A NOTE ON FRACTIONAL SUPERSOLUTIONS

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Abstract. We deal with a class of equations driven by nonlocal, possibly degenerate, integro-differential operators of differentiability order $s \in (0,1)$ and summability growth $p > 1$, whose model is the fractional p-Laplacian with measurable coefficients. We prove that the minimum of the corresponding weak supersolutions is a weak supersolution as well.

1. Introduction

In this note we are interested in a very general class of nonlinear nonlocal equations, which include as a particular case some fractional Laplacian-type equations; that is, those related to the operator $\mathcal{L}$ defined on suitable fractional Sobolev functions by

(1.1)

$$\mathcal{L}u(x) = P.\,V.\int_{\mathbb{R}^n} K(x,y)|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\,\mathrm{d}y, \qquad x \in \mathbb{R}^n.$$

The nonlinear nonlocal operator $\mathcal{L}$ in the display above is driven by its symmetric *kernel* $K : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$, which is a measurable function of differentiability order $s \in (0,1)$ and summability exponent $p > 1$,

$$\Lambda^{-1} \leq K(x,y)|x-y|^{n+sp} \leq \Lambda \quad \text{for a.e. } x, y \in \mathbb{R}^n,$$

for some $\Lambda \geq 1$. In order to simplify, one can just keep in mind the model case when the kernel $K = K(x,y)$ does coincide with the Gagliardo kernel $|x-y|^{-n-sp}$; that is, when the corresponding equation reduces to

$$(-\Delta)_p^s\, u = 0 \quad \text{in } \mathbb{R}^n,$$

where the symbol $(-\Delta)_p^s$ denotes the usual fractional p-Laplacian operator, though in such a case the difficulties arising from having merely measurable coefficients disappear.

We prove the following basic result which concerns the minimum of two fractional weak supersolutions, and constitutes a natural and fundamental result for approaching the development of a fractional nonlinear Potential Theory.

2010 *Mathematics Subject Classification.* Primary 35D10, 35B45; Secondary 35B05, 35R05, 47G20, 60J75.

Key words and phrases. Quasilinear nonlocal operators, fractional Sobolev spaces, fractional Laplacian, nonlocal tail, fractional superharmonic functions.

Theorem 1.1. *Suppose that u and v are fractional weak supersolutions in Ω. Then the function $w := \min\{u, v\}$ is a fractional weak supersolution in Ω as well.*

In contrast with respect to the classic local case of the p-Laplace equation (that is, when $s = 1$; see for instance Theorem 3.23 in [10]), here the proof that the function $w := \min\{u, v\}$ is a weak supersolution tangles up significantly in the nonlocality of the involved operators $\mathcal{L}$. Indeed, the main difficulty into the treatment of the operators $\mathcal{L}$ in (1.1) lies in their very definition, which combines the typical issues given by its *nonlocal* feature together with the ones given by its *nonlinear* growth behavior; also, further efforts are needed due to the presence of merely measurable coefficients in the kernel K. For this, some very important tools recently introduced in the nonlocal theory, as the by-now classic s-harmonic extension ([4]), the strong three-term commutators estimates to deduce the regularity of weak fractional harmonic maps ([5]), the pseudo-differential commutator compactness in [20, 21, 22], the energy density estimates in [23, 24], and many other successful tricks seem not to be trivially adaptable to the nonlinear framework considered here. Moreover, increased difficulties are due to the non-Hilbertian structure of the involved fractional Sobolev spaces $W^{s,p}$ when $p \neq 2$. In spite of that, some related regularity results have been very recently achieved in this context, in [9, 17, 18, 25, 15, 2, 1, 3] and many others, where often a fundamental role to understand the nonlocality of the nonlinear operators $\mathcal{L}$ has been played by the nonlocal tail defined by forthcoming formula (2.1) in order to obtain fine quantitative controls on the long-range interactions.

