

Weighted Sobolev Inequalities in $CD(0, N)$ spaces

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Abstract

In this note, we prove global weighted Sobolev inequalities on non-compact $CD(0, N)$ spaces satisfying a suitable growth condition, extending to possibly non-smooth and non-Riemannian structures a previous result from [M09] stated for Riemannian manifolds with non-negative Ricci curvature and a suitable reverse doubling condition. We use this result in the context of $RCD(0, N)$ spaces to get a uniform bound of the corresponding weighted heat kernel via a weighted Nash inequality.

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1 Introduction

Riemannian manifolds with non-negative Ricci curvature have strong analytic properties. Indeed, the doubling condition and the local L^2 -Poincaré inequality are satisfied on such spaces, and they imply many important results, like the well-known Li-Yau Gaussian estimates for a class of Green functions including the heat kernel [LY86] or powerful local Sobolev inequalities and parabolic Harnack inequalities (see e.g. [SC02]).

In the recent years, several classes of possibly non-smooth metric measure spaces containing the collection of Riemannian manifolds with non-negative Ricci curvature have been under investigation, both from a geometric and an analytic point of view. For instance, in the context of measure spaces endowed with a suitable Dirichlet form, Sturm proved existence and uniqueness of the fundamental solution of parabolic operators along with Gaussian estimates and parabolic Harnack inequalities [St95, St96], provided the doubling and Poincaré properties hold. Afterwards, general doubling spaces with Poincaré-type

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inequalities were studied at length by Hajlasz and Koskela [HK00] who proved local Sobolev-type inequalities, a Trudinger inequality, a Rellich-Kondrachov theorem, and many related results.

Approximately a decade ago, Sturm [St06] and Lott and Villani [LV09] independently proposed the curvature-dimension condition $\text{CD}(0, N)$, for $N \in [1, +\infty)$, as an extension of non-negativity of the Ricci curvature and bound from above by N of the dimension for possibly non-smooth metric measure spaces. Coupled with the infinitesimal Hilbertianness introduced later on by Ambrosio, Gigli and Savaré [AGS14b] to rule out non-Riemannian structures, the $\text{CD}(0, N)$ condition leads to the stronger $\text{RCD}(0, N)$ condition, where R stands for Riemannian.

The classes of $\text{CD}(0, N)$ and $\text{RCD}(0, N)$ spaces have been extensively studied over the past few years, and it is by now well-known that they both contain the measured Gromov-Hausdorff closure of the class of Riemannian manifolds with non-negative Ricci curvature and dimension lower than N , as well as Alexandrov spaces with non-negative generalized sectional curvature and locally finite and non-zero n -dimensional Hausdorff measure, n being lower than N . Moreover, $\text{CD}(0, N)$ spaces satisfy the doubling and Poincaré properties, and $\text{RCD}(0, N)$ spaces are, in addition, endowed with a regular and strongly local Dirichlet form called Cheeger energy (see Section 2). Therefore, the works of Sturm [St95, St96] imply existence and uniqueness of a heat kernel, which by the way satisfies Gaussian estimates, on $\text{RCD}(0, N)$ spaces.

One of the interest of the $\text{CD}(0, N)$ and $\text{RCD}(0, N)$ conditions, and of the more general $\text{CD}(K, N)$ and $\text{RCD}(K, N)$ conditions for arbitrary $K \in \mathbb{R}$, is the possibility of proving classical functional inequalities on spaces with rather loose structure thanks to optimal transport or gradient flow arguments. In this regard, Lott and Villani obtained in [LV07, Th. 5.29] a global Sobolev-type inequality for $\text{CD}(K, N)$ spaces with $K > 0$ and $N \in (2, +\infty)$. Later on, in their striking work [CM17, Th. 1.11], Cavaletti and Mondino proved a global Sobolev-type inequality with sharp constant for bounded essentially non-branching $\text{CD}^*(K, N)$ spaces with $K \in \mathbb{R}$ and $N \in (1, +\infty)$; in case $K > 0$ and $N > 2$, they get the classical Sobolev inequality with sharp constant. This last inequality had been previously justified on $\text{RCD}^*(K, N)$ spaces with $K > 0$ and $N > 2$ by Profeta [P15].

The aim of this note is to provide a new related analytic result, namely a global weighted Sobolev inequality, for certain non-compact $\text{CD}(0, N)$ spaces with $N > 2$. It is worth underlying that our result does not require the Riemannian synthetic condition $\text{RCD}(0, N)$.

Theorem 1 (Weighted Sobolev inequalities). *Let $(X, \mathbf{d}, \mathbf{m})$ be a $\text{CD}(0, N)$ space with $N > 2$. Assume that there exists $1 < \eta \leq N$ such that*

$$0 < \Theta_{inf} := \liminf_{r \rightarrow +\infty} \frac{V(o, r)}{r^\eta} \leq \Theta_{sup} := \limsup_{r \rightarrow +\infty} \frac{V(o, r)}{r^\eta} < +\infty \quad (1.1)$$

for some $o \in X$. Then for any $1 \leq p < \eta$, there exists a constant $C > 0$, depending only on $N, \eta, \Theta_{inf}, \Theta_{sup}$ and p , such that for any continuous function $u : X \rightarrow \mathbb{R}$ admitting an upper gradient $g \in L^p(X, \mathbf{m})$,

$$\left(\int_X |u|^{p^*} d\mu \right)^{\frac{1}{p^*}} \leq C \left(\int_X g^p d\mathbf{m} \right)^{\frac{1}{p}}$$

where $p^* = Np/(N - p)$ and μ is the measure absolutely continuous with respect to \mathbf{m} with density $w_o = V(o, \mathbf{d}(o, \cdot))^{p/(N-p)} \mathbf{d}(o, \cdot)^{-Np/(N-p)}$.

Note that the above number η might not be an integer: for instance, for any $\eta \in (3, 4)$ there exists a four dimensional hyperkähler (in particular Ricci-flat) manifold satisfying (1.1), see [Ha11, Th. 1.2].

Theorem 1 extends a result due to Minerbe stated in the case $p = 2$ on Riemannian manifolds with non-negative Ricci curvature [M09, Th. 0.1]. In this Riemannian setting, the weighted L^2 -Sobolev inequality appears as a good replacement of the classical unweighted L^2 -Sobolev inequality which cannot hold if $\eta < N$, see [M09, Prop. 2.21] and note that the proof of this last fact relies on the locally Euclidean structure of manifolds.

Our proof is an adaptation of Minerbe's proof to the setting of $CD(0, N)$ spaces and is based upon ideas of Grigor'yan and Saloff-Coste introduced in the smooth category [GS05] which extend easily to the setting of metric measure spaces. More precisely, we apply an abstract process (Theorem 2) which permits to patch local inequalities into a global one by means of an appropriate discrete Poincaré inequality.

In [M09], Minerbe deduced from the weighted L^2 -Sobolev inequality a finiteness result in L^2 -cohomology. The L^2 -cohomology of $RCD(0, N)$ spaces being well-defined [G18, Sec. 3.5.2], it is natural to ask whether an extension of Minerbe's finiteness result holds on such spaces. However, two harsh difficulties arise:

- there is presently no synthetic formulation of the crucial integrability assumption $\int_M |\text{Rm}|^{n/2} < +\infty$, where Rm denotes the Riemann curvature tensor of a Riemannian manifold (M^n, g) ;
- up to now, no Weitzenböck formula is known in the RCD context (in this regard, see the discussion before Prop. 3.6.9 in [G18]); in particular, while for Riemannian manifolds the difference between the Hodge Laplacian and the connection Laplacian on 1-forms is given by the Ricci curvature, the way this difference relates with curvature for general $RCD(0, N)$ spaces is not known yet.

Instead of this, we follow a classical approach (see e.g. [BCLS95]) which was neither considered in [M09] nor in the subsequent related work [He11] to deduce a weighted Nash inequality (Theorem 4) for $CD(0, N)$ spaces satisfying the growth assumption (1.1) provided $\eta > 2$. Let us mention that in the context of non-reversible Finsler manifolds, Ohta put forward an unweighted Nash inequality [Oh17] and that Bakry, Bolley, Gentil and Maheux introduced weighted Nash inequalities in the study of possibly non-ultracontractive Markov semigroups [BBGL12]; nevertheless, these inequalities seem presently unrelated to our.

We conclude this note with a natural consequence in the setting of $RCD(0, N)$ spaces satisfying a uniform local Ahlfors regularity property, namely a uniform bound for the (weighted) heat kernel associated with a suitable modification of the Cheeger energy. This weighted heat kernel would certainly require further investigation in the future.

The paper is organized as follows. In Section 2, we introduce the tools of non-smooth analysis that we shall use throughout the article. We also define the $CD(0, N)$ and $RCD(0, N)$ conditions, and present the aforementioned patching process. Section 3 is devoted to the proof of Theorem 1. Section 4 deals with the weighted Nash inequality and the uniform bound on the weighted heat kernel we mentioned earlier.

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2 Preliminaries

Unless otherwise mentioned, in the whole article $(X, \mathbf{d}, \mathbf{m})$ denotes a triple where (X, \mathbf{d}) is a proper, complete and separable metric space and \mathbf{m} is a Borel measure, positive and finite on balls with finite and non-zero radius, such that $\text{supp}(\mathbf{m}) = X$. We use the standard notations for functional spaces: $C(X)$ for the space of \mathbf{d} -continuous functions, $\text{Lip}(X)$ for the space of \mathbf{d} -Lipschitz functions and $L^p(X, \mathbf{m})$ (respectively $L^p_{loc}(X, \mathbf{m})$) for the space of p -integrable (respectively locally p -integrable) functions, for any $1 \leq p \leq +\infty$. If A is a subset of X , we denote by $C_c(A)$ the space of continuous functions on X compactly supported in A . We also write $L^0(X, \mathbf{m})$ (respectively $L^0_+(X, \mathbf{m})$) for the space of \mathbf{m} -measurable (respectively non-negative \mathbf{m} -measurable) functions. We write $B_r(x)$ for the ball of radius $r > 0$ centered at $x \in X$, $S_r(x)$ for $\overline{B_r(x)} \setminus B_r(x)$, and $V(x, r)$ for $\mathbf{m}(B_r(x))$. For any $\lambda > 0$, if B denotes a ball of radius $r > 0$, we write λB for the ball with same center as B and of radius λr . For any bounded Borel set $A \subset X$ and any locally integrable function $u : X \rightarrow \mathbb{R}$, we denote by u_A or $\int_A u \, d\mathbf{m}$ the mean value $\frac{1}{\mathbf{m}(A)} \int_A u \, d\mathbf{m}$, and by $\langle u \rangle_A$ the mean value $\frac{1}{\mu(A)} \int_A u \, d\mu$, where μ is as in Theorem 1.

