DISTRIBUTIONAL SOLUTIONS OF BURGERS' EQUATION AND INTRINSIC REGULAR GRAPHS IN HEISENBERG GROUPS

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ABSTRACT. In the present paper we will characterize the continuous distributional solutions of Burgers' equation such as those which induce intrinsic regular graphs in the first Heisenberg group $\mathbb{H}^1 \equiv \mathbb{R}^3$, endowed with a left- invariant metric d_∞ equivalent to its Carnot- Carathéodory metric. We will also extend the characterization to higher Heisenberg groups $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$.

1. INTRODUCTION AND RESULTS.

In the present paper we deal with continuous distributional solutions (also named weak solutions) of Burgers' equation

$$\frac{\partial}{\partial y}u + \frac{1}{2}\frac{\partial}{\partial t}(u^2) = g \quad \text{in}\, \omega\,, \tag{1.1}$$

where $\omega \subset \mathbb{R}^2$ is an open set and $g : \omega \to \mathbb{R}$ is a prescribed continuous function. We will characterize each distributional solution $u \in C^0(\omega)$ of (1.1) such as those solutions which induce intrinsic regular graphs in the first Heisenberg group $\mathbb{H}^1 \equiv \mathbb{R}^3$, endowed with a left- invariant metric d_∞ equivalent to its Carnot- Carathéodory (CC) metric. We will also extend the characterization to higher Heisenberg groups $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ (Theorem 1.2).

By an intrinsic regular graph we mean a submanifold which has, in the CC geometry of the Heisenberg group, the same role a C^1 regular graph has in the Euclidean geometry. The notion of intrinsic regular graph have had several applications so far within the theory of rectifiable sets and minimal surfaces in CC geometry. Besides their own geometric interest, rectifiable sets in Lie groups appear in several applications, such as theoretical computer science, geometry of Banach spaces, mathematical models in neurosciences We recommend to the reader the monograph [6] and

Date: November 26, 2009.
Key words and phrases. Burgers' equation, Heisenberg group, Carnot-Carathéodory metric, intrinsic graph.
F.B. is supported by MIUR, Italy, GNAMPA of INDAM and University of Trento, Italy.
F.S.C. is supported by MIUR, Italy, GNAMPA of INDAM and University of Trento, Italy, the project GALA (Sixth Framework Programme, NEST, EU).

the references therein for a complete introduction to the Heisenberg group and the afore-mentioned arguments (see also [5]) .

Before stating the main result, we need to recall some preliminary facts. We denote the points of $\mathbb{H}^n \equiv \mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$ by

$$p = [z,t] = [x+iy,t] = (x,y,t), \qquad z \in \mathbb{C}^n,\ x,y \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

If $p = [z,t]$, $q = [\zeta,\tau] \in \mathbb{H}^n$ and $r > 0$, the group operation reads as follows

$$p \cdot q := \left[z+\zeta, t+\tau - \frac{1}{2}\Im m(\langle z, \bar\zeta\rangle) \right]. \tag{1.2}$$

The group identity is the origin 0 and one has $[z,t]^{-1} = [-z,-t]$. In $\mathbb{H}^n$ there is a natural one parameter group of non isotropic dilations $\delta_r(p) := [e^r z, e^{2r} t]$, $r > 0$.

The group $\mathbb{H}^n$ can be endowed with the homogeneous norm

$$\|p\|_\infty := \max\{|z|, |t|^{1/2}\} \tag{1.3}$$

and with the left-invariant and homogeneous distance

$$d_\infty(p,q) := \|p^{-1} \cdot q\|_\infty. \tag{1.4}$$

The metric d_∞ is equivalent to the standard CC distance ([6] or [5]). It follows that the Hausdorff dimension of $(\mathbb{H}^n, d_\infty)$ is $2n+2$, whereas its topological dimension is $2n+1$.

The Lie algebra $\mathfrak{h}_n$ of left invariant vector fields is (linearly) generated by

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}, \quad j = 1,\dots,n, \quad T = \frac{\partial}{\partial t} \tag{1.5}$$

and the only nonvanishing commutators are

$$[X_j, Y_j] = T, \qquad j = 1,\dots n. \tag{1.6}$$

We also use the notation $X_j := Y_{j-n}$ for $j = n+1,\dots,2n$.

A real valued function f, defined on an open set $\Omega \subset \mathbb{H}^n$, is said to be of class $C^1_{\mathbb{H}}(\Omega)$ if $f \in C^0(\Omega)$ and the distribution

$$\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_{2n} f)$$

is represented by a continuous function. We say that $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-*regular surface* if, for every $p \in S$, there exist a neighbourhood U of p and a function $f \in C^1_{\mathbb{H}}(U)$ such that $\nabla_{\mathbb{H}} f \neq 0$ and $S \cap U = \{q \in U : f(q) = 0\}$ ([11] and [1]). The *horizontal normal* to S at p is

$$\nu_S(p) := -\frac{\nabla_{\mathbb{H}} f(p)}{|\nabla_{\mathbb{H}} f(p)|}.$$

This not being restrictive, we deal merely with surfaces S which are level sets of functions $f \in C^1_{\mathbb{H}}$ with $X_1 f \neq 0$.

If $n \geq 2$, we identify the maximal subgroup $\mathbb{W} = \{(x,y,t) \in \mathbb{H}^n : x_1 = 0\}$ with $\mathbb{R}^{2n}$ by writing $(x_2,\dots,x_n,y_1,\dots,y_n,t)$ instead of $(0,x_2,\dots,x_n,y_1,\dots,y_n,t)$;

similarly $\mathbb{W} = \{(0, y, t) \in \mathbb{H}^1 : y, t \in \mathbb{R}\} \equiv \mathbb{R}^2$ if $n = 1$. Moreover, for $s \in \mathbb{R}$ we denote by se_1 the point $\exp(sX_1) = (s, 0, \dots, 0) \in \mathbb{H}^n$.