2. Preliminaries

In this section, we recall the definition of weak supersolutions to nonlinear integro-differential equations driven by the operator $\mathcal{L}$ in (1.1). For this, we need first to recall the definition of *the nonlocal tail* $\mathrm{Tail}(f; z, r)$ *of a function f in the ball of radius $r > 0$ centered in* $z \in \mathbb{R}^n$; see [6, 7]. For any function f initially defined in $L^{p-1}_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\mathrm{Tail}(f; z, r) := \left(r^{sp} \int_{\mathbb{R}^n \setminus B_r(z)} |f(x)|^{p-1} |x - z|^{-n-sp} \, \mathrm{d}x \right)^{\frac{1}{p-1}}. \tag{2.1}$$

In accordance, we recall the definition of the corresponding *tail space* $L^{p-1}_{sp}(\mathbb{R}^n)$,

$$L^{p-1}_{sp}(\mathbb{R}^n) := \left\{ f \in L^{p-1}_{\mathrm{loc}}(\mathbb{R}^n) \;:\; \mathrm{Tail}(f; z, r) < \infty \quad \forall z \in \mathbb{R}^n, \forall r \in (0, \infty) \right\};$$

see [15], and also [13, Section 2] for an equivalent definition and related properties. As expected, one can check that $L^\infty(\mathbb{R}^n) \subset L^{p-1}_{sp}(\mathbb{R}^n)$ and

$W^{s,p}(\mathbb{R}^n) \subset L^{p-1}_{sp}(\mathbb{R}^n)$, where we denoted by $W^{s,p}(\mathbb{R}^n)$ the usual fractional Sobolev space of order (s,p), defined by the norm

$$\|v\|_{W^{s,p}(\mathbb{R}^n)} := \|v\|_{L^p(\mathbb{R}^n)} + [v]_{W^{s,p}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |v|^p \,\mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}.$$

Similarly, one can define the fractional Sobolev spaces $W^{s,p}(\Omega)$ in a domain $\Omega \subset \mathbb{R}^n$. By $W^{s,p}_0(\Omega)$ we denote the closure of $C^\infty_0(\Omega)$ in $W^{s,p}(\mathbb{R}^n)$. For the basic properties of these spaces and some related topics, we refer the reader to [8] and the references therein.

Let us denote the negative part of a real valued function u by $u_- := \max\{-u, 0\}$. We are now ready to provide the definitions of sub- and supersolutions u to

$$\mathcal{L}u = 0 \quad \text{in } \mathbb{R}^n. \tag{2.2}$$

Definition 2.1. A function $u \in W^{s,p}_{\mathrm{loc}}(\Omega)$ such that u_- belongs to $L^{p-1}_{sp}(\mathbb{R}^n)$ is a *fractional weak p-supersolution* of (2.2) if

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(\eta(x)-\eta(y)\big)K(x,y)\,\mathrm{d}x\mathrm{d}y \,\geq\, 0 \tag{2.3}$$

for every nonnegative $\eta \in C^\infty_0(\Omega)$. A function u is a *fractional weak p-subsolution*[1] if $-u$ is a fractional weak p-supersolution; and u is a *fractional weak p-solution* if it is both fractional weak p-sub- and p-supersolution.

Remark 2.2. The function $\eta \in C^\infty_0(\Omega)$ in the definition above can be replaced by $\eta \in W^{s,p}_0(D)$ with every $D \Subset \Omega$. Furthermore, it can be extended to a $W^{s,p}$-function in the whole $\mathbb{R}^n$ (see, e. g., Section 5 in [8]). We also notice that the summability assumption of u_- belonging to the tail space $L^{p-1}_{sp}(\mathbb{R}^n)$ is what one expects in the general nonlocal framework considered here. This is one of the novelty with respect to the analog of the definition of supersolutions in the local case (i. e., when $s = 1$), and it is necessary since here one has to use in a precise way the definition of nonlocal tail given in (2.1) in order to deal with the interactions coming from far; see [13, Remark 2.3], and also, the regularity estimates in [6, 7, 12, 15, 16].

