On the optimality of stripes in a variational model with non-local interactions

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Abstract

We study pattern formation for a variational model displaying competition between a local term penalizing interfaces and a non-local term favoring oscillations. By means of a Γ-convergence analysis, we show that as the parameter J converges to a critical value J_c, the minimizers converge to periodic one-dimensional stripes. A similar analysis has been previously performed by other authors for related discrete systems. In that context, a central point is that each "angle" comes with a strictly positive contribution to the energy. Since this is not anymore the case in the continuous setting, we need to overcome this difficulty by slicing arguments and a rigidity result.

1 Introduction

Motivated by recent works [19, 20, 22] on striped patterns in Ising models with competing interactions, we consider for $d \geq 2$, $J, L > 0$ and $p > 2d$ the functional

$$\widetilde{\mathcal{F}}_{J,L}(E) := \frac{1}{L^d}\left(J\int_{\partial E\cap[0,L)^d}|\nu^E|_1\,\mathrm{d}\mathcal{H}^{d-1} - \int_{[0,L)^d\times\mathbb{R}^d}\frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^p+1}\,\mathrm{d}x\,\mathrm{d}y\right), \tag{1}$$

where E is a $[0,L)^d$-periodic set, ν^E is its external normal and $|\cdot|_1$ denotes the 1-norm. The first term on the right-hand side of (1) represents the (anisotropic) perimeter of E and penalizes oscillations while the second term is a repulsive non-local term so that the two terms are in competition. As in the discrete case [19], it can be shown (see Proposition 3.5) that for $J \geq J_c := \int_{\mathbb{R}^d}\frac{|\zeta_1|}{1+|\zeta|^p}\,\mathrm{d}\zeta$, the energy is always non-negative and thus minimizers are the uniform states while for $J < J_c$, there exists non trivial minimizers. We are interested here in the behavior of these minimizers as $J \uparrow J_c$. Building on the computations made in [19], it is expected that for $\tau := J_c - J$ small enough, minimizers are periodic striped patterns. A simple computation (see (30)) shows that the optimal stripes have width of order $\tau^{-1/(p-d-1)}$ and energy of order $-\tau^{(p-d)/(p-d-1)}$. This motivates the

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rescaling given in (11) which yields stripes of width and energy of order one as τ goes to zero. After this rescaling, we are led to study the minimizers of

$$\mathcal{F}_{\tau,L}(E) := \frac{1}{L^d}\Bigg(-\int_{\partial E\cap[0,L)^d} |\nu^E|_1 \,\mathrm{d}\mathcal{H}^{d-1} \\ + \int_{\mathbb{R}^d} \frac{1}{|\zeta|^p + \tau^{p/(p-d-1)}} \left[\int_{\partial E\cap[0,L)^d} \sum_{i=1}^d |\nu_i^E||\zeta_i| \,\mathrm{d}\mathcal{H}^{d-1} - \int_{[0,L)^d} |\chi_E(x) - \chi_E(x+\zeta)| \,\mathrm{d}x \right] \mathrm{d}\zeta \Bigg).$$

Our main theorem is a Γ-convergence [7] result for $\mathcal{F}_{\tau,L}$.

Theorem 1.1. *For $p > 2d$ and $L > 1$, the functionals $\mathcal{F}_{\tau,L}$ Γ-converge as τ goes to zero with respect to the L^1-convergence to the functional $\mathcal{F}_{0,L}$ defined for sets E which are (up to permutation of coordinates) of the form $E = \widehat{E}\times\mathbb{R}^{d-1}$ where $\widehat{E}$ is L-periodic with $\sharp\{\partial\widehat{E}\cap[0,L)\} < +\infty$, as*

$$\mathcal{F}_{0,L}(E) := \frac{1}{L}\left(-\sharp\{\partial\widehat{E}\cap[0,L)\} + \int_{\mathbb{R}^d} \frac{1}{|\zeta|^p} \left[\sum_{x\in\partial\widehat{E}\cap[0,L)} |\zeta_1| - \int_0^L |\chi_{\widehat{E}}(x) - \chi_{\widehat{E}}(x+\zeta_1)| \,\mathrm{d}x \right] \mathrm{d}\zeta \right), \tag{2}$$

and $\mathcal{F}_{0,L}(E) = +\infty$ otherwise. Finally, if E^τ is such that $\sup_\tau \mathcal{F}_{\tau,L}(E^\tau) < +\infty$, then up to a relabeling of the coordinate axes, there is a subsequence which converges in L^1 to some set E with $E = \widehat{E}\times\mathbb{R}^{d-1}$ and $\sharp\{\partial\widehat{E}\cap[0,L)\} < +\infty$.

This theorem is a restatement of Theorem 5.1. Using the method of reflection positivity, we can then compute the minimizers of the limiting energy (see Theorem 6.4).

Theorem 1.2. *There exist $h^\star > 0$ (whose value is given in Lemma 6.1) and $C > 0$ such that for every $L > 1$, minimizers of $\mathcal{F}_{0,L}$ are periodic stripes of width h with*

$$|h - h^\star| \le CL^{-1}.$$

As a direct consequence of Theorem 1.1 and Theorem 1.2, we obtain the following corollary

Corollary 1.3. *Let E^τ be minimizers of $\mathcal{F}_{\tau,L}$. Then, up to a subsequence, they converge as τ goes to zero to stripes of width h with $|h - h^\star| \le CL^{-1}$.*

Let us notice that for most values of L, the minimizer of $\mathcal{F}_{0,L}$ is unique. In this case, up to a rotation, the whole sequence E^τ converges.

We now give an outline of the proofs of Theorem 1.1 and Theorem 1.2, and discuss the relations and main differences with those in the discrete setting [20, 22]. The main ingredients in the proof of Theorem 1.1 are Lemma 3.2 which permit to identify the part of the energy which penalizes non straight boundaries, the slicing formula (3), the crucial (but simple) estimate (9) which leads to the estimate (14) of the non-local part of the energy by the local widths and gaps and then to an estimate of the perimeter by the total energy (which gives strong compactness), and finally a rigidity result (see Proposition 4.3) which proves that in the limit, sets of finite energy must be one-dimensional. This last ingredient is based on the study of a functional which is somewhat reminiscent of integral characterizations of Sobolev spaces that have recently received a lot of attention [5, 8, 15]. This connection will be further explored in a future work [24]. As in the proofs of [20, 22], a central point is to estimate the cost of "angles" and "holes". However, this is where the biggest difference between the discrete and the continuous settings appears. Indeed, the geometry of a set in $\mathbb{R}^d$ can not be characterized only in terms of "angles" and "holes". Moreover, in the discrete setting these are quantized and thus carry a positive energy which forbids their presence on a scale which is much larger than the typical width of the stripes (see for instance [22, Lemma 2]). On the contrary, in the continuous setting, "angles" or "holes" may be extremely small and give almost no contribution to the energy. Finallly, in the discrete case, the number of "angles" gives an upper bound on the perimeter, which is again not the case in the continuous setting. The same observations also hold for the local widths and gaps (which in [20, 22] appear in the form of the distance between bonds facing each others). For all these reasons, it is difficult in our setting to estimate the contributions of the "angles" for $\tau > 0$. This is the main reason why we need to pass to the limit $\tau \to 0$. Since we use a compactness argument (given by a BV-bound), a caveat of our approach is that it does not yield a rate of convergence of the minimizers of $\mathcal{F}_{\tau,L}$ to the optimal stripes. In particular, it does not exclude that this rate might depend on L.

In the discrete case, as far as minimizers are concerned, the results in [22] are much stronger. Indeed, the authors of [22] were able to prove that for τ small enough but positive, periodic stripes are minimizers under compact perturbations of the discrete energy in $\mathbb{Z}^d$ [1]. Yet, we believe that our approach based on slicing and on the splitting estimate (5), gives a good insight on some of the more combinatorial proofs of [20, 22]. In fact, building on the results obtained in this paper combined with several new ingredients, it has been proven recently by the second author together with S. Daneri in [14] that for small enough τ but uniformly in L, the minimizers of $\mathcal{F}_{\tau,L}$ are periodic stripes for the wider range of exponents $p \geq d + 2$.

Let us point our that one advantage of our Γ-convergence result compared to a bare analysis of the minimizers is that our proof easily extends to more general integrable kernels behaving like $|\zeta|^{-p}$ at infinity or to model containing for instance volume constraints. We should add the technical observation that due to the minus sign in front of the perimeter in the definition of $\mathcal{F}_{\tau,L}$, the lower-bound (25) can look surprising at first sight. In order to obtain it, we need to combine the bound (15) on the gaps with the fact that the limiting objects are one dimensional. The proof of Theorem 1.2 is based on reflection positivity [16] and does not differ much for instance from the proofs in [17, 21]. Since this technique is not so well known in the Calculus of Variation community and since besides [18, 21] we are not aware of many examples where reflection positivity has been used in a continuous context, we decided to include the proof of Theorem 1.2 for the reader's convenience. Indeed, one of the purposes of this work is to advertise this powerful method.

Let us now comment on some choices we made. The functional (1) is arguably the simplest example of a variational problem with competition between a local term penalizing interfaces and a repulsive

[1] A. Giuliani has pointed out to us that from the proofs in [22], it follows that stripes of width $h^\star$ are also minimizers under periodic boundary conditions in cubes of arbitrary size proportional to $h^\star$.

non-local term, leading to a complex pattern formation. Even though stripe patterns are expected in numerous continuous systems of this kind (see the discussion below) this paper is to the best of our knowledge the first where minimality of the stripes is rigorously obtained. We thus chose to work in the simplest possible setting. First, as in the discrete case, we are restricting ourselves to $p > 2d$. This condition ensures that the energy of stripes scales differently compared to the energy of a checkerboard (see [19]). This is reflected in the fact that for the limiting functional, only striped patterns are admissible. A long time after the first version of this paper was available, it has been observed in [24, 14] that this condition can be relaxed to $p \geq d+2$. Unfortunately, this still leaves out the most interesting cases $d = 3$ and $p = 1$ or $d = 2$ and $p = 3$, corresponding to Coulombic or Dipole interactions. The specific choice of the kernel is not important as long as it is integrable and behaves as $|\zeta|^{-p}$ at infinity. Second, we decided to work with the 1-perimeter instead of the usual Euclidean perimeter. This choice makes the splitting and slicing arguments work better by identifying the preferred axes of periodicity. The extension of this work to the classical perimeter will be the subject of further investigations.

The closest model to (1) is certainly the sharp interface version of the Ohta-Kawasaki functional which has been used to model diblock copolymers [30, 11, 25, 29] or nuclear matter [31]. Minimizers of this type of variational problems are expected to be periodic (see for instance [9] for some numerics). However, besides the one-dimensional situation [28] (see also [27, 33] for an almost one-dimensional case) and the low volume fraction limit [10, 23, 26, 6] not much is known. To the best of our knowledge, the only results available on periodicity of minimizers for intermediate volume fractions are [21] and the uniform local energy distribution [12, 2] as well as minimality in the perimeter dominant regime [34, 1, 32, 13]. We refer to [20, 22] for more references in particular on the discrete setting and to the review paper [4] for a discussion of the related issue of crystallization.

The paper is organized as follows. In Section 2, we set some notation and recall basic facts about sets of finite perimeter. In Section 3, we derive the functional $\mathcal{F}_{\tau,L}$ from (1) and prove some useful estimates. In Section 4, we prove the rigidity result Proposition 4.3. Then, in Section 5, we prove our Γ-convergence result. Finally in Section 6, we study the minimizers of the limiting problem.

2 Notation

In the paper we will use the following notation. The symbols $\sim$, $\gtrsim$, $\lesssim$ indicate estimates that hold up to a global constant depending only on the dimension and possibly on $p > 2d$. For instance, $f \lesssim g$ denotes the existence of a constant $C > 0$ such that $f \leq Cg$, $f \sim g$ means $f \lesssim g$ and $g \lesssim f$. We let $(e_1, \cdots, e_d)$ be the canonical basis of $\mathbb{R}^d$. For $(x,\zeta) \in (\mathbb{R}^d)^2$ and $i \in \{1, \ldots, d\}$, we let $x + \zeta_i := x + \zeta_i e_i$, and then $x_i^{\perp} := x - x_i$. We will denote by $|x|$ the Euclidean norm of x and by $|x|_1 := \sum_{i=1}^d |x_i|$ its 1-norm. For $L > 0$, we will let $Q_L := [0, L)^d$. We let $\partial_i f := \frac{\partial f}{\partial x_i}$. For a k-dimensional set $E \subset \mathbb{R}^k$, we let $|E|$ be its Lebesgue measure. For $z \in \mathbb{R}$, we let $z_{\pm}$ be the positive and negative parts of z.

