ON THE SMOOTHNESS OF THE VALUE FUNCTION FOR AFFINE OPTIMAL CONTROL PROBLEMS

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Abstract. We prove that the value function associated with an affine optimal control problem with quadratic cost plus a potential always admits points of smoothness. The result is obtained by a careful analysis of points of continuity of the value function, without assuming any condition on singular minimizers.

Contents

1. Introduction 1
2. Preliminaries 3
3. Tame points 8
4. Smooth points 17
Acknowledgments 19
References 19

1. Introduction

The regularity of the value function associated with an optimal control problem is a classical topic of investigation in control theory and has been deeply studied in the last decades, extensively using tools from nonsmooth analysis. It is indeed well-known that the value function associated with an optimal control problem fails to be everywhere differentiable, in general. This is typically the case at those points where the uniqueness of minimizers is not guaranteed. Actually, the value function associated with affine optimal control problem is not even continuous in general, as soon as singular minimizers are allowed (see for instance [4, 19]).

In this paper we investigate the regularity of the value function associated with affine optimal control problems whose cost is written as a quadratic term plus a potential. Inspired by the results and the arguments of [1, 15], we prove that in such cases the value function always admits smoothness points. Indeed, we show that the value function is smooth on an open dense subset of the interior of the attainable set, without assuming any condition on singular minimizers. Let us introduce briefly the setting and the main results.

1.1. Setting and main results. Let $M$ be a smooth, connected $m$-dimensional manifold and let $T > 0$ be a given fixed final time. A smooth affine control system is a dynamical system which can be written in the form:

$$
\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{d} u_i(t)X_i(x(t)),
$$

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where $X_0, X_1, \ldots, X_d$ are smooth vector fields on $M$, and the map $t \mapsto u(t) = (u_1(t), \ldots, u_d(t))$ belongs to the Hilbert space $L^2([0, T], \mathbb{R}^d)$.

Given $x_0 \in M$ we define:

(i) the set of admissible controls $\Omega^T_{x_0}$ as the subset of $u \in L^2([0, T], \mathbb{R}^d)$ such that the solution $x_u(\cdot)$ to (1.1) satisfying $x_u(0) = x_0$ is defined on the interval $[0, T]$. If $u \in \Omega^T_{x_0}$ we say that $x_u(\cdot)$ is an admissible trajectory. By classical results of ODE theory, the set $\Omega^T_{x_0}$ is open.

(ii) the attainable set $A^T_{x_0}$ (in time $T > 0$, from the point $x_0$) as the set of points of $M$ that can be reached from $x_0$ by admissible trajectories in time $T$, i.e.,

$$A^T_{x_0} = \{ x_u(T) \mid u \in \Omega^T_{x_0} \}.$$

For a given smooth function $Q : M \to \mathbb{R}$ we are interested in those trajectories that minimize the cost given by

$$C_T : \Omega^T_{x_0} \to \mathbb{R}, \quad C_T(u) = \frac{1}{2} \int_0^T \left( \sum_{i=1}^k u_i(t)^2 - Q(x_u(t)) \right) dt. \quad (1.2)$$

More precisely, given $x_0 \in M$ and $T > 0$ we are interested in the regularity properties of the value function $S^T_{x_0} : M \to \mathbb{R}$ defined as follows

$$S^T_{x_0}(x) = \inf \left\{ C_T(u) \mid u \in \Omega^T_{x_0}, x_u(T) = x \right\}; \quad (1.3)$$

with the understanding that $S^T_{x_0}(x) = +\infty$ if $x$ cannot be attained by admissible curves in time $T$. We call optimal control any control $u$ which solves the optimal control problem (1.3).

**Main assumptions.** For the rest of the paper we will assume the following hypotheses:

(H1) The weak Hörmander condition holds on $M$. Namely, we require for every point $x \in M$ the equality

$$\text{Lie}_x \left\{ (\text{ad } X_0)^j X_i \mid j \geq 0, \ i = 1, \ldots, d \right\} = T_x M. \quad (1.4)$$

where $(\text{ad } X) Y = [X, Y]$ and $\text{Lie}_x \mathcal{F} \subset T_x M$ denotes the values at the point $x$ of vector fields that belong to the Lie algebra generated by a family of vector fields $\mathcal{F}$.

(H2) For every bounded family $\mathcal{U} \subset \Omega^T_{x_0}$ of admissible controls, there exists a compact subset $K_T \subset M$ (depending on $\mathcal{U}$) such that $x_u(t) \in K_T$, for every $u \in \mathcal{U}$ and $t \in [0, T]$.

(H3) The potential $Q$ is a smooth function bounded from above.

The assumption (H1) is needed to guarantee that the attainable set has at least nonempty interior, i.e., $\text{int} \left( A^T_{x_0} \right) \neq \emptyset$ (cf. [11] Ch. 3, Thm. 3)). The second assumption (H2) is a completeness/compactness assumption on the dynamical system that, together with (H3), is needed to guarantee the existence of optimal controls. We stress that (H2) and (H3) are automatically satisfied when $M$ is compact. When $M$ is not compact, (H2) holds true under a sublinear growth condition on the vector fields $X_0, \ldots, X_d$. We refer to Section 2 for more details on the role of these assumptions.

The main result of this paper reads as follows.

**Theorem 1.** Fix $x_0 \in M$ and let $S^T_{x_0}$ be the value function associated with an optimal control problem of the form (1.1)-(1.2) satisfying assumptions (H1)-(H3). Then $S^T_{x_0}$ is smooth on a non-empty open and dense subset of $\text{int} \left( A^T_{x_0} \right)$.

In the paper [11] it is proved the analogous result for the value functions associated with sub-Riemannian optimal control problems, i.e., driftless systems with zero potential. In this case (H1) reduces to the classical Hörmander condition and the value function (at time $T$) coincides...
with one half of the square of the sub-Riemannian distance (divided by $T$) associated with the family of vector fields $X_1, \ldots, X_d$.

A key point in our proof of Theorem 1 is the characterization of points where the value function is continuous. It turns out that the continuity of the value function at a point $s$ is strictly related with the openness of the end-point map on the optimal control steering to $x$. Here by end-point map we mean the map that with every control $u$ associates the final point of the corresponding trajectory (cf. Section 2 for precise definitions).

In the sub-Riemannian case (more precisely, when $X_0 = 0$) the end-point map is open at every point (even if it is not a submersion in presence of singular minimizers) and this implies that the sub-Riemannian distance is continuous everywhere. This is no more true in presence of a drift field and the characterization of the set of points where the end-point is open is more delicate.

In the affine case, in [17, 19] the authors assumes that there are no abnormal optimal controls, which gives openness of the end-point at first order, while in [4] the authors obtain the openness of the end-point map on optimal controls with second order techniques assuming no optimal Goh abnormals controls exists. For more details on Goh abnormals we refer the reader to [5, Chapter 20] (see also [3, 16]). Let us mention that in [8] the authors proved that the system (1.1) admits no Goh optimal trajectories for the generic choice of the $(d + 1)$-tuple $X_0, \ldots, X_d$ (in the Whitney topology). Finally in [13] the author proves the continuity (and other Hölder properties) of the $L^1$ cost under a strong bracket generating assumption.

