A PDE approach to a 2-dimensional matching problem

Luigi Ambrosio [*]   Federico Stra [†]   Dario Trevisan [‡]

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Abstract

We prove asymptotic results for 2-dimensional random matching problems. In particular, we obtain the leading term in the asymptotic expansion of the expected quadratic transportation cost for empirical measures of two samples of independent uniform random variables in the square. Our technique is based on a rigorous formulation of the challenging PDE ansatz by S. Caracciolo et al. (Phys. Rev. E, **90** 012118, 2014) that linearize the Monge-Ampère equation.

Contents

[*]Scuola Normale Superiore, `luigi.ambrosio@sns.it`.

[†]Scuola Normale Superiore, `federico.stra@sns.it`.

[‡]Università di Pisa, `dario.trevisan@unipi.it`. Partially supported by Project PRA_2016_41.

1 Introduction

Optimal matching problems are random variational problems widely investigated in the mathematical and physical literature. Many variants are possible, for instance the monopartite problem, dealing with the optimal coupling of an even number n of i.i.d. points X_i, the grid matching problem, where one looks for the optimal matching of an empirical measure $\sum_i \frac{1}{n} \delta_{X_i}$ to a deterministic and "equally spaced" grid, the closely related problem of optimal matching to the common law $\mathfrak{m}$ of X_i and the bipartite problem, dealing with the optimal matching of $\sum_i \frac{1}{n} \delta_{X_i}$ to $\sum_i \frac{1}{n} \delta_{X_i}$, with (X_i, Y_i) i.i.d. See the monographs [Y98] and [T14] for many more informations on this subject. In addition to these problems, one may study the optimal assignment problem, [C04], where the optimization involves also the weights of the Dirac masses δ_{X_i} and the closely related problem of transporting Lebesgue measure to a Poisson point process [HS13], which involves in the limit measures with infinite mass.

In this paper we focus on two of these problems, namely optimal matching to the reference measure and the bipartite problem. Denoting by D the d-dimensional domain and by $\mathfrak{m} \in \mathscr{P}(D)$ the law of the points X_i, Y_i, the problem is to estimate the rate of convergence to 0 of

$$\mathbb{E}\left[W_p^p\left(\sum_{i=1}^n \frac{1}{n}\delta_{X_i}, \mathfrak{m}\right)\right], \qquad \mathbb{E}\left[W_p^p\left(\sum_{i=1}^n \frac{1}{n}\delta_{X_i}, \sum_{i=1}^n \frac{1}{n}\delta_{Y_i}\right)\right], \qquad (1.1)$$

where $p \in [1, \infty)$ is the power occurring in the transportation cost $c = \mathsf{d}^p$ (also the case $p = \infty$ is considered in the literature, see for instance [SY91] and the references therein), finding tight upper and lower bounds and, possibly, proving existence of the limit of the renormalized quantities as $n \to \infty$.

When $\mathfrak{m}$ is the uniform measure, the typical distance between points is expected to be of order $n^{-1/d}$, and therefore it is natural to guess that the quantities $c_{n,p,d}$ introduced in (1.1) behave as $n^{-p/d}$. Indeed, e.g. when $D = (0,1)^d$ and $\mathfrak{m}$ is the uniform measure, the 1-Lipschitz test function $\phi(x) = \min_i |x - X_i|$ easily provides the deterministic estimate $W_p(\sum_i \frac{1}{n}\delta_{X_i}, \mathfrak{m}) \geq W_1(\sum_i \frac{1}{n}\delta_{X_i}, \mathfrak{m}) \geq c(d)n^{-1/d}$. However, it is by now well known that the scaling $n^{-p/d}$ is correct for $d \geq 3$, while it is false for $d = 1$ and $d = 2$. Despite plenty of heuristic arguments and numerical results, these are (as far as we know) the main results that have been rigorously proved (we focus here on the model case when $\mathfrak{m}$ is the uniform measure and we do not distinguish between optimal matching to $\mathfrak{m}$ and bipartite), denoting $a_n \sim b_n$ if $\limsup_n a_n/b_n < \infty$ and $\limsup_n b_n/a_n < \infty$:

- when $D = [0,1]$ or $D = \mathbb{T}^1$, then $c_{n,p,1}/n^{-p} \sim n^{p/2}$ and, when $p = 2$, $\lim_{n\to\infty} nc_{n,2,1}$ can be explicitly computed, see [CS14];

- when $D = [0,1]^2$, then $c_{n,p,2}/n^{-p/2} \sim (\log n)^{p/2}$, see [AKT84];

- when $D = [0,1]^d$ with $d \geq 3$, then $c_{n,1,d}/n^{-1/d} \sim 1$ and the limit exists [BM02], [DY95], for $p \in [1, d/2)$ one has $c_{n,p,d}/n^{-p/d} \sim 1$ and the limit exists [BB13], it is not known whether the limit exists for $p \in [d/2, \infty)$. In the more recent paper [FG15] also non-asymptotic upper bounds have been provided.

In the case $d = 2$, the convergence of $(\log n)^{-p/2} c_{n,p,2}/n^{-p/2}$ as $n \to \infty$ and the characterization of the limit are still open problems, particularly in the case $p = 1$ [T14, Research problem 4.3.3].

Our interest in this subject has been motivated by the recent work [CLPS14] where, on the basis of an ansatz, very specific predictions on the expansion of

$$n^{-p/d} \, \mathbb{E} \left[W_p^p \left(\sum_{i=1}^n \frac{1}{n} \delta_{X_i}, \sum_{i=1}^n \frac{1}{n} \delta_{Y_i} \right) \right]$$

have been made on $\mathbb{T}^d$, for all ranges of dimensions d and powers p. See also [CS15] for the analysis of correlations. In brief, the ansatz of [CLPS14] is based on a linearisation ($\rho_i \sim 1$ in C^1 topology, $\psi \sim f + \frac{1}{2}|x|^2$ in C^2 topology) of the Monge-Ampère equation

$$\rho_1(\nabla\psi)\det\nabla^2\psi = \rho_0,$$

(which describes the optimal transport map $T = \nabla\psi$ from the measures having probability densities ρ_0 to ρ_1) leading to Poisson's equation $-\Delta f = \rho_1 - \rho_0$.

This ansatz is very appealing, but on the mathematical side it poses several challenges, because the energies involved are infinite for $d \geq 2$ (the measures being Dirac masses), because this procedure does not provide an exact matching between the measures (due to the linearisation) and because the necessity of giving lower bounds persists, as matchings provide only upper bounds. While we are still very far from justifying rigorously all predictions of [CLPS14], see also Section 6 for a discussion on this topic, we have been able to use this idea to prove existence of the limit and compute explicitly it in the case $p = d = 2$, in agreement with [CLPS14]:

Theorem 1.1 (Main result). *Assume that either $D = [0,1]^2$ or that D is a compact 2-dimensional Riemannian manifold with no boundary and set $\mu^n = \sum_i \frac{1}{n} \delta_{X_i}$, $\nu^n = \sum_i \frac{1}{n} \delta_{Y_i}$, being X_i, Y_i i.i.d. with law $\mathfrak{m}_D = \mathfrak{m}/\mathfrak{m}(D)$. Then,*

$$\lim_{n \to \infty} \frac{n}{\log n} \, \mathbb{E} \left[W_2^2(\mu^n, \mathfrak{m}_D) \right] = \frac{\mathfrak{m}(D)}{4\pi}. \tag{1.2}$$

In the bipartite case, if either $D = [0,1]^2$ or $D = \mathbb{T}^2$, one has

$$\lim_{n \to \infty} \frac{n}{\log n} \, \mathbb{E} \left[W_2^2(\mu^n, \nu^n) \right] = \frac{1}{2\pi}. \tag{1.3}$$

Finally, in the case $D = [0,1]^2$, if T^{μ^n} denotes the optimal transport map from $\mathfrak{m}$ to μ^n, one has

$$\lim_{n \to \infty} \frac{n}{\log n} \int_D \left| \mathbb{E}[T^{\mu^n}(x) - x] \right|^2 \mathrm{d}\mathfrak{m}(x) = 0. \tag{1.4}$$

By the invariance of the statements under rescaling of the measure, we always assume in the following that $\mathfrak{m}(D) = 1$. Using the spectral gap, standard results related to the phenomenon of concentration of measure (see [GM83], [L01], [BL16]) also yield that the random variables $n(\log n)^{-1} W_2^2(\mu^n, \mathfrak{m})$ converge in law to the Dirac mass at $1/(4\pi)$, more precisely (1.2) and exponential concentration yield

$$\lim_{n \to \infty} \mathbb{E} \left[F \left(\frac{n}{\log n} W_2^2(\mu^n, \mathfrak{m}) \right) \right] = F \left(\frac{1}{4\pi} \right) \tag{1.5}$$

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for any $F : \mathbb{R} \to \mathbb{R}$ continuous and with linear growth. See Remark 4.7 for more precise results, whose proof can also be adapted to cover the bipartite case.

In our proof the geometry of the domain D enters only through the (asymptotic) properties of the spectrum of the Laplacian with homogeneous Neumann boundary conditions; for this reason we are able to cover also abstract manifolds, as the two-dimensional sphere or the two-dimensional nonflat torus embedded in $\mathbb{R}^3$. Even though in dimension $d = 1$ (but mostly for the case $D = [0,1]$) a much more detailed analysis can be made, see Remark 4.2 and [BL16] for much more on the subject, we include proofs and statements of the 1-d case, to illustrate the flexibility of our synthetic method.

Let us give some heuristic ideas on the strategy of proof, starting from the upper bound. In order to obtain finite energy solutions to Poisson's equation we study the regularized PDE

$$- \Delta f^{n,t} = (u^{n,t} - 1) \tag{1.6}$$

where $u^{n,t}$ is the density of $P_t^*(\mu^n - \mathfrak{m})$ and P_t^* is the heat semigroup with Neumann boundary conditions, acting on measures. Then, choosing $t = \gamma n^{-1} \log n$ with γ small, we have a small error in the estimation from above of $c_{n,2,2}$ if we replace μ^n by its regularization $P_t^* \mu^n$. Eventually, we use Dacorogna-Moser's technique (see Proposition 2.3) to provide an exact coupling between $P_t^* \mu^n$ and $\mathfrak{m}$, leading to an estimate of the form

$$W_2^2(P_t^* \mu^n, \mathfrak{m}) \leq \int_D \left(\int_0^1 \frac{1}{(1-s) + s u^{n,t}} \, \mathrm{d}s \right) |\nabla f^{n,t}|^2 \, \mathrm{d}\mathfrak{m}.$$

To conclude, we have to estimate very carefully how much the factor in front of $|\nabla f^{n,t}|^2$ differs from 1; this requires in particular higher integrability estimates on $|\nabla f^{n,t}|$.

Let us consider now the lower bound. The duality formula

$$\frac{1}{2} W_2^2(\mu, \nu) = \sup_{\phi(x) + \psi(y) \leq \mathsf{d}^2(x,y)/2} - \int_D \phi \, \mathrm{d}\mu + \int_D \psi \, \mathrm{d}\nu$$

is the standard way to provide lower bounds on W_2; given ϕ, the best possible $\psi = Q_1 \phi$ compatible with the constraint is given by the Hopf-Lax formula (2.2). Choosing again $\phi = f^{n,t}$ as in the ansatz, we are led to estimate carefully

$$\frac{1}{2} \int |\nabla f^{n,t}|^2 - \left(- \int_D f^{n,t} u^{n,t} \, \mathrm{d}\mathfrak{m} + \int_D Q_1 f^{n,t} \, \mathrm{d}\mathfrak{m} \right)$$

in events of the form $\{\sup_D |u^{n,t} - 1| \leq \eta\}$ (whose probabilities tend to 1). We do this using Laplacian estimates and the viscosity approximation of the Hopf-Lax semigroup provided by the Hopf-Cole transform; in these estimates, lower bound on the Ricci curvature play an important role.

In the bipartite case, the result can be obtained from the previous ones playing with independence. Heuristically, the random "vectors" pointing from $\mathfrak{m}$ to μ^n and from $\mathfrak{m}$ to ν^n are independent, and since $\mathscr{P}(D)$ is "Riemannian" on small scales when endowed with the distance W_2, we obtain a factor 2, as in the identity $\mathbb{E}[(X - Y)^2] = 2 \operatorname{Var}(X)$ when X, Y are i.i.d. random variables. Interestingly, the rigorous proof of this fact provides also the information (1.4) on the mean displacement as function of the position.

The paper is organized as follows. In Section 2 we first recall preliminary results on the Wasserstein distance and the main tools (Dacorogna-Moser interpolation, duality, Hopf-Lax semigroup) involved in the proof of the upper and lower bounds. Then, we provide moment estimates for $\sqrt{n}(\mu^n - \mathfrak{m})$.

In Section 3 we introduce the heat semigroup P_t and, in a quantitative way, the regularity properties of P_t needed for our scheme to work. We also provide estimates on the canonical regularization of the Hamilton-Jacobi equation provided by the Hopf-Cole transform $-\sigma(\log P_t e^{-f/\sigma})$. The most delicate part of our proof involves bounds on the probability of the events

$$\left\{ \sup_{x \in D} |u^{n,t}(x) - 1| > \eta \right\}, \qquad \eta > 0$$

which ensure that the probability of these events has a power like decay as $n \to \infty$ if $t = \gamma n^{-1} \log n$, with γ sufficiently large (this plays a role in the proof of the lower bound). Finally, in light of the ansatz of [CLPS14], we provide a formula for

$$\mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m} \right],$$

where $f^{n,t}$ solves the random PDE (1.6), and prove convergence of the renormalized quantity as $n \to \infty$, if $t \sim n^{-1} \log n$.

Section 4 provides the proof of our main result, together with Theorem 4.1 dealing with the simpler case $d = 1$. We first deal with the optimal matching to $\mathfrak{m}$, and then we deal with the bipartite case.

In Section 5 we recover the result found in [AKT84] as a consequence of our estimates via a Lipschitz approximation argument.

Finally, Section 6 covers extensions to more general classes of domains and open problems, pointing out some potential developments.

2 Notation and preliminary results

2.1 Wasserstein distance

Let (D, d) be a complete and separable metric space. We recall (see e.g. [AGS08]) that the quadratic Wasserstein distance $W_2(\mu, \nu)$ between Borel probability measures μ, ν in D with finite quadratic moments is defined by

$$W_2^2(\mu, \nu) = \min \left\{ \int_{D \times D} \mathsf{d}(x, y)^2 \, d\Sigma(x, y) : \Sigma \in \Gamma(\mu, \nu) \right\},$$

where $\Gamma(\mu, \nu)$ is the class of transport plans (couplings in Probability) between μ and ν, namely Borel probability measures Σ in $D \times D$ having μ and ν as first and second

marginals, respectively. We say that a Borel map T pushing μ to ν is optimal if

$$W_2^2(\mu, \nu) = \int_D \mathsf{d}^2(T(x), x) \, \mathrm{d}\mu(x).$$

This means that the plan $\Sigma = (\mathrm{Id} \times T)_{\#}\mu$ induced by T is optimal.

The following duality formula will play a key role, both in the proof of the upper and lower bound of the matching cost:

$$\frac{1}{2} W_2^2(\mu, \nu) = \sup_{\phi \in \mathrm{Lip}_b(D)} - \int_D \phi \, \mathrm{d}\mu + \int_D Q_1 \phi \, \mathrm{d}\nu. \tag{2.1}$$

In (2.1) above, $\mathrm{Lip}_b(D)$ stands for the class of bounded Lipschitz functions on D and, for $t > 0$, $Q_t \phi$ is provided by the Hopf-Lax formula

$$Q_t \phi(y) = \inf_{x \in D} \phi(x) + \frac{1}{2t} \mathsf{d}(x, y)^2. \tag{2.2}$$

This formula also provides a semigroup if (X, d) is a length space, and $Q_t \phi \uparrow \phi$ as $t \downarrow 0$.

We recall a few basic properties of Q_t, whose proof is elementary: if $\phi \in \mathrm{Lip}_b(D)$ then $\inf \phi \le Q_t \phi \le \sup \phi$ and (where Lip stands for the Lipschitz constant)

$$\mathrm{Lip}(Q_t \phi) \le 2 \, \mathrm{Lip}(\phi), \qquad \left\| \frac{\mathrm{d}}{\mathrm{d}t} Q_t \phi(x) \right\|_\infty \le 2 \big[\mathrm{Lip}(\phi) \big]^2 \quad \text{for all } x \in D.$$

In particular $\mathrm{Lip}_b(D)$ is invariant under the action of Q_t. For $\phi \in \mathrm{Lip}_b(D)$, the key property of $Q_t \phi$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_t \phi + \frac{1}{2} |\nabla Q_t \phi|^2 \le 0 \quad \mathfrak{m}\text{-a.e. in } X, \text{ for all } t > 0, \tag{2.3}$$

with equality if (D, d) is a length space (but we will only need the inequality). In (2.3), $|\nabla Q_t \phi|$ is the metric slope of $Q_t \phi$, which corresponds to the norm of the gradient in the Riemannian setting.

