ON THE BOUNDARY BEHAVIOR FOR THE BLOW UP SOLUTIONS OF THE SINH-GORDON EQUATION AND $\mathbf{B}_2, \mathbf{G}_2$ TODA SYSTEMS IN BOUNDED DOMAIN

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ABSTRACT. In this paper we study the boundary behavior of blow up solutions for the sinh-Gordon equation and $\mathbf{B}_2, \mathbf{G}_2$ Toda systems in a bounded domain. The fact that the blow up solution may not concentrate makes the analysis delicate: nevertheless, we prove that there is no boundary blow up point with a detailed blow-up analysis and in particular without using any concentration property.

1. INTRODUCTION

In this paper the first problem we are interested in is the following mean field equation

$$\begin{cases} \Delta u_k + \rho_{1k} \dfrac{h_1 e^{u_k}}{\int_\Omega h_1 e^{u_k}} - \rho_{2k} \dfrac{h_2 e^{-u_k}}{\int_\Omega h_2 e^{-u_k}} = 0 & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and Δ is the Euclidean Laplace operator, ρ_{1k}, ρ_{2k} are two positive parameters, h_1, h_2 are two smooth positive functions in Ω.

Equation (1.1) arises in mathematical physics as a mean-field equation of the equilibrium turbulence with arbitrarily signed vortices and it was first deriver by Joyce and Montgomery [20] and by Pointin and Lundgren [39]. For more discussions concerning the physical background we refer for example to [5, 30, 33, 34, 36] and references therein. The case $h_1 = h_2$, $\rho_{1k} = \rho_{2k}$ has a close relationship with geometry and is related to the study of constant mean curvature surfaces, see [47, 48].

Before we study the equation (1.1), let us first consider the case $\rho_{1k} = \rho_k$, $\rho_{2k} = 0$, i.e.,

$$\begin{cases} \Delta u_k + \rho_k \dfrac{h e^{u_k}}{\int_\Omega h e^{u_k}} = 0 & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Equation (1.2) has been extensively studied in the literature since it is related to the prescribed Gaussian curvature problem and Euler flows, see [1, 44] and [4, 21], respectively. We refer the interested readers to the survey [45]. Let us first recall some results on (1.2). Suppose (u_k, ρ_k) is a sequence of blow up solutions to (1.2)

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with ρ_k uniformly bounded, then it is known that there is no boundary blow up, namely that u_k is uniformly bounded near a neighborhood of $\partial\Omega$, see [31, 35]. Furthermore, we have a clear understanding of the blowing up solution: it holds $\rho_k \to 8m\pi$, $m \in \mathbb{N}$ and, after passing to a subsequence

$$u_k(x) \to 8\pi \sum_{j=1}^{m} G(x, p_j) \qquad \text{in } C^2_{\text{loc}}\big(\Omega \setminus \{p_1, \cdots, p_m\}\big), \tag{1.3}$$

where G is the Green function of $-\Delta$ with Dirichlet boundary condition, see [3, 22, 23, 31, 35]. Roughly speaking, the latter property denotes a concentration property of the blowing up solution, see the discussion later on.

Let us now return to equation (1.1). The latter problem has attracted a lot of attention in the last decades: we refer to [16, 17, 19, 36] for blow up analysis and to [2, 10, 11, 12] for what concerns existence results. In this paper we shall consider the blow up analysis of solutions to (1.1) and we shall address the existence of possible boundary bubbles. Such study is of independent interest: moreover, the exclusion of boundary blow up allows to exploit the analysis developed for the internal region and hence to extend some results from the compact surface case to the bounded domain setting. For instance, one example could be the topological degree counting introduced in [17].

In the case of equation (1.2), boundary blow up is excluded by the method of moving planes and the use of Kelvin's transform, see for example [31]. We point out such argument can be suitably used to treat cooperative systems, see [46], where we say

$$\begin{cases} \Delta u + f(x, u, v) = 0, \\ \Delta v + g(x, u, v) = 0, \end{cases} \tag{1.4}$$

is cooperative if $\frac{\partial f(x,u,v)}{\partial v} \geq 0$, $\frac{\partial g(x,u,v)}{\partial u} \geq 0$. Observe that we can uniquely decompose $u_k = u_{k1} - u_{k2}$, where u_{k1} and u_{k2} satisfy

$$\begin{cases} \Delta u_{k1} + \rho_1 \dfrac{h_1 e^{u_{k1} - u_{k2}}}{\int_\Omega h_1 e^{u_{k1} - u_{k2}}} = 0, \quad u_{k1} = 0 \text{ on } \partial\Omega \\ \Delta u_{k2} + \rho_2 \dfrac{h_2 e^{u_{k2} - u_{k1}}}{\int_\Omega h_2 e^{u_{k2} - u_{k1}}} = 0, \quad u_{k2} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.5}$$

However, the system (1.5) is not cooperative and the method of moving planes does not apply. Our strategy is to use the Pohozaev identity and a detailed analysis of the behavior of the blow up solutions. Similar arguments have been used by Robert-Wei [40] in the fourth order mean field equation and then by Lin-Wei-Zhao [25] in the $SU(3)$ Toda system.

Some remarks are needed here. In the works [25, 40] the concentration property plays an important role. More precisely, for equation (1.2) we say that a blowing up solution u_k has the concentration property if

$$\rho_k \frac{h e^{u_k}}{\int_M h e^{u_k}} \to \sum_{p \in \mathcal{B}} \beta_p \delta_p,$$

in the sense of measures, where $\mathcal{B}$ is the blow up set of u_k. In the seminar work [3], Brezis and Merle study the blow up behavior of the standard Liouville equation alike (1.2) and they showed the "bubbling implies mass concentration" result, i.e., if the blow up phenomena happens the concentration property holds. For the

fourth order equation it is not difficult to get this property by adapting the similar argument of [3]. However, as for the sinh-Gordon equation we can not expect to get the corresponding property. Indeed, since the work by [7], it seems that there exists a sequence of bubbling solutions of (1.1) which blows up at some point $p \in \Omega$ and the concentration property may not hold, in other words, we may get

$$\frac{h_1 e^{u_k}}{\int_\Omega h_1 e^{u_k}}, \frac{h_2 e^{-u_k}}{\int_\Omega h_2 e^{-u_k}} \not\to 0 \qquad \text{for } x \in \Omega \setminus \{p\}. \tag{1.6}$$

For the question whether concentration property hold or not, we shall pursue this in the forthcoming paper. Therefore, we can not conclude the bubbling solutions u_k of (1.1) converge to some function which is the summation of Green functions away from the blow up points as in (1.3). This makes the study of equation (1.1) more complicate. In order to overcome this difficulty, based on the idea from [40] and [25], we have to apply some delicate analysis for the bubbling solutions around the blow-up point, see Lemma 3.6. However, we need a further refinement of the latter analysis suitable to treat our case. Indeed, the first part of our argument follows the one used in [40, 25] but the final part substantially differ from the previous strategies. *To the best of our knowledge, this is the first paper in treating this kind of problem without using the concentration property* (1.6) and this is the main contribution of this work.

Our first main result is the following.