Several constants appear in this work. For better readability, if a constant C depends only on parameters a_1, a_2, \dots we always write $C = C(a_1, a_2, \dots)$ for its first occurrence, and then write more simply C if there is no ambiguity.

Non-smooth analysis.

Let us recall that a continuous function $\gamma : [0, L] \rightarrow X$ is called a rectifiable curve if its length

$$L(\gamma) := \sup \left\{ \sum_{i=1}^n \mathbf{d}(\gamma(x_i), \gamma(x_{i-1})) : 0 = x_0 < \dots < x_n = L, n \in \mathbb{N} \setminus \{0\} \right\}$$

is finite. For any rectifiable curve $\gamma : [0, L] \rightarrow X$, there exists a continuous function $\bar{\gamma} : [0, L(\gamma)] \rightarrow X$, called arc-length parametrization of γ , such that $\mathbf{d}(\bar{\gamma}(s), \bar{\gamma}(t)) = |t - s|$ for all $0 \leq t \leq s \leq L(\gamma)$, and a non-decreasing continuous map $\varphi : [0, L] \rightarrow [0, L(\gamma)]$, such that $\gamma = \bar{\gamma} \circ \varphi$ (see e.g. [BBI01, Prop. 2.5.9]). When $\gamma = \bar{\gamma}$, we say that γ is parametrized by arc-length.

In the context of metric analysis, a weak notion of norm of the gradient of a function is available and due to Heinonen and Koskela [HeK98].

Definition 1 (Upper gradients). *Let $u : X \rightarrow [-\infty, +\infty]$ be an extended real-valued function. A Borel function $g : X \rightarrow [0, +\infty]$ is called upper gradient of u if for any rectifiable curve $\gamma : [0, L] \rightarrow X$ parametrized by arc-length,*

$$|u(\gamma(L)) - u(\gamma(0))| \leq \int_0^L g(\gamma(s)) \, ds.$$

Building on this, Cheeger defined in [Ch99] the Sobolev spaces $H^{1,p}(X, \mathbf{d}, \mathbf{m})$, where $p \in [1, +\infty)$, in the following way:

Definition 2 (Cheeger's Sobolev spaces). *Let $1 \leq p < +\infty$. The Sobolev $(1, p)$ norm of a function $u \in L^p(X, \mathbf{m})$ is set as*

$$\|u\|_{H^{1,p}} := \left(\|u\|_{L^p}^p + \inf \liminf_{i \rightarrow \infty} \|g_i\|_{L^p}^p \right)^{1/p}$$

where the infimum is taken over all the sequences $\{u_i\}_i \subset L^p(X, \mathbf{m})$ and $\{g_i\}_i \subset L^p_+(X, \mathbf{m})$ such that g_i is an upper gradient of u_i and $\|u_i - u\|_{L^p} \rightarrow 0$. The Sobolev space $H^{1,p}(X, \mathbf{d}, \mathbf{m})$ is then defined as the closure of $\text{Lip}(X) \cap L^p(X, \mathbf{m})$ with respect to $\|\cdot\|_{H^{1,p}}$.

The above relaxation process can be performed with slopes of bounded Lipschitz functions instead of upper gradients, see Lemma 3. Recall that the slope of a Lipschitz function f is defined as

$$|\nabla f|(x) := \begin{cases} \limsup_{y \rightarrow x} \frac{|f(x) - f(y)|}{d(x,y)} & \text{if } x \in X \text{ is not isolated,} \\ 0 & \text{otherwise,} \end{cases}$$

and that it satisfies the chain rule, namely $|\nabla(fg)| \leq f|\nabla g| + g|\nabla f|$ for any $f, g \in \text{Lip}(X)$.

Let us recall that $(X, \mathbf{d}, \mathbf{m})$ is called doubling if there exists $C_D \geq 1$ such that

$$V(x, 2r) \leq C_D V(x, r) \quad \forall x \in X, \forall r > 0, \quad (2.1)$$

and that it satisfies a uniform weak local L^p -Poincaré inequality, where $p \in [1, +\infty)$, if there exists $\lambda > 1$ and $C_P > 0$ such that

$$\int_B |u - u_B|^p \, \mathbf{d}\mathbf{m} \leq C_P r^p \int_{\lambda B} g^p \, \mathbf{d}\mathbf{m} \quad (2.2)$$

holds for any ball B of arbitrary radius $r > 0$, any $u \in L^1_{loc}(X, \mathbf{m})$ and any upper gradient $g \in L^p(X, \mathbf{m})$ of u . If (2.2) holds with $\lambda = 1$, we say that a uniform *strong* local L^p -Poincaré inequality holds.

The next notion serves to turn weak inequalities into strong inequalities, see e.g. [HK00, Sect. 9].

Definition 3 (John domain). *A bounded open set $\Omega \subset X$ is called a John domain if there exists $x_o \in \Omega$ and $C_J > 0$ such that for every $x \in \Omega$, there exists a Lipschitz curve $\gamma : [0, L] \rightarrow \Omega$ parametrized by arc-length such that $\gamma(0) = x$, $\gamma(L) = x_o$ and $t^{-1} \mathbf{d}(\gamma(t), X \setminus \Omega) \geq C_J$ for any $t \in [0, L]$.*

Finally let us introduce a technical property taken from [HK00]. For any $v \in L^0(X, \mathbf{m})$ and $0 < t_1 < t_2 < +\infty$, we denote by $v_{t_1}^{t_2}$ the truncated function $\min(\max(0, v - t_1), t_2 - t_1) + t_1$. We write χ_A for the characteristic function of a set $A \subset X$.

Definition 4 (Truncation property). *We say that a pair of \mathbf{m} -measurable functions (u, g) such that for some $p \in [1, +\infty)$, $C_P > 0$ and $\lambda > 1$, the inequality (2.2) holds for any ball B of arbitrary radius $r > 0$, has the truncation property if for any $0 < t_1 < t_2 < +\infty$, $b \in \mathbb{R}$ and $\varepsilon \in \{-1, 1\}$, there exists $C > 0$ such that (2.2) holds for any ball B of arbitrary radius $r > 0$ with u, g and C_P replaced by $(\varepsilon(u - b))_{t_1}^{t_2}$, $g\chi_{\{t_1 < u < t_2\}}$ and C respectively.*

The next proposition is a particular case of [HK00, Th. 10.3].

Proposition 1. *If $(X, \mathbf{d}, \mathbf{m})$ satisfies a uniform weak local L^1 -Poincaré inequality, any pair (u, g) where $u \in C(X)$ and $g \in L^1_{loc}(X, \mathbf{m})$ is an upper gradient of u has the truncation property.*

The $\text{CD}(0, N)$ and $\text{RCD}(0, N)$ conditions.

Let us give the definition of the curvature-dimension conditions $\text{CD}(0, N)$ and $\text{RCD}(0, N)$. For the general condition $\text{CD}(K, N)$ with $K \in \mathbb{R}$, we refer to [Vi09, Chap. 29 & 30].

Recall that a curve $\gamma : [0, 1] \rightarrow X$ is called a geodesic if $\mathbf{d}(\gamma(s), \gamma(t)) = |t - s| \mathbf{d}(\gamma(0), \gamma(1))$ for any $s, t \in [0, 1]$. The space (X, \mathbf{d}) is called geodesic if for any couple of points $(x_0, x_1) \in X^2$ there exists a geodesic γ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. We denote by $\mathcal{P}(X)$ the set of probability measures on X and by $\mathcal{P}_2(X)$ the set of probability

measures μ on X with finite second moment, i.e. such that there exists $x_o \in X$ for which $\int_X d^2(x_o, x) d\mu(x) < +\infty$. The Wasserstein distance between two measures $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ is by definition

$$W_2(\mu_0, \mu_1) := \inf \left(\int_{X \times X} d(x_0, x_1)^2 d\pi(x_0, x_1) \right)^{1/2}$$

where the infimum is taken among all the probability measures π on $X \times X$ with first marginal equal to μ_0 and second marginal equal to μ_1 . A standard result of optimal transport theory states that if the space (X, d) is geodesic, then the metric space (\mathcal{P}_2, W_2) is geodesic too. Let us introduce the Rényi entropies.

Definition 5 (Rényi entropies). *Given $N \in (1, +\infty)$, the N -Rényi entropy relative to \mathbf{m} , denoted by $S_N(\cdot|\mathbf{m})$, is defined as follows:*

$$S_N(\mu|\mathbf{m}) := - \int_X \rho^{1-\frac{1}{N}} d\mathbf{m} \quad \forall \mu \in \mathcal{P}(X),$$

where $\mu = \rho\mathbf{m} + \mu^{sing}$ is the Lebesgue decomposition of μ with respect to \mathbf{m} .

We are now in a position to introduce the $CD(0, N)$ condition, which could be summarized as weak geodesical convexity of all the N' -Rényi entropies with $N' \geq N$.