We recall that W_2^2 is jointly convex, namely if $\mu_i, \nu_i \in \mathscr{P}(D)$, $t_i \ge 0$, $\sum_{i=1}^k t_i = 1$,

$$\mu = \sum_{i=1}^k t_i \mu_i, \qquad \nu = \sum_{i=1}^k t_i \nu_i$$

then

$$W_2^2(\mu, \nu) \le \sum_{i=1}^k t_i W_2^2(\mu_i, \nu_i). \tag{2.4}$$

This easily follows by the linear dependence w.r.t. Σ in the cost function, and by the linearity of the marginal constraint. More generally, the same argument shows that, for a generic index set I,

$$W_2^2 \left(\int_I \mu_i \, \mathrm{d}\Theta(i), \int_I \nu_i \, \mathrm{d}\Theta(i) \right) \le \int_I W_2^2(\mu_i, \nu_i) \, \mathrm{d}\Theta(i) \tag{2.5}$$

with μ_i, ν_i and Θ probability measures, under appropriate measurability assumptions that are easily checked in all cases when we are going to apply this formula.

The following result is by now well known, we detail for the reader's convenience some steps of the proof from [AGS08] (see also [TU93] for more refined results).

Proposition 2.1 (Existence and stability of optimal maps)**.** *Let $D \subset \mathbb{R}^d$ be a compact set, $\mu, \nu \in \mathscr{P}(D)$ with μ absolutely continuous w.r.t. Lebesgue d-dimensional measure. Then:*

(a) *there exists a unique optimal transport map T_μ^ν from μ to ν.*

(b) *if $\nu_h \to \nu$ weakly in $\mathscr{P}(D)$, then $T_\mu^{\nu_h} \to T_\mu^\nu$ in $L^2(D, \mu; D)$.*

Proof. Statement *(a)* is a simple generalization of Brenier's theorem, see for instance [AGS08, Theorem 6.2.4] for a proof. The proof of statement *(b)* is typically obtained by combining the stability w.r.t. weak convergence of the optimal plans $\nu \mapsto (\mathrm{Id} \times T_\mu^\nu)$ (see [AGS08, Proposition 7.1.3]) with a general criterion (see [AGS08, Lemma 5.4.1]) which allows to deduce convergence in μ-measure of the maps T_h to T from the weak convergence of the plans $(\mathrm{Id} \times T_h)_{\#}\mu$ to $(\mathrm{Id} \times T)_{\#}\mu$. $\square$

2.2 Transport estimate

Assume in this section that D is a connected Riemannian manifold, possibly with boundary, whose finite Riemannian volume measure is denoted by $\mathfrak{m}$, with d equal to the Riemannian distance. The estimate from above on W_2^2 provided by Proposition 2.3 below is closely related to the Benamou-Brenier formula [BB00], [AGS08, Theorem 8.3.1], which provides a representation of W_2^2 in terms of the minimization of the action $\int_0^1 \int_D |\mathbf{b}_t|^2 \, d\mu_t \, dt$, among all solutions to the continuity equation $\frac{d}{dt}\mu_t + \mathrm{div}(\mathbf{b}_t \mu_t) = 0$. It is also related to the Dacorogna-Moser scheme, which provides constructively (under suitable smoothness assumptions) a transport map between $\mu_0 = u_0\mathfrak{m}$ and $\mu_1 = u_1\mathfrak{m}$ by solving the PDE

$$\begin{cases} \Delta f = u_1 - u_0 & \text{in } D, \\ \nabla f \cdot n_D = 0 & \text{on } \partial D \end{cases} \tag{2.6}$$

and then using the flow of the vector field $\mathbf{b}_t = u_t^{-1}\nabla f$ at time 1, with $u_t = (1-t)u_1 + tu_0$, to provide the map. We provide here the estimate without building explictly a coupling, in the spirit of [K10] (see also, in an abstract setting [AMS15, Theorem 6.6]), using the duality formula (2.1). This has the advantage to avoid smoothness issues and, moreover, uses (2.6) only in the weak sense, namely

$$-\int_D \langle \nabla\phi, \nabla f \rangle \, d\mathfrak{m} = \int_D \phi(u_1 - u_0) \, d\mathfrak{m} \qquad \forall \phi \in \mathrm{Lip}_b(D).$$

Notice that uniqueness of f in (2.6) is obvious, up to additive constants. Existence is guaranteed for $u_i \in L^2(\mathfrak{m})$ with $\int_D(u_1 - u_0) \, d\mathfrak{m} = 0$ under a spectral gap assumption, thanks to the variational interpretation provided by Lax-Milgram theorem. Notice also that with the choice $\mathbf{b}_t = u_t^{-1}\nabla f$ the continuity equation $\frac{d}{dt}u_t + \mathrm{div}(\mathbf{b}_t u_t) = 0$ holds, in weak form.

We will also need this definition.

Definition 2.2 (Logarithmic mean)**.** Given $a, b > 0$, we define the logarithmic mean

$$M(a,b) = \frac{a-b}{\log a - \log b} = \left(\int_0^1 \frac{1}{(1-s)a + sb} \, ds\right)^{-1}.$$

This can be extended to $a, b \geq 0$ by continuity, so that $M(a,0) = M(0,b) = M(0,0) = 0$.

Proposition 2.3. *Let $u_0, u_1 \in L^2(\mathfrak{m})$ be probability densities with $u_0 > 0$ $\mathfrak{m}$-a.e. in X and let $f \in H^{1,2}(D, \mathfrak{m})$ be any solution to* (2.6). *Then*

$$W_2^2(u_0 \mathfrak{m}, u_1 \mathfrak{m}) \leq \int_0^1 \int_D \frac{|\nabla f|^2}{(1-s)u_0 + su_1} \, \mathrm{d}\mathfrak{m} \, \mathrm{d}s = \int_D \frac{|\nabla f|^2}{M(u_0, u_1)} \, \mathrm{d}\mathfrak{m}.$$

Proof. Let $\phi \in \mathrm{Lip}_b(D)$, set $u_s = (1-s)u_0 + su_1$ and notice that $u_s > 0$ $\mathfrak{m}$-a.e. in X for all $s \in [0, 1)$. We interpolate, then use Leibniz's rule and (2.3) to get

$$\begin{aligned}
\int_D (u_1 Q_1 \phi - u_0 \phi) \, \mathrm{d}\mathfrak{m} &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \int_D u_s Q_s \phi \, \mathrm{d}\mathfrak{m} \, \mathrm{d}s \\
&= \int_0^1 \int_D u_s \frac{\mathrm{d}}{\mathrm{d}s} Q_s \phi + (u_1 - u_0) Q_s \phi \, \mathrm{d}\mathfrak{m} \, \mathrm{d}s \\
&\leq \int_0^1 \int_D -\frac{1}{2} |\nabla Q_s \phi|^2 u_s - \langle \nabla f, \nabla Q_s \phi \rangle \, \mathrm{d}\mathfrak{m} \, \mathrm{d}s \\
&\leq \frac{1}{2} \int_0^1 \int_D \frac{|\nabla f|^2}{u_s} \, \mathrm{d}\mathfrak{m} \, \mathrm{d}s.
\end{aligned}$$

Since ϕ is arbitrary, the statement follows from the duality formula (2.1). $\square$

2.3 Bounds for moments and tails

In this subsection (D, d) is a complete and separable metric space equipped with a Borel probability measure $\mathfrak{m}$. We assume $\mathrm{diam}\, D < \infty$.

For $n \in \mathbb{N}_+$, let $X_1, \ldots, X_n$ be independent and uniformly distributed random variables in D, whose common law is $\mathfrak{m}$. Let $\mu^n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$ be the random empirical measure. We define the measures $r^n = \sqrt{n}(\mu^n - \mathfrak{m})$, where we use the natural scaling provided by the central limit theorem. Our goal is to derive upper bounds for the exponential moments $\exp(\lambda \int_D f \, \mathrm{d}r^n)$ and, as a consequence, tail estimates for $\int_D f \, \mathrm{d}r^n$, related to classical concentration inequalities (Bernstein inequality), see e.g. [T14, Lemma 4.3.4]. For the reader's convenience, we provide a complete proof in the form that we need for our purposes.

Definition 2.4. For $k \in \mathbb{N}$ and $f \in C_b(D)$, define the k-moments $\llbracket \cdot \rrbracket_k$ by

$$\llbracket f \rrbracket_k^k = \int_D \left(f(x) - \int_D f \, \mathrm{d}\mathfrak{m} \right)^k \mathrm{d}\mathfrak{m}(x)$$

and

$$\llbracket f \rrbracket_\infty = \left\| f - \int_D f \, \mathrm{d}\mathfrak{m} \right\|_{L^\infty(\mathfrak{m})}.$$

Notice that $\llbracket f \rrbracket_0 = 1$, $\llbracket f \rrbracket_1 = 0$, $\llbracket f \rrbracket_2 \leq \|f\|_2$ and $\left| \llbracket f \rrbracket_{k+2}^{k+2} \right| \leq \llbracket f \rrbracket_2^2 \llbracket f \rrbracket_\infty^k$. Moreover, $\llbracket \cdot \rrbracket_2^2$ is a quadratic form, therefore we introduce also the associated bilinear form

$$m_2(f, g) = \int_D \left(f(x) - \int_D f \, \mathrm{d}\mathfrak{m} \right) \left(g(x) - \int_D g \, \mathrm{d}\mathfrak{m} \right) \mathrm{d}\mathfrak{m}(x).$$

Analogously, we consider also the following quantity

$$m_4(f, g) = \int_D \left(f(x) - \int_D f \, \mathrm{d}\mathfrak{m} \right)^2 \left(g(x) - \int_D g \, \mathrm{d}\mathfrak{m} \right)^2 \mathrm{d}\mathfrak{m}(x),$$

so that $m_4(f, f) = \llbracket f \rrbracket_4^4$.

Lemma 2.5 (Moment generating function). *Let $f \in C_b(D)$ and $\lambda \in \mathbb{R}$. Then*

$$\mathbb{E}\left[\exp\left(\lambda \int_D f \, dr^n\right)\right] = \left\{\int_D \exp\left[\frac{\lambda}{\sqrt{n}}\left(f(x) - \int_D f \, d\mathfrak{m}\right)\right] d\mathfrak{m}(x)\right\}^n$$

$$= \left(1 + \sum_{k=2}^{\infty} \frac{\lambda^k [\![f]\!]_k^k}{k! n^{k/2}}\right)^n.$$

As a consequence

$$\mathbb{E}\left[\exp\left(\lambda \int_D f \, dr^n\right)\right] \leq \exp\left[\frac{\lambda^2 [\![f]\!]_2^2}{2} \exp\left(\frac{|\lambda| [\![f]\!]_\infty}{\sqrt{n}}\right)\right]. \tag{2.7}$$

Proof. It is sufficient to show the result for $\lambda = 1$. The general statement then follows by taking λf in place of f. By the definition of empirical measure we have

$$\mathbb{E}\left[\exp\left(\int_D f \, dr^n\right)\right] = \mathbb{E}\left[\exp\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n f(X_i) - \sqrt{n}\int_D f \, d\mathfrak{m}\right)\right]$$

$$= \mathbb{E}\left[\exp\left(\sum_{i=1}^n \frac{1}{\sqrt{n}}\left\{f(X_i) - \int_D f \, d\mathfrak{m}\right\}\right)\right]$$

$$= \mathbb{E}\left[\exp\left(\frac{1}{\sqrt{n}}\left\{f(X_1) - \int_D f \, d\mathfrak{m}\right\}\right)\right]^n$$

$$= \left\{\int_D \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{1}{\sqrt{n}}\left(f(x) - \int_D f \, d\mathfrak{m}\right)\right]^k d\mathfrak{m}(x)\right\}^n.$$

The equality above gives

$$\mathbb{E}\left[\exp\left(\int_D f \, dr^n\right)\right] = \left\{1 + \sum_{k=2}^{\infty} \frac{[\![f]\!]_k^k}{k! n^{k/2}}\right\}^n = \left\{1 + \sum_{k=0}^{\infty} \frac{[\![f]\!]_{k+2}^{k+2}}{(k+2)! n^{k/2+1}}\right\}^n$$

$$\leq \left\{1 + \frac{[\![f]\!]_2^2}{2n}\sum_{k=0}^{\infty} \frac{[\![f]\!]_\infty^k}{k! n^{k/2}}\right\}^n = \left\{1 + \frac{[\![f]\!]_2^2}{2n}\exp\left(\frac{[\![f]\!]_\infty}{\sqrt{n}}\right)\right\}^n$$

$$\leq \exp\left[\frac{[\![f]\!]_2^2}{2}\exp\left(\frac{[\![f]\!]_\infty}{\sqrt{n}}\right)\right]. \qquad \square$$

Lemma 2.6. *Let $f, g \in C_b(D)$. Then*

$$\mathbb{E}\left[\left(\int_D f \, dr^n\right)^2\right] = [\![f]\!]_2^2, \qquad \mathbb{E}\left[\left(\int_D f \, dr^n\right)\left(\int_D g \, dr^n\right)\right] = m_2(f, g), \tag{2.8}$$

and

$$\mathbb{E}\left[\left(\int_D f \, dr^n\right)^2 \left(\int_D g \, dr^n\right)^2\right] = \frac{n-1}{n}\left[[\![f]\!]_2^2 [\![g]\!]_2^2 + 2m_2(f, g)^2\right] + \frac{1}{n}m_4(f, g)$$

$$\leq 3\frac{n-1}{n}[\![f]\!]_2^2 [\![g]\!]_2^2 + \frac{1}{n}[\![f]\!]_4^2 [\![g]\!]_4^2.$$

Proof. Since

$$\mathbb{E}\left[\exp\left(\lambda\int_D f\,\mathrm{d}r^n\right)\right] = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mathbb{E}\left[\left(\int_D f\,\mathrm{d}r^n\right)^k\right],$$

it is sufficient to compute the second and fourth derivatives with respect to λ at $\lambda = 0$ in the expression for $\mathbb{E}\left[\exp\left(\lambda\int_D f\,\mathrm{d}r^n\right)\right]$ provided by Lemma 2.5 to obtain, respectively, the first identity in (2.8) and

$$\mathbb{E}\left[\left(\int_D f\,\mathrm{d}r^n\right)^4\right] = 3\frac{n-1}{n}[\![f]\!]_2^4 + \frac{1}{n}[\![f]\!]_4^4.$$

The remaining two identities follow by polarization. $\qquad\square$

For $c,\eta > 0$, define the function

$$F(c,\eta) = \sup_{\lambda>0}\left\{\lambda\eta - \frac{\lambda^2}{2}\exp(c\lambda)\right\} > 0. \tag{2.9}$$

Notice that $F(c,\eta)$ is decreasing in c, increasing in η and that the formula

$$cF(c,\eta) = \sup_{\lambda>0}\left\{c\lambda\eta - c\frac{\lambda^2}{2}\exp(c\lambda)\right\} = \sup_{\lambda'>0}\left\{\lambda'\eta - \frac{(\lambda')^2}{2c}\exp(\lambda')\right\}$$

shows that $cF(c,\eta)$ is increasing in c. We will use the function F to estimate the tails of $\int_D f\,\mathrm{d}r^n$.

Lemma 2.7 (Tail bound)**.** *Let X be a real random variable such that, for some $c_1, c_2 > 0$,*

$$\mathbb{E}[\exp(\lambda X)] \leq \exp\left[\frac{\lambda^2 c_1}{2}\exp(|\lambda|c_2)\right] \qquad \forall\lambda \in \mathbb{R}.$$

Then for every $\eta \geq 0$ we have

$$\mathbb{P}(|X| > \eta) \leq 2\exp\left[-\frac{1}{c_1}F\left(\frac{c_2}{c_1},\eta\right)\right].$$

Proof. We have $\mathbb{P}(|X| > \eta) \leq \mathbb{P}(X > \eta) + \mathbb{P}(X < -\eta)$. For the first term and $\lambda > 0$

$$\mathbb{P}(X > \eta) = \mathbb{P}\big(\exp(\lambda X) > \exp(\lambda\eta)\big)$$

$$\leq \mathbb{E}[\exp(\lambda X)]\exp(-\lambda\eta) \leq \exp\left[\frac{\lambda^2 c_1}{2}\exp(\lambda c_2) - \lambda\eta\right].$$

Hence

$$\mathbb{P}(X > \eta) \leq \exp\left[\inf_{\lambda>0}\left(\frac{\lambda^2 c_1}{2}\exp(\lambda c_2) - \lambda\eta\right)\right] = \exp\left[\frac{1}{c_1}\inf_{\lambda>0}\left\{\frac{\lambda^2}{2}\exp\left(\lambda\frac{c_2}{c_1}\right) - \lambda\eta\right\}\right].$$

For the other term, we use the fact that $\mathbb{P}(X < -\eta) = \mathbb{P}(-X > \eta)$ and $-X$ satisfies the same hypothesis. $\qquad\square$

3 Heat semigroup

In this section we add more structure to D, assuming that (D, d) is a connected Riemannian manifold (possibly with boundary) endowed with the Riemannian distance, and that D has finite diameter and volume. Then, we can and will normalize (D, d) in such a way that the volume is unitary, and let $\mathfrak{m}$ be the volume measure of (D, d). The typical examples we have in mind are the flat d-dimensional torus $\mathbb{T}^d$ and the d-dimensional cube $[0, 1]^d$, see also Section 6 for more general setups.