Theorem 1.1. *Let u_k be a sequence of solutions to (1.1) such that as $k \to \infty$, it holds $\rho_{1k}, \rho_{2k} \leq C$ and*

$$\max_{x \in \Omega} |u_k| \to +\infty.$$

Then the blow-up set of $|u_k|$ is finite and in the interior of $\overline{\Omega}$.

Remark 1. Our method can be also applied to the following asymmetric sinh-Gordon equation:

$$\begin{cases} \Delta u_k + \rho_{1k} \dfrac{h_1 e^{u_k}}{\int_\Omega h_1 e^{u_k}} - \rho_{2k} \dfrac{h_2 e^{-a u_k}}{\int_\Omega h_2 e^{-a u_k}} = 0 & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

with $a \in (0,1)$. The latter equation arises in the context of the statistical mechanics description of $2D$-turbulence under a deterministic assumption on the vortex intensities, see [38, 43]. Concerning equation (1.7) we refer the interested readers to the recent results [13, 15, 41, 42].

The second class of problems we consider is the following:

$$\begin{cases} \Delta u_{1k} + K_{11}\rho_{1k} \dfrac{h_1 e^{u_{1k}}}{\int_\Omega h_1 e^{u_{1k}}} + K_{12}\rho_{2k} \dfrac{h_2 e^{u_{2k}}}{\int_\Omega h_2 e^{u_{2k}}} = 0, \quad u_{1k} = 0 \text{ on } \partial\Omega, \\ \Delta u_{2k} + K_{21}\rho_{1k} \dfrac{h_1 e^{u_{1k}}}{\int_\Omega h_1 e^{u_{1k}}} + K_{22}\rho_{2k} \dfrac{h_2 e^{u_{2k}}}{\int_\Omega h_2 e^{u_{2k}}} = 0, \quad u_{2k} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.8}$$

where $\Omega, \rho_{1k}, \rho_{2k}, h_1$ and h_2 are as the ones introduced in (1.1), $\mathbf{K} = (K_{ij})_{2\times 2}$ represents the Cartan matrix of rank 2. As we know, there are three types of the corresponding Cartan matrices of rank 2:

$$\mathbf{A}_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \ \mathbf{B}_2 = (\mathbf{C}_2) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \ \mathbf{G}_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

Corresponding to each one there is a Toda system. When $\mathbf{K} = \mathbf{A}_2$, Lin, Wei and Zhao [25] proved there is no boundary blow up. In our second result, we shall study the left two cases by using the same argument in treating (1.1). For more background of (1.8) with $\mathbf{K}$ Cartan matrix of rank 2, one can see [8, 49] and the references therein. For what concerns the analytical studies about Toda-type systems we refer to [18, 26, 27] for blow-up analysis, to [28] for classification issues and to [2, 14, 32] for existence results.

The second main result is the following.

Theorem 1.2. *Let $\mathbf{K}$ denote the Cartan matrix $\mathbf{B}_2$ or $\mathbf{G}_2$, and (u_{1k}, u_{2k}) be a sequence of solutions to (1.8) such that as $k \to \infty$, it holds $\rho_{1k}, \rho_{2k} \leq C$ and*

$$\max_{x\in\Omega} \max\{u_{1k}, u_{2k}\} \to +\infty.$$

Then the blow-up set of u_{1k}, u_{2k} is finite and in the interior of $\overline{\Omega}$.

Since the proofs of Theorem 1.1 and Theorem 1.2 are very similar, we shall only give the details of Theorem 1.1 and state the differences in the proof if necessary. The present paper is organized as follows. In Section 2 we give some useful lemmas and the Pohozaev identity. Theorems 1.1 and 1.2 are proved in Section 3.

Notation

Throughout this paper, without other explanations, the constant C will denote some generic constant which is independent of k and the value of C might change from one line to the other. The quantity $B = O(A)$ means that there exists $C > 0$ such that $|B| \leq CA$. All the convergence results are stated by passing to a subsequence. The symbol $B_r(p)$ will denote the open ball of radius r and center p.

2. Useful facts

In this section we list some useful results which will be used in the sequel. First, we collect some properties of the Green function in the following lemma.

Lemma 2.1. *Let $G(x, y)$ be the Green function of $-\Delta$ with Dirichlet boundary condition. There exists $C > 0$ such that for all $x, y \in \Omega$, $x \neq y$, we have*

$$|G(x,y)| \leq C \log\left(2 + \frac{1}{|x-y|}\right), \ |\nabla G(x,y)| \leq C|x-y|^{-1}.$$

Proof. We refer the readers to [6] and [9] for a proof. □

We set

$$\alpha_{1k} = \ln\left(\frac{\int_\Omega h_1 e^{u_k}}{\rho_{1k}}\right), \quad \alpha_{2k} = \ln\left(\frac{\int_\Omega h_2 e^{-u_k}}{\rho_{2k}}\right),$$

and

$$v_k = u_k - \alpha_{1k}, \quad w_k = -u_k - \alpha_{2k}. \tag{2.1}$$

Then, we write equation (1.1) into the following form

$$\begin{cases} \Delta v_k + h_1 e^{v_k} - h_2 e^{w_k} = 0, & v_k = -\alpha_{1k} \text{ on } \partial\Omega, \\ \Delta w_k - h_1 e^{v_k} + h_2 e^{w_k} = 0, & w_k = -\alpha_{2k} \text{ on } \partial\Omega. \end{cases} \tag{2.2}$$

For equation (2.2), let

$$\mathfrak{S}_1 = \Big\{ p \in \overline{\Omega} \mid \exists \{x_k\}, x_k \to p,\ v_k(x_k) \to +\infty \Big\},$$
$$\mathfrak{S}_2 = \Big\{ p \in \overline{\Omega} \mid \exists \{x_k\}, x_k \to p,\ w_k(x_k) \to +\infty \Big\}$$

and

$$\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2. \tag{2.3}$$

Concerning the set $\mathfrak{S}$ we have the following result.

Lemma 2.2. *The set $\mathfrak{S}$ is finite.*

Proof. The finiteness of blow-up points is a very standard result for the mean field type equation. It has been done several times for similar problems by using the celebrated result of [3, Corollary 4], see [26, Lemma 2.1, Lemma 2.2]. For the Sinh-Gordon type equation it was never proved in full details, we shall provide a complete proof to make this paper more self-contained.

For any $p \in \mathfrak{S}$ we define the local mass for u_k and $-u_k$ by

$$\sigma_{1,p} = \frac{1}{2\pi} \lim_{\delta \to 0} \lim_{k \to 0} \int_{B_\delta(p)} \frac{\rho_1 h_1 e^{u_k}}{\int_\Omega h_1 e^{u_k}} \quad \text{and} \quad \sigma_{2,p} = \frac{1}{2\pi} \lim_{\delta \to 0} \lim_{k \to 0} \int_{B_\delta(p)} \frac{\rho_2 h_2 e^{-u_k}}{\int_\Omega h_2 e^{-u_k}}.$$

Step 1. We claim that if $\sigma_{1,p}, \sigma_{2,p} < \frac{1}{3}$, then $p \notin \mathfrak{S}$.

