INTRINSIC GEOMETRY AND DE GIORGI CLASSES
FOR CERTAIN ANISOTROPIC PROBLEMS

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ABSTRACT. We analyze a natural approach to the regularity of solutions of problems related to some anisotropic Laplacian operators, and a subsequent extension of the usual De Giorgi classes, by investigating the relation of the functions in such classes with the weak solutions to some anisotropic elliptic equations as well as with the quasi-minima of the corresponding functionals with anisotropic polynomial growth.

CONTENTS

1. **Introduction.** In this note we deal with nonlinear operators and functional classes modeled after the anisotropic Laplacian operator,

$$\Delta_{(p_i)} u := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \right), \quad 1 < p_1 \le p_2 \le \cdots \le p_n, \tag{1}$$

for any u in the corresponding anisotropic Sobolev spaces $W^{1,(p_i)}(\Omega)$, $\Omega \subset \mathbb{R}^n$ being a bounded domain. For the involved definitions and some basic properties, we immediately refer the reader to Section 2. Clearly, the operator in the display above is an extension of the usual pseudo p-Laplacian (by taking $p_i \equiv p$ for any $1 \le i \le n$) as well as the classical Laplacian operator (by taking $p_i \equiv 2$ for any $1 \le i \le n$).

Anisotropic operators appear in many problems in several branches of applied sciences; especially in Physics, when directional derivatives with distinct weights create distortions in the ambient space. During the last decades, the anisotropic Laplacian operator has been a classical topic in both Calculus of Variations and Partial Differential Equations analyses, in various problems when a nonstandard growth penalization in the exponents naturally arises. The involved literature is really too wide to attempt any reasonable comprehensive treatment in a single paper. However, for what concerns related aspects in regularity theory, it is worth mentioning the pioneering works by Marcellini (see [30, 31]), the subsequent higher integrability result by Leonetti (see [26]) and by Esposito, Leonetti and Mingione (see [19, 20]). Also, we refer to the paper [1] by Acerbi and Fusco, and to other recent contributions given in [7, 2, 27, 28, 3, 23, 25].

We analyze here a natural approach to the regularity of solutions of problems related to (1) and a subsequent extension of the usual De Giorgi classes for functions in the anisotropic Sobolev spaces; this note is inspired by the recent results in [29, 16] relying exactly on this approach, in turn borrowing the idea of the so-called intrinsic (parabolic) scaling proposed by DiBenedetto [11] in order to overcome the problematic anisotropic character of the problem.

As in the classic parabolic case involving the p-Laplace operator, one is lead to consider an "intrinsic" distance (and the corresponding intrinsic cubes). This means that the cubes have to be suitably re-parameterized along the coordinate directions, as well as the corresponding distances have to be appropriately dilated; the scaling parameter will be related with the solution itself. For instance, when considering Harnack-type inequalities, some computations suggest to stretch the cubes exactly of a factor $u(x_0)$ ($a = u(x_0)$ in the definition in (2) below), where u is the solution considered, and x_0 is the center point of the cube. This is exactly what happens for the parabolic p-Laplacian equation: we refer the reader to [12, 24], and also to [14, 15] for a related approach in a fractional setting.

According with this, we introduce the anisotropic De Giorgi classes $DG_{(p_i)}(\Omega)$, as in Definitions 3.1, 3.2 and 3.3 below, relate to this family of cubes: for $x_0 \in \Omega$ and $a > 0$, we set

$$Q_{x_0;\rho,a} = \left\{ x \in \mathbb{R}^n : |x_i - x_{0,i}| < a^{\frac{p_i - p_n}{p_i}} \rho^{\frac{p_n}{p_i}}, \ \forall\, i = 1, \dots, n \right\}. \tag{2}$$

Even if this is a more general class than standard Euclidean ball/cubes (note that the scaling parameter a can vary in Ω, as for instance mentioned above when $a = u(x_0)$; moreover, in such a case, the cubes do not derive from a metric), we find this family extremely natural for the problems considered. Note also that in order to prove

boundedness of solutions, it is sufficient to consider energy inequalities on standard balls, see [21].

Our definition of anisotropic De Giorgi classes is consistent with respect to the usual properties holding in the classical isotropic case when $p_i \equiv 2$, for any $i = 1, \ldots, n$. In particular, we will show the corresponding invariance by dilation and translation (see Remark 2) and we consider the homogeneous classes $DGO_{(p_i)}(\Omega)$; see Definitions 3.4, 3.5 and 3.6. As expected, we will prove that the weak solutions to the equation

$$- \Delta_{(p_i)} u = f \text{ in } \Omega, \text{ with } f \in L^m(\Omega), \tag{3}$$

for appropriate exponents m, belong to $DG_{(p_i)}(\Omega)$; see Theorem 3.7. Moreover, again in clear accordance with the classic isotropic case, we prove that the anisotropic De Giorgi classes contain the quasi-minima u of any energy functional $\mathcal{F}$ of the form

$$\mathcal{F}(u) = \int F(x, u, Du) \, \mathrm{d}x, \tag{4}$$

when the energy F satisfies p_i-growth assumptions; see Theorem 3.8.