2.1 Sets of finite perimeter

The purpose of this section is to recall the definition of sets of finite perimeter. For a general introduction we refer to [3]. Let us start with the one-dimensional case. We say that a L-periodic set $E \subset \mathbb{R}$ is of finite perimeter if $E \cap [0, L) = \cup_{i=1}^{N} I_i$ up to Lebesgue null-sets, for some $N \in \mathbb{N}$ and some disjoint intervals I_i. It is immediate to notice that the representation as a finite union of open intervals is unique. Moreover, as all the statements are independent under modifications on a null-set we will identify the set of finite perimeter with its representant.

We then let

$$\mathrm{Per}(E, [0, L)) := 2N.$$

By periodicity, we will often assume that $E \cap [0, L] = \cup_{i=1}^{N}(s_i, t_i)$ with $s_1 > 0$ and $t_N < L$. For any interval I, we let $\mathrm{Per}(E, I) := \sharp\{\partial E \cap I\}$.

We can now turn to higher dimensions.

Definition 2.1. *A Q_L-periodic set E is said to be of finite perimeter if $D\chi_E$, the distributional derivative of χ_E, is a locally finite measure. For such a set, we let ∂E be the collection of all points $x \in \mathrm{spt}(D\chi_E)$ such that the limit*

$$\nu^E(x) := -\lim_{\rho \downarrow 0} \frac{D\chi_E(B(x, \rho))}{|D\chi_E|(B(x, \rho))}$$

exists and satisfies $|\nu^E(x)| = 1$. We call ν^E the generalized outer normal to E. We then have $D\chi_E = -\nu^E \mathcal{H}^{d-1} \llcorner \partial E$, where $\mathcal{H}^{d-1} \llcorner \partial E$ is the restriction of the $(d-1)$-dimensional Hausdorff measure to ∂E.

We then define

$$\mathrm{Per}_1(E, Q_L) := \int_{\partial E \cap Q_L} |\nu^E(x)|_1 \, \mathrm{d}\mathcal{H}^{d-1}(x) = \int_{\partial E \cap Q_L} \sum_{i=1}^{d} |\nu_i^E(x)| \, \mathrm{d}\mathcal{H}^{d-1}(x).$$

As for the one-dimensional case, by periodicity we will always assume that $|D\chi_E|(\partial Q_L) = 0$ so that $\mathrm{Per}_1(E, Q_L)$ coincides with the 1-perimeter of E in $(0, L)^d$.

For $i \in \{1, \ldots, d\}$, $x_i^{\perp} \in [0, L)^{d-1}$ and $E \subset Q_L$, we define the one-dimensional slices

$$E_{x_i^{\perp}} := \left\{ x_i \in [0, L) : \; x_i + x_i^{\perp} \in E \right\}.$$

Note that in the above definition there is an abuse of notation as the information on the direction of the slice is contained in the index $x_i^{\perp}$. As it would be always clear from the context which is the direction of the slicing, we hope this will not cause confusion to the reader.

Let $d \geq 2$ and $i \in \{1, \ldots, d\}$. Given a Q_L-periodic set of finite perimeter $E \subset \mathbb{R}^d$, one can show that for almost every $x_i^{\perp}$ the slice $E_{x_i^{\perp}}$ is a one-dimensional set of finite perimeter. Moreover, for every $i \in \{1, \dots, d\}$, the following slicing formula holds

$$\int_{\partial E \cap Q_L} |\nu_i^E| \, \mathrm{d}\mathcal{H}^{d-1} = \int_{[0,L)^{d-1}} \mathrm{Per}(E_{x_i^\perp}, [0, L)) \, \mathrm{d}x_i^\perp. \tag{3}$$

We refer to [3, § 3.7] for this and finer properties of sets of finite perimeter.

3 The functional and preliminary results

We recall that we are considering the functional

$$\widetilde{\mathcal{F}}_{J,L}(E) := \frac{1}{L^d} \left(J \, \mathrm{Per}_1(E, Q_L) - \int_{Q_L \times \mathbb{R}^d} K_1(x - y) |\chi_E(x) - \chi_E(y)| \, \mathrm{d}x \, \mathrm{d}y \right),$$

where $K_1(\zeta) := \frac{1}{|\zeta|^p + 1}$ for some $p > 2d$. More generally, for $\tau \geq 0$, we let $K_\tau(\zeta) := \frac{1}{|\zeta|^p + \tau^{p/(p-d-1)}}$. Notice that the functional $\widetilde{\mathcal{F}}_{J,L}$ can also be written as

$$\widetilde{\mathcal{F}}_{J,L}(E) = \frac{1}{L^d} \left(J \, \mathrm{Per}_1(E, Q_L) - \int_{Q_L \times \mathbb{R}^d} K_1(\zeta) |\chi_E(x) - \chi_E(x + \zeta)| \, \mathrm{d}x \, \mathrm{d}\zeta \right). \tag{4}$$

The aim of this section is to give some first properties of $\widetilde{\mathcal{F}}_{J,L}$. We will in particular show that there exists a positive constant J_c such that for $J > J_c$ all minimizers of $\widetilde{\mathcal{F}}_{J,L}$ are trivial while for $J < J_c$ they are not. This will lead us to study the behavior of these minimizers in term of the parameter $\tau := J_c - J$ after suitable rescaling.
Let us point out that in this section, by approximation we can always work with polygonal sets having only horizontal and vertical edges. For these sets, both the definition of $\mathrm{Per}_1(E, Q_L)$ and (3) can be obtained without referring to the theory of sets of finite perimeter.

Remark 3.1. *Since E is periodic, we could also write the functional in a more geometric way as*

$$\int_{Q_L \times \mathbb{R}^d} K_1(x - y) |\chi_E(x) - \chi_E(y)| \, \mathrm{d}x \, \mathrm{d}y = 2 \int_{E \cap Q_L \times E^c} K_1(x - y) \, \mathrm{d}x \, \mathrm{d}y.$$

Indeed, since $|\chi_E(x) - \chi_E(x + \zeta)| = \chi_E(x)\chi_{E^c}(x + \zeta) + \chi_E(x + \zeta)\chi_{E^c}(x)$, it follows from (4) *and*

$$\begin{aligned}
\int_{Q_L \times \mathbb{R}^d} K_1(\zeta) \chi_E(x) \chi_{E^c}(x + \zeta) \, \mathrm{d}x \, \mathrm{d}\zeta &= \int_{\mathbb{R}^d} K_1(\zeta) \left(\int_{Q_L + \zeta} \chi_E(\widetilde{x} - \zeta) \chi_{E^c}(\widetilde{x}) \, \mathrm{d}\widetilde{x} \right) \mathrm{d}\zeta \\
&= \int_{\mathbb{R}^d} K_1(\zeta) \left(\int_{Q_L} \chi_E(\widetilde{x} - \zeta) \chi_{E^c}(\widetilde{x}) \, \mathrm{d}\widetilde{x} \right) \mathrm{d}\zeta \\
&= \int_{Q_L \times \mathbb{R}^d} K_1(\zeta) \chi_{E^c}(x) \chi_E(x + \zeta) \, \mathrm{d}x \, \mathrm{d}\zeta.
\end{aligned}$$

We recall that for $(x, \zeta) \in (\mathbb{R}^d)^2$, we let $x + \zeta_i := x + \zeta_i e_i$ and $\zeta_i^\perp := \zeta - \zeta_i$. The following lemma will allow us to split the various contributions of the energy

Lemma 3.2. *For every Q_L-periodic set E and every $\tau > 0$, there holds*

$$\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)|\chi_E(x)-\chi_E(x+\zeta)|\,\mathrm{d}x\,\mathrm{d}\zeta \le \int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)\sum_{i=1}^d|\chi_E(x)-\chi_E(x+\zeta_i)|\,\mathrm{d}x\,\mathrm{d}\zeta \\ -\frac{2}{d}\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)\sum_{i=1}^d|\chi_E(x)-\chi_E(x+\zeta_i)||\chi_E(x)-\chi_E(x+\zeta_i^\perp)|\,\mathrm{d}x\,\mathrm{d}\zeta. \quad (5)$$

Proof. We claim that for every $i\in\{1,\dots,d\}$,

$$\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)|\chi_E(x)-\chi_E(x+\zeta)|\,\mathrm{d}x\,\mathrm{d}\zeta \le \int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)\sum_{i=1}^d|\chi_E(x)-\chi_E(x+\zeta_i)|\,\mathrm{d}x\,\mathrm{d}\zeta \\ -2\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)|\chi_E(x)-\chi_E(x+\zeta_i)||\chi_E(x)-\chi_E(x+\zeta_i^\perp)|\,\mathrm{d}x\,\mathrm{d}\zeta. \quad (6)$$

Summing (6) over i and dividing by d, would then yield (5). Without loss of generality, we may assume that $i=1$. By disjunction of cases, it can be seen that for every $x\in Q_L$ and $\zeta\in\mathbb{R}^d$,

$$|\chi_E(x)-\chi_E(x+\zeta)| = |\chi_E(x)-\chi_E(x+\zeta_1)| + |\chi_E(x+\zeta_1)-\chi_E(x+\zeta)| \\ -2|\chi_E(x)-\chi_E(x+\zeta_1)||\chi_E(x+\zeta_1)-\chi_E(x+\zeta)|. \quad (7)$$

We thus have by integration and using the periodicity of E as in Remark 3.1,

$$\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)|\chi_E(x)-\chi_E(x+\zeta)|\,\mathrm{d}x\,\mathrm{d}\zeta \\ = \int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)(|\chi_E(x)-\chi_E(x+\zeta_1)| + |\chi_E(x)-\chi_E(x+\zeta_1^\perp)|)\,\mathrm{d}x\,\mathrm{d}\zeta \\ -2\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)|\chi_E(x)-\chi_E(x+\zeta_1)||\chi_E(x)-\chi_E(x+\zeta_1^\perp)|\,\mathrm{d}x\,\mathrm{d}\zeta.$$

Using that by the triangle inequality,

$$|\chi_E(x)-\chi_E(x+\zeta_1^\perp)| \le \sum_{k=2}^d\left|\chi_E\left(x+\sum_{j=2}^{k-1}\zeta_j\right)-\chi_E\left(x+\sum_{j=2}^{k}\zeta_j\right)\right|,$$

and using again periodicity of E, we get (6).

□

Remark 3.3. *Using (7) recursively, one could get an equality in (5) by replacing the term $\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)\sum_{i=1}^d|\chi_E(x)-\chi_E(x+\zeta_i)||\chi_E(x)-\chi_E(x+\zeta_i^\perp)|\,\mathrm{d}x\,\mathrm{d}\zeta$ by a more complex one. However, for our purpose this simpler bound is sufficient. Notice also that if E depends only on one variable, then equality holds.*

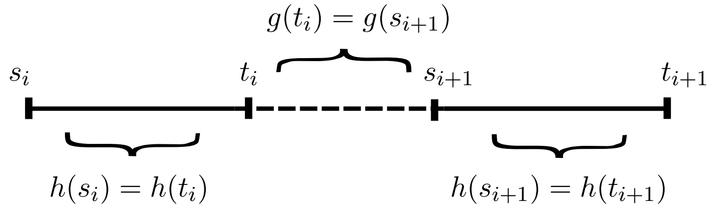

Figure 1

For a L-periodic set $E = \cup_{i \in \mathbb{Z}}(s_i, t_i) \subset \mathbb{R}$ of finite perimeter, we let for $i \in \mathbb{Z}$, the widths and gaps be defined by (see also Figure 1)

$$h(s_i) := h(t_i) := t_i - s_i \qquad g(t_i) := s_{i+1} - t_i \qquad g(s_{i+1}) := s_{i+1} - t_i. \tag{8}$$

We then define for $z \in \mathbb{R}$, and $i \in \mathbb{Z}$,

$$\eta(t_i, z) := \min(z_+, h(t_i)) + \min(z_-, g(t_i))$$

and

$$\eta(s_i, z) := \min(z_+, g(s_i)) + \min(z_-, h(s_i)).$$

Notice that of course, we always have $\eta \leq |z|$. For a Q_L-periodic set E of finite perimeter, the functions h_{x_i}, g_{x_i} and η_{x_i} are defined by slicing. For instance, for $x_1 \in E_{x_1^\perp}$ such that $\nu_1^E(x_1, x_1^\perp) > 0$ (so that $x_1 = t_i$ for some i) and $\zeta \in \mathbb{R}^d$,

$$\eta_{x_1^\perp}(x_1, \zeta_1) := \min((\zeta_1)_+, h_{x_1^\perp}(x_1)) + \min((\zeta_1)_-, g_{x_1^\perp}(x_1)).$$

We may now prove a simple but crucial estimate relating the non-local part of the energy with the quantity η.