Since $\sigma_{ip} < \frac{1}{3}$, we can choose small r_0 such that in $B_{r_0}(p)$ the following holds

$$\int_{B_{r_0}(p)} \rho_i h_i e^{\widetilde{u}_{ik}} < \pi, \qquad i = 1, 2, \tag{2.4}$$

where

$$\widetilde{u}_{1k} = u_k - \ln \int_\Omega h_1 e^{u_k} \quad \text{and} \quad \widetilde{u}_{2k} = -u_k - \ln \int_\Omega h_2 e^{-u_k}. \tag{2.5}$$

From (2.4) we get $\int_{B_{r_0}(p)} \widetilde{u}_{ik}^+ \le C$. We decompose $\widetilde{u}_{1k} = \sum_{j=1}^3 \widetilde{u}_{1k,j}$, where $\widetilde{u}_{1k,j}$ satisfy the following equation

$$\begin{cases} -\Delta \widetilde{u}_{1k,1} = \rho_1 h_1 e^{\widetilde{u}_{1k}} - \rho_2 h_2 e^{\widetilde{u}_{2k}} & \text{in } B_{r_0}(p), \quad \widetilde{u}_{1k,1} = 0 \quad \text{on } \partial B_{r_0}(p), \\ -\Delta \widetilde{u}_{1k,2} = -\rho_1 + \rho_2 & \text{in } B_{r_0}(p), \quad \widetilde{u}_{1k,2} = 0 \quad \text{on } \partial B_{r_0}(p), \\ -\Delta \widetilde{u}_{1k,3} = 0 & \text{in } B_{r_0}(p), \quad \widetilde{u}_{1k,3} = \widetilde{u}_{1k} \quad \text{on } \partial B_{r_0}(p). \end{cases} \tag{2.6}$$

For the first equation in (2.6), since

$$\int_{B_{r_0}(p)} \left| \rho_1 h_1 e^{\widetilde{u}_{1k}} - \rho_2 h_2 e^{\widetilde{u}_{2k}} \right| < 3\pi,$$

by [3, Theorem 1], we have

$$\int_{B_{r_0}(p)} \exp\big((1+\delta)|\widetilde{u}_{1k,1}|\big) dx \le C, \tag{2.7}$$

where $\delta \in (0, \frac{1}{3})$. Therefore, we have

$$\int_{B_{r_0}(p)} |\widetilde{u}_{1k,1}| \le C. \tag{2.8}$$

For the second equation in (2.6), we can easily get

$$\int_{B_{r_0}(p)} |\widetilde{u}_{1k,2}| \leq C \quad \text{and} \quad |\widetilde{u}_{1k,2}| \leq C. \tag{2.9}$$

For the third equation in (2.6), by the mean value theorem for harmonic function we have

$$\begin{aligned} \|\widetilde{u}^+_{1k,3}\|_{L^\infty(B_{r_0/2}(p))} &\leq C\|\widetilde{u}^+_{1k,3}\|_{L^1(B_{r_0}(p))} \\ &\leq C\Big[\|\widetilde{u}^+_{1k}\|_{L^1(B_{r_0}(p))} + \|\widetilde{u}_{1k,1}\|_{L^1(B_{r_0}(p))} + \|\widetilde{u}_{1k,2}\|_{L^1(B_{r_0}(p))}\Big] \\ &\leq C. \end{aligned} \tag{2.10}$$

From (2.9)-(2.10), we have

$$\rho_1 h_1 e^{\widetilde{u}_{1k,2}+\widetilde{u}_{1k,3}} \leq C \qquad \text{in } B_{r_0/2}(p). \tag{2.11}$$

By (2.7), (2.11) and Hölder inequality, we obtain

$$e^{\widetilde{u}_{1k}} \in L^{1+\delta_1}(B_{r_0}(p)),$$

with $\delta_1 > 0$ independent of k. Similarly, we have

$$e^{\widetilde{u}_{2k}} \in L^{1+\delta_2}(B_{r_0}(p)),$$

with $\delta_2 > 0$ independent of k. By using the standard elliptic estimates for the first equation in (2.6), we get $\|\widetilde{u}_{1k,1}\|_{L^\infty}(B_{r_0/2}(p))$ is uniformly bounded. Combined with (2.9) and (2.10), we have $\widetilde{u}_{1k}$ is uniformly bounded above in $B_{\frac{r_0}{2}}(p)$. Following the same process we can also obtain $\widetilde{u}_{2k}$ is uniformly bounded above in $B_{\frac{r_0}{2}}(p)$. On the other hand, we note that

$$\widetilde{u}_{1k} = v_k + \ln \rho_{1k}, \qquad \widetilde{u}_{2k} = w_k + \ln \rho_{2k}.$$

As a consequence, we get $p \notin \mathfrak{S}$ as claimed.

Step 2. It follows that if $p \in \mathfrak{S}$, either $\sigma_{1p} \geq \frac{1}{3}$ or $\sigma_{2p} \geq \frac{1}{3}$: together with the fact that ρ_{1k}, ρ_{2k} are finite, we deduce $|\mathfrak{S}| < \infty$. Hence, we finish the proof of the lemma. □

For the terms α_{1k}, α_{2k} the following holds.

Lemma 2.3. *There exists a constant $C \in \mathbb{R}$ independent of k such that $\alpha_{ik} \geq C$, $i = 1, 2$.*

Proof. Note that v_k, w_k satisfy

$$\begin{cases} \Delta v_k + h_1 e^{v_k} - h_2 e^{w_k} = 0, & v_k = -\alpha_{1k} \text{ on } \partial\Omega, \\ \Delta w_k - h_1 e^{v_k} + h_2 e^{w_k} = 0, & w_k = -\alpha_{2k} \text{ on } \partial\Omega. \end{cases}$$

Using Green's representation formula, we have

$$v_k = \int_\Omega G(x,z)\left(h_1 e^{v_k}(z) - h_2 e^{w_k}(z)\right) \mathrm{d}z - \alpha_{1k}, \tag{2.12}$$

$$w_k = \int_\Omega G(x,z)\left(h_2 e^{w_k}(z) - h_1 e^{v_k}(z)\right) \mathrm{d}z - \alpha_{2k}. \tag{2.13}$$

Thus we get

$$\|v_k + \alpha_{1k}\|_{L^1(\Omega)} \leq C, \quad \|w_k + \alpha_{2k}\|_{L^1(\Omega)} \leq C. \tag{2.14}$$

It is known that $\mathfrak{S} \subset \overline{\Omega}$ is finite, and both v_k and w_k are uniformly bounded from above in any compact subset of $\overline{\Omega} \setminus \mathfrak{S}$. Therefore, from (2.14) we see that α_{1k}, α_{2k} are bounded from below, which proves the lemma. □

We state now the Pohozaev identity which is one of the key ingredients in our argument.

