On the optimality of stripes in a variational model with non-local interactions

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Abstract

We study pattern formation for a variational model displaying competition between a local term penalizing interfaces and a non-local term favoring oscillations. By means of a $\Gamma-$convergence analysis, we show that as the parameter J converges to a critical value J_c, the minimizers converge to periodic one-dimensional stripes. A similar analysis has been previously performed by other authors for related discrete systems. In that context, a central point is that each "angle" comes with a strictly positive contribution to the energy. Since this is not anymore the case in the continuous setting, we need to overcome this difficulty by slicing arguments and a rigidity result.

1 Introduction

Motivated by recent works [18, 19, 21] on striped patterns in Ising models with competing interactions, we consider for $d \geq 2$, $J, L > 0$ and $p > 2d$ the functional

$$\widetilde{\mathcal{F}}_{J,L}(E) := \frac{1}{L^d}\left(J\int_{\partial E\cap[0,L)^d} |\nu^E|_1 d\mathcal{H}^{d-1} - \int_{[0,L)^d\times\mathbb{R}^d} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^p+1}dxdy\right), \tag{1}$$

where E is a $[0,L)^d-$periodic set, ν^E is its external normal and $|\cdot|_1$ denotes the $1-$norm. As in the discrete case, it can be shown (see Proposition 3.5) that for $J \geq J_c := \int_{\mathbb{R}^d} \frac{|\zeta_1|}{1+|\zeta|^p}$, the energy is always non-negative and thus minimizers are the uniform states while for $J < J_c$, there exists non trivial minimizers. We are interested here in the behavior of these minimizers as $J \uparrow J_c$. Building on the computations made in [18], it is expected that for $\tau := J_c - J$ small enough, minimizers are periodic striped patterns. A simple computation (see (30)) shows that the optimal stripes have width of order $\tau^{-1/(p-d-1)}$ and energy of order $-\tau^{(p-d)/(p-d-1)}$. This motivates the rescaling given

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in (9) which yields stripes of width and energy of order one as τ goes to zero. After this rescaling, we are led to study the minimizers of

$$\mathcal{F}_{\tau,L}(E) := \frac{1}{L^d}\left(-\int_{\partial E\cap[0,L)^d} |\nu^E|_1 d\mathcal{H}^{d-1} \right.$$
$$\left. + \int_{\mathbb{R}^d} \frac{1}{|\zeta|^p + \tau^{p/(p-d-1)}} \left[\int_{\partial E\cap[0,L)^d} \sum_{i=1}^d |\nu_i^E||\zeta_i| d\mathcal{H}^{d-1} - \int_{[0,L)^d} |\chi_E(x) - \chi_E(x+\zeta)| dx \right] d\zeta \right).$$

Our main theorem is a $\Gamma-$convergence [7] result for $\mathcal{F}_{\tau,L}$.

Theorem 1.1. *For $p > 2d$ and $L > 0$, the functionals $\mathcal{F}_{\tau,L}$ $\Gamma-$converge as τ goes to zero with respect to the L^1-convergence to the functional defined for sets $E = \widehat{E}\times\mathbb{R}^{d-1}$ where $\widehat{E}$ is $L-$periodic with $\sharp\{\partial\widehat{E}\cap[0,L)\} < +\infty$, by*

$$\mathcal{F}_{0,L}(E) := \frac{1}{L}\left(-\sharp\{\partial\widehat{E}\cap[0,L)\} + \int_{\mathbb{R}^d} \frac{1}{|\zeta|^p} \left[\sum_{x\in\partial\widehat{E}\cap[0,L)} |\zeta_1| - \int_0^L |\chi_{\widehat{E}}(x) - \chi_{\widehat{E}}(x+\zeta_1)| dx \right] d\zeta \right),$$

and $\mathcal{F}_{0,L}(E) := +\infty$ otherwise. Moreover, if E^τ is such that $\sup_\tau \mathcal{F}_{\tau,L}(E^\tau) < +\infty$, then up to a relabeling of the coordinate axes, there is a subsequence which converges in L^1 to some set E with $E = \widehat{E}\times\mathbb{R}^{d-1}$ and $\sharp\{\partial\widehat{E}\cap[0,L)\} < +\infty$.

This theorem is a restatement of Theorem 5.1. Using the method of reflection positivity, we can then compute the minimizers of the limiting energy (see Theorem 6.4).

Theorem 1.2. *There exists $h^\star > 0$ (whose value is given in Lemma 6.1) such that for every $L > 0$, minimizers of $\mathcal{F}_{0,L}$ are periodic stripes of width h with*

$$|h - h^\star| \lesssim \frac{1}{L}.$$

As a direct consequence of Theorem 1.1 and Theorem 1.2, we obtain the following corollary

Corollary 1.3. *Let E^τ be minimizers of $\mathcal{F}_{\tau,L}$. Then, up to a subsequence, they converge as τ goes to zero to stripes of width h with $|h - h^\star| \lesssim \frac{1}{L}$.*

Let us notice that as pointed out in Remark 6.5, for most values of L, the minimizer of $\mathcal{F}_{0,L}$ is unique. In this case, up to a rotation, the whole sequence E^τ converges.

We now give an outline of the proofs of Theorem 1.1 and Theorem 1.2, and discuss the relations and main differences with those in the discrete setting [19, 21]. The main ingredients in the proof of Theorem 1.1 are Lemma 3.2 which permits to identify the part of the energy which penalizes non straight boundaries, the slicing formula (2), the crucial (but simple) estimate (7) which leads to the estimate (12) of the non-local part of the energy by the local widths and gaps and then to an estimate of the perimeter by the total energy, and finally a rigidity result (see Proposition

4.3) which proves that in the limit, sets of finite energy must be one-dimensional. Let us point out that this last ingredient is based on the study of a functional which is somewhat reminiscent of integral characterizations of Sobolev spaces that have recently received a lot of attention [5, 8, 14]. This connection will be further explored in a future work. As in the proofs of [19, 21], a central point is to estimate the cost of "angles". However, this is where the biggest difference between the discrete and the continuous settings appears. In fact, in the discrete one, angles are quantized and thus carry a positive energy (see for instance [21, Lem. 2]) while in the continuous one, they can be extremely small and give almost no contribution to the energy. Moreover, in the discrete case, the number of "angles" gives an upper bound on the perimeter, which is again not the case in the continuous setting. The same observations also hold for the local widths and gaps (which in [19, 21] appear in the form of the distance between bonds facing each others). For all these reasons, it seems difficult in our setting to estimate the contributions of the "angles" for $\tau > 0$. This is the main reason why we need to pass to the limit $\tau \to 0$. This can be compared to the striking result of [21] where the authors were recently able to prove that for τ small enough but positive, periodic stripes are actually local minimizers of the discrete energy. Yet, we believe that our approach based on slicing and on the splitting estimate (4), gives a good insight on some of the more combinatorial proofs of [19, 21]. The proof of Theorem 1.2 is based on reflection positivity [15] and does not differ much for instance from the proofs in [16, 20]. Nevertheless, there are some subtleties coming from the non-integrability of $|z|^{-(p-d+1)}$ around zero. Let us finally point out that besides [17, 20] we are not aware of many examples where reflection positivity has been used in a continuous context.

Let us now comment on some choices we made in (1). First, as in the discrete case, we are restricting ourselves to $p > 2d$. This condition ensures that the energy of stripes scales differently compared to the energy of a checkerboard (see [18]). This is reflected in the fact that for the limiting functional, only striped patterns are admissible. Unfortunately, this leaves out the most interesting cases $d = 3$ and $p = 1$ or $p = 3$, corresponding to Coulombic or Dipole interactions. Second, we decided to work with the $1-$perimeter instead of the usual Euclidean perimeter. This choice makes the splitting and slicing arguments work better by identifying the preferred axes of periodicity. The extension of this work to the classical perimeter will be the subject of further investigations.

The functional (1) is one of the simplest examples of a variational problem with competition between a local term penalizing interfaces and a repulsive non-local term. This competition leads to a complex pattern formation. The closest model to (1) is certainly the sharp interface version of the Ohta-Kawasaki functional which has been used to model diblock copolymers [28, 11, 23, 27] or nuclear matter [29]. Minimizers of this type of variational problems often exhibit periodicity (see for instance [9] for some numerics). However, besides the one-dimensional situation [26] (see also [25, 31] for an almost one-dimensional case) and the low volume fraction limit [10, 22, 24, 6] not much is known. To the best of our knowledge, the only results available on periodicity of minimizers for intermediate volume fractions are [20] and the uniform local energy distribution [12, 2] as well as minimality in the perimeter dominant regime [32, 1, 30, 13]. We refer to [19, 21] for more references in particular on the discrete setting and to the review paper [4] for a discussion of the related issue of crystallization.

The paper is organized as follows. In Section 2, we set some notation and recall basic facts about sets of finite perimeter. In Section 3, we derive the functional $\mathcal{F}_{\tau,L}$ from (1) and prove some useful estimates. In Section 4, we prove the rigidity result Proposition 4.3. Then, in Section 5, we prove our $\Gamma-$convergence result. Finally in Section 6, we study the minimizers of the limiting problem.

2 Notation

In the paper we will use the following notation. The symbols $\sim$, $\gtrsim$, $\lesssim$ indicate estimates that hold up to a global constant. For instance, $f \lesssim g$ denotes the existence of a constant $C > 0$ such that $f \leq Cg$, $f \sim g$ means $f \lesssim g$ and $g \lesssim f$. We let $(e_1, \cdots, e_d)$ be the canonical basis of $\mathbb{R}^d$. For $(x, \zeta) \in (\mathbb{R}^d)^2$ and $i \in \{1, \dots, d\}$, we let $x + \zeta_i := x + \zeta_i e_i$, and then $x_i^{\perp} := x - x_i$. We will denote by $|x|$ the Euclidean norm of x and by $|x|_1 := \sum_{i=1}^d |x_i|$ its $1-$norm. For $L > 0$, we will let $Q_L := [0, L)^d$. We let $\partial_i f := \frac{\partial f}{\partial x_i}$. When it is clear from the context, we will not specify the measure of integration in the integrals. We take as a convention that whenever we integrate over (x, ζ) (respectively (x, z)), the integral on the unbounded domain always concerns ζ (respectively z). For a $k-$dimensional set $E \subset \mathbb{R}^k$, we let $|E|$ be its Lebesgue measure.