We conclude by mentioning possible natural developments in this framework and some open questions. As far as we know, at the present the Hölder regularity of solutions to (3) in the general case is still an open problem (except for the special case where $p_1 = 2$, $p_i = p > 2$ for all $i > 1$; see [29]), as well as the validity of Harnack type inequalities and their possible independence.

As well-known since the breakthrough result of De Giorgi, the Caccioppoli estimates do contain all the information needed to prove basic regularity results; on the other hand, the proofs in [29, 16] rely on logarithmic estimates, allowing to "move informations" along the coordinate axes. More precisely, they allow – once known that the solution is bounded from below by a positive constant in an $(n-1)$-dimensional ball – to quantify the decay of the solution in the n-dimensional cylinder having the aforementioned ball as base. The drawback of this approach is that it is not clear how to adapt it to prove regularity of quasi-minima of (4), for instance, even in the simplest cases where Hölder continuity of solutions to (3) is now known.

A related issue is the quest for Harnack inequalities; here for instance the logarithmic estimate seems unavoidable – the proof without logarithmic estimate in [32] is unstable as $p \to 2$ and heavily relies on the parabolic structure. Related Harnack inequalities for quasi-minima and De Giorgi classes in a special (p, q)-case have been provided in [4, 5].

A better comprehension of the role of energy inequalities driving the definition of De Giorgi classes would allow to extend those possible results to a wider class of problems and to get a better understanding of these anisotropic operators.

2. **Notation.** Let $\{p_i\}_{i=1,\ldots,n} \subset \mathbb{R}$ such that $p_i > 1$ for any $i = 1, \ldots, n$. Without loss of generality, we can assume

$$1 < p_1 \leq \cdots \leq p_{n-1} \leq p_n. \tag{5}$$

It will be useful to denote by $\bar{p}$ the harmonic mean of the p_i's, that is,

$$\frac{1}{\bar{p}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}. \tag{6}$$

Throughout all the paper the set Ω will always denote a bounded domain in $\mathbb{R}^n$. Given a continuous function $u : \Omega \to \mathbb{R}$ and two real numbers $k, l \in \mathbb{R}$, we denote by

$$A_{k,S}^{(u)} := \{x \in S : u(x) > k\} \tag{7}$$

$$B_{l,S}^{(u)} := \{x \in S : u(x) < l\} = S \setminus \overline{A_{l,S}^{(u)}} \tag{8}$$

for some set $S \subseteq \Omega$, where, as usual, $\overline{S}$ denotes the closure of the set S, and $|S|$ is the measure of the set S. When no misunderstanding can occur, we will use the simpler notation $A_{k,S}$. When using $\inf_A f$ we shall always mean the essential infimum of f over A; pointwise values will be intended in the sense of precise representatives. For a real number $s \in \mathbb{R}$, s_+ is its positive part: $s_+ := \max\{s, 0\}$ while its negative part is defined as $s_- := (-s)_+$.

For any $\rho > 0$, let $Q_{x_0;\rho} = Q(x_0; \rho)$ be the cube in $\mathbb{R}^n$ of side 2ρ and center at x_0, whose sides are parallel to the coordinate axes, that is

$$Q_{x_0;\rho} = \big\{x \in \mathbb{R}^n : |x_i - x_{0,i}| < \rho, \ \forall\, i = 1, \ldots, n\big\}. \tag{9}$$

For any $\rho, a > 0$, the (anisotropic) parallelepiped in $\mathbb{R}^n$ $Q_{x_0;\rho,a} = Q(x_0; \rho, a)$, whose sides are parallel to the coordinate axes, are defined in (2). We note that, when $p_i = p$ for any $i = 1, \ldots, n$, the anisotropic parallelepiped $Q_{x_0;\rho,a}$ coincides with the cube $Q_{x_0;\rho}$; whereas $Q_{x_0;\rho,1} \neq Q_{x_0;\rho}$ if $p_i \neq p_j$ for some $i \neq j$. Also, for any $x_0 \in \mathbb{R}^n$ and $\rho > 0$,

$$|Q(x_0; \rho)| = 2^n \rho^n, \quad \text{and} \quad |Q(x_0; \rho, a)| = 2^n \rho^{\frac{n p_n}{\bar{p}}} a^{n - \frac{n p_n}{\bar{p}}}, \tag{10}$$

where $\bar{p}$ is defined by (6). When not important in the context, or when all the parallelepipeds have the same center, we shall simply denote $Q_{x_0;\rho} \equiv Q_\rho$ and $Q_{x_0;\rho,a} \equiv Q_{\rho,a}$. Finally, it is worth underlining some plain consequences of the definitions in (9) and (2). For any $a > 0$, $\rho_2 > \rho_1 > 0$,

$$Q_{\rho_1,a} \subset Q_{\rho_2,a};$$

for any ρ and $a_1 > a_2 > 0$,

$$Q_{\rho,a_1} \subset Q_{\rho,a_2};$$

for any $a > \rho$,

$$Q_{\rho,a} \subset Q_\rho.$$

Now, in connection with the anisotropic operators that we are considering, we need to recall the definitions of the anisotropic Sobolev spaces:

$$W_{\mathrm{loc}}^{1,(p_i)}(\Omega) = \left\{u \in W_{\mathrm{loc}}^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L_{\mathrm{loc}}^{p_i}(\Omega) \ \forall\, i = 1, \ldots, n\right\}$$

and

$$W_0^{1,(p_i)}(\Omega) = \left\{u \in W_0^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega) \ \forall\, i = 1, \ldots, n\right\}.$$