Definition 6 ($CD(0, N)$ condition). *Given $N \in (1, +\infty)$, (X, d, \mathbf{m}) satisfies the $CD(0, N)$ condition if for any $N' \geq N$, the N' -Rényi entropy is weakly geodesically convex, meaning that for any couple of measures $(\mu_0, \mu_1) \in \mathcal{P}_2(X)^2$, there exists a W_2 -geodesic $(\mu_t)_{t \in [0,1]}$ between μ_0 and μ_1 such that for any $t \in [0, 1]$,*

$$S_N(\mu_t|\mathbf{m}) \leq (1-t)S_N(\mu_0|\mathbf{m}) + tS_N(\mu_1|\mathbf{m}).$$

Any space satisfying the $CD(0, N)$ condition is called a $CD(0, N)$ space.

The Bishop-Gromov theorem holds on $CD(0, N)$ spaces [Vi09, Th. 30.11], and as a direct consequence, the doubling condition (2.1) holds too, with $C_D = 2^N$. Moreover, Rajala proved the following uniform weak local L^1 -Poincaré inequality [Raj12, Th. 1.1].

Proposition 2. *Assume that (X, d, \mathbf{m}) is a $CD(0, N)$ space. Then for any function $u \in C(X)$ and any upper gradient $g \in L^1_{loc}(X, \mathbf{m})$ of u , for any ball $B \subset X$ of arbitrary radius $r > 0$,*

$$\int_B |u - u_B| d\mathbf{m} \leq 4r \int_{2B} g d\mathbf{m}.$$

The $CD(0, N)$ condition does not distinguish between Riemannian-like and non-Riemannian-like structures: for instance, \mathbb{R}^n equipped with the distance induced by the L^∞ -norm and the Lebesgue measure satisfies the $CD(0, N)$ condition [Vi09, p. 912], though it is not a Riemannian structure because the L^∞ -norm is not induced by any scalar product. To focus on Riemannian-like structures, Ambrosio, Gigli and Savaré added to the theory the notion of infinitesimal Hilbertianity, leading to the so-called RCD condition, R standing for *Riemannian* [AGS14b].

Definition 7 (RCD(0, N) condition). *Define the Cheeger energy of a function $f \in H^{1,2}(X, d, \mathbf{m})$ as*

$$\text{Ch}(f) := \frac{1}{2} (\|f\|_{H^{1,2}}^2 - \|f\|_{L^2}^2).$$

(X, d, \mathbf{m}) is called infinitesimally Hilbertian if Ch is a quadratic form. If in addition (X, d, \mathbf{m}) is a $CD(0, N)$ space, it is said to satisfy the RCD(0, N) condition, or more simply it is called a RCD(0, N) space.

Let us provide some standard facts taken from [AGS14b, G18]. First, note that (X, d, \mathbf{m}) is infinitesimally Hilbertian if and only if $H^{1,2}(X, d, \mathbf{m})$ is a Hilbert space, whence the terminology. Moreover, for infinitesimally Hilbertian spaces, a suitable diagonal argument justifies for any $f \in H^{1,2}(X, d, \mathbf{m})$ the existence of a function $|\nabla f|_* \in L^2(X, \mathbf{m})$, called *minimal relaxed slope* or *minimal generalized upper gradient* of f , which gives integral representation of Ch, meaning:

$$\text{Ch}(f) = \int_X |\nabla f|_*^2 \, d\mathbf{m} \quad \forall f \in H^{1,2}(X, d, \mathbf{m}).$$

The minimal relaxed slope is a local object, meaning that $|\nabla f|_* = |\nabla g|_*$ \mathbf{m} -a.e. on $\{f = g\}$ for any $f, g \in H^{1,2}(X, d, \mathbf{m})$, and it satisfies the chain rule, namely $|\nabla(fg)|_* \leq f|\nabla g|_* + g|\nabla f|_*$ \mathbf{m} -a.e. on X for all $f, g \in H^{1,2}(X, d, \mathbf{m})$. In addition, the function

$$\langle \nabla f_1, \nabla f_2 \rangle := \lim_{\epsilon \rightarrow 0} \frac{|\nabla(f_1 + \epsilon f_2)|_*^2 - |\nabla f_1|_*^2}{2\epsilon}$$

provides a symmetric bilinear form on $H^{1,2}(X, d, \mathbf{m}) \times H^{1,2}(X, d, \mathbf{m})$ with values in $L^1(X, \mathbf{m})$, and

$$\text{Ch}(f_1, f_2) := \int_X \langle \nabla f_1, \nabla f_2 \rangle \, d\mathbf{m} \quad \forall f_1, f_2 \in H^{1,2}(X, d, \mathbf{m}),$$

defines a strongly local, regular and symmetric Dirichlet form. Finally, the infinitesimally Hilbertian condition allows to apply the general theory of gradient flows on Hilbert spaces, ensuring the existence of the L^2 -gradient flow $(h_t)_{t \geq 0}$ of the convex and lower semicontinuous functional Ch, called *heat flow* of (X, d, \mathbf{m}) . This heat flow is a linear, continuous, self-adjoint and Markovian contraction semigroup in $L^2(X, \mathbf{m})$. The terminology ‘heat flow’ comes from the characterization of $(h_t)_{t \geq 0}$ as the only semigroup of operators such that $t \mapsto h_t f$ is locally absolutely continuous in $(0, +\infty)$ with values in $L^2(X, \mathbf{m})$ and

$$\frac{d}{dt} h_t f = \Delta h_t f \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, +\infty)$$

holds for any $f \in L^2(X, \mathbf{m})$, the Laplace operator Δ being defined in this context by:

$$f \in D(\Delta) \iff \exists h := \Delta f \in L^2(X, \mathbf{m}) \text{ s.t. } \text{Ch}(f, g) = - \int_X h g \, d\mathbf{m} \quad \forall g \in H^{1,2}(X, d, \mathbf{m}).$$

Patching process

Let us present now the patching process [GS05, M09] that we shall apply to get Theorem 1. In the whole paragraph, (X, d) is a metric space equipped with two Borel measures \mathbf{m}_1 and \mathbf{m}_2 both finite and nonzero on balls with finite and nonzero radius and such that $\text{supp}(\mathbf{m}_1) = \text{supp}(\mathbf{m}_2) = X$. For any bounded Borel set $A \subset X$ and any locally \mathbf{m}_2 -integrable function $u : X \rightarrow \mathbb{R}$, we denote by $\{u\}_A$ the mean value $\frac{1}{\mathbf{m}_2(A)} \int_A u \, d\mathbf{m}_2$. For any given set S , we denote by $\text{Card}(S)$ its cardinality.

Definition 8 (Good covering). *Let $A \subset A^\# \subset X$ be two Borel sets. A countable family $(U_i, U_i^*, U_i^\#)_{i \in I}$ of triples of Borel subsets of X with finite \mathbf{m}_j -measure for any $j \in \{1, 2\}$ is called a good covering of $(A, A^\#)$ with respect to $(\mathbf{m}_1, \mathbf{m}_2)$ if:*

1. for every $i \in I$, $U_i \subset U_i^* \subset U_i^\#$;
2. there exists a Borel set $E \subset X$ such that $E \subset \bigcup_i U_i$ and $\mathbf{m}_1(X \setminus E) = \mathbf{m}_2(X \setminus E) = 0$;
3. there exists $Q_1 > 0$ such that $\text{Card}(\{i \in I : U_{i_0}^\# \cap U_i^\# \neq \emptyset\}) \leq Q_1$ for any $i_0 \in I$;

4. for any $(i, j) \in I \times I$ such that $\overline{U}_i \cap \overline{U}_j \neq \emptyset$, there exists $k(i, j) \in I$ such that $U_i \cup U_j \subset U_{k(i, j)}^*$;

5. there exists $Q_2 > 0$ such that for any $(i, j) \in I \times I$ satisfying $\overline{U}_i \cap \overline{U}_j \neq \emptyset$,

$$\mathbf{m}_2(U_{k(i, j)}^*) \leq Q_2 \min(\mathbf{m}_2(U_i), \mathbf{m}_2(U_j)).$$

For the sake of clarity, we call condition 3. the *overlapping condition*, condition 4. the *embracing condition* and condition 5. the *measure control condition* of the good covering.

When $A = A^\# = X$, we say that $(U_i, U_i^*, U_i^\#)_{i \in I}$ is a good covering of (X, \mathbf{d}) with respect to $(\mathbf{m}_1, \mathbf{m}_2)$.

From now on, we consider two numbers $p, q \in [1, +\infty)$ and two Borel sets $A \subset A^\# \subset X$. We assume that a good covering $(U_i, U_i^*, U_i^\#)_{i \in I}$ of $(A, A^\#)$ with respect to $(\mathbf{m}_1, \mathbf{m}_2)$ exists.

Let us explain how to define from $(U_i, U_i^*, U_i^\#)_{i \in I}$ a canonical weighted graph $(\mathcal{V}, \mathcal{E}, \nu)$, where \mathcal{V} is the set of vertices of the graph, \mathcal{E} is the set of edges, and ν is a weight on the graph (i.e. a function $\nu : \mathcal{V} \sqcup \mathcal{E} \rightarrow \mathbb{R}$). We define \mathcal{V} by associating to each U_i a vertex i (informally, we put a point i on each U_i). Then we set $\mathcal{E} := \{(i, j) \in \mathcal{V} \times \mathcal{V} : i \neq j \text{ and } \overline{U}_i \cap \overline{U}_j \neq \emptyset\}$. Finally we weight the vertices of the graph by setting $\nu(i) := \mathbf{m}_2(U_i)$ for every $i \in \mathcal{V}$ and the edges by setting $\nu(i, j) := \max(\nu(i), \nu(j))$ for every $(i, j) \in \mathcal{E}$.

The patching theorem (Theorem 2) states that if some local inequalities are true on the pieces of the good covering and if a discrete inequality holds on the associated canonical weighted graph, then the local inequalities can be patched into a global one. Let us give the precise definitions.