2.1 Sets of finite perimeter

The purpose of this section is to recall the definition of sets of finite perimeter. For a general introduction we refer to [3]. Let us start with the one-dimensional case. We say that a $L-$periodic set $E \subset \mathbb{R}$ is of finite perimeter if $E \cap [0, L) = \cup_{i=1}^N I_i$ for some $N \in \mathbb{N}$ and some disjoint intervals I_i. We then let

$$\mathrm{Per}(E, [0, L)) := 2N.$$

By periodicity, we will often assume that $E \cap [0, L] = \cup_{i=1}^N (s_i, t_i)$ with $s_1 > 0$ and $t_N < L$. For any interval I, we let $\mathrm{Per}(E, I) := \sharp\{\partial E \cap I\}$.

We can now turn to higher dimension

Definition 2.1. *A Q_L-periodic set E is said to be of finite perimeter if $D\chi_E$, the distributional derivative of χ_E, is a locally finite measure. For such a set, we let ∂E be the collection of all points $x \in \mathrm{spt}(D\chi_E)$ such that the limit*

$$\nu^E(x) := -\lim_{\rho \downarrow 0} \frac{D\chi_E(B(x, \rho))}{|D\chi_E|(B(x, \rho))}$$

exists and satisfies $|\nu^E(x)| = 1$. We call ν^E the generalized outer normal to E. We then have $D\chi_E = -\nu^E \mathcal{H}^{d-1} \llcorner \partial E$, where $\mathcal{H}^{d-1}$ is the $(d-1)-$dimensional Hausdorff measure.

We then define

$$\mathrm{Per}_1(E, Q_L) := \int_{\partial E \cap Q_L} |\nu^E(x)|_1 \, \mathrm{d}\mathcal{H}^{d-1}(x) = \int_{\partial E \cap Q_L} \sum_{i=1}^d |\nu_i^E(x)| \, \mathrm{d}\mathcal{H}^{d-1}(x).$$

As for the one-dimensional case, by periodicity we will always assume that $|D\chi_E|(\partial Q_L) = 0$ so that $\mathrm{Per}_1(E, Q_L)$ coincides with the $1-$perimeter of E in $(0, L)^d$.

For $i \in \{1, \dots, d\}$, $x_i^\perp \in [0, L)^{d-1}$ and $E \subset Q_L$, we define the one-dimensional slices

$$E_{x_i^\perp} := \left\{ x_i \in [0, L) : \ x_i + x_i^\perp \in E \right\}.$$

Note that in the above definition there is an abuse of notation as the information on the direction of the slice is contained in the index $x_i^\perp$. As it would be always clear from the context which is the direction of the slicing, we hope this will not cause confusion to the reader.

Then, for every $i \in \{1, \dots, d\}$, the following slicing formula holds (see [3, § 3.7])

$$\int_{\partial E \cap Q_L} |\nu_i^E| = \int_{[0,L)^{d-1}} \mathrm{Per}(E_{x_i^\perp}, [0, L)). \tag{2}$$

Let us point out that with the exception of Section 4, by approximation we can always work with polygonal sets having only horizontal and vertical edges. For these sets, both the definition of $\mathrm{Per}_1(E, Q_L)$ and (2) can be obtained without referring to the theory of sets of finite perimeter.

3 The functional and preliminary results

We recall that we are considering the functional

$$\widetilde{\mathcal{F}}_{J,L}(E) := \frac{1}{L^d} \left(J \,\mathrm{Per}_1(E, Q_L) - \int_{Q_L \times \mathbb{R}^d} K_1(x - y) |\chi_E(x) - \chi_E(y)| \right),$$

where $K_1(\zeta) := \frac{1}{|\zeta|^p + 1}$ for some $p > 2d$. More generally, for $\tau \geq 0$, we let $K_\tau(\zeta) := \frac{1}{|\zeta|^p + \tau^{p/(p-d-1)}}$. Notice that the functional $\widetilde{\mathcal{F}}_{J,L}$ can also be written as

$$\widetilde{\mathcal{F}}_{J,L}(E) = \frac{1}{L^d} \left(J \,\mathrm{Per}_1(E, Q_L) - \int_{Q_L \times \mathbb{R}^d} K_1(\zeta) |\chi_E(x) - \chi_E(x + \zeta)| \right). \tag{3}$$

The aim of this section is to give some first properties of $\widetilde{\mathcal{F}}_{J,L}$. We will in particular show that there exists a positive constant J_c such that for $J > J_c$ all minimizers of $\widetilde{\mathcal{F}}_{J,L}$ are trivial while for $J < J_c$ they are not. This will lead us to study the behavior of these minimizers in term of the parameter $\tau := J_c - J$ after suitable rescaling.

Remark 3.1. *Since E is periodic,*

$$\int_{Q_L \times \mathbb{R}^d} K_1(x - y) |\chi_E(x) - \chi_E(y)| = 2 \int_{Q_L \times \mathbb{R}^d} K_1(x - y) \chi_E(x) \chi_{E^c}(y).$$

Indeed, since $|\chi_E(x)-\chi_E(x+\zeta)| = \chi_E(x)\chi_{E^c}(x+\zeta)+\chi_E(x+\zeta)\chi_{E^c}(x)$*, it follows from* (3) *and*

$$\begin{aligned}\int_{Q_L\times\mathbb{R}^d} K_1(\zeta)\chi_E(x)\chi_{E^c}(x+\zeta) &= \int_{\mathbb{R}^d} K_1(\zeta)\int_{Q_L+\zeta}\chi_E(\widetilde{x}-\zeta)\chi_{E^c}(\widetilde{x})\\ &= \int_{\mathbb{R}^d} K_1(\zeta)\int_{Q_L}\chi_E(\widetilde{x}-\zeta)\chi_{E^c}(\widetilde{x})\\ &= \int_{Q_L\times\mathbb{R}^d} K_1(\zeta)\chi_{E^c}(x)\chi_E(x+\zeta).\end{aligned}$$

We recall that for $(x,\zeta)\in(\mathbb{R}^d)^2$, we let $x+\zeta_i := x+\zeta_i e_i$ and $\zeta_i^\perp := \zeta-\zeta_i$.

Lemma 3.2. *For every* Q_L-*periodic set* E *and every* $\tau>0$*, it holds*

$$\begin{aligned}\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)|\chi_E(x)-\chi_E(x+\zeta)| \le \int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)\sum_{i=1}^d|\chi_E(x)-\chi_E(x+\zeta_i)|\\ -\frac{2}{d}\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)\sum_{i=1}^d|\chi_E(x)-\chi_E(x+\zeta_i)||\chi_E(x)-\chi_E(x+\zeta_i^\perp)|. \quad (4)\end{aligned}$$

Proof. We claim that for every $i\in\{1,\dots,d\}$,

$$\begin{aligned}\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)|\chi_E(x)-\chi_E(x+\zeta)| \le \int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)\sum_{i=1}^d|\chi_E(x)-\chi_E(x+\zeta_i)|\\ -2\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)|\chi_E(x)-\chi_E(x+\zeta_i)||\chi_E(x)-\chi_E(x+\zeta_i^\perp)|. \quad (5)\end{aligned}$$

Summing (5) over i and dividing by d, would then yield (4). Without loss of generality, we can assume that $i=1$. By disjunction of cases, it can be seen that for every x,ζ,

$$\begin{aligned}|\chi_E(x)-\chi_E(x+\zeta)| = |\chi_E(x)-\chi_E(x+\zeta_1)|+|\chi_E(x+\zeta_1)-\chi_E(x+\zeta)|\\ -2|\chi_E(x)-\chi_E(x+\zeta_1)||\chi_E(x+\zeta_1)-\chi_E(x+\zeta)|. \quad (6)\end{aligned}$$

We thus have by integration and using the periodicity of E as in Remark 3.1,

$$\begin{aligned}\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)|\chi_E(x)-\chi_E(x+\zeta)| = \int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)(|\chi_E(x)-\chi_E(x+\zeta_1)|+|\chi_E(x)-\chi_E(x+\zeta_1^\perp)|)\\ -2\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)|\chi_E(x)-\chi_E(x+\zeta_1)||\chi_E(x)-\chi_E(x+\zeta_1^\perp)|.\end{aligned}$$

Using that by the triangle inequality,

$$|\chi_E(x)-\chi_E(x+\zeta_1^\perp)| \le \sum_{k=2}^d\left|\chi_E\left(x+\sum_{j=2}^{k-1}\zeta_j\right)-\chi_E\left(x+\sum_{j=2}^{k}\zeta_j\right)\right|,$$

and using again periodicity of E, we get (5).

□

Remark 3.3. *Using* (6) *recursively, one could get an equality in* (4) *by replacing the term* $\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)\sum_{i=1}^d |\chi_E(x)-\chi_E(x+\zeta_i)||\chi_E(x)-\chi_E(x+\zeta_i^\perp)|$ *by a more complex one. However, for our purpose this simpler bound is sufficient. Notice also that if* E *depends only on one variable, then equality holds.*

For a $L-$periodic set $E=\cup_{i\in\mathbb{Z}}(s_i,t_i)\subset\mathbb{R}$ of finite perimeter, we let for $i\in\mathbb{Z}$,

$$h(t_i):=h(s_i):=t_i-s_i \qquad g(t_i):=s_{i+1}-t_i \qquad g(s_i):=s_i-t_{i-1}.$$

We then define for $z\in\mathbb{R}$, and $i\in\mathbb{Z}$,

$$\eta(t_i,z):=\min(z_+,h(t_i))+\min(z_-,g(t_i))$$

and

$$\eta(s_i,z):=\min(z_+,g(s_i))+\min(z_-,h(s_i)).$$

For a Q_L-periodic set E of finite perimeter, the functions h_{x_i}, g_{x_i} and η_{x_i} are defined by slicing. For instance, for $x_1\in E_{x_1^\perp}$ such that $\nu_1^E(x_1,x_1^\perp)>0$ (so that $x_1=t_i$ for some i) and $\zeta\in\mathbb{R}^d$,

$$\eta_{x_1^\perp}(x_1,\zeta_1):=\min((\zeta_1)_+,h_{x_1^\perp}(x_1))+\min((\zeta_1)_-,g_{x_1^\perp}(x_1)).$$

We can now prove a simple but crucial estimate

Lemma 3.4. *For every* $L-$*periodic set* E *of finite perimeter and every* $z\in\mathbb{R}$*, it holds*

$$\int_0^L |\chi_E(x)-\chi_E(x+z)|dx\le\sum_{x\in\partial E\cap[0,L)}\eta(x,z). \tag{7}$$

By integration, for every $\tau\ge 0$*, every* Q_L-*periodic set* E *of finite perimeter, every* $\zeta\in\mathbb{R}^d$ *and* $i\in\{1,\dots,d\}$*,*

$$\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)|\chi_E(x)-\chi_E(x+\zeta_i)|\le\int_{\partial E\cap Q_L}|\nu_i^E|\int_{\mathbb{R}^d}K_\tau(\zeta)\eta_{x_i^\perp}(x_i,\zeta_i). \tag{8}$$