Lemma 3.4. *For every L-periodic set $E \subset \mathbb{R}$ of finite perimeter and every $z \in \mathbb{R}$, there holds*

$$\int_0^L |\chi_E(x) - \chi_E(x+z)| \,\mathrm{d}x \leq \sum_{x \in \partial E \cap [0,L)} \eta(x, z). \tag{9}$$

By integration, for every $\tau \geq 0$, every Q_L-periodic set E of finite perimeter and $i \in \{1, \ldots, d\}$,

$$\int_{Q_L \times \mathbb{R}^d} K_\tau(\zeta) |\chi_E(x) - \chi_E(x + \zeta_i)| \,\mathrm{d}x \,\mathrm{d}\zeta \leq \int_{\partial E \cap Q_L} |\nu_i^E| \left(\int_{\mathbb{R}^d} K_\tau(\zeta) \eta_{x_i^\perp}(x_i, \zeta_i) \,\mathrm{d}\zeta \right) \mathrm{d}\mathcal{H}^{d-1}. \tag{10}$$

Proof. Let us prove (9). We consider only the case $z \geq 0$ since the case $z \leq 0$ can be treated in a similar way. Up to a translation, we can assume that $E \cap [0, L) = \cup_{i=1}^N (s_i, t_i)$ for some $N \in \mathbb{N}$. We then have by periodicity of E,

$$\begin{aligned}\int_0^L |\chi_E(x)-\chi_E(x+z)|\,\mathrm{d}x &= \sum_{i=1}^N \int_{s_i}^{t_i} \chi_{E^c}(x+z)\,\mathrm{d}x + \int_0^{s_1} \chi_E(x+z)\,\mathrm{d}x + \int_{t_N}^{L} \chi_E(x+z)\,\mathrm{d}x \\ &\quad + \sum_{i=2}^N \int_{t_{i-1}}^{s_i} \chi_E(x+z)\,\mathrm{d}x \\ &= \sum_{i=1}^N \int_{s_i}^{t_i} \chi_{E^c}(x+z)\,\mathrm{d}x + \sum_{i=1}^N \int_{t_{i-1}}^{s_i} \chi_E(x+z)\,\mathrm{d}x.\end{aligned}$$

For every $i \in [1,N]$, if $x \in (s_i,t_i)$ and $x+z \in E^c$, then $x+z \geq t_i$ and thus $|x-t_i| \leq \min(z,h(t_i)) = \eta(t_i,z)$. Similarly, for $i \in [1,N-1]$, if $x \in (t_{i-1},s_i)$ and $x+z \in E$, then $|x-s_i| \leq \eta(s_i,z)$. Therefore,

$$\int_0^L |\chi_E(x)-\chi_E(x+z)|\,\mathrm{d}x \leq \sum_{i=1}^N \left[\eta(s_i,z)+\eta(t_i,z)\right],$$

which proves (9). □

We now show that this quickly implies that for $J \geq J_c$ the minimizers of $\widetilde{\mathcal{F}}_{J,L}$ are trivial. A somewhat similar proof in the discrete setting may be found in [19].

Proposition 3.5. *For $J \geq J_c := \int_{\mathbb{R}^d} K_1(\zeta)|\zeta_1|\,\mathrm{d}\zeta$ and every Q_L-periodic set E, $\widetilde{\mathcal{F}}_{J,L}(E) \geq 0$.*

Proof. Putting Lemma 3.2 together with (10), we get

$$\begin{aligned}\widetilde{\mathcal{F}}_{J,L}(E) &\geq \frac{1}{L^d}\left(J\,\mathrm{Per}_1(E,Q_L) - \sum_{i=1}^d \int_{\partial E\cap Q_L} |\nu_i^E| \left[\int_{\mathbb{R}^d} K_1(\zeta)\eta_{x_i^\perp}(x_i,\zeta_i)\,\mathrm{d}\zeta\right]\mathrm{d}\mathcal{H}^{d-1}\right) \\ &\geq \frac{1}{L^d}\sum_{i=1}^d \int_{\partial E\cap Q_L} |\nu_i^E|\left[J - \int_{\mathbb{R}^d} K_1(\zeta)|\zeta_i|\,\mathrm{d}\zeta\right]\mathrm{d}\mathcal{H}^{d-1} \\ &= \frac{1}{L^d}\int_{\partial E\cap Q_L} |\nu^E|_1 \left[J - \int_{\mathbb{R}^d} K_1(\zeta)|\zeta_1|\,\mathrm{d}\zeta\right]\mathrm{d}\mathcal{H}^{d-1},\end{aligned}$$

which proves the claim. □

Letting $\tau := J_c - J$ for $J < J_c$, it is possible (see for instance the proof of (30) below) to compute the energy of periodic stripes E_h of period h to get

$$\widetilde{\mathcal{F}}_{J,L}(E_h) \simeq -\frac{\tau}{h} + h^{-(p-d)}.$$

Optimizing in h, we find that the optimal stripes have a width of order $\tau^{-1/(p-d-1)}$ and energy of order $-\tau^{(p-d)/(p-d-1)}$. Letting $\beta := p-d-1$, this motivates the rescaling

$$x := \tau^{-1/\beta}\widehat{x}, \quad L := \tau^{-1/\beta}\widehat{L} \quad \text{and} \quad \widetilde{\mathcal{F}}_{J,L}(E) := \tau^{(p-d)/\beta}\mathcal{F}_{\tau,\widehat{L}}(\widehat{E}). \tag{11}$$

In these variables, the optimal stripes have width of order one. From now on, when there is no ambiguity, we will drop the hats. Let us define $\widehat{K}_\tau(z) := \int_{\mathbb{R}^{d-1}} K_\tau(z,\xi)d\xi$. Since for $z \in \mathbb{R}$,

$$\frac{1}{|z|^{p-d+1} + \tau^{(p-d+1)/\beta}} \lesssim \int_{\mathbb{R}^{d-1}} \frac{1}{(|z|^2+|\xi|^2)^{p/2} + \tau^{p/\beta}} \,\mathrm{d}\xi \lesssim \frac{1}{|z|^{p-d+1} + \tau^{(p-d+1)/\beta}}, \tag{12}$$

we have $\widehat{K}_\tau(z) \simeq \frac{1}{|z|^q + \tau^{q/\beta}}$, where we have let $q := p-d+1$. We then let for $i \in \{1,\dots,d\}$,

$$\mathcal{G}^i_{\tau,L}(E) := \frac{1}{L^d}\int_{\mathbb{R}} \widehat{K}_\tau(\zeta_i)\left[\int_{\partial E\cap Q_L} |\nu^E_i||\zeta_i|\,\mathrm{d}\mathcal{H}^{d-1} - \int_{Q_L} |\chi_E(x)-\chi_E(x+\zeta_i)|\,\mathrm{d}x\right]\mathrm{d}\zeta_i,$$

and

$$I_{\tau,L}(E) := \frac{2}{dL^d}\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)\sum_{i=1}^d |\chi_E(x)-\chi_E(x+\zeta_i)||\chi_E(x)-\chi_E(x+\zeta_i^\perp)|\,\mathrm{d}x\,\mathrm{d}\zeta.$$

Notice that combining (9), the fact that $\eta(x,z) \le |z|$ together with (3) and Fubini, in particular imply that for every ζ and $i \in \{1,\dots,d\}$,

$$\int_{\partial E\cap Q_L} |\nu^E_i||\zeta_i|\,\mathrm{d}\mathcal{H}^{d-1} - \int_{Q_L} |\chi_E(x)-\chi_E(x+\zeta_i)|\,\mathrm{d}x \ge 0. \tag{13}$$

We define also for $E \subset \mathbb{R}$, L-periodic and of finite perimeter, the one-dimensional functionals

$$\mathcal{G}^{1d}_{\tau,L}(E) := \int_{\mathbb{R}} \widehat{K}_\tau(z)\left(\mathrm{Per}(E,[0,L))|z| - \int_0^L |\chi_E(x)-\chi_E(x+z)|\,\mathrm{d}x\right)\mathrm{d}z,$$

so that by Fubini,

$$\mathcal{G}^i_{\tau,L}(E) = \frac{1}{L^d}\int_{[0,L]^{d-1}} \mathcal{G}^{1d}_{\tau,L}(E_{x_i^\perp})\,\mathrm{d}x_i^\perp.$$

In the next lemma we carry out the rescaling and use Lemma 3.2 to split the various contributions of the energy.

Lemma 3.6. *For every Q_L-periodic set E of finite perimeter, we have*

$$\mathcal{F}_{\tau,L}(E) = \frac{1}{L^d}\Bigg(-\mathrm{Per}_1(E,Q_L) + \int_{\mathbb{R}^d} K_\tau(\zeta)\left[\int_{\partial E\cap Q_L}\sum_{i=1}^d |\nu^E_i||\zeta_i|\,\mathrm{d}\mathcal{H}^{d-1} - \int_{Q_L}|\chi_E(x)-\chi_E(x+\zeta)|\,\mathrm{d}x\right]\mathrm{d}\zeta\Bigg).$$

By Lemma 3.2, this yields

$$\mathcal{F}_{\tau,L}(E) \ge -\frac{1}{L^d}\mathrm{Per}_1(E,Q_L) + \sum_{i=1}^d \mathcal{G}^i_{\tau,L}(E) + I_{\tau,L}(E).$$

Proof. By writing that $J\,\mathrm{Per}_1(E,Q_L) = -\tau\,\mathrm{Per}_1(E,Q_L) + \int_{\mathbb{R}^d} K_1(\zeta)\int_{\partial E\cap Q_L}\sum_{i=1}^d |\nu_i^E||\zeta_i|\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}\zeta$, we get

$$\widetilde{\mathcal{F}}_{J,L}(E) = \frac{1}{L^d}\Bigg(-\tau\,\mathrm{Per}_1(E,Q_L) + \int_{\mathbb{R}^d} K_1(\zeta)\left[\int_{\partial E\cap Q_L}\sum_{i=1}^d |\nu_i^E||\zeta_i|\,\mathrm{d}\mathcal{H}^{d-1} - \int_{Q_L}|\chi_E(x)-\chi_E(x+\zeta)|\,\mathrm{d}x\right]\mathrm{d}\zeta\Bigg).$$

Making the change of variables given in (11) and letting also $\zeta := \tau^{-1/\beta}\widehat{\zeta}$, we obtain

$$\begin{aligned}
\widetilde{\mathcal{F}}_{J,L}(E) =& \frac{\tau^{d/\beta}}{\widehat{L}^d}\Big(-\tau^{(p-2d)/\beta}\,\mathrm{Per}_1(\widehat{E},Q_{\widehat{L}}) \\
&+ \int_{\mathbb{R}^d}\frac{\tau^{-2d/\beta}}{\tau^{-p/\beta}|\widehat{\zeta}|^p+1}\left[\int_{\partial\widehat{E}\cap Q_{\widehat{L}}}\sum_{i=1}^d |\nu_i^{\widehat{E}}||\widehat{\zeta}_i|\,\mathrm{d}\mathcal{H}^{d-1} - \int_{Q_{\widehat{L}}}|\chi_{\widehat{E}}(\widehat{x})-\chi_{\widehat{E}}(\widehat{x}+\widehat{\zeta})|\,\mathrm{d}\widehat{x}\right]\mathrm{d}\widehat{\zeta}\Big) \\
=& \frac{\tau^{(p-d)/\beta}}{\widehat{L}^d}\Bigg(-\mathrm{Per}_1(\widehat{E},Q_{\widehat{L}}) \\
&+ \int_{\mathbb{R}^d} K_\tau(\widehat{\zeta})\left[\int_{\partial\widehat{E}\cap Q_{\widehat{L}}}\sum_{i=1}^d |\nu_i^{\widehat{E}}||\widehat{\zeta}_i|\,\mathrm{d}\mathcal{H}^{d-1} - \int_{Q_{\widehat{L}}}|\chi_{\widehat{E}}(\widehat{x})-\chi_{\widehat{E}}(\widehat{x}+\widehat{\zeta})|\,\mathrm{d}\widehat{x}\right]\mathrm{d}\widehat{\zeta}\Bigg).
\end{aligned}$$

□

Before closing this section, we prove several estimates which are consequences of (9). These estimates show that in dimension one, the non-local part of the energy is controlled from below by a negative power of the local gaps and widths. Since there is direct connection between these and the perimeter, this allows to bound from above the perimeter by the non-local part of the energy.