We denote by P_t the heat semigroup associated to $(D, \mathsf{d}, \mathfrak{m})$, with Neumann boundary conditions. In one of the many equivalent representations, it can be viewed as the $L^2(\mathfrak{m})$ gradient flow of the Dirichlet energy $\frac{1}{2} \int_D |\nabla f|^2 \, \mathrm{d}\mathfrak{m}$. Standard results (see for instance [W14]) ensure that P_t is a Markov semigroup, so that it is a contraction semigroup in all $L^p \cap L^2(\mathfrak{m})$ spaces, $1 \le p \le \infty$; thanks to this property it has a unique extension to all $L^p(\mathfrak{m})$ spaces even when $p \in [1, 2)$. Moreover, the finiteness of volume and boundary conditions ensure that P_t is mass-preserving, i.e. $t \mapsto \int_D P_t f \, \mathrm{d}\mathfrak{m}$ is constant in $[0, \infty)$ for all $f \in L^1(\mathfrak{m})$ and thus it can be viewed as an operator in the class of probability densities (which correspond to the measures absolutely continuous w.r.t. $\mathfrak{m}$). More generally, we can use the Feller property (i.e. that P_t maps $C_b(D)$ into $C_b(D)$) to define the adjoint semigroup P_t^* on the class $\mathcal{M}$ of Borel measures in D with finite total variation by

$$\int_D f \, \mathrm{d}P_t^* \mu = \int_D P_t f \, \mathrm{d}\mu$$

and to regularize with the aid of P_t^* singular measures to absolutely continuous measures, under appropriate additional assumptions on P_t. Since P_t is selfadjoint, the operator P_t^* can also be viewed as the extension of P_t from $L^1(\mathfrak{m})$ to $\mathcal{M}$.

We denote by $p_t(x, y)$ the transition probabilities of the semigroup, characterized by the formula

$$P_t f(x) = \int_D p_t(x, y) f(y) \, \mathrm{d}\mathfrak{m}(y),$$

so that

$$P_t^* \delta_{x_0} = p_t(x_0, \, \cdot \,)\mathfrak{m} \qquad \text{for all } x_0 \in D, \, t > 0.$$

We denote by Δ the infinitesimal generator of P_t, namely the extension of the Laplace-Beltrami operator on D. Besides the "qualitative" properties of P_t mentioned above, our proof depends on several quantitative estimates related to P_t.

Quantitative estimates on P_t. We assume throughout the validity of the following properties: there are positive constants d, C_{sg}, C_{uc}, C_{ge}, C_{rt}, C_{dr} and K such that

(SG) spectral gap: $\|P_t f\|_2 \le e^{-C_{\mathrm{sg}} t} \|f\|_2$ for any f with $\int_D f \, \mathrm{d}\mathfrak{m} = 0$,

(UC) ultracontractivity: $|p_t(x, y) - 1| \le C_{\mathrm{uc}} t^{-d/2}$ for all $x, y \in D$,

(GE) gradient estimate: $\mathrm{Lip}(p_t(x, \, \cdot \,)) \le C_{\mathrm{ge}} t^{-(d+1)/2}$,

(RT) Riesz transform bound:

$$\int_D |\nabla f|^4 \, \mathrm{d}\mathfrak{m} \le C_{\mathrm{rt}} \int_D \left| (-\Delta)^{1/2} f \right|^4 \mathrm{d}\mathfrak{m},$$

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(DR) dispersion rate: $\int_D \mathsf{d}^2(x,y) p_t(x,y) \, \mathrm{d}\mathfrak{m}(y) \leq C_{\mathrm{dr}} t$,

(GC) gradient contractivity: $|\nabla P_t f| \leq e^{Kt} P_t |\nabla f|$.

In the sequel, since many parameters and constants will be involved, in some statements we call a constant *geometric* if it depends only on D through C_{sg}, C_{uc}, C_{ge}, C_{rt}, C_{dr} and K. The validity of these properties on a wide class of compact Riemannian manifolds is known, we refer e.g. to the monograph [W14], chapter 1, for a detailed discussion of the case where D has no boundary, and chapter 2 for that with (convex) boundary. Here we notice that (SG) is equivalent to a Poincaré inequality, (GC) encodes a lower bound on Ricci curvature, (RT) holds if the Ricci curvature is bounded from below [B87], while (UC) is related to Sobolev inequalities and (DR) follows from an upper bound on the Laplacian of the distance squared from any fixed point. Property (GE) is in fact a consequence of the others, see (3.4). Let us draw now some easy consequences of these assumptions.

Spectral gap implies that for $f \in L^2(D, \mathfrak{m})$ with $\Delta f \in L^2(D, \mathfrak{m})$ we have the representation

$$f(x) = \int_D f \, \mathrm{d}\mathfrak{m} + \int_0^\infty (P_t \Delta f)(x) \, \mathrm{d}t. \tag{3.1}$$

Ultracontractivity entails that $P_t : L^1 \to L^\infty$ continuously for $t > 0$, because

$$|P_t f(x)| \leq \left| \int_D p_t(x,y) f(y) \, \mathrm{d}\mathfrak{m}(y) - \int_D f(y) \, \mathrm{d}\mathfrak{m}(y) \right| + \|f\|_1 \leq (C_{\mathrm{uc}} t^{-d/2} + 1) \|f\|_1.$$

Hence, by interpolation $P_t : L^p \to L^q$ for any $1 \leq p \leq q \leq \infty$ with norms bounded from above by geometric constants. If $p = 1$, by approximation we also get $P_t^* : \mathcal{M} \to L^q$ continuously for $1 \leq q \leq \infty$. Notice also that

$$[\![p_t(\,\cdot\,,y)]\!]_2^2 \leq [\![p_t(\,\cdot\,,y)]\!]_\infty \leq C_{\mathrm{uc}} t^{-d/2} \tag{3.2}$$

because

$$[\![p_t(\,\cdot\,,y)]\!]_2^2 = \int_D \big(p_t(x,y) - 1 \big)^2 \, \mathrm{d}\mathfrak{m}(x) = \int_D p_t(x,y) \big(p_t(x,y) - 1 \big) \, \mathrm{d}\mathfrak{m}(x)$$

$$\leq [\![p_t(\,\cdot\,,y)]\!]_\infty \int_D p_t(x,y) \, \mathrm{d}\mathfrak{m}(x) = [\![p_t(\,\cdot\,,y)]\!]_\infty \leq C_{\mathrm{uc}} t^{-d/2}.$$

Writing $\mu = \int_D \delta_x \, \mathrm{d}\mu(x)$ and (DR) in the form $W_2^2(P_t^* \delta_x, \delta_x) \leq C_{\mathrm{dr}} t$, from the joint convexity of W_2^2 (2.5) we obtain

$$W_2^2(P_t^* \mu, \mu) \leq C_{\mathrm{dr}} t.$$

By duality, see [K10], the gradient contractivity property leads to contractivity w.r.t. W_2 distance

$$W_2^2(P_t^* \mu, P_t^* \nu) \leq e^{2Kt} W_2^2(\mu, \nu).$$

Moreover, it implies that for some geometric constant C we have

$$\|\nabla f\|_\infty \leq C \|\Delta f\|_\infty \tag{3.3}$$

for every $f \in L^2(D, \mathfrak{m})$ with $\Delta f \in L^\infty(D, \mathfrak{m})$. In fact, from Bakry's work (see [BGL14] and [W11]), it is known that (GC) gives the reverse Poincaré inequality

$$|\nabla P_t g|^2 \leq \frac{K}{e^{2Kt} - 1} \left[P_t(g^2) - (P_t g)^2 \right] \leq \frac{K}{e^{2Kt} - 1} \|g\|_\infty^2,$$

hence the bound $\|\nabla P_t g\|_\infty \leq c t^{-1/2} \|g\|_\infty$ for $t \in (0, 1]$ and some geometric constant c. Using the representation formula (3.1) and the previous estimate with $g = \Delta f$ and $g = P_{t-1} \Delta f$ we obtain (3.3) as

$$\|\nabla f\|_\infty \leq \int_0^\infty \|\nabla P_t \Delta f\|_\infty \, \mathrm{d}t = \int_0^2 \|\nabla P_t \Delta f\|_\infty \, \mathrm{d}t + \int_2^\infty \|\nabla P_1(P_{t-1} \Delta f)\|_\infty \, \mathrm{d}t$$

$$\leq c \left(\|\Delta f\|_\infty \int_0^2 t^{-1/2} \, \mathrm{d}t + \|P_1\|_{L^2 \to L^\infty} \int_2^\infty \|P_{t-2} \Delta f\|_2 \, \mathrm{d}t \right)$$

$$\leq c \left(2\sqrt{2} + \|P_1\|_{L^2 \to L^\infty} \int_2^\infty e^{-C_{\mathrm{sg}}(t-2)} \, \mathrm{d}t \right) \|\Delta f\|_\infty.$$

Using the same bound, we also have, for $x \in D$, $t \in (0, 1]$,

$$\mathrm{Lip}(p_t(x, \cdot)) = \|\nabla P_t^* \delta_x\|_\infty \leq c(t/2)^{-1/2} \|P_{t/2}^*(\delta_x - \mathfrak{m})\|_\infty \leq c\, C_{\mathrm{uc}}(t/2)^{-(d+1)/2}, \quad (3.4)$$

which actually shows that (GE) above is a consequence of (SG), (UC) and (GC) (for $t > 1$, one uses (SG) to obtain even exponential decay).

In the following lemma we collect some further consequences of the gradient contractivity.

Lemma 3.1. *For every $s \geq 0$ and $g \in C_b(D)$ one has*

$$\min g \leq -\log \left(P_s e^{-g} \right) \leq \max g, \quad \text{hence} \quad \left\| -\log \left(P_s e^{-g} \right) \right\|_\infty \leq \|g\|_\infty, \quad (3.5)$$

$$\left\| \nabla \log \left(P_s e^{-g} \right) \right\|_\infty \leq e^{K^- s} \|\nabla g\|_\infty. \quad (3.6)$$

Proof. Write $G = e^{-g}$. Inequality (3.5) follows from the fact that P_s is Markov and the inequalities $e^{-\max g} \leq G \leq e^{-\min g}$. In order to prove (3.6) we use (GC) to get

$$\left| \nabla \log \left(P_s e^{-g} \right) \right| = \frac{|\nabla P_s e^{-g}|}{P_s e^{-g}} \leq e^{K^- s} \frac{P_s \left(|\nabla g| e^{-g} \right)}{P_s e^{-g}} \leq e^{K^- s} \|\nabla g\|_\infty. \qquad \square$$

Lemma 3.2 (Viscous Hamilton-Jacobi)**.** *Assume that D is a compact Riemannian manifold without boundary. Let $\sigma > 0$, $f \in C(D)$, and define, for $t \geq 0$,*

$$\phi_t^\sigma = -\sigma \log \left(P_{(\sigma t)/2} e^{-f/\sigma} \right).$$

Then $\phi_t^\sigma \in C\left([0, +\infty) \times D\right) \cap C^\infty\left((0, +\infty) \times D\right)$ solves

$$\begin{cases} \partial_t \phi_t^\sigma = -\dfrac{|\nabla \phi_t^\sigma|^2}{2} + \dfrac{\sigma}{2} \Delta \phi_t^\sigma & \text{in } (0, +\infty) \times D, \\ \phi_0^\sigma = f & \text{in } D. \end{cases} \quad (3.7)$$

Moreover

$$\min f \le \phi_t^\sigma \le \max f, \qquad \|\phi_t^\sigma\|_\infty \le \|f\|_\infty, \tag{3.8}$$

$$\|\nabla \phi_t^\sigma\|_\infty \le e^{K^- t}\|\nabla f\|_\infty, \tag{3.9}$$

$$\|(\Delta \phi_t^\sigma)^+\|_\infty \le \|(\Delta f)^+\|_\infty + \frac{e^{2K^- t} - 1}{2}\|\nabla f\|_\infty^2, \tag{3.10}$$

$$\phi_1^\sigma(y) - \phi_0^\sigma(x) \le \frac{\mathsf{d}(x,y)^2}{2} + \frac{\sigma}{2}\|(\Delta f)^+\|_\infty + \frac{\sigma}{4}(e^{2K^-} - 1)\|\nabla f\|_\infty^2 \tag{3.11}$$

$$\int_D (\phi_0^\sigma - \phi_1^\sigma)\,\mathrm{d}\mathfrak{m} \le e^{\|(\Delta f)^+\|_\infty + \frac{1}{2}(e^{2K^-} - 1)\|\nabla f\|_\infty^2} \int_D \frac{|\nabla f|^2}{2}\,\mathrm{d}\mathfrak{m}. \tag{3.12}$$

Proof. The smoothness of ϕ_t^σ for positive times follows by the chain rule and standard (linear) parabolic theory. To check that ϕ^σ solves (3.7), it is sufficient to compare

$$\partial_t \phi_t^\sigma = -\frac{\sigma^2}{2}\frac{\Delta P_{(\sigma t)/2}e^{-f/\sigma}}{P_{(\sigma t)/2}e^{-f/\sigma}}$$

with the terms arising from the application of the diffusion chain rule

$$\begin{aligned}
\frac{\sigma}{2}\Delta \phi_t^\sigma &= -\frac{\sigma^2}{2}\Delta \log\left(P_{(\sigma t)/2}e^{-f/\sigma}\right) \\
&= -\frac{\sigma^2}{2}\frac{\Delta P_{(\sigma t)/2}e^{-f/\sigma}}{P_{(\sigma t)/2}e^{-f/\sigma}} + \frac{\sigma^2}{2}\left|\frac{\nabla P_{(\sigma t)/2}e^{-f/\sigma}}{P_{(\sigma t)/2}e^{-f/\sigma}}\right|^2 \\
&= \partial_t \phi_t^\sigma + \frac{1}{2}|\nabla \phi_t^\sigma|^2.
\end{aligned} \tag{3.13}$$

Inequalities (3.8) and (3.9) follow in a straightforward way, respectively from (3.5) and (3.6) of Lemma 3.1, with $s = (\sigma t)/2$ and $g = f/\sigma$.

To prove (3.10) we use Bochner's inequality

$$\Delta \frac{|\nabla \phi_t^\sigma|^2}{2} \ge K|\nabla \phi_t^\sigma|^2 + \langle \nabla \phi_t^\sigma, \nabla \Delta \phi_t^\sigma\rangle,$$

which encodes the bound from below on Ricci curvature, and, setting $\xi_t = \Delta \phi_t^\sigma$, we get

$$\partial_t \xi_t \le -K|\nabla \phi_t^\sigma|^2 - \langle \nabla \phi_t^\sigma, \nabla \xi_t\rangle + \frac{\sigma}{2}\Delta \xi_t \le K^- e^{2K^- t}\|\nabla f\|_\infty^2 - \langle \nabla \phi_t^\sigma, \nabla \xi_t\rangle + \frac{\sigma}{2}\Delta \xi_t$$

which, by the maximum principle (here we use that D has no boundary), leads to (3.10).

To prove (3.11), let $\gamma \in C^1([0,1], D)$, with $\gamma(0) = x$, $\gamma(1) = y$, and compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^\sigma(\gamma(t)) &= (\partial_t \phi_t^\sigma)(\gamma(t)) + \langle(\nabla \phi_t^\sigma)(\gamma(t)), \dot\gamma(t)\rangle \\
&= -\frac{1}{2}|\nabla \phi_t^\sigma(\gamma(t))|^2 + \frac{\sigma}{2}\Delta \phi_t^\sigma(\gamma(t)) + \langle(\nabla \phi_t^\sigma)(\gamma(t)), \dot\gamma(t)\rangle \\
&\le \frac{1}{2}|\dot\gamma(t)|^2 + \frac{\sigma}{2}\|(\Delta \phi_t^\sigma)^+\|_\infty.
\end{aligned} \tag{3.14}$$

Integrating over $t \in (0, 1)$ and using (3.10), we obtain

$$\phi_1^\sigma(y) - \phi_0^\sigma(x) \leq \frac{1}{2} \int_0^1 |\dot{\gamma}(t)|^2 \, \mathrm{d}t + \frac{\sigma}{2} \|(\Delta f)^+\|_\infty + \frac{\sigma}{4}(e^{2K^-} - 1)\|\nabla f\|_\infty^2,$$

which yields (3.11) after we take the infimum with respect to γ.

