}(k_2)^{2\alpha}} \mu \left(D_{k_2/5124} \right) \leq 5^{2n+1} C'(n) \mathbf{e}(k_2)^{1-2\alpha}, \end{aligned}$$

which proves (5.1).

Step 3: proof of (5.3). By Lemma 5.2 and by [8, Proposition 4.4], we have

$$\begin{aligned} \mu_\varphi(A)^2 &= \left(\int_A |\nabla^\varphi \varphi| d\mathcal{L}^{2n} \right)^2 \\ &\leq \sqrt{1 + \|\nabla^\varphi \varphi\|_{L^\infty(D_{k_2/5124})}^2} \mathcal{L}^{2n}(A) \int_{\text{gr}(\varphi|_A)} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} d\mu_{E_\varphi} \\ &\leq C(n) \mathcal{L}^{2n}(A) \int_{\text{gr}(\varphi|_A)} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} d\mu_{E_\varphi} \end{aligned}$$

for all Borel sets $A \subset D_1$, where $C(n)$ is a dimensional constant. Moreover, for any $x \in K$ and $8r < \frac{k_2}{5124} - \|x\|_\infty$, by (5.5) in Lemma 5.3, by (2.7) and by (5.8), we have

$$\begin{aligned} \int_{\Phi(U_\varphi(x,r))} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} d\mu_{E_\varphi} &\leq \int_{\Gamma \cap B_{2r}(\Phi(x))} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} d\mu_{E_\varphi} \\ &\leq \int_{\Gamma \cap C_{4r}(\Phi(x))} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} d\mu_{E_\varphi} \\ &\leq 2 \int_{M \cap D_{8r}(x) * I} \frac{|\nu_E - \nu|_g^2}{2} d\mu_E + \int_{(\Gamma \setminus M) \cap D_{8r}(x) * I} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} d\mu_{E_\varphi} \\ &= \mu(D_{8r}(x)). \end{aligned}$$

Therefore, for any $x \in K$ and $8r < \frac{k_2}{5124} - \|x\|_\infty$, we get

$$(5.12) \quad \mu_\varphi(U_\varphi(x, r))^2 \leq C(n) \mathcal{L}^{2n}(U_\varphi(x, r)) \mu(D_{8r}(x)).$$

We now apply Lemma 4.3. We choose the parameter $s > 0$ in Lemma 4.3 such that

$$D_1 \subset U_\varphi(0, s) \quad \text{and} \quad U_\varphi(0, \rho(n)s) \subset D_{k_2},$$

where $\rho(n)$ is the dimensional constant defined in (4.14). Since $\text{Lip}_H(\varphi) \leq L(n) < 1$, where $L(n)$ is the dimensional constant defined in (3.7), possibly choosing $\varepsilon_2(n, \alpha)$ smaller, we can directly assume that $L(n) \leq \ell(n)$ as in (4.11). In particular, the constant $c(n, \text{Lip}_H(\varphi))$ appearing in (4.16) of Lemma 4.3, is controlled from above by a dimensional constant. Since $\sup_{\mathbb{W}} |\varphi| < 1$, by Lemma 5.4 we can choose $s = 3$ provided that we also choose

$$k_2(n) \geq 3\rho(n) + 2\sqrt{3\rho(n)}.$$

We then have

$$r_\varphi(x, 3) = \frac{3\rho(n)}{c_L} - d_\varphi(x, 0) \leq 3\rho(n),$$

where $r_\varphi(x, s)$ was defined in (4.13). By (5.12) and (4.10), for any $x \in K$ we have

$$\begin{aligned} [\mu_\varphi](x)^2 &= \sup_{0 < r < r_\varphi(x, 3)} \frac{\mu_\varphi(U_\varphi(x, r))^2}{\mathcal{L}^{2n}(U_\varphi(x, r))^2} \leq C(n) \sup_{0 < r < 3\rho(n)} \frac{\mu(D_{8r}(x))}{\mathcal{L}^{2n}(U_\varphi(x, r))} \\ &\leq \frac{C(n) 8^{2n+1} \kappa_n}{c_1^L} \sup_{0 < r < 3\rho(n)} \frac{\mu(D_{8r}(x))}{\kappa_n (8r)^{2n+1}} \\ &\leq C'(n) \sup_{0 < \rho < 24\rho(n)} \frac{\mu(D_\rho(x))}{\kappa_n \rho^{2n+1}} \end{aligned}$$

where $C'(n)$ is a dimensional constant. Now we can choose

$$k_2 > 122976\rho(n) + 5124,$$

so that $24\rho(n) \leq \frac{k_2}{5124} - \|x\|_\infty$ for any $x \in D_1$. Therefore, for any $x \in K$, we get