Lemma 2.4. *For any solution of equation (1.1) and any bounded domain $D \subset \mathbb{R}^2$, it holds that*

$$\begin{aligned}
&\int_D 2\left(h_1 e^{u_k-\alpha_{1k}} + h_2 e^{-u_k-\alpha_{2k}}\right) + \int_D \langle x-\xi, \nabla h_1\rangle e^{u_k-\alpha_{1k}} + \int_D \langle x-\xi, \nabla h_2\rangle e^{-u_k-\alpha_{2k}} \\
&= \int_{\partial D} \left(h_1 e^{u_k-\alpha_{1k}} + h_2 e^{-u_k-\alpha_{2k}}\right) \langle x-\xi, \nu\rangle + \int_{\partial D} \frac{\partial u_k}{\partial \nu} \langle x-\xi, \nabla u_k\rangle \\
&\quad - \frac{1}{2}\int_{\partial D} |\nabla u_k|^2 \langle x-\xi, \nu\rangle
\end{aligned} \tag{2.15}$$

for any $\xi \in \mathbb{R}^2$, where ν stands for the normal vector on ∂D. While, for any solution of equation (1.8) and any bounded domain in $D \subset \mathbb{R}^2$, we have

$$\int_D \left(2K^{21} h_1 e^{\widetilde{u}_{1k}} + 2K^{12} h_2 e^{\widetilde{u}_{2k}} + K^{21} e^{\widetilde{u}_{1k}} \langle x-\xi, \nabla h_1\rangle + K^{12} e^{\widetilde{u}_{2k}} \langle x-\xi, \nabla h_2\rangle\right) \tag{2.16}$$

$$\begin{aligned}
&= \int_{\partial D} \sum_{j=1}^{2} K^{21} K^{1i} \frac{\partial u_{ik}}{\partial \nu} \langle x-\xi, \nabla u_{1k}\rangle + \int_{\partial D} \sum_{i=1}^{2} K^{12} K^{2i} \frac{\partial u_{ik}}{\partial \nu} \langle x-\xi, \nabla u_{2k}\rangle \\
&- \int_{\partial D} \left[\frac{1}{2} K^{21} K^{11} |\nabla u_{1k}|^2 + K^{21} K^{12} \langle \nabla u_{1k}, \nabla u_{2k}\rangle + \frac{1}{2} K^{12} K^{22} |\nabla u_{2k}|^2\right] \langle x-\xi, \nu\rangle \\
&+ \int_{\partial D} K^{21} h_1 e^{\widetilde{u}_{1k}} \langle x-\xi, \nu\rangle + \int_{\partial D} K^{12} h_2 e^{\widetilde{u}_{2k}} \langle x-\xi, \nu\rangle
\end{aligned} \tag{2.17}$$

where $(K^{ij})_{2\times 2}$ is the inverse matrix of $\mathbf{K}$, $\xi \in \mathbb{R}^2$, ν stands for the normal vector on ∂D and $\widetilde{u}_{ik} = u_{ik} - \ln\left(\frac{\int_\Omega h_i e^{u_{ik}}}{\rho_i}\right)$, $i = 1, 2$.

Proof. Multiplying the equation (1.1) by $\langle x-\xi, \nabla u_k\rangle$, and integrating by parts, we can obtain (2.15). For (1.8), we write it into

$$\sum_{j=1}^{2} K^{ij} \Delta u_{jk} + h_i e^{\widetilde{u}_{ik}} = 0, \qquad i = 1, 2. \tag{2.18}$$

Multiplying the $i-$th equation by $\langle x-\xi, \nabla u_{ik}\rangle$ and integrating by parts it is easy to derive (2.16) after some straightforward computations. □

3. Proof of the no boundary blow up

In this section we shall prove Theorem 1.1 (the same argument can be used in the proof of Theorem 1.2). We collect the blow up points for $|u_k|$, that is, let

$$\mathcal{S} = \left\{ p \in \overline{\Omega} \mid \exists \{x_k\},\ x_k \to p,\ |u_k|(x_k) \to +\infty \right\}. \tag{3.1}$$

Recall the definitions of v_k, w_k in (2.1) and the definition of $\mathfrak{S}$ in (2.3). We have the following.

Lemma 3.1. *It holds that*

$$\mathfrak{S} = \mathcal{S}. \tag{3.2}$$

Proof. First we prove $\mathcal{S} \subseteq \mathfrak{S}$. It suffices to show that if any $x \notin \mathfrak{S}$, then $x \notin \mathcal{S}$. Suppose $x \notin \mathfrak{S}$, we have $|u_k(x)| \leq C$ by some constant C uniformly. Using Lemma 2.3, we have

$$v_k(x) = u_k(x) - \alpha_{1k} \leq C \quad \text{and} \quad w_k(x) = -u_k(x) - \alpha_{2k} \leq C,$$

where C is a constant independent of k. Consequently $x \notin \mathcal{S}$. Therefore we have $\mathcal{S} \subseteq \mathfrak{S}$.

To prove the other direction, we shall show that u_k is uniformly bounded in any compact subset of $\overline{\Omega} \setminus \mathfrak{S}$. More precisely, for any compact subset $K \subset\subset \Omega \setminus \mathcal{S}$, we shall prove that there is a constant $C_K > 0$ that depends on the compact set K such that

$$|u_k| \leq C_K, \qquad \forall x \in K.$$

By Green's representation formula we have

$$\begin{aligned} u_k(x) &= \int_\Omega G(x,z)\Big(\rho_1(h_1 e^{\widetilde{u}_{1k}} - 1) - \rho_2(h_2 e^{\widetilde{u}_{2k}} - 1)\Big) \\ &= \int_{\Omega_1} G(x,z)\Big(\rho_1(h_1 e^{\widetilde{u}_{1k}} - 1) - \rho_2(h_2 e^{\widetilde{u}_{2k}} - 1)\Big) \\ &\quad + \int_{\Omega\setminus\Omega_1} G(x,z)\Big(\rho_1(h_1 e^{\widetilde{u}_{1k}} - 1) - \rho_2(h_2 e^{\widetilde{u}_{2k}} - 1)\Big), \end{aligned} \tag{3.3}$$

where $\widetilde{u}_{ik},\ i = 1,2$ are defined in (2.5), $\Omega_1 = \cup_{p\in\mathfrak{S}} B_{r_0}(p)$ and r_0 is small enough to make $K \subset\subset \Omega \setminus \Omega_1$. It is easy to see that

$$\int_{\Omega_1} G(x,z)\Big(\rho_1(h_1 e^{\widetilde{u}_{1k}} - 1) - \rho_2(h_2 e^{\widetilde{u}_{2k}} - 1)\Big) = O(1),$$

because $G(x,z)$ is bounded due to the distance $d(x,z) \geq \delta_0 > 0$ for $z \in \Omega_1$, and $x \in K$. In $\Omega \setminus \Omega_1$, we can see that $\widetilde{u}_{ik}$ are bounded above by some constant that depends on r_0. Then it is not difficult to obtain that

$$\int_{\Omega\setminus\Omega_1} G(x,z)\Big(\rho_1(h_1 e^{\widetilde{u}_{1k}} - 1) - \rho_2(h_2 e^{\widetilde{u}_{2k}} - 1)\Big) = O(1).$$

Therefore, we can conclude that $|u_k(x)| \leq C$ from (3.3), where C depends on K only. Hence $|u_k(x)|$ is bounded away from $\mathfrak{S}$ and $\mathfrak{S} \subseteq \mathcal{S}$. This completes the proof. □

In Lemma 2.2, we have proved $\mathfrak{S}$ is finite. We may assume that

$$\mathfrak{S} = \{p_1, p_2, \cdots, p_m\}. \tag{3.4}$$

For any $p_i \in \overline{\Omega}$, let r be a positive number such that $B_r(p_i) \cap B_r(p_j) = \emptyset$ for $i \neq j$. We assume $p_{i,k} \in \overline{\Omega}$ satisfies $M_k(p_{i,k}) = \max_{\overline{\Omega}\cap B_r(p_i)} M_k(x)$, where

$$M_k(x) = \max\{v_k(x), w_k(x)\}. \tag{3.5}$$

Define $\mu_{i,k}$ by

$$-2\ln \mu_{i,k} = M_k(p_{i,k}).$$

At first, we note that $\mu_{i,k} \to 0$. On the other hand, $p_{i,k} \notin \partial\Omega$ because we know that $M_k(x)\ |_{\partial\Omega} \leq C$ from Lemma 2.3. Furthermore, we can estimate more precisely the distance of the point $p_{i,k}$ from the boundary.