To show (3.12), we notice first that

$$\begin{aligned}
\int_D (\phi_0^\sigma - \phi_1^\sigma) \, \mathrm{d}\mathfrak{m} &= -\int_D \int_0^1 \partial_t \phi_t^\sigma \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} \\
&= \frac{1}{2} \int_0^1 \int_D |\nabla \phi_t^\sigma|^2 \, \mathrm{d}\mathfrak{m} - \frac{\sigma}{2} \int_0^1 \int_D \Delta \phi_t^\sigma \, \mathrm{d}\mathfrak{m} \qquad (3.15) \\
&= \frac{1}{2} \int_0^1 \int_D |\nabla \phi_t^\sigma|^2 \, \mathrm{d}\mathfrak{m}.
\end{aligned}$$

where the second term vanishes because D is without boundary. For $t \in (0, 1)$, one has

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_D |\nabla \phi_t^\sigma|^2 \, \mathrm{d}\mathfrak{m} &= -\frac{\mathrm{d}}{\mathrm{d}t} \int_D (\Delta \phi_t^\sigma) \, \phi_t^\sigma \, \mathrm{d}\mathfrak{m} = -2 \int_D (\Delta \phi_t^\sigma) \, \partial_t \phi_t^\sigma \, \mathrm{d}\mathfrak{m} \\
&= \int_D (\Delta \phi_t^\sigma) |\nabla \phi_t^\sigma|^2 - \sigma \int_D (\Delta \phi_t^\sigma)^2 \, \mathrm{d}\mathfrak{m} \qquad (3.16) \\
&\leq \|(\Delta \phi_t^\sigma)^+\|_\infty \int_D |\nabla \phi_t^\sigma|^2 \, \mathrm{d}\mathfrak{m}.
\end{aligned}$$

Combining (3.15) and (3.16) and taking into account the estimate (3.10) on $\Delta \phi_t^\sigma$, inequality (3.12) follows by Gronwall's inequality. $\qquad \square$

Corollary 3.3 (Dual potential). *Assume that D is a compact Riemannian manifold without boundary. For every Lipschitz function f with $\|(\Delta f)^-\|_\infty < \infty$, there exists $g \in C_b(D)$ such that*

$$f(x) + g(y) \leq \frac{\mathrm{d}(x, y)^2}{2}, \quad \int_D (f + g) \, \mathrm{d}\mathfrak{m} \geq -e^{\|(\Delta f)^-\|_\infty + \frac{1}{2}(e^{2K^-}-1)\|\nabla f\|_\infty^2} \int_D \frac{|\nabla f|^2}{2} \, \mathrm{d}\mathfrak{m}.$$

Proof. For $\sigma > 0$, consider the functions $g^\sigma = \phi_1^\sigma$ solving the initial value problem (3.7) with f replaced by $-f$. Inequalities (3.5) and (3.6) entail that g^σ are uniformly bounded in the space of Lipschitz functions: as $\sigma \to 0$, we can extract a subsequence (g^{σ_h}) pointwise converging to some bounded Lipschitz function g. Inequality (3.11) gives in the limit the first inequality of the thesis, while (3.12) yields the second one, by dominated convergence. $\qquad \square$

Remark 3.4 (On the equality $g = Q_1(-f)$). Recall that the theory of viscosity solutions [CL83], [BC97] is specifically designed to deal with equations, as the Hamilton-Jacobi equations, for which the distributional point of view fails. This theory can be carried out also on manifolds, see [F] for a nice presentation of this subject. Since one can prove (using also apriori estimates on the time derivatives, arguing as in Corollary 3.3) the existence of a function ϕ_t, uniform limit of a subsequence of ϕ_t^σ, since classical solutions are viscosity solutions and since locally uniform limits of viscosity solutions are viscosity solutions, the function ϕ_t is a viscosity solution to the HJ equation $\partial_t u + \frac{1}{2}|\nabla u|^2 = 0$. Then, if the initial condition is $-f$, the uniqueness theory of first order

viscosity solutions applies, and gives that ϕ_t is precisely given by $Q_t(-f)$, as defined in (2.2). Setting $t = 1$, this argument proves that actually the function g of Corollary 3.3 coincides with $Q_1(-f)$, and that there is full convergence as $\sigma \to 0$ (see also [C03] for a proof of the convergence, in Euclidean spaces, based on the theory of large deviations). We preferred a more elementary and self-contained presentation, because the weaker statement $g \leq Q_1(-f)$ provided by the Corollary is sufficient for our purposes, and because our argument works also in the more abstract setting described in Section 6 (in which neither large deviations nor theory of viscosity solutions are yet available), emphasizing the role played by the lower Ricci curvature bounds.

3.1   Density fluctuation bounds

Recalling the notation $\mu^n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$, $r^n = \sqrt{n}(\mu^n - \mathfrak{m})$, we now define our regularized empirical measures.

Definition 3.5 (Regularized empirical measures). For $t \geq 0$ define

$$\mu^{n,t} = P_t^* \mu^n, \qquad r^{n,t}\mathfrak{m} = P_t^* r^n = \sqrt{n}(\mu^{n,t} - \mathfrak{m}),$$

so that for $t > 0$ one has

$$r^{n,t}(y) = \int_D (p_t(\cdot, y) - 1) \, \mathrm{d}r^n.$$

The goal of this subsection is to collect apriori estimates on the deviation of $r^{n,t}$ from 0.

Lemma 3.6 (Pointwise bound). *For $y \in D$ and $\eta > 0$ one has*

$$\mathbb{P}\left(\frac{|r^{n,t}(y)|}{\sqrt{n}} > \eta\right) \leq 2 \exp\left(-\frac{nt^{d/2}}{2C_{\mathrm{uc}}} F(1, \eta)\right), \tag{3.17}$$

where F is defined in (2.9).

Proof. Consider the random variable $X = r^{n,t}(y)/\sqrt{n} = \int_D p_t(\cdot, y)/\sqrt{n} \, \mathrm{d}r^n$. By (2.7) with $f = p_t(\cdot, y)/\sqrt{n}$ we have

$$\mathbb{E}[\exp(\lambda X)] \leq \exp\left[\frac{\lambda^2 [\![p_t(\cdot, y)]\!]_2^2}{2n} \exp\left(|\lambda| \frac{[\![p_t(\cdot, y)]\!]_\infty}{n}\right)\right]$$

$$\leq \exp\left[\frac{\lambda^2}{2} \cdot \frac{C_{\mathrm{uc}}}{nt^{d/2}} \exp\left(|\lambda| \frac{C_{\mathrm{uc}}}{nt^{d/2}}\right)\right],$$

where in the second inequality we used (3.2). Then Lemma 2.7 with $c_1 = c_2 = C_{\mathrm{uc}}/(nt^{d/2})$ implies (3.17). $\square$

Lemma 3.7 (Deterministic bound). *With probability 1 one has*

$$\frac{|r^{n,t}(y) - r^{n,t}(z)|}{\sqrt{n}} \leq \frac{2C_{\mathrm{ge}}}{t^{(d+1)/2}} \mathsf{d}(y, z).$$

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Proof. Using (GE) and the fact that the total variation of the measures r^n is $2\sqrt{n}$, we get

$$\frac{|r^{n,t}(y) - r^{n,t}(z)|}{\sqrt{n}} = \left| \int_D (p_t(x,y) - p_t(x,z)) \frac{\mathrm{d} r^n(x)}{\sqrt{n}} \right|$$

$$\leq \mathsf{d}(y,z) \int_D \mathrm{Lip}(p_t(x,\cdot)) \frac{\mathrm{d}|r^n|(x)}{\sqrt{n}} \leq \frac{2C_{\mathrm{ge}}}{t^{(d+1)/2}} \mathsf{d}(y,z). \qquad \square$$

We shall need another geometric function related to D.

Definition 3.8 (Minimal δ-cover). In the sequel, for $\delta > 0$ we denote by $N_D(\delta)$ be smallest cardinality of a δ-net of D, namely a set whose closed δ-neighbourhood contains D.

Proposition 3.9 (Uniform bound, $d = 1$). *Assume that ultracontractivity holds with $d = 1$ and that $N_D(\delta) \leq \max\{1, C_D \delta^{-1}\}$ for all $\delta > 0$. Then there exists a constant $C = C(C_{\mathrm{ge}}, C_D)$ with the following property: for all $\eta \in (0,1)$, $q \in (0,1)$ and $\eta^{-1} n^{-2q} \leq t \leq 4 C_{\mathrm{ge}} C_D$, we have*

$$\mathbb{P}\left(\sup_{y \in D} \frac{|r^{n,t}(y)|}{\sqrt{n}} > \eta \right) \leq C \exp\left(-\gamma n^{1-q} \right)$$

with $\gamma = \gamma(\eta, C_{\mathrm{uc}})$ and $n \geq n(\eta, q, C_{\mathrm{uc}})$.

Proof. We pick $\delta = \frac{\eta}{4 C_{\mathrm{ge}}} t$, so that, by Lemma 3.7, with probability 1 we have

$$\frac{|r^{n,t}(y) - r^{n,t}(z)|}{\sqrt{n}} \leq \frac{\eta}{2} \qquad \text{for any } y, z \in D \text{ with } \mathsf{d}(y,z) \leq \delta. \qquad (3.18)$$

Let T be a minimal δ-net. Then the condition $t \leq 4 C_{\mathrm{ge}} C_D$ implies $C_D \delta^{-1} \geq 1$, hence

$$|T| \leq C_D \delta^{-1} = \frac{4 C_{\mathrm{ge}} C_D}{\eta} t^{-1} \leq 4 C_{\mathrm{ge}} C_D n^{2q}.$$

From an application of Lemma 3.6 with $\eta/2$ instead of η we get

$$\mathbb{P}\left(\sup_{y \in T} \frac{|r^{n,t}(y)|}{\sqrt{n}} > \frac{\eta}{2} \right) \leq 2|T| \exp\left(-\frac{n t^{1/2}}{2 C_{\mathrm{uc}}} F(1, \eta/2) \right)$$

$$\leq 8 C_{\mathrm{ge}} C_D \exp\left(2q \log n - \frac{n^{1-q}}{2\eta^{1/2} C_{\mathrm{uc}}} F(1, \eta/2) \right)$$

$$\leq 8 C_{\mathrm{ge}} C_D \exp\left(-\gamma n^{1-q} \right),$$

where the last inequality holds with $\gamma = F(1, \eta/2)/(4\eta^{1/2} C_{\mathrm{uc}})$ and $n \geq n(\eta, q, C_{\mathrm{uc}})$, absorbing the logarithm $\log n$ into the power n^{1-q}. We conclude since

$$\mathbb{P}\left(\sup_{y \in D} \frac{|r^{n,t}(y)|}{\sqrt{n}} > \eta \right) \leq \mathbb{P}\left(\sup_{y \in T} \frac{|r^{n,t}(y)|}{\sqrt{n}} > \frac{\eta}{2} \right). \qquad \square$$

Proposition 3.10 (Uniform bound, $d = 2$). *Assume that ultracontractivity holds with $d = 2$ and that $N_D(\delta) \leq \max\{1, C_D \delta^{-2}\}$ for every $\delta > 0$. Then there exists a constant $C = C(C_{\mathrm{ge}}, C_D)$ with the following property: for all $\eta > 0$ there exists $\gamma = \gamma(\eta, C_{\mathrm{uc}})$ such that*

$$\mathbb{P}\left(\sup_{y \in D} \frac{|r^{n,t}(y)|}{\sqrt{n}} > \eta\right) \leq \frac{C}{n}$$

holds for $(16 C_D C_{\mathrm{ge}}^2)^{1/3} \geq t \geq \gamma n^{-1} \log n$ and $n \geq n(\eta, C_{\mathrm{uc}})$.

Proof. Given $\eta > 0$, we choose γ in such a way that $\gamma F(1, \eta/2)/(2 C_{\mathrm{uc}}) = 4$. Then, we define $n(\eta, C_{\mathrm{uc}})$ in such a way that $\gamma \log n \geq \eta^{-2/3}$ for $n \geq n(\eta, C_{\mathrm{uc}})$.

We pick $\delta = \frac{\eta}{4 C_{\mathrm{ge}}} t^{3/2}$, so that, by Lemma 3.7, with probability 1 we have (3.18). Let T be a minimal δ-net. Then the condition $t^3 \leq 16 C_D C_{\mathrm{ge}}^2$ implies $C_D \delta^{-2} \geq 1$,

$$|T| \leq C_D \delta^{-2} = \frac{16 C_D C_{\mathrm{ge}}^2}{\eta^2} t^{-3} \leq 16 C_D C_{\mathrm{ge}}^2 n^3,$$

where we used also the inequality $t \geq \gamma n^{-1} \log n \geq \eta^{-2/3}/n$. From an application of Lemma 3.6 with $\eta/2$ instead of η we get

$$\mathbb{P}\left(\sup_{y \in T} \frac{|r^{n,t}(y)|}{\sqrt{n}} > \frac{\eta}{2}\right) \leq 2|T| \exp\left(-\frac{nt}{2 C_{\mathrm{uc}}} F(1, \eta/2)\right)$$

$$\leq 32 C_D C_{\mathrm{ge}}^2 \exp\left(3 \log n - \gamma \frac{F(1, \eta/2)}{2 C_{\mathrm{uc}}} \log n\right).$$

Our choice of γ then gives

$$\mathbb{P}\left(\sup_{y \in T} \frac{|r^{n,t}(y)|}{\sqrt{n}} > \frac{\eta}{2}\right) \leq \frac{32 C_D C_{\mathrm{ge}}^2}{n}. \qquad \square$$

We now report some estimates on the logarithmic mean.

Lemma 3.11. *For $a, b \geq 0$ and $q > 0$ we have*

$$q(ab)^{q/2} \frac{a - b}{a^q - b^q} \leq M(a, b) \leq q \frac{a^q + b^q}{2} \cdot \frac{a - b}{a^q - b^q}.$$

Proof. It is known that

$$\sqrt{ab} \leq M(a, b) \leq \frac{a + b}{2}. \tag{3.19}$$

The thesis follows by applying these inequalities to a^q and b^q. $\qquad \square$

In the following lemma we estimate the logarithmic mean of the densities of $\mu^{n,t,c}$ obtained by a further regularization, i.e. by adding to $\mu^{n,t}$ a small multiple of $\mathfrak{m}$.

Lemma 3.12 (Integral bound). *Define $\mu^{n,t,c} = (1 - c)\mu^{n,t} + c\mathfrak{m}$ and let $u^{n,t,c} = (1 - c)u^{n,t} + c$ be its probability density, with $c = c(n) \in (0, 1]$. If $t^{d/2} = t^{d/2}(n) \geq \gamma n^{-1} \log n$ and $nc(n) \to \infty$ as $n \to \infty$, then*

$$\lim_{n \to \infty} \mathbb{E}\left[\int_D \left(M(u^{n,t,c}, 1)^{-1} - 1\right)^2 \mathrm{d}\mathfrak{m}\right] = 0.$$

Proof. Fix $x \in D$ and $\eta \in (0, 1)$. By Lemma 3.6 we have

$$\mathbb{P}\left(|u^{n,t}(x) - 1| > \eta\right) \leq 2n^{-q},$$

where $q \in (0, 1)$ depends only on d, C_{uc}, γ and η. In the event $\{|u^{n,\varepsilon}(x) - 1| > \eta\}$, using the first inequality in Lemma 3.11 we can estimate the squared difference with the sum of squares to get

$$\left(M(u^{n,t,c}(x), 1)^{-1} - 1\right)^2 \leq \frac{2}{q^2} \cdot \frac{1}{u^{n,t,c}(x)^q} \left(\frac{u^{n,t,c}(x)^q - 1}{u^{n,t,c}(x) - 1}\right)^2 + 2 \leq \frac{2}{q^2 c^q} + 2.$$

In the complementary event $\{|u^{n,t}(x) - 1| \leq \eta\}$, we have $|u^{n,t,c}(x) - 1| \leq (1 - c)\eta \leq \eta$ and, expanding the squares and using both inequalities in (3.19), we get

$$\left(M(u^{n,t,c}(x), 1)^{-1} - 1\right)^2 \leq \frac{1}{u^{n,t,c}(x)} - \frac{4}{u^{n,t,c}(x) + 1} + 1 \leq \frac{1}{1 - \eta} - \frac{4}{2 + \eta} + 1.$$

Therefore

$$\mathbb{E}\left[\int_D \left(M(\mu^{n,t,c}, 1)^{-1} - 1\right)^2 d\mathfrak{m}\right] \leq 2n^{-q}\left(\frac{2}{q^2 c(n)^q} + 2\right) + \frac{1}{1 - \eta} - \frac{4}{2 + \eta} + 1,$$

hence the growth condition on c gives

$$\limsup_{n \to \infty} \mathbb{E}\left[\int_D \left(M(u^{n,t,c}, 1)^{-1} - 1\right)^2 d\mathfrak{m}\right] \leq \frac{1}{1 - \eta} - \frac{4}{2 + \eta} + 1.$$

Letting $\eta \to 0$ we obtain the result. $\qquad\square$

3.2 Energy estimates

Retaining Definition 3.5 of $r^{n,t}$ from the previous subsection, here we derive energy bounds for the solutions to the following random PDE:

$$\begin{cases} \Delta f^{n,t} = r^{n,t} & \text{in } D, \\ \nabla f^{n,t} \cdot n_D = 0 & \text{on } \partial D \end{cases} \tag{3.20}$$

which are uniquely determined up to a (random) additive constant. As we will see (particularly in Section 6), these estimates involve either the trace of Δ or sums indexed by the spectrum $\sigma(\Delta)$ (which contains $\{0\}$ and, by the spectral gap assumption, satisfies $\sigma(\Delta) \subset (-\infty, -C_{\text{sg}}^2] \cup \{0\}$); it is understood that the eigenvalues in these sums are counted with multiplicity.