The space $W_0^{1,(p_i)}(\Omega)$ also denotes the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,(p_i)} = \sum_{i=1}^n \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}.$$

As expected, the Sobolev spaces defined above enjoy some natural embeddings, at least when the domain has some (strong) structural properties. For instance, we

are going to consider the rectangular case, so $\Omega = R = \prod_{i=1}^{n}(a_i - b_i)$. In the case $\bar{p} < n$, let

$$\bar{p}^* = \frac{\bar{p}n}{n - \bar{p}};$$

then

$$\begin{cases} \text{if} \quad p_n < \bar{p}^*, & W_0^{1,(p_i)}(R) \hookrightarrow L^{\bar{p}^*}(R), \\ \text{if} \quad p_n = \bar{p}^*, & W_0^{1,(p_i)}(R) \hookrightarrow L^q(R) \quad \text{for all } q \in [1, \bar{p}^*), \end{cases}$$

see for instance [17, 22]. If $\bar{p} = n$, then $W_0^{1,(p_i)}(R) \hookrightarrow L^q(R)$ for all $q \geq 1$. If $\bar{p} > n$, then $W_0^{1,(p_i)}(R) \hookrightarrow L^\infty(R)$. The following Sobolev type inequality also holds: there exists a universal constant $c = c(p_i, n)$ such that

$$\|u\|_{L^q(R)} \leq c \prod_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(R)}^{\frac{1}{n}} \tag{11}$$

$\forall q \in [1, \bar{p}^*], \forall q \in [1, \bar{p}^*), \forall q \in [1, \infty), q = \infty$, respectively.

We conclude this section by recalling a classic, useful lemma.

Lemma 2.1. *Let $\Phi(t)$ be a bounded nonnegative function in the interval $[\sigma, \rho]$. Assume that for $\sigma \leq t < s \leq \rho$ we have*

$$\Phi(t) \leq [Z_1(s - t)^{-\beta_1} + Z_2(s - t)^{-\beta_2} + Z_3] + \vartheta\Phi(s)$$

with $Z_i \geq 0$, $i = 1, 2, 3$, $\beta_1 > \beta_2 > 0$ and $0 \leq \vartheta < 1$. Then

$$\Phi(\sigma) \leq c(\beta_1, \vartheta)[Z_1(\rho - \sigma)^{-\beta_1} + Z_2(\rho - \sigma)^{-\beta_2} + Z_3].$$

In the following we will denote $\partial_i := \partial/\partial x_i$. Moreover we will follow the usual convention of denoting by c a general positive constant, the value of which may vary from line to line and depending on the data, that is they will be fixed in the assumptions we will make, as the dimension n, the set Ω, the exponents p_i, and so on. Relevant dependencies on parameters will be emphasized by using parentheses.

3. **Anisotropic De Giorgi classes and related problems.** This section is devoted to the definition of the anisotropic De Giorgi classes together with the investigation of some related properties. Let u be in $W_{\text{loc}}^{1,(p_i)}(\Omega)$ and $\epsilon, \chi \geq 0$.

Definition 3.1. We say that u belongs to *the anisotropic De Giorgi class $DG_{(p_i)}^{+}(\Omega)$* if there exists a constant C such that for every $Q_{\sigma,a} \subset Q_{\rho,a} \subset\subset \Omega$ and for every $k \in \mathbb{R}$

$$\sum_{i=1}^{n} \int_{A_{k,Q_{\sigma,a}}} |\partial_i u|^{p_i} \, \mathrm{d}x \leq C \left(\sum_{i=1}^{n} \frac{a^{p_n - p_i}}{(\rho^{\frac{p_n}{p_i}} - \sigma^{\frac{p_n}{p_i}})^{p_i}} \int_{A_{k,Q_{\rho,a}}} (u - k)^{p_i} \, \mathrm{d}x \right.$$
$$\left. + \left(\chi^{p_n} + |k|^{p_n} \rho^{-p_n} \right) |A_{k,Q_{\rho,a}}|^\epsilon \right), \quad (12)$$

with $A_{k,Q_{\cdot,a}}$ defined by (7) and $Q_{\cdot,a}$ by (2).