Now we come to the definition of intrinsic graph. A set $S \subset \mathbb{H}^n$ is called the X_1-*graph*, induced by a function $\phi : \omega \subset \mathbb{W} \to \mathbb{R}$, if

$$S = \{A \cdot \phi(A)\, e_1 : A \in \omega\}. \tag{1.7}$$

The notion of X_1-graph is not a pointless generalization: for a more complete introduction see [12]. Given ϕ, we denote by $\Phi : \omega \to \mathbb{H}^n$ the corresponding parametric map defined as

$$\Phi(A) = A \cdot \phi(A)\, e_1 = \exp(\phi(A) X_1)(A), \quad A \in \omega. \tag{1.8}$$

In [1] it has been proved that each $\mathbb{H}$- regular graph $\Phi(\omega)$ admits an intrinsic gradient $\nabla^\phi \phi \in C^0(\omega; \mathbb{R}^{2n})$, in the sense of distributions, which shares a lot of properties with the Euclidean gradient. Indeed, since $\mathbb{W} = \exp(\mathrm{span}\{X_2, \dots, X_n, Y_1, \dots, Y_n, T\})$, it is possible to define the differential operators given, in the sense of distributions, by

$$\begin{aligned} W^\phi \phi &:= Y_1 \phi + \frac{1}{2} T(\phi^2), \\ \nabla^\phi \phi &:= \begin{cases} (X_2\phi, \dots, X_n\phi, W^\phi\phi, Y_2\phi, \dots, Y_n\phi) & \text{if } n \ge 2 \\ W^\phi \phi & \text{if } n = 1 \end{cases}. \end{aligned} \tag{1.9}$$

We also denote by $\nabla^\phi := (\nabla^\phi_2, \dots, \nabla^\phi_{2n})$ the family of vector fields on $\mathbb{R}^{2n}$, $\nabla^\phi_j := X_j$ for $j \neq n+1$ and $\nabla^\phi_{n+1} = W^\phi := Y_1 + \phi T$.

Let $n \ge 2$, $A_0 = (x_2^0, \dots, x_n^0, y_1^0, \dots, y_n^0, t^0) \in \mathbb{R}^{2n}$ and define

$$\begin{aligned} I_r(A_0) \ &:= \big\{(x_2, \dots, x_n, y_1, \dots, y_n, t) \in \mathbb{R}^{2n} : |y_1 - y_1^0| < r, \\ &\textstyle\sum_{i=2}^n [(x_i - x_i^0)^2 + (y_i - y_i^0)^2] < r^2,\ |t - t^0| < r\big\}. \end{aligned}$$

When $n = 1$ and $A_0 = (y^0, t^0) \in \mathbb{R}^2$ let

$$I_r(A_0) := \big\{(y, t) \in \mathbb{R}^2 : |y - y^0| < r,\ |t - t^0| < r\big\}.$$

Let us recall the following characterizations of $\mathbb{H}$-regular graphs $\Phi(\omega)$, given respectively in [1], Theorem 1.3 and [4], Theorem 1.2 (see also [7], [2] and [14], respectively, for a characterization in general Carnot groups, for higher codimensional surfaces and for sets of finite $\mathbb{H}$-perimeter).

Theorem 1.1. *Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ be a continuous function. The following conditions are equivalent:*

(i) *The set $S := \Phi(\omega)$ is an $\mathbb{H}$-regular surface and $\nu_S^1(p) < 0$ for all $p \in S$, where $\nu_S(p) = (\nu_S^1(p), \dots, \nu_S^{2n}(p))$ is the horizontal normal to S at p.*

(ii) *There exist $w = (w_2, \dots, w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$ and a family $(\phi_\epsilon)_{\epsilon > 0} \subset C^1(\omega)$ such that, as $\epsilon \to 0^+$,*

$$\phi_\epsilon \to \phi \quad \textit{and} \quad \nabla^{\phi_\epsilon} \phi_\epsilon \to w \quad \textit{in } L^\infty_{loc}(\omega), \tag{1.10}$$

and

$$\nabla^{\phi}\phi = w \qquad \text{in } \ \omega\,, \tag{1.11}$$

in the sense of distributions.

(iii) *There exists* $w = (w_2, \dots, w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$ *such that* ϕ *is a broad* solution of the system* (1.11). *Namely, if for every* $A_0 \in \omega$, $\forall\, j = 2, ..., 2n$, *there exists a map, we will call exponential map,*

$$\exp_{A_0}(\cdot\nabla^{\phi}_j)(\cdot) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A_0)} \to \overline{I_{\delta_1}(A_0)} \Subset \omega\,, \tag{1.12}$$

where $0 < \delta_2 < \delta_1$ *such that, if* $\gamma^A_j(s) = \exp_{A_0}(s\nabla^{\phi}_j)(A)$,

(E.1): $\gamma^A_j \in C^1([-\delta_2, \delta_2]; \mathbb{R}^{2n})$,

(E.2): $\begin{cases} \dot\gamma^A_j = \nabla^{\phi}_j \circ \gamma^A_j \\ \gamma^A_j(0) = A \end{cases}$,

(E.3): $\phi\left(\gamma^A_j(s)\right) - \phi\left(\gamma^A_j(0)\right) = \int_0^s w_j\left(\gamma^A_j(\sigma)\right)\, d\sigma \quad \forall s \in [-\delta_2, \delta_2]\,,$

$\forall\, A \in I_{\delta_2}(A_0)$, $\forall\, j = 2, ..., 2n$.