We recall the so-called trace formula

$$\int_D p_s(x, x) \, d\mathfrak{m}(x) = \sum_{\lambda \in \sigma(\Delta)} e^{s\lambda} \tag{3.21}$$

which follows easily by integration of the representation formula

$$p_s(x, y) = \sum_{\lambda \in \sigma(\Delta)} e^{s\lambda} u_\lambda(x) u_\lambda(y),$$

where $\{u_\lambda\}_{\lambda \in \sigma(\Delta)}$ is an $L^2(\mathfrak{m})$ orthonormal basis of eigenvalues of Δ.

The following expansion (3.22) of the trace formula as $s \to 0$ will be useful. In this paper we will only use the leading term in (3.22).

Proposition 3.13 (Expansion of the trace formula)**.** *Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ with unit volume. Then*

$$\int_D p_s(x,x)\,\mathrm{d}\mathfrak{m}(x) = (4\pi s)^{-d/2}\left(1 + \frac{\sqrt{\pi s}}{2}\mathscr{H}^{n-1}(\partial D) + o(\sqrt{s})\right) \quad \text{as } s \to 0. \quad (3.22)$$

The same holds if D is a smooth, compact d-dimensional Riemannian manifold with a smooth boundary (possibly empty).

Proof. The first statement is proved in [B93]. The second one, also with additional terms in the expansion, in [MKS67]. $\qquad\square$

Lemma 3.14 (Representation formula)**.** *Let $f^{n,t}$ be the solution to (3.20). For all $t > 0$ one has*

$$\mathbb{E}\left[\int_D |\nabla f^{n,t}|^2\,\mathrm{d}\mathfrak{m}\right] = 2\int_t^\infty \left(\int_D p_{2s}(x,x)\,\mathrm{d}\mathfrak{m}(x) - 1\right)\mathrm{d}s = -\sum_{\lambda\in\sigma(\Delta)\setminus\{0\}}\frac{e^{2t\lambda}}{\lambda}. \quad (3.23)$$

Proof. Using the representation formula $g = -\int_0^\infty P_s\Delta g\,\mathrm{d}s$ with $g = f^{n,t}$ we get $f^{n,t} = -\int_0^\infty P_s r^{n,t}\,\mathrm{d}s$, so that

$$\int_D |\nabla f^{n,t}|^2\,\mathrm{d}\mathfrak{m} = -\int_D f^{n,\varepsilon}\Delta f^{n,t}\,\mathrm{d}\mathfrak{m} = \int_D \left(\int_0^\infty P_s r^{n,t}\,\mathrm{d}s\right) r^{n,t}\,\mathrm{d}\mathfrak{m}$$

$$= \int_0^\infty \int_D P_s r^{n,t} r^{n,t}\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s = \int_0^\infty \int_D P_{s/2} r^{n,t} P_{s/2} r^{n,t}\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \quad (3.24)$$

$$= 2\int_t^\infty \int_D (P_s^* r^n)^2\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s.$$

Now, notice that the symmetry and semigroup properties of the transition probabilities give

$$\int_D\int_D p_s(x,y)^2\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) = \int_D\int_D p_s(x,y)p_s(y,x)\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}\mathfrak{m}(x)$$

$$= \int_D p_{2s}(x,x)\,\mathrm{d}\mathfrak{m}(x).$$

Hence, by Lemma 2.6 with $f = p_s(\,\cdot\,,y)$ we can compute

$$\mathbb{E}\left[\int_D (P_s^* r^n)^2\,\mathrm{d}\mathfrak{m}\right] = \int_D \mathbb{E}\left[(P_s^* r^n(y))^2\right]\mathrm{d}\mathfrak{m}(y) = \int_D \mathbb{E}\left[\left(\int_D p_s(x,y)\,\mathrm{d}r^n(x)\right)^2\right]\mathrm{d}\mathfrak{m}(y)$$

$$= \int_D \llbracket p_s(\,\cdot\,,y)\rrbracket_2^2\,\mathrm{d}\mathfrak{m}(y) = \int_D\int_D (p_s(x,y)-1)^2\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)$$

$$= \int_D\int_D p_s(x,y)^2\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) - 1 = \int_D p_{2s}(x,x)\,\mathrm{d}\mathfrak{m}(x) - 1.$$

By the trace formula (3.21), (3.23) follows. $\qquad\square$

The following lemma basically applies only to 1-dimensional domains, in view of the ultracontractivity assumption with $d = 1$.

Lemma 3.15 (Energy estimate and convergence, $d = 1$)**.** *Let $f^{n,t}$ be the solution to* (3.20). *If $t = t(n) \to 0$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right] = \int_0^\infty \left(\int_D p_s(x,x) \, d\mathfrak{m}(x) - 1\right) ds = - \sum_{\lambda \in \sigma(\Delta)\setminus\{0\}} \frac{1}{\lambda}. \quad (3.25)$$

If ultracontractivity holds with $d = 1$ we have also

$$\limsup_{n\to\infty} \mathbb{E}\left[\left(\int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right)^2\right] < \infty \quad (3.26)$$

and, in particular, the limit in (3.25) *is finite.*

Proof. The identities (3.25) follow by (3.23) by taking the limit as $n \to \infty$. If ultracontractivity holds with $d = 1$, we show that the $\limsup$ in (3.25) is finite by splitting the integration in $(t, 1)$ and $(1, \infty)$ in the identity

$$\mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right] = 2\int_t^\infty \int_D \int_D (p_s(x,y) - 1)^2 \, d\mathfrak{m}(x) \, d\mathfrak{m}(y) \, ds, \quad (3.27)$$

which is a by-product of the intermediate computations made in the proof of Lemma 3.14. For $s \in (t, 1)$ we estimate

$$\int_D \int_D (p_s(x,y) - 1)^2 \, d\mathfrak{m}(x) \, d\mathfrak{m}(y) \le C_{\mathrm{uc}} s^{-1/2} \int_D \int_D |p_s(x,y) - 1| \, d\mathfrak{m}(x) \, d\mathfrak{m}(y)$$

$$\le 2 C_{\mathrm{uc}} s^{-1/2}.$$

For $s \in (1, \infty)$ instead

$$\int_D \int_D (p_s(x,y) - 1)^2 \, d\mathfrak{m}(x) \, d\mathfrak{m}(y) = \int_D \|P_{s-1} P_1^*(\delta_y - \mathfrak{m})\|_2^2 \, d\mathfrak{m}(y)$$

$$\le e^{-2C_{\mathrm{sg}}(s-1)} \int_D \|P_1^*(\delta_y - \mathfrak{m})\|_2^2 \, d\mathfrak{m}(y) \le 4\|P_1^*\|_{\mathcal{M}\to L^2}^2 e^{-2C_{\mathrm{sg}}(s-1)}.$$

In conclusion, for some geometric constant C, one has

$$\mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right] \le C\left(\int_t^1 s^{-1/2} \, ds + \int_1^\infty e^{-2C_{\mathrm{sg}}s} \, ds\right),$$

from which the finiteness of (3.25) readily follows.

To show (3.26), we start from (3.24) and estimate with the aid of Lemma 2.6

$$\mathbb{E}\left[\left(\int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right)^2\right] = \mathbb{E}\left[\left(2\int_t^\infty \int_D (P_s^* r^n)^2 \, d\mathfrak{m} \, ds\right)^2\right]$$

$$= 4\int_t^\infty \int_t^\infty \int_D \int_D \mathbb{E}\left[(P_s^* r^n(y))^2 (P_{s'}^* r^n(z))^2\right] d\mathfrak{m}(y) \, d\mathfrak{m}(z) \, ds \, ds'$$

$$\le 4\int_t^\infty \int_t^\infty \int_D \int_D \Big(3[\![p_s(\cdot, y)]\!]_2^2 [\![p_{s'}(\cdot, z)]\!]_2^2$$

$$+ \frac{1}{n}[\![p_s(\cdot, y)]\!]_4^2 [\![p_{s'}(\cdot, z)]\!]_4^2\Big) d\mathfrak{m}(y) \, d\mathfrak{m}(z) \, ds \, ds'$$

$$= 3\left(2\int_t^\infty \int_D [\![p_s(\cdot, y)]\!]_2^2 \, d\mathfrak{m}(y) \, ds\right)^2 + \frac{1}{n}\left(2\int_t^\infty \int_D [\![p_s(\cdot, y)]\!]_4^2 \, d\mathfrak{m}(y) \, ds\right)^2$$

$$= 3\left(\sum_{\lambda \in \sigma(\Delta)\setminus\{0\}} \frac{e^{2t\lambda}}{\lambda}\right)^2 + \frac{1}{n}\left(2\int_t^\infty \int_D [\![p_s(\cdot, y)]\!]_4^2 \, d\mathfrak{m}(y) \, ds\right)^2.$$

In order to show that the lim sup of last integral is finite we split the integration in $(t, 1)$ and $(1, \infty)$. For $s \in (t, 1)$ we use

$$\llbracket p_s(\,\cdot\,, y) \rrbracket_4^4 \leq \int_D (p_s(x, y) - 1)^4 \, \mathrm{dm}(x) \leq C_{\mathrm{uc}}^3 s^{-3/2} \int_D |p_s(x, y) - 1| \, \mathrm{dm}(x) \leq 2 C_{\mathrm{uc}}^3 s^{-3/2}.$$

For $s \in (1, \infty)$ instead

$$\llbracket p_s(\,\cdot\,, y) \rrbracket_4^4 \leq C_{\mathrm{uc}}^2 s^{-1} \llbracket p_s(\,\cdot\,, y) \rrbracket_2^2 \leq C_{\mathrm{uc}}^2 s^{-1} \| P_s^*(\delta_y - \mathfrak{m}) \|_2^2 \leq C_{\mathrm{uc}}^2 e^{-2 C_{\mathrm{sg}}(s-1)} \| P_1^*(\delta_y - \mathfrak{m}) \|_2^2.$$

Putting these estimates together,

$$\int_t^\infty \llbracket p_s(\,\cdot\,, y) \rrbracket_4^2 \, \mathrm{d}s \leq \sqrt{2} C_{\mathrm{uc}}^{3/2} \int_t^1 s^{-3/4} \, \mathrm{d}s + C_{\mathrm{uc}} \| P_1^*(\delta_y - \mathfrak{m}) \|_2 \int_1^\infty e^{-C_{\mathrm{sg}}(s-1)} \, \mathrm{d}s,$$

which is bounded, uniformly in y and t, because $\| P_1^*(\delta_y - \mathfrak{m}) \|_2 \leq 2 \| P_1^* \|_{\mathcal{M} \to L^2}$. $\qquad \square$

Lemma 3.16 (Renormalized energy estimate and convergence, $d = 2$)**.** *Assume that ultracontractivity holds with $d = 2$. Let $f^{n,t}$ be the solution to (3.20). If $t = t(n) \to 0$ as $n \to \infty$ and $t \geq C/n$ for some $C > 0$, then*

$$\limsup_{n \to \infty} \frac{1}{(\log t)^2} \, \mathbb{E}\left[\int_D |\nabla f^{n,t}|^4 \, \mathrm{dm} \right] < \infty. \tag{3.28}$$

In particular

$$\limsup_{n \to \infty} \frac{1}{|\log t|} \, \mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, \mathrm{dm} \right] < \infty. \tag{3.29}$$

Moreover, under the assumptions on D of [Proposition 3.13](#), one has

$$\lim_{n \to \infty} \frac{1}{|\log t|} \, \mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, \mathrm{dm} \right] = \frac{1}{4\pi}. \tag{3.30}$$

Proof. We will prove first (3.29) as an intermediate step in the proof of (3.28), starting from the representation formula (3.27). For $s \in (t, 1)$ we estimate

$$\int_D \int_D (p_s(x, y) - 1)^2 \, \mathrm{dm}(x) \, \mathrm{dm}(y) \leq C_{\mathrm{uc}} s^{-1} \int_D \int_D |p_s(x, y) - 1| \, \mathrm{dm}(x) \, \mathrm{dm}(y) \leq 2 C_{\mathrm{uc}} s^{-1}.$$

For $s \in (1, \infty)$ instead

$$\int_D \int_D (p_s(x, y) - 1)^2 \, \mathrm{dm}(x) \, \mathrm{dm}(y) = \int_D \| P_{s-1} P_1(\delta_y - \mathfrak{m}) \|_2^2 \, \mathrm{dm}(y)$$

$$\leq e^{-2 C_{\mathrm{sg}}(s-1)} \int_D \| P_1(\delta_y - \mathfrak{m}) \|_2^2 \, \mathrm{dm}(y) \leq 4 \| P_1 \|_{L^1 \to L^2}^2 e^{-2 C_{\mathrm{sg}}(s-1)}.$$

In conclusion, for some geometric constant C, one has

$$\mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, \mathrm{dm} \right] \leq C \left(\int_t^1 s^{-1} \, \mathrm{d}s + \int_1^\infty e^{-2 C_{\mathrm{sg}} s} \, \mathrm{d}s \right) \leq C(|\log t| + 1),$$

from which (3.29) readily follows.

In order to prove (3.30), we notice that the estimates given in the proof of (3.29) show that

$$\frac{1}{|\log t|} \mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, \mathrm{d}\mathfrak{m}\right] - \frac{2}{|\log t|} \int_t^1 \left(\int_D p_{2s}(x,x) \, \mathrm{d}\mathfrak{m}(x) - 1\right) \mathrm{d}s$$

is infinitesimal as $n \to \infty$. Combining this information with (3.22) of Proposition 3.13, we obtain (3.30).

To deal with (3.28), we introduce the Paley-Littlewood function

$$S(g) = \left(\int_0^\infty (s\partial_s P_s g)^2 \, \frac{\mathrm{d}s}{s}\right)^{1/2}.$$

Using the Riesz transform bound and the fundamental theorem [S70] $\|g\|_p^p \leq c_p \|S(g)\|_p^p$ for any $p \in (1, \infty)$ and g with $\int_D g \, \mathrm{d}\mathfrak{m} = 0$, we obtain

$$\int_D |\nabla f^{n,t}|^4 \, \mathrm{d}\mathfrak{m} \leq C_r \int_D \left|(-\Delta)^{1/2} f^{n,t}\right|^4 \, \mathrm{d}\mathfrak{m} \leq C_r c_4 \int_D S\left((-\Delta)^{1/2} f^{n,t}\right)^4 \, \mathrm{d}\mathfrak{m}$$

$$= C_r c_4 \int_0^\infty \int_0^\infty \int_D s\left(\partial_s P_s (-\Delta)^{1/2} f^{n,t}\right)^2 s'\left(\partial_{s'} P_{s'} (-\Delta)^{1/2} f^{n,t}\right)^2 \, \mathrm{d}\mathfrak{m} \, \mathrm{d}s \, \mathrm{d}s'.$$

Using the fact that $\partial_t P_t = \Delta P_t$ and that the operators Δ, P_t and $(-\Delta)^{1/2}$ commute we have

$$\partial_\tau P_\tau (-\Delta)^{1/2} f^{n,t} = (-\Delta)^{1/2} P_{\tau+t}^* r^n,$$

so that

$$\int_D |\nabla f^{n,t}|^4 \, \mathrm{d}\mathfrak{m} \leq C_r c_4 \int_0^\infty \int_0^\infty \int_D \left((-s\Delta)^{1/2} P_{s+t}^* r^n\right)^2 \left((-s'\Delta)^{1/2} P_{s'+t}^* r^n\right)^2 \, \mathrm{d}\mathfrak{m} \, \mathrm{d}s \, \mathrm{d}s'.$$