Definition 3.2. We say that u belongs to *the anisotropic De Giorgi class* $DG^-_{(p_i)}(\Omega)$ if there exists a constant C such that for every $Q_{\sigma,a} \subset Q_{\rho,a} \subset\subset \Omega$ and for every $k \in \mathbb{R}$

$$\sum_{i=1}^n \int_{B_{k,Q_{\sigma,a}}} |\partial_i u|^{p_i} \, \mathrm{d}x \le C \left(\sum_{i=1}^n \frac{a^{p_n - p_i}}{(\rho^{\frac{p_n}{p_i}} - \sigma^{\frac{p_n}{p_i}})^{p_i}} \int_{B_{k,Q_{\rho,a}}} (k-u)^{p_i} \, \mathrm{d}x \right.$$
$$\left. + \left(\chi^{p_n} + |k|^{p_n} \rho^{-p_n} \right) |B_{k,Q_{\rho,a}}|^\epsilon \right), \quad (13)$$

with $B_{k,Q_{\cdot,a}}$ defined by (8) and $Q_{\cdot,a}$ by (2); equivalently, if $-u \in DG^+_{(p_i)}(\Omega)$.

Now, we are in position to define *the anisotropic De Giorgi class* $DG_{(p_i)}(\Omega)$.

Definition 3.3. We say that u belongs to $DG_{(p_i)}(\Omega)$, if it satisfies both inequalities (12) and (13); that is, if

$$u \in DG^+_{(p_i)}(\Omega) \cap DG^-_{(p_i)}(\Omega).$$

Remark 1. In order to simplify the proofs in the rest of the paper, we note that one can easily get rid of the constant χ in the definitions above. Indeed, if we set $v = u + \chi\rho$ and $l = k + \chi\rho$ in (12) and (13), then, for every $Q_{\sigma,a} \subset Q_{\rho,a} \subset\subset \Omega$, we get respectively

$$\sum_{i=1}^n \int_{A^{(v)}_{l,Q_{\sigma,a}}} |\partial_i v|^{p_i} \, \mathrm{d}x \le C \left(\sum_{i=1}^n \frac{a^{p_n - p_i}}{(\rho^{\frac{p_n}{p_i}} - \sigma^{\frac{p_n}{p_i}})^{p_i}} \int_{A^{(v)}_{l,Q_{\rho,a}}} (v-l)^{p_i} \, \mathrm{d}x \right.$$
$$\left. + |l|^{p_n} \rho^{-p_n} |A^{(v)}_{l,Q_{\rho,a}}|^\epsilon \right), \quad (14)$$

and

$$\sum_{i=1}^n \int_{B^{(v)}_{l,Q_{\sigma,a}}} |\partial_i v|^{p_i} \, \mathrm{d}x \le C \left(\sum_{i=1}^n \frac{a^{p_n - p_i}}{(\rho^{\frac{p_n}{p_i}} - \sigma^{\frac{p_n}{p_i}})^{p_i}} \int_{B^{(v)}_{l,Q_{\rho,a}}} (l-v)^{p_i} \, \mathrm{d}x \right.$$
$$\left. + |l|^{p_n} \rho^{-p_n} |B^{(v)}_{l,Q_{\rho,a}}|^\epsilon \right). \quad (15)$$

Of course the relations above are valid for $l \in \mathbb{R}$, and C is the same constant that appears in (12) and (13).

Remark 2. The relations in Definitions 3.1–3.2 enjoy a useful *invariance by dilations and translations*, as it is shown in the example below.

For any $x \in Q_{x_0;\sigma,a} \subset\subset \Omega$, consider the variable $y = (y_1, y_2, \ldots, y_n)$ defined by

$$y_i = \sigma^{-\frac{p_n}{p_i}} M^{\frac{p_n - p_i}{p_i}} (x_i - x_{0,i}) \quad \forall i = 1, \ldots, n, \quad (16)$$

for $M > 0$. We claim that the function v defined by

$$v(y) = \frac{u(y)}{M},$$

for any y as in (16), satisfies the inequalities in (12)-(13) in $Q_{0;1,a/M} \subset Q_{0;\rho/\sigma,a/M}$, with the same constant C of the function u. Indeed,

$$\sum_{i=1}^{n} \int_{B_{k,Q_{0;1,\frac{a}{M}}}^{(v)}} |\partial_{y_i} v|^{p_i} \, \mathrm{d}y$$

$$= \sum_{i=1}^{n} \frac{1}{M^{p_i}} \cdot \frac{\sigma^{p_n}}{M^{p_n - p_i}} \int_{B_{k,Q_{0;1,\frac{a}{M}}}^{(v)}} \left| \partial_{x_i} u \left(\sigma^{-\frac{p_n}{p_i}} M^{\frac{p_n - p_i}{p_i}} (x - x_0) \right) \right|^{p_i} \, \mathrm{d}y$$

$$= \sum_{i=1}^{n} \frac{\sigma^{p_n}}{M^{p_n}} \cdot \frac{M^{(\frac{p_n}{\bar{p}}-1)n}}{\sigma^{\frac{np_n}{\bar{p}}}} \int_{B_{kM,Q_{x_0;\sigma,a}}^{(u)}} |\partial_i u|^{p_i} \, \mathrm{d}x, \tag{17}$$

where we used the chain rule and the change of variable formula.