Moreover, for all $p \in S$ *we have*

$$\nu_S(p) = \left(-\frac{1}{\sqrt{1 + |\nabla^{\phi}\phi|^2}}, \frac{\nabla^{\phi}\phi}{\sqrt{1 + |\nabla^{\phi}\phi|^2}} \right) (\Phi^{-1}(p))\,. \tag{1.13}$$

Here we will prove the following new characterization of $\mathbb{H}$- regular graphs $\Phi(\omega)$.

Theorem 1.2. *Let* $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ *be an open set and let* $\phi : \omega \to \mathbb{R}$ *be a continuous function. The following conditions are equivalent:*

(i) *The surface* $S := \Phi(\omega)$ *is* $\mathbb{H}$*-regular and* $\nu^1_S(P) < 0$ *for all* $P \in S$, *where* $\nu_S(P) = (\nu^1_S(P), \dots, \nu^{2n}_S(P))$ *is the horizontal normal to* S *at* P.

(ii) *There exists* $w = (w_2, \dots, w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$ *such that* ϕ *is a distributional solution of the system* (1.11), *i.e. for each* $\varphi \in C^\infty_c(\omega)$

$$\int_\omega \phi\, X_j\varphi\, d\mathcal{L}^{2n} = -\int_\omega w_j\varphi\, d\mathcal{L}^{2n} \quad \forall\, j \neq n+1 \tag{1.14}$$

and

$$\int_\omega \left(\phi\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}\phi^2\frac{\partial\varphi}{\partial t} \right) d\mathcal{L}^{2n} = -\int_\omega w_{n+1}\varphi\, d\mathcal{L}^{2n} \tag{1.15}$$

Let us stress that, in the case $n = 1$, the system (1.11) reduces to Burgers' equation (1.1) and then (1.15) has only to be considered.

Theorem 1.2 provides a simple criterion in constructing intrinsic regular graphs in $\mathbb{H}^n$ and extends a previous criterion only to be applied when $n = 1$ and ϕ depends on t, i.e. $\phi = \phi(t)$ ([1, Corollary 5.11]). Moreover, it is the exact counterpart of the characterization in the Euclidean setting. In fact, a function $\phi \in C^1(\omega)$ can be understood as a continuous distributional solution of $\nabla\phi = w$ in ω, with

$w \in C^0(\omega; \mathbb{R}^m)$ and $\omega \subset \mathbb{R}^m$ open set. We also point out that the intrinsic gradient shares with the Euclidean gradient a uniqueness property. In fact, Theorems 1.1, 1.2 and a local uniqueness result for the broad* solutions of system (1.11), contained in [4], yield the following.

Corollary 1.3. *Let* $M > 0$, $A_0 = (y^0, t^0) \in \mathbb{R}^2$ *if* $n = 1$, $A_0 = (x_2^0, \dots, x_n^0, y_1^0, \dots, y_n^0, t^0) \in \mathbb{R}^{2n}$ *if* $n \geq 2$, $r_0 > 0$, $w = (w_2, ..., w_{2n}) \in C^0(\overline{I_{r_0}(A_0)}; \mathbb{R}^{2n-1})$ *be given. Let* $\phi_i \in C^0(\overline{I_{r_0}(A_0)})$ *(i=1,2) verifying* $|\phi_i(A)| \leq M \quad \forall A \in \overline{I_{r_0}(A_0)}$.

- **i:** *Let* $n = 1$ *and assume* ϕ_i $(i = 1, 2)$ *are distributional solutions of* (1.11) *in* $\omega = I_{r_0}(A_0)$ *such that* $\phi_1(y^0, t) = \phi_2(y^0, t)$ *for each* $t \in [t^0 - r_0, t^0 + r_0]$. *Then* $\phi_1 = \phi_2$ *in* $I_r(A_0)$, *if* $0 < r < \frac{r_0}{1+M}$.
- **ii:** *Let* $n \geq 2$ *and assume* ϕ_i $(i = 1, 2)$ *are distributional solutions of* (1.11) *in* $\omega = I_{r_0}(A_0)$ *such that* $\phi_1(A_0) = \phi_2(A_0)$.*Then* $\phi_1 = \phi_2$ *in* $I_r(A_0)$, *if* $0 < r < \frac{r_0}{1+M}$.

Observe that the strong approximation assumption (1.10) is not required in the statement of Theorem 1.2 (ii). Its equivalence to the statement of Theorem 1.1 (ii) is not immediate. Our strategy will be to prove the equivalence between Theorem 1.2 (ii) and Theorem 1.1 (iii).

On the other hand we are not aware if the approximation (1.10) can be directly obtained by utilising technical devices, like mollification or approximation by vanishing viscosity, of the continuous distributional solutions of the system (1.11). A very deep study of vanishing viscosity techniques for nonlinear hyperbolic systems has been carried out in [3]. However, this study does not seem to imply the strong approximation (1.10).

Obviously the approximation (1.10) will follow. Indeed, Theorems 1.1, 1.2 and the regularity results contained in [4] yield the following approximation and regularity properties for continuous distributional solutions of (1.11).

Corollary 1.4. *Let* $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ *be an open set, let* $\phi \in C^0(\omega)$ *and* $w = (w_2, \dots, w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$. *Assume that* ϕ *is a distributional solution of system* (1.11). *Then*

(i) *there exists a family* $(\phi_\epsilon)_{\epsilon>0} \subset C^1(\omega)$ *such that, as* $\epsilon \to 0^+$,

$$\phi_\epsilon \to \phi \quad \text{and} \quad \nabla^{\phi_\epsilon}\phi_\epsilon \to w \quad \text{in } L^\infty_{loc}(\omega)\,.$$

(ii) ϕ *belongs to the class of locally little Hölder continuous functions in* ω *of order 1/2, denoted by* $h^{\frac{1}{2}}_{loc}(\omega)$, *i.e. for every open set* $\omega' \Subset \omega$,

$$\lim_{r\to 0^+} \sup\left\{\frac{|\phi(A) - \phi(B)|}{\sqrt{|A-B|}} : A, B \in \omega', 0 < |A-B| < r\right\} = 0\,,$$

where $|A|$ *denotes the Euclidean norm of* $A \in \mathbb{W} \equiv \mathbb{R}^{2n}$.