Lemma 3.7. *Let $E \subset \mathbb{R}$ be a L-periodic set of finite perimeter and let $\beta := p - d - 1$. Then, for every $\tau \geq 0$ (recall the definition of h and g from (8)),*

$$\mathcal{G}^{1d}_{\tau,L}(E) \gtrsim \sum_{x\in\partial E\cap[0,L)} \min(h(x)^{-\beta},\tau^{-1}) + \min(g(x)^{-\beta},\tau^{-1}). \tag{14}$$

As a consequence, if $\mathcal{G}^{1d}_{\tau,L}(E) \lesssim \tau^{-1}$, then

$$\min_{x\in\partial E\cap[0,L)} \min(h(x),g(x)) \gtrsim \mathcal{G}^{1d}_{\tau,L}(E)^{-1/\beta}. \tag{15}$$

For $L \geq r > 0$ and $t \in [0,L)$, let $I_t(r) := (t - r/2, t + r/2)$. Then, for every $\delta \geq \tau^{1/\beta}$,

$$\mathrm{Per}(E, I_t(r)) - 1 \lesssim r\delta^{-1} + \delta^\beta \mathcal{G}^{1d}_{\tau,L}(E). \tag{16}$$

In particular, for $r = L$, after optimizing in δ this implies

$$L^{-1}\,\mathrm{Per}(E,[0,L)) \lesssim L^{-1} + \max(\tau L^{-1}\mathcal{G}^{1d}_{\tau,L}(E), (L^{-1}\mathcal{G}^{1d}_{\tau,L}(E))^{1/(p-d)}). \tag{17}$$

Proof. We start by proving (14). By (9) and recalling that $q = p - d + 1$, we have

$$\begin{aligned}
\mathcal{G}^{1d}_{\tau,L}(E) &= \int_{\mathbb{R}} \widehat{K}_\tau(z) \left(|z| \operatorname{Per}(E, [0, L)) - \int_0^L |\chi_E(x) - \chi_E(x+z)| \,\mathrm{d}x \right) \mathrm{d}z \\
&\geq \int_{\mathbb{R}} \widehat{K}_\tau(z) \sum_{x \in \partial E \cap [0,L)} (|z| - \eta(x,z)) \,\mathrm{d}z \\
&\gtrsim \sum_{x \in \partial E \cap [0,L)} \int_{\mathbb{R}} \frac{1}{|z|^q + \tau^{q/\beta}} (|z| - \eta(x,z)) \,\mathrm{d}z,
\end{aligned}$$

where in the last line, we have used (12). Assume now that $E = \cup_{i \in \mathbb{Z}} (s_i, t_i)$ and that $x = s_i$ for some $i \in \mathbb{Z}$ (the case $x = t_i$ can be treated analogously). Then, for $g(x) \geq z \geq -h(x)$, $|z| - \eta(x,z) = 0$. Therefore,

$$\begin{aligned}
\mathcal{G}^{1d}_{\tau,L}(E) &\gtrsim \sum_{x \in \partial E \cap [0,L)} \int_{h(x)}^{+\infty} \frac{z - h(x)}{z^q + \tau^{q/\beta}} \,\mathrm{d}z + \int_{g(x)}^{+\infty} \frac{z - g(x)}{z^q + \tau^{q/\beta}} \,\mathrm{d}z \\
&\gtrsim \sum_{x \in \partial E \cap [0,L)} \min(h(x)^{-\beta}, \tau^{-1}) + \min(g(x)^{-\beta}, \tau^{-1}),
\end{aligned}$$

where we have used that $q - 2 = \beta$. Estimate (15) follows directly from (14). Let us finally prove (16). Let

$$A_\delta := \{x \in \partial E \cap I_t(r) \ : \min(h(x), g(x)) \geq \delta\},$$

so that in particular $\sharp A_\delta \leq 1 + Cr\delta^{-1}$ and for $x \in A_\delta^c \cap \partial E \cap I_t(r)$, since $\delta \geq \tau^{1/\beta}$,

$$1 \lesssim \delta^\beta \left(\min(h(x)^{-\beta}, \tau^{-1}) + \min(g(x)^{-\beta}, \tau^{-1}) \right).$$

Using (14), we obtain

$$\begin{aligned}
\operatorname{Per}(E, I_t(r)) - 1 &= -1 + \sum_{x \in A_\delta} 1 + \sum_{x \in A_\delta^c \cap \partial E \cap I_t(r)} 1 \\
&\lesssim r\delta^{-1} + \sum_{x \in A_\delta^c \cap \partial E \cap I_t(r)} \delta^\beta \left(\min(h(x)^{-\beta}, \tau^{-1}) + \min(g(x)^{-\beta}, \tau^{-1}) \right) \\
&\lesssim r\delta^{-1} + \delta^\beta \mathcal{G}^{1d}_{\tau,L}(E),
\end{aligned}$$

which proves (16).

□

Remark 3.8. *Estimate* (17) *shows that for every* i*, the function* $x_i^\perp \to \operatorname{Per}(E_{x_i^\perp}, [0, L))$ *is almost in* $L^{p-d}([0, L)^{d-1})$.

A simple consequence of (16) and slicing is that the perimeter and the energy are controlled by the non-local terms.

Lemma 3.9. *For every $L \gtrsim 1$ and $\tau \lesssim 1$, and every Q_L-periodic set E,*

$$\mathcal{F}_{\tau,L}(E) \gtrsim -1 + \sum_{i=1}^{d} \mathcal{G}^i_{\tau,L}(E) + I_{\tau,L}(E) \tag{18}$$

and

$$\mathrm{Per}_1(E, Q_L) \lesssim L^d \max(1, \mathcal{F}_{\tau,L}(E)). \tag{19}$$

Proof. Estimate (18) follows from integrating (17) and Young's inequality. Estimate (19) then follows by integrating (16) applied to $r = L$ and $\delta = 1$. $\square$

Remark 3.10. *For $L \gtrsim 1$ and $\tau \lesssim 1$, since $\mathcal{G}^i_{\tau,L}(E)$ and $I_{\tau,L}$ are non-negative,* (18) *yields the uniform lower bound*

$$\min_E \mathcal{F}_{\tau,L}(E) \gtrsim -1.$$

The corresponding upper bound can be readily obtained by computing the energy of periodic stripes with width of order one (see Section 6 for instance).

We finally give another consequence of (16) which resembles (17). Since we are going to use it only for $\tau = 0$, we state it only in that case but an analogous statement holds for $\tau > 0$.

Lemma 3.11. *For every Q_L-periodic set E of finite perimeter and every $m \in \mathbb{N}$ with $m \geq 2$, $t \in [0, L)$, $L \geq r > 0$ and $i \in \{1, \dots, d\}$,*

$$|\{x_i^\perp \in [0, L)^{d-1} \ : \ \mathrm{Per}(E_{x_i^\perp}, I_t(r)) = m\}| \lesssim \mathcal{G}^i_{0,L}(E) L^d r^\beta \left(\frac{1}{m-1}\right)^{p-d}. \tag{20}$$

Proof. Let $\mathcal{B}_m(I_t(r)) := \{x_i^\perp \in [0, L)^{d-1} \ : \ \mathrm{Per}(E_{x_i^\perp}, I_t(r)) = m\}$ and let $x_i^\perp \in \mathcal{B}_m(I_t(r))$. Thanks to (16), for every $\delta > 0$,

$$m - 1 \lesssim r\delta^{-1} + \delta^\beta \mathcal{G}^{1d}_{0,L}(E_{x_i^\perp}).$$

Optimizing in δ, we get $m - 1 \lesssim r^{\beta/(\beta+1)} \mathcal{G}^{1d}_{0,L}(E_{x_i^\perp})^{1/(\beta+1)}$, which can be equivalently written as

$$\mathcal{G}^{1d}_{0,L}(E_{x_i^\perp}) \gtrsim r^{-\beta}(m-1)^{p-d}.$$

We can thus conclude that

$$\mathcal{G}^i_{0,L}(E) \geq \frac{1}{L^d} \int_{\mathcal{B}_m(I_t(r))} \mathcal{G}^{1d}_{0,L}(E_{x_i^\perp}) \, \mathrm{d}x_i^\perp \gtrsim |\mathcal{B}_m(I_t(r))| L^{-d} r^{-\beta} (m-1)^{p-d},$$

from which (20) follows.

$\square$

4 A rigidity result

In this section, we prove that in the limit $\tau = 0$, sets of finite energy must be stripes. For a set E of finite perimeter, we introduce the measures

$$\mu_i := |\partial\chi_{E_{x_i^\perp}}| \otimes dx_i^\perp,$$

so that actually by the slicing formula [3, Th. 3.108] (see also [3, Cor. 2.29]), $|\partial_i \chi_E| = \mu_i$ for $i \in \{1, \dots, d\}$.

We then define for $x \in Q_L$ and $i \in \{1, \dots, d\}$ the "cubic" upper $(d-1)$-dimensional densities

$$\Theta_i(x) := \limsup_{r \to 0} \frac{\mu_i(Q_x(r))}{r^{d-1}},$$

where $Q_x(r) := x + [-r/2, r/2)^d$. We recall that the classical upper $(d-1)$-dimensional densities are defined by [3, Def. 2.55]

$$\Theta_i^*(x) := \limsup_{r \to 0} \frac{\mu_i(B_r(x))}{\omega_{d-1} r^{d-1}},$$

where $B_r(x)$ is the ball of radius r centered in x and where ω_{d-1} is the volume of the unit ball of $\mathbb{R}^{d-1}$. Notice that of course for every $x \in Q_L$,

$$\Theta_i(x) \sim \Theta_i^*(x). \tag{21}$$

Lemma 4.1. *Let E be a Q_L- periodic set of finite perimeter and such that $\sum_{i=1}^d \mathcal{G}^i_{0,L}(E) + I_{0,L}(E) < +\infty$. Then, for every $x \in Q_L$ and every $i \in \{1, \dots, d\}$, $\Theta_i(x) \in \{0, 1\}$.*

Proof. For definiteness, we prove the assertion for Θ_1. Let us first show that $\Theta_1 \le 1$. We recall that $\beta = p - d - 1 \ge d - 1$.