Proof. Let us prove (7). We consider only the case $z\ge 0$ since the case $z\le 0$ can be treated in a similar way. Up to a translation, we can assume that $E\cap[0,L)=\cup_{i=1}^N(s_i,t_i)$ for some $N\in\mathbb{N}$. We then have by periodicity of E,

$$\begin{aligned}\int_0^L|\chi_E(x)-\chi_E(x+z)| &= \sum_{i=1}^N\int_{s_i}^{t_i}\chi_{E^c}(x+z)+\int_0^{s_1}\chi_E(x+z)+\int_{t_N}^L\chi_E(x+z)+\sum_{i=2}^N\int_{t_{i-1}}^{s_i}\chi_E(x+z)\\ &= \sum_{i=1}^N\int_{s_i}^{t_i}\chi_{E^c}(x+z)+\sum_{i=1}^N\int_{t_{i-1}}^{s_i}\chi_E(x+z).\end{aligned}$$

For every $i\in[1,N]$, if $x\in(s_i,t_i)$ and $x+z\in E^c$, then $x+z\ge t_i$ and thus $|x-t_i|\le\min(z,h(t_i))=\eta(t_i,z)$. Similarly, for $i\in[1,N-1]$, if $x\in(t_{i-1},s_i)$ and $x+z\in E$, then $|x-s_i|\le\eta(s_i,z)$. Therefore,

$$\int_0^L|\chi_E(x)-\chi_E(x+z)|dx\le\sum_{i=1}^N\eta(s_i,z)+\eta(t_i,z),$$

which proves (7). □

Proposition 3.5. *For $J \geq J_c := \int_{\mathbb{R}^d} K_1(\zeta)|\zeta_1|$ and every Q_L-periodic set E, $\widetilde{\mathcal{F}}_{J,L}(E) \geq 0$.*

Proof. Putting Lemma 3.2 together with (8), we get

$$\begin{aligned}\widetilde{\mathcal{F}}_{J,L}(E) &\geq \frac{1}{L^d}\left(J\,\mathrm{Per}_1(E,Q_L) - \sum_{i=1}^d \int_{\partial E\cap Q_L} |\nu_i^E|\left[\int_{\mathbb{R}^d} K_1(\zeta)\eta_{x_i^\perp}(x_i,\zeta_i)\right]\right)\\ &\geq \frac{1}{L^d}\sum_{i=1}^d \int_{\partial E\cap Q_L} |\nu_i^E|\left[J - \int_{\mathbb{R}^d} K_1(\zeta)|\zeta_i|\right]\\ &= \frac{1}{L^d}\int_{\partial E\cap Q_L} |\nu^E|_1\left[J - \int_{\mathbb{R}^d} K_1(\zeta)|\zeta_1|\right],\end{aligned}$$

which proves the claim. □

Letting $\tau := J_c - J$ for $J < J_c$, it is possible (see for instance the proof of (30) below) to compute the energy of periodic stripes E_h of period h to get

$$\widetilde{\mathcal{F}}_{J,L}(E_h) \simeq -\frac{\tau}{h} + h^{-(p-d)}.$$

Optimizing in h, we find that the optimal stripes have a width of order $\tau^{-1/(p-d-1)}$ and energy of order $-\tau^{(p-d)/(p-d-1)}$. Letting $\beta := p-d-1$, this motivates the rescaling

$$x := \tau^{-1/\beta}\widehat{x}, \quad L := \tau^{-1/\beta}\widehat{L} \quad \text{and} \quad \widetilde{\mathcal{F}}_{J,L}(E) := \tau^{(p-d)/\beta}\mathcal{F}_{\tau,\widehat{L}}(\widehat{E}). \tag{9}$$

In these variables, the optimal stripes have width of order one. From now on, when there is no ambiguity, we will drop the hats. Let us define $\widehat{K}_\tau(z) := \int_{\mathbb{R}^{d-1}} K_\tau(z,\xi)d\xi$. Since for $z \in \mathbb{R}$,

$$\frac{1}{|z|^{p-d+1} + \tau^{(p-d+1)/\beta}} \lesssim \int_{\mathbb{R}^{d-1}} \frac{1}{(|z|^2+|\xi|^2)^{p/2} + \tau^{p/\beta}}\,\mathrm{d}\xi \lesssim \frac{1}{|z|^{p-d+1} + \tau^{(p-d+1)/\beta}}, \tag{10}$$

we have $\widehat{K}_\tau(z) \simeq \frac{1}{|z|^q + \tau^{q/\beta}}$, where we have let $q := p-d+1$. We then let for $i \in \{1,\dots,d\}$,

$$\mathcal{G}^i_{\tau,L}(E) := \frac{1}{L^d}\int_{\mathbb{R}} \widehat{K}_\tau(\zeta_i)\left[\int_{\partial E\cap Q_L} |\nu_i^E||\zeta_i| - \int_{Q_L} |\chi_E(x) - \chi_E(x+\zeta_i)|\right],$$

and

$$I_{\tau,L}(E) := \frac{2}{dL^d}\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)\sum_{i=1}^d |\chi_E(x) - \chi_E(x+\zeta_i)||\chi_E(x) - \chi_E(x+\zeta_i^\perp)|.$$

Notice that (8) in particular implies that for every ζ and $i \in \{1,\dots,d\}$,

$$\int_{\partial E\cap Q_L} |\nu_i^E||\zeta_i| - \int_{Q_L} |\chi_E(x) - \chi_E(x+\zeta_i)| \geq 0. \tag{11}$$

We define also for $E \subset \mathbb{R}$, $L-$periodic and of finite perimeter, the one-dimensional functionals

$$\mathcal{G}^{1d}_{\tau,L}(E) := \int_{\mathbb{R}} \widehat{K}_\tau(z)\left(\mathrm{Per}(E,[0,L))|z| - \int_0^L |\chi_E(x) - \chi_E(x+z)|\right),$$

so that by Fubini,

$$\mathcal{G}^i_{\tau,L}(E) = \frac{1}{L^d}\int_{[0,L]^{d-1}} \mathcal{G}^{1d}_{\tau,L}(E_{x_i^\perp})dx_i^\perp.$$

Lemma 3.6. *For every Q_L-periodic set E of finite perimeter, we have*

$$\mathcal{F}_{\tau,L}(E) = \frac{1}{L^d}\left(-\operatorname{Per}_1(E,Q_L) + \int_{\mathbb{R}^d} K_\tau(\zeta)\left[\int_{\partial E\cap Q_L}\sum_{i=1}^d |\nu_i^E||\zeta_i| - \int_{Q_L}|\chi_E(x)-\chi_E(x+\zeta)|\right]\right).$$

By Lemma 3.2, this yields

$$\mathcal{F}_{\tau,L}(E) \geq -\frac{1}{L^d}\operatorname{Per}_1(E,Q_L) + \sum_{i=1}^d \mathcal{G}^i_{\tau,L}(E) + I_{\tau,L}(E).$$

Proof. By writing that $J\operatorname{Per}_1(E,Q_L) = -\tau\operatorname{Per}_1(E,Q_L) + \int_{\mathbb{R}^d} K_1(\zeta)\int_{\partial E\cap Q_L}\sum_{i=1}^d |\nu_i^E||\zeta_i|$, we get

$$\widetilde{\mathcal{F}}_{J,L}(E) = \frac{1}{L^d}\left(-\tau\operatorname{Per}_1(E,Q_L) + \int_{\mathbb{R}^d} K_1(\zeta)\left[\int_{\partial E\cap Q_L}\sum_{i=1}^d |\nu_i^E||\zeta_i| - \int_{Q_L}|\chi_E(x)-\chi_E(x+\zeta)|\right]\right).$$

Making the change of variables given in (9) and letting also $\zeta := \tau^{-1/\beta}\widehat{\zeta}$, we obtain

$$\begin{aligned}\widetilde{\mathcal{F}}_{J,L}(E) =& \frac{\tau^{d/\beta}}{\widehat{L}^d}\Big(-\tau^{(p-2d)/\beta}\operatorname{Per}_1(\widehat{E},Q_{\widehat{L}}) \\ &+ \int_{\mathbb{R}^d}\frac{\tau^{-2d/\beta}}{\tau^{-p/\beta}|\widehat{\zeta}|^p+1}\left[\int_{\partial\widehat{E}\cap Q_{\widehat{L}}}\sum_{i=1}^d |\nu_i^{\widehat{E}}||\widehat{\zeta}_i| - \int_{Q_{\widehat{L}}}|\chi_{\widehat{E}}(\widehat{x})-\chi_{\widehat{E}}(\widehat{x}+\widehat{\zeta})|\right]\Big) \\ =& \frac{\tau^{(p-d)/\beta}}{\widehat{L}^d}\left(-\operatorname{Per}_1(\widehat{E},Q_{\widehat{L}}) + \int_{\mathbb{R}^d} K_\tau(\widehat{\zeta})\left[\int_{\partial\widehat{E}\cap Q_{\widehat{L}}}|\nu_1^{\widehat{E}}||\widehat{\zeta}_1| + |\nu_2^{\widehat{E}}||\widehat{\zeta}_2| - \int_{Q_{\widehat{L}}}|\chi_{\widehat{E}}(\widehat{x})-\chi_{\widehat{E}}(\widehat{x}+\widehat{\zeta})|\right]\right).\end{aligned}$$

□

Before closing this section, we prove several estimates which are consequences of (7).