[1] In the rest of the paper we suppress p from notation, by simply say that u is a *weak supersolution* in Ω. For related properties in the linear case without coefficients when the operator in (1.1) does reduce to the usual fractional Laplacian operator $(-\Delta)^s$ we refer to [27, 26, 19].

3. Proof of Theorem 1.1

Firstly, in order to simplify the notation in the weak formulation (2.3), from now on we denote by

$$L(a,b) := |a-b|^{p-2}(a-b), \quad a,b \in \mathbb{R}. \tag{3.1}$$

Notice that $L(a,b)$ is increasing with respect to a and decreasing with respect to b.

Take a nonnegative test function $\phi \in C_0^\infty(\Omega)$. For any $0<\varepsilon<1/4$ we consider the marker function θ_ε defined by

$$\theta_\varepsilon := \min\left\{1, \frac{(u-v)_+}{\varepsilon}\right\}.$$

We now choose $\eta_1 = (1-\theta_\varepsilon)\phi$ as a test function in the formulation in (2.3) for u, and $\eta_2 = \theta_\varepsilon\phi$ for v, respectively. Then, by summing up the corresponding integrals for u and v, we obtain

(3.2)

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(\eta_1(x)-\eta_1(y)\big)K(x,y)\,\mathrm{d}x\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |v(x)-v(y)|^{p-2}\big(v(x)-v(y)\big)\big(\eta_2(x)-\eta_2(y)\big)K(x,y)\,\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} L(u(x),u(y))\big((1-\theta_\varepsilon(x))\phi(x)-(1-\theta_\varepsilon(y))\phi(y)\big)K(x,y)\,\mathrm{d}x\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} L(v(x),v(y))\big(\theta_\varepsilon(x)\phi(x)-\theta_\varepsilon(y)\phi(y)\big)K(x,y)\,\mathrm{d}x\mathrm{d}y \\
&=: \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Psi(x,y)K(x,y)\,\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where we denoted by L the function defined in (3.1), and by $\Psi = \Psi(x,y)$ the function defined as follows:

$$\begin{aligned}
\Psi(x,y) &:= L(u(x),u(y))\big((1-\theta_\varepsilon(x))\phi(x)-(1-\theta_\varepsilon(y))\phi(y)\big) \\
&\qquad + L(v(x),v(y))\big(\theta_\varepsilon(x)\phi(x)-\theta_\varepsilon(y)\phi(y)\big), \qquad x,y\in\mathbb{R}^n.
\end{aligned}$$

Our goal is now to provide some suitable estimates in order to control the function Ψ,

$$\Psi(x,y) \leq L(w(x),w(y))\big(\phi(x)-\phi(y)\big) \tag{3.3}$$

in the limit as ε goes to 0. The latter estimate together with (3.2) will give us the desired result.

We start by noticing that the presence of the marker function θ_ε in the definition of the function Ψ suggests us to consider separately the following three cases, $\theta_\varepsilon = 0$, $0<\theta_\varepsilon<1$, and $\theta_\varepsilon = 1$, which actually reduce to the cases when $u \leq v$, $\; v<u<v+\varepsilon$, $\;$ and $u \geq v+\varepsilon$, respectively. Also,

we have to take into account the nonlocality of the involved integrals, by considering the cases above with respect to both $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. As a consequence, we will have to deal with nine different cases, each of them corresponding to the set $A_{i,j}$, $i, j = 1, 2, 3$, as shown in the table below.