(iii) *ϕ is locally Lipschitz continuous in ω, provided that each w_i $(i = 2, \dots, 2n)$ is Lipschitz continuous.*

The regularity thresholds in Corollary 1.4 (ii) and (iii) are optimal. For instance, let $n = 1$, $\omega := \{(y,t) \in \mathbb{R}^2 : y \in \mathbb{R}, t \in (-1,1)\}$ and $\phi : \omega \to \mathbb{R}$ be the continuous function defined as $\phi(y,t) = \dfrac{\sqrt{|t|}}{\log|t|}$ if $t \neq 0$ and $\phi(y,0) = 0$. By a simple computation, ϕ turns out to be a distributional solution of (1.11) with $w_2 \in C^0(\omega)$, defined as $w_2(y,t) = \dfrac{t\,(1 + 2\log|t|)}{2\,|t|\,\log^2|t|}$ if $t \neq 0$ and $w_2(y,0) = 0$. Furthermore, $\phi \in h^{\frac{1}{2}}_{loc}(\omega)$ from Corollary 1.4 (ii) , but it is easy to see that $\phi \notin \bigcup_{\alpha > \frac{1}{2}} C^{0,\alpha}_{loc}(\omega)$. The function $\phi : \omega \to \mathbb{R}$ with $\omega := \{(y,t) \in \mathbb{R}^2 : y \in (-1,1), t \in \mathbb{R}\}$, defined as $\phi(y,t) = \dfrac{t}{y + \frac{t}{|t|}}$ if $t \neq 0$ and $\phi(y,0) = 0$, is no better than locally Lipschitz continuous in ω as well as it is a distributional solution of (1.11) with $w_2(y,t) \equiv 0$.

Finally, let us recall there are examples of $\mathbb{H}$- regular graphs $\Phi(\omega) \subset \mathbb{H}^1$ which look like fractal sets in $\mathbb{R}^3$ from the Euclidean metric point of view ([13]).

Acknowledgements. We warmly thank F. Ancona and S. Bianchini for enlightening discussions on the weak solutions of Burgers' equation during the CIME course in Cetraro (Cosenza, Italy) in June 2008. We also thank G. Crippa and L. Spinolo for useful discussions on the subject. Eventually we thank the referee for his careful reading and suggestions.

2. Proof of the results.

We will prove in this section Theorem 1.2, Corollaries 1.3 and 1.4

A key result for the proof of Theorem 1.2 will be the following regularity result for the continuous distributional solutions of (1.1), contained in [10], Theorem 3 (see also [9], Theorem 3.3).

Theorem 2.1. *Let $\omega := (-\delta_1, \delta_1) \times (-\delta_1, \delta_1)$ and let $g : \omega \to \mathbb{R}$ be bounded and measurable such that $g(\eta, \cdot)$ is continuous on $(-\delta_1, \delta_1)$ for any $\eta \in (-\delta_1, \delta_1)$. Let $\xi : [-\delta_2, \delta_2] \to (-\delta_1, \delta_1)$ $(0 < \delta_2 < \delta_1)$ be a characteristic associated with a continuous distributional solution $u : \omega \to \mathbb{R}$ of (1.1), i.e.,by definition, $\xi \in C^1([-\delta_2, \delta_2])$ and*

$$\dot{\xi}(s) = u(s, \xi(s))\, \forall s \in [-\delta_2, \delta_2]\,.$$

Define $\nu(s) := u(s, \xi(s))$ for $s \in [-\delta_2, \delta_2]$, then $(\xi(s), \nu(s))$ satisfies the system

$$\begin{cases} \dot{\xi}(s) = \nu(s) \\ \dot{\nu}(s) = g(s, \xi(s)) \end{cases}$$

on $[-\delta_2, \delta_2]$. In particular ν and $\dot{\xi}$ are Lipschitz continuous on $[-\delta_2, \delta_2]$.

Theorem 2.1 amounts to the observation that, even though the solution u may be merely continuous, its restriction along characteristics is differentiable.

Also recall the following characterization of the space $C^1_{\mathbb{H}}(\Omega)$ ([11], section 5).

Proposition 2.2. *Let $\Omega \subset \mathbb{H}^n$ be an open set and let $f \in C^0(\Omega)$. Then the following conditions are equivalent:*

(i) $f \in C^1_{\mathbb{H}}(\Omega)$;

(ii) *there exist $g_j \in C^0(\Omega)$ $(j = 1, \dots, 2n)$ such that f is differentiable along X_j in Ω with derivative g_j. More precisely, for each $p \in \Omega$ there exists $\delta_p > 0$ such that $(-\delta_p, \delta_p) \ni s \to \exp(s\,X_j)(p) = p \cdot s\,e_j \in \Omega$, $(-\delta_p, \delta_p) \ni s \to f(\exp(s\,X_j)(p))$ is C^1 and*

$$\frac{d}{ds} f(\exp(s\,X_j)(p)) = g_j(exp(s\,X_j)(p)) \quad \forall s \in (-\delta_p, \delta_p)\,,$$

where the set $\{e_1, \dots, e_{2n+1}\}$ denotes the canonical basis in $\mathbb{R}^{2n+1}$.

Proof of Theorem 1.2. **i** $\Rightarrow$ **ii**: It follows at once by Theorem 1.1 (ii).

ii$\Rightarrow$ **i**: We are going to prove that each continuous distributional solution of (1.11) is a broad* solution. Then Theorem 1.1 (iii) completes the proof. We will divide up the proof in two steps.