Lemma 3.7. *Let $E\subset\mathbb{R}$ be a $L-$periodic set of finite perimeter and let $\beta := p-d-1$. Then, for every $\tau\geq 0$,*

$$\mathcal{G}^{1d}_{\tau,L}(E) \gtrsim \sum_{x\in\partial E\cap[0,L)} \min(h(x)^{-\beta},\tau^{-1}) + \min(g(x)^{-\beta},\tau^{-1}). \tag{12}$$

As a consequence, if $\mathcal{G}^{1d}_{\tau,L}(E)\lesssim\tau^{-1}$, then

$$\min_{x\in\partial E\cap[0,L)}\min(h(x),g(x)) \gtrsim \mathcal{G}^{1d}_{\tau,L}(E)^{-1/\beta}. \tag{13}$$

For $L \geq r > 0$ and $t \in [0, L)$, let $I_t(r) := (t - r/2, t + r/2)$. Then, for every $\delta \geq \tau^{1/\beta}$,

$$\mathrm{Per}(E, I_t(r)) - 1 \lesssim r\delta^{-1} + \delta^{\beta} \mathcal{G}^{1d}_{\tau,L}(E). \tag{14}$$

In particular, for $r = L$, after optimizing in δ this implies

$$\mathrm{Per}(E, [0, L)) \lesssim 1 + \max(\tau \mathcal{G}^{1d}_{\tau,L}(E), (L\mathcal{G}^{1d}_{\tau,L}(E))^{1/(p-d)}). \tag{15}$$

Proof. We start by proving (12). By (7),

$$\begin{aligned}
\mathcal{G}^{1d}_{\tau,L}(E) &= \int_{\mathbb{R}} \widehat{K}_\tau(z) \left(|z| \,\mathrm{Per}(E, [0, L)) - \int_0^L |\chi_E(x) - \chi_E(x + z)| \right) \\
&\geq \int_{\mathbb{R}} \widehat{K}_\tau(z) \sum_{x \in \partial E \cap [0,L)} (|z| - \eta(x, z)) \\
&\gtrsim \sum_{x \in \partial E \cap [0,L)} \int_{\mathbb{R}} \frac{1}{|z|^q + \tau^{q/\beta}} \left(|z| - \eta(x, z) \right),
\end{aligned}$$

where in the last line, we have used (10). Assume now that $E = \cup_{i \in \mathbb{Z}} (s_i, t_i)$ and that $x = s_i$ for some $i \in \mathbb{Z}$ (the case $x = t_i$ can be treated analogously). Then, for $g(x) \geq z \geq -h(x)$, $|z| - \eta(x, z) = 0$. Therefore,

$$\begin{aligned}
\mathcal{G}^{1d}_{\tau,L}(E) &\gtrsim \sum_{x \in \partial E \cap [0,L)} \int_{h(x)}^{+\infty} \frac{z - h(x)}{z^q + \tau^{q/\beta}} + \int_{g(x)}^{+\infty} \frac{z - g(x)}{z^q + \tau^{q/\beta}} \\
&\gtrsim \sum_{x \in \partial E \cap [0,L)} \min(h(x)^{-\beta}, \tau^{-1}) + \min(g(x)^{-\beta}, \tau^{-1}),
\end{aligned}$$

where we have used that $q - 2 = \beta$. Estimate (13) follows directly from (12). Let us finally prove (14). Let

$$A_\delta := \{x \in \partial E \cap I_t(r) \ : \min(h(x), g(x)) \geq \delta\},$$

so that in particular $\sharp A_\delta \leq 1 + Cr\delta^{-1}$ and for $x \in A_\delta^c \cap \partial E \cap I_t(r)$, since $\delta \geq \tau^{1/\beta}$,

$$1 \lesssim \delta^{\beta} \left(\min(h(x)^{-\beta}, \tau^{-1}) + \min(g(x)^{-\beta}, \tau^{-1}) \right).$$

Using (12), we obtain

$$\begin{aligned}
\mathrm{Per}(E, I_t(r)) - 1 &= -1 + \sum_{x \in A_\delta} 1 + \sum_{x \in A_\delta^c \cap \partial E \cap I_t(r)} 1 \\
&\lesssim r\delta^{-1} + \sum_{x \in A_\delta^c \cap \partial E \cap I_t(r)} \delta^{\beta} \left(\min(h(x)^{-\beta}, \tau^{-1}) + \min(g(x)^{-\beta}, \tau^{-1}) \right) \\
&\lesssim r\delta^{-1} + \delta^{\beta} \mathcal{G}^{1d}_{\tau,L}(E),
\end{aligned}$$

which proves (14).

□

Remark 3.8. *Estimate* (15) *shows that for every* i*, the function* $x_i^\perp \to \mathrm{Per}(E_{x_i^\perp}, [0,L))$ *is almost in* $L^{p-d}([0,L)^{d-1})$*.*

A simple consequence of (14) is that the perimeter and the energy are controlled by the non-local terms.

Lemma 3.9. *For every* Q_L-*periodic set* E*,*

$$\mathcal{F}_{\tau,L}(E) \gtrsim -1 + \sum_{i=1}^{d} \mathcal{G}^i_{\tau,L}(E) + I_{\tau,L}(E) \tag{16}$$

and

$$\mathrm{Per}_1(E, Q_L) \lesssim L^d \max(1, \mathcal{F}_{\tau,L}(E)). \tag{17}$$

Proof. Estimate (16) follows from integrating (15) and Young's inequality. Estimate (17) then follows by integrating (14) applied to $r = L$ and $\delta = 1$. □

We finally give another consequence of (14) which resembles (15). Since we are going to use it only for $\tau = 0$, we state it only in that case but an analogous statement holds for $\tau > 0$.

Lemma 3.10. *For every* Q_L-*periodic set* E *of finite perimeter and every* $m \in \mathbb{N}$ *with* $m \geq 2$*,* $t \in [0,L)$*,* $L \geq r > 0$ *and* $i \in \{1, \dots, d\}$*,*

$$|\{x_i^\perp \in [0,L)^{d-1} \ : \ \mathrm{Per}(E_{x_i^\perp}, I_t(r)) = m\}| \lesssim \mathcal{G}^i_{0,L}(E) L^d r^\beta \left(\frac{1}{m-1}\right)^{p-d}. \tag{18}$$

Proof. Let $\mathcal{B}_m(I_t(r)) := \{x_i^\perp \in [0,L)^{d-1} \ : \ \mathrm{Per}(E_{x_i^\perp}, I_t(r)) = m\}$ and let $x_i^\perp \in \mathcal{B}_m(I_t(r))$. Thanks to (14), for every $\delta > 0$,

$$m - 1 \lesssim r\delta^{-1} + \delta^\beta \mathcal{G}^{1d}_{0,L}(E_{x_i^\perp}).$$

Optimizing in δ, we get $m - 1 \lesssim r^{\beta/(\beta+1)} \mathcal{G}^{1d}_{0,L}(E_{x_i^\perp})^{1/(\beta+1)}$, which can be equivalently written as

$$\mathcal{G}^{1d}_{0,L}(E_{x_i^\perp}) \gtrsim r^{-\beta}(m-1)^{p-d}.$$

We can thus conclude that

$$\mathcal{G}^i_{0,L}(E) \geq \frac{1}{L^d} \int_{\mathcal{B}_m(I_t(r))} \mathcal{G}^{1d}_{0,L}(E_{x_i^\perp}) \gtrsim |\mathcal{B}_m(I_t(r))| L^{-d} r^{-\beta} (m-1)^{p-d},$$

from which (18) follows.

□

4 A rigidity result

In this section, we prove that in the limit $\tau = 0$, sets of finite energy must be stripes. For a set E of finite perimeter, we introduce the measures

$$\mu_i := |\partial_i \chi_{E_{x_i^\perp}}| \otimes dx_i^\perp,$$

so that actually by the slicing formula [3, Th. 3.107] (see also [3, Cor. 2.29]), $|\partial_i \chi_E| = \mu_i$ for $i \in \{1, \dots, d\}$.

We then define for $x \in Q_L$ and $i \in \{1, \dots, d\}$ the lower $(d-1)-$dimensional densities

$$\Theta_i(x) := \liminf_{r \to 0} \frac{\mu_i(Q_x(r))}{r^{d-1}},$$

where $Q_x(r) := x + [-r/2, r/2)^d$. In particular, $\Theta_i = |\nu_i^E|$, $\mathcal{H}^{d-1}-$a.e. on ∂E.

Lemma 4.1. *Let E be a Q_L- periodic set of finite perimeter and such that $\sum_{i=1}^d \mathcal{G}_{0,L}^i(E) + I_{0,L}(E) < +\infty$. Then, for every $x \in Q_L$ and every $i \in \{1, \dots, d\}$, $\Theta_i(x) \in \{0, 1\}$.*

Proof. For definiteness, we prove the assertion for Θ_1. Let us first show that $\Theta_1 \leq 1$. We recall that $\beta = p - d - 1 \geq d - 1$.

For $r > 0$, let

$$\mathcal{S}_r := \{x_1^\perp \in [0, L)^{d-1} \ : \ \min_{x_1 \in \partial E_{x_1^\perp}} \min(g_{x_1^\perp}(x_1), h_{x_1^\perp}(x_1)) > r\}.$$

For $x_1^\perp \in \mathcal{S}_r^c$, by (13), $r \gtrsim \mathcal{G}_{0,L}^{1d}(E_{x_1^\perp})^{-1/\beta}$, that is $\mathcal{G}_{0,L}^{1d}(E_{x_1^\perp}) \gtrsim r^{-\beta}$. Integrating this, we get that

$$|\mathcal{S}_r^c| \lesssim \mathcal{G}_{0,L}^1(E) L^d r^\beta. \tag{19}$$

We claim that for $\bar{x} = (\bar{x}_1, \bar{x}_1^\perp) \in Q_L$,

$$\Theta_1(\bar{x}) = \liminf_{r \to 0} \frac{1}{r^{d-1}} \int_{Q'_{\bar{x}_1^\perp}(r) \cap \mathcal{S}_r} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r)) dx_1^\perp, \tag{20}$$

where for $(x_1, x_1^\perp) \in [0, L)^d$ and $r > 0$, $Q'_{\bar{x}_1^\perp}(r) := \bar{x}_1^\perp + [-r/2, r/2)^{d-1}$ and $I_{\bar{x}_1}(r) := \bar{x}_1 + [-r/2, r/2)$. Indeed, letting as in the proof of (18), $\mathcal{B}_m(I_{\bar{x}_1}(r)) := \{x_1^\perp \in [0, L)^{d-1} \ : \ \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r)) = m\}$, for $r > 0$ we have

$$\int_{Q'_{\bar{x}_1^\perp}(r) \cap \mathcal{S}_r^c} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r)) dx_1^\perp = \int_{Q'_{\bar{x}_1^\perp}(r) \cap \mathcal{S}_r^c \cap \mathcal{B}_2^c(I_{\bar{x}_1}(r))} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r)) dx_1^\perp$$
$$+ \int_{Q'_{\bar{x}_1^\perp}(r) \cap \mathcal{S}_r^c \cap \mathcal{B}_2(I_{\bar{x}_1}(r))} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r)) dx_1^\perp.$$

Since on the one hand, from (19)

$$\int_{Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r^c\cap\mathcal{B}_2(I_{\bar{x}_1}(r))} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r))dx_1^\perp \le 2|\mathcal{S}_r^c| \lesssim \mathcal{G}^1_{0,L}(E)L^d r^\beta,$$

and on the other hand, thanks to (18) and $\beta > d-1 \ge 1$,

$$\int_{Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r^c\cap\mathcal{B}_2^c(I_{\bar{x}_1}(r))} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r))dx_1^\perp \le \sum_{m=3}^{+\infty} m|\mathcal{B}_m(I_{\bar{x}_1}(r))| \lesssim \mathcal{G}^1_{0,L}(E)L^d r^\beta,$$

we get

$$\int_{Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r^c} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r))dx_1^\perp \lesssim \mathcal{G}^1_{0,L}(E)L^d r^\beta. \tag{21}$$

From this (20) follows. Since for $x_1^\perp \in \mathcal{S}_r$ and $I \subset [0,L)$ with $|I| \le r$, $\mathrm{Per}(E_{x_1^\perp}, I) \in \{0,1\}$, (20) implies that $\Theta_1 \le 1$.