Now we apply the inequality in (13) to u in $Q_{x_0;\sigma,a} \subset Q_{x_0;\rho,a}$. For the sake of simplicity, and also in view of Remark 1, we suppose that $\chi = 0$. We get

$$\sum_{i=1}^{n} \int_{B_{kM,Q_{x_0;\sigma,a}}^{(u)}} |\partial_i u|^{p_i} \, \mathrm{d}x$$

$$\leq C \sum_{i=1}^{n} \frac{a^{p_n - p_i}}{(\rho^{\frac{p_n}{p_i}} - \sigma^{\frac{p_n}{p_i}})^{p_i}} \int_{B_{kM,Q_{x_0;\rho,a}}^{(u)}} (kM - u)^{p_i} \, \mathrm{d}x$$

$$+ C |k\,M|^{p_n} \rho^{-p_n} \left| B_{kM,Q_{x_0;\rho,a}}^{(u)} \right|^{\epsilon}$$

$$= C \sum_{i=1}^{n} \frac{\left(\frac{a}{M}\right)^{p_n - p_i}}{\left(\left(\frac{\rho}{\sigma}\right)^{\frac{p_n}{p_i}} - 1\right)^{p_i}} \cdot \frac{M^{p_n}}{\sigma^{p_n}} \int_{B_{k,Q_{0;\frac{\rho}{\sigma},\frac{a}{M}}}^{(v)}} (k - v)^{p_i} \left(\frac{M^{(\frac{p_n}{\bar{p}}-1)n}}{\sigma^{\frac{np_n}{\bar{p}}}} \right)^{-1} \mathrm{d}y$$

$$+ C |k\,M|^{p_n} \left(\frac{M^{(\frac{p_n}{\bar{p}}-1)n}}{\sigma^{\frac{np_n}{\bar{p}}}} \right)^{-1} \rho^{-p_n} \left| B_{k,Q_{0;\frac{\rho}{\sigma},\frac{a}{M}}}^{(v)} \right|^{\epsilon},$$

where we used again the change of variable formula. Finally, by combining the previous inequality with (17), we obtain

$$\sum_{i=1}^{n} \int_{B_{k,Q_{0;1,\frac{a}{M}}}^{(v)}} |\partial_{y_i} v(y)|^{p_i} \, \mathrm{d}y$$

$$\leq C \left(\sum_{i=1}^{n} \frac{\left(\frac{a}{M}\right)^{p_n - p_i}}{\left(\left(\frac{\rho}{\sigma}\right)^{\frac{p_n}{p_i}} - 1\right)^{p_i}} \right.$$

$$\left. \times \int_{B_{k,Q_{0;\frac{\rho}{\sigma},\frac{a}{M}}}^{(v)}} (k - v)^{p_i} \, \mathrm{d}y + |k|^{p_n} \left(\frac{\rho}{\sigma} \right)^{-p_n} \left| B_{k,Q_{0;\frac{\rho}{\sigma},\frac{a}{M}}}^{(v)} \right|^{\epsilon} \right).$$

Similarly, one can show that the function v satisfies inequality (12).

In the same spirit of the previous definitions, one can consider to investigate the functions u that belong to the anisotropic analog of the *homogeneous De Giorgi classes*. Let $u \in W_{\text{loc}}^{1,(p_i)}(\Omega)$.

Definition 3.4. We say that u belongs to $DGO^+_{(p_i)}(\Omega)$ if there exists a constant C such that

$$\sum_{i=1}^{n} \int_{A_{k,Q_{\sigma,a}}} |\partial_i u|^{p_i} \, \mathrm{d}x \leq C \sum_{i=1}^{n} \frac{a^{p_n - p_i}}{(\rho^{\frac{p_n}{p_i}} - \sigma^{\frac{p_n}{p_i}})^{p_i}} \int_{A_{k,Q_{\rho,a}}} (u - k)^{p_i} \, \mathrm{d}x, \qquad (18)$$

for every $Q_{\sigma,a} \subset Q_{\rho,a} \subset\subset \Omega$, and for every $k \in \mathbb{R}$.

Definition 3.5. We say that $u \in DGO^-_{(p_i)}(\Omega)$ if there exists a constant C such that

$$\sum_{i=1}^{n} \int_{B_{k,Q_{\sigma,a}}} |\partial_i u|^{p_i} \, \mathrm{d}x \leq C \sum_{i=1}^{n} \frac{a^{p_n - p_i}}{(\rho^{\frac{p_n}{p_i}} - \sigma^{\frac{p_n}{p_i}})^{p_i}} \int_{B_{k,Q_{\rho,a}}} (k - u)^{p_i} \, \mathrm{d}x, \qquad (19)$$

for every $Q_{\sigma,a} \subset Q_{\rho,a} \subset\subset \Omega$, and for every $k \in \mathbb{R}$.

Definition 3.6. We say that $u \in DGO_{(p_i)}(\Omega)$ if both the relations (18) and (19) are satisfied.

Now we show that the classes $DG_{(p_i)}(\Omega)$ and $DGO_{(p_i)}(\Omega)$ contain the solutions to some anisotropic elliptic equations in divergence form and the quasi-minima of related energy functionals.