In our setting, we admit the presence of abnormal optimal minimizers of any kind, without any further assumption on their nature. We introduce the set $\Sigma_t \subset \text{int} (A_T^{x_0})$ of points $x$ such that the end-point map is open and a submersion on every optimal control steering to $x$ (called tame points in what follows). The core of the proof of Theorem 1, which leads to a non trivial modification of the argument in the driftless case, is that we can find a large set of tame points.

**Theorem 2.** The set $\Sigma_t$ of tame points is open and dense in $\text{int} (A_T^{x_0})$. Moreover $S_T^{x_0}$ is continuous on $\Sigma_t$.

Let us mention that, even in the sub-Riemannian case, it is an open question whether the set of smoothness points of the value function has full measure in $\text{int} (A_T^{x_0})$ or not.

For different approaches investigating the regularity of the value function through techniques of nonsmooth analysis, one can see for instance the monographs [6, 9, 7, 10].

1.2. **Structure of the paper.** In Section 2 we recall some properties of the end-point map, we prove the existence of minimizers in our setting and recall their characterization in terms of an Hamiltonian equation. Section 3 is devoted to the study of tame points and the proof of Theorem 2. In the last Section 4 we complete the proof of Theorem 1.

2. **Preliminaries**

For a fixed admissible control $u \in \Omega_T^{x_0}$, it is well-defined the family of diffeomorphisms

$$P_{0,t}^u : U_{x_0} \subset M \to M, \quad t \in [0, T],$$

definite on some neighborhood $U_{x_0}$ of $x_0$ by $P_{0,t}^u(y) = x_{u,y}(t)$, where $x_{u,y}(t)$ is the solution of the equation (1.1) with initial condition $x_{u,y}(0) = y$. It is a classical fact that this family is Lipschitz with respect to $t$. Similarly, given $u \in \Omega_T^{x_0}$ it is possible to define the family of diffeomorphism flow $P_{s,t}^u : U_{x_0} \to M$ by solving (1.1) with initial condition $x_{u,y}(s) = y$; notice then that $P_{t,t}^u = \text{Id}$, and that the following composition formulas hold true (at those points where
all terms are defined):

\[ P_{s,t}^u \circ P_{r,s}^u = P_{r,t}^u \quad \text{and} \quad (P_{s,t}^u)^{-1} = P_{t,s}^u. \]

2.1. The end-point map. In what follows we fix a point \( x_0 \in M \) and a time \( T > 0 \).

**Definition 3** (end-point map). The end-point map at time \( T \) is the map

\[ E^T_{x_0} : \Omega^T_{x_0} \to M, \quad E^T_{x_0}(u) = x_u(T), \]

where \( x_u(\cdot) \) is the admissible trajectory driven by the control \( u \).

The end-point map is smooth on \( \Omega^T_{x_0} \subset L^2([0,T], \mathbb{R}^d) \). The computation of its Fréchet differential is classical and can be found for example in [3, 16, 19]:

**Proposition 4.** The differential \( d_u E^T_{x_0} : L^2([0,T], \mathbb{R}^d) \to T_{x_u(T)}M \) of the end-point map at \( u \in \Omega^T_{x_0} \) is given by the formula:

\[
(2.1) \quad d_u E^T_{x_0}(v) = \int_0^T \sum_{i=1}^d v_i(s) \left( P_{s,T}^u \right)_s X_i(x_u(s)) \, ds.
\]

Let us consider a sequence of admissible controls \( \{u_n\}_{n \in \mathbb{N}} \), which weakly converges to some element \( u \in L^2([0,T], \mathbb{R}^d) \). Then the sequence \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( L^2 \) and, thanks to our assumption (H2), there exists a compact set \( K_T \) such that \( x_{u_n}(t) \in K_T \) for all \( n \in \mathbb{N} \) and \( t \in [0,T] \). This yields that the family of trajectories \( \{x_{u_n}(\cdot)\}_{n \in \mathbb{N}} \) is uniformly bounded, and from here it is a classical fact to deduce that the weak limit \( u \) is an admissible control, and that \( x_u(\cdot) = \lim_{n \to \infty} x_{u_n}(\cdot) \) (in the uniform topology) is its associated trajectory (see for example [20]).

This proves that the end-point map \( E^T_{x_0} \) is *weakly continuous*. Indeed one can prove that the same holds true for its differential \( d_u E^T_{x_0} \). More precisely: if \( \{u_n\}_{n \in \mathbb{N}} \) is a sequence of admissible controls which weakly converges in \( L^2([0,T], \mathbb{R}^d) \) to \( u \) (which is admissible by the previous discussion), we have that

\[
\lim_{n \to \infty} E^T_{x_0}(u_n) = E^T_{x_0}(u) \quad \text{and} \quad \lim_{n \to \infty} d_u E^T_{x_0} = d_u E^T_{x_0},
\]

where the last convergence is in the (strong) operator norm (see [19]).

**Remark.** There are other possible assumptions to ensure that the weak limit of a sequence of admissible controls is again an admissible control; for example, as suggested in [17], one could ask a sublinear growth condition on the vector fields \( X_0, \ldots, X_d \). In this case the uniform bound on the trajectories (equivalent to (H2)) follows as a consequence of the Gronwall inequality, and the observation that a weakly convergent sequence in \( L^2 \) is necessarily bounded.

**Definition 5** (Attainable set). For a fixed final time \( T > 0 \), we denote by \( A^T_{x_0} \) the image of the end-point map at time \( T \), and we call it the *attainable set* (from the point \( x_0 \)).

In general the inclusion \( A^T_{x_0} \subset M \) can be proper, that is the end-point map \( E^T_{x_0} \) may not be surjective on \( M \); nevertheless, the weak Hörmander condition \([1.4]\) implies that for every initial point \( x_0 \) one has \( \text{int} (A^T_{x_0}) \neq \emptyset \) [11] Ch. 3, Thm. 3].

2.2. Value function and optimal trajectories. If we introduce the Tonelli Lagrangian

\[
L : M \times \mathbb{R}^d \to \mathbb{R}, \quad L(x, u) = \frac{1}{2} \left( \sum_{i=1}^d u_i^2 - Q(x) \right),
\]
Then the cost $C_T : \Omega^T_{x_0} \to \mathbb{R}$ is written as

$$C_T(u) = \int_0^T L(x_u(t), u(t)) dt = \frac{1}{2} \int_0^T \left( \sum_{i=1}^d u_i(t)^2 - Q(x_u(t)) \right) dt.$$ 

The differential $d_u C_T$ of the cost can be recovered similarly as for the differential of the end-point map, and is given, for every $v \in L^2([0,T], \mathbb{R}^d)$, by the formula

$$d_u C_T(v) = \int_0^T \langle u(t), v(t) \rangle dt - \frac{1}{2} \int_0^T Q'(x_u(t)) \left( \int_0^t \sum_{i=1}^d v_i(s)(P_{s,t}^u X_i) ds \right) (x_u(t)) dt.$$ 

that is obtained by writing $x_u(t) = E^t_{x_0}(u)$ and applying (2.1).

Fix two points $x_0$ and $x$ in $M$. The problem of describing optimal trajectories steering $x_0$ to $x$ in time $T$ can be naturally reformulated in the following way: introducing the value function $S^T_{x_0} : M \to \mathbb{R}$, we write in fact

$$S^T_{x_0}(x) = \inf \left\{ C_T(u) \mid u \in \Omega^T_{x_0} \cap (E^T_{x_0})^{-1}(x) \right\};$$

moreover, we set $S^T_{x_0}(x) = +\infty$ if the preimage $(E^T_{x_0})^{-1}(x)$ is empty. Then, for any fixed $x \in M$, the optimal control problem consists into looking for elements $u \in L^2([0,T], \mathbb{R}^d)$ realizing the infimum in (2.2); accordingly, from now on we will call optimal control any admissible control $u$ which solves the optimal control problem.