Definition 9 (Local continuous $L^{q,p}$ -Sobolev-Neumann inequalities). *We say that the good covering $(U_i, U_i^*, U_i^\#)_{i \in I}$ satisfies local continuous $L^{q,p}$ -Sobolev-Neumann inequalities if there exists a constant $S_c > 0$ such that for all $i \in I$,*

$$\left(\int_{U_i} |u - \{u\}_{U_i}|^q \, \mathbf{d}\mathbf{m}_2 \right)^{\frac{1}{q}} \leq S_c \left(\int_{U_i^*} g^p \, \mathbf{d}\mathbf{m}_1 \right)^{\frac{1}{p}} \quad (2.3)$$

for all $u \in L^1(U_i, \mathbf{m}_2)$ and all upper gradients $g \in L^p(U_i^*, \mathbf{m}_1)$, and

$$\left(\int_{U_i^*} |u - \{u\}_{U_i^*}|^q \, \mathbf{d}\mathbf{m}_2 \right)^{\frac{1}{q}} \leq S_c \left(\int_{U_i^\#} g^p \, \mathbf{d}\mathbf{m}_1 \right)^{\frac{1}{p}} \quad (2.4)$$

for all $u \in L^1(U_i^*, \mathbf{m}_2)$ and all upper gradients $g \in L^p(U_i^\#, \mathbf{m}_1)$.

Definition 10 (Discrete L^q -Poincaré inequality). *We say that the weighted graph $(\mathcal{V}, \mathcal{E}, \nu)$ satisfies a discrete L^q -Poincaré inequality if there exists $S_d > 0$ such that:*

$$\left(\sum_{i \in \mathcal{V}} |f(i)|^q \nu(i) \right)^{\frac{1}{q}} \leq S_d \left(\sum_{\{i, j\} \in \mathcal{E}} |f(i) - f(j)|^q \nu(i, j) \right)^{\frac{1}{q}} \quad \forall f \in L^q(\mathcal{V}, \nu). \quad (2.5)$$

Remark 1. *Here we differ a bit from Minerbe's terminology. Indeed, in [M09], the following discrete L^q Sobolev-Dirichlet inequalities of order k were introduced for any $k \in (1, +\infty]$ and any $q \in [1, k)$:*

$$\left(\sum_{i \in \mathcal{V}} |f(i)|^{\frac{qk}{k-q}} \nu(i) \right)^{\frac{k-q}{qk}} \leq S_d \left(\sum_{\{i, j\} \in \mathcal{E}} |f(i) - f(j)|^q \nu(i, j) \right)^{\frac{1}{q}} \quad \forall f \in L^q(\mathcal{V}, \nu).$$

In the present paper we only need the case $k = +\infty$, in which we recover (2.5): here is why we have chosen the terminology "Poincaré" which seems, in our setting, more appropriate.

We are now in a position to state the patching theorem.

Theorem 2 (Patching theorem). *Let (X, \mathbf{d}) be a metric space equipped with two Borel measures \mathbf{m}_1 and \mathbf{m}_2 , both finite and nonzero on balls with finite and nonzero radius, such that $\text{supp}(\mathbf{m}_1) = \text{supp}(\mathbf{m}_2) = X$. Let $A \subset A^\# \subset X$ be two Borel sets, and $p, q \in [1, +\infty)$ be such that $q \geq p$. Assume that $(A, A^\#)$ admits a good covering $(U_i, U_i^*, U_i^\#)$ with respect to $(\mathbf{m}_1, \mathbf{m}_2)$ which satisfies the local $L^{q,p}$ -Sobolev-Neumann inequalities (2.3) and (2.4) and whose associated weighted graph $(\mathcal{V}, \mathcal{E}, \nu)$ satisfies the discrete L^q -Poincaré inequality (2.5). Then there exists a constant $C = C(p, q, Q_1, Q_2, S_c, S_d) > 0$ such that for any function $u \in C_c(A^\#)$ and any upper gradient $g \in L^p(A^\#, \mathbf{m}_1)$ of u ,*

$$\left(\int_A |u|^q \, \mathbf{d}\mathbf{m}_2 \right)^{\frac{1}{q}} \leq C \left(\int_{A^\#} g^p \, \mathbf{d}\mathbf{m}_1 \right)^{\frac{1}{p}}.$$

Although the proof of Theorem 2 is a straightforward adaptation of [M09, Th. 1.8], we provide it for the reader's convenience.

Proof. Let us consider $u \in C_c(A^\#)$. Then

$$\int_A |u|^q \, \mathbf{d}\mathbf{m}_2 \leq \sum_{i \in \mathcal{V}} \int_{U_i} |u|^q \, \mathbf{d}\mathbf{m}_2.$$

From convexity of the function $t \mapsto |t|^q$, we deduce $|u|^q \leq 2^{q-1}(|u - \{u\}_{U_i}|^q + |\{u\}_{U_i}|^q)$ \mathbf{m}_2 -a.e. on each U_i , and then

$$\int_A |u|^q \, \mathbf{d}\mathbf{m}_2 \leq 2^{q-1} \sum_{i \in \mathcal{V}} \int_{U_i} |u - \{u\}_{U_i}|^q \, \mathbf{d}\mathbf{m}_2 + 2^{q-1} \sum_{i \in \mathcal{V}} |\{u\}_{U_i}|^q \nu(i). \quad (2.6)$$

From (2.3) and the fact that $\sum_j x_j^{q/p} \leq (\sum_j x_j)^{q/p}$ for any finite family of real numbers $\{x_j\}$ (since $q \geq p$), we get

$$\begin{aligned} \sum_{i \in \mathcal{V}} \int_{U_i} |u - \{u\}_{U_i}|^q \, \mathbf{d}\mathbf{m}_2 &\leq S_c^{q/p} \left(\sum_{i \in \mathcal{V}} \int_{U_i^*} g^p \, \mathbf{d}\mathbf{m}_1 \right)^{q/p} \\ &\leq S_c^{q/p} Q_1^{q/p} \left(\int_{A^\#} g^p \, \mathbf{d}\mathbf{m}_1 \right)^{q/p}, \end{aligned} \quad (2.7)$$

this last inequality being a direct consequence of the overlapping condition. Now the discrete L^q -Poincaré inequality (2.5) implies

$$\sum_{i \in \mathcal{V}} |\{u\}_{U_i}|^q \nu(i) \leq S_d \sum_{(i,j) \in \mathcal{E}} |\{u\}_{U_i} - \{u\}_{U_j}|^q \nu(i,j). \quad (2.8)$$

For any $(i,j) \in \mathcal{E}$, a double application of Hölder's inequality yields to

$$|\{u\}_{U_i} - \{u\}_{U_j}|^q \nu(i,j) \leq \frac{\nu(i,j)}{\mathbf{m}_2(U_i) \mathbf{m}_2(U_j)} \int_{U_i} \int_{U_j} |u(x) - u(y)|^q \, \mathbf{d}\mathbf{m}_2(x) \, \mathbf{d}\mathbf{m}_2(y),$$

and as the measure control condition ensures $\nu(i,j) = \max(\mathbf{m}_2(U_i), \mathbf{m}_2(U_j)) \leq Q_2 \mathbf{m}_2(U_{k(i,j)})$, the embracing condition implies

$$|\{u\}_{U_i} - \{u\}_{U_j}|^q \nu(i,j) \leq \frac{Q_2}{\mathbf{m}_2(U_{k(i,j)})} \int_{U_{k(i,j)}^*} \int_{U_{k(i,j)}^*} |u(x) - u(y)|^q \, \mathbf{d}\mathbf{m}_2(x) \, \mathbf{d}\mathbf{m}_2(y).$$

Convexity of $t \mapsto |t|^q$ leads to $|u(x) - u(y)|^q \leq 2^{q-1}(|u(x) - \{u\}_{U_{k(i,j)}^*}| + |\{u\}_{U_{k(i,j)}^*} - u(y)|)$ for any $x, y \in U_{k(i,j)}^*$ and any $(i, j) \in \mathcal{E}$, so that

$$|\{u\}_{U_i} - \{u\}_{U_j}|^q \nu(i, j) \leq Q_2 2^q \int_{U_{k(i,j)}^*} |u - \{u\}_{U_{k(i,j)}^*}|^q \, \mathrm{d}\mathbf{m}.$$

Summing over $(i, j) \in \mathcal{E}$, we get

$$\sum_{(i,j) \in \mathcal{E}} |\{u\}_{U_i} - \{u\}_{U_j}|^q \nu(i, j) \leq Q_2 2^q \sum_{(i,j) \in \mathcal{E}} \int_{U_{k(i,j)}^*} |u - \{u\}_{U_{k(i,j)}^*}|^q \, \mathrm{d}\mathbf{m}_2. \quad (2.9)$$

Then (2.4) yields to

$$\sum_{(i,j) \in \mathcal{E}} |\{u\}_{U_i} - \{u\}_{U_j}|^q \nu(i, j) \leq Q_2 2^q S_c^{q/p} \left(\sum_{(i,j) \in \mathcal{E}} \int_{U_{k(i,j)}^\#} g^p \, \mathrm{d}\mathbf{m}_1 \right)^{q/p}. \quad (2.10)$$

Finally, a simple counting argument shows that

$$\sum_{(i,j) \in \mathcal{E}} \int_{U_{k(i,j)}^\#} g^p \, \mathrm{d}\mathbf{m}_1 \leq Q_1^3 \int_{A^\#} g^p \, \mathrm{d}\mathbf{m}. \quad (2.11)$$

The result follows from combining (2.6), (2.7), (2.8), (2.9), (2.10) and (2.11). \square

A similar statement holds if we replace the discrete L^q -Poincaré inequality by a discrete “ L^q -Poincaré-Neumann” version:

$$\left(\sum_{i \in \mathcal{V}} |f(i) - \nu(f)|^q \nu(i) \right)^{\frac{1}{q}} \leq S_d \left(\sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)|^q \nu(i, j) \right)^{\frac{1}{q}} \quad (2.12)$$

for all compactly supported $f : \mathcal{V} \rightarrow \mathbb{R}$, where $\nu(f) = \left(\sum_{i: f(i) \neq 0} \nu(i) \right)^{-1} \sum_i f(i) \nu(i)$. The terminology “Poincaré-Neumann” comes from the mean-value in the left-hand side of (2.12) and the analogy with the local Poincaré inequality used in the study of the Laplacian on bounded Euclidean domains with Neumann boundary conditions, see [SC02, Sect. 1.5.2].