Assume now for the sake of contradiction that there exists $\bar{x} \in Q_L$ such that $0 < \Theta_1(\bar{x}) < 1$. Let

$$A := \{x_1^\perp \in Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r \ : \ \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r)) = 1\} \qquad \text{and}$$
$$B := \{x_1^\perp \in Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r \ : \ \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r)) = 0\}.$$

Then, thanks to (21), there exists $\delta > 0$ such that for all $\bar{r} > 0$ there exists $0 < r \le \bar{r}$, with

$$\delta r^{d-1} \le |A| \le (1-\delta)r^{d-1} \qquad \text{and} \qquad \delta r^{d-1} \le |B| \le (1-\delta)r^{d-1}.$$

Letting

$$\widetilde{A} := \{x_1^\perp \in A \ : E_{x_1^\perp} \cap I_{\bar{x}_1}(r) = (s(x_1^\perp), \bar{x}_1 + r/2)\} \quad \text{and} \quad \widetilde{B} := \{x_1^\perp \in B : \ E_{x_1^\perp} \cap I_{\bar{x}_1}(r) = I_{\bar{x}_1}(r)\},$$

we can assume without loss of generality that

$$\delta r^{d-1}/2 \le |\widetilde{A}| \le (1-\delta)r^{d-1} \qquad \text{and} \qquad \delta r^{d-1}/2 \le |\widetilde{B}| \le (1-\delta)r^{d-1}. \tag{22}$$

Indeed, the case when (22) holds with $\widetilde{A}^c$ (respectively $\widetilde{B}^c$) instead of $\widetilde{A}$ (respectively $\widetilde{B}$) can be similarly treated. Reducing the integral defining $I_{0,L}(E)$ to the set

$$\{x_1^\perp \in \widetilde{A},\ x_1 \in (s(x_2)-r, s(x_2)) \subset E^c_{x_1^\perp},\ x_1+\zeta_1 \in (s(x_1^\perp), s(x_1^\perp)+r) \subset E_{x_1^\perp} \text{ and } x_1^\perp + \zeta_1^\perp \in \widetilde{B}\},$$

we can now estimate

$$\begin{aligned} I_{0,L}(E) &\ge \frac{2}{L^d}\int_{\widetilde{A}}\int_{s(x_1^\perp)-r}^{s(x_1^\perp)}\int_{s(x_1^\perp)-x_1}^{s(x_1^\perp)-x_1+r}\int_{|\zeta_1^\perp|\le 2\sqrt{d-1}r} \frac{\chi_{\widetilde{B}}(x_1^\perp+\zeta_1^\perp)}{|\zeta|^p}d\zeta_1^\perp d\zeta_1 dx_1 dx_1^\perp \\ &\gtrsim \frac{1}{L^d r^p}\int_{\widetilde{A}}\int_{s(x_1^\perp)-r}^{s(x_1^\perp)}\int_{s(x_1^\perp)-x_1}^{s(x_1^\perp)-x_1+r} |\widetilde{B}| d\zeta_1 dx_1 dx_1^\perp \\ &\gtrsim \frac{\delta^2}{L^d r^{p-2d}}, \end{aligned}$$

which using that r^{2d-p} is unbounded as r goes to zero, contradicts the fact that $I_{0,L}(E)$ is finite. □

Lemma 4.2. *Let E be a Q_L-periodic set of finite perimeter such that $\sum_{i=1}^d \mathcal{G}^i_{0,L}(E)+I_{0,L}(E) < +\infty$. For $i \in \{1,\dots,d\}$, if $\bar{x} \in Q_L$ is such that $\Theta_i(\bar{x}) = 1$ then $\Theta_i(\bar{x} + \zeta_i^\perp) = 1$ for all $\zeta \in \mathbb{R}^d$.*

Proof. Assume for definiteness that $\Theta_1(\bar{x}) = 1$ and let $\bar{\zeta}_1^\perp \in \mathbb{R}^{d-1}$ be such that $\Theta_1(\bar{x} + \bar{\zeta}_1^\perp) = 0$. For all $\bar{r} > 0$, there exists $r < \bar{r}$ such that

$$\mu_1(Q_{\bar{x}}(r)) \geq \frac{3}{4} r^{d-1} \qquad \text{and} \qquad \mu_1(Q_{\bar{x}+\bar{\zeta}_1^\perp}(r)) \leq \frac{1}{4} r^{d-1}.$$

Since $\zeta_1^\perp \to \mu_1(Q_{\bar{x}+\zeta_1^\perp}(r))$ is continuous (this is a consequence of $x_1^\perp \to \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r)) \in L^1((0,L)^{d-1})$), there exists $\zeta_1^\perp \in (0, \bar{\zeta}_1^\perp)$ such that

$$\mu_1(Q_{\bar{x}+\zeta_1^\perp}(r)) = \frac{1}{2} r^{d-1}.$$

Arguing exactly as in the proof of Lemma 4.1, we reach a contradiction. □

We are finally in position to prove a rigidity result for sets of finite energy.

Proposition 4.3. *Let E be a Q_L-periodic set of finite perimeter such that $\sum_{i=1}^d \mathcal{G}^i_{0,L}(E)+I_{0,L}(E) < +\infty$. Then, E is one-dimensional i.e. up to permutation of the coordinates, $E = \widehat{E} \times \mathbb{R}^{d-1}$ for some $L-$periodic set $\widehat{E}$.*

Proof. Let us prove that there can be at most one Θ_i which is not identically zero. Since $\Theta_i = |\nu_i^E|$ $\mathcal{H}^{d-1}-$a.e. on ∂E, this would prove the claim. By Lemma 4.2, if $\bar{x}$ is such that $\Theta_i(\bar{x}) = 1$ for some $i \in \{1,\dots,d\}$, then for every $\zeta_i^\perp$, $\Theta_i(\bar{x} + \zeta_i^\perp) = 1$, which in turn implies that $\bar{x} + \zeta_i^\perp \subset \partial E$. Since E has finite perimeter (in Q_L), it can contain at most a finite number of such hyperplanes. If now Q is a cube which does not intersect any of these hyperplanes, then $\Theta_i = 0$ in Q for every $i \in \{1,\dots,d\}$ and therefore $|D\chi_E|(Q) = 0$ so that either $Q \subset E$ or $Q \subset E^c$. In Q_L, the set E is thus made of a finite union of hyperrectangles, which constitute a checkerboard structure. Arguing as in the last part of the proof of Lemma 4.1, we obtain that $I_{0,L}(E) = +\infty$ unless this checkerboard is one-dimensional. □

Remark 4.4. *For a set E of finite perimeter it can be readily seen that $I_{0,L}(E)$ finite, implies that every blow-up of E is an hyperplane orthogonal to some coordinate axis. This in particular implies that for $\mathcal{H}^{d-1}$-a.e. $x \in \partial E$, $\nu^E = e_i$ for some $i \in \{1,\dots,d\}$ (with i depending on x). However, it does not seems to be easy to conclude from this fact that E is one dimensional. In a work in progress, we will show that the conclusion of Proposition 4.3 actually holds without assuming that E is of finite perimeter or that $\mathcal{G}^i_{0,L}(E)$ is finite.*

5 The Gamma-convergence result

In this section, we prove our main result, which is the $\Gamma-$convergence of $\mathcal{F}_{\tau,L}$ to $\mathcal{F}_{0,L}$. Recall that

$$\mathcal{F}_{0,L}(E) := \begin{cases} \frac{1}{L}\left(-\mathrm{Per}(\widehat{E},[0,L)) + \mathcal{G}^{1d}_{0,L}(\widehat{E})\right) & \text{if } E = \widehat{E} \times \mathbb{R}^{d-1} \text{ for some } L-\text{periodic} \\ & \text{set } \widehat{E} \text{ of finite perimeter,} \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 5.1. *It holds:*

i) [Compactness and lower bound] Let E^τ be a sequence of Q_L-periodic sets such that $\sup_\tau \mathcal{F}_{\tau,L}(E^\tau) < +\infty$, then up to a subsequence and a relabelling of the coordinate axes, E^τ converges strongly in L^1 to some one-dimensional Q_L-periodic set E of finite perimeter. Moreover,

$$\liminf_{\tau\to 0} \mathcal{F}_{\tau,L}(E^\tau) \geq \mathcal{F}_{0,L}(E). \tag{23}$$

ii) [Upper bound] For every set E with $\mathcal{F}_{0,L}(E) < +\infty$, there exists a sequence $E^\tau \to E$ with

$$\limsup_{\tau\to 0} \mathcal{F}_{\tau,L}(E^\tau) \leq \mathcal{F}_{0,L}(E). \tag{24}$$

Proof. Let us start by proving i). Let E^τ be such that $\sup_\tau \mathcal{F}_{\tau,L}(E^\tau) \lesssim 1$. Then, by (17), $\sup_\tau \mathrm{Per}_1(E^\tau, Q_L) \lesssim 1$ so that we can extract a subsequence converging in L^1 to some Q_L-periodic set E of finite perimeter. Let us first prove that

$$\liminf_{\tau\to 0} \mathcal{G}^i_{\tau,L}(E^\tau) \geq \mathcal{G}^i_{0,L}(E) \quad \text{for } i \in \{1,\dots,d\} \qquad \text{and} \qquad \liminf_{\tau\to 0} I_{\tau,L}(E^\tau) \geq I_{0,L}(E). \tag{25}$$