Lemma 3.2. *It holds that*

$$\operatorname{dist}(p_{i,k}, \partial\Omega)/\mu_{i,k} \to \infty.$$

Proof. Suppose the result is not true, one can find a sequence $(p_{i,k}, \mu_{i,k})$, such that $\operatorname{dist}(p_{i,k}, \partial\Omega) = O(\mu_{i,k})$. Let

$$\Omega_{i,k} = (\Omega - p_{i,k})/\mu_{i,k}.$$

We may assume that $\Omega_{i,k} \to (-\infty, t_0) \times \mathbb{R}$. Without loss of generality, we may further assume $v_k(p_{i,k}) = -2\ln\mu_{i,k}$ and define

$$\begin{aligned} \hat{v}_k(y) &= v_k(p_{i,k} + \mu_{i,k}y) + 2\ln\mu_{i,k} + \ln h_1(p_{i,k}), \\ \hat{w}_k(y) &= w_k(p_{i,k} + \mu_{i,k}y) + 2\ln\mu_{i,k} + \ln h_2(p_{i,k}). \end{aligned} \tag{3.6}$$

We note that

$$\begin{aligned} \hat{v}_k + \hat{w}_k &= v_k + w_k + 4\ln\mu_{i,k} + \ln h_1(p_{i,k}) + \ln h_2(p_{i,k}) \\ &= 4\ln\mu_{i,k} - \alpha_{1k} - \alpha_{2k} + C \le 4\ln\mu_{i,k} + C. \end{aligned}$$

Let $R > 0$ and $y \in B_R(0) \cap \Omega_{i,k}$. By the representation formula, with a little abuse of notation we have

$$\begin{aligned} |\nabla\hat{v}_k| &= |\mu_{i,k}\nabla v_k(p_k + \mu_{i,k}y)| \\ &= \mu_{i,k}\left|\int_\Omega \nabla G(p_k + \mu_{i,k}y, z)\left[h_1 e^{v_k}(z) - h_2 e^{w_k}(z)\right]\mathrm{d}z\right| \\ &\le C\mu_{i,k}\left[\int_{B_{2R\mu_{i,k}}(p_k)} + \int_{\Omega\setminus B_{2R\mu_{i,k}}(p_k)}\right]\frac{\left|h_1 e^{v_k(z)} - h_2 e^{w_k(z)}\right|}{|p_k + \mu_{i,k}y - z|}\mathrm{d}z. \end{aligned} \tag{3.7}$$

In $B_{2R\mu_{i,k}}$, we have $e^{v_k}, e^{w_k} \le e^{v_k(p_k)} = \mu_{i,k}^{-2}$. In $\Omega_k \setminus B_{2R\mu_{i,k}}$, we have

$$|p_k + \mu_{i,k}y - z| \ge |z - p_k| - \mu_{i,k}|y| \ge R\mu_{i,k}.$$

Hence,

$$|\nabla\hat{v}_k| \le C\mu_{i,k}\int_{B_{2R\mu_{i,k}}(p_k)}\frac{|h_1 e^{v_k} - h_2 e^{w_k}|}{|p_k + \mu_{i,k}y - z|} + C(R)\int_\Omega |h_1 e^{v_k} - h_2 e^{w_k}| \le C(R).$$

Therefore, we get $|\nabla\hat{v}_k| = |\nabla\hat{w}_k| \le C(R)$ in $B_R(0) \cap \Omega_{i,k}$, which implies

$$|\hat{v}_k(y) - \hat{v}_k(0)| \le C|y| \le C \qquad \text{for any } y \in \overline{B_R(0) \cap \Omega_{i,k}}.$$

Choosing y_0 in $\partial\Omega_{i,k}$, we obtain

$$|v_k(p_k) + \alpha_{1k}| = |\hat{v}_k(y_0) - \hat{v}_k(0)| \le C.$$

Then, we have

$$-2\ln\mu_{i,k} + \alpha_{1k} = O(1),$$

from which we get a contradiction to Lemma 2.3 and the fact $\mu_{i,k} \to 0$. Thus we prove the lemma. □

Next, we study the behavior of the blow up solutions around each p_i. In the following result we shall introduce a selection process for (2.2), one can see [16, Proposition 2.1] for a proof.

Proposition 3.1. *Let u_k be a sequence of bubbling solutions of (1.1) and v_k, w_k be defined in (2.1). Let $\mathfrak{S}$ be defined in* (2.3)*. Then around each point $p_i \in \mathfrak{S}$ there exists a finite sequence of points*

$$\Sigma_{k,i} := \left\{p_{i,1}^k, p_{i,2}^k, \cdots, p_{i,m_i}^k\right\} \qquad \left(\lim_{k\to\infty} p_{i,j}^k \to p_i,\ j = 1, \cdots, m_i\right)$$

and positive numbers $l_{i,0}^k, \cdots, l_{i,m_i}^k \to 0$ such that

(1) $\max\left\{v_k(p_{i,j}^k), w_k(p_{i,j}^k)\right\} = \max_{x\in B_{l_{i,j}^k}(p_{i,j}^k)} \max\left\{v_k(x), w_k(x)\right\}$, *where $j = 1, \ldots, m_i$, $i = 1, \cdots, m$.*

(2) $\exp\left(\frac{1}{2}\max\left\{v_k(p_{i,j}^k), w_k(p_{i,j}^k)\right\}\right) l_{i,j}^k \to \infty$ *for $j = 1, \ldots, m_i$, $i = 1, \cdots, m$.*

(3) *Let $\varepsilon_{i,j,k} = e^{-\frac{1}{2}\max\{v_k(p_{i,j}^k), w_k(p_{i,j}^k)\}}$. In each $B_{l_{i,j}^k}(p_{i,j}^k)$ we define the dilated functions*

$$\begin{aligned} \hat{v}_{k,i,j}(y) &= v_k(p_{i,j}^k + \varepsilon_{i,j,k} y) + 2\log \varepsilon_{i,j,k}, \\ \hat{w}_{k,i,j}(y) &= w_k(p_{i,j}^k + \varepsilon_{i,j,k} y) + 2\log \varepsilon_{i,j,k}. \end{aligned} \tag{3.8}$$

Then it holds that one of the $v_{k,i,j}, w_{k,i,j}$ converges to a solution of Liouville equation, while the other one tends to minus infinity over all compact subsets of $\mathbb{R}^2$.