Theorem 3 (Patching theorem - Neumann version). *Let (X, \mathbf{d}) be a metric space equipped with two Borel measures \mathbf{m}_1 and \mathbf{m}_2 , both finite and nonzero on balls with finite and nonzero radius, such that $\mathrm{supp}(\mathbf{m}_1) = \mathrm{supp}(\mathbf{m}_2) = X$. Let $A \subset A^\# \subset X$ be two Borel sets such that $0 < \mathbf{m}(A) < +\infty$ and $p, q \in [1, +\infty)$ such that $q \geq p$. Assume that $(A, A^\#)$ admits a good covering $(U_i, U_i^*, U_i^\#)$ with respect to $(\mathbf{m}_1, \mathbf{m}_2)$ which satisfies the local $L^{q,p}$ -Sobolev-Neumann inequalities (2.3) and (2.4) and whose associated weighted graph $(\mathcal{V}, \mathcal{E}, \nu)$ satisfies the discrete L^q -Poincaré-Neumann inequality (2.12). Then there exists a constant $C = C(p, q, Q_1, Q_2, S_c, S_d) > 0$ such that for any $u \in C_c(A^\#)$ and any upper gradient $g \in L^p(A^\#, \mathbf{m}_1)$,*

$$\left(\int_A |u - \{u\}_A|^q \, \mathrm{d}\mathbf{m}_2 \right)^{\frac{1}{q}} \leq C \left(\int_{A^\#} g^p \, \mathrm{d}\mathbf{m}_1 \right)^{\frac{1}{p}}.$$

The proof of Theorem 3 is similar to the proof of Theorem 2 and writes exactly as [M09, Th. 1.10] with upper gradients instead of norms of gradients, so we skip it.

3 Proof of the main result

In this section, we prove Theorem 1 after a few preliminary results.

As already pointed out in [M09], the local continuous L^{2^*} -Sobolev-Neumann inequalities on Riemannian manifolds (where $2^* = 2n/(n-2)$ and n is the dimension of the manifold) can be derived from the doubling condition and the uniform strong local L^2 -Poincaré inequality which are both implied by non-negativity of the Ricci curvature. However, the discrete L^{2^*} -Poincaré inequality requires an additional reverse doubling condition which is an immediate consequence of the growth condition (1.1), as shown in the next lemma.

Lemma 1. *Let (Y, d_Y, \mathbf{m}_Y) be a metric measure space such that*

$$0 < \Theta_{inf} := \liminf_{r \rightarrow +\infty} \frac{\mathbf{m}_Y(B_r(y_o))}{r^\alpha} \leq \Theta_{sup} := \limsup_{r \rightarrow +\infty} \frac{\mathbf{m}_Y(B_r(y_o))}{r^\alpha} < +\infty \quad (3.1)$$

for some $y_o \in Y$ and $\alpha > 0$. Then there exists $A > 0$ and $C_{RD} = C_{RD}(\Theta_{inf}, \Theta_{sup}) > 0$ such that

$$\frac{\mathbf{m}_Y(B_R(o))}{\mathbf{m}_Y(B_r(o))} \geq C_{RD} \left(\frac{R}{r} \right)^\alpha \quad \forall A < r \leq R. \quad (3.2)$$

Proof. The growth condition (3.1) implies the existence of $A > 0$ such that for any $R \geq r > A$, $\Theta_{inf}/2 \leq r^{-\alpha} \mathbf{m}_Y(B_r(y_o)) \leq 2\Theta_{sup}$ and $R^{-\alpha} \mathbf{m}_Y(B_R(y_o)) \geq \Theta_{inf}/2$, whence (3.2) with $C_{RD} = \Theta_{inf}/(4\Theta_{sup})$. \square

Remark 2. *Note that the doubling condition (2.1) easily implies (3.4): see for instance [GHL09, p.9] for a proof giving $C_{RD} = (1 + C_D^{-4})^{-1}$ and $\alpha = \log_2(1 + C_D^{-4})$. But in this case, $\alpha > 1$ if and only if $C_D < 1$ which is impossible. So we emphasize that in our context, in which we want the segment $(1, \alpha)$ to be non-empty, doubling and reverse doubling must be thought as complementary hypotheses.*

The next result, a strong local L^p -Sobolev inequality for $CD(0, N)$ spaces, is an important technical tool for our purposes. In the context of Riemannian manifolds, it was proved by Maheux and Saloff-Coste [MS95].

Lemma 2. *Let (Y, d_Y, \mathbf{m}_Y) be a $CD(0, N)$ space. Then for any $p \in [1, N)$ there exists $C = C(N, p) > 0$ such that for any $u \in C(Y)$, any upper gradient $g \in L^1_{loc}(Y, \mathbf{m}_Y)$, and any ball B with arbitrary radius $r > 0$,*

$$\left(\int_B |u - u_B|^{p^*} d\mathbf{m}_Y \right)^{\frac{1}{p^*}} \leq C \frac{r}{\mathbf{m}_Y(B)^{1/N}} \left(\int_B g^p d\mathbf{m}_Y \right)^{\frac{1}{p}}, \quad (3.3)$$

where $p^* = Np/(N - p)$.

Proof. Let u be a continuous function on Y , $g \in L^1_{loc}(Y, \mathbf{m}_Y)$ be an upper gradient of u , B be a ball with arbitrary radius $r > 0$, and $p \in [1, N)$. In this proof u_B stands for $\mathbf{m}_Y(B)^{-1} \int_B u d\mathbf{m}_Y$. Thanks to Hölder's inequality and the doubling property, Proposition 2 implies

$$\int_B |u - u_B| d\mathbf{m}_Y \leq 2^{N+2} r \left(\int_{2B} g^p d\mathbf{m}_Y \right)^{1/p}.$$

Let $x_0, x_1 \in Y$ and $r_0, r_1 > 0$ be such that $x_1 \in B_{r_0}(x_0)$ and $r_1 \leq r_0$. Then

$$\frac{\mathbf{m}_Y(B_{r_1}(x_1))}{\mathbf{m}_Y(B_{r_0}(x_0))} \geq \frac{\mathbf{m}_Y(B_{r_1}(x_1))}{\mathbf{m}_Y(B_{r_0+d_Y(x_0, x_1)}(x_1))} \geq 2^{-N} \left(\frac{r_1}{r_0 + d_Y(x_0, x_1)} \right)^N \geq 2^{-2N} \left(\frac{r_1}{r_0} \right)^N$$

by the doubling condition. Moreover, we know from Proposition 1 that (u, g) satisfies the truncation property, so that [HK00, Th. 5.1, 1.] applies and gives

$$\left(\int_B |u - u_B|^{p^*} \, d\mathbf{m}_Y \right)^{1/p^*} \leq \tilde{C}r \left(\int_{10B} g^p \, d\mathbf{m}_Y \right)^{1/p}$$

where \tilde{C} depends only on p and the doubling and Poincaré constants of (Y, d_Y, \mathbf{m}_Y) which depend only on N . As (Y, d_Y, \mathbf{m}_Y) is a $CD(0, N)$ space, the metric structure (Y, d_Y) is proper and geodesic, so it follows from [HK00, Cor. 9.5] that all the balls in Y are John domains with a universal constant $C_J > 0$. Then [HK00, Th. 9.7] applies and yields to the result since $1/p^* - 1/p = 1/N$. \square

Finally, let us state a result whose proof - omitted here - can be deduced from [M09, Prop. 2.8] by using Proposition 2. Note that even if Proposition 2 provides only a weak inequality, one can harmlessly substitute it to the strong one used in the proof of [M09, Prop. 2.8], because it is applied there to a function f which is Lipschitz on a ball B and extended by 0 outside of B . Note also that Proposition 2 being a L^1 -Poincaré inequality, we can assume $\alpha > 1$ (a L^2 -Poincaré inequality would have only permit $\alpha > 2$).

Proposition 3. *Let (Y, d_Y, \mathbf{m}_Y) be a $CD(0, N)$ space satisfying the growth condition (3.1) with $\alpha > 1$. Then there exists $\kappa_0 = \kappa_0(N, \alpha) > 1$ such that for any $R > 0$ such that $S_R(y_o)$ is non-empty, for any couple of points $(x, x') \in S_R(y_o)^2$, there exists a rectifiable curve from x to x' that remains inside $B_R(y_o) \setminus B_{\kappa_0^{-1}R}(y_o)$.*

Remark 3. *It is worth pointing out that Proposition 3 implies a strong connectedness property for (Y, d_Y, \mathbf{m}_Y) ; notably, all the annuli $B_{\kappa_0^{i+2}}(y_o) \setminus B_{\kappa_0^{i-1}}(y_o)$, $i \in \mathbb{N}$, are connected.*

Let us prove now Theorem 1. Let (X, d, \mathbf{m}) be a non-compact $CD(0, N)$ space with $N \geq 3$ satisfying the growth condition (1.1) with parameter $\eta \in (1, N]$, and $p \in [1, \eta)$. We recall that μ is the measure absolutely continuous with respect to \mathbf{m} with density $w_o = V(o, d(o, \cdot))^{p/(N-p)} d(o, \cdot)^{-Np/(N-p)}$, and that $p^* = Np/(N-p)$. Note that Lemma 1 applied to (X, d, \mathbf{m}) , assuming with no loss of generality that $A = 1$, implies:

$$\frac{V(o, R)}{V(o, r)} \geq C_{RD} \left(\frac{R}{r} \right)^\eta \quad \forall 1 < r < R. \quad (3.4)$$

STEP 1: The good covering.