By (16) and Proposition 4.3, this would prove that E is one-dimensional. For definiteness, let us prove the inequality concerning $\mathcal{G}^1_{\tau,L}$. The proof of the related lower bound for $I_{\tau,L}$ is similar (and actually simpler). For $\tau > \tau'$, since $\widehat{K}_\tau \leq \widehat{K}_{\tau'}$ and recalling (11),

$$\mathcal{G}^1_{\tau,L}(E^{\tau'}) \leq \mathcal{G}^1_{\tau',L}(E^{\tau'}).$$

Now, if τ is fixed, by Fatou and (11),

$$\begin{aligned}
\liminf_{\tau'\to 0} \mathcal{G}^1_{\tau,L}(E^{\tau'}) &\geq \frac{1}{L^d}\int_{\mathbb{R}} \widehat{K}_\tau(\zeta_1) \liminf_{\tau'\to 0} \left[\int_{\partial E^{\tau'}\cap Q_L} |\nu_1^{E^{\tau'}}||\zeta_1| - \int_{Q_L} |\chi_{E^{\tau'}}(x) - \chi_{E^{\tau'}}(x+\zeta_1)|\right] \\
&\geq \frac{1}{L^d}\int_{\mathbb{R}} \widehat{K}_\tau(\zeta_1) \left[\int_{\partial E\cap Q_L} |\nu_1^{E}||\zeta_1| - \int_{Q_L} |\chi_{E}(x) - \chi_{E}(x+\zeta_1)|\right] \\
&= \mathcal{G}^1_{\tau,L}(E),
\end{aligned}$$

where we have used that for fixed ζ_1, $\int_{\partial E\cap Q_L} |\nu_1^E||\zeta_1| - \int_{Q_L} |\chi_E(x) - \chi_E(x+\zeta_1)|$ is lower semicontinuous with respect to L^1 convergence. Finally, using again (11) and the monotone convergence theorem, we have

$$\liminf_{\tau'\to 0} \mathcal{G}^1_{\tau',L}(E^{\tau'}) \geq \lim_{\tau\to 0} \mathcal{G}^1_{\tau,L}(E) = \mathcal{G}^1_{0,L}(E),$$

which proves (25). From this point, in order to show the lower bound (23), we are left to check that

$$\liminf_{\tau\to 0} -\mathrm{Per}_1(E^\tau, Q_L) \geq -\mathrm{Per}_1(E, Q_L).$$

This is not straightforward since we have a minus sign in front of the perimeter. By slicing it is enough to prove that for $i \in \{1,\dots,d\}$,

$$\liminf_{\tau\to 0} -\int_{[0,L)^{d-1}} \mathrm{Per}(E^\tau_{x_i^\perp}, [0,L)) \geq -\int_{[0,L)^{d-1}} \mathrm{Per}(E_{x_i^\perp}, [0,L)). \tag{26}$$

Thanks to (16), for a.e. $x_i^\perp \in [0,L)^{d-1}$, $\mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp})$ is bounded (by a constant depending on $x_i^\perp$). Therefore, by (13),

$$\min_{x_i \in \partial E^\tau_{x_i^\perp}} \min(h^\tau_{x_i^\perp}(x_i), g^\tau_{x_i^\perp}(x_i)) \geq c_{x_i^\perp},$$

for some constant $c_{x_i^\perp} > 0$. This implies that the L^1 convergence of $E^\tau_{x_i^\perp}$ to $E_{x_i^\perp}$ improves to Hausdorff convergence of the boundaries (by choosing opportune representatives in the L^1 class) and in particular, since $E^\tau_{x_i^\perp} \subset \mathbb{R}$, that for τ small enough, $\mathrm{Per}(E^\tau_{x_i^\perp}, [0,L)) = \mathrm{Per}(E_{x_i^\perp}, [0,L))$. Letting

$$f_\tau(x_i^\perp) := \mathrm{Per}(E^\tau_{x_i^\perp}, [0,L)) \qquad \text{and} \qquad f(x_i^\perp) := \mathrm{Per}(E_{x_i^\perp}, [0,L)),$$

we have just proven that $f_\tau(x_i^\perp)$ converges to $f(x_i^\perp)$ for a.e. $x_i^\perp$. Notice that since E is one dimensional, f is actually constant. Let $\lambda > 0$ be larger than f. Then, the functions $f_\tau^\lambda := \min(\lambda, f_\tau)$ still converge almost everywhere to f. Being bounded, we can apply the dominated convergence theorem to f_τ^λ. We thus have

$$\begin{aligned}
\liminf_{\tau\to 0} -\int_{[0,L)^{d-1}} f_\tau &= \liminf_{\tau\to 0} -\int_{[0,L)^{d-1}} f_\tau^\lambda + \int_{[0,L)^{d-1}} f_\tau^\lambda - f_\tau \\
&\geq \liminf_{\tau\to 0} -\int_{[0,L)^{d-1}} f_\tau^\lambda - \limsup_{\tau\to 0} \int_{\{f_\tau > \lambda\}} f_\tau \qquad (27)\\
&= -\int_{[0,L)^{d-1}} f - \limsup_{\tau\to 0} \int_{\{f_\tau>\lambda\}} f_\tau.
\end{aligned}$$

Let us now estimate $|\{f_\tau > \lambda\}|$. By (15),

$$\lambda|\{f_\tau > \lambda\}| \leq \int_{\{f_\tau>\lambda\}} f_\tau \lesssim \tau L^d \mathcal{G}^i_{\tau,L}(E^\tau) + L^d \int_{[0,L)^{d-1}} (L\mathcal{G}^{1d}_{\tau,L}(E_{x_i^\perp}))^{1/(p-d)}. \qquad (28)$$

Using then Hölder's inequality and the bound $\sup_\tau \mathcal{F}_{\tau,L}(E^\tau) \lesssim 1$, we get[1]

$$\lambda|\{f_\tau > \lambda\}| \lesssim \tau L^d + L^{2d-1}\left(\int_{[0,L)^{d-1}} \mathcal{G}^{1d}_{\tau,L}(E_{x_i^\perp})\right)^{1/(p-d)} \lesssim \tau L^d + L^{2d-1} \lesssim L^{2d-1}.$$

This leads to

$$|\{f_\tau > \lambda\}| \lesssim \frac{L^{2d-1}}{\lambda}.$$

Arguing as in (28), and using Hölder's inequality again, we obtain,

$$\begin{aligned}
\int_{\{f_\tau>\lambda\}} f_\tau &\lesssim \tau L^d \mathcal{G}^i_{\tau,L}(E^\tau) + L^d \int_{\{f_\tau>\lambda\}} (L\mathcal{G}^{1d}_{\tau,L}(E_{x_i^\perp}))^{1/(p-d)} \\
&\lesssim L^d \left(\tau + |\{f_\tau > \lambda\}|^{(p-d-1)/(p-d)} L^{1/(p-d)} \left(\int_{[0,L)^{d-1}} \mathcal{G}^{1d}_{\tau,L}(E_{x_i^\perp})\right)^{1/(p-d)}\right) \\
&\lesssim L^d \left(\tau + \frac{L^{2(d-1)}}{\lambda^{(p-d-1)/(p-d)}}\right).
\end{aligned}$$

[1] applying Hölder's inequality to $\int_{\{f_\tau>\lambda\}} (L\mathcal{G}^{1d}_{\tau,L}(E_{x_i^\perp}))^{1/(p-d)}$ instead, one could get a better decay rate for $\{f_\tau > \lambda\}$.

Thanks to (27), we conclude that

$$\liminf_{\tau\to 0} -\int_{[0,L)^{d-1}} f_\tau \geq \int_{[0,L)^{d-1}} f - C\frac{L^{3d-2}}{\lambda^{(p-d-1)/(p-d)}}.$$

Letting finally $\lambda \to +\infty$, we obtain (26).

We can now turn to the proof of (24). Let E be such that $\mathcal{F}_{0,L}(E) < +\infty$. Without loss of generality, we can assume that $E = \widehat{E} \times \mathbb{R}$ for some $L-$periodic set $\widehat{E}$ of finite perimeter. Since $\widehat{E}$ is of finite perimeter, we have that[2]

$$c_0 := \min_{x\in\partial\widehat{E}} \min(h(x), g(x)) > 0.$$

Arguing as in the proof of (12), we see that

$$\mathcal{G}^{1d}_{0,L}(\widehat{E}) = \int_{|z|\geq c_0} \widehat{K}_0(z)\left(\mathrm{Per}(\widehat{E},[0,L))|z| - \int_0^L |\chi_{\widehat{E}}(x) - \chi_{\widehat{E}}(x+z)|\right].$$

Since $\widehat{K}_0$ is integrable in $\{|z|\geq c_0\}$, by the dominated convergence theorem,

$$\lim_{\tau\to 0} \mathcal{G}^{1d}_{\tau,L}(\widehat{E}) = \mathcal{G}^{1d}_{0,L}(\widehat{E}),$$

so that we can use $E = \widehat{E}\times\mathbb{R}$ itself as a recovery sequence. □

6 Minimizers of the one-dimensional problem

In this section, we prove that minimizers of $\mathcal{F}_{0,L}$ are periodic stripes of period essentially not depending on L. For a set E with $\mathcal{F}_{0,L}(E) < +\infty$, we identify by a slight abuse of notation, the set E and corresponding one-dimensional set $\widehat{E}$. That is for a $L-$periodic set E of finite perimeter, we consider

$$\mathcal{F}_{0,L}(E) = \frac{1}{L}\left(-\mathrm{Per}(E,[0,L)) + C_q\int_{\mathbb{R}}\frac{1}{|z|^q}\left[\mathrm{Per}(E,[0,L))|z| - \int_0^L |\chi_E(x)-\chi_E(x+z)|\right]\right), \quad (29)$$

where $q := p-d+1 > d+1$ and where we have used that for $z\in\mathbb{R}$, $\widehat{K}_0(z) = C_q|z|^{-q}$ for some constant $C_q > 0$. Since E is of finite perimeter, we can write it as $E = \cup_{i\in\mathbb{Z}}(s_i,t_i)$. As usually, we can assume that $E\cap[0,L) = \cup_{i=1}^N(s_i,t_i)$ for some $N\in\mathbb{N}$, $s_1 > 0$ and $t_N < L$.

For $h > 0$, let $E_h := \cup_{k\in\mathbb{Z}}[(2k)h,(2k+1)h]$. Then, we define

$$e_\infty(h) := \mathcal{F}_{0,2h}(E_h) = \lim_{L\to+\infty}\mathcal{F}_{0,L}(E_h).$$

We can now compute

[2] In fact by (13), we can even get a quantitative estimate of c_0 in term of the energy.

Lemma 6.1. *Letting*

$$\overline{C}_q := \frac{4C_q(1-2^{-(q-3)})}{(q-2)(q-1)} \sum_{k\geq 1} \frac{1}{k^{q-2}},$$

for every $h>0$, it holds

$$e_\infty(h) = -\frac{1}{h} + \overline{C}_q h^{-(q-1)}. \tag{30}$$

Therefore, $h^\star := ((q-1)\overline{C}_q)^{-1/(q-2)}$ is the unique (positive) minimizer of $e_\infty(h)$.