(4) *There exits a constant $C > 0$ independent of k such that*

$$\max\{v_k(x), w_k(x)\} + 2\log \operatorname{dist}(x, \Sigma_{k,i}) \le C, \tag{3.9}$$

for all $x \in B_r(p_i)$, $i = 1, 2, \cdots, m$.

We can also formulate (3.9) as the following result.

Lemma 3.3. *Let $\Sigma_k = \cup_{i=1}^m \Sigma_{k,i}$. Then there exists a constant $C > 0$ such that*

$$\max\{v_k(x), w_k(x)\} + 2\log \operatorname{dist}(x, \Sigma_k) \le C, \quad \forall x \in \Omega. \tag{3.10}$$

Moreover, we derive the following estimates.

Lemma 3.4. *There exists a $C > 0$ such that*

$$\operatorname{dist}(x, \Sigma_k)|\nabla v_k(x)| \le C, \qquad \operatorname{dist}(x, \Sigma_k)|\nabla w_k(x)| \le C, \quad \forall x \in \Omega.$$

Proof. By Green's representation formula, we have

$$|\nabla v_k| \le C \int_\Omega \frac{1}{|x-z|}\left[h_1 e^{v_k}(z) - h_2 e^{w_k}(z)\right] \mathrm{d}z.$$

To simplify our notation, we rewrite

$$\Sigma_k = \{q_{k,1}, \cdots, q_{k,n}\}.$$

Let

$$R_k(x) := \inf_{i=1,\cdots,n} |x - q_{k,i}|,$$

$$\Omega_{k,i} = \left\{x \in \Omega : |x - q_{k,i}| = R_k(x)\right\}, \qquad i = 1, \cdots, n.$$

It is easy to see that $\Omega = \cup_{i=1}^n \Omega_{k,i}$. By using (3.10), for any $z \in \Omega_{k,i} \setminus B_{\frac{|x-q_{k,i}|}{2}}(q_{k,i})$,

$$|x-z|^{-1} e^{v_k(z)} \le \frac{C}{|x-z||z-q_{k,i}|^2} \le \frac{C}{|x-z||x-q_{k,i}|^2}.$$

Then,

$$\int_{\Omega_{k,i}\setminus B_{\frac{|x-q_{k,i}|}{2}}(q_{k,i})} \frac{h_1 e^{v_k}(z)}{|x-z|}\mathrm{d}z \leq \frac{C}{|x-q_{k,i}|}. \tag{3.11}$$

On the other hand, for $z \in \Omega_{k,i} \cap B_{\frac{|x-q_{k,i}|}{2}}(q_{k,i})$, we have $|x-z| \geq \frac{1}{2}|x-q_{k,i}|$ and hence

$$\int_{\Omega_{k,i}\cap B_{\frac{|x-q_{k,i}|}{2}}(q_{k,i})} \frac{h_1 e^{v_k}(z)}{|x-z|}\mathrm{d}z \leq \frac{C}{|x-q_{k,i}|}. \tag{3.12}$$

By (3.11) and (3.12), we have

$$\int_{\Omega_{k,i}} \frac{h_1 e^{v_k}(z)}{|x-z|}\mathrm{d}z \leq \frac{C}{|x-q_{k,i}|}. \tag{3.13}$$

Similarly,

$$\int_{\Omega_{k,i}} \frac{h_2 e^{w_k}(z)}{|x-z|}\mathrm{d}z \leq \frac{C}{|x-q_{k,i}|}. \tag{3.14}$$

From (3.13) and (3.14), we can easily obtain that

$$\inf_{i=1,\cdots,n} |x-q_{k,i}||\nabla v_k(x)| \leq C.$$

Equivalently, we get

$$\mathrm{dist}(x,\Sigma_k)|\nabla v_k(x)| \leq C.$$

We note that $|\nabla v_k(x)| = |\nabla w_k(x)|$. Therefore, we get the other inequality. □

Remark 2. A similar selection process as in Proposition 3.1 can be carried out also for the Toda-type system in (1.8). The only difference is that the dilated functions considered in point (3) may both converge to an entire solution of the Toda system in $\mathbb{R}^2$. We refer for example to [24]. Nevertheless, similar estimates as in Proposition 3.1 and Lemmas 3.3, 3.4 hold true for (1.8).

Concerning the asymptotic behavior of the blowing up solutions we have the following.

Lemma 3.5. *Let u_k be a sequence of solutions to (1.1) and v_k, w_k are defined in Section 2. Then, in the sense of measures, we have*

$$h_1 e^{v_k} dx \to r_1(x)dx + \sum_{p\in\mathcal{S}\cap\Omega} m_1(p)\delta_p \text{ in } \Omega, \tag{3.15}$$

$$h_2 e^{w_k} dx \to r_2(x)dx + \sum_{p\in\mathcal{S}\cap\Omega} m_2(p)\delta_p \text{ in } \Omega, \tag{3.16}$$

where $r_i(x) \in L^1(\overline{\Omega}) \cap C^\infty_{\mathrm{loc}}(\overline{\Omega}\setminus\mathcal{S})$ and $m_i(p)$ are multiple of 8π for $i = 1, 2$. Moreover, u_k converges to $\mathcal{G}+\mathcal{U}$ in $C^\infty_{\mathrm{loc}}(\overline{\Omega}\setminus\mathcal{S})$ and in $W^{1,q}_0(\Omega)$ for any $q < 2$. Here $\mathcal{G}$ and $\mathcal{U}$ are defined by

$$\Delta\mathcal{G}(x) + \sum_{p\in\mathcal{S}\cap\Omega}\big(m_1(p)-m_2(p)\big)\delta_p = 0 \quad \text{in } \Omega, \qquad \mathcal{G}(x) = 0 \quad \text{on } \partial\Omega,$$

and

$$\Delta\mathcal{U}(x) + r_1(x) - r_2(x) = 0 \quad \text{in } \Omega, \qquad \mathcal{U}(x) = 0 \quad \text{on } \partial\Omega.$$

Proof. By [19, Lemma 3.4], we can get (3.15)-(3.16). Using the quantization result [19, Theorem 1.1] (one can see [16] for a different proof), we have

$$\big(m_1(p), m_2(p)\big) = \big(4\pi(l+1)l, 4\pi(l-1)l\big) \text{ or } \big(4\pi(l-1)l, 4\pi(l+1)l\big),$$

for $l \in \mathbb{N}$ when $p_j \in \Omega$. The left conclusion of the Lemma 3.5 is a direct consequence of classical elliptic regularity theory and Lemma 3.3. □

Remark 3. For equation (1.8), let u_{1k}, u_{2k} be a sequence of solutions to (1.8) and $\widetilde{u}_{1k}, \widetilde{u}_{2k}$ be defined in Section 2. Then, when $\mathbf{K} = \mathbf{B}_2$, we have

$$h_i e^{\widetilde{u}_{ik}} dx \to f_i(x) + \sum_{p \in \mathcal{S} \cap \Omega} n_i(p)\delta_p \qquad \text{in } \Omega,\ i = 1, 2,$$

where $f_1, f_2 \in L^1(\Omega) \cap C^\infty_{\text{loc}}(\overline{\Omega} \setminus \mathcal{S})$ and $(n_1(p), n_2(p))$ can only be one of the following,

$$\Big\{(4\pi, 0), (0, 4\pi), (8\pi, 4\pi), (4\pi, 12\pi), (8\pi, 16\pi), (12\pi, 12\pi), (12\pi, 16\pi)\Big\}.$$