Let us briefly explain how to construct a good covering on (X, d, \mathbf{m}) , referring to [M09, Section 2.3.1] for additional details. Define κ as the square-root of the constant κ_0 given by Proposition 3. Then for any $R > 0$, two connected components X_1 and X_2 of $B_{\kappa R}(o) \setminus B_R(o)$ are always contained in one component of $B_{\kappa R}(o) \setminus B_{\kappa^{-1}R}(o)$: otherwise, linking $x \in \overline{X_1} \cap S_{\kappa R}(o)$ and $x' \in \overline{X_2} \cap S_{\kappa R}(o)$ by a curve remaining inside $B_{\kappa R}(o) \setminus B_{\kappa^{-1}R}(o)$ would not be possible.

Let γ be a line starting at o , i.e. a continuous function $\gamma : [0, +\infty) \rightarrow X$ such that $\gamma(o) = 0$ and $d(\gamma(t), \gamma(s)) = |t - s|$ for any $s, t \geq 0$. Such a line can be obtained as follows. For $x_1 \in S_1(o)$, let $\gamma_1 : [0, 1] \rightarrow X$ be a geodesic between o and x_1 . Define then recursively $x_n := \operatorname{argmin}\{d(x_{n-1}, x) : x \in S_n(o)\}$ and γ_n geodesic between x_{n-1} and x_n for any $n \geq 1$. The concatenation of all the γ_n provides the desired γ .

For any $i \in \mathbb{N}$, let us write $A_i = B_{\kappa^i}(o) \setminus B_{\kappa^{i-1}}(o)$ and denote by $(U'_{i,a})_{0 \leq a \leq h'_i}$ the connected components of A_i , $U'_{i,0}$ being set as the one intersecting γ . The next simple result was used without a proof in [M09].

Claim 1. *There exists a constant $h = h(N, \kappa) < \infty$ such that $\sup_i h'_i \leq h$.*

Proof. Take $i \in \mathbb{N}$. For every $0 \leq a \leq h'_i$, pick x_a in $U_{i,a} \cap S_{(\kappa^i + \kappa^{i-1})/2}(o)$. As the balls $(B_a := B_{(\kappa^i - \kappa^{i-1})/4}(x_a))_{0 \leq a \leq h'_i}$ are disjoint and all included in $B_{\kappa^i}(o)$, we have

$$h'_i \min_{0 \leq a \leq h'_i} \mathbf{m}(B_a) \leq \sum_{0 \leq a \leq h'_i} \mathbf{m}(B_a) \leq V(o, \kappa^i).$$

With no loss of generality, we can assume that $\min_{0 \leq a \leq h'_i} \mathbf{m}(B_a) = \mathbf{m}(B_0)$. Notice that $d(o, x_0) \leq \kappa^i$. Then

$$h'_i \leq \frac{V(o, \kappa^i)}{\mathbf{m}(B_0)} \leq \frac{V(x_0, \kappa^i + d(o, x_0))}{\mathbf{m}(B_0)} \leq \left(\frac{8\kappa^i}{\kappa^i - \kappa^{i-1}} \right)^N$$

by the doubling condition. This yields to the result with $h := \left(\frac{8\kappa}{\kappa-1} \right)^N$. \square

Define then the covering $(U'_{i,a}, U'^*_{i,a}, U'^{\#}_{i,a})_{i \in \mathbb{N}, 0 \leq a \leq h'_i}$ where $U'^*_{i,a}$ is by definition the union of the sets $U'_{j,b}$ such that $\overline{U'_{j,b}} \cap \overline{U'_{i,a}} \neq \emptyset$, and $U'^{\#}_{i,a}$ is by definition the union of the sets $U'^*_{j,b}$ such that $\overline{U'^*_{j,b}} \cap \overline{U'^*_{i,a}} \neq \emptyset$. Note that $(U'_{i,a}, U'^*_{i,a}, U'^{\#}_{i,a})_{i \in \mathbb{N}, 0 \leq a \leq h'_i}$ is not necessarily a good covering, as pieces $U'_{i,a}$ might be arbitrary small compared to their neighbors: in this case, the measure control of the overlapping condition would not be true. So whenever $\overline{U'_{i,a}} \cap S_{\kappa}(o) = \emptyset$ (this condition being satisfied by all “small” pieces), we set $U_{i-1,a} := U'_{i,a} \cup U'_{i-1,a'}$ where a' is the integer such that $\overline{U'_{i,a}} \cap \overline{U'_{i-1,a'}} \neq \emptyset$; otherwise we set $U_{i,a} := U'_{i,a}$. See Figure 1 below.

Then we define $U^*_{i,a}$ and $U^{\#}_{i,a}$ in a similar way than $U'^*_{i,a}$ and $U'^{\#}_{i,a}$. Using the doubling condition, one can easily show that $(U_{i,a}, U^*_{i,a}, U^{\#}_{i,a})_{i \in \mathbb{N}, 0 \leq a \leq h_i}$ is a good covering of (X, \mathbf{d}) with respect to (μ, \mathbf{m}) , with constants Q_1 and Q_2 depending only on N .

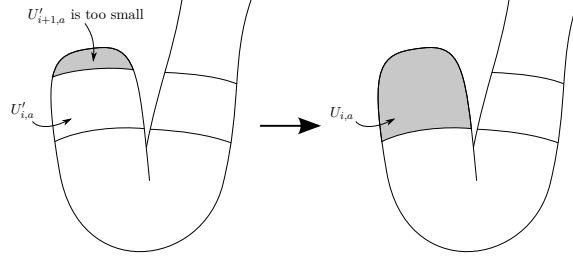


Figure 1: we glue the small piece $U'_{i+1,a}$ to the adjacent piece $U'_{i-1,a'}$ to form $U_{i,a}$.

STEP 2: The discrete L^{p^*} -Poincaré inequality.

Let $(\mathcal{V}, \mathcal{E}, \nu)$ be the weighted graph obtained from $(U_{i,a}, U^*_{i,a}, U^{\#}_{i,a})_{i \in \mathbb{N}, 0 \leq a \leq h_i}$. Define the degree $\deg(i)$ of a vertex i as the number of vertices j such that $i \sim j$. As a consequence of Claim 1, $\sup\{\deg(i) : i \in \mathcal{V}\} \leq 2h$. Moreover, the doubling condition easily justifies the existence of a number $C \geq 1$ such that $C^{-1}\nu(i) \leq \nu(j) \leq C\nu(i)$ for every $(i, j) \in \mathcal{E}$. Thus by [M09, Prop. 1.12], the discrete L^1 -Poincaré inequality implies the L^q one for any given $q \geq 1$. But the discrete L^1 -Poincaré inequality is equivalent to the isoperimetric inequality

([M09, Prop. 1.14]): there exists a constant $\mathcal{I} > 0$ such that for any $\Omega \subset \mathcal{V}$ with finite measure,

$$\frac{\nu(\Omega)}{\nu(\partial\Omega)} \leq \mathcal{I}$$

where $\partial\Omega := \{(i, j) \in \mathcal{E} : i \in \Omega, j \notin \Omega\}$. The only ingredients to prove this isoperimetric inequality are the doubling and reverse doubling conditions, see Section 2.3.3 in [M09]. Then the discrete L^q -Poincaré inequality holds for any $q \geq 1$, with a constant S_d depending only on $q, \eta, \Theta_{inf}, \Theta_{sup}$ and on the doubling and Poincaré constants of $(X, \mathbf{d}, \mathbf{m})$, i.e. on N . In case $q = p^*$, we have $S_d = S_d(N, \eta, p, \Theta_{inf}, \Theta_{sup})$.

STEP 3: The local continuous L^{p^*} -Sobolev-Neumann inequalities.

Let us explain how to get the local continuous L^{p^*} -Sobolev-Neumann inequalities. We start by deriving from the strong local L^p -Sobolev inequality (3.3) a L^p -Sobolev-type inequality on connected Borel subsets of annuli.

Claim 2. *Let $R > 0$ and $\alpha > 1$. Let A be a connected Borel subset of $B_{\alpha R}(o) \setminus B_R(o)$. For $0 < \delta < 1$, denote by $[A]_\delta$ the δ -neighborhood of A , i.e. $[A]_\delta = \bigcup_{x \in A} B_\delta(x)$. Then there exists a constant $C = C(N, \delta, \alpha, p) > 0$ such that for any function $u \in C(X)$ and any upper gradient $g \in L^p([A]_\delta, \mathbf{m})$ of u ,*

$$\left(\int_A |u - u_A|^{p^*} \, \mathbf{d}\mathbf{m} \right)^{1/p^*} \leq C \frac{R^p}{V(o, R)^{p/N}} \left(\int_{[A]_\delta} g^p \, \mathbf{d}\mathbf{m} \right)^{1/p}.$$

Proof. Define $s = \delta R$ and choose an s -lattice of A (i.e. a maximal set of points whose distance between two of them is at least s) $(x_j)_{j \in J}$. Set $V_i = B(x_i, s)$ and $V_i^* = V_i^\# = B(x_i, 3s)$. Using the doubling condition, there is no difficulty in proving that $(V_i, V_i^*, V_i^\#)$ is a good covering of (X, \mathbf{d}) with respect to (\mathbf{m}, \mathbf{m}) . A discrete L^{p^*} -Poincaré inequality holds on the associated weighted graph, as one can easily check following the lines of [M09, Lem. 2.10]. The local continuous L^{p^*} -Sobolev-Neumann inequalities stem from the proof of [M09, Lem. 2.11], where we replace (14) there by (3.3). Then Theorem 3 gives the result. \square