	$u(y) \le v(y)$	$v(y) < u(y) < v(y) + \varepsilon$	$u(y) \ge v(y) + \varepsilon$
$u(x) \le v(x)$	$A_{1,1}$	$A_{1,2}$	$A_{1,3}$
$v(x) < u(x) < v(x) + \varepsilon$	$A_{2,1}$	$A_{2,2}$	$A_{2,3}$
$u(x) \ge v(x) + \varepsilon$	$A_{3,1}$	$A_{3,2}$	$A_{3,3}$

In the set $A_{1,1} := \big\{u(x) \le v(x),\, u(y) \le v(y)\big\}$, we can use the fact that $\theta_\varepsilon(x) = \theta_\varepsilon(y) = 0$, which plainly yields

$$\Psi(x, y) = L(u(x), u(y))\big(\phi(x) - \phi(y)\big) = L(w(x), w(y))\big(\phi(x) - \phi(y)\big). \tag{3.4}$$

In the set $A_{1,2} := \big\{u(x) \le v(x),\, v(y) < u(y) < v(y) + \varepsilon\big\}$, the situation is more delicate and one has to precisely estimate each of the contributions in (3.2). First of all, we can use that $\theta_\varepsilon(x) = 0$ yields

$$\begin{aligned}
\Psi(x, y) &= L(u(x), u(y))\big(\phi(x) - (1 - \theta_\varepsilon(y))\phi(y)\big) - L(v(x), v(y))\theta_\varepsilon(y)\phi(y)\\
&= L(u(x), u(y))\big(\phi(x) - \phi(y)\big)\\
&\quad + \big(L(u(x), u(y)) - L(v(x), v(y))\big)\theta_\varepsilon(y)\phi(y)\\
&\le L(w(x), w(y))\big(\phi(x) - \phi(y)\big) \qquad (3.5)\\
&\quad + |L(u(x), u(y)) - L(u(x), v(y))||\phi(x) - \phi(y)|.
\end{aligned}$$

Then, we estimate the second term in the right-hand side of (3.5) by distinguishing two complementary cases; i. e., when $|u(x) - u(y)| \ge \varepsilon^{1/2}$ and $|u(x) - u(y)| < \varepsilon^{1/2}$. In the former case, we denote by

$$f(t) = L\big(u(x),\, tu(y) + (1 - t)v(y)\big), \qquad 0 \le t \le 1,$$

so that, by the chain rule and by the fact that

$$\partial_b L(a, b) = -(p - 1)|a - b|^{p-2}, \qquad \forall a, b \in \mathbb{R},$$

we obtain

$$\begin{aligned}
\big|L(u(x), u(y)) - L(u(x), v(y))\big| &= |f(1) - f(0)| = \Big|\int_0^1 f'(t)\,\mathrm{d}t\Big|\\
&= (p - 1)|u(y) - v(y)| \int_0^1 |u(x) - u(y) + (1 - t)(u(y) - v(y))|^{p-2}\,\mathrm{d}t\\
&\le (p - 1)\varepsilon \max\Big\{\Big(\frac{1}{2}\Big)^{p-2}, \Big(\frac{3}{2}\Big)^{p-2}\Big\}|u(x) - u(y)|^{p-2}\\
&\le c\,\varepsilon^{1/2}|u(x) - u(y)|^{p-1},
\end{aligned}$$

where we also used the inequalities

$$|u(y)-v(y)|<\varepsilon\le\frac{1}{2}\varepsilon^{1/2}\le\frac{1}{2}|u(x)-u(y)|.$$

Hence, the contribution from the term

$$|L(u(x),u(y))-L(u(x),v(y))||\phi(x)-\phi(y)|$$

vanishes by the dominated convergence theorem as $\varepsilon\to 0$.

Let us move to the remaining case, that is, when $|u(x)-u(y)|<\varepsilon^{1/2}$. In this case the second term in the right-hand side of (3.5) can be estimated from above as follows

$$\begin{aligned}&|L(u(x),u(y))-L(u(x),v(y))||\phi(x)-\phi(y)|\\ &\quad\le|L(u(x),u(y))||\phi(x)-\phi(y)|+|L(w(x),w(y))||\phi(x)-\phi(y)|.\end{aligned}\tag{3.6}$$

Now, notice that both the terms in the right-hand side of the preceding inequality can be treated in the same way because $w\in W^{s,p}_{\rm loc}(\Omega)$ as the minimum of two supersolutions and

$$|w(x)-w(y)|=|u(x)-v(y)|\le|u(x)-u(y)|+|u(y)-v(y)|\le\varepsilon^{1/2}+\varepsilon\le 2\,\varepsilon^{1/2}.$$