When $\mathbf{K} = \mathbf{G}_2$, we have

$$h_i e^{\widetilde{u}_{ik}} dx \to g_i(x) + \sum_{p \in \mathcal{S} \cap \Omega} l_i(p)\delta_p \qquad \text{in } \Omega,\ i = 1, 2,$$

where $g_1, g_2 \in L^1(\Omega) \cap C^\infty_{\text{loc}}(\overline{\Omega} \setminus \mathcal{S})$ and $(l_1(p), l_2(p))$ can only be one of the following,

$$\Big\{(4\pi, 0), (0, 4\pi), (4\pi, 16\pi), (8\pi, 4\pi), (24\pi, 36\pi), (24\pi, 40\pi),$$
$$(8\pi, 24\pi), (16\pi, 16\pi), (16\pi, 36\pi), (20\pi, 24\pi), (20\pi, 40\pi)\Big\}.$$

The readers are referred to [29, Theorem 1.1] and [24, Theorem 1.3] for the proof of the above quantization result.

We prove now the main result concerning the no boundary blow up.

Proof of Theorem 1.1. We have to prove that $S \cap \partial\Omega = \emptyset$. We argue by contradiction. Let $x_0 \in \mathcal{S} \cap \partial\Omega$. Since $|\mathcal{S}|$ is finite, we may assume further that $\mathcal{S} \cap B_r(x_0) = \{x_0\}$. Let $z_k = x_0 + \Theta_{k,r}\nu(x_0)$ with

$$\Theta_{k,r} = \frac{\int_{\partial\Omega \cap B_r(x_0)} \langle x - x_0, \nu\rangle \left|\frac{\partial u_k}{\partial \nu}\right|^2}{\int_{\partial\Omega \cap B_r(x_0)} \langle \nu(x_0), \nu\rangle \left|\frac{\partial u_k}{\partial \nu}\right|^2}, \tag{3.17}$$

where r is small such that $\frac{1}{2} \le \langle \nu(x_0), \nu\rangle \le 1$ for $x \in \partial\Omega \cap B_r(x_0)$. Here $\nu(x)$ is the unit outer normal at $x \in \partial\Omega$. It is then easy to check that $|\Theta_{k,r}| \le 2r$ for $|\langle x - x_0, \nu\rangle| \le r$. Observing

$$x - z_k = x - x_0 - \Theta_{k,r}\nu(x_0),$$

we know that

$$\int_{\partial\Omega \cap B_r(x_0)} \langle x - z_k, \nu\rangle \left|\frac{\partial u_k}{\partial \nu}\right|^2 = 0. \tag{3.18}$$

Now, applying the Pohozaev identity of Lemma 2.4 in $\Omega \cap B_r(x_0)$ with $\xi = z_k$, we have that

$$\begin{aligned}
&\int_{\Omega\cap B_r(x_0)} 2h_1 e^{u_k-\alpha_{1k}} + \int_{\Omega\cap B_r(x_0)} 2h_2 e^{-u_k-\alpha_{2k}} + \int_{\Omega\cap B_r(x_0)} e^{u_k-\alpha_{1k}}\langle x-z_k, \nabla h_1\rangle \\
&\quad + \int_{\Omega\cap B_r(x_0)} e^{-u_k-\alpha_{2k}}\langle x-z_k, \nabla h_2\rangle \\
&= \int_{\partial(\Omega\cap B_r(x_0))} \left(h_1 e^{u_k-\alpha_{1k}} + h_2 e^{-u_k-\alpha_{2k}}\right)\langle x-z_k, \nu\rangle + \int_{\partial(\Omega\cap B_r(x_0))} \frac{\partial u_k}{\partial \nu}\langle x-z_k, \nabla u_k\rangle \\
&\quad - \frac{1}{2}\int_{\partial(\Omega\cap B_r(x_0))} |\nabla u_k|^2 \langle x-z_k, \nu\rangle. \qquad (3.19)
\end{aligned}$$

In view of the boundary conditions, it is easy to see that

$$\lim_{k\to+\infty} \int_{\partial\Omega\cap B_r(x_0)} \left(h_1 e^{u_k-\alpha_{1k}} + h_2 e^{-u_k-\alpha_{2k}}\right)\langle x-z_k, \nu\rangle = O(r^2)$$

and, by (3.18),

$$\begin{aligned}
&\int_{\partial\Omega\cap B_r(x_0)} \frac{\partial u_k}{\partial \nu}\langle x-z_k, \nabla u_k\rangle - \frac{1}{2}\int_{\partial\Omega\cap B_r(x_0)} |\nabla u_k|^2 \langle x-z_k, \nu\rangle \\
=&\frac{1}{2}\int_{\partial\Omega\cap B_r(x_0)} \langle x-z_k, \nu\rangle |\nabla u_k|^2 = 0.
\end{aligned}$$

From the proof of Lemma (3.1) and standard elliptic regularity theory, we have $u_k \to \mathcal{G} + \mathcal{U}$ in $C^\infty_{\mathrm{loc}}(\Omega \setminus \mathcal{S})$. Then, we claim

$$\lim_{k\to+\infty} \int_{\Omega\cap\partial B_r(x_0)} \left(h_1 e^{u_k-\alpha_{1k}} + h_2 e^{-u_k-\alpha_{2k}}\right)\langle x-z_k, \nu\rangle = O(\varepsilon(r)),$$

and

$$\int_{\Omega\cap\partial B_r(x_0)} \frac{\partial u_k}{\partial \nu}\langle x-z_k, \nabla u_k\rangle - \frac{1}{2}\int_{\Omega\cap\partial B_r(x_0)} |\nabla u_k|^2 \langle x-z_k, \nu\rangle = O(\varepsilon(r)),$$

where $\varepsilon(r) \to 0$ as $r \to 0$ and we leave the proof of this claim in next Lemma 3.6. On the other hand, since $\int_\Omega h_1 e^{u_k-\alpha_{1k}} \le C$ and $\int_\Omega h_2 e^{-u_k-\alpha_{2k}} \le C$, it holds that

$$\lim_{k\to+\infty} \int_{\Omega\cap B_r(x_0)} e^{u_k-\alpha_{1k}}\langle x-z_k, \nabla h_1\rangle = O(r)$$

and

$$\lim_{k\to+\infty} \int_{\Omega\cap B_r(x_0)} e^{-u_k-\alpha_{2k}}\langle x-z_k, \nabla h_2\rangle = O(r).$$

Then we have,

$$\lim_{r\to 0}\lim_{k\to\infty} \int_{\Omega\cap B_r(x_0)} h_1 e^{u_k-\alpha_{1k}} + h_2 e^{-u_k-\alpha_{2k}} = 0,$$

this is a contradiction to (3.3). We prove the lemma. □

We are left with the proof of the following estimate.