Let us prove that Claim 2 implies the local continuous L^{p^*} -Sobolev-Neumann inequalities with a constant S_c depending only on N, η and p . Take a piece of the good covering $U_{i,a}$. Choose $\delta = (1 - \kappa^{-1})/2$ so that $[U_{i,a}]_\delta \subset U_{i,a}^*$. Take a function $u \in C(X)$ and an upper gradient $g \in L^p([U_{i,a}]_\delta, \mathbf{m})$ of u . Since $|u - \langle u \rangle_{U_{i,a}}| \leq |u - c| + |c - \langle u \rangle_{U_{i,a}}|$ for any $c \in \mathbb{R}$, and since $|x + y|^{p^*} \leq 2^{p^*-1}(|x| + |y|)^{p^*}$ holds for any $x, y \in \mathbb{R}$ by convexity of $t \mapsto |t|^{p^*}$, we have

$$\int_{U_{i,a}} |u - \langle u \rangle_{U_{i,a}}|^{p^*} \, \mathbf{d}\mu \leq 2^{p^*} \inf_{c \in \mathbb{R}} \int_{U_{i,a}} |u - c|^{p^*} \, \mathbf{d}\mu \leq 2^{p^*} \int_{U_{i,a}} |u - u_{U_{i,a}}|^{p^*} w_o \, \mathbf{d}\mathbf{m}.$$

As w_o is a radial function, we can set $\bar{w}_o(r) := w_o(x)$ for any $r > 0$ and any $x \in X$ such that $\mathbf{d}(o, x) = r$. Note that by the Bishop-Gromov theorem, \bar{w}_o is a decreasing function, so

$$\int_{U_{i,a}} |u - \langle u \rangle_{U_{i,a}}|^{p^*} \, \mathbf{d}\mu \leq 2^{p^*} \bar{w}_o(\kappa^{i-1}) \int_{U_{i,a}} |u - u_{U_{i,a}}|^{p^*} \, \mathbf{d}\mathbf{m}.$$

Applying Claim 2 with $A = U_{i,a}$, $R = \kappa^{i-1}$ and $\alpha = \kappa^2$ yields to

$$\begin{aligned} \int_{U_{i,a}} |u - \langle u \rangle_{U_{i,a}}|^{p^*} d\mu &\leq C^{p^*} 2^{p^*} \frac{\kappa^{p^*(i-1)}}{V(o, \kappa^{i-1})^{p^*/N}} \bar{w}_o(\kappa^{i-1}) \left(\int_{U_{i,a}^*} g^p d\mathbf{m} \right)^{p^*/p} \\ &\leq C \left(\int_{U_{i,a}^*} g^p d\mathbf{m} \right)^{p^*/p} \end{aligned}$$

where we used the same letter C to denote different constants depending only on N , p , and κ . As κ depends only on N , η and p , we get the result.

An analogous argument implies the inequalities between levels 2 and 3.

STEP 4: Conclusion.

Apply Theorem 2 to get the result.

4 Weighted Nash inequality and bound of the corresponding heat kernel

In this section, we deduce from Theorem 1 a weighted Nash inequality. We use this result in the context of $\text{RCD}(0, N)$ spaces to get a uniform bound on a corresponding weighted heat kernel.

Theorem 4 (Weighted Nash inequality). *Let $(X, \mathbf{d}, \mathbf{m})$ be a $\text{CD}(0, N)$ space with $N > 2$ satisfying (1.1) with $\eta > 2$. Then there exists a constant $C_{Na} = C_{Na}(N, \eta, \Theta_{inf}, \Theta_{sup}) > 0$ such that:*

$$\|u\|_{L^2(X, \mu)}^{2+\frac{4}{N}} \leq C_{Na} \|u\|_{L^1(X, \mu)}^{\frac{4}{N}} \text{Ch}(u) \quad \forall u \in L^1(X, \mu) \cap H^{1,2}(X, \mathbf{d}, \mathbf{m}),$$

where $\mu \ll \mathbf{m}$ has density $w_o = V(o, \mathbf{d}(o, \cdot))^{2/(N-2)} \mathbf{d}(o, \cdot)^{-2N/(N-2)}$.

To prove this theorem, we need a standard lemma, consequence of [ACDM15, Section 3], which states that the relaxation procedure defining Ch can be performed with slopes of Lipschitz functions with bounded support (we write $\text{Lip}_{bs}(X)$ in the sequel for the space of such functions) instead of upper gradients of L^2 -functions. We provide a proof for the reader's convenience. Note that here and until the end of this section we write $L^p(\mathbf{m})$, $L^p(\mu)$ instead of $L^p(X, \mathbf{m})$, $L^p(X, \mu)$ respectively for any $1 \leq p \leq +\infty$.

Lemma 3. *Let $(X, \mathbf{d}, \mathbf{m})$ be a complete and separable metric measure space, and $u \in H^{1,2}(X, \mathbf{d}, \mathbf{m})$. Then*

$$\text{Ch}(u) = \inf \left\{ \liminf_{n \rightarrow \infty} \int_X |\nabla u_n|^2 d\mathbf{m} : (u_n)_n \subset \text{Lip}_{bs}(X), \|u_n - u\|_{L^2(\mathbf{m})} \rightarrow 0 \right\}.$$

In particular, for any $u \in H^{1,2}(X, \mathbf{d}, \mathbf{m})$, there exists a sequence $(u_n)_n \subset \text{Lip}_{bs}(X)$ such that $\|u - u_n\|_{L^2(\mathbf{m})} \rightarrow 0$ and $\|\nabla u_n\|_{L^2(\mathbf{m})}^2 \rightarrow \text{Ch}(u)$ when $n \rightarrow +\infty$.

Proof. Choose a point $o \in X$ and for every $n \in \mathbb{N} \setminus \{0\}$, let χ_n be a Lipschitz function constant equal to 1 on $B(o, n)$, to 0 on $X \setminus B(o, n+1)$ and such that $|\nabla \chi_n| \leq 2$. Take $f \in \text{Lip}(X) \cap L^2(\mathbf{m})$ and define, for every $n \in \mathbb{N} \setminus \{0\}$, $f_n = f \chi_n$. Using the chain rule and

Young's inequality for some $\varepsilon > 0$, denoting by $\text{Lip}(\chi_n) (\leq 2)$ the Lipschitz constant of χ_n , we get

$$\begin{aligned} |\nabla f_n|^2 &\leq \left(\chi_n |\nabla f| + f \text{Lip}(\chi_n) 1_{B(0,n+1) \setminus B(0,n)} \right)^2 \\ &\leq (1 + \varepsilon) |\nabla f|^2 + 4(1/(1 + \varepsilon)) f^2 1_{B(0,n+1) \setminus B(0,n)}. \end{aligned}$$

Integrating over X and taking the limit superior, it implies

$$\limsup_{n \rightarrow \infty} \int_X |\nabla f_n|^2 \, \text{d}\mathbf{m} \leq (1 + \varepsilon) \int_X |\nabla f|^2 \, \text{d}\mathbf{m},$$

and letting ε go to 0 leads to

$$\limsup_{n \rightarrow \infty} \int_X |\nabla f_n|^2 \, \text{d}\mathbf{m} \leq \int_X |\nabla f|^2 \, \text{d}\mathbf{m}.$$

Then for $u \in H^{1,2}(X, \mathbf{d}, \mathbf{m})$, for any sequence $(u_k)_k \subset \text{Lip}(X) \cap L^2(\mathbf{m})$ L^2 -converging to u , considering for any $k \in \mathbb{N}$ a sequence $(v_{k,n})_n \subset \text{Lip}_{bs}(X)$ built as above, a diagonal argument provides a sequence $(v_{k,n(k)})_k$ such that

$$\liminf_{k \rightarrow \infty} \int_X |\nabla v_{k,n(k)}|^2 \, \text{d}\mathbf{m} \leq \liminf_{k \rightarrow \infty} \int_X |\nabla u_k|^2 \, \text{d}\mathbf{m}.$$

Taking the infimum among all sequences $(u_k)_k$ L^2 -converging to u leads to the result. \square

We are now in a position to prove Theorem 4.

Proof. By the previous lemma it is sufficient to prove the result for $u \in \text{Lip}_{bs}(X)$. By Hölder's inequality,

$$\|u\|_{L^2(\mu)} \leq \|u\|_{L^1(\mu)}^\theta \|u\|_{L^{2^*}(\mu)}^{1-\theta}$$

where $\frac{1}{2} = \frac{\theta}{1} + \frac{1-\theta}{2^*}$ i.e. $\theta = \frac{2}{N+2}$. Then by Theorem 1 applied in the case $p = 2 < \eta$,

$$\|u\|_{L^2(\mu)} \leq C \|u\|_{L^1(\mu)}^{\frac{2}{N+2}} \|\nabla u\|_{L^2(\mathbf{m})}^{\frac{N}{N+2}}.$$

As $u \in \text{Lip}_{bs}(X)$, $\text{Ch}(u) = \|\nabla u\|_{L^2(\mathbf{m})}^2$, so the result follows by raising the previous inequality to the power $2(N+2)/N$. \square

Let us consider now a $\text{RCD}(0, N)$ space $(X, \mathbf{d}, \mathbf{m})$ satisfying the growth condition (1.1) for some $\eta > 2$ and the uniform local N -Ahlfors regularity property :

$$C_o^{-1} \leq \frac{V(x, r)}{r^N} \leq C_o \quad \forall x \in X, \quad \forall 0 < r < r_o \quad (4.1)$$

for some $C_o > 1$ and $r_o > 0$. Note that it follows from [BS18] that N is an integer which coincides with the essential dimension of $(X, \mathbf{d}, \mathbf{m})$.

We take the weight $w_o = V(o, \mathbf{d}(o, \cdot))^{2/(N-2)} \mathbf{d}(o, \cdot)^{-2N/(N-2)}$ which corresponds to the case $p = 2$ in Theorem 1; recall that $\mu = w_o \mathbf{m}$, and note that $L^2(\mathbf{m}) \subset L^2(\mu)$ as w_o is a bounded function (this follows from Bishop-Gromov's theorem and (4.1)).