For this, we focus only on the first term. Denote by

$$U_\varepsilon:=\Big\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n\,:\,|u(x)-u(y)|<\varepsilon^{1/2}\Big\}$$

and let Ω' be an open set satisfying $\operatorname{supp}\phi\subset\Omega'\Subset\Omega$ with

$$d:=\operatorname{dist}(\operatorname{supp}\phi,\partial\Omega')>0.$$

We have

$$\begin{aligned}&\iint_{U_\varepsilon}|L(u(x),u(y))||\phi(x)-\phi(y)|K(x,y)\,{\rm d}x{\rm d}y\\ &\quad=\iint_{U_\varepsilon\cap(\Omega'\times\Omega')}|u(x)-u(y)|^{p-1}|\phi(x)-\phi(y)|K(x,y)\,{\rm d}x{\rm d}y\\ &\quad\quad+2\iint_{U_\varepsilon\cap(\Omega'\times\mathbb{R}^n\setminus\Omega')}|u(x)-u(y)|^{p-1}\phi(x)K(x,y)\,{\rm d}x{\rm d}y\\ &\quad=:I_1+2I_2.\end{aligned}\tag{3.7}$$

Denoting by

$$\sigma:=\min\left\{\frac{1-s}{2s},\frac{p-1}{2}\right\}>0,$$

we can estimate by Hölder's Inequality

$$I_1\le\Lambda\|D\phi\|_\infty\iint_{U_\varepsilon\cap(\Omega'\times\Omega')}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp}}|x-y|\,{\rm d}x{\rm d}y$$

$$
\begin{aligned}
&\le c\,\varepsilon^{\sigma/2}\int_{\Omega'}\int_{\Omega'}\frac{|u(x)-u(y)|^{p-1-\sigma}}{|x-y|^{s(p-1-\sigma)}}|x-y|^{1-s(1+\sigma)}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n}\\
&\le c\,\varepsilon^{\sigma/2}\left(\int_{\Omega'}\int_{\Omega'}\frac{|u(x)-u(y)|^{p}}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\right)^{(p-1-\sigma)/p} \qquad (3.8)\\
&\qquad\times\left(\int_{\Omega'}\int_{\Omega'}|x-y|^{p(1-s(1+\sigma))/(1+\sigma)}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n}\right)^{(1+\sigma)/p}.
\end{aligned}
$$

The former integral is finite since $u \in W^{s,p}_{\mathrm{loc}}(\Omega)$, whereas the latter is finite by the choice of σ and boundedness of Ω'. Hence, $I_1 \to 0$ as $\varepsilon \to 0$. Now, we estimate I_2. Take $z \in \operatorname{supp}\phi$. By Hölder's Inequality, and using the definition of the set U_ε, we can write

$$
\begin{aligned}
I_2 &\le \Lambda \iint\limits_{U_\varepsilon\cap(\Omega'\times\mathbb{R}^n\setminus\Omega')}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp}}\phi(x)\,\mathrm{d}x\mathrm{d}y\\
&\le \Lambda\|\phi\|_\infty\left(\frac{\operatorname{diam}(\Omega')}{d}\right)^{n+sp}\iint\limits_{U_\varepsilon\cap(\Omega'\times\mathbb{R}^n\setminus\Omega')}\frac{|u(x)-u(y)|^{p-1}}{|z-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\\
&\le c\,\varepsilon^{(p-1)/2}\int_{\mathbb{R}^n\setminus\Omega'}\int_{\Omega'}\frac{1}{|z-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\\
&\le c\,|\Omega'|\varepsilon^{(p-1)/2}, \qquad (3.9)
\end{aligned}
$$

which goes to 0 as $\varepsilon \to 0$. As a consequence, the contributions of

$$
|L(u(x),u(y))-L(u(x),v(y))||\phi(x)-\phi(y)| \qquad (3.10)
$$

in (3.5) vanish as $\varepsilon \to 0$ also in the case when $|u(x)-u(y)| < \varepsilon^{1/2}$, completing the estimates for the whole $A_{1,2}$ case and, in addition, for the case $A_{2,1} := \big\{v(x) < u(x) < v(x)+\varepsilon,\ u(y) \le v(y)\big\}$, which can be treated analogously by exchanging the roles of x and y.