In this paper we will always concentrate on the case that the final point $x$ of an admissible trajectory belongs to the interior of the attainable set $A^T_{x_0}$. Indeed, it is a general fact that $\text{int} (A^T_{x_0})$ is densely contained in $A^T_{x_0}$, and the weak Hörmander condition ensures that $\text{int} (A^T_{x_0})$ is nonempty; moreover, for every point $x \in \text{int} (A^T_{x_0})$, we trivially have that $S^T_{x_0}(x) < +\infty$, since by definition there exists at least one admissible control $v$ steering $x_0$ to $x$.

Existence of minimizers under our main assumptions (H1)-(H3) is classical. For the reader’s convenience we give a proof below.

**Proposition 6** (Existence of minimizers). Let $x \in A^T_{x_0}$. Then there exists an optimal control $u \in \Omega^T_{x_0}$ satisfying:

$$E^T_{x_0}(u) = x, \quad \text{and} \quad C_T(u) = S^T_{x_0}(x).$$

**Proof.** Let $c > 0$ be any real constant satisfying $Q \leq c$ (the existence of $c$ follows from (H3)). Then, for every admissible control $v \in \Omega^T_{x_0}$ such that $E^T_{x_0}(v) = x$, we have:

$$C_T(v) = \frac{1}{2} \int_0^T \left( \sum_{i=1}^d u_i(t)^2 - Q(x_v(t)) \right) dt \geq -\frac{cT}{2},$$

therefore, taking the infimum on both sides, we deduce the lower bound $S^T_{x_0}(x) \geq -cT/2 > -\infty$.

Consider a minimizing sequence $\{u_n\}_{n \in \mathbb{N}}$ such that $\liminf_{n \to \infty} C_T(u_n) = S^T_{x_0}(x)$; for $n$ large enough, the relation $C_T(u_n) \leq S^T_{x_0}(x) + 1$ holds, and yields that the sequence of $L^2$-norms $\{\|u_n\|_{L^2}\}_{n \in \mathbb{N}}$ is uniformly bounded:

$$\|u_n\|_{L^2}^2 = \int_0^T \sum_{i=1}^d u_i(t)^2 dt \leq 2(S^T_{x_0}(x) + 1) + cT.$$

Since balls are weakly compact for the $L^2$-topology, there exist a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}} \subset \{u_n\}_{n \in \mathbb{N}}$ and $u \in \Omega^T_{x_0}$ such that $u_{n_k} \rightharpoonup u$, and then we call $x_u(\cdot)$ the admissible trajectory associated with $u$. 

The weak continuity of the end-point map $E^T_{x_0}$ ensures that $x = \lim_{k \to \infty} E^T_{x_0}(u_{n_k}) = E^T_{x_0}(u)$ and moreover, since $x_{u_{n_k}}(\cdot) \to x_u(\cdot)$ uniformly as trajectories (cf. Section 2.2), the integral terms strongly converge:

$$\lim_{k \to \infty} \int_0^T Q(x_{u_{n_k}}(t))dt = \int_0^T Q(x_u(t))dt.$$ 

Finally, since the $L^2$-norm is weakly lower semicontinuous, we have the inequalities:

$$C_T(u) \leq \liminf_{k \to \infty} \frac{1}{2} \int_0^T \left( \sum_{i=1}^d u_{i,n_k}(t)^2 - Q(x_{u_{n_k}}(t)) \right) dt$$

$$= \liminf_{k \to \infty} C_T(u_{n_k}) = S^T_{x_0}(x) \leq C_T(u),$$

which imply that $S^T_{x_0}(x) = C_T(u)$, as claimed. \hfill \Box

**Remark.** The assumptions (H2)-(H3) play a crucial role for the existence of optimal control. An equivalent approach could be to work directly inside a given compact set (see [2]) or with $M$ itself a compact manifold. For some specific cases, as in the classical case of the harmonic oscillator, one is able to integrate directly Hamilton’s equations (cf. Section 2.4), and the existence of optimal trajectories could be proved with ad hoc arguments.

**Proposition 7.** The map $S^T_{x_0} : \text{int}(A^T_{x_0}) \to \mathbb{R}$ is lower semicontinuous.

**Proof.** Let $x \in \text{int}(A^T_{x_0})$. To prove the claim it is sufficient to show that for every sequence $x_n \to x$ there holds:

$$S^T_{x_0}(x) \leq \liminf_{n \to \infty} S^T_{x_0}(x_n).$$

If $\liminf_{n \to \infty} S^T_{x_0}(x_n) = +\infty$ the statement is trivially true, and therefore we can assume that $\liminf_{n \to \infty} S^T_{x_0}(x_n) = L < +\infty$; in particular we can extract a subsequence $(x_{n_k})_{k \in \mathbb{N}} \subset \{ x_n \}_{n \in \mathbb{N}}$ such that $\lim_{k \to \infty} S^T_{x_0}(x_{n_k}) = L$.

Let $c > 0$ be an upper bound for the map $Q$. If we denote by $u_{n_k}$ an optimal control associated with $x_{n_k}$, then for $k$ large enough $S^T_{x_0}(x_{n_k}) \leq L + 1$ and, in turn, this yields

$$\| u_{n_k} \|_{L^2}^2 \leq 2(L + 1) + \int_0^T Q(x_{u_{n_k}}(t))dt \leq 2(L + 1) + cT.$$ 

It is therefore not restrictive to assume, up to subsequences, that $u_{n_k} \rightharpoonup u$ weakly in $L^2$. As in Proposition 6, the weak convergence $u_{n_k} \rightharpoonup u$ implies both that $x = \lim_{k \to \infty} E^T_{x_0}(u_{n_k}) = E^T_{x_0}(u)$ and also that $x_{u_{n_k}}(\cdot) \to x_u(\cdot)$ uniformly in the space of trajectories. Since the $L^2$-norm is weakly sequentially lower semicontinuous, we can write:

$$S^T_{x_0}(x) \leq C_T(u) = \frac{1}{2} \| u \|_{L^2}^2 - \frac{1}{2} \int_0^T Q(x_u(t))dt$$

$$\leq \liminf_{k \to \infty} \frac{1}{2} \| u_{n_k} \|_{L^2}^2 - \frac{1}{2} \int_0^T Q(x_{u_{n_k}}(t))dt = \liminf_{k \to \infty} S^T_{x_0}(x_{n_k}) \leq L,$$

which completes the proof. \hfill \Box

### 2.3. Lagrange multipliers rule

In this section, we briefly recall the classical necessary condition satisfied by optimal controls $u$ realizing the infimum in (2.2). It is indeed a restatement of the classical Lagrange multipliers’ rule (see [5, 6, 12]).