Set $H_{loc}^{1,2}(X, \mathbf{d}, \mathbf{m}) = \{f \in L_{loc}^2(\mathbf{m}) : \varphi f \in H^{1,2}(X, \mathbf{d}, \mathbf{m}) \quad \forall \varphi \in \text{Lip}_{bs}(X)\}$ and note that as an immediate consequence of the boundedness of w_o , we have $f \in L_{loc}^2(\mathbf{m})$ if and only if $f \in L_{loc}^2(\mu)$.

Define the Dirichlet form Q on $L^2(\mu)$ as follows:

$$Q(f) = \begin{cases} \int_X |\nabla f|_*^2 \, d\mathbf{m} & \text{if } f \in H_{loc}^{1,2}(X, \mathbf{d}, \mathbf{m}) \text{ with } |\nabla f|_* \in L^2(\mathbf{m}) \\ +\infty & \text{otherwise.} \end{cases}$$

Q is easily seen to be convex. Moreover, since convergence in $L_{loc}^2(\mathbf{m})$ and in $L_{loc}^2(\mu)$ are equivalent, Q is a $L^2(\mu)$ -lower semicontinuous functional on $L^2(\mu)$, so we can apply the general theory of gradient flow to define the semigroup $(h_t^\mu)_{t>0}$ associated to Q which is characterized by the property that for any $f \in L^2(X, \mu)$, $t \rightarrow h_t^\mu f$ is locally absolutely continuous on $(0, +\infty)$ with values in $L^2(X, \mu)$, and

$$\frac{d}{dt} h_t^\mu f = -A h_t^\mu f \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, +\infty),$$

where the self-adjoint operator $-A$ associated to Q is defined on a dense subset $\mathcal{D}(A)$ of $\mathcal{D}(Q) = \{Q < +\infty\}$ and characterized by:

$$Q(f, g) = \int_X (Af)g \, d\mu \quad \forall f \in \mathcal{D}(A), \forall g \in \mathcal{D}(Q).$$

Be aware that although Q is defined by integration with respect to \mathbf{m} , it is a Dirichlet form on $L^2(\mu)$, whence the involvement of μ in the above characterization.

Note that by the Markov property, each h_t^μ can be uniquely extended from $L^2(X, \mu) \cap L^1(X, \mu)$ to a contraction from $L^1(X, \mu)$ to itself.

We start with a preliminary lemma stating that a weighted Nash inequality also holds on the appropriate functional space when Ch is replaced by Q .

Lemma 4. *Let $(X, \mathbf{d}, \mathbf{m})$ be a $\text{RCD}(0, N)$ space with $N > 3$ satisfying (1.1) and (4.1) for some $\eta > 2$, $C_o > 1$ and $r_o > 0$. Then there exists a constant $C = C(N, \eta, \Theta_{inf}, \Theta_{sup}) > 0$ such that:*

$$\|u\|_{L^2(\mu)}^{2+\frac{4}{N}} \leq C \|u\|_{L^1(\mu)}^{\frac{4}{N}} Q(u) \quad \forall u \in L^1(\mu) \cap \mathcal{D}(Q).$$

Proof. Let $u \in L^1(\mu) \cap \mathcal{D}(Q)$. Then $u \in L_{loc}^2(\mathbf{m})$, $\varphi u \in H^{1,2}(X, \mathbf{d}, \mathbf{m})$ for any $\varphi \in \text{Lip}_{bs}(X)$ and $|\nabla u|_* \in L^2(\mu)$. In particular, if we take $(\chi_n)_n$ as in the proof of Lemma 3, for any $n \in \mathbb{N}$ we get that $\chi_n u \in H^{1,2}(X, \mathbf{d}, \mathbf{m})$ and consequently there exists a sequence $(u_{n,k})_k \subset \text{Lip}_{bs}(X)$ such that $u_{n,k} \rightarrow \chi_n u$ in $L^2(\mathbf{m})$ and $\int_X |\nabla u_{n,k}|^2 \, d\mathbf{m} \rightarrow \int_X |\nabla(\chi_n u)|_*^2 \, d\mathbf{m}$. Apply Theorem 4 to the functions $u_{n,k}$ to get

$$\|u_{n,k}\|_{L^2(\mu)}^{2+\frac{4}{N}} \leq C \|u_{n,k}\|_{L^1(\mu)}^{\frac{4}{N}} \int_X |\nabla u_{n,k}|^2 \, d\mathbf{m} \quad (4.2)$$

for any $k \in \mathbb{N}$. As the $u_{n,k}$ and $\chi_n u$ have bounded support, and thanks to (4.1) which ensures boundedness of w_o , the $L^2(\mathbf{m})$ convergence $u_{n,k} \rightarrow \chi_n u$ is equivalent to the $L_{loc}^2(\mathbf{m})$, $L_{loc}^2(\mu)$, $L^2(\mu)$ and $L^1(\mu)$ convergences. Therefore, passing to the limit $k \rightarrow +\infty$ in (4.2), we get

$$\|\chi_n u\|_{L^2(\mu)}^{2+\frac{4}{N}} \leq C \|\chi_n u\|_{L^1(\mu)}^{\frac{4}{N}} \int_X |\nabla(\chi_n u)|_*^2 \, d\mathbf{m}.$$

By an argument similar to the proof of Lemma 3, we can show that

$$\limsup_{n \rightarrow +\infty} \int_X |\nabla(\chi_n u)|_*^2 \, d\mathbf{m} \leq \int_X |\nabla u|_*^2 \, d\mathbf{m}.$$

And monotone convergence ensures that $\|\chi_n u\|_{L^2(\mu)} \rightarrow \|u\|_{L^2(\mu)}$ and $\|\chi_n u\|_{L^1(\mu)} \rightarrow \|u\|_{L^1(\mu)}$, whence the result. \square

Let us apply Lemma 4 to get a bound on the heat kernel of Q .

Theorem 5 (Bound of the weighted heat kernel). *Let $(X, \mathbf{d}, \mathbf{m})$ be a $\text{RCD}(0, N)$ space with $N > 3$ satisfying the growth condition (1.1) for some $\eta > 2$ and the uniform local N -Ahlfors regular property (4.1) for some $C_o > 1$ and $r_o > 0$. Then there exists $C = C(N, \eta, \Theta_{inf}, \Theta_{sup}) > 0$ such that*

$$\|h_t^\mu\|_{L^1(\mu) \rightarrow L^\infty(\mu)} \leq \frac{C}{t^{N/2}}, \quad \forall t > 0,$$

or equivalently, for any $t > 0$, h_t^μ admits a kernel p_t^μ with respect to μ such that:

$$p_t^\mu(x, y) \leq \frac{C}{t^{N/2}} \quad \forall x, y \in X.$$

To prove this theorem we follow closely the lines of [SC02, Th. 4.1.1]. The constant C may differ from line to line, note however that it will always depend only on η , N , Θ_{inf} and Θ_{sup} .

Proof. Let $u \in L^1(\mu)$ be such that $\|u\|_{L^1(\mu)} = 1$. Let us show that $\|h_t^\mu u\|_{L^2(\mu)} \leq Ct^{-N/4}$ for any $t > 0$. First of all, by density of $\text{Lip}_{bs}(X)$ in $L^1(\mu)$, we can assume $u \in \text{Lip}_{bs}(X)$ with $\|u\|_{L^1(\mu)} = 1$. Furthermore, since for any $t > 0$, the Markov property ensures that the operator $h_t^\mu : L^1(\mu) \cap L^2(\mu) \rightarrow \mathcal{D}(Q)$ extends uniquely to a contraction operator from $L^1(\mu)$ to itself, we have $h_t^\mu u \in L^1(\mu) \cap \mathcal{D}(Q)$ and $\|h_t^\mu u\|_{L^1(\mu)} \leq 1$. Therefore, we can apply Lemma 4 to get:

$$\|h_t^\mu u\|_{L^2(\mu)}^{2+\frac{4}{N}} \leq CQ(h_t^\mu u) \quad \forall t > 0.$$

As $\int_X |\nabla h_t^\mu u|_*^2 \, \mathbf{d}\mathbf{m} = \int_X (Ah_t^\mu u) h_t^\mu u \, \mathbf{d}\mu = - \int_X \left(\frac{\mathbf{d}}{\mathbf{d}t} h_t^\mu u \right) h_t^\mu u \, \mathbf{d}\mu = -\frac{1}{2} \frac{\mathbf{d}}{\mathbf{d}t} \|h_t^\mu u\|_{L^2(\mu)}^2$, we finally end up with the following differential inequality:

$$\|h_t^\mu u\|_{L^2(\mu)}^{2+4/N} \leq -\frac{C}{2} \frac{\mathbf{d}}{\mathbf{d}t} \|h_t^\mu u\|_{L^2(\mu)}^2 \quad \forall t > 0.$$

Writing $\varphi(t) = \|h_t^\mu u\|_{L^2(\mu)}^2$ and $\psi(t) = \frac{N}{2} \varphi(t)^{-2/N}$ for any $t > 0$, we get $\frac{2}{C} \leq \psi'(t)$ and thus $\frac{2}{C}t \leq \psi(t) - \psi(0)$. As $\psi(0) = \frac{N}{2} \|u\|_{L^2(\mu)}^{-4/N} \geq 0$, we obtain $\frac{2}{C}t \leq \psi(t)$, leading to

$$\|h_t^\mu u\|_{L^2(\mu)} \leq \frac{C}{t^{N/4}}.$$

We have consequently $\|h_t^\mu\|_{L^1(\mu) \rightarrow L^2(\mu)} \leq \frac{C}{t^{N/4}}$. Using the self-adjointness of h_t^μ , we deduce $\|h_t^\mu\|_{L^2(\mu) \rightarrow L^\infty(\mu)} \leq \frac{C}{t^{N/4}}$ by duality. Finally the semigroup property

$$\|h_t^\mu\|_{L^1(\mu) \rightarrow L^\infty(\mu)} \leq \|h_{t/2}^\mu\|_{L^1(\mu) \rightarrow L^2(\mu)} \|h_{t/2}^\mu\|_{L^2(\mu) \rightarrow L^\infty(\mu)}$$

implies the result. \square

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