3.1. **Weak solutions to nonlinear Dirichlet equations.** We consider the equation

$$-\sum_{i=1}^{n} \partial_i \left(|\partial_i u|^{p_i - 2} \partial_i u \right) = f \quad \text{in } \Omega, \qquad (20)$$

where f is a given function such that

$$f \in \begin{cases} L^{n/\bar{p}} & \text{if } \bar{p} < n \text{ and } p_n < \bar{p}^*, \\ L^m \quad \forall m > n/\bar{p} & \text{if } \bar{p} < n \text{ and } p_n = \bar{p}^*, \\ L^m \quad \forall m > 1 & \text{if } \bar{p} = n, \\ L^1 & \text{if } \bar{p} > n, \end{cases} \qquad (21)$$

and $\bar{p}$ is defined by (6). If in addition Dirichlet boundary conditions are assumed, then there exists a unique weak solution u to (20); that is, a function $u \in W_0^{1,(p_i)}(\Omega)$ such that

$$\sum_{i=1}^{n} \int_{\Omega} |\partial_i u|^{p_i - 2} \partial_i u \, \partial_i \varphi \, \mathrm{d}x = \int_{\Omega} f \varphi \, \mathrm{d}x, \quad \forall \varphi \in W_0^{1,(p_i)}(\Omega),$$

(see [13, 6] for instance); moreover, $u \in L^\infty_{\mathrm{loc}}(\Omega)$. Also, it is worth mentioning that, in the case when we do not assume further boundary conditions, the boundedness of the solutions u to (20) is guaranteed for instance when $p_n < \bar{p}^*$ (i.e., the anisotropy is concentrated; see [17, 9]). As a matter of fact, in [31] it is shown that the equation

$$-\sum_{i=1}^{n} \partial_i \left[|\partial_i u|^{p_i - 2} \partial_i u \right] = 0 \quad \text{in } \Omega \qquad (22)$$

may have unbounded weak solutions in the case when $p_n > \bar{p}^*$. For related results, including the borderline case $p_n = \bar{p}^*$, see also the paper [10].

Let us just consider the first case in (21), that is when the datum f belongs to $L^{n/\bar{p}}$ with $\bar{p} < n$ and $p_n < \bar{p}^*$: the other ones can be treated similarly. One can

prove that the weak solution u to (20) satisfies the following energy estimates, for any $k \in \mathbb{R}$:

$$\sum_{i=1}^{n} \int_{\Omega} |\partial_i (u-k)_+|^{p_i} \xi \, dx \le c \left(\sum_{i=1}^{n} \int_{\Omega} (u-k)_+^{p_i} |\xi_i'|^{p_i} \, dx + \sum_{i=1}^{n} \|f\|_{L^{n/\bar{p}}(\Omega)}^{\frac{p_i}{p_i-1}} \right), \quad (23)$$

where

$$\xi = \prod_{i=1}^{n} \xi_i^{p_i}, \quad \text{with } 0 \le \xi_i = \xi(x_i) \le 1, \quad \forall \, i = 1, \ldots, n \quad (24)$$

is a piecewise smooth cut-off function in $Q_{\rho,a} \subset \Omega$, vanishing outside of the set $Q_{\rho,a}$. The estimates in (23) can be deduced by testing the weak formulation of the equation in (20) with the test functions $\varphi^+ = (u-k)_+\xi$. Notice that the support of φ^+ is contained in the parallelepiped $Q_{\rho,a}$. The only non completely trivial term is estimated as follows: noting that $n/\bar{p} = \bar{p}^*$ and using the Hölder inequality and then the Sobolev embedding in (11) and the inequality between geometric and arithmetic mean yields

$$\int_{\Omega} f(u-k)_+ \xi \, dx$$

$$\le \|f\|_{L^{n/\bar{p}}(\Omega)} \left(\int_{A_{k,Q_{\rho,a}}} (u-k)_+^{\bar{p}^*} \xi^{\bar{p}^*} \, dx \right)^{1/\bar{p}^*}$$

$$\le c(n,p_i) \|f\|_{L^{n/\bar{p}}(\Omega)} \prod_{i=1}^{n} \left\| \frac{\partial}{\partial x_i}((u-k)_+\xi) \right\|_{L^{p_i}(Q_{\rho,a})}^{\frac{1}{n}}$$

$$\le \frac{1}{2} \sum_{i=1}^{n} \int_{Q_{\rho,a}} |\partial_i(u-k)_+|^{p_i} \xi \, dx + c \sum_{i=1}^{n} \int_{Q_{\rho,a}} (u-k)_+^{p_i} |\xi_i'|^{p_i} \, dx$$

$$+ c \sum_{i=1}^{n} \|f\|_{L^{n/\bar{p}}(\Omega)}^{\frac{p_i}{p_i-1}};$$

we can then reabsorb the anisotropic Sobolev norm in the left-hand side.

In addition, the weak solution u satisfies

$$\sum_{i=1}^{n} \int_{\Omega} |\partial_i(u-k)_-|^{p_i} \xi \, dx \le c \left(\sum_{i=1}^{n} \int_{\Omega} (u-k)_-^{p_i} |\xi_i'|^{p_i} \, dx + \sum_{i=1}^{n} \|f\|_{L^{n/\bar{p}}(\Omega)}^{\frac{p_i}{p_i-1}} \right), \quad (25)$$

for all $k \in \mathbb{R}$, where $(u-k)_- = (k-u)_+$ and ξ is as before. The estimate in (25) can be as well deduced by testing the equation (20) with $\varphi^- = -(u-k)_-\xi$.