**Proposition 8.** Let $u \in L^2([0,T],\mathbb{R}^d)$ be an optimal control with $x = E^T_{x_0}(u)$. Then at least one of the following statements is true:
of the notation, we extend this language even to the associated optimal trajectories
abnormal mutually exclusive, and we define accordingly a control $u$

\[ u \in \mathbb{T}^* \mathbb{M} \text{ with } \lambda_T \neq 0 \text{ such that } \lambda_T d_u E^{T}_{x_0} = 0. \]

Here $\lambda_T d_u E^{T}_{x_0} : L^2([0, T]) \rightarrow \mathbb{R}$ denotes the composition of the linear maps $d_u E^{T}_{x_0} : L^2([0, T]) \rightarrow T_x \mathbb{M}$ and $\lambda_T : T_x \mathbb{M} \rightarrow \mathbb{R}$.

A control $u$, satisfying the necessary conditions for optimality stated in Proposition [8] is said normal in case (a) and abnormal in case (b); moreover, directly from the definition we see that $d_u E^{T}_{x_0}$ is not surjective in the abnormal case. We stress again that the two possibilities are not mutually exclusive, and we define accordingly a control $u$ to be strictly normal (resp. strictly abnormal) if it is normal but not abnormal (resp. abnormal but not normal). Slightly abusing of the notation, we extend this language even to the associated optimal trajectories $t \mapsto x_u(t)$.

2.4. Normal extremals and exponential map. Let us denote by $\pi : T^* \mathbb{M} \rightarrow \mathbb{M}$ the canonical projection of the cotangent bundle, and by $(\lambda, v)$ the duality pairing between a covector $\lambda \in T^*_x \mathbb{M}$ and a vector $v \in T_x \mathbb{M}$. In canonical coordinates $(p, x)$ on the cotangent space, we can express the Liouville form $s = \sum_{i=1}^{m} p_i dx_i$ and the standard symplectic form $\sigma = ds = \sum_{i=1}^{m} dp_i \wedge dx_i$. We denote by $\vec{h}$ the Hamiltonian vector field associated with a smooth function $h : T^* \mathbb{M} \rightarrow \mathbb{R}$, defined by the identity:

\[
\vec{h} = \sum_{i=1}^{m} \frac{\partial h}{\partial p_i} \frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i} \frac{\partial}{\partial p_i}.
\]

Consider the (control-dependent) Hamiltonian $\mathcal{H} : \mathbb{R}^{d} \times T^* \mathbb{M} \rightarrow \mathbb{R}$:

\[
\mathcal{H}(u, p, x) = \langle p, X_0(x) \rangle + \sum_{i=1}^{d} u_i \langle p, X_i(x) \rangle - \frac{1}{2} \sum_{i=1}^{d} u_i^2 + \frac{1}{2} Q(x)
\]

associated with the system [方程式]. The Pontryagin Maximum Principle [12] tells us that normal optimal trajectories are integral curves of the maximized Hamiltonian

\[
H(p, x) = \max_{u \in \mathbb{R}^{d}} \mathcal{H}(u, p, x);
\]

in particular, since $\mathcal{H}$ is smooth, they satisfy the relation $\partial_u \mathcal{H}(u, p, x) = 0$ so that the optimal control $u$ can be recovered using the equality

\[
(2.3) \quad u_i = \langle p, X_i(x) \rangle, \quad \text{for } i = 1, \ldots, d.
\]

Notice that if $p' \in T^*_x \mathbb{M}$ can be written as $p' = p + \xi$, where $\xi$ satisfies $\langle \xi, X_i(x) \rangle = 0$ for $i = 1, \ldots, d$, then we have $u_i = \langle p, X_i(x) \rangle = \langle p', X_i(x) \rangle$ for all the indexes $i$. The maximized Hamiltonian takes the expression:

\[
H(p, x) = \langle p, X_0(x) \rangle + \frac{1}{2} \sum_{i=1}^{d} \langle p, X_i(x) \rangle^2 + \frac{1}{2} Q(x),
\]

and we see that, along normal trajectories, the relation $u_i(t) = \langle p(t), X_i(x(t)) \rangle$ holds. Since the pair $\dot{x}(t) = \frac{\partial H}{\partial p}(p(t), x(t))$ and $\dot{p}(t) = -\frac{\partial H}{\partial x}(p(t), x(t))$ is smooth, being the solution to a smooth autonomous system of differential equations, we deduce that the normal control $u(t) = \langle p(t), X_i(x(t)) \rangle$ is smooth as well.

**Definition 9** (Exponential map). The exponential map $\mathcal{E}$ with base point $x_0$ is defined as

\[
\mathcal{E}_{x_0} : [0, T] \times T^*_x \mathbb{M} \rightarrow \mathbb{M}, \quad \mathcal{E}_{x_0}(s, \lambda) = \pi(e^{s \vec{h}}(\lambda)).
\]
When the first argument is fixed, we employ the notation \( \mathcal{E}_{x_0}^s : T^*_{x_0} M \to M \) to denote the exponential map with base point \( x_0 \) at time \( s \); that is to say, we set \( \mathcal{E}_{x_0}^s(\lambda) := \mathcal{E}_{x_0}(s, \lambda) \).

The exponential map parametrizes normal trajectories. It is well known that, under our assumptions, small pieces of normal trajectories are optimal among all the admissible curves that connect their end-points (see for instance [5]). We end this section by defining the notion of conjugate points along a normal trajectory.

**Definition 10.** We say that a point \( x = \mathcal{E}_{x_0}(\lambda, s) \) is conjugate to \( x_0 \) along the normal trajectory \( t \mapsto \mathcal{E}_{x_0}(\lambda, t) \) if \( (s, \lambda) \) is a critical point of \( \mathcal{E}_{x_0} \).

### 3. Tame points

In this section we study fine properties of the value function on different subsets of \( \text{int} (A^T_{x_0}) \).

#### 3.1. Fair points

We start by introducing the set of fair points.

**Definition 11.** A point \( x \in \text{int} (A^T_{x_0}) \) is said to be a fair point if there exists a unique optimal trajectory steering \( x_0 \) to \( x \), which admits a normal lift. We call \( \Sigma_j \) the set of all fair points contained in the attainable set.

We stress that only the uniqueness of the optimal trajectory matters in the definition of a fair point; abnormal lifts are as well admitted for the moment.

The lower semicontinuity of \( S^T_{x_0} \) permits to find a great abundance of fair points; their existence is related to the notion of proximal subdifferential (see for instance [9, 15] for more details).

**Definition 12.** Let \( F : \text{int} (A^T_{x_0}) \to \mathbb{R} \) be a lower semicontinuous function. For every \( x \in \text{int} (A^T_{x_0}) \) we call the **proximal subdifferential** at \( x \) the subset of \( T^*_x M \) defined by:

\[
\partial_P F(x) = \{ \lambda = d_x \phi \in T^*_x M \mid \phi \in C^\infty(\text{int} (A^T_{x_0})) \text{ and } F - \phi \text{ attains a local minimum at } x \}.
\]

The proximal subdifferential is a convex subset of \( T^*_x M \) which is nonempty for a lower semicontinuous function [9, Theorem 3.1].

**Proposition 13.** Let \( F : \text{int} (A^T_{x_0}) \to \mathbb{R} \) be a lower semicontinuous function. Then the proximal subdifferential \( \partial_P F(x) \) is not empty for a dense set of points \( x \in \text{int} (A^T_{x_0}) \).