$$[\mu_\varphi](x) \leq \sqrt{C'(n) \eta} = C''(n) \mathbf{e}(k_2)^\alpha,$$

where $C''(n)$ is a positive dimensional constant. Thus $K \subset U_\varphi(0, 3) \setminus J_\theta^\varphi$, where J_θ^φ is as in (4.15) and $\theta = C''(n) \mathbf{e}(k_2)^\alpha$. Therefore, by (4.16) in Lemma 4.3, we conclude that for all $x, y \in K$ we have

$$|\varphi(x) - \varphi(y)| \leq C(n) \mathbf{e}(k_2)^\alpha d_\varphi(x, y).$$

This proves (5.3) and the proof of Theorem 5.1 is complete. \square

Theorem 5.1 leads to the following result, which contains Theorem 1.2 in the Introduction as a particular case.

Corollary 5.5. *Let $n \geq 2$ and $\alpha \in (0, \frac{1}{2})$. There exist positive constants $C_3(n)$, $\varepsilon_3(\alpha, n)$ and $k_3 = k_3(n)$ with the following property. For any set $E \subset \mathbb{H}^n$ that is a (Λ, r_0) -minimizer of H -perimeter in C_{k_3} with $\mathbf{e}(k_3) \leq \varepsilon_3(\alpha, n)$, $\Lambda r_0 \leq 1$, $r_0 > k_3$ and $0 \in \partial E$, there exist a set $K \subset D_1$ and an intrinsic Lipschitz function $\varphi: \mathbb{W} \rightarrow \mathbb{R}$ such that:*

$$(5.13) \quad \begin{aligned} \mathcal{L}^{2n}(D_1 \setminus K) &\leq C_3(n) \mathbf{e}(k_3)^{1-2\alpha}, \\ \text{gr}(\varphi|_K) &= \partial E \cap K * (-1, 1), \end{aligned}$$

$$(5.14) \quad \mathcal{S}^{2n+1}\left((\partial E \triangle \text{gr}(\varphi)) \cap C_1\right) \leq C_3(n) \mathbf{e}(k_3)^{1-2\alpha},$$

$$\text{Lip}_H(\varphi) \leq C_3(n) \mathbf{e}(k_3)^\alpha,$$

$$(5.15) \quad \int_{D_1} |\nabla^\varphi \varphi|^2 d\mathcal{L}^{2n} \leq C_3(n) \mathbf{e}(k_3).$$

Proof. Let $\alpha \in (0, \frac{1}{2})$ be fixed and assume that $\varepsilon_3(n, \alpha) \leq \varepsilon_2(n, \alpha)$ and $k_3 = k_2$. Let K and φ be as in Theorem 5.1. Recall that, by construction, $\text{Lip}_H(\varphi) < 1$ and $\sup_{\mathbb{W}} |\varphi| < 1$. Moreover, by (5.3), we have

$$\text{Lip}_H(\varphi|_K) \leq C_2(n) \mathbf{e}(k_2)^\alpha.$$

Thus, according to Proposition 2.6, choosing $\varepsilon_3(n, \alpha) \leq \varepsilon_2(n, \alpha)$ sufficiently small, we can extend φ outside K to the whole \mathbb{W} in such a way that $\sup_{\mathbb{W}} |\varphi| < 1$ and

$$\text{Lip}_H(\varphi) \leq C(n) \mathbf{e}(k_3)^\alpha,$$

where $C(n)$ is a dimensional constant. Thus we only need to prove (5.14) and (5.15).

We prove (5.14). Let $J = D_1 \setminus K$, $I = (-1, 1)$, and note that, by (5.13), we have

$$\begin{aligned} \mathcal{S}^{2n+1}\left((\partial E \triangle \text{gr}(\varphi)) \cap C_1\right) &= \mathcal{S}^{2n+1}\left((\partial E \setminus \text{gr}(\varphi)) \cap J * I\right) \\ &\quad + \mathcal{S}^{2n+1}\left((\text{gr}(\varphi) \setminus \partial E) \cap J * I\right) \\ &\leq \mathcal{S}^{2n+1}(\partial E \cap J * I) + \mathcal{S}^{2n+1}(\text{gr}(\varphi) \cap J * I). \end{aligned}$$