For $r > 0$, let

$$\mathcal{S}_r := \{x_1^\perp \in [0, L)^{d-1} \ : \ \min_{x_1 \in \partial E_{x_1^\perp}} \min(g_{x_1^\perp}(x_1), h_{x_1^\perp}(x_1)) > r\}.$$

For $x_1^\perp \in \mathcal{S}_r^c$, by (15), $r \gtrsim \mathcal{G}^{1d}_{0,L}(E_{x_1^\perp})^{-1/\beta}$, that is $\mathcal{G}^{1d}_{0,L}(E_{x_1^\perp}) \gtrsim r^{-\beta}$. Integrating this, we get that

$$|\mathcal{S}_r^c| \lesssim \mathcal{G}^1_{0,L}(E) L^d r^\beta. \tag{22}$$

We claim that for $\bar{x} = (\bar{x}_1, \bar{x}_1^\perp) \in Q_L$,

$$\Theta_1(\bar{x}) = \limsup_{r \to 0} \frac{1}{r^{d-1}} \int_{Q'_{\bar{x}_1^\perp}(r) \cap \mathcal{S}_r} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r)) \, \mathrm{d}x_1^\perp, \tag{23}$$

where for $(x_1, x_1^\perp) \in [0, L)^d$ and $r > 0$, $Q'_{\bar{x}_1^\perp}(r) := \bar{x}_1^\perp + [-r/2, r/2)^{d-1}$ and $I_{\bar{x}_1}(r) := \bar{x}_1 + [-r/2, r/2)$. Indeed, letting as in the proof of (20), $\mathcal{B}_m(I_{\bar{x}_1}(r)) := \{x_1^\perp \in [0, L)^{d-1} \ : \ \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r)) = m\}$, for

$r > 0$ we have

$$\int_{Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r^c} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r))\,\mathrm{d}x_1^\perp = \int_{Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r^c\cap\mathcal{B}_2^c(I_{\bar{x}_1}(r))} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r))\,\mathrm{d}x_1^\perp + \int_{Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r^c\cap\mathcal{B}_2(I_{\bar{x}_1}(r))} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r))\,\mathrm{d}x_1^\perp.$$

Since on the one hand, from (22)

$$\int_{Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r^c\cap\mathcal{B}_2(I_{\bar{x}_1}(r))} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r))\,\mathrm{d}x_1^\perp \le 2|\mathcal{S}_r^c| \lesssim \mathcal{G}^1_{0,L}(E)L^d r^\beta,$$

and on the other hand, thanks to (20) and $\beta > d-1 \ge 1$,

$$\int_{Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r^c\cap\mathcal{B}_2^c(I_{\bar{x}_1}(r))} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r))\,\mathrm{d}x_1^\perp \le \sum_{m=3}^{+\infty} m|\mathcal{B}_m(I_{\bar{x}_1}(r))| \lesssim \mathcal{G}^1_{0,L}(E)L^d r^\beta,$$

we get

$$\int_{Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r^c} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r))\,\mathrm{d}x_1^\perp \lesssim \mathcal{G}^1_{0,L}(E)L^d r^\beta.$$

From this,

$$\limsup_{r\to 0} \frac{1}{r^{d-1}} \int_{Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r^c} \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r))\,\mathrm{d}x_1^\perp \lesssim \limsup_{r\to 0} \mathcal{G}^1_{0,L}(E)L^d r^{\beta-(d-1)} = 0$$

and (23) follows. Since for $x_1^\perp \in \mathcal{S}_r$ and $I \subset [0,L)$ with $|I| \le r$, $\mathrm{Per}(E_{x_1^\perp}, I) \in \{0,1\}$, (23) implies that $\Theta_1 \le 1$.

Assume now for the sake of contradiction that there exists $\bar{x} \in Q_L$ such that $0 < \Theta_1(\bar{x}) < 1$. Let

$$A := \{x_1^\perp \in Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r \ :\ \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r)) = 1\} \qquad \text{and} \qquad B := \{x_1^\perp \in Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r \ :\ \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r)) = 0\},$$

so that $A \cup B = Q'_{\bar{x}_1^\perp}(r)\cap\mathcal{S}_r$. Then, thanks to (23), there exists $\delta > 0$ such that for all $\bar{r} > 0$ there exists $0 < r \le \bar{r}$, with

$$\delta r^{d-1} \le |A| \le (1-\delta)r^{d-1} \qquad \text{and} \qquad \delta r^{d-1} \le |B| \le (1-\delta)r^{d-1}.$$

Letting

$$\widetilde{A} := \{x_1^\perp \in A\ : E_{x_1^\perp} \cap I_{\bar{x}_1}(r) = (s(x_1^\perp), \bar{x}_1 + r/2)\} \quad \text{and} \quad \widetilde{B} := \{x_1^\perp \in B :\ E_{x_1^\perp} \cap I_{\bar{x}_1}(r) = I_{\bar{x}_1}(r)\},$$

we may assume without loss of generality that

$$\delta r^{d-1}/2 \le |\widetilde{A}| \le (1-\delta) r^{d-1} \qquad \text{and} \qquad \delta r^{d-1}/2 \le |\widetilde{B}| \le (1-\delta) r^{d-1}. \tag{24}$$

Indeed, the case when (24) holds with $\widetilde{A}^c$ (respectively $\widetilde{B}^c$) instead of $\widetilde{A}$ (respectively $\widetilde{B}$) can be similarly treated. Since $(-r/2, r/2)^{d-1} \subset B_{\frac{r}{2}\sqrt{d-1}}(0)$, for every $x_1^\perp \in Q'_{\bar{x}_1^\perp}(r)$,

$$Q'_{\bar{x}_1^\perp}(r) \subset x_1^\perp + B_{\frac{3r}{2}\sqrt{d-1}}(0)$$

so that reducing the integral defining $I_{0,L}(E)$ to the set

$$\{x_1^\perp \in \widetilde{A},\ x_1 \in (s(x_1^\perp) - r, s(x_1^\perp)) \subset E^c_{x_1^\perp},\ x_1 + \zeta_1 \in (s(x_1^\perp), s(x_1^\perp) + r) \subset E_{x_1^\perp} \text{ and } x_1^\perp + \zeta_1^\perp \in \widetilde{B}\},$$

we may now estimate

$$\begin{aligned}
I_{0,L}(E) &\ge \frac{2}{dL^d} \int_{\widetilde{A}} \int_{s(x_1^\perp)-r}^{s(x_1^\perp)} \int_{s(x_1^\perp)-x_1}^{s(x_1^\perp)-x_1+r} \int_{|\zeta_1^\perp| \le \frac{3r}{2}\sqrt{d-1}} \frac{\chi_{\widetilde{B}}(x_1^\perp + \zeta_1^\perp)}{|\zeta|^p} \,\mathrm{d}\zeta_1^\perp \,\mathrm{d}\zeta_1 \,\mathrm{d}x_1 \,\mathrm{d}x_1^\perp \\
&\gtrsim \frac{1}{L^d r^p} \int_{\widetilde{A}} \int_{s(x_1^\perp)-r}^{s(x_1^\perp)} \int_{s(x_1^\perp)-x_1}^{s(x_1^\perp)-x_1+r} \int_{Q'_{\bar{x}_1^\perp}(r)} \chi_{\widetilde{B}}(\zeta_1^\perp) \,\mathrm{d}\zeta_1^\perp \,\mathrm{d}\zeta_1 \,\mathrm{d}x_1 \,\mathrm{d}x_1^\perp \\
&\gtrsim \frac{1}{L^d r^p} \int_{\widetilde{A}} \int_{s(x_1^\perp)-r}^{s(x_1^\perp)} \int_{s(x_1^\perp)-x_1}^{s(x_1^\perp)-x_1+r} |\widetilde{B}| \,\mathrm{d}\zeta_1 \,\mathrm{d}x_1 \,\mathrm{d}x_1^\perp \\
&\gtrsim \frac{\delta^2}{L^d r^{p-2d}},
\end{aligned}$$

which using that r^{2d-p} is unbounded as r goes to zero, contradicts the fact that $I_{0,L}(E)$ is finite. $\square$

Lemma 4.2. *Let E be a Q_L-periodic set of finite perimeter such that $\sum_{i=1}^d \mathcal{G}^i_{0,L}(E) + I_{0,L}(E) < +\infty$. For $i \in \{1, \dots, d\}$, if $\bar{x} \in Q_L$ is such that $\Theta_i(\bar{x}) = 1$ then $\Theta_i(\bar{x} + \zeta_i^\perp) = 1$ for all $\zeta \in \mathbb{R}^d$.*

Proof. Assume for definiteness that $\Theta_1(\bar{x}) = 1$ and let $\bar{\zeta}_1^\perp \in \mathbb{R}^{d-1}$ be such that $\Theta_1(\bar{x} + \bar{\zeta}_1^\perp) = 0$. For all $\bar{r} > 0$, there exists $r < \bar{r}$ such that

$$\mu_1(Q_{\bar{x}}(r)) \ge \frac{3}{4} r^{d-1} \qquad \text{and} \qquad \mu_1(Q_{\bar{x}+\bar{\zeta}_1^\perp}(r)) \le \frac{1}{4} r^{d-1}.$$

Since $\zeta_1^\perp \mapsto \mu_1(Q_{\bar{x}+\zeta_1^\perp}(r))$ is continuous (this is a consequence of $x_1^\perp \mapsto \mathrm{Per}(E_{x_1^\perp}, I_{\bar{x}_1}(r)) \in L^1((0,L)^{d-1})$), there exists $\zeta_1^\perp \in (0, \bar{\zeta}_1^\perp)$ such that

$$\mu_1(Q_{\bar{x}+\zeta_1^\perp}(r)) = \frac{1}{2} r^{d-1}.$$

Arguing exactly as in the proof of Lemma 4.1, we reach a contradiction. $\square$

We are finally in position to prove a rigidity result for sets of finite energy.

Proposition 4.3. *Let E be a Q_L-periodic set of finite perimeter such that $\sum_{i=1}^d \mathcal{G}^i_{0,L}(E)+I_{0,L}(E)<+\infty$. Then, E is one-dimensional i.e. up to permutation of the coordinates, $E=\widehat{E}\times\mathbb{R}^{d-1}$ for some L-periodic set $\widehat{E}$.*

Proof. By Lemma 4.2, if $\bar{x}$ is such that $\Theta_i(\bar{x})=1$ for some $i\in\{1,\dots,d\}$, then for every $\zeta_i^\perp$, $\Theta_i(\bar{x}+\zeta_i^\perp)=1$, which in turn by (21) and [3, Thm. 2.56] implies that $\bar{x}+\zeta_i^\perp\subset\partial E$. Since E has finite perimeter (in Q_L), it may contain at most a finite number of such hyperplanes. If now Q is a cube which does not intersect any of these hyperplanes, then by (21), $\Theta_i^*=\Theta_i=0$ in Q for every $i\in\{1,\dots,d\}$ and therefore by [3, Thm. 2.56] again, $|D\chi_E|(Q)=0$ so that either $Q\subset E$ or $Q\subset E^c$. In Q_L, the set E is thus made of a finite union of hyperrectangles, which constitute a checkerboard structure. Arguing as in the last part of the proof of Lemma 4.1, we obtain that $I_{0,L}(E)=+\infty$ unless this checkerboard is one-dimensional. $\square$

Remark 4.4. *For a set E of finite perimeter it can be readily seen that $I_{0,L}(E)$ finite, implies that every blow-up of E is an hyperplane orthogonal to some coordinate axis. This in particular implies that for $\mathcal{H}^{d-1}$-a.e. $x\in\partial E$, $\nu^E=e_i$ for some $i\in\{1,\dots,d\}$ (with i depending on x). However, it does not seems to be easy to conclude from this fact that E is one dimensional. Indeed, sets of finite perimeter can be very badly behaved (see [3, Ex. 3.53] for instance). In the work in progress [24], we will show that the conclusion of Proposition 4.3 actually holds without assuming that E is of finite perimeter or that $\mathcal{G}^i_{0,L}(E)$ is finite.*

5 The Gamma-convergence result

In this section, we prove our main result, which is the Γ-convergence of $\mathcal{F}_{\tau,L}$ to $\mathcal{F}_{0,L}$. Recall that

$$\mathcal{F}_{0,L}(E):=\begin{cases}\frac{1}{L}\left(-\operatorname{Per}(\widehat{E},[0,L))+\mathcal{G}^{1d}_{0,L}(\widehat{E})\right) & \text{if } E=\widehat{E}\times\mathbb{R}^{d-1} \text{ for some } L\text{-periodic}\\ & \text{ set } \widehat{E} \text{ of finite perimeter,}\\ +\infty & \text{otherwise,}\end{cases}$$

where for $\tau\geq 0$

$$\mathcal{G}^{1d}_{\tau,L}(E)=\int_{\mathbb{R}}\widehat{K}_\tau(z)\left(\operatorname{Per}(E,[0,L))|z|-\int_0^L|\chi_E(x)-\chi_E(x+z)|\,\mathrm{d}x\right)\mathrm{d}z,$$

and for every direction e_i,

$$\begin{aligned}\mathcal{G}^i_{\tau,L}(E)&=\frac{1}{L^d}\int_{[0,L]^{d-1}}\mathcal{G}^{1d}_{\tau,L}(E_{x_i^\perp})\,\mathrm{d}x_i^\perp\\ &=\frac{1}{L^d}\int_{\mathbb{R}}\widehat{K}_\tau(\zeta_i)\left[\int_{\partial E\cap Q_L}|\nu_i^E||\zeta_i|\,\mathrm{d}\mathcal{H}^{d-1}-\int_{Q_L}|\chi_E(x)-\chi_E(x+\zeta_i)|\,\mathrm{d}x\right]\mathrm{d}\zeta_i.\end{aligned}$$