Proof. Since the contribution of the perimeter to the energy is clear, we just need to compute the non-local interaction. Denote by

$$A := \int_{\mathbb{R}} \frac{1}{|z|^q} \left[\mathrm{Per}(E_h, [0,2h))|z| - \int_0^{2h} |\chi_{E_h}(x) - \chi_{E_h}(x+z)| \right].$$

We start by noting that

$$\begin{aligned} A =& \int_{\mathbb{R}} \frac{1}{|z|^q} \left(|z| - \int_0^h \chi_{E_h^c}(x+z) \right) + \int_{\mathbb{R}} \frac{1}{|z|^q} \left(|z| - \int_h^{2h} \chi_{E_h}(x+z) \right) \\ =& 2\int_{\mathbb{R}} \frac{1}{|z|^q} \left(|z| - \int_0^h \chi_{E_h^c}(x+z) \right) = 4\int_{\mathbb{R}^+} \frac{1}{z^q} \left(z - \int_0^h \chi_{E_h^c}(x+z) \right), \end{aligned}$$

where we have first made the change of variables $x = y+h$ and used that $x+z \in E_h$ is equivalent to $y+z \in E_h^c$ and then, for $z<0$, we have let $z' = -s$ and $x' = h-x$ (so that if $x+z \in E_h^c$, also $x'+z' \in E_h^c$). Hence, we want to show that

$$\int_{\mathbb{R}^+} \frac{1}{z^q} \left(z - \int_0^h \chi_{E_h^c}(x+z) \right) = \frac{2(1-2^{-(q-3)})}{(q-2)(q-1)} \sum_{k\geq 1} \frac{1}{k^{q-2}} h^{-(q-2)}. \tag{31}$$

Since in $\mathbb{R}^+$, $\chi_{E_h^c} = \chi_{[0,h]^c} - \sum_{k\geq 1} \chi_{[(2k)h,(2k+1)h]}$,

$$\begin{aligned} \int_{\mathbb{R}^+} z^{-q} \left(z - \int_0^h \chi_{E_h^c}(x+z) \right) =& \int_{\mathbb{R}^+} z^{-q} \left(z - \int_0^h \chi_{[0,h]^c}(x+z) \right) \\ &+ \sum_{k\geq 1} \int_{\mathbb{R}^+} z^{-q} \int_0^h \chi_{[(2k)h,(2k+1)h]}(x+z). \end{aligned}$$

The first term on the right-hand side can be computed as

$$\int_{\mathbb{R}^+} z^{-q} \left(z - \int_0^h \chi_{[0,h]^c}(x+z) \right) = \int_h^{+\infty} z^{-q}(z-h) = \frac{h^{-(q-2)}}{(q-2)(q-1)},$$

while for the second term we can use that for $k \geq 1$,

$$\begin{aligned} \int_{\mathbb{R}^+} z^{-q} \int_0^h \chi_{[(2k)h,(2k+1)h]}(x+z) =& \int_{(2k)h}^{(2k+1)h} \int_0^h \frac{dxdy}{(y-x)^q} \\ =& \frac{h^{-(q-2)}}{(q-2)(q-1)} \left(\frac{1}{(2k-1)^{q-2}} + \frac{1}{(2k+1)^{q-2}} - \frac{2}{(2k)^{q-2}} \right). \end{aligned}$$

Putting this together, we get

$$\int_{\mathbb{R}^+} \frac{1}{z^q}\left(z - \int_0^h \chi_{E_h^c}(x+z)\right) = \frac{h^{-(q-2)}}{(q-2)(q-1)}\left(1 + \sum_{k\geq 1} \frac{1}{(2k-1)^{q-2}} + \frac{1}{(2k+1)^{q-2}} - \frac{2}{(2k)^{q-2}}\right).$$

Since

$$2(1-2^{-(q-3)})\sum_{k\geq 1}\frac{1}{k^{q-2}} = 1 + \sum_{k\geq 1} \frac{1}{(2k-1)^{q-2}} + \frac{1}{(2k+1)^{q-2}} - \frac{2}{(2k)^{q-2}}, \tag{32}$$

this concludes the proof of (31). □

Remark 6.2. *The sum $\sum_{k\geq 1} k^{-(q-2)}$ is finite since $q > d+1 > 3$. Notice that actually, the sum in the right-hand side of (32) is finite for every $q > 1$. Therefore, the energy of periodic stripes is finite for every $q > 1$ i.e. for $p > d$.*

The main estimate of this section is the following chessboard estimate:

Lemma 6.3. *For every $L-$periodic set E of finite perimeter, it holds*

$$\mathcal{F}_{0,L}(E) \geq \frac{1}{2L}\sum_{x\in\partial E\cap[0,L)} h(x)e_\infty(h(x)) + g(x)e_\infty(g(x)). \tag{33}$$

The proof of Lemma 6.3 will occupy the rest of this section. Before turning to its proof, let us state its main consequence

Theorem 6.4. *For every $L > 0$, the minimizers of $\mathcal{F}_{0,L}$ are periodic stripes E_h for some $h > 0$ satisfying*

$$|h - h^\star| \lesssim \frac{1}{L}. \tag{34}$$

Moreover, for $L \in 2h^\star\mathbb{N}$, $E_{h^\star}$ is the unique minimizer.

Proof. Let us first prove the last claim. Let $L > 0$ and let E be any L periodic set then by (33), and the minimality of $h^\star$ for e_∞,

$$\mathcal{F}_{0,L}(E) \geq \frac{1}{2L}\sum_{x\in\partial E\cap[0,L)} h(x)e_\infty(h(x)) + g(x)e_\infty(g(x)) \geq \frac{e_\infty(h^\star)}{2L}\sum_{x\in\partial E\cap[0,L)} h(x) + g(x) = e_\infty(h^\star). \tag{35}$$

For $L \in 2h^\star\mathbb{N}$, $E_{h^\star}$ is admissible thus we have equalities in (35) for minimizers of $\mathcal{F}_{0,L}$. Since $h^\star$ is the unique minimizer of $e_\infty(h)$, this implies that $h(x) = g(x) = h^\star$ for every $x \in \partial E$, proving the claim.

If now L is arbitrary, using the first inequality in (35) and (30), we get

$$L\mathcal{F}_{0,L}(E) \geq -\operatorname{Per}(E,[0,L)) + \frac{1}{2}\overline{C}_q \sum_{x\in\partial E\cap[0,L)} h(x)^{-(q-2)} + g(x)^{-(q-2)}.$$

If $\mathrm{Per}(E,[0,L))=2N$ is fixed, then, from the supperadditivity of $x^{-(q-2)}$,

$$\min_{\sum_{i=1}^{2N} h_i+g_i=2L} \sum_{i=1}^{2N} h_i^{-(q-2)}+g_i^{-(q-2)} = 4N\left(\frac{L}{2N}\right)^{-(q-2)},$$

and the minimum is attained only at $h_i=g_i=\frac{L}{2N}$. Since $E_{\frac{L}{2N}}$ is admissible and satisfies

$$L\mathcal{F}_{0,L}(E_{\frac{L}{2N}}) = -2N + 2\overline{C}_q N\left(\frac{L}{2N}\right)^{-(q-2)},$$

we obtain as before that every minimizer has to be equal to $E_{\frac{L}{2N}}$ for some $N\in\mathbb{N}$. Letting $2N^\star := \frac{L}{h^\star}$, we see that the function $x\to -x+\overline{C}_q L^{-(q-2)}x^{q-1}$ is minimized at $x=2N^\star$. This implies that letting $h^+ := L(2\lceil L/(2h^\star)\rceil)^{-1}$ and $h_- := L(2\lfloor L/(2h^\star)\rfloor)^{-1}$, the only possible minimizers of $\mathcal{F}_{0,L}$ are $E_{h^\pm}$. Since $|h^\pm - h^\star|\lesssim L^{-1}$, this concludes the proof of (34).

□

Remark 6.5. *From the proof of* (34)*, it is not hard to see that for most values of L, the minimizer of $\mathcal{F}_{0,L}$ is actually unique and equal to E_{h^+} or E_{h^-}.*

We now turn to the proof of (33). The idea is, as in [19, 20], to use the method of reflection positivity. As in these papers, the main point is to prove it for the non-local part of the energy. However, we face here the technical difficulty that the kernel $|s|^{-q}$ is not integrable around zero and thus we cannot directly split the integral in (29) into two pieces. Let us start by recalling that thanks to (13), this integral is actually less singular than it appears.

Lemma 6.6. *There exists $c_0(L)>0$ such that every minimizer E of $\mathcal{F}_{0,L}$ satisfies $\min_{x\in\partial E}\min(h(x),g(x))\geq c_0(L)$. For such sets, it holds*

$$\mathcal{F}_{0,L}(E) = \frac{1}{L}\left(\left(-1+\frac{2C_q}{(q-2)c_0^{q-2}}\right)\mathrm{Per}(E,[0,L)) - C_q\int_{[0,L]\times\mathbb{R}}\frac{\chi_{\{|z|\geq c_0\}}(x-y)}{|x-y|^q}|\chi_E(x)-\chi_E(y)|dxdy\right). \tag{36}$$

Proof. The claim follows by combining (13) with (16) and the fact that for a minimizer E, $\mathcal{F}_{0,L}(E)\leq 0$. □

Remark 6.7. *By working a bit harder, it can be proven that actually the constant c_0 does not depend on L. Of course, this also follows a posteriori from Theorem 6.4.*

Since L is fixed, we will not explicit anymore the dependence on L of c_0. We can now proceed as usually when using reflection positivity, and use the Laplace transform.

Lemma 6.8. *Let $\rho\geq 0$ be such that $\int_0^{+\infty}\rho=1$ and let*

$$\rho_{c_0}(\alpha) := \left(-1+\frac{2C_q}{(q-2)c_0^{q-2}}\right)\rho(\alpha) + \frac{2C_q\alpha^{q-2}}{\Gamma(q)}\left(\alpha^{-1}(1-e^{-c_0\alpha}) - c_0e^{-\alpha c_0}\right),$$

where Γ *is Euler's Gamma function. Then, for every* E *with* $\min_{x\in\partial E}\min(h(x),g(x))\geq c_0$*,*

$$\mathcal{F}_{0,L}(E)=\int_0^{+\infty}\frac{1}{L}\left(\rho_{c_0}(\alpha)\operatorname{Per}(E,[0,L))-\frac{C_q\alpha^{q-1}}{\Gamma(q)}\int_{[0,L]\times\mathbb{R}}|\chi_E(x)-\chi_E(y)|e^{-\alpha|x-y|}\right)d\alpha. \quad (37)$$

Proof. Since for $s>0$, $s^{-q}=\frac{1}{\Gamma(q)}\int_0^{+\infty}\alpha^{q-1}e^{-\alpha s}$, we have

$$\begin{aligned}\int_{[0,L]\times\mathbb{R}}&\frac{\chi_{\{|z|\geq c_0\}}(x-y)}{|x-y|^q}|\chi_E(x)-\chi_E(y)|\\&=\frac{1}{\Gamma(q)}\int_0^{+\infty}\alpha^{q-1}\int_{[0,L]\times\mathbb{R}}\chi_{\{|z|\geq c_0\}}(x-y)|\chi_E(x)-\chi_E(y)|e^{-\alpha|x-y|}d\alpha.\end{aligned}$$