On the one hand, by definition of excess and by (3.56) in [21, Lemma 3.4], we have

$$\begin{aligned} \mathcal{S}^{2n+1}(\partial E \cap J * I) &= \int_{\partial E \cap J * I} 1 + \langle \nu_E, X_1 \rangle_g d\mathcal{S}^{2n+1} - \int_{\partial E \cap J * I} \langle \nu_E, X_1 \rangle_g d\mathcal{S}^{2n+1} = \\ &= \delta(n)^{-1} \int_{\partial E \cap J * I} \frac{|\nu_E - \nu|_g^2}{2} d\mu_E + \mathcal{L}^{2n}(J) \\ (5.16) \quad &\leq \delta(n)^{-1} \mathbf{e}(1) + \mathcal{L}^{2n}(J), \end{aligned}$$

thus, by the scaling property of the excess and by (5.1), we can estimate

$$(5.17) \quad \mathcal{S}^{2n+1}(\partial E \cap J * I) \leq \delta(n)^{-1} k_3^{2n+1} \mathbf{e}(k_3) + C_2(n) \mathbf{e}(k_3)^{1-2\alpha} \leq C(n) \mathbf{e}(k_3)^{1-2\alpha},$$

where $C(n)$ is a dimensional constant. On the other hand, by the area formula (2.15), we have

$$\begin{aligned} \mathcal{S}^{2n+1}(\text{gr}(\varphi) \cap J * I) &= \delta(n)^{-1} \int_J \sqrt{1 + |\nabla^\varphi \varphi|^2} d\mathcal{L}^{2n} \\ (5.18) \quad &\leq \delta(n)^{-1} \sqrt{1 + \|\nabla^\varphi \varphi\|_{L^\infty(D_1)}^2} \mathcal{L}^{2n}(J), \end{aligned}$$

and thus, by [8, Proposition 4.4] and again by (5.1), we can estimate

$$\mathcal{S}^{2n+1}(\text{gr}(\varphi) \cap J * I) \leq C(n) \mathbf{e}(k_3)^{1-2\alpha},$$

where $C(n)$ is a dimensional constant. Combining (5.16) with (5.17) and (5.18), we prove (5.14).

Finally, we prove (5.15). Since $D_1 = K \cup J$ with disjoint union, we can split

$$(5.19) \quad \int_{D_1} |\nabla^\varphi \varphi|^2 d\mathcal{L}^{2n} = \int_K |\nabla^\varphi \varphi|^2 d\mathcal{L}^{2n} + \int_J |\nabla^\varphi \varphi|^2 d\mathcal{L}^{2n}.$$

On the one hand, by [8, Proposition 4.4] and by (5.2), we have

$$\begin{aligned}
 \int_K |\nabla^\varphi \varphi|^2 d\mathcal{L}^{2n} &= \int_{\text{gr}(\varphi|_K)} \frac{|\nabla^\varphi \varphi|^2}{\sqrt{1 + |\nabla^\varphi \varphi|^2}} d\mu_{E_\varphi} \\
 &\leq \sqrt{1 + \|\nabla^\varphi \varphi\|_{L^\infty(D_1)}^2} \int_{\text{gr}(\varphi|_K)} \frac{|\nabla^\varphi \varphi|^2}{1 + |\nabla^\varphi \varphi|^2} d\mu_{E_\varphi} \\
 (5.20) \quad &\leq C(n) \int_{M \cap K * I} \frac{|\nu_E - \nu|_g^2}{2} d\mu_E \leq C(n) \mathbf{e}(1) \leq C'(n) \mathbf{e}(k_3),
 \end{aligned}$$

where $C(n)$ and $C'(n)$ are dimensional constants. On the other hand, again by [8, Proposition 4.4] and by (5.3), we have

$$\begin{aligned}
 \int_J |\nabla^\varphi \varphi|^2 d\mathcal{L}^{2n} &\leq \|\nabla^\varphi \varphi\|_{L^\infty(D_1)}^2 \mathcal{L}^{2n}(J) \\
 (5.21) \quad &\leq C(n) \text{Lip}_H(\varphi)^2 \mathcal{L}^{2n}(J) \leq C'(n) \mathbf{e}(k_3).
 \end{aligned}$$