Theorem 5.1. *There holds:*

i) [Compactness and lower bound] Let E^τ be a sequence of Q_L-periodic sets such that $\sup_\tau \mathcal{F}_{\tau,L}(E^\tau) < +\infty$, then up to a subsequence and a relabeling of the coordinate axes, E^τ converges strongly in L^1 to some one-dimensional Q_L-periodic set E of finite perimeter. Moreover,

$$\liminf_{\tau\to 0} \mathcal{F}_{\tau,L}(E^\tau) \geq \mathcal{F}_{0,L}(E). \tag{25}$$

ii) [Upper bound] For every set E with $\mathcal{F}_{0,L}(E) < +\infty$, there exists a sequence $E^\tau \to E$ with

$$\limsup_{\tau\to 0} \mathcal{F}_{\tau,L}(E^\tau) \leq \mathcal{F}_{0,L}(E). \tag{26}$$

Proof. We start by proving i). Let E^τ be such that $\sup_\tau \mathcal{F}_{\tau,L}(E^\tau) < +\infty$. Then, by (19), $\sup_\tau \mathrm{Per}_1(E^\tau, Q_L) < +\infty$ so that we may extract a subsequence converging in L^1 to some Q_L-periodic set E of finite perimeter. Let us first prove that

$$\liminf_{\tau\to 0} \mathcal{G}^i_{\tau,L}(E^\tau) \geq \mathcal{G}^i_{0,L}(E) \quad \text{for } i \in \{1,\ldots,d\} \qquad \text{and} \qquad \liminf_{\tau\to 0} I_{\tau,L}(E^\tau) \geq I_{0,L}(E). \tag{27}$$

By (18) and Proposition 4.3, this would prove that E is one-dimensional. For definiteness, let us prove the inequality concerning $\mathcal{G}^1_{\tau,L}$. The proof of the related lower bound for $I_{\tau,L}$ is similar (and actually simpler). For $\tau > \tau'$, since $\widehat{K}_\tau \leq \widehat{K}_{\tau'}$ and recalling (13),

$$\mathcal{G}^1_{\tau,L}(E^{\tau'}) \leq \mathcal{G}^1_{\tau',L}(E^{\tau'}).$$

Now, if τ is fixed, by Fatou and (13),

$$\begin{aligned}
&\liminf_{\tau'\to 0} \mathcal{G}^1_{\tau,L}(E^{\tau'}) \\
&\geq \frac{1}{L^d}\int_{\mathbb{R}} \widehat{K}_\tau(\zeta_1) \liminf_{\tau'\to 0} \left[\int_{\partial E^{\tau'}\cap Q_L} |\nu_1^{E^{\tau'}}||\zeta_1| \,\mathrm{d}\mathcal{H}^{d-1} - \int_{Q_L} |\chi_{E^{\tau'}}(x) - \chi_{E^{\tau'}}(x+\zeta_1)| \,\mathrm{d}x\right] \mathrm{d}\zeta_1 \\
&\geq \frac{1}{L^d}\int_{\mathbb{R}} \widehat{K}_\tau(\zeta_1) \left[\int_{\partial E\cap Q_L} |\nu_1^{E}||\zeta_1| \,\mathrm{d}\mathcal{H}^{d-1} - \int_{Q_L} |\chi_{E}(x) - \chi_{E}(x+\zeta_1)| \,\mathrm{d}x\right] \mathrm{d}\zeta_1 \\
&= \mathcal{G}^1_{\tau,L}(E),
\end{aligned}$$

where we have used that for fixed ζ_1, $\int_{\partial E\cap Q_L} |\nu_1^E||\zeta_1| \,\mathrm{d}\mathcal{H}^{d-1} - \int_{Q_L} |\chi_E(x) - \chi_E(x+\zeta_1)| \,\mathrm{d}x$ is lower semicontinuous with respect to L^1 convergence. Finally, using again (13) and the monotone convergence theorem, we have

$$\liminf_{\tau'\to 0} \mathcal{G}^1_{\tau',L}(E^{\tau'}) \geq \lim_{\tau\to 0} \mathcal{G}^1_{\tau,L}(E) = \mathcal{G}^1_{0,L}(E),$$

which proves the first part of (27). From this point, in order to show the lower bound (25), we are left to check that

$$\limsup_{\tau\to 0} \mathrm{Per}_1(E^\tau, Q_L) \leq \mathrm{Per}_1(E, Q_L).$$

This is not straightforward since the perimeter is not upper-semicontinuous. By slicing it is enough to prove that for $i \in \{1,\ldots,d\}$,

$$\limsup_{\tau\to 0} \int_{[0,L)^{d-1}} \mathrm{Per}(E^\tau_{x_i^\perp}, [0,L)) \,\mathrm{d}x_i^\perp \leq \int_{[0,L)^{d-1}} \mathrm{Per}(E_{x_i^\perp}, [0,L)) \,\mathrm{d}x_i^\perp. \tag{28}$$

To ease a bit the notation, we write $f_\tau(x_i^\perp) := \mathrm{Per}(E^\tau_{x_i^\perp}, [0,L))$ and $f(x_i^\perp) := \mathrm{Per}(E_{x_i^\perp}, [0,L))$. Let now $\delta > 0$ be fixed. By (15), if $\mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp}) \lesssim \delta^{-\beta}$ then

$$\min_{x_i \in \partial E^\tau_{x_i^\perp}} \min(h^\tau_{x_i^\perp}(x_i), g^\tau_{x_i^\perp}(x_i)) \geq \delta.$$

Therefore if also $|E^\tau_{x_i^\perp} \Delta E_{x_i^\perp}| \leq \delta$, then $f_\tau(x_i^\perp) = f(x_i^\perp)$. We can thus compute

$$\begin{aligned}
&\limsup_{\tau\to 0} \int_{[0,L)^{d-1}} f_\tau \,\mathrm{d}x_i^\perp \\
&\leq \limsup_{\tau\to 0} \int_{\{\mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp})\,\mathrm{d}x_i^\perp \lesssim \delta^{-\beta}\}\cap\{|E^\tau_{x_i^\perp}\Delta E_{x_i^\perp}|\leq\delta\}} f_\tau \,\mathrm{d}x_i^\perp + \limsup_{\tau\to 0} \int_{\{|E^\tau_{x_i^\perp}\Delta E_{x_i^\perp}|\geq\delta\}} f_\tau \,\mathrm{d}x_i^\perp \\
&\quad + \limsup_{\tau\to 0} \int_{\{\mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp})\gtrsim\delta^{-\beta}\}} f_\tau \,\mathrm{d}x_i^\perp \\
&= \limsup_{\tau\to 0} \int_{\{\mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp})\lesssim\delta^{-\beta}\}\cap\{|E^\tau_{x_i^\perp}\Delta E_{x_i^\perp}|\leq\delta\}} f \,\mathrm{d}x_i^\perp + \limsup_{\tau\to 0} \int_{\{|E^\tau_{x_i^\perp}\Delta E_{x_i^\perp}|\geq\delta\}} f_\tau \,\mathrm{d}x_i^\perp \\
&\quad + \limsup_{\tau\to 0} \int_{\{\mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp})\gtrsim\delta^{-\beta}\}} f_\tau \,\mathrm{d}x_i^\perp \\
&\leq \int_{[0,L)^{d-1}} f \,\mathrm{d}x_i^\perp + \limsup_{\tau\to 0} \int_{\{|E^\tau_{x_i^\perp}\Delta E_{x_i^\perp}|\geq\delta\}} f_\tau \,\mathrm{d}x_i^\perp + \limsup_{\tau\to 0} \int_{\{\mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp})\gtrsim\delta^{-\beta}\}} f_\tau \,\mathrm{d}x_i^\perp.
\end{aligned}$$

By (17) and Hölder inequality,

$$\begin{aligned}
\limsup_{\tau\to 0} \int_{\{|E^\tau_{x_i^\perp}\Delta E_{x_i^\perp}|\geq\delta\}} f_\tau \,\mathrm{d}x_i^\perp &\lesssim \limsup_{\tau\to 0} \Bigg[|\{|E^\tau_{x_i^\perp}\Delta E_{x_i^\perp}| \geq \delta\}| + \tau \int_{[0,L)^{d-1}} \mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp}) \,\mathrm{d}x_i^\perp \\
&\qquad + L \int_{\{|E^\tau_{x_i^\perp}\Delta E_{x_i^\perp}|\geq\delta\}} (L^{-1}\mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp}))^{1/(p-d)} \,\mathrm{d}x_i^\perp \Bigg] \\
&\lesssim \limsup_{\tau\to 0} \Bigg[|\{|E^\tau_{x_i^\perp}\Delta E_{x_i^\perp}| \geq \delta\}| \\
&\qquad + L \left(\int_{[0,L)^{d-1}} L^{-1}\mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp}) \,\mathrm{d}x_i^\perp \right)^{1/(p-d)} |\{|E^\tau_{x_i^\perp}\Delta E_{x_i^\perp}| \geq \delta\}|^{\frac{p-d-1}{p-d}} \Bigg] \\
&= 0,
\end{aligned}$$

where in the last line we used that by Fubini, since $|E^\tau \Delta E| \to 0$, also $|\{|E^\tau_{x_i^\perp}\Delta E_{x_i^\perp}| \geq \delta\}| \to 0$. Analogously, using that

$$|\{\mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp}) \gtrsim \delta^{-\beta}\}| \lesssim \delta^\beta \int_{[0,L)^{d-1}} \mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp}) \,\mathrm{d}x_i^\perp \leq C\delta^\beta$$

where now C depends on L and on $\sup_\tau \mathcal{F}_{\tau,L}(E^\tau)$, we obtain for $\delta < 1$,

$$\begin{aligned}
&\limsup_{\tau\to 0} \int_{\{\mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp})\gtrsim\delta^{-\beta}\}} f_\tau \,\mathrm{d}x_i^\perp \\
&\lesssim \limsup_{\tau\to 0} \Bigg[|\{\mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp}) \gtrsim \delta^{-\beta}\}| + \tau \int_{[0,L)^{d-1}} \mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp}) \,\mathrm{d}x_i^\perp \\
&\qquad + L \left(\int_{[0,L)^{d-1}} L^{-1} \mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp}) \,\mathrm{d}x_i^\perp \right)^{1/(p-d)} |\{\mathcal{G}^{1d}_{\tau,L}(E^\tau_{x_i^\perp}) \gtrsim \delta^{-\beta}\}|^{\frac{p-d-1}{p-d}} \Bigg] \\
&\le C \left(\delta^\beta + \delta^{\beta\frac{p-d-1}{p-d}} \right) \le C \delta^{\beta\frac{p-d-1}{p-d}},
\end{aligned}$$

where C depends on L and on $\sup_\tau \mathcal{F}_{\tau,L}(E^\tau)$. Putting these together, we get

$$\limsup_{\tau\to 0} \int_{[0,L)^{d-1}} f_\tau \,\mathrm{d}x_i^\perp \le \int_{[0,L)^{d-1}} f \,\mathrm{d}x_i^\perp + C\delta^{\beta\frac{p-d-1}{p-d}}.$$

Letting finally $\delta \to 0$, we get (28).