Now, for fixed $\alpha>0$,

$$\begin{aligned}\int_{[0,L]\times\mathbb{R}}\chi_{\{|z|\geq c_0\}}(x-y)|\chi_E(x)-\chi_E(y)|e^{-\alpha|x-y|}&=\int_{[0,L]\times\mathbb{R}}|\chi_E(x)-\chi_E(y)|e^{-\alpha|x-y|}\\&\quad-\int_{[0,L]\times\mathbb{R}}\chi_{\{|z|<c_0\}}(x-y)|\chi_E(x)-\chi_E(y)|e^{-\alpha|x-y|}.\end{aligned}$$

Since $\min_{x\in\partial E}(\min h(x),\min g(x))\geq c_0$, we have

$$\begin{aligned}\int_{[0,L]\times\mathbb{R}}\chi_{\{|z|<c_0\}}(x-y)|\chi_E(x)-\chi_E(y)|e^{-\alpha|x-y|}&=\sum_{x\in\partial E}\int_{|x-y|\leq c_0}|x-y|e^{-\alpha|x-y|}dy\\&=2\operatorname{Per}(E,[0,L))\int_0^{c_0}ze^{-\alpha z}\\&=2\alpha^{-1}\left(\alpha^{-1}(1-e^{-c_0\alpha})-c_0e^{-\alpha c_0}\right)\operatorname{Per}(E,[0,L)).\end{aligned}$$

Putting all this together, we get

$$\begin{aligned}&\int_{[0,L]\times\mathbb{R}}\frac{\chi_{\{|z|\geq c_0\}}(x-y)}{|x-y|^q}|\chi_E(x)-\chi_E(y)|\\&=\int_0^{+\infty}\frac{\alpha^{q-1}}{\Gamma(q)}\left[\int_{[0,L]\times\mathbb{R}}|\chi_E(x)-\chi_E(y)|e^{-\alpha|x-y|}-2\alpha^{-1}\left(\alpha^{-1}(1-e^{-c_0\alpha})-c_0e^{-\alpha c_0}\right)\operatorname{Per}(E,[0,L))\right]d\alpha.\end{aligned}$$

Plugging this into (36) yields (37). □

Remark 6.9. *Notice that in* (37)*, we cannot split the integral in two since each of the term is separately not integrable.*

For $\alpha,h>0$, let

$$e_{\alpha,\infty}(h):=-\frac{1}{2h}\int_0^{2h}\int_{\mathbb{R}}|\chi_{E_h}(x)-\chi_{E_h}(y)|e^{-\alpha|x-y|}=\lim_{L\to+\infty}-\frac{1}{L}\int_{[0,L]\times\mathbb{R}}|\chi_{E_h}(x)-\chi_{E_h}(y)|e^{-\alpha|x-y|}.$$

Up to noticing that in (33), the interfacial terms are the same on both sides, thanks to (37) and integration in α, Lemma 6.3 is proven provided we can show

Lemma 6.10. *For every $\alpha > 0$ and every $L-$periodic set E*[3]*,*

$$-\int_{[0,L]\times\mathbb{R}} |\chi_E(x) - \chi_E(y)| e^{-\alpha|x-y|} dxdy \geq \frac{1}{2} \sum_{x\in\partial E\cap[0,L)} h(x) e_{\alpha,\infty}(h(x)) + g(x) e_{\alpha,\infty}(g(x)). \quad (38)$$

Proof. Since α is fixed, in order to lighten notation, we will assume that $\alpha = 1$.

As pointed out in [16, Ap. A], periodic boundary conditions are not well suited for the application of reflection positivity. We will thus prove a statement similar to (38) under free boundary conditions. For this, we notice that since the kernel $e^{-|s|}$ is integrable and since E is periodic,

$$-\int_{[0,L]\times\mathbb{R}} |\chi_E(x) - \chi_E(y)| e^{-|x-y|} dxdy = \lim_{k\to+\infty} -\frac{1}{k} \int_{[0,kL]\times[0,kL]} |\chi_E(x) - \chi_E(y)| e^{-|x-y|}.$$

We are thus left to prove that for every set $E \subset [0, L)$ of finite perimeter,

$$-\int_{[0,L]^2} |\chi_E(x) - \chi_E(y)| e^{-|x-y|} \geq \frac{1}{2} \sum_{x\in\partial E} h(x) e_{1,\infty}(h(x)) + g(x) e_{1,\infty}(g(x)). \quad (39)$$

In order to prove (39) we need to introduce some further notation. For $L_1, L_2 > 0$, and two sets $E_1 \subset [0, L_1)$, $E_2 \subset (L_1, L_1 + L_2)$, we let $L := L_1 + L_2$, $(E_1, E_2) := E_1 \cup E_2$ and

$$\mathcal{J}(E_1, E_2) := -\int_{[0,L]\times[0,L]} |\chi_{(E_1,E_2)}(x) - \chi_{(E_1,E_2)}(y)| e^{-|x-y|}.$$

We then define the set $(E_1, \theta E_1)$ in $[0, 2L_1]$ by

$$\chi_{(E_1,\theta E_1)}(x) := \begin{cases} \chi_{E_1}(x) & \text{for } x \in [0, L_1] \\ 1 - \chi_{E_1}(2L_1 - x) & \text{for } x \in (L_1, 2L_1]. \end{cases}$$

Letting $L_2 = L - L_1$, we similarly define, $(\theta E_2, E_2)$ as a subset of $[L_1 - L_2, L]$ by

$$\chi_{(\theta E_2,E_2)}(x) := \begin{cases} \chi_{E_2}(x) & \text{for } x \in [L_1, L] \\ 1 - \chi_{E_2}(2L_1 - x) & \text{for } x \in (L_1 - L_2, L_1]. \end{cases}$$

The key estimate is

$$\mathcal{J}(E_1, E_2) \geq \frac{1}{2} \left(\mathcal{J}(E_1, \theta E_1) + \mathcal{J}(\theta E_2, E_2) \right). \quad (40)$$

Once (40) is established, (39) follows by multiple reflections. We refer the reader to [20, Lem. A.1] or to [16, Ap. A] for instance for a proof of this fact.
Let us prove (40). We start by computing $\mathcal{J}(E_1, E_2)$. By definition,

$$\begin{aligned} \mathcal{J}(E_1, E_2) = &-\int_{[0,L_1]^2} |\chi_{E_1}(x) - \chi_{E_1}(y)| e^{-|x-y|} - \int_{[L_1,L]^2} |\chi_{E_2}(x) - \chi_{E_2}(y)| e^{-|x-y|} \\ &- 2\int_{[0,L_1]\times[L_1,L]} |\chi_{E_1}(x) - \chi_{E_2}(y)| e^{-|x-y|}. \end{aligned}$$

[3]Notice that here since α is fixed, we do not need to assume that $\min_{\partial E} \min(h(x), g(x)) \geq c_0$

Using that $|\chi_{E_1}(x)-\chi_{E_2}(y)| = \chi_{E_1}(x)\chi_{E_2^c}(y)+\chi_{E_1^c}(x)\chi_{E_2}(y)$ and that for $x \in [0, L_1]$ and $y \in [L_1, L]$, $|x-y| = y-x$, we obtain

$$\begin{aligned}\mathcal{J}(E_1, E_2) = &- \int_{[0,L_1]^2} |\chi_{E_1}(x) - \chi_{E_1}(y)|e^{-|x-y|} - \int_{[L_1,L]^2} |\chi_{E_2}(x) - \chi_{E_2}(y)|e^{-|x-y|} \\ &- 2\left(\int_0^{L_1} \chi_{E_1} e^x\right)\left(\int_{L_1}^{L} \chi_{E_2^c} e^{-y}\right) - 2\left(\int_0^{L_1} \chi_{E_1^c} e^x\right)\left(\int_{L_1}^{L} \chi_{E_2} e^{-y}\right).\end{aligned}$$

Using the definition of $(E_1, \theta E_1)$, we compute similarly

$$\begin{aligned}\mathcal{J}(E_1, \theta E_1) = &- \int_{[0,L_1]^2} |\chi_{E_1}(x) - \chi_{E_1}(y)|e^{-|x-y|} - \int_{[L_1,2L_1]} |\chi_{\theta E_1}(x) - \chi_{(\theta E_1)}(y)|e^{-|x-y|} \\ &- 2\int_{[0,L_1]\times[L_1,2L_1]} (\chi_{E_1}(x)\chi_{(\theta E_1)^c}(y) + \chi_{E_1^c}(x)\chi_{\theta E_1}(y))e^{x-y} \\ = &- 2\int_{[0,L_1]^2} |\chi_{E_1}(x) - \chi_{E_1}(y)|e^{-|x-y|} \\ &- 2\int_{[0,L_1]\times[L_1,2L_1]} (\chi_{E_1}(x)\chi_{E_1}(2L_1-y) + \chi_{E_1^c}(x)\chi_{E_1^c}(2L_1-y))e^{x-y} \\ = &- 2\int_{[0,L_1]^2} |\chi_{E_1}(x) - \chi_{E_1}(y)|e^{-|x-y|} - 2e^{-2L_1}\left[\left(\int_0^{L_1} \chi_{E_1} e^x\right)^2 + \left(\int_0^{L_1} \chi_{E_1^c} e^x\right)^2\right].\end{aligned}$$

Analogously, we get

$$\mathcal{J}(\theta E_2, E_2) = -2\int_{[L_1,L]^2} |\chi_{E_2}(x) - \chi_{E_2}(y)|e^{-|x-y|} - 2e^{2L_1}\left[\left(\int_{L_1}^{L} \chi_{E_2} e^{-x}\right)^2 + \left(\int_{L_1}^{L} \chi_{E_2^c} e^{-x}\right)^2\right].$$

This concludes the proof of (40) since

$$\begin{aligned}e^{2L_1}\left[\left(\int_{L_1}^{L} \chi_{E_2} e^{-x}\right)^2 + \left(\int_{L_1}^{L} \chi_{E_2^c} e^{-x}\right)^2\right] + e^{-2L_1}\left[\left(\int_0^{L_1} \chi_{E_1} e^x\right)^2 + \left(\int_0^{L_1} \chi_{E_1^c} e^x\right)^2\right] \geq \\ 2\left(\int_0^{L_1} \chi_{E_1} e^x\right)\left(\int_{L_1}^{L} \chi_{E_2^c} e^{-x}\right) + 2\left(\int_0^{L_1} \chi_{E_1^c} e^x\right)\left(\int_{L_1}^{L} \chi_{E_2} e^{-x}\right).\end{aligned}$$

□

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