We proved in Proposition 7 that the value function \( S^T_{x_0} : \text{int} (A^T_{x_0}) \to \mathbb{R} \) is lower semicontinuous. Then the proximal subdifferential machinery yields the following result (cf. also [15, 1]):

**Proposition 14.** Let \( x \in \text{int} (A^T_{x_0}) \) be such that \( \partial_P S^T_{x_0}(x) \neq \emptyset \). Then there exists a unique optimal trajectory \( x_u(\cdot) : [0, T] \to M \) steering \( x_0 \) to \( x \), which admits a normal lift. In particular \( x \) is a fair point.

**Proof.** Fix any \( \lambda \in \partial_P S^T_{x_0}(x) \). Let us prove that every optimal trajectory steering \( x_0 \) to \( x \) admits a normal lift having \( \lambda \) as final covector.

Indeed, if \( \phi \) is a smooth function such that \( \lambda = d_x \phi \in \partial_P S^T_{x_0}(x) \), by definition the map

\[
\psi : \text{int} (A^T_{x_0}) \to \mathbb{R}, \quad \psi(y) = S^T_{x_0}(y) - \phi(y)
\]

has a local minimum at \( x \), i.e. there exists an open neighborhood \( O \subset \text{int} (A^T_{x_0}) \) of \( x \) such that \( \psi(y) \geq \psi(x) \) for every \( y \in O \). Then, let \( t \mapsto x_u(t), t \in [0, T] \) be an optimal trajectory from \( x_0 \) to \( x \), let \( u \) be the associated optimal control, and define the smooth map:

\[
\Phi : \Omega^T_{x_0} \to \mathbb{R}, \quad \Phi(v) = C_T(v) - \phi(E^T_{x_0}(v)).
\]
There exists a neighborhood \( \mathcal{V} \subset \Omega^T_{x_0} \) of \( u \) such that \( E^T_{x_0}(\mathcal{V}) \subset O \), and since \( C_T(v) \geq S^T_{x_0}(E^T_{x_0}(v)) \) we have the following chain of inequalities:

\[
\Phi(v) = C_T(v) - \phi(E^T_{x_0}(v)) \geq S^T_{x_0}(E^T_{x_0}(v)) - \phi(E^T_{x_0}(v)) \\
\geq S^T_{x_0}(E^T_{x_0}(u)) - \phi(E^T_{x_0}(u)) = C_T(u) - \phi(E^T_{x_0}(u)) = \Phi(u), \quad \forall v \in \mathcal{V}.
\]

Then

\[
0 = d_u \Phi = d_u C_T - (d_u \phi) d_u E^T_{x_0},
\]

and therefore we see that the curve \( \lambda(t) = e^{(t-T)H}(\lambda) \) is the desired normal lift of the trajectory \( x_u(\cdot) \).

In particular, since any two normal extremal lifts having \( \lambda \) as common final point have to coincide, we see that there can only be one optimal trajectory between \( x_0 \) and \( x \), which precisely means that \( x \in \Sigma_f \) is a fair point. \( \square \)

Remark. Notice that from the previous proof it follows that, when \( \partial P S^T_{x_0}(x) \neq \emptyset \), then the unique normal trajectory steering \( x_0 \) to \( x \) is strictly normal if and only if \( \partial P S^T_{x_0}(x) \) is a singleton.

**Corollary 15** (Density of fair points). The set \( \Sigma_f \) of fair points is dense in \( \text{int}(A^T_{x_0}) \).

In particular we have that all differentiability points of \( S^T_{x_0} \) are fair points.

**Corollary 16.** Suppose that \( S^T_{x_0} \) is differentiable at some point \( x \in \text{int}(A^T_{x_0}) \). Then \( x \) is a fair point, and its normal covector is \( \lambda = d_x S^T_{x_0} \in T^*_x M \).

**Proof.** Indeed, let \( u \) be any optimal control steering \( x_0 \) to \( x \); then it is sufficient to consider the non negative map

\[
v \mapsto C_T(v) - S^T_{x_0}(E^T_{x_0}(v)),
\]

which has by definition a local minimum at \( u \) (equal to zero). Then

\[
0 = d_u C_T - (d_x S^T_{x_0}) d_u E^T_{x_0},
\]

and the uniqueness of \( u \) (hence the claim) follows as in the previous proof. \( \square \)

### 3.2. Points of continuity.

We are also interested in the subset \( \Sigma_c \) of the points of continuity for the value function. The first result we give concerns their existence: again the lower semicontinuity of \( S^T_{x_0} \) permits to find plenty of them.

**Proposition 17.** The set \( \Sigma_c \) is a residual subset of \( \text{int}(A^T_{x_0}) \).

**Proof.** This is a general fact; however, since the proof for real-valued functions is easy, we report it here for the sake of completeness.

We will show that the complement of \( \Sigma_c \) is a meager set, i.e. it can be written as a countable union of closed, nowhere dense subsets of \( \text{int}(A^T_{x_0}) \). Then the claim will follow from the classical Baire category theorem.

Let then \( x \) be a discontinuity point of \( S^T_{x_0} \). This implies that \( S^T_{x_0} \) is not upper semicontinuous at \( x \), i.e. there exists \( \varepsilon > 0 \) and a sequence \( x_n \to x \) such that for all \( n \)

\[
S^T_{x_0}(x) + \varepsilon \leq S^T_{x_0}(x_n).
\]

For any \( q \in \mathbb{Q} \) define the set

\[
K_q = \{ x \in \text{int}(A^T_{x_0}) \mid S^T_{x_0}(x) \leq q \};
\]
the lower semicontinuity of $S^T_{x_0}$ implies that $K_r$ is closed. Moreover, let us choose $r \in \mathbb{Q}$ such that $S^T_{x_0}(x) < r < S^T_{x_0}(x) + \varepsilon$; then by construction $x \in K_r \setminus \text{int}(K_r)$, which means that

$$\text{int}(A^T_{x_0}) \setminus \Sigma_c \subset \bigcup_{r \in \mathbb{Q}} (K_r \setminus \text{int}(K_r)),$$

as desired. \qed

The existence of points of continuity is tightly related to the compactness of optimal controls, as it is shown in the next lemma.

**Lemma 18.** Let $x \in \text{int}(A^T_{x_0})$ be a continuity point of $S^T_{x_0}$. Let \( \{x_n\}_{n \in \mathbb{N}} \subset \text{int}(A^T_{x_0}) \) be a sequence converging to $x$ and let $u_n$ be an optimal control steering $x_0$ to $x_n$. Then there exists a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}} \subset \{x_n\}_{n \in \mathbb{N}}$, whose associated sequence of optimal controls $\{u_{n_k}\}_{k \in \mathbb{N}}$, strongly converges in $L^2([0, T], \mathbb{R}^d)$ to some optimal control $u$ which steers $x_0$ to $x$.