Step 1. Assume $n = 1$. In this case ϕ is a distributional solution of Burgers' equation (1.1). Fix $A_0 = (y^0, t^0) \in \omega$ and $I_{2\delta_1}(A_0) \Subset \omega$. Let $M = \sup_{I_{2\delta_1}(A_0)} |\phi|$ and $\delta_2 = \min\left\{\frac{\delta_1}{4}, \frac{\delta_1}{2\,M}\right\}$, then Peano's theorem yields that, $\forall\, A = (y, t) \in \overline{I_{\delta_2}(A_0)}$, there exists a function $\xi^A \in C^1([-\delta_2, \delta_2])$ such that

$$\gamma^A(s) := (y + s,\, t + \xi^A(s)) \in I_{\delta_1}(A_0) \quad \forall s \in [-\delta_2, \delta_2]\,, \tag{2.1}$$

and ξ^A can be chosen as the maximum solution of the Cauchy problem

$$\begin{cases} \dot{\xi}(s) = u(s, \xi(s)) \\ \xi(0) = 0 \end{cases} \tag{2.2}$$

in the interval $[-\delta_2, \delta_2]$, with $u : (-\delta_1, \delta_1) \times (-\delta_1, \delta_1) \to \mathbb{R}$ defined as $u(\tilde{y}, \tilde{t}) := \phi(y + \tilde{y},\, t + \tilde{t})$ (see, for instance, [8, Theorem 1.2, Chapter 1 and Theorem 1.2, Chapter 2]). On the other hand u is a continuous distributional solution of (1.1) with $g(\tilde{y}, \tilde{t}) := w_2(y + \tilde{y},\, t + \tilde{t})$ in $\omega = (-\delta_1, \delta_1) \times (-\delta_1, \delta_1)$. From Theorem 2.1, $\big(\xi^A(s), \nu^A(s)\big)$ satisfies on $[-\delta_2, \delta_2]$ the system

$$\begin{cases} \dot{\xi}^A(s) = \nu^A(s) \\ \dot{\nu}^A(s) = g(s, \xi^A(s)) \end{cases}, \tag{2.3}$$

where $\nu^A(s) = u(s, \xi^A(s))$. In particular ν^A and $\dot{\xi}^A \in C^1([-\delta_2, \delta_2])$. Therefore the curve $\gamma^A : [-\delta_2, \delta_2] \to I_{\delta_1}(A_0)$ fulfills (E_1), (E_2) and (E_3), for each $A \in I_{\delta_2}(A_0)$; the map $\exp_{A_0}(\cdot\nabla_2^{\phi})(\cdot) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A_0)} \to I_{\delta_1}(A_0)$, defined as $\exp_{A_0}(s\nabla_2^{\phi})(A) := \gamma^A(s)$, is an exponential map. This completes the proof.

Step 2. Assume $n \geq 2$. We are going to consider the $2n-1$ vector fields $X_2, \dots, X_n$, $Y_2, \dots, Y_n$, T as the canonical basis of the Lie algebra $\mathfrak{h}_{n-1}$, associated with $\mathbb{H}^{n-1}$. A point $p \in \mathbb{H}^{n-1}$ can be denoted, by means of the usual identification with $\mathbb{R}^{2n-1}$, as $p = (x_2, \dots, x_n, y_2, \dots, y_n, t)$. Moreover we will characterize the exponential maps, defined in (1.12), as suitable lifting of integral curves, respectively, of the $2(n-1)$ horizontal vector fields on $\mathbb{H}^{n-1}$, $X_2, \dots, X_n$, $Y_2, \dots, Y_n$, and of the vector field on $\mathbb{R}^2$, $\frac{\partial}{\partial y_1} + \phi(x_2, \dots, x_n, \cdot, y_2, \dots, y_n, \cdot)\,\frac{\partial}{\partial t}$.

Let us introduce the following diffeomorphism

$$\iota : \mathbb{R}^{2n} \equiv \mathbb{R} \times \mathbb{H}^{n-1} \to \mathbb{W} \equiv \mathbb{R}^{2n},$$

$$\iota\big(\eta, (v, \tau)\big) := (v_2, \dots, v_n, \eta, v_{n+2}, \dots, v_{2n}, \tau)$$

if $(\eta, (v, \tau)) \in \mathbb{R}^{2n} \equiv \mathbb{R}_\eta \times (\mathbb{R}^{2n-2}_v \times \mathbb{R}_\tau)$ and $v = (v_2, \dots, v_n, v_{n+2}, \dots, v_{2n}) \in \mathbb{R}^{2n-2}$.

We can define the vector fields $\widetilde{X}_2 \dots, \widetilde{X}_n, \widetilde{Y}_1, \dots, \widetilde{Y}_n$, and $\widetilde{T}$ on $\mathbb{R}^{2n}$ given by $\widetilde{X}_j := (\iota^{-1})_* X_j$ and $\widetilde{Y}_j := (\iota^{-1})_* Y_j, \widetilde{T} := (\iota^{-1})_* T$, where $(\iota^{-1})_*$ is the usual push-forward of vector fields after the diffeomorphism ι^{-1}. In coordinates, they can be represented as follows

$$\begin{aligned}
\widetilde{X}_j(\eta, (v, \tau)) &= \frac{\partial}{\partial v_j} - \frac{v_{j+n}}{2}\frac{\partial}{\partial \tau} \qquad \text{for } j = 2, \dots, n \\
\widetilde{Y}_1(\eta, (v, \tau)) &= \frac{\partial}{\partial \eta} \\
\widetilde{Y}_j(\eta, (v, \tau)) &= \frac{\partial}{\partial v_{j+n}} + \frac{v_j}{2}\frac{\partial}{\partial \tau} \qquad \text{for } j = 2, \dots, n \\
\widetilde{T}(\eta, (v, \tau)) &= \frac{\partial}{\partial \tau}
\end{aligned} \tag{2.4}$$