Combining (5.19) with (5.20) and (5.21), we prove (5.15). \square

REFERENCES

- [1] F. J. Almgren Jr., *Almgren's big regularity paper*, World Scientific Monograph Series in Mathematics, vol. 1, World Scientific Publishing Co., Inc., River Edge, NJ, 2000.
- [2] L. Ambrosio, *Regularity theory for mass-minimizing currents (after Almgren-De Lellis-Spadaro)*, *Astérisque* **380** (2016), 139–169.
- [3] L. Ambrosio, C. De Lellis, and T. Schmidt, *Partial regularity for area-minimizing currents in Hilbert spaces*, *J. Reine Angew. Math.*, to appear, available at <http://cvgmt.sns.it/paper/2112/>.
- [4] L. Ambrosio and M. Scienza, *Locality of the perimeter in Carnot groups and chain rule*, *Ann. Mat. Pura Appl. (4)* **189** (2010), no. 4, 661–678.
- [5] L. Capogna, G. Citti, and M. Manfredini, *Regularity of non-characteristic minimal graphs in the Heisenberg group \mathbb{H}^1* , *Indiana Univ. Math. J.* **58** (2009), no. 5, 2115–2160.
- [6] ———, *Smoothness of Lipschitz minimal intrinsic graphs in Heisenberg groups \mathbb{H}^n , $n > 1$* , *J. Reine Angew. Math.* **648** (2010), 75–110.
- [7] J.-H. Cheng, J.-F. Hwang, and P. Yang, *Regularity of C^1 smooth surfaces with prescribed p -mean curvature in the Heisenberg group*, *Math. Ann.* **344** (2009), no. 1, 1–35.
- [8] G. Citti, M. Manfredini, A. Pinamonti, and F. Serra Cassano, *Smooth approximation for intrinsic Lipschitz functions in the Heisenberg group*, *Calc. Var. Partial Differential Equations* **49** (2014), no. 3-4, 1279–1308.
- [9] ———, *Poincaré-type inequality for Lipschitz continuous vector fields*, *J. Math. Pures Appl. (9)* **105** (2016), no. 3, 265–292.
- [10] D. Danielli, N. Garofalo, and D. M. Nhieu, *Sub-Riemannian calculus and monotonicity of the perimeter for graphical strips*, *Math. Z.* **265** (2010), no. 3, 617–637.
- [11] C. De Lellis and E. Spadaro, *Center manifold: a case study*, *Discrete Contin. Dyn. Syst.* **31** (2011), no. 4, 1249–1272.
- [12] B. Franchi, R. Serapioni, and F. Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*, *Math. Ann.* **321** (2001), no. 3, 479–531.
- [13] ———, *Intrinsic Lipschitz graphs in Heisenberg groups*, *J. Nonlinear Convex Anal.* **7** (2006), no. 3, 423–441.
- [14] ———, *Regular submanifolds, graphs and area formula in Heisenberg groups*, *Adv. Math.* **211** (2007), no. 1, 152–203.
- [15] ———, *Differentiability of intrinsic Lipschitz functions within Heisenberg groups*, *J. Geom. Anal.* **21** (2011), no. 4, 1044–1084.
- [16] B. Kirchheim and F. Serra Cassano, *Rectifiability and parameterization of intrinsic regular surfaces in the Heisenberg group*, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **3** (2004), no. 4, 871–896.

- [17] F. Maggi, *Sets of finite perimeter and geometric variational problems*, Cambridge Studies in Advanced Mathematics, vol. 135, Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.
- [18] V. Magnani, *A new differentiation, shape of the unit ball and perimeter measure*, Indiana Univ. Math. J. (2016), to appear, available at <http://cvgmt.sns.it/paper/2519/>.
- [19] R. Monti, *Lipschitz approximation of \mathbb{H} -perimeter minimizing boundaries*, Calc. Var. Partial Differential Equations **50** (2014), no. 1-2, 171–198.
- [20] ———, *Minimal surfaces and harmonic functions in the Heisenberg group*, Nonlinear Anal. **126** (2015), 378–393.
- [21] R. Monti and D. Vittone, *Height estimate and slicing formulas in the Heisenberg group*, Anal. PDE **8** (2015), no. 6, 1421–1454.
- [22] S. D. Pauls, *H-minimal graphs of low regularity in \mathbb{H}^1* , Comment. Math. Helv. **81** (2006), no. 2, 337–381.
- [23] M. Ritoré, *Examples of area-minimizing surfaces in the sub-Riemannian Heisenberg group \mathbb{H}^1 with low regularity*, Calc. Var. Partial Differential Equations **34** (2009), no. 2, 179–192.
- [24] R. Schoen and L. Simon, *A new proof of the regularity theorem for rectifiable currents which minimize parametric elliptic functionals*, Indiana Univ. Math. J. **31** (1982), no. 3, 415–434.
- [25] F. Serra Cassano and D. Vittone, *Graphs of bounded variation, existence and local boundedness of non-parametric minimal surfaces in Heisenberg groups*, Adv. Calc. Var. **7** (2014), no. 4, 409–492.

E-mail address: `monti@math.unipd.it`, `giorgio.stefani.2@studenti.unipd.it`

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI PADOVA, VIA TRIESTE 63, 35121 PADOVA, ITALY