**Proof.** Let $\{x_n\}_{n \in \mathbb{N}} \subset \text{int}(A^T_{x_0})$ be a sequence converging to $x$ and let $\{u_n\}_{n \in \mathbb{N}}$ the corresponding sequence of optimal controls. Since $x$ is a continuity point for the value function, it is not restrictive to assume that the sequence of norms $\{\|u_n\|_{L^2}\}_{n \in \mathbb{N}}$ remains uniformly bounded, and thus we can suppose to extract a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}} \subset \{u_n\}_{n \in \mathbb{N}}$ such that $u_{n_k} \rightharpoonup u$ weakly in $L^2([0, T], \mathbb{R}^d)$, which in turn implies

$$\lim_{k \to \infty} \int_0^T Q(x_{u_{n_k}}(t))dt = \int_0^T Q(x_u(t))dt.$$

Then we have

$$\frac{1}{2}\|u\|^2_{L^2} - \frac{1}{2} \int_0^T Q(x_u(t))dt \leq \liminf_{k \to \infty} \frac{1}{2}\|u_{n_k}\|^2_{L^2} - \frac{1}{2} \int_0^T Q(x_{u_{n_k}}(t))dt \leq \limsup_{k \to \infty} \frac{1}{2}\|u_{n_k}\|^2_{L^2} - \frac{1}{2} \int_0^T Q(x_{u_{n_k}}(t))dt$$

$$= \lim_{k \to \infty} S^T_{x_0}(E^T_{x_0}(u_{n_k})) = \lim_{k \to \infty} S^T_{x_0}(x_{n_k}) = S^T_{x_0}(x) = S^T_{x_0}(E^T_{x_0}(u)) \leq \frac{1}{2}\|u\|^2_{L^2} - \frac{1}{2} \int_0^T Q(x_u(t))dt,$$

which readily means both that $\lim_{k \to \infty} \|u_{n_k}\|_{L^2} = \|u\|_{L^2}$ (from which the convergence in $L^2$ follows), and that $C_T(u) = S^T_{x_0}(E^T_{x_0}(u)) = S^T_{x_0}(x)$ \(\square\).

### 3.3. From regularity to continuity.

We have introduced so far two subsets of $\text{int}(A^T_{x_0})$, namely the sets $\Sigma_c$ of the continuity points of $S^T_{x_0}$, and the set $\Sigma_f$ of fair points, which are well-parametrized by the exponential map. Both are dense in $\text{int}(A^T_{x_0})$, still their intersection can be empty. To see that this is not actually the case, we need to introduce one more class of points.

**Definition 19** (Tame Points). Let $x \in \text{int}(A^T_{x_0})$. We say that $x$ is a **tame point** if for every optimal control $u$ steering $x_0$ to $x$ there holds

$$\text{rank } d_u E^T_{x_0} = \text{dim } M = m.$$

We call $\Sigma_t$ the set of tame points.
The importance of tame points resides in that they locate open sets on which the value function \( S^{T}_{x_{0}} \) is continuous. This is crucial in the affine case where, as opposed to driftless one, one does not have the Chow-Rashevskii \([13, 3]\) theorem. The precise statement is contained in the following (cf. also the arguments of \[19\] Theorem 4.6):

**Lemma 20.** Let \( x \in \text{int}(A^{T}_{x_{0}}) \) be a tame point. Then

(i) \( x \) is a point of continuity of \( S^{T}_{x_{0}} \);

(ii) there exists a neighborhood \( O_{x} \) of \( x \) such that every \( y \in O_{x} \) is a tame point. In particular, the restriction \( S^{T}_{x_{0}}|_{O_{x}} \) is a continuous map.

**Proof.** To prove (i) we will show that, for every sequence \( \{ x_{n} \}_{n \in \mathbb{N}} \) converging to \( x \), there holds \( \lim_{n \to +\infty} S^{T}_{x_{0}}(x_{n}) = S^{T}_{x_{0}}(x) \); in particular we will prove the latter equality by showing that \( S^{T}_{x_{0}}(x) \) is the unique cluster point for all such sequences \( \{ S^{T}_{x_{0}}(x_{n}) \}_{n \in \mathbb{N}} \).

Let \( u \) be any optimal control steering \( x_{0} \) to \( x \); by hypothesis \( d_{u}E^{T}_{x_{0}} \) is surjective, and therefore \( E^{T}_{x_{0}} \) is locally open at \( u \), which means that there exists a neighborhood \( \mathcal{V}_{u} \subset \Omega^{T}_{x_{0}} \) of \( u \) such that the image \( E^{T}_{x_{0}}(\mathcal{V}_{u}) \) covers a full neighborhood of \( x \) in \( \text{int}(A^{T}_{x_{0}}) \). This implies that, for \( n \) large enough, the \( L^{2} \)-norms \( \{ \| u_{n} \|_{L^{2}} \}_{n \in \mathbb{N}} \) of optimal controls steering \( x_{0} \) to \( x_{n} \) remain uniformly bounded by some positive constant \( C \).

Let now \( a \) be a cluster point for the sequence \( \{ S^{T}_{x_{0}}(x_{n}) \}_{n \in \mathbb{N}} \). Then, it is not restrictive to assume that \( \lim_{n \to +\infty} S^{T}_{x_{0}}(x_{n}) = a \). Moreover, our previous point implies that we can find a subsequence \( \{ x_{n_{k}} \}_{k \in \mathbb{N}} \), whose associated sequence of optimal controls \( \{ u_{n_{k}} \}_{k \in \mathbb{N}} \) weakly converge in \( L^{2}([0, T], \mathbb{R}^{d}) \) to some admissible control \( u \) steering \( x_{0} \) to \( x \), which in turn yields the inequality

\[
S^{T}_{x_{0}}(x) \leq C_{T}(u) \leq \liminf_{k \to +\infty} C_{T}(u_{n_{k}}) = \liminf_{k \to +\infty} S^{T}_{x_{0}}(x_{n_{k}}) = a.
\]

Let us assume by contradiction that \( S^{T}_{x_{0}}(x) = b < a \), and let \( \varepsilon > 0 \) be such that \( b + \varepsilon < a \); moreover, let \( v \) be an optimal control attaining that cost. By the tameness assumption, the endpoint map \( E^{T}_{x_{0}} \) is open in a (strong) neighborhood \( \mathcal{V}_{v} \subset \Omega^{T}_{x_{0}} \) of \( v \), which means that all points \( y \) sufficiently close to \( x \) can be reached by admissible (but not necessarily optimal) trajectories, driven by controls \( w \in \mathcal{V}_{v} \) satisfying \( C_{T}(w) \leq b + \varepsilon < a \). But this gives a contradiction since \( S^{T}_{x_{0}}(x_{n_{k}}) \) must become arbitrarily close to \( a \), as \( k \) goes to infinity.

To prove (ii), assume by contradiction that such a neighborhood \( O_{x} \) does not exist; then we can find a sequence \( \{ x_{n} \}_{n \in \mathbb{N}} \) convergent to \( x \), and such that for every \( n \in \mathbb{N} \) there exists a choice of an abnormal optimal control \( u_{n} \), steering \( x_{0} \) to \( x_{n} \), that is for every \( n \in \mathbb{N} \) there exists a norm-one covector \( \lambda_{n} \) such that:

\[
\lambda_{n} d_{u_{n}}E^{T}_{x_{0}} = 0.
\]

By Lemma 18 there exists a subsequence \( u_{n_{k}} \) which converges strongly in \( L^{2}([0, T], \mathbb{R}^{d}) \) to some optimal control \( u \) reaching \( x \); moreover, since we assumed \( |\lambda_{n}| = 1 \) for all \( n \in \mathbb{N} \), it is not restrictive to suppose that \( \overline{X} = \lim_{k \to +\infty} \lambda_{n_{k}} \) exists. Thus, passing to the limit as \( k \) tend to infinity in \( 3.1 \), we see that \( u \) is forced to be abnormal, and thus we have a contradiction, as \( x \) is tame. It follows then from point (i) that \( S^{T}_{x_{0}}|_{O_{x}} \) is indeed a continuous map. \( \square \)

**Corollary 21.** The set \( \Sigma_{t} \) of tame points is open. Moreover \( \Sigma_{t} \subset \Sigma_{c} \).

3.4. **Density of tame points.** We start with the somewhat trivial observation that the set of optimal controls reaching a fixed point \( x \) is compact in the \( L^{2} \)-topology.