For $y \in D$ fixed, consider the operators

$$(T_s^t \mu)(y) = \left((-s\Delta)^{1/2} P_{s+t}^* \mu\right)(y) = \int_D K_s^t(x,y) \, \mathrm{d}\mu(x)$$

and notice that

$$\int_D K_s^t(x,y) \, \mathrm{d}\mathfrak{m}(x) = T_s^t \mathfrak{m}(y) = 0.$$

In addition, since $T_s^t : L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$ is self-adjoint, the kernel K_s^t is symmetric and

$$T_s^t \delta_x(y) = K_s^t(x,y) = K_s^t(y,x). \tag{3.31}$$

Taking the expectation of the integrand,

$$\mathbb{E}\left[\left((-s\Delta)^{1/2} P_{s+t}^* r^n\right)^2 (y) \left((-s'\Delta)^{1/2} P_{s'+t}^* r^n\right)^2 (y)\right] = \mathbb{E}\left[(T_s^t r^n)^2(y)(T_{s'}^t r^n)^2(y)\right]$$

$$= \mathbb{E}\left[\left(\int_D K_s^t(x,y) \, \mathrm{d}r^n(x)\right)^2 \left(\int_D K_{s'}^t(x',y) \, \mathrm{d}r^n(x')\right)^2\right]$$

$$\leq 3\frac{n-1}{n} [\![K_s^t(\,\cdot\,,y)]\!]_2^2 [\![K_{s'}^t(\,\cdot\,,y)]\!]_2^2 + \frac{1}{n} [\![K_s^t(\,\cdot\,,y)]\!]_4^2 [\![K_{s'}^t(\,\cdot\,,y)]\!]_4^2.$$

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Integrating in s and s' we obtain

$$\int_0^\infty \int_0^\infty \mathbb{E}\left[\left((-s\Delta)^{1/2} P_{s+t}^* r^n\right)^2 (y) \left((-s'\Delta)^{1/2} P_{s'+t}^* r^n\right)^2 (y)\right] \mathrm{d}s\,\mathrm{d}s'$$

$$\leq 3\frac{n-1}{n}\left(\int_0^\infty [\![K_s^t(\cdot,y)]\!]_2^2\,\mathrm{d}s\right)^2 + \frac{1}{n}\left(\int_0^\infty [\![K_{s'}^t(\cdot,y)]\!]_4^2\,\mathrm{d}s'\right)^2.$$

Since $(P_t)_{t\geq 0}$ is a bounded analytic semigroup, complex interpolation yields that, for $p \in (1,\infty)$, $(-\tau\Delta)^{1/2} P_{\tau/2} : L^p \to L^p$ is continuous with norms uniformly bounded for $\tau \geq 0$ [Y80, Sections X.10-11], hence we have the estimate

$$[\![K_s^t(\cdot,y)]\!]_p = \|T_s^t \delta_y\|_p = \|T_s^t(\delta_y - \mathfrak{m})\|_p$$

$$= \left\|(-s\Delta)^{1/2} P_{s/2} P_{s/2+t}^* (\delta_y - \mathfrak{m})\right\|_p$$

$$\leq C_p \|P_{s/2+t}^* (\delta_y - \mathfrak{m})\|_p,$$

where in the first equality we used (3.31). We consider

$$\int_0^\infty [\![K_s^t(\cdot,y)]\!]_p^2\,\mathrm{d}s \leq C_p \int_0^\infty \|P_{s/2+t}^* (\delta_y - \mathfrak{m})\|_p^2\,\mathrm{d}s = 2C_p \int_t^\infty \|P_s^* (\delta_y - \mathfrak{m})\|_p^2\,\mathrm{d}s.$$

Now we split the integrals for $s \in (t,2)$ and $s \in (2,\infty)$. In the former interval we use the estimate

$$\|P_s^* (\delta_y - \mathfrak{m})\|_p = \left(\int_D |p_s(x,y) - 1|^p\,\mathrm{d}\mathfrak{m}(x)\right)^{1/p}$$

$$\leq \left(\int_D (C_{\mathrm{uc}} s^{-1})^{p-1} |p_s(x,y) - 1|\,\mathrm{d}\mathfrak{m}(x)\right)^{1/p}$$

$$\leq 2^{1/p} C_{\mathrm{uc}}^{(p-1)/p} s^{-(p-1)/p}.$$

In the latter interval we use the estimate

$$\|P_s^* (\delta_y - \mathfrak{m})\|_p = \|P_1 P_{(s-2)} P_1^* (\delta_y - \mathfrak{m})\|_p$$

$$\leq \|P_1\|_{L^2 \to L^p} e^{-C_{\mathrm{sg}}(s-2)} \|P_1^* (\delta_y - \mathfrak{m})\|_2$$

$$\leq \|P_1\|_{L^2 \to L^p} \|P_1^*\|_{\mathcal{M} \to L^2} e^{-C_{\mathrm{sg}}(s-2)} \|\delta_y - \mathfrak{m}\|_{\mathcal{M}}$$

$$\leq 2e^{2C_{\mathrm{sg}}} \|P_1\|_{L^2 \to L^p} \|P_1^*\|_{\mathcal{M} \to L^2} e^{-C_{\mathrm{sg}}s}.$$

Putting these estimates together, in the case $p = 2$ we have,

$$\int_t^\infty \|P_s^* (\delta_y - \mathfrak{m})\|_p^2\,\mathrm{d}s \leq C\left(\int_t^2 s^{-1}\,\mathrm{d}s + \int_2^\infty e^{-2C_{\mathrm{sg}}s}\,\mathrm{d}s\right) \leq C(|\log t| + 1)$$

for some geometric constant C. In the case $p = 4$ we have also

$$\int_t^\infty \|P_s^* (\delta_y - \mathfrak{m})\|_p^2\,\mathrm{d}s \leq C\left(\int_t^2 s^{-3/2}\,\mathrm{d}s + \int_2^\infty e^{-2C_{\mathrm{sg}}s}\,\mathrm{d}s\right) \leq C\left(\frac{1}{\sqrt{t}} + 1\right).$$

This yields

$$3\frac{n-1}{n}\left(\int_0^\infty [\![K_s^t(\cdot,y)]\!]_2^2\,\mathrm{d}s\right)^2 + \frac{1}{n}\left(\int_0^\infty [\![K_s^t(\cdot,y)]\!]_4^2\,\mathrm{d}s\right)^2$$

$$\leq C\frac{n-1}{n}(\log t)^2 + C\left(\frac{n}{t} + 1\right).$$

In conclusion

$$\mathbb{E}\left[\int_D |\nabla f^{n,t}|^4 \, d\mathfrak{m}\right] \frac{1}{(\log t)^2} \le \frac{C}{(\log t)^2} \int_D \left(\frac{n-1}{n}(\log t)^2 + \frac{n}{t} + 1\right) d\mathfrak{m}(y)$$

$$\le C\left[1 + \frac{n}{(\log t)^2 t} + \frac{1}{(\log t)^2}\right]$$

is uniformly bounded as $n \to \infty$ by the assumptions on $t = t(n)$. $\qquad\square$

4 Proof of the main result

In this section we prove Theorem 1.1. In the proof of the upper bound we need only to assume the regularizing properties of P_t listed in Section 3; in particular this inequality covers also the case $D = [0,1]^2$ and compact 2-dimensional Riemannian manifolds with smooth boundary. In the proof of the lower bound we need also to assume that D has no boundary; by a comparison argument, since the distance in $\mathbb{T}^2$ is smaller than the distance in $[0,1]^2$, we recover also the lower bound for $D = [0,1]^2$.

We include also the 1-dimensional case (whose proofs are a bit simpler), which covers the case of the interval and the case of the circle. For brevity we state the result only in the Riemannian case, but the strength of this method relies in the fact that it can be extended to more general 1-dimensional spaces (see also Section 6).

Theorem 4.1. *Assume that either $D = [0,1]$ or $D = \mathbb{T}^1$. Then*

$$\lim_{n\to\infty} n\,\mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\right] = - \sum_{\lambda \in \sigma(\Delta)\setminus\{0\}} \frac{1}{\lambda}.$$

In particular, from Euler's formula $\pi^2 = 6\sum_{k\ge 1} k^{-2}$, the limit equals $1/6$ for $D = [0,1]$ and $1/12$ for $D = \mathbb{T}^1$.

Remark 4.2. In the case $D = [0,1]$ and $\mathfrak{m} = \mathscr{L}^1 \llcorner D$ we can explicitly compute $n\,\mathbb{E}[W_2^2(\mu^n, \mathfrak{m})]$ and $n\,\mathbb{E}[W_2^2(\mu^n, \nu^n)]$ as follows (and in particular, the former is identically equal to $1/6$). For any fixed $n \in \mathbb{N}$, let $X_{(k)}$ and $Y_{(k)}$ denote the order statistics of the random variables $(X_i)_{i=1}^n$ and $(Y_i)_{i=1}^n$. It is well known that $X_{(k)}$ and $Y_{(k)}$ are distributed according to the beta distribution $X_{(k)} \sim Y_{(k)} \sim \mathrm{B}(k, n+1-k)$.

The optimal map is given by the monotone rearrangement of the mass, therefore

$$\mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\right] = \mathbb{E}\left[\sum_{k=1}^n \int_{(k-1)/n}^{k/n} (X_{(k)} - t)^2 \, dt\right] = \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \mathbb{E}[(X_{(k)} - t)^2]\, dt$$

$$= \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left(\mathrm{Var}(X_{(k)}) + (\mathbb{E}[X_{(k)}] - t)^2\right) dt$$

$$= \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left[\frac{(k+1)k}{(n+2)(n+1)} - 2t\frac{k}{n+1} + t^2\right] dt$$

$$= \sum_{k=1}^n \left[\frac{(k+1)k}{(n+2)(n+1)n} - \frac{(2k-1)k}{(n+1)n^2} + \frac{3k^2 - 3k + 1}{3n^3}\right] = \frac{1}{6n}.$$

Similarly, in the bipartite case we have

$$\mathbb{E}\left[W_2^2(\mu^n, \nu^n)\right] = \mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n}(X_{(k)} - Y_{(k)})^2\right] = \frac{1}{n}\sum_{k=1}^{n}\left(\mathbb{E}[X_{(k)}^2] - \mathbb{E}[X_{(k)}^2]\right)$$

$$= \frac{2}{n}\sum_{k=1}^{n}\text{Var}(X_{(k)}) = \frac{2}{n}\sum_{k=1}^{n}\frac{k(n+1-k)}{(n+1)^2(n+2)} = \frac{1}{3(n+1)}.$$

4.1 Upper bound

Theorem 4.3 (Upper bound, $d = 1$). *Assume that ultracontractivity holds with $d = 1$. Then*

$$\limsup_{n\to\infty} n\,\mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\right] \leq -\sum_{\lambda\in\sigma(\Delta)\setminus\{0\}}\frac{1}{\lambda}.$$

Proof. Fix $q \in (1/2, 1)$, $\eta \in (0, 1)$ and let $t = t(n) = n^{-2q}$. For $\eta \in (0, 1)$ consider the event

$$A_\eta = A_{\eta,n} = \left\{\sup_{y\in D}\frac{|r^{n,t}(y)|}{\sqrt{n}} \leq \eta\right\}.$$

By Proposition 3.9, since $W_2^2(\mu^n, \mathfrak{m}) \leq (\text{diam } D)^2$, for n large enough we have

$$n\,\mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\right] = n\,\mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\chi_{A_\eta}\right] + n\,\mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\chi_{A_\eta^c}\right]$$

$$\leq n\,\mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\chi_{A_\eta}\right] + C(\text{diam } D)^2 n\exp\left(-\gamma n^{1-q}\right)$$

with $C = C(C_D, C_{\text{ge}})$ and $\gamma = \gamma(\eta, C_{\text{uc}}) > 0$.

Using the Young inequality for products with $\alpha > 0$ and $W_2^2(\mu^n, \mu^{n,t}) \leq C_{\text{dr}}t$ we have

$$W_2^2(\mu^n, \mathfrak{m}) \leq \left(W_2(\mu^{n,t}, \mathfrak{m}) + W_2(\mu^n, \mu^{n,t})\right)^2$$

$$\leq (1+\alpha)W_2^2(\mu^{n,t}, \mathfrak{m}) + (1+\alpha^{-1})W_2^2(\mu^n, \mu^{n,t})$$

$$\leq (1+\alpha)W_2^2(\mu^{n,t}, \mathfrak{m}) + (1+\alpha^{-1})C_{\text{dr}}t.$$

Therefore, since $nt \to 0$, it is sufficient to estimate

$$\limsup_{n\to\infty} n\,\mathbb{E}\left[W_2^2(\mu^{n,t}, \mathfrak{m})\chi_{A_\eta}\right].$$

To this end, we apply Proposition 2.3 with $u_0 = u^{n,t}$ and $u_1 = 1$. Since $f^{n,t}$ solves (3.20) from Proposition 2.3 we get

$$W_2^2(\mu^{n,t}, \mathfrak{m}) \leq \frac{1}{n}\int_D \frac{|\nabla f^{n,t}|^2}{M(u^{n,t}, 1)}\,\mathrm{d}\mathfrak{m}.$$

In the event A_η we have $u^{n,t} \geq 1 - \eta$ in D, hence the first inequality in (3.19) gives

$$\frac{1}{M(u^{n,t}, 1)} \leq \frac{1}{\sqrt{u^{n,t}}} \leq \frac{1}{\sqrt{1-\eta}}.$$

The previous two inequalities and Lemma 3.15 give

$$\limsup_{n \to \infty} n \, \mathbb{E}\left[W_2^2(\mu^{n,t}, \mathfrak{m})\chi_{A_\eta}\right] \leq \lim_{n \to \infty} \frac{1}{\sqrt{1-\eta}} \mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right]$$

$$= -\frac{1}{\sqrt{1-\eta}} \sum_{\lambda \in \sigma(\Delta)\setminus\{0\}} \frac{1}{\lambda}.$$

In conclusion we have

$$\limsup_{n \to \infty} n \, \mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\right] \leq -\frac{1+\alpha}{\sqrt{1-\eta}} \sum_{\lambda \in \sigma(\Delta)\setminus\{0\}} \frac{1}{\lambda}$$

and we obtain the thesis by letting first $\alpha \to 0$ and then $\eta \to 0$. $\qquad\square$

Theorem 4.4 (Upper bound, $d = 2$). *Assume that ultracontractivity holds with $d = 2$ and that D is as in Proposition 3.13. Then*

$$\limsup_{n \to \infty} \frac{n}{\log n} \mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\right] \leq \frac{1}{4\pi}.$$

Proof. Fix $\gamma > 0$ and let $t(n) = c(n) = \gamma n^{-1}\log n$. Let us set

$$\mu^{n,t,c} = (1-c)\mu^{n,t} + c\mathfrak{m}, \qquad u^{n,t,c} = (1-c)u^{n,t} + c$$

as in Lemma 3.12. From the joint convexity of W_2^2 (see (2.4)) we immediately get

$$W_2^2(\mu^{n,t}, \mu^{n,t,c}) \leq (\mathrm{diam}\, D)^2 c$$

Using the Young inequality for products with $\alpha > 0$ and $W_2^2(\mu^n, \mu^{n,t}) \leq C_{\mathrm{dr}}t$, we have

$$W_2^2(\mu^n, \mathfrak{m}) \leq \left(W_2(\mu^{n,t,c}, \mathfrak{m}) + W_2(\mu^n, \mu^{n,t}) + W_2(\mu^{n,\varepsilon}, \mu^{n,t,c})\right)^2$$
$$\leq (1+\alpha)W_2^2(\mu^{n,t,c}, \mathfrak{m}) + 2(1+\alpha^{-1})\left[W_2^2(\mu^n, \mu^{n,t}) + W_2^2(\mu^{n,t}, \mu^{n,t,c})\right]$$
$$\leq (1+\alpha)W_2^2(\mu^{n,t,c}, \mathfrak{m}) + 2(1+\alpha^{-1})[C_{\mathrm{dr}}t + (\mathrm{diam}\, D)^2 c].$$

We start by estimating the contribution of the first term.

To this end we apply Proposition 2.3 with $u_0 = u^{n,t,c}$ and $u_1 = 1$. Recalling that $f^{n,t}$ solves the PDE $\Delta f^{n,t} = \sqrt{n}(u^{n,t} - 1)$ with homogeneous Neumann boundary conditions, we get

$$\Delta \frac{(1-c)}{\sqrt{n}} f^{n,t} = (u^{n,t,c} - 1),$$

hence Proposition 2.3 gives

$$W_2^2(\mu^{n,t,c}, \mathfrak{m}) \leq \frac{(1-c)^2}{n}\int_D \frac{|\nabla f^{n,t}|^2}{M(u^{n,t,c}, 1)} \, d\mathfrak{m} \leq \frac{1}{n}\int_D \frac{|\nabla f^{n,t}|^2}{M(u^{n,t,c}, 1)} \, d\mathfrak{m}. \qquad (4.1)$$

Adding and subtracting $|\nabla f^{n,t}|^2$ to the integrand we obtain

$$\mathbb{E}\left[\int_D \frac{|\nabla f^{n,t}|^2}{M(u^{n,t,c}, 1)} \, d\mathfrak{m}\right] = \mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right] + \mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \left(M(u^{n,t,c}, 1)^{-1} - 1\right) d\mathfrak{m}\right]$$

We deal with the two addends separately. For the former, since the function $f^{n,t}$ solves (3.20), $t \geq C/n$ and $|\log t|/\log n \to 1$ as $n \to \infty$, Lemma 3.16 gives

$$\lim_{n \to \infty} \frac{1}{\log n} \mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, \mathrm{d}\mathfrak{m}\right] = \frac{1}{4\pi}.$$

For the latter, by Cauchy-Schwarz inequality we have

$$\limsup_{n \to \infty} \frac{1}{\log n} \mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \left(M(u^{n,t,c}, 1)^{-1} - 1\right) \mathrm{d}\mathfrak{m}\right] \leq$$

$$\leq \left(\limsup_{n \to \infty} \frac{1}{\log n} \mathbb{E}\left[\int_D |\nabla f^{n,t}|^4 \, \mathrm{d}\mathfrak{m}\right]^{1/2}\right) \left(\lim_{n \to \infty} \mathbb{E}\left[\int_D \left(M(u^{n,t,c}, 1)^{-1} - 1\right)^2 \mathrm{d}\mathfrak{m}\right]^{1/2}\right)$$

which converges to 0 by Lemma 3.16 and Lemma 3.12.