For $n+1 \leq j \leq 2n$ we will also use the notation $\widetilde{X}_j := \widetilde{Y}_{j-n}$. Notice that the set of vector fields $\widetilde{X}_2 \dots, \widetilde{X}_n, \widetilde{Y}_2, \dots, \widetilde{Y}_n$ and $\widetilde{T}$ is the standard basis of the Lie algebra $\mathfrak{h}_{n-1}$ associated with $\mathbb{H}^{n-1} \equiv \mathbb{R}^{2n-2}_v \times \mathbb{R}_\tau$. It is easy to see that

$$|\det J(\iota)(\eta, (v, \tau))| = 1 \quad \forall (\eta, (v, \tau)) \in \mathbb{R}^{2n}\,, \tag{2.5}$$

where $J(\iota)(\eta, (v, \tau))$ denotes the Jacobian matrix of ι at $(\eta, (v, \tau))$. Fix $I_r(A_0) \Subset \omega$, and denote $\widetilde{I}_r(\tilde{A}_0) := \iota^{-1}(I_r(A_0))$, if $A_0 = (x_2^0, \dots, x_n^0, y_1^0, \dots, y_n^0, t^0) \in \mathbb{W}$, and $\tilde{A}_0 := \iota^{-1}(A_0)$. Then

$$\widetilde{I}_r(\tilde{A}_0) = (y_1^0 - r, y_1^0 + r) \times B(v_0, r) \times (t^0 - r, t^0 + r)\,, \tag{2.6}$$

where $v^0 = (x_2^0, \dots, x_n^0, y_2^0, \dots, y_n^0) \in \mathbb{R}^{2n-2}$ and $B(v^0, r)$ denotes the Euclidean open ball in $\mathbb{R}^{2n-2}$ centered at v^0 with radius r. Denote also

$$\widetilde{I}_{\mathbb{H}^{n-1},r}((v^0, t^0)) := B(v^0, r) \times (t^0 - r, t^0 + r) \subset \mathbb{H}^{n-1},$$

and

$$\widetilde{I}_{\mathbb{R}^2,r}((y_1^0, t^0)) := (y_1^0 - r, y_1^0 + r) \times (t^0 - r, t^0 + r) \subset \mathbb{R}^2\,.$$

Let $\psi : I_r(A_0) \to \mathbb{R}$ be a given function. Then, for fixed $\bar\eta \in (y_1^0 - r, y_1^0 + r)$ and $\bar v \in B(v^0, r)$, let $\psi_{1,\bar\eta} : \widetilde{I}_{\mathbb{H}^{n-1},r}((v^0,t^0)) \to \mathbb{R}$, $\psi_{2,\bar v} : \widetilde{I}_{\mathbb{R}^2,r}((y_1^0,t^0)) \to \mathbb{R}$ be the functions defined as follows

$$\psi_{1,\bar\eta}\big((v,\tau)\big) := \psi\big(\iota(\bar\eta,(v,\tau))\big), \quad \psi_{2,\bar v}\big((\eta,\tau)\big) := \psi\big(\iota(\eta,(\bar v,\tau))\big)\,. \tag{2.7}$$

Observe that, when $j \neq n+1$ and $A = \iota\big(\eta,(v,\tau)\big)$, a C^1 curve $\gamma : [-\delta_2, \delta_2] \to \overline{I_{\delta_1}(A_0)}$ satisfies (E_2) and (E_3) if and only if

$$\gamma(s) = \iota\big(\eta, \exp(s\,\tilde X_j)((v,\tau))\big) \quad \forall s \in [-\delta_2,\delta_2]\,, \tag{2.8}$$

$[-\delta_2, \delta_2] \ni s \to \phi_{1,\eta}(\exp(s\,\widetilde{X}_j)(v,\tau))$ is C^1 and

$$\frac{d}{ds}\phi_{1,\eta}\big(\exp(s\,\tilde X_j)((v,\tau))\big) = w_{j,1,\eta}\big(\exp(s\,\tilde X_j)((v,\tau))\big) \quad \forall s \in [-\delta_2,\delta_2]\,. \tag{2.9}$$

When $j = n+1$ and $A = \iota\big(\eta,(v,\tau)\big)$, $\gamma : [-\delta_2,\delta_2] \to \overline{I_{\delta_1}(A_0)}$ satisfies (E_2) and (E_3) if and only if there exists a C^1 function $\xi : [-\delta_2,\delta_2] \to \mathbb{R}$ such that

$$\gamma(s) = \iota\big(\eta+s,(v,\tau+\xi(s))\big) \in \overline{I_{\delta_1}(A_0)}, \quad \begin{cases} \dot\xi(s) = \phi_{2,v}(\eta+s,\tau+\xi(s)) \\ \xi(0) = 0 \end{cases} \quad \forall s \in [-\delta_2,\delta_2]\,, \tag{2.10}$$

$[-\delta_2, \delta_2] \ni s \to \phi_{2,v}\big(\eta+s,\tau+\xi(s)\big)$ is C^1 and

$$\frac{d}{ds}\phi_{2,v}\big(\eta+s,\tau+\xi(s)\big) = w_{j,2,v}\big(\eta+s,\tau+\xi(s)\big) \quad \forall s \in [-\delta_2,\delta_2]\,, \tag{2.11}$$

$\forall v \in B(v^0, \delta_1)$, where $\phi_{i,\eta}$, $w_{j,i,\eta}$, $\phi_{i,v}$ and $w_{j,i,v}$ are the functions defined according to (2.7), respectively, with $\psi \equiv \phi$ and $\psi \equiv w_j$.