**Lemma 22.** For every \( x \in A^{T}_{x_{0}} \), the set

\[
\mathcal{U}_{x} = \{ u \in \Omega^{T}_{x_{0}} \mid u \text{ is an optimal control steering } x_{0} \text{ to } x \}
\]
is strongly compact in $L^2([0, T], \mathbb{R}^d)$.

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{U}_x$. Then we have $S^T_{x_0}(x) = C_T(u_n)$ for every $n \in \mathbb{N}$, and consequently there exists $C > 0$ such that $\|u_n\|_{L^2} \leq C$ for every $n \in \mathbb{N}$. Thus we may assume that there exists a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}} \subset \{u_n\}_{n \in \mathbb{N}}$, and a control $u$ steering $x_0$ to $x$, such that $u_{n_k} \rightharpoonup u$ weakly in $L^2([0, T], \mathbb{R}^d)$. This, on the other hand, implies that

\[
\frac{1}{2} \left\| u \right\|_{L^2}^2 - \frac{1}{2} \int_0^T Q(x_u(t))dt \leq \liminf_{k \to \infty} \frac{1}{2} \left\| u_{n_k} \right\|_{L^2}^2 - \frac{1}{2} \int_0^T Q(x_{u_{n_k}}(t))dt
\]

\[
= \liminf_{k \to \infty} C_T(u_{n_k}) = S^T_{x_0}(x)
\]

\[
= C_T(u) = \frac{1}{2} \left\| u \right\|_{L^2}^2 - \frac{1}{2} \int_0^T Q(x_u(t))dt,
\]

therefore $\|u\|_{L^2} = \lim_{k \to \infty} \|u_{n_k}\|_{L^2}$, and the claim is proved. \hfill \Box

We introduce now the notion of class of a point. Heuristically, the class of a point $x \in \text{int} \left( \mathcal{A}^T_{x_0} \right)$ measures how much that point “fails” to be tame (see Definition 19).

**Definition 23.** Let $x \in \mathcal{A}^T_{x_0}$. We define

\[\text{class}(x) = \min_{u \in \mathcal{U}_x} \text{rank } d_u \mathcal{E}^T_{x_0} .\]

Any point $x \in \text{int} \left( \mathcal{A}^T_{x_0} \right)$ satisfying $\text{class}(x) = m$ is necessarily a tame point.

**Definition 24.** We also define the subset $\mathcal{U}^\text{min}_x \subset \mathcal{U}_x$ as follows:

\[\mathcal{U}^\text{min}_x = \{ u \in \mathcal{U}_x \mid \text{rank } d_u \mathcal{E}^T_{x_0} = \text{class}(x) \} .\]

By the lower semicontinuity of the rank function, the set $\mathcal{U}^\text{min}_x$ is closed in $\mathcal{U}_x$, hence (strongly) compact in $L^2([0, T], \mathbb{R}^d)$.

We can now state and prove the main result of this section.

**Theorem 25.** The set $\Sigma_t$ of tame points is dense in $\text{int} \left( \mathcal{A}^T_{x_0} \right)$.

We postpone the proof of Theorem 25 at the end of the section, since we need first a series of preliminary results.

**Definition 26.** Pick $x$ in $\text{int} \left( \mathcal{A}^T_{x_0} \right)$ and let $u \in \mathcal{U}^\text{min}_x$. If $u$ is not strictly abnormal, then we choose a normal covector $\eta_x \in T^*_x M$ associated to $u$ and we define

\[\hat{\Xi}_x^u = \{ \xi \in T^*_x M \mid \xi d_u \mathcal{E}^T_{x_0} = \eta_x d_u \mathcal{E}^T_{x_0} \} = \eta_x + \ker (d_u \mathcal{E}^T_{x_0})^* \subset T^*_x M .\]

If instead $u$ is strictly abnormal, we simply set $\hat{\Xi}_x^u = \ker (d_u \mathcal{E}^T_{x_0})^* \subset T^*_x M .\]

Notice that whenever $u$ is strictly abnormal, then $\hat{\Xi}_x^u$ is a linear subspace, while if $u$ admits at least one normal lift, $\hat{\Xi}_x^u$ is affine; also, the dimension of these subspaces equals $m - \text{class}(x) \geq 0$.

We call $\hat{Z}_u \subset T^*_x M$ the orthogonal subspace to $\ker (d_u \mathcal{E}^T_{x_0})^*$, of dimension equal to $\text{class}(x)$, for which:

\[
T^*_x M = \ker (d_u \mathcal{E}^T_{x_0})^* \oplus \hat{Z}_u ;
\]

moreover we let $\pi_{\hat{Z}_u} : T^*_x M \to \hat{Z}_u$ to be the orthogonal projection subordinated to this splitting, that is satisfying:

\[\ker(\pi_{\hat{Z}_u}) = \ker (d_u \mathcal{E}^T_{x_0})^* .\]
Finally, by means of the adjoint map \((P_{0,T}^u)\)∗, we can pull the spaces \(\hat{\Xi}_x^u\) “back” to \(T_{x_0}M\), and set

\[
\Xi_x^u := (P_{0,T}^u)^* \hat{\Xi}_x^u \subset T_{x_0}^* M.
\]

The following estimate will be crucial in what follows.

**Proposition 27.** Let \(O \subset \text{int} \left( A^T_{x_0} \right)\) be an open set, and assume that:

\[
\text{class} \left( z \right) \equiv k_O < m, \quad \text{for every } z \in O.
\]

Let \(x \in O\) and \(u \in \mathcal{U}_{x}^{\text{min}}\). Then there exists a neighborhood \(\mathcal{V}_u \subset \Omega_{x_0}^T\) of \(u\) such that, for every \(\lambda_u \in \Xi_x^u \subset T_{x_0}^* M\), there exists a constant \(K = K(\lambda_u) > 1\) such that, for every \(v \in \mathcal{V}_u \cap \mathcal{U}_{y}^{\text{min}}\) \((P_{y}^T)\), there is \(\xi_v \in \Xi_{E_{x_0}^y(v)}^u \subset T_{x_0}^* M\) satisfying:

\[
|\lambda_u - \xi_v| \leq K.
\]

**Proof.** Let us choose a neighborhood \(\mathcal{V}_u \subset \Omega_{x_0}^T\) of \(u\), such that all the endpoints of admissible trajectories driven by controls in \(\mathcal{V}_u\) belong to \(O\).

Then, if \(y = E_{x_0}^T(v)\) for some \(v \in \mathcal{V}_u\), it follows that \(y \in O\); moreover, if also \(v \in \mathcal{U}_{y}^{\text{min}}\), we can define the \((m - k_O)\)-dimensional subspace \(\Xi_{y}^u \subset T_{x_0}^* M\) as in Definition 26 Therefore we can assume from the beginning that all such subspaces \(\Xi_{y}^u\) have dimension constantly equal to \(m - k_O > 0\).