Lemma 3.6. *For any small $\varepsilon > 0$, we can always choose $r > 0$ small such that,*

$$\lim_{k\to\infty} \int_{\Omega\cap\partial B_r(x_0)} \left(h_1 e^{u_k-\alpha_{1k}} + h_2 e^{-u_k-\alpha_{2k}}\right)|\langle x-z_k, \nu\rangle| = O(\varepsilon), \qquad (3.20)$$

and

$$\int_{\Omega\cap\partial B_r(x_0)} \left| \frac{\partial u_k}{\partial \nu}\langle x-z_k, \nabla u_k\rangle - \frac{1}{2}|\nabla u_k|^2\langle x-z_k,\nu\rangle \right| = O(\varepsilon). \tag{3.21}$$

Proof. Set $r \in \left(0, \frac{1}{2}\mathrm{dist}(x_0, \mathcal{S}\setminus\{x_0\})\right)$. Then, on $\Omega\cap\partial B_r(x_0)$, we can get $\|\mathcal{G}\|_{C^2(\Omega\cap\partial B_r(x_0))} \le C$ for some C independent of r. Note that

$$|x-z_k| = |x-x_0-\Theta_{k,r}\nu(x_0)| \le |x-x_0| + |\Theta_{k,r}| = O(r) \text{ for } x\in\partial B_r(x_0)\cap\Omega.$$

By the above facts and Lemma 2.3, in order to get (3.20) and (3.21), it suffices to show that for any ε, we can choose r such that

$$\int_{\Omega\cap\partial B_r(x_0)} re^{|\mathcal{U}|} = O(\varepsilon) \quad \text{and} \quad \int_{\Omega\cap\partial B_r(x_0)} r|\nabla\mathcal{U}|^2 = O(\varepsilon). \tag{3.22}$$

We recall that

$$\Delta\mathcal{U}(x) + r_1(x) - r_2(x) = 0 \quad \text{in } \Omega, \qquad \mathcal{U}(x) = 0 \quad \text{on } \partial\Omega.$$

By Green's representation formula and Lemma 2.1, for any $x\in\partial B_r(x_0)\cap\Omega$,

$$\begin{aligned}
|\mathcal{U}(x)| &= \int_\Omega \left|G(x,y)\big(r_1(y)-r_2(y)\big)\right| \mathrm{d}y \\
&\le C\int_\Omega \log\left(2+\frac{1}{|x-y|}\right)\big(|r_1(y)|+|r_2(y)|\big)\mathrm{d}y \\
&\le C\int_{\Omega\cap B_{r'}(x_0)} \log\left(2+\frac{1}{|x-y|}\right)\big(|r_1(y)|+|r_2(y)|\big)\mathrm{d}y \\
&\quad + C\int_{\Omega\setminus B_{r'}(x_0)} \log\left(2+\frac{1}{|x-y|}\right)\big(|r_1(y)|+|r_2(y)|\big)\mathrm{d}y,
\end{aligned} \tag{3.23}$$

where r' is chosen such that $B_{3r'}(x_0)\cap(\mathcal{S}\setminus\{x_0\}) = \emptyset$ and

$$\int_{\Omega\cap B_{r'}(x_0)} \big(|r_1(y)|+|r_2(y)|\big)\mathrm{d}y \le \delta,$$

with δ to be determined later. Here we note that the choice of r', δ and the constants C in (3.23) are independent of r. For the last term on the right hand side of (3.23), we have

$$\int_{\Omega\setminus B_{r'}(x_0)} \log\left(2+\frac{1}{|x-y|}\right)\big(|r_1(y)|+|r_2(y)|\big)\mathrm{d}y \le C\log\left(1+\frac{1}{r'}\right), \tag{3.24}$$

where C depends only on $\|r_i\|_{L^1(\Omega)}$, $i=1,2$ and the shape of the domain Ω, but independent of r. For the other term in (3.23), we write

$$\begin{aligned}
&\int_{\Omega\cap B_{r'}(x_0)} \log\left(2+\frac{1}{|x-y|}\right)\big(|r_1(y)|+|r_2(y)|\big)\mathrm{d}y \\
&= \int_{\Omega\cap B_{r'}(x_0)\cap B_{\frac{r}{N}}(x)} \log\left(2+\frac{1}{|x-y|}\right)\big(|r_1(y)|+|r_2(y)|\big)\mathrm{d}y \\
&\quad + \int_{(\Omega\cap B_{r'}(x_0))\setminus B_{\frac{r}{N}}(x)} \log\left(2+\frac{1}{|x-y|}\right)\big(|r_1(y)|+|r_2(y)|\big)\mathrm{d}y \\
&= I_1 + I_2,
\end{aligned}$$

where N is determined later. From Lemma 3.3 and the fact $B_{3r'}(x_0) \cap (\mathcal{S} \setminus \{x_0\}) = \emptyset$, we have

$$|x - x_0|^2 \max\{|r_1(x)|, |r_2(x)|\} \leq C \qquad \text{in } \Omega \cap B_{r'}(x_0). \tag{3.25}$$

Let us further restrict $r < \min\left\{\frac{r'}{4}, \frac{1}{2}\text{dist}(x_0, \mathcal{S} \setminus \{x_0\})\right\}$. For any $y \in B_{\frac{r}{N}}(x)$, we have

$$|y - x_0| \geq |x - x_0| - |x - y| = \frac{N-1}{N} r.$$

Then,

$$\max\{|r_1(y)|, |r_2(y)|\} \leq C\Big(\frac{N}{N-1}\Big)^2 \frac{1}{r^2}. \tag{3.26}$$

By using (3.26), we can get

$$I_1 \leq C\Big(\frac{N}{N-1}\Big)^2 \Big(\frac{1}{N}\Big)^2 \log\left(2 + \frac{N}{r}\right) = C\Big(\frac{1}{N-1}\Big)^2 \log\left(2 + \frac{N}{r}\right). \tag{3.27}$$

For $y \in \big(\Omega \cap B_{r'}(x_0)\big) \setminus B_{\frac{r}{N}}(x)$, we have

$$\log\left(2 + \frac{1}{|x-y|}\right) \leq \log\left(2 + \frac{N}{r}\right).$$

Therefore,

$$I_2 \leq C \log\left(2 + \frac{N}{r}\right) \int_{\Omega \cap B_{r'}(x_0)} \big(|r_1(y)| + |r_2(y)|\big) \mathrm{d}y \leq C\delta \log\left(2 + \frac{N}{r}\right). \tag{3.28}$$

From (3.23)-(3.28), we have

$$|\mathcal{U}|(x) \leq C \log\left(1 + \frac{1}{r'}\right) + C\left(\frac{1}{(N-1)^2} + \delta\right) \log\left(2 + \frac{N}{r}\right)$$

and

$$e^{|\mathcal{U}|}(x) \leq \left(1 + \frac{1}{r'}\right)^C \left(2 + \frac{N}{r}\right)^{C\left(\frac{1}{(N-1)^2} + \delta\right)}.$$

For the above term, we choose N sufficiently large and then r' small such that $C\big(\frac{1}{(N-1)^2} + \delta\big) < 2$. Here we note the choices of N and r' are independent of r. In the end, for any ε, we choose r sufficiently small such that

$$r^2 \left(1 + \frac{1}{r'}\right)^C \left(2 + \frac{N}{r}\right)^{C\left(\frac{1}{(N-1)^2} + \delta\right)} \leq \varepsilon.$$

As a conclusion, we obtain the first one in (3.22): for the left one we can use a similar way to get it. This finishes the proof of the lemma. □

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