Recalling (4.1), we deduce

$$\limsup_{n \to \infty} \frac{n}{\log n} \mathbb{E}[W_2^2(\mu^{n,t,c}, \mathfrak{m})] \leq \frac{1}{4\pi}.$$

In conclusion

$$\limsup_{n \to \infty} \frac{n}{\log n} \mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\right] \leq \frac{(1+\alpha)}{4\pi} + 2(1+\alpha^{-1})[C_{\mathrm{dr}} + (\mathrm{diam}\, D)^2]\gamma$$

and the thesis follows letting first $\gamma \to 0$ and then $\alpha \to 0$. $\square$

4.2 Lower bound

Theorem 4.5 (Lower bound, $d = 1$). *Assume that ultracontractivity holds with $d = 1$ and that $N_D(\delta) \leq \max\{1, C_D \delta^{-1}\}$ for every $\delta > 0$. Then*

$$\liminf_{n \to \infty} n \, \mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\right] \geq - \sum_{\lambda \in \sigma(\Delta) \setminus \{0\}} \frac{1}{\lambda}.$$

Proof. Fix $q \in (0,1)$, $\eta \in (0,1)$ and let $t = t(n) = \eta^{-1} n^{-2q}$. By Proposition 3.9 the complement of the event

$$A_\eta = A_{\eta,n} = \left\{\sup_{y \in D} \frac{|r^{n,t}(y)|}{\sqrt{n}} \leq \eta\right\} \tag{4.2}$$

has infinitesimal probability as $n \to \infty$. By the contractivity assumption we have

$$W_2^2(\mu^n, \mathfrak{m}) \geq e^{2Kt} W_2^2(\mu^{n,t}, \mathfrak{m}).$$

Therefore it is sufficient to estimate $\liminf_{n \to \infty} n \, \mathbb{E}[W_2^2(\mu^{n,t}, \mathfrak{m})\chi_{A_\eta}]$ from below. By duality,

$$\frac{1}{2} W_2^2(\mu^{n,t}, \mathfrak{m}) \geq \sup\left\{\int_D f \, \mathrm{d}\mu^{n,t} + \int_D g \, \mathrm{d}\mathfrak{m} \,\bigg|\, f(x) + g(y) \leq \frac{\mathsf{d}(x,y)^2}{2}\right\}$$

$$= \sup\left\{\int_D f \frac{\mathrm{d}r^{n,t}}{\sqrt{n}} + \int_D (f+g) \, \mathrm{d}\mathfrak{m} \,\bigg|\, f(x) + g(y) \leq \frac{\mathsf{d}(x,y)^2}{2}\right\}. \tag{4.3}$$

Let $f^{n,t}$ be the solution to (3.20) and define $f = -f^{n,t}/\sqrt{n}$, so that $\|\Delta f\|_\infty \le \eta$ in the event A_η, and we can estimate thanks to (3.3)

$$\|(\Delta f)^+\|_\infty + \frac{e^{2K^- t} - 1}{2} \|\nabla f\|_\infty^2 \le \omega(\eta) \tag{4.4}$$

with $\omega(\eta) \to 0$ as $\eta \to 0$. To this function f we associate the potential g given by Corollary 3.3, hence we get (still in the event A_η)

$$\begin{aligned}
\frac{1}{2} W_2^2(\mu^{n,\varepsilon}, \mathfrak{m}) &\ge \int_D (f + g)\, \mathrm{d}\mathfrak{m} + \int_D f \frac{\mathrm{d} r^{n,t}}{\sqrt{n}} \\
&\ge -e^{\omega(\eta)} \int_D \frac{|\nabla f|^2}{2}\, \mathrm{d}\mathfrak{m} - \int_D f \Delta f\, \mathrm{d}\mathfrak{m} = \\
&= \left(1 - \frac{e^{\omega(\eta)}}{2}\right) \frac{1}{n} \int_D |\nabla f^{n,t}|^2\, \mathrm{d}\mathfrak{m}.
\end{aligned}$$

Thus, by Lemma 3.15,

$$\begin{aligned}
\frac{1}{2 - e^{\omega(\eta)}} \liminf_{n\to\infty} n\, \mathbb{E}\left[W_2^2(\mu^{n,t}, \mathfrak{m}) \chi_{A_\eta} \right] &\ge \liminf_{n\to\infty} \mathbb{E}\left[\chi_{A_\eta} \int_D |\nabla f^{n,t}|^2\, \mathrm{d}\mathfrak{m} \right] \\
&\ge \lim_{n\to\infty} \mathbb{E}\left[\int_D |\nabla f^{n,t}|^2\, \mathrm{d}\mathfrak{m} \right] - \limsup_{n\to\infty} \mathbb{E}\left[\chi_{A_\eta^c} \int_D |\nabla f^{n,t}|^2\, \mathrm{d}\mathfrak{m} \right] \\
&= -\sum_{\lambda \in \sigma(\Delta)\setminus\{0\}} \frac{1}{\lambda}
\end{aligned}$$

because

$$\limsup_{n\to\infty} \mathbb{E}\left[\chi_{A_\eta^c} \int_D |\nabla f^{n,t}|^2\, \mathrm{d}\mathfrak{m} \right] \le \limsup_{n\to\infty} \mathbb{P}(A_{\eta,n}^c)^{1/2} \left(\mathbb{E}\left[\left(\int_D |\nabla f^{n,t}|^2\, \mathrm{d}\mathfrak{m} \right)^2 \right] \right)^{1/2} = 0$$

by Hölder inequality and (3.26). The thesis follows letting $\eta \to 0$. $\qquad\square$

Theorem 4.6 (Lower bound, $d = 2$). *Assume that ultracontractivity holds with $d = 2$ and that $N_D(\delta) \le C\delta^{-2}$ for every $\delta > 0$. Then*

$$\liminf_{n\to\infty} \mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m}) \right] \frac{n}{\log n} \ge \frac{1}{4\pi}.$$

Proof. By Proposition 3.10, for any $\eta \in (0,1)$ there is $\gamma > 0$ such that, if we let $t = t(n) = \gamma n^{-1} \log n$, the event A_η in (4.2) satisfies $\mathbb{P}(A_\eta^c) \le C/n$, for n large enough and some $C > 0$ independent of n. As in the previous proof, thanks to contractivity it is sufficient to estimate from below

$$\liminf_{n\to\infty} \frac{n}{\log n} \mathbb{E}\left[W_2^2(\mu^{n,t}, \mathfrak{m}) \chi_{A_\eta} \right].$$

Let $f^{n,t}$ be the solution to (3.20) and define $f = -f^{n,t}/\sqrt{n}$, so that $\|\Delta f\|_\infty \le \eta$ in the event A_η. To this function f we associate the potential g given by Corollary 3.3, hence

thanks to the duality formula (4.3) we can estimate (in the event A_η)

$$
\begin{aligned}
\frac{1}{2} W_2^2(\mu^{n,t}, \mathfrak{m}) &\geq \int_D (f+g) \, d\mathfrak{m} + \int_D f \frac{dr^{n,t}}{\sqrt{n}} \\
&\geq -e^{\omega(\eta)} \int_D \frac{|\nabla f|^2}{2} \, d\mathfrak{m} - \int_D f \Delta f \, d\mathfrak{m} \\
&= \left(1 - \frac{e^{\omega(\eta)}}{2}\right) \frac{1}{n} \int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}
\end{aligned}
$$

with $\omega(\eta)$ as in (4.4). Since $t \geq C/n$ for some positive constant C and $|\log t|/\log n \to 1$, from Lemma 3.16 we get

$$
\begin{aligned}
\frac{1}{2 - e^{\omega(\eta)}} \liminf_{n\to\infty} \frac{n}{\log n} \mathbb{E}\left[W_2^2(\mu^{n,t}, \mathfrak{m}) \chi_{A_\eta}\right] &\geq \liminf_{n\to\infty} \mathbb{E}\left[\chi_A \int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right] \frac{1}{\log n} \\
&\geq \lim_{n\to\infty} \mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right] \frac{1}{\log n} - \limsup_{n\to\infty} \mathbb{E}\left[\chi_{A^c} \int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right] \frac{1}{\log n} \\
&= \frac{1}{4\pi}
\end{aligned}
$$

because

$$
\limsup_{n\to\infty} \mathbb{E}\left[\chi_{A_\eta^c} \int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right] \frac{1}{\log n} \leq
$$

$$
\leq \limsup_{n\to\infty} \mathbb{P}(A_\eta^c)^{1/2} \left(\mathbb{E}\left[\int_D |\nabla f^{n,t}|^4 \, d\mathfrak{m}\right] \frac{1}{(\log n)^2}\right)^{1/2} = 0
$$

by Hölder inequality and (3.28). In conclusion we have

$$
\liminf_{n\to\infty} \frac{n}{\log n} \mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\right] \geq \frac{1}{4\pi}
$$

and the thesis follows letting $\eta \to 0$. $\qquad\square$

Remark 4.7. In this remark we sketch the proof of the concentration property (1.5), see also [L01], [BL16] for more details and references. Notice also that the concentration property, when written in terms of deviation from the expectation, holds independently of the convergence of the renormalized expectations as $n \to \infty$.

It is well-known since the seminal paper [GM83] that the spectral gap assumption is stable under tensorization and implies exponential concentration, more precisely

$$
\alpha_{\mathfrak{m}^n}(r) \leq C e^{-2\sqrt{C_{\mathrm{sg}}} r},
$$

with C numerical constant. Here $\alpha_{\mathfrak{m}^n}$ is the supremum of $1 - \mathfrak{m}^n(A^r)$ among all Borel sets A with $\mathfrak{m}^n(A) \geq 1/2$, where A^r is the open r neighbourhood of A. Now, the basic observation is that the mapping L_n associating to $(x_1, \ldots, x_n) \in D^n$ the empirical measure is $n^{-1/2}$-Lipschitz, if in the target space $\mathscr{P}(D)$ we use the distance W_2. By looking at the inclusion $L_n^{-1}(A^r) \supset \left(L_n^{-1}(A)\right)^{rn^{1/2}}$ between enlargements, this gives that the concentration function α_{Q^n} of $(L_n)_\# \mathfrak{m}^n$ satisfies

$$
\alpha_{Q^n}(r) \leq C e^{-2\sqrt{C_{\mathrm{sg}} n} r}.
$$

By standard arguments (passing through the deviation from the median), this exponential concentration leads to the inequality

$$Q^n(\{\nu : |G(\nu) - \mathbb{E}[G]| > r\} \leq C'e^{-2\sqrt{C_{sg}nr}}.$$

for some new numerical constant C', for any 1-Lipschitz function $G : \mathscr{P}(D) \to \mathbb{R}$. Using $G(\nu) = W_2(\nu, \mathfrak{m})$ and Cavalieri's formula we then get that $Z_n = W_2(\mu^n, \mathfrak{m})$ satisfy

$$\mathbb{E}[|Z_n - \mathbb{E}[Z_n]|^2] \leq \frac{C'}{2C_{sg}n} \int_0^\infty se^{-s} \, \mathrm{d}s,$$

so that

$$\mathbb{E}\left[\left|\sqrt{\frac{n}{\log n}} Z_n - \sqrt{\frac{n}{\log n}} \mathbb{E}[Z_n]\right|^2\right] \leq \frac{C'}{2C_{sg}\log n} \int_0^\infty se^{-s} \, \mathrm{d}s \qquad (4.5)$$

and, in particular $n(\log n)^{-1}(\mathbb{E}[Z_n^2] - \mathbb{E}[Z_n]^2) \to 0$. Since $n(\log n)^{-1}\mathbb{E}[Z_n^2] \to (4\pi)^{-1}$ we obtain that $n(\log n)^{-1}\mathbb{E}[Z_n]^2 \to (4\pi)^{-1}$, hence (4.5) gives

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\sqrt{\frac{n}{\log n}} Z_n - \frac{1}{\sqrt{4\pi}}\right|^2\right] = 0,$$

which immediately gives (1.5).

Under the stronger assumption that $\mathfrak{m}$ satisfies the dimension-free Gaussian concentration property, namely

$$\alpha_{\mathfrak{m}^n}(r) \leq Ce^{-cr^2}$$

with c, $C > 0$, the concentration estimate (4.5) can be improved to $O((\log n)^{-2})$. The dimension-free Gaussian concentration property is implied by the logarithmic Sobolev inequality, and the latter holds for compact Riemannian manifolds without boundary and with a positive lower bound on Ricci curvature (see for instance [L01]), as the 2-dimensional sphere.

4.3 The bipartite case

We prove now the bipartite part of Theorem 1.1. It will be convenient to introduce a notation $(\Omega, \mathbb{P})$ for the underlying probability space.

Lemma 4.8. *Let $D \subset \mathbb{R}^d$ be a compact set and assume that $\mathfrak{m} \in \mathscr{P}(D)$ is absolutely continuous w.r.t. the Lebesgue measure. The $L^2(D, \mathfrak{m}; D)$-valued maps*

$$\Omega \ni \omega \mapsto T^{\mu^n(\omega)}, \quad \Omega \ni \omega \mapsto T^{\nu^n(\omega)}$$

providing the optimal maps from $\mathfrak{m}$ to $\mu^n(\omega)$ and $\nu^n(\omega)$ are measurable and independent.

Proof. The independence of (X_i, Y_i) easily implies that the two measure-valued random variables $\mu^n(\omega)$, $\nu^n(\omega)$ are measurable and independent, where in $\mathscr{P}(D)$ we consider the Borel σ-algebra induced by the topology of weak convergence in duality with $C(D)$. Now, recalling Proposition 2.1, since independence is stable under composition with continuous functions the statement follows. $\square$

Proposition 4.9. *Let $D \subset \mathbb{R}^d$ be a bounded domain. For all $n \geq 1$ one has*

$$\mathbb{E}\left[W_2^2(\mu^n, \nu^n)\right] = 2\,\mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\right] - 2\int_D \left|\mathbb{E}\left[T^{\mu^n}(x) - x\right]\right|^2 \mathrm{d}\mathfrak{m}(x). \qquad (4.6)$$

Proof. If $S, T : \Omega \to L^2(D, \mathfrak{m}; \mathbb{R}^d)$ are independent, one has the identity

$$\mathbb{E}\left[\int_D \langle S^\omega(x), T^\omega(x)\rangle \,\mathrm{d}\mathfrak{m}(x)\right] = \int_D \langle \mathbb{E}\left[S^\omega(x)\right], \mathbb{E}\left[T^\omega(x)\right]\rangle \,\mathrm{d}\mathfrak{m}(x). \qquad (4.7)$$

We sketch the argument of the proof: if $S = \lambda e$, $T = \lambda' e'$, with $\lambda, \lambda' : \Omega \to \mathbb{R}^d$ and e, e' orthogonal unit vectors of $L^2(D, \mathfrak{m})$, then λ and λ' are independent and (4.7) reduces to $\mathbb{E}[\langle \lambda, \lambda'\rangle] = \langle \mathbb{E}[\lambda], \mathbb{E}[\lambda']\rangle$. By bilinearity, (4.7) still holds if S and T take their values in the vector space generated on $\mathbb{R}^d$ by a finite orthonormal set $\{e_1, \ldots, e_k\}$ of $L^2(D, \mathfrak{m})$. By a standard projection argument, and by approximation, we recover the general result.