Notice that by choosing $\xi \in C_0^\infty(Q_{\rho,a})$ such that $\xi \equiv 1$ in $Q_{\sigma,a}$, $0 < \sigma < \rho$ and

$$|\xi_i'| \le \frac{c}{(\rho^{\frac{p_n}{p_i}} - \sigma^{\frac{p_n}{p_i}})a^{\frac{p_i-p_n}{p_i}}}, \quad \forall \, i = 1, \ldots, n, \quad (26)$$

the estimates (23) and (25) plainly become

$$\sum_{i=1}^{n} \int_{A_{k,Q_{\sigma,a}}} |\partial_i u|^{p_i} \, dx \le C \left(\sum_{i=1}^{n} \frac{a^{p_n-p_i}}{(\rho^{\frac{p_n}{p_i}} - \sigma^{\frac{p_n}{p_i}})^{p_i}} \int_{A_{k,Q_{\rho,a}}} (u-k)^{p_i} \, dx \right.$$

$$\left. + \sum_{i=1}^{n} \|f\|_{L^{n/\bar{p}}(\Omega)}^{\frac{p_i}{p_i-1}} \right), \quad (27)$$

and

$$\sum_{i=1}^{n} \int_{B_{k,Q_{\sigma,a}}} |\partial_i u|^{p_i} \, \mathrm{d}x \le C \left(\sum_{i=1}^{n} \frac{a^{p_n - p_i}}{(\rho^{\frac{p_n}{p_i}} - \sigma^{\frac{p_n}{p_i}})^{p_i}} \int_{B_{k,Q_{\rho,a}}} (k - u)^{p_i} \, \mathrm{d}x \right.$$
$$\left. + \sum_{i=1}^{n} \|f\|_{L^{n/\bar{p}}(\Omega)}^{\frac{p_i}{p_i - 1}} \right). \quad (28)$$

Therefore, we deduce the following

Theorem 3.7. *Let u be a weak solutions to problem* (20) *with f as in* (21). *Then $u \in DG_{(p_i)}(\Omega)$.*

Proof. The proof is immediate, because of the estimates in (27) and (28), which imply (12) and (13), respectively. Hence, we can conclude that $u \in DG_{(p_i)}(\Omega)$ with χ given by

$$\chi = \left(\sum_{i=1}^{n} \|f\|_{L^{n/\bar{p}}(\Omega)}^{\frac{p_i}{p_i - 1}} \right)^{\frac{1}{p_n}}, \qquad \epsilon = 0.$$

Modifications in the other cases are straightforward. □

Clearly, if u is the solution to the equation in (20) with $f \equiv 0$, then it plainly follows that u belongs to $DGO_{(p_i)}(\Omega)$.

Remark 3. One can see also that if u is a weak solution to (20) in $Q_{x_0;R,a}$, then the rescaled function

$$v(y) := \frac{u(y)}{M}, \qquad y_i := M^{\frac{p_n - p_i}{p_i}} \left(\frac{\rho}{R} \right)^{\frac{p_n}{p_i}} (x_i - x_{0,1}), \quad i = 1, \dots, n,$$

for some $M, \rho > 0$, is a solution to (20) with f replaced by $f M^{1-p_n} (\rho^{-1} R)^{p_n}$ in $Q_{0;\rho,a/M}$. Indeed,

$$\partial_{y_i} v(y) = M^{-\frac{p_n}{p_i}} \left(\frac{\rho}{R} \right)^{-\frac{p_n}{p_i}} \partial_{x_i} u(y),$$

$$|\partial_{y_i} v(y)|^{p_i - 2} \partial_{y_i} v(y) = M^{-p_n + \frac{p_n}{p_i}} \left(\frac{\rho}{R} \right)^{-p_n + \frac{p_n}{p_i}} |\partial_{x_i} u(y)|^{p_i - 2} \partial_{x_i} u(y),$$

$$\partial_{y_i} \left(|\partial_{y_i} v(y)|^{p_i - 2} \partial_{y_i} v(y) \right) = M^{-p_n + 1} \left(\frac{\rho}{R} \right)^{-p_n} \partial_{x_i} \left(|\partial_{x_i} u(y)|^{p_i - 2} \partial_{x_i} u(y) \right).$$

3.2. Quasi-minima of functionals of the Calculus of Variations. We prove here that the anisotropic De Giorgi classes contain the quasi-minima u of the energy functional $\mathcal{F}$ given by

$$\mathcal{F}(u) = \int_\Omega F(x, u, Du) \, \mathrm{d}x,$$

where $F = F(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty)$ is a Carathéodory function satisfying, for some constants $L > 0$ and $M \ge 0$,

$$\sum_{i=1}^{n} |\xi_i|^{p_i} - M \le F(x, s, \xi) \le L \sum_{i=1}^{n} |\xi_i|^{p_i} + M. \quad (29)$$