From (1.14), (1.15),(2.4) and (2.5), for each $\varphi \in C_c^\infty(I_r(A_0))$, it follows that

$$\int_{\widetilde{I}_r(\tilde A_0)} (\phi\circ\iota)\,\widetilde{X}_j(\varphi\circ\iota)\,d\mathcal{L}^{2n} = -\int_{\widetilde{I}_r(\tilde A_0)} (w_j\circ\iota)\,(\varphi\circ\iota)\,d\mathcal{L}^{2n} \quad \forall\, j \neq n+1 \tag{2.12}$$

and

$$\int_{\widetilde{I}_r(\tilde A_0)} \left((\phi\circ\iota)\frac{\partial(\varphi\circ\iota)}{\partial\eta} + \frac12(\phi\circ\iota)^2\frac{\partial(\varphi\circ\iota)}{\partial\tau}\right) d\mathcal{L}^{2n} = -\int_{\widetilde{I}_r(\tilde A_0)} (w_{n+1}\circ\iota)(\varphi\circ\iota)\,d\mathcal{L}^{2n}\,. \tag{2.13}$$

From (2.6) and by selecting test functions in (2.12) and (2.13) of the type, respectively,

$$\varphi(x_2,\dots,x_n,y_1,\dots,y_n,t) = \varphi_1(y_1)\,\varphi_2(x_2,\dots,x_n,y_2,\dots,y_n,t)$$

with $\mathrm{supp}(\varphi_1) \Subset (y_1^0 - r, y_1^0 + r)$, $\mathrm{supp}(\varphi_2) \Subset \widetilde{I}_{\mathbb{H}^{n-1},r}((v^0,t^0))$ and

$$\varphi(x_2,\dots,x_n,y_1,\dots,y_n,t) = \varphi_1(y_1,t)\,\varphi_2(x_2,\dots,x_n,y_2,\dots,y_n)$$

with $\mathrm{supp}(\varphi_1) \Subset \widetilde{I}_{\mathbb{R}^2,r}((y_1^0,t^0))$, $\mathrm{supp}(\varphi_2) \Subset B(v^0,r)$, we obtain

$$(2.14)\quad \int_{\widetilde{I}_{\mathbb{H}^{n-1},r}((v^0,t^0))} \phi_{1,\eta}\,\widetilde{X}_j\varphi\,d\mathcal{L}^{2n-1} = -\int_{\widetilde{I}_{\mathbb{H}^{n-1},r}((v^0,t^0))} w_{j,1,\eta}\,\varphi\,d\mathcal{L}^{2n-1}\quad \forall\, j\neq n+1$$

for each $\varphi\in C^\infty_c\big(\widetilde{I}_{\mathbb{H}^{n-1},r}((v^0,t^0))\big)$ and $\eta\in(y^0_1-r,y^0_1+r)$,

$$(2.15)\qquad \int_{\widetilde{I}_{\mathbb{R}^2,r}((y^0_1,t^0))}\left(\phi_{2,v}\frac{\partial\varphi}{\partial\eta}+\frac{1}{2}\phi^2_{2,v}\frac{\partial\varphi}{\partial\tau}\right)d\mathcal{L}^2 = -\int_{\widetilde{I}_{\mathbb{R}^2,r}((y^0_1,t^0))} w_{n+1,2,v}\,\varphi\,d\mathcal{L}^2$$

for each $\varphi\in C^\infty_c\big(\widetilde{I}_{\mathbb{R}^2,r}((y^0_1,t^0))\big)$ and $v\in B(v^0,r)$.

Notice that (2.14) entails, by definition, that $\phi_{1,\eta}\in C^1_{\mathbb{H}}\big(\widetilde{I}_{\mathbb{H}^{n-1},r}((v^0,t^0))\big)$, with respect to the horizontal differentiable structure of $\mathbb{H}^{n-1}$, for each $\eta\in(y^0_1-r,y^0_1+r)$. Meanwhile, (2.15) and the previous step 1 yield that the intrinsic graph $\Phi\big(\widetilde{I}_{\mathbb{R}^2,r}((y^0_1,t^0))\big)\subset\mathbb{H}^1$, induced by the function $\phi\equiv\phi_{2,v}$, is $\mathbb{H}$-regular for each $v\in B(v^0,r)$.

We need to show, in order to complete the proof, the existence, for every $j=2,\dots,2n$, of an exponential map

$$\exp_{A_0}\big(\,\cdot\,\nabla^\phi_j\big)(\cdot):[-\delta_2,\delta_2]\times\overline{I_{\delta_2}(A_0)}\to\overline{I_{\delta_1}(A_0)}\Subset\omega\,.$$

Namely, for fixed $A\in\overline{I_{\delta_2}(A_0)}$, the existence of a curve $\gamma^A_j(s):=\exp_{A_0}\big(s\,\nabla^\phi_j\big)(A)$ satisfying (2.8) and (2.9), when $j\neq n+1$, and (2.10) and (2.11) when $j=n+1$.

If $j\neq n+1$, by classical ODEs results, there exist $0<\delta_2<\delta_1$ such that

$$\exp(\cdot\,\widetilde{X}_j)(\cdot):\ [-\delta_2,\delta_2]\times\overline{\widetilde{I}_{\mathbb{H}^{n-1},\delta_2}((v^0,t^0))}\to\overline{\widetilde{I}_{\mathbb{H}^{n-1},\delta_1}((v^0,t^0))}\,.$$

Let $A\in\overline{I_{\delta_2}(A_0)}$ and $\tilde{A}=\iota^{-1}(A)=(\eta,(v,\tau))\in\overline{\widetilde{I}_{\delta_2}(\tilde{A}_0)}$. Then $\gamma^A_j(s):=\iota\big(\eta,\exp(s\,\widetilde{X}_j)((v,\tau))\big)$ satisfies (2.8), by construction, and (2.9), because of Proposition 2.2 (ii) with $\Omega\equiv\widetilde{I}_{\mathbb{H}^{n-1},\delta_1}((v^0,t^0))$, $f\equiv\phi_{1,\eta}$ and $g_j\equiv w_{j,1,\eta}$, since $\phi_{1,\eta}\in C^1_{\mathbb{H}}\big(\widetilde{I}_{\mathbb{H}^{n-1},\delta_1}((v^0,t^0))\big)$.