We may now turn to the proof of (26). Let E be such that $\mathcal{F}_{0,L}(E) < +\infty$. Without loss of generality, we may assume that $E = \widehat{E} \times \mathbb{R}^{d-1}$ for some L-periodic set $\widehat{E}$ of finite perimeter. Since $\widehat{E}$ is of finite perimeter, we have that[2]

$$c_0 := \min_{x\in\partial\widehat{E}} \min(h(x), g(x)) > 0.$$

Arguing as in the proof of (14), we see that

$$\mathcal{G}^{1d}_{0,L}(\widehat{E}) = \int_{|z|\ge c_0} \widehat{K}_0(z) \left(\mathrm{Per}(\widehat{E}, [0,L))|z| - \int_0^L |\chi_{\widehat{E}}(x) - \chi_{\widehat{E}}(x+z)| \,\mathrm{d}x \right] \mathrm{d}z.$$

Since $\widehat{K}_0$ is integrable in $\{|z| \ge c_0\}$, by the dominated convergence theorem,

$$\lim_{\tau\to 0} \mathcal{G}^{1d}_{\tau,L}(\widehat{E}) = \mathcal{G}^{1d}_{0,L}(\widehat{E}),$$

so that we can use $E = \widehat{E} \times \mathbb{R}^{d-1}$ itself as a recovery sequence. □

6 Minimizers of the one-dimensional problem

In this section, we prove that minimizers of $\mathcal{F}_{0,L}$ are periodic stripes of period essentially not depending on L. For a set E with $\mathcal{F}_{0,L}(E) < +\infty$, we identify by a slight abuse of notation, the set

[2] In fact by (15), we can even get a quantitative estimate of c_0 in term of the energy.

E and corresponding one-dimensional set $\widehat{E}$. That is for a L-periodic set E of finite perimeter, we consider

$$\mathcal{F}_{0,L}(E) = \frac{1}{L}\left(-\operatorname{Per}(E,[0,L)) + C_q \int_{\mathbb{R}} \frac{1}{|z|^q}\left[\operatorname{Per}(E,[0,L))|z| - \int_0^L |\chi_E(x) - \chi_E(x+z)|\,\mathrm{d}x\right]\mathrm{d}z\right), \tag{29}$$

where $q := p - d + 1 > d + 1$ and where we have used that for $z \in \mathbb{R}$, $\widehat{K}_0(z) = C_q|z|^{-q}$ for some constant $C_q > 0$. Since E is of finite perimeter, we can write it as $E = \cup_{i\in\mathbb{Z}}(s_i, t_i)$. As usually, we may assume that $E \cap [0, L) = \cup_{i=1}^N (s_i, t_i)$ for some $N \in \mathbb{N}$, $s_1 > 0$ and $t_N < L$.

For $h > 0$, let $E_h := \cup_{k\in\mathbb{Z}}[(2k)h, (2k+1)h]$. Then, we define

$$e_\infty(h) := \mathcal{F}_{0,2h}(E_h) = \lim_{L\to+\infty} \mathcal{F}_{0,L}(E_h).$$

We can now compute

Lemma 6.1. *Letting*

$$\overline{C}_q := \frac{4C_q(1 - 2^{-(q-3)})}{(q-2)(q-1)} \sum_{k\geq 1} \frac{1}{k^{q-2}},$$

for every $h > 0$, there holds

$$e_\infty(h) = -\frac{1}{h} + \overline{C}_q h^{-(q-1)}. \tag{30}$$

Therefore, $h^\star := ((q-1)\overline{C}_q)^{-1/(q-2)}$ is the unique (positive) minimizer of $e_\infty(h)$.

Proof. Since the contribution of the perimeter to the energy is clear, we just need to compute the non-local interaction. Denote by

$$A := \int_{\mathbb{R}} \frac{1}{|z|^q}\left[\operatorname{Per}(E_h,[0,2h))|z| - \int_0^{2h} |\chi_{E_h}(x) - \chi_{E_h}(x+z)|\,\mathrm{d}x\right]\mathrm{d}z.$$

We start by noting that

$$\begin{aligned} A =& \int_{\mathbb{R}} \frac{1}{|z|^q}\left(|z| - \int_0^h \chi_{E_h^c}(x+z)\,\mathrm{d}x\right)\mathrm{d}z + \int_{\mathbb{R}} \frac{1}{|z|^q}\left(|z| - \int_h^{2h} \chi_{E_h}(x+z)\,\mathrm{d}x\right)\mathrm{d}z \\ =& 2\int_{\mathbb{R}} \frac{1}{|z|^q}\left(|z| - \int_0^h \chi_{E_h^c}(x+z)\,\mathrm{d}x\right)\mathrm{d}z = 4\int_{\mathbb{R}^+} \frac{1}{z^q}\left(z - \int_0^h \chi_{E_h^c}(x+z)\,\mathrm{d}x\right)\mathrm{d}z, \end{aligned}$$

where we have first made the change of variables $x = y + h$ and used that $x + z \in E_h$ is equivalent to $y + z \in E_h^c$ and then, for $z < 0$, we have let $z' = -s$ and $x' = h - x$ (so that if $x + z \in E_h^c$, also $x' + z' \in E_h^c$). Hence, we want to show that

$$\int_{\mathbb{R}^+} \frac{1}{z^q}\left(z - \int_0^h \chi_{E_h^c}(x+z)\,\mathrm{d}x\right)\mathrm{d}z = \frac{2(1 - 2^{-(q-3)})}{(q-2)(q-1)} \sum_{k\geq 1} \frac{1}{k^{q-2}} h^{-(q-2)}. \tag{31}$$

Since in $\mathbb{R}^+$, $\chi_{E_h^c} = \chi_{[0,h]^c} - \sum_{k\geq 1} \chi_{[(2k)h,(2k+1)h]}$,

$$\int_{\mathbb{R}^+} z^{-q} \left(z - \int_0^h \chi_{E_h^c}(x+z)\,\mathrm{d}x \right) \mathrm{d}z = \int_{\mathbb{R}^+} z^{-q} \left(z - \int_0^h \chi_{[0,h]^c}(x+z)\,\mathrm{d}x \right) \mathrm{d}z$$
$$+ \sum_{k\geq 1} \int_{\mathbb{R}^+} z^{-q} \int_0^h \chi_{[(2k)h,(2k+1)h]}(x+z)\,\mathrm{d}x\,\mathrm{d}z.$$

The first term on the right-hand side can be computed as

$$\int_{\mathbb{R}^+} z^{-q} \left(z - \int_0^h \chi_{[0,h]^c}(x+z)\,\mathrm{d}x \right) \mathrm{d}z = \int_h^{+\infty} z^{-q}(z-h)\,\mathrm{d}z = \frac{h^{-(q-2)}}{(q-2)(q-1)},$$

while for the second term we can use that for $k \geq 1$,

$$\int_{\mathbb{R}^+} z^{-q} \int_0^h \chi_{[(2k)h,(2k+1)h]}(x+z)\,\mathrm{d}x\,\mathrm{d}z = \int_{(2k)h}^{(2k+1)h} \int_0^h \frac{\mathrm{d}x\,\mathrm{d}y}{(y-x)^q}$$
$$= \frac{h^{-(q-2)}}{(q-2)(q-1)} \left(\frac{1}{(2k-1)^{q-2}} + \frac{1}{(2k+1)^{q-2}} - \frac{2}{(2k)^{q-2}} \right).$$

Putting this together, we get

$$\int_{\mathbb{R}^+} \frac{1}{z^q} \left(z - \int_0^h \chi_{E_h^c}(x+z)\,\mathrm{d}x \right) \mathrm{d}z$$
$$= \frac{h^{-(q-2)}}{(q-2)(q-1)} \left(1 + \sum_{k\geq 1} \left[\frac{1}{(2k-1)^{q-2}} + \frac{1}{(2k+1)^{q-2}} - \frac{2}{(2k)^{q-2}} \right] \right).$$

Since

$$2(1-2^{-(q-3)}) \sum_{k\geq 1} \frac{1}{k^{q-2}} = 1 + \sum_{k\geq 1} \left[\frac{1}{(2k-1)^{q-2}} + \frac{1}{(2k+1)^{q-2}} - \frac{2}{(2k)^{q-2}} \right], \tag{32}$$

this concludes the proof of (31). $\square$

Remark 6.2. *The sum $\sum_{k\geq 1} k^{-(q-2)}$ is finite since $q > d+1 > 3$. Notice that actually, the sum in the right-hand side of (32) is finite for every $q > 1$. Therefore, the energy of periodic stripes is finite for every $q > 1$ i.e. for $p > d$.*

The main estimate of this section is the following chessboard estimate:

Lemma 6.3. *For every L-periodic set E of finite perimeter, there holds*

$$\mathcal{F}_{0,L}(E) \geq \frac{1}{2L} \sum_{x\in\partial E\cap[0,L)} \left(h(x)e_\infty(h(x)) + g(x)e_\infty(g(x)) \right). \tag{33}$$

The proof of Lemma 6.3 will occupy the rest of this section. Before turning to its proof, let us state its main consequence

Theorem 6.4. *For every $L > 1$, the minimizers of $\mathcal{F}_{0,L}$ are periodic stripes E_h for some $h > 0$ satisfying*

$$|h - h^\star| \lesssim \frac{1}{L}. \tag{34}$$

Moreover, for $L \in 2h^\star\mathbb{N}$, $E_{h^\star}$ is the unique minimizer.

Proof. Let us first prove the last claim. Let $L > 1$ and let E be any L periodic set then by (33), and the minimality of $h^\star$ for e_∞,

$$\begin{aligned}\mathcal{F}_{0,L}(E) \geq \frac{1}{2L} \sum_{x\in\partial E\cap[0,L)} (h(x)e_\infty(h(x)) + g(x)e_\infty(g(x))) \\ \geq \frac{e_\infty(h^\star)}{2L} \sum_{x\in\partial E\cap[0,L)} (h(x) + g(x)) = e_\infty(h^\star).\end{aligned} \tag{35}$$

For $L \in 2h^\star\mathbb{N}$, $E_{h^\star}$ is admissible thus we have equalities in (35) for minimizers of $\mathcal{F}_{0,L}$. Since $h^\star$ is the unique minimizer of $e_\infty(h)$, this implies that $h(x) = g(x) = h^\star$ for every $x \in \partial E$, proving the claim.

If now L is arbitrary, using the first inequality in (35) and (30), we get

$$L\mathcal{F}_{0,L}(E) \geq -\operatorname{Per}(E, [0, L)) + \frac{1}{2}\overline{C}_q \sum_{x\in\partial E\cap[0,L)} \left(h(x)^{-(q-2)} + g(x)^{-(q-2)}\right).$$

If $\operatorname{Per}(E, [0, L)) = 2N$ is fixed, then, from the supperadditivity of $x^{-(q-2)}$,

$$\min_{\sum_{i=1}^{2N}(h_i+g_i)=2L} \sum_{i=1}^{2N} \left(h_i^{-(q-2)} + g_i^{-(q-2)}\right) = 4N \left(\frac{L}{2N}\right)^{-(q-2)},$$

and the minimum is attained only at $h_i = g_i = \frac{L}{2N}$. Since $E_{\frac{L}{2N}}$ is admissible and satisfies

$$L\mathcal{F}_{0,L}(E_{\frac{L}{2N}}) = -2N + 2\overline{C}_q N \left(\frac{L}{2N}\right)^{-(q-2)},$$

we obtain as before that every minimizer has to be equal to $E_{\frac{L}{2N}}$ for some $N \in \mathbb{N}$. Letting $2N^\star := \frac{L}{h^\star}$, we see that the function $x \to -x + \overline{C}_q L^{-(q-2)} x^{q-1}$ is minimized at $x = 2N^\star$. This implies that letting $h^+ := L(2\lceil L/(2h^\star)\rceil)^{-1}$ and $h_- := L(2\lfloor L/(2h^\star)\rfloor)^{-1}$, the only possible minimizers of $\mathcal{F}_{0,L}$ are $E_{h^\pm}$. Since $|h^\pm - h^\star| \lesssim L^{-1}$, this concludes the proof of (34).

□

Remark 6.5. *From the proof of (34), it is not hard to see that for most values of L, the minimizer of $\mathcal{F}_{0,L}$ is actually unique and equal to E_{h^+} or E_{h^-}.*

We now turn to the proof of (33). The idea is, as in [20, 21], to use the method of reflection positivity. As in these papers (which we mostly follow), the main point is to prove it for the non-local part of the energy. However, we face here the slight technical difficulty that the kernel $|s|^{-q}$ is not integrable around zero and thus we cannot directly split the integral in (29) into two pieces but will use the Laplace transform first.