We recall that $u \in W^{1,(p_i)}_{\mathrm{loc}}(\Omega)$ is a *quasi-minimum* of the functional $\mathcal{F}$, with constant $Q \ge 1$, briefly a *Q-minimum*, if for every $v \in W^{1,(p_i)}_{\mathrm{loc}}(\Omega)$, with $K := \mathrm{supp}(u-v) \subset\subset \Omega$, we have

$$\mathcal{F}(u, K) \le Q\mathcal{F}(v, K). \quad (30)$$

Theorem 3.8. *Let u be a Q-minimum of the functional $\mathcal{F}(u) = \int_\Omega F(x, u, Du)\, dx$, where F satisfies* (29). *Then $u \in DG_{(p_i)}(\Omega)$.*

Proof. We extend to the anisotropic framework the proof of Theorem 2.1 in [18]. Consider $v = u - (u - k)_+ \xi$, where the cut-off functions $\xi \in C_0^\infty(Q_{\rho,a})$ are defined as in the previous section, satisfying in particular (24) and (26).

We want to use v as test function in (30). Noticing that $\operatorname{supp}(u - v) = A_{k,Q_{\rho,a}}$ and using the growth assumptions in (29), we obtain

$$\sum_{i=1}^n \int_{A_{k,Q_{\rho,a}}} |\partial_i u|^{p_i}\, dx \leq Q\left(\sum_{i=1}^n \int_{A_{k,Q_{\rho,a}}} |\partial_i v|^{p_i}\, dx + 2M|A_{k,Q_{\rho,a}}|\right).$$

For any $x \in A_{k,Q_{\rho,a}}$ we have $v = u(1 - \xi) + \xi k$. Hence, since for any $i = 1, \ldots, n$

$$\partial_i v = \partial_i u(1 - \xi) - (u - k)\partial_i \xi \text{ and } |\partial_i v|^{p_i} \leq c\big(|\partial_i u|^{p_i}(1 - \xi)^{p_i} + (u - k)^{p_i}|\xi|^{p_i}\big)$$

in $A_{k,Q_{\rho,a}}$, we get

$$\sum_{i=1}^n \int_{A_{k,Q_{\rho,a}}} |\partial_i u|^{p_i}\, dx \leq c\bigg(\sum_{i=1}^n \int_{A_{k,Q_{\rho,a}}} |\partial_i u|^{p_i}(1 - \xi)^{p_i}\, dx$$
$$+ \sum_{i=1}^n \frac{a^{p_n - p_i}}{(\rho^{\frac{p_n}{p_i}} - \sigma^{\frac{p_n}{p_i}})^{p_i}} \int_{A_{k,Q_{\rho,a}}} (u - k)^{p_i}\, dx + M|A_{k,Q_{\rho,a}}|\bigg).$$

Recalling that $\xi = 1$ in $Q_{\sigma,a}$, we arrive at

$$\sum_{i=1}^n \int_{A_{k,Q_{\sigma,a}}} |\partial_i u|^{p_i}\, dx \leq \tilde{c}\bigg(\sum_{i=1}^n \int_{A_{k,Q_{\rho,a}} \setminus A_{k,Q_{\sigma,a}}} |\partial_i u|^{p_i}\, dx$$
$$+ \sum_{i=1}^n \frac{a^{p_n - p_i}}{(\rho^{\frac{p_n}{p_i}} - \sigma^{\frac{p_n}{p_i}})^{p_i}} \int_{A_{k,Q_{\rho,a}}} (u - k)^{p_i}\, dx + M|A_{k,Q_{\rho,a}}|\bigg),$$

that implies, "filling the hole",

$$\sum_{i=1}^n \int_{A_{k,Q_{\sigma,a}}} |\partial_i u|^{p_i}\, dx \leq \frac{\tilde{c}}{\tilde{c} + 1} \sum_{i=1}^n \int_{A_{k,Q_{\rho,a}}} |\partial_i u|^{p_i}\, dx$$
$$+ c\bigg(\sum_{i=1}^n \frac{a^{p_n - p_i}}{(\rho^{\frac{p_n}{p_i}} - \sigma^{\frac{p_n}{p_i}})^{p_i}} \int_{A_{k,Q_{\rho,a}}} (u - k)^{p_i}\, dx + M|A_{k,Q_{\rho,a}}|\bigg).$$

At this point, we can apply Lemma 2.1 to get estimate (12) with $\chi = M^{\frac{1}{p_n}}$ and $\epsilon = 1$.

In order to obtain estimate (13), we can proceed in the same way as above, by using $v = u + (u - k)_- \xi$ instead of $v = u - (u - k)_+ \xi$, and thus a Q-minimum belongs to $DG_{(p_i)}(\Omega)$. Finally, we notice that if $M = 0$ then $u \in DGO_{(p_i)}(\Omega)$. $\square$

Remark 4. We stress that local boundedness of quasi-minima of general functionals with anisotropic growth conditions has been recently proven by Cupini, Marcellini and Mascolo in [9, 10].

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