If $j=n+1$, let $I_{2\delta_1}(A_0)\Subset\omega$, $M=\sup_{I_{2\delta_1}(A_0)}|\phi|$ and $\delta_2=\min\big\{\frac{\delta_1}{4},\frac{\delta_1}{2\,M}\big\}$. Then we can verbatim repeat the construction in the step 1 with the function $\phi\equiv\phi_{2,v}:\widetilde{I}_{\mathbb{R}^2,2\,\delta_1}((y^0_1,t^0))\to\mathbb{R}$, for each $v\in\overline{B(v^0,\delta_2)}$. Indeed, let $A\in\overline{I_{\delta_2}(A_0)}$ and $\tilde{A}=\iota^{-1}(A)=(\eta,(v,\tau))\in\overline{\widetilde{I}_{\delta_2}(\tilde{A}_0)}$, then there exists a C^1 function $\xi:\ [-\delta_2,\delta_2]\to\mathbb{R}$ such that the curve $\gamma^A_{n+1}(s):=\iota\big(\eta+s,(v,\tau+\xi(s)\big)\in I_{\delta_1}(A_0)\,\forall s\in[-\delta_2,\delta_2]$ and ξ is the maximum solution of the Cauchy problem (2.2) in $[-\delta_2,\delta_2]$, with $u(\tilde{\eta},\tilde{\tau})=\phi_{2,v}(\eta+\tilde{\eta},\tau+\tilde{\tau})$ $(\tilde{\eta},\tilde{\tau})\in(-\delta_1,\delta_1)\times(-\delta_1,\delta_1)$, for each $v\in\overline{B(v^0,\delta_2)}$. In particular γ^A_{n+1} satisfies (2.10). Moreover, (2.11) also holds since the function $\nu^A(s):=u(s,\xi(s))=\phi_{2,v}(\eta+s,\tau+\xi(s))\in Lip([-\delta_2,\delta_2])$ and satisfies the system (2.3) with $g(\tilde{\eta}.\tilde{\tau}):=w_{n+1,2,v}(\eta+\tilde{\eta},\tau+\tilde{\tau})$, for each $v\in\overline{B(v^0,\delta_2)}$.

□

Proof of Corollary 1.3. Theorems 1.1 (iii) and 1.2 yield that ϕ_i $(i = 1, 2)$ are broad* solutions of the system (1.11) in $\omega = I_{r_0}(A_0)$. We complete the proof applying the uniqueness result for broad* solution [4, Theorem 3.8]. □

Proof of Corollary 1.4. Applying Theorem 1.2, $S = \Phi(\omega) \subset \mathbb{H}^n$ turns out to be a $\mathbb{H}$-regular surface, where Φ is the parametrization induced by ϕ and defined in (1.8). Thus the statement (i) follows from Theorem 1.1 (ii).

Theorem 1.1 (iii) yields that ϕ is a broad* solution of (1.11) too. Then we obtain the statement (ii) from [4, Theorem 3.2] .

Finally the statement (iii) follows from [4, Theorem 1.3]. □

References

[1] L.Ambrosio, F.Serra Cassano, D.Vittone, *Intrinsic Regular Hypersurfaces in Heisenberg Groups,* J. Geom. Anal. **16** (2006), no. 2, 187–232.

[2] G. Arena, R. Serapioni, *Intrinsic regular submanifolds in Heisenberg groups are differentiable graphs*, Calc. Var. PDEs **35** (2009), no. 4, 17–49.

[3] S.Bianchini, A.Bressan, *Vanishing viscosity solutions of nonlinear hyperbolic systems*, Ann. of Math. **161** (2005), 223-342

[4] F.Bigolin, F.Serra Cassano, *Intrinsic regular graphs in Heisenberg groups vs. weak solutions of non linear first-order PDEs,* to appear on Adv. Calc. Var., available at http://cvgmt.sns.it/papers/bigser07/

[5] A.Bonfiglioli, E.Lanconelli, F.Uguzzoni, *Stratified Lie Groups and Potential Theory for their Sub-Laplacians*, Springer, 2007.

[6] L.Capogna, D.Danielli, S.D.Pauls, J.T. Tyson, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, PM 259, Birkhäuser, 2007.

[7] G. Citti, M. Manfredini, *Implicit function theorem in Carnot-Carathéodory spaces*, Commun. Contemp. Math. 8 (2006), no. 5, 657–680.

[8] E.A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw- Hill, 1955.

[9] C.M.Dafermos, *Generalized Characteristics and the Structure of Solutions of Hyperbolic Conservation laws*, Ind. U. Math. J. 26, No.6 1097–1119 (1997).

[10] C.M.Dafermos, *Continuous solutions for balance laws*, Ric. Mat **55** (2006), 79–91 .

[11] B.Franchi, R.Serapioni, F.Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*, Math.Ann., **321** (2001), 479–531.

[12] B.Franchi, R.Serapioni, F.Serra Cassano, *Regular submanifolds, graphs and area formula in Heisenberg groups*, Advances in Math. 211 (2007) 157-203.

[13] B.Kirchheim, F.Serra Cassano, *Rectifiability and parametrization of intrinsic regular surfaces in the Heisenberg group*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **III** (2004),871–896.

[14] R.Monti, D. Vittone, *Sets with finite $\mathbb{H}$-perimeter and controlled normal*, Preprint 2009.

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