Consider now the set $A_{1,3} := \big\{u(x) \le v(x),\ u(y) \ge v(y)+\varepsilon\big\}$. Since $\theta_\varepsilon(x) = 0$ and $\theta_\varepsilon(y) = 1$, one can plainly write

$$
\begin{aligned}
\Psi(x,y) &= L(u(x),u(y))\phi(x) - L(v(x),v(y))\phi(y)\\
&\le L(u(x),v(y))\phi(x) - L(u(x),v(y))\phi(y) \qquad (3.11)\\
&= L(w(x),w(y))(\phi(x)-\phi(y)),
\end{aligned}
$$

and the same occurs for the case $A_{3,1} := \big\{u(x) \ge v(x)+\varepsilon,\ u(y) \le v(y)\big\}$.

In the set $A_{2,2} := \big\{v(x) < u(x) < v(x)+\varepsilon,\, v(y) < u(y) < v(y)+\varepsilon\big\}$, we have

$$
\begin{aligned}
\Psi(x,y) &= \big(L(u(x),u(y)) - L(v(x),v(y))\big)\big((1-\theta_\varepsilon(x))\phi(x) - (1-\theta_\varepsilon(y))\phi(y)\big)\\
&\quad + L(v(x),v(y))\big(\phi(x)-\phi(y)\big)\\
&= \big(L(u(x),u(y)) - L(v(x),v(y))\big)\big(1-\theta_\varepsilon(x)\big)\big(\phi(x)-\phi(y)\big)\\
&\quad + \big(L(u(x),u(y)) - L(v(x),v(y))\big)\big(\theta_\varepsilon(y)-\theta_\varepsilon(x)\big)\phi(y)\\
&\quad + L(w(x),w(y))\big(\phi(x)-\phi(y)\big).
\end{aligned} \tag{3.12}
$$

The contribution from the first term in the right-hand side of (3.12) vanishes as $\varepsilon \to 0$ by the dominated convergence theorem since $\theta_\varepsilon(x) \to 1$. In turn, the second term reduces to

$$
-\frac{1}{\varepsilon}\big(L(u(x),u(y)) - L(v(x),v(y))\big)\big((u(x)-u(y)) - (v(x)-v(y))\big)\phi(y),
$$

which is nonpositive by the very definition of L, and the following algebraic inequality

$$
\big(|a|^{p-2}a - |b|^{p-2}b\big)(a-b) \geq 0, \quad \forall a,b \in \mathbb{R}.
$$

Thus, in the limit $\varepsilon \to 0$, we have

$$
\Psi(x,y) \leq L(w(x),w(y))\big(\phi(x)-\phi(y)\big). \tag{3.13}
$$

In the set $A_{2,3} := \big\{v(x) < u(x) < v(x)+\varepsilon,\, u(y) \geq v(y)+\varepsilon\big\}$, we use the fact that $\theta_\varepsilon(y) = 1$ and we get

$$
\begin{aligned}
\Psi(x,y) &= L(u(x),u(y))\big(1-\theta_\varepsilon(x)\big)\phi(x) + L(v(x),v(y))\big(\theta_\varepsilon(x)\phi(x) - \phi(y)\big)\\
&= \big(L(u(x),u(y)) - L(v(x),v(y))\big)\big(1-\theta_\varepsilon(x)\big)\phi(x)\\
&\quad + L(v(x),v(y))\big(\phi(x)-\phi(y)\big)\\
&\leq \big(L(v(x)+\varepsilon,\, v(y)+\varepsilon) - L(v(x),v(y))\big)\big(1-\theta_\varepsilon(x)\big)\phi(x) \qquad (3.14)\\
&\quad + L(v(x),v(y))\big(\phi(x)-\phi(y)\big)\\
&= L(w(x),w(y))\big(\phi(x)-\phi(y)\big).
\end{aligned}
$$