Lemma 6.6. *Let $\rho \geq 0$ be such that $\int_0^{+\infty} \rho \,\mathrm{d}\alpha = 1$ and let*

$$\hat{\rho}(\alpha) := -\rho(\alpha) + \frac{2C_q \alpha^{q-3}}{\Gamma(q)},$$

where Γ is Euler's Gamma function. Then, for every L-periodic set E of finite perimeter,

$$\mathcal{F}_{0,L}(E) = \int_0^{+\infty} \frac{1}{L}\left(\hat{\rho}(\alpha)\operatorname{Per}(E,[0,L)) - \frac{C_q\alpha^{q-1}}{\Gamma(q)}\int_{[0,L]\times\mathbb{R}} |\chi_E(x)-\chi_E(y)|e^{-\alpha|x-y|}\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}\alpha. \tag{36}$$

Proof. Since for $s>0$, $s^{-q} = \frac{1}{\Gamma(q)}\int_0^{+\infty}\alpha^{q-1}e^{-\alpha s}\,\mathrm{d}\alpha$, we have

$$\begin{aligned}\mathcal{F}_{0,L}(E) = \frac{1}{L}\Bigg(&\int_0^{+\infty} -\rho(\alpha)\operatorname{Per}(E,[0,L))\,\mathrm{d}\alpha\\ &+\frac{C_q}{\Gamma(q)}\int_{\mathbb{R}}\int_0^{+\infty}\alpha^{q-1}e^{-\alpha|z|}\left[\operatorname{Per}(E,[0,L))|z| - \int_0^L|\chi_E(x)-\chi_E(x+z)|\,\mathrm{d}x\right]\mathrm{d}\alpha\,\mathrm{d}z\Bigg).\end{aligned}$$

The set E being of finite perimeter, it is a finite union of intervals from which arguing as in the proof of (14) we see that the function $\alpha^{q-1}e^{-\alpha|z|}\left[\operatorname{Per}(E,[0,L))|z| - \int_0^L|\chi_E(x)-\chi_E(x+z)|\,\mathrm{d}x\right]$ is integrable in (α,z) so that we can apply Fubini to obtain

$$\begin{aligned}\mathcal{F}_{0,L}(E) = \frac{1}{L}\int_0^{+\infty}\Bigg(&-\rho(\alpha)\operatorname{Per}(E,[0,L))\\ &+\frac{C_q}{\Gamma(q)}\alpha^{q-1}\int_{\mathbb{R}}e^{-\alpha|z|}\Big[\operatorname{Per}(E,[0,L))|z| - \int_0^L|\chi_E(x)-\chi_E(x+z)|\,\mathrm{d}x\Big]\,\mathrm{d}z\Bigg)\mathrm{d}\alpha.\end{aligned}$$

Using that $\int_{\mathbb{R}}|z|e^{-\alpha|z|}dz = 2\alpha^{-2}$ we conclude the proof of (36). $\square$

For $\alpha, h > 0$, let

$$\begin{aligned}e_{\alpha,\infty}(h) &:= -\frac{1}{2h}\int_0^{2h}\int_{\mathbb{R}}|\chi_{E_h}(x)-\chi_{E_h}(y)|e^{-\alpha|x-y|}\,\mathrm{d}x\,\mathrm{d}y\\ &= \lim_{L\to+\infty} -\frac{1}{L}\int_{[0,L]\times\mathbb{R}}|\chi_{E_h}(x)-\chi_{E_h}(y)|e^{-\alpha|x-y|}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Up to noticing that in (33), the interfacial terms are the same on both sides, thanks to (36) and integration in α, Lemma 6.3 is proven provided we can show

Lemma 6.7. *For every $\alpha > 0$ and every L-periodic set E of finite perimeter,*

$$-\int_{[0,L]\times\mathbb{R}} |\chi_E(x) - \chi_E(y)| e^{-\alpha|x-y|}\, \mathrm{d}x\, \mathrm{d}y \geq \frac{1}{2} \sum_{x\in\partial E\cap[0,L)} \left[h(x)e_{\alpha,\infty}(h(x)) + g(x)e_{\alpha,\infty}(g(x))\right]. \quad (37)$$

Proof. Since α is fixed, in order to lighten notation, we will assume that $\alpha = 1$.

As pointed out in [17, Ap. A], periodic boundary conditions are not well suited for the application of reflection positivity. We will thus prove a statement similar to (37) under free boundary conditions. For this, we notice that since the kernel $e^{-|s|}$ is integrable and since E is periodic,

$$-\int_{[0,L]\times\mathbb{R}} |\chi_E(x) - \chi_E(y)| e^{-|x-y|}\, \mathrm{d}x\, \mathrm{d}y = \lim_{k\to+\infty} -\frac{1}{k} \int_{[0,kL]\times[0,kL]} |\chi_E(x) - \chi_E(y)| e^{-|x-y|}\, \mathrm{d}x\, \mathrm{d}y.$$

We are thus left to prove that for every set $E \subset [0, L)$ of finite perimeter,

$$-\int_{[0,L]^2} |\chi_E(x) - \chi_E(y)| e^{-|x-y|}\, \mathrm{d}x\, \mathrm{d}y \geq \frac{1}{2} \sum_{x\in\partial E} \left[h(x)e_{1,\infty}(h(x)) + g(x)e_{1,\infty}(g(x))\right]. \quad (38)$$

In order to prove (38) we need to introduce some further notation. For $L_1, L_2 > 0$, and two sets $E_1 \subset [0, L_1)$, $E_2 \subset (L_1, L_1 + L_2)$, we let $L := L_1 + L_2$, $(E_1, E_2) := E_1 \cup E_2$ and

$$\mathcal{J}(E_1, E_2) := -\int_{[0,L]\times[0,L]} |\chi_{(E_1,E_2)}(x) - \chi_{(E_1,E_2)}(y)| e^{-|x-y|}\, \mathrm{d}x\, \mathrm{d}y.$$

We then define the set $(E_1, \theta E_1)$ in $[0, 2L_1]$ by

$$\chi_{(E_1,\theta E_1)}(x) := \begin{cases} \chi_{E_1}(x) & \text{for } x \in [0, L_1] \\ 1 - \chi_{E_1}(2L_1 - x) & \text{for } x \in (L_1, 2L_1]. \end{cases}$$

Letting $L_2 = L - L_1$, we similarly define, $(\theta E_2, E_2)$ as a subset of $[L_1 - L_2, L]$ by

$$\chi_{(\theta E_2,E_2)}(x) := \begin{cases} \chi_{E_2}(x) & \text{for } x \in [L_1, L] \\ 1 - \chi_{E_2}(2L_1 - x) & \text{for } x \in (L_1 - L_2, L_1]. \end{cases}$$

The key estimate is

$$\mathcal{J}(E_1, E_2) \geq \frac{1}{2} \left(\mathcal{J}(E_1, \theta E_1) + \mathcal{J}(\theta E_2, E_2)\right). \quad (39)$$

Once (39) is established, (38) follows by multiple reflections. We refer the reader to [21, Lem. A.1] or to [17, Ap. A] for instance for a proof of this fact.
Let us prove (39). We start by computing $\mathcal{J}(E_1, E_2)$. By definition,

$$\begin{aligned}\mathcal{J}(E_1, E_2) = &-\int_{[0,L_1]^2} |\chi_{E_1}(x) - \chi_{E_1}(y)| e^{-|x-y|}\, \mathrm{d}x\, \mathrm{d}y - \int_{[L_1,L]^2} |\chi_{E_2}(x) - \chi_{E_2}(y)| e^{-|x-y|}\, \mathrm{d}x\, \mathrm{d}y \\ &- 2\int_{[0,L_1]\times[L_1,L]} |\chi_{E_1}(x) - \chi_{E_2}(y)| e^{-|x-y|}\, \mathrm{d}x\, \mathrm{d}y.\end{aligned}$$

Using that $|\chi_{E_1}(x)-\chi_{E_2}(y)| = \chi_{E_1}(x)\chi_{E_2^c}(y)+\chi_{E_1^c}(x)\chi_{E_2}(y)$ and that for $x \in [0, L_1]$ and $y \in [L_1, L]$, $|x-y| = y-x$, we obtain

$$\begin{aligned}\mathcal{J}(E_1, E_2) = &-\int_{[0,L_1]^2} |\chi_{E_1}(x) - \chi_{E_1}(y)|e^{-|x-y|}\,\mathrm{d}x\,\mathrm{d}y - \int_{[L_1,L]^2} |\chi_{E_2}(x) - \chi_{E_2}(y)|e^{-|x-y|}\,\mathrm{d}x\,\mathrm{d}y \\ &- 2\left(\int_0^{L_1} \chi_{E_1}e^x\,\mathrm{d}x\right)\left(\int_{L_1}^{L} \chi_{E_2^c}e^{-y}\,\mathrm{d}y\right) - 2\left(\int_0^{L_1} \chi_{E_1^c}e^x\,\mathrm{d}x\right)\left(\int_{L_1}^{L} \chi_{E_2}e^{-y}\,\mathrm{d}y\right).\end{aligned}$$

Using the definition of $(E_1, \theta E_1)$, we compute similarly

$$\begin{aligned}\mathcal{J}(E_1, \theta E_1) = &-\int_{[0,L_1]^2} |\chi_{E_1}(x) - \chi_{E_1}(y)|e^{-|x-y|}\,\mathrm{d}x\,\mathrm{d}y - \int_{[L_1,2L_1]^2} |\chi_{\theta E_1}(x) - \chi_{(\theta E_1)}(y)|e^{-|x-y|}\,\mathrm{d}x\,\mathrm{d}y \\ &- 2\int_{[0,L_1]\times[L_1,2L_1]} (\chi_{E_1}(x)\chi_{(\theta E_1)^c}(y) + \chi_{E_1^c}(x)\chi_{\theta E_1}(y))e^{x-y}\,\mathrm{d}x\,\mathrm{d}y \\ = &-2\int_{[0,L_1]^2} |\chi_{E_1}(x) - \chi_{E_1}(y)|e^{-|x-y|}\,\mathrm{d}x\,\mathrm{d}y \\ &- 2\int_{[0,L_1]\times[L_1,2L_1]} (\chi_{E_1}(x)\chi_{E_1}(2L_1 - y) + \chi_{E_1^c}(x)\chi_{E_1^c}(2L_1 - y))e^{x-y}\,\mathrm{d}x\,\mathrm{d}y \\ = &-2\int_{[0,L_1]^2} |\chi_{E_1}(x) - \chi_{E_1}(y)|e^{-|x-y|}\,\mathrm{d}x\,\mathrm{d}y \\ &- 2e^{-2L_1}\left[\left(\int_0^{L_1} \chi_{E_1}e^x\,\mathrm{d}x\right)^2 + \left(\int_0^{L_1} \chi_{E_1^c}e^x\,\mathrm{d}x\right)^2\right].\end{aligned}$$

Analogously, we get

$$\begin{aligned}\mathcal{J}(\theta E_2, E_2) = &-2\int_{[L_1,L]^2} |\chi_{E_2}(x) - \chi_{E_2}(y)|e^{-|x-y|}\,\mathrm{d}x\,\mathrm{d}y \\ &- 2e^{2L_1}\left[\left(\int_{L_1}^{L} \chi_{E_2}e^{-x}\,\mathrm{d}x\right)^2 + \left(\int_{L_1}^{L} \chi_{E_2^c}e^{-x}\,\mathrm{d}x\right)^2\right].\end{aligned}$$

This concludes the proof of (39) since

$$\begin{aligned}e^{2L_1}&\left[\left(\int_{L_1}^{L} \chi_{E_2}e^{-x}\,\mathrm{d}x\right)^2 + \left(\int_{L_1}^{L} \chi_{E_2^c}e^{-x}\,\mathrm{d}x\right)^2\right] + e^{-2L_1}\left[\left(\int_0^{L_1} \chi_{E_1}e^x\,\mathrm{d}x\right)^2 + \left(\int_0^{L_1} \chi_{E_1^c}e^x\,\mathrm{d}x\right)^2\right] \\ &\geq 2\left(\int_0^{L_1} \chi_{E_1}e^x\,\mathrm{d}x\right)\left(\int_{L_1}^{L} \chi_{E_2^c}e^{-x}\,\mathrm{d}x\right) + 2\left(\int_0^{L_1} \chi_{E_1^c}e^x\,\mathrm{d}x\right)\left(\int_{L_1}^{L} \chi_{E_2}e^{-x}\,\mathrm{d}x\right).\end{aligned}$$

□

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