For all $\omega \in \Omega$ the plan $(T^{\mu^n(\omega)}, T^{\nu^n(\omega)})_{\#}\mathfrak{m}$ is a coupling between $\mu^n(\omega)$ and $\nu^n(\omega)$. Hence (omitting for simplicity the dependence on ω) and using (4.7) with $S = T^{\mu_n}$, $T = T^{\nu^n}$ one has

$$\mathbb{E}\left[W_2^2(\mu^n, \nu^n)\right] \leq \mathbb{E}\left[\int_D \left|T^{\mu^n} - T^{\nu^n}\right|^2 \mathrm{d}\mathfrak{m}\right]$$

$$= \mathbb{E}\left[\int_D \left(\left|T^{\mu^n}(x) - x\right|^2 + \left|T^{\nu^n}(x) - x\right|^2 - 2\langle T^{\mu^n}(x) - x, T^{\nu^n}(x) - x\rangle\right) \mathrm{d}\mathfrak{m}(x)\right]$$

$$= 2\,\mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\right] - 2\,\mathbb{E}\left[\int_D \langle T^{\mu^n}(x) - x, T^{\nu^n}(x) - x\rangle \,\mathrm{d}\mathfrak{m}(x)\right]$$

$$= 2\,\mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\right] - 2\int_D \left|\mathbb{E}\left[T^{\mu^n}(x) - x\right]\right|^2 \mathrm{d}\mathfrak{m}(x),$$

where we used that $\mathbb{E}\left[W_2^2(\mu^n, \mathfrak{m})\right] = \mathbb{E}\left[W_2^2(\nu^n, \mathfrak{m})\right]$ since μ^n and ν^n have the same law. $\qquad\square$

In particular, combining the inequality in (4.6) (neglecting for a moment the negative term in the right hand side) with the first part of Theorem 1.1, we obtain

$$\limsup_{n\to\infty} \frac{n}{\log n}\,\mathbb{E}\left[W_{2,\mathbb{T}^2}^2(\mu^n, \nu^n)\right] \leq \limsup_{n\to\infty} \frac{n}{\log n}\,\mathbb{E}\left[W_{2,[0,1]^2}^2(\mu^n, \nu^n)\right] \leq \frac{1}{2\pi}. \qquad (4.8)$$

Next, we deal with lower bounds. It will be sufficient, by a comparison argument, to provide the lower bound only in the flat torus.

Proposition 4.10. *Let $\mathrm{D} = \mathbb{T}^2$. Then*

$$\liminf_{n\to\infty} \frac{n}{\log n}\,\mathbb{E}\left[W_2^2(\mu^n, \nu^n)\right] \geq \frac{1}{2\pi}. \qquad (4.9)$$

Proof. Similarly to the proof of Theorem 4.6, for $\eta \in (0, 1)$ we introduce the event

$$A_\eta = A_{\eta,n} = \left\{\sup_{y\in D} \frac{|r^{n,t}(y)|}{\sqrt{n}} \leq \frac{\eta}{2},\ \sup_{y\in D} \frac{|s^{n,t}(y)|}{\sqrt{n}} \leq \frac{\eta}{2}\right\},$$

whose probability tends to 1 as $n \to \infty$. By the contractivity assumption in W_2 we have $W_2^2(\mu^n, \nu^n) \geq e^{2Kt} W_2^2(\mu^{n,t}, \nu^{n,t})$, therefore it is sufficient to study the asymptotic behaviour of

$$\mathbb{E}\left[W_2^2(\mu^{n,t}, \nu^{n,t})\chi_{A_\eta}\right].$$

To this end, we let $f^{n,t}$ be the solution to (3.20), $g^{n,t}$ the solution to the same equation with $s^{n,t}$ in place of $r^{n,t}$ and $h^{n,t} = f^{n,t} - g^{n,t}$. Define $h = -h^{n,t}/\sqrt{n}$, so that $\Delta h = -(r^{n,t} - s^{n,t})/\sqrt{n}$ and $\|\Delta h\|_\infty \leq \eta$ in the event A_η. To this function h we associate the potential k given by Corollary 3.3, hence we can estimate (in the event A_η, with $\omega(\eta)$ defined as in (4.4) with f replaced by h)

$$\begin{aligned}
\frac{1}{2}W_2^2(\mu^{n,t}, \nu^{n,t}) &\geq \int_D h \, d\mu^{n,t} + \int_D k \, d\nu^{n,t} \\
&= \int_D (h+k) \, d\mathfrak{m} + \int_D h \frac{d(r^{n,t} - s^{n,t})}{\sqrt{n}} + \int_D (h+k)\frac{ds^{n,t}}{\sqrt{n}} \\
&\geq -e^{\omega(\eta)} \int_D \frac{|\nabla h|^2}{2} \, d\mathfrak{m} + \int_D |\nabla h|^2 \, d\mathfrak{m} + \int_D (h+k)\frac{ds^{n,t}}{\sqrt{n}} \\
&\geq \left(1 - \frac{e^{\omega(\eta)}}{2}\right)\frac{1}{n}\int_D |\nabla h^{n,t}|^2 \, d\mathfrak{m} - \left|\int_D (h+k)\frac{ds^{n,t}}{\sqrt{n}}\right|.
\end{aligned}$$

Since $h + k \leq 0$, we have

$$\left|\int_D (h+k)\frac{ds^{n,t}}{\sqrt{n}}\right| \leq -\eta\int_D (h+k) \, d\mathfrak{m} \leq \eta e^{\omega(\eta)}\int_D \frac{|\nabla h|^2}{2} \, d\mathfrak{m},$$

therefore, still in the event A_η,

$$\frac{1}{2}W_2^2(\mu^{n,t}, \nu^{n,t}) \geq \left(1 - (1+\eta)\frac{e^{\omega(\eta)}}{2}\right)\frac{1}{n}\int_D |\nabla h^{n,t}|^2 \, d\mathfrak{m}.$$

The proof now concludes as before, noticing that, by independence of μ^n and ν^n,

$$\begin{aligned}
&\mathbb{E}\left[\int_D |\nabla h^{n,t}|^2 \, d\mathfrak{m}\right] \\
&= \mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right] + \mathbb{E}\left[\int_D |\nabla g^{n,t}|^2 \, d\mathfrak{m}\right] + 2\mathbb{E}\left[\int_D \langle \nabla f^{n,t}, \nabla g^{n,t}\rangle \, d\mathfrak{m}\right] \\
&= 2\mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right] - 2\mathbb{E}\left[\int_D f^{n,t}\Delta g^{n,t} \, d\mathfrak{m}\right] \\
&= 2\mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right] - 2\int_D \mathbb{E}[f^{n,t}]\,\mathbb{E}[s^{n,t}] \, d\mathfrak{m} = 2\mathbb{E}\left[\int_D |\nabla f^{n,t}|^2 \, d\mathfrak{m}\right]
\end{aligned}$$

and

$$\mathbb{E}\left[\int_D |\nabla h^{n,t}|^4 \, d\mathfrak{m}\right] \leq 16\,\mathbb{E}\left[\int_D |\nabla f^{n,t}|^4 \, d\mathfrak{m}\right]. \qquad \square$$

From the previous result we get

$$\liminf_{n\to\infty}\frac{n}{\log n}\mathbb{E}\left[W_{2,[0,1]^2}^2(\mu^n, \nu^n)\right] \geq \liminf_{n\to\infty}\frac{n}{\log n}\mathbb{E}\left[W_{2,\mathbb{T}^2}^2(\mu^n, \nu^n)\right] \geq \frac{1}{2\pi}$$

which, combined with (4.8), concludes the proof (1.3). By looking at (4.6) we see also that (1.4) holds, and this concludes the proof of Theorem 1.1.

5 A new proof of the AKL lower bound

In this section we see how a minor modification of the ansatz of [CLPS14] provides a new proof of the lower bound in [AKT84], written in terms of expectations; the upper bound follows immediately from Theorem 1.1 and Hölder inequality.

The following real analysis lemma is well known, we state it for the case of the flat torus. Its proof (see for instance [AF84]) can be obtained by considering the sublevel sets of the maximal function of $|\nabla h|$.

Lemma 5.1 (Lusin approximation of Sobolev functions). *For all $p > 1$, $h \in H^{1,p}(\mathbb{T}^d)$ and all $\lambda > 0$ there exists a λ-Lipschitz function $\phi : \mathbb{T}^d \to \mathbb{R}$ with*

$$\mathfrak{m}(\{h \neq \phi\}) \leq \frac{C(d,p)}{\lambda^p} \int_{\mathbb{T}^d} |\nabla h|^p \, \mathrm{d}\mathfrak{m}. \tag{5.1}$$

Theorem 5.2. *If $D = \mathbb{T}^2$ one has*

$$\liminf_{n \to \infty} \sqrt{\frac{n}{\log n}} \, \mathbb{E}\left[W_1(\mu^n, \nu^n)\right] > 0.$$

By the triangle inequality, the same holds for the matching to the reference measure.

Proof. As in the proof of the lower bound for $p = 2$ we can use contractivity, reducing ourselves to estimating from below the Wasserstein distance between the regularized measures $\mu^{n,t} = u^{n,t}\mathfrak{m}$, $\nu^{n,t} = v^{n,t}\mathfrak{m}$. Let $M > 0$ be fixed and set $c(n) = M\sqrt{n^{-1}\log n}$. Let $t = t(n) = \gamma n^{-1}\log n$ with γ sufficiently large and let $h^{n,t}$ be as in the proof of the lower bound in the case $p = 2$, so that $h = h^{n,t}/\sqrt{n}$ satisfies

$$\lim_{n \to \infty} \frac{n}{\log n} \, \mathbb{E}\left[\int_D |\nabla h|^2 \, \mathrm{d}\mathfrak{m}\right] = \frac{1}{2\pi}. \tag{5.2}$$

Denote by ϕ the $c(n)$-Lipschitz function provided by Lemma 5.1. We denote by E_n the set $\{h \neq \phi\}$ and from (5.1) and (5.2) we obtain the estimates

$$\mathbb{E}\left[\mathfrak{m}(E_n)\right] \leq \frac{C(2,2)}{c(n)^2} \, \mathbb{E}\left[\int_D |\nabla h|^2 \, \mathrm{d}\mathfrak{m}\right] \leq \frac{2C(2,2)}{2\pi M^2}$$

for n large enough, so that we can use Hölder inequality and (3.28) to get, for some positive constant $C > 0$,

$$\mathbb{E}\left[\int_{E_n} |\nabla h|^2 \, \mathrm{d}\mathfrak{m}\right] \leq \frac{C}{M} \frac{\log n}{n} \tag{5.3}$$

for n large enough. Another application of Hölder's inequality yields

$$\mathbb{E}\left[\int_{E_n} |\nabla h| \, \mathrm{d}\mathfrak{m}\right] \leq \mathbb{E}\left[\int_{E_n} |\nabla h|^2 \, \mathrm{d}\mathfrak{m}\right]^{1/2} \mathbb{E}\left[\mathfrak{m}(E_n)\right]^{1/2} \leq \frac{C}{M^{3/2}} \sqrt{\frac{\log n}{n}} \tag{5.4}$$

again for some positive constant $C > 0$ (possibly larger than the one in (5.3)).

From Kantorovich's duality formula we get

$$c(n)W_1(\mu^{n,t}, \nu^{n,t}) \geq \left|\int_D \phi(u^{n,t} - v^{n,t}) \, \mathrm{d}\mathfrak{m}\right| = \left|\int_D \langle \nabla h, \nabla \phi \rangle \, \mathrm{d}\mathfrak{m}\right|,$$

where we used the PDE $\Delta h = u^{n,t} - v^{n,t}$ solved by h. Now we estimate

$$W_1(\mu^{n,t}, \nu^{n,t}) \geq \frac{1}{c(n)} \int_D |\nabla h|^2 \, d\mathfrak{m} - \frac{1}{c(n)} \left| \int_{E_n} \langle \nabla h, \nabla h - \nabla \phi \rangle \, d\mathfrak{m} \right|.$$

By (5.2), the first term is asymptotic to $(2\pi M)^{-1} \sqrt{n^{-1} \log n}$. We will see that, for M sufficiently large, the first term dominates the second one. Indeed, we have

$$\frac{1}{c(n)} \left| \int_{E_n} \langle \nabla h, \nabla h - \nabla \phi \rangle \, d\mathfrak{m} \right| \leq \frac{1}{c(n)} \left[\int_{E_n} |\nabla h|^2 + |\nabla h| c(n) \, d\mathfrak{m} \right]$$

Taking expectation, using (5.3) and (5.4) we have the inequality, for n sufficiently large,

$$\frac{1}{c(n)} \mathbb{E} \left[\int_{E_n} |\nabla h|^2 + |\nabla h| c(n) \, d\mathfrak{m} \right] \leq C \left(\frac{1}{M^2} + \frac{1}{M^{3/2}} \right) \sqrt{\frac{\log n}{n}}. \qquad \square$$

6 Open problems and extensions

In this section we discuss open problems, the present limitations of our technique, and some potential generalizations.

Improvements in the case $p = d = 2$. In this case, the more demanding prediction of [CLPS14] is

$$\lim_{n \to \infty} \left(\frac{n}{\log n} \mathbb{E} \left[W_2^2(\mu^n, \nu^n) \right] - \frac{1}{2\pi} \right) \log n \in \mathbb{R}.$$

This is still open, in this connection notice also that our technique for the lower bound requires $t = \gamma n^{-1} \log n$ with γ sufficiently large, while necessarily in the upper bound one is forced to take $t = \gamma n^{-1} \log n$ with γ small. Other open problems regard the distribution of the random variables $\frac{n}{\log n} W_2^2(\mu^n, \nu^n)$ and the matching problem involving more general reference measures $\mathfrak{m}$ (the Gaussian case could be interesting, replacing the heat semigroup with the Ornstein-Uhlenbeck semigroup).

Different powers and dimensions. Our proof in the case $d = 2$ exploits the extra room given by the logarithmic correction to the "natural" scale $n^{-1/d}$. Let us discuss the difficulties coming from $p \neq 2$ and $d > 2$ separately, of course the problem is even more challenging if both things happen.

If $d = 2$ and $p = 1$, we have already seen in Section 5 that the proof can be adapted to obtain the tight lower bound of [AKT84]. Via Hölder's inequality, one obtains the tight upper and lower bounds also for $1 < p < 2$, and we believe that also the case $p > 2$ could be covered, by estimating $\mathbb{E}\left[|\nabla f^{n,t}|^k \right]$ with k large integer (we did this for $k = 2, 4$). On the other hand, proving convergence of the renormalized expectations seems to require a more precise scheme, since the gradients of solutions to the Monge-Ampère equation describe the optimal transport map T only when $p = 2$; in this vein, one could consider (see [AGS08, Theorem 6.2.4]) the linearizations of

$$T = \mathrm{Id} - |\nabla \varphi|^{\frac{2-p}{p-1}} \nabla \varphi, \qquad \rho_1(T) \det(\nabla T) = \rho_0.$$

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In the case $p = 1$, an alternative PDE possibility could be given by the construction of the transport density via a q-laplacian approximation in [EG99], $q \to \infty$, which led to the first rigorous proof of the optimal transport map for Monge's problem.

If $p = 2$ and $d > 2$, the prediction of [CLPS14] is that

$$n^{2/d} \, \mathbb{E}\left[W_2^2(\mu^n, \nu^n)\right] - c_d = \frac{\xi}{2\pi^2} n^{-1+2/d} + o\left(n^{-1+2/d}\right),$$

where c_d is not conjectured and the coefficient ξ is explicitly given in terms of the Epstein function. However, our regularization technique seems to fail, at least for the purpose of computing c_d (namely proving convergence of the renormalized expectations). For instance, from (3.22) we get $\mathbb{E}\left[|\nabla f^{n,t}|^2\right] \sim t^{1-d/2}$, and therefore choosing $t = \gamma n^{-2/3}$ one is led to the estimate (even assuming that the factor due the logarithmic mean can be neglected) from above of $\mathbb{E}[W_2^2(\mu^n, \mathfrak{m})]$ with a term behaving like

$$C_{\mathrm{dr}} \gamma n^{-2/3} + C \gamma^{1-d/2} n^{-2/3}.$$

Therefore, it is not clear how to let $\gamma \to 0$ to eliminate the cost due to the short time regularization.

A class of abstract metric measure spaces. We already noticed that in our proof the geometry of the domain enters only through the properties of the heat semigroup P_t with homogeneous Neumann boundary conditions. As a matter of fact, let us briefly indicate how our proof works, still in the case $N = 2$, for the class $RCD^*(K, N)$ of "Riemannian" metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$, extensively studied and characterized in [AGS15], [AMS15], [EKS15]. This class of possibly nonsmooth metric measure spaces, includes for instance all compact Riemannian manifolds without boundary, or "convex" manifolds with boundary, namely manifolds having the property that geodesics between any two points do not touch the boundary (as it happens for compact convex domains in $\mathbb{R}^d$). The class $RCD^*(K, N)$ can be characterized either in terms of suitable K-convexity properties w.r.t. W_2-geodesics (of the logarithmic entropy for $N = \infty$ [AGS15], of power entropy [EKS15] or nonlinear diffusion semigroups [AMS15] in the case $N < \infty$), or in terms of Bochner's inequality, very much in the spirit of the Bakry-Émery theory (see [BGL14] for a nice introduction to the subject). In the very recent work [JLZ14], all regularizing properties of P_t needed for our proof to work have been proved in the context of $RCD^*(K, N)$ spaces. The only missing ingredient in this more abstract framework is the asymptotic expansion of the trace formula provided by Proposition 3.13, but thanks to (3.23) our results can be stated in terms of the limit

$$\lim_{t \to 0^+} \sum_{\lambda \in \sigma(\Delta) \setminus \{0\}} \frac{e^{2\lambda t}}{\lambda \log t}$$

whenever it exists.

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