Analogously, we can obtain the same estimate as in (3.14) holding in the case $A_{3,2} := \big\{u(x) \geq v(x)+\varepsilon,\, v(y) < u(y) < v(y)+\varepsilon\big\}$.

In the set $A_{3,3} := \big\{u(x) \geq v(x)+\varepsilon,\, u(y) \geq v(y)+\varepsilon\big\}$, $\theta_\varepsilon(x) = \theta_\varepsilon(y) = 1$ yield

$$
\Psi(x,y) = L(v(x),v(y))\big(\phi(x)-\phi(y)\big) = L(w(x),w(y))\big(\phi(x)-\phi(y)\big). \tag{3.15}
$$

All in all, putting together the estimates in (3.4), (3.5), (3.10), (3.11), (3.13), (3.14), and (3.15), and letting $\varepsilon \to 0$, we obtain the estimate in (3.3) and thus

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} L(w(x),w(y))\big(\phi(x)-\phi(y)\big)K(x,y)\,\mathrm{d}x\mathrm{d}y \geq 0,$$

in turn implying that w is a weak supersolution in Ω. □

Remark 3.1. We finally observe that the assumptions on K can be weakened as follows

$$\Lambda^{-1} \leq K(x,y)|x-y|^{n+sp} \leq \Lambda \quad \text{for a.e. } x,y \in \mathbb{R}^n \text{ s.t. } |x-y| \leq 1, \tag{3.16}$$

$$0 \leq K(x,y)|x-y|^{n+\eta} \leq M \quad \text{for a.e. } x,y \in \mathbb{R}^n \text{ s.t. } |x-y| > 1, \tag{3.17}$$

for some s, p, Λ as above, $\eta > 0$ and $M \geq 1$, as seen, e.g., in the recent series of papers by Kassmann (see for instance the more general assumptions in the important paper [11]). In the same sake of generalizing, one can also consider the operator $\mathcal{L} = \mathcal{L}_\Phi$ defined by

$$\mathcal{L}_\Phi u(x) = \text{P.V.}\int_{\mathbb{R}^n} K(x,y)\Phi(u(x)-u(y))\,\mathrm{d}y, \quad x \in \Omega, \tag{3.18}$$

where the real function Φ is assumed to be continuous, satisfying $\Phi(0) = 0$ together with the monotonicity property

$$\lambda^{-1}|t|^p \leq \Phi(t)t \leq \lambda|t|^p \quad \text{for every } t \in \mathbb{R}\setminus\{0\},$$

for some $\lambda > 1$, and some p as above (see, for instance, [15]).

However, for the sake of simplicity, we took $\Phi(t) = |t|^{p-2}t$ and we worked under the assumptions given in the introduction, since the assumptions in (3.16)–(3.18) would have brought no relevant differences in the proof. Moreover, let us remark that we assumed that the kernel K is symmetric, and once again this is not restrictive, in view of the weak formulation presented in Definition 2.1, since one may always define the corresponding symmetric kernel K_{sym} given by

$$K_{\text{sym}}(x,y) := \frac{1}{2}\Big(K(x,y)+K(y,x)\Big).$$

On the contrary, such a symmetry may be restrictive in other frameworks, as in viscosity for nondivergence form equations, where for instance $K(x,y) = K(x,-y)$ is a common assumption; see [14].

Acknowledgments. The first author has been supported by the Magnus Ehrnrooth Foundation (grant no. ma2014n1, ma2015n3). The third author member of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of Istituto Nazionale di Alta Matematica "F. Severi" (INdAM), whose support is acknowledged.

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