Fix \(\lambda_u \in \Xi_{x}^u\), and set

\[
\hat{\lambda}_{\lambda}^u = (P_{T,0}^u)^* \lambda_u \in T_{y}^* M, \quad v \in \mathcal{V}_u \cap \mathcal{U}_{y}^{\text{min}}, \quad y = E_{x_0}^T(v).
\]

The intersection \((\hat{\lambda}_{\lambda}^u + \hat{Z}_v) \cap \hat{\Xi}_y^u\) (cf. with \(3.2\) and Figure 1) consists of the single point \(\hat{\xi}_v\); since both \(\hat{\lambda}_{\lambda}^u\) and \(\hat{\xi}_v\) belong to the affine subspace \(\hat{\lambda}_{\lambda}^u + \hat{Z}_v\), in order to estimate the norm \(|\hat{\lambda}_{\lambda}^u - \hat{\xi}_v|\) it is sufficient to evaluate the norm \(|\pi_{\hat{Z}_v}(\hat{\lambda}_{\lambda}^u) - \pi_{\hat{Z}_v}(\hat{\xi}_v)|\) of the projections onto the linear space \(\hat{Z}_v = (\ker(d_v E_{x_0}^T)^* \perp).\) The key point is the computation of the norm of \(|\pi_{\hat{Z}_v}(\hat{\xi}_v)|\): in fact, since
We deduce immediately from (3.3) that, whenever $v$ is strictly abnormal, then $\pi_{Z_v}(\hat{\xi}_v) = 0$, while from the expression for the normal control (2.3)

$$v_i(t) = \langle \hat{\xi}_v(t), X_i(x_v(t)) \rangle = \langle \hat{\xi}_v, (P_{T,t}^v)_*X_i(x_v(t)) \rangle,$$

we see that $\langle v, w \rangle_{L^2} = \langle \hat{\xi}_v, d_v E_{x_0}^T(w) \rangle$, and we can continue from (3.3) as follows ($W_v$ denotes the $k_O$-dimensional subspace of $L^2([0,T],\mathbb{R}^d)$ on which the restriction $d_v E_{x_0}^T|_{W_v}$ is invertible):

$$(3.4) \quad \frac{|\pi_{Z_v}(\hat{\xi}_v)|}{\sup \{\|\xi_v, d_v E_{x_0}^T(w)\| \}} \leq \frac{|\langle \hat{\xi}_v, d_v E_{x_0}^T(w) \rangle|}{\|w\|_{L^2}} \|(d_v E_{x_0}^T|_{W_v})^{-1}\| \leq \frac{|\langle v, w \rangle|}{\|w\|_{L^2}} \|(d_v E_{x_0}^T|_{W_v})^{-1}\| \leq |\langle v, w \rangle|_{L^2} \|(d_v E_{x_0}^T|_{W_v})^{-1}\|. $$

It is not restrictive to assume that the $L^2$-norm of any element $v \in \mathcal{V}_u \cap \mathcal{U}^{\min}$ remains bounded; moreover, since all the subspaces have the same dimension, the map $v \mapsto W_v$ is continuous, which implies that so is the map $v \mapsto (d_v E_{x_0}^T|_{W_v})^{-1}$. This, on the other hand, guarantees that the operator norm $\|(d_v E_{x_0}^T|_{W_v})^{-1}\|$ remains bounded for all $v \in \mathcal{V}_u \cap \mathcal{U}^{\min}_v$, and then from (3.4) we conclude that for some $C > 1$, the estimate $|\pi_{Z_v}(\hat{\xi}_v)| \leq C$ holds true, which implies as well, by the triangular inequality, that:

$$|\hat{\lambda}_u - \hat{\xi}_v| \leq |\hat{\lambda}_u| + C.$$ 

Finally, the continuity of both the map $v \mapsto P_{0,T}^v$ and its inverse, implies that for another real constant $C > 1$ we have:

$$\sup \{\|(P_{0,T}^v)^*\|, \|(P_{T,0}^v)^*\| \} \leq C.$$ 

Thus, setting $\xi_v = (P_{0,T}^v)^*\tilde{\xi}_v \in T_{x_0}^* M$ (cf. Figure [1]) we can compute (here $C$ denotes a constant that can change from line to line):

$$|\lambda_u - \xi_v| \leq C|\hat{\lambda}_u - \hat{\xi}_v| \leq C|\hat{\lambda}_u| + C^2 \leq C^2 (|\lambda_u| + 1) \leq 2C^2 \max\{|\lambda_u|, 1\}.$$ 

Setting $K(\lambda_u) := 2C^2 \max\{|\lambda_u|, 1\}$ the claim is proved. \hfill \Box

**Remark.** Let us fix $\lambda_u \in \Xi_u^v \subset T_{x_0}^* M$ and consider the $k_O$-dimensional affine subspace

$$(P_{0,T}^v)^*(\hat{\lambda}_u + \tilde{Z}_v) = \lambda_u + (P_{0,T}^v)^*\tilde{Z}_v,$$

$$\ker(d_v E_{x_0}^T)^* = (\text{Im } d_v E_{x_0}^T)^\perp,$$ this amounts to evaluate

$$(3.3) \quad |\pi_{Z_v}(\hat{\xi}_v)| = \sup_{f \in \text{Im } d_v E_{x_0}^T} \frac{|\langle \hat{\xi}_v, f \rangle|}{|f|}.$$
with \( \tilde{Z}_u \) defined as in (3.2). Then if we call \( Z_v := (P_{u,T}^v)\ast \tilde{Z}_v \subset T^*_x M \), the map

\[
v \mapsto \lambda_u + Z_v, \quad v \in \mathcal{V}_u \cap \mathcal{U}^\text{min}_y, \quad y = E_{x_0}^T(v)
\]
is continuous; moreover, \( Z_v \) is by construction transversal to \( \Xi^u_v \), and \( \xi_v \in (\lambda_u + Z_v) \cap \Xi^u_v \).

Having in mind this remark, we deduce the following:

**Corollary 28.** Let \( O \subset \text{int} (A^T_{x_0}) \) be an open set, and assume that

\[
\text{class } (z) = k_O < m, \quad \text{for every } z \in O.
\]

Let \( x \in O, u \in \mathcal{U}^\text{min}_x \), and consider \( \mathcal{V}_u \subset \Omega^T \) as in Proposition 27. Then, for every \( \lambda_u \in \Xi^u_x \), there exists a \( k_O \)-dimensional compact ball \( A_u \), centered at \( \lambda_u \) and transversal to \( \Xi^u_x \), such that:

\[
A_u \cap \Xi^u_y \neq \emptyset \quad \text{for every } v \in \mathcal{V}_u \cap \mathcal{U}^\text{min}_y, \quad \text{where } y = E_{x_0}^T(v).
\]

![Figure 2](attachment:image.png)

**Figure 2.** On the fiber \( T^*_x M \), the point \( \eta \) denotes the intersection between \( T_v \) and the affine space \( \lambda_u + Z_u \).

**Proof.** Let \( \lambda_u \in \Xi^u_x \) be chosen, and assume without loss of generality that \( \mathcal{V}_u \) is relatively compact. For every \( v \in \mathcal{V}_u \), we can construct an \( m \)-dimensional ball \( B^u_v \), of radius \( C^u_0 \) strictly greater than \( K = K(\lambda_u) \) (given by Proposition 27), and centered at \( \lambda_u \).

Then, the existence of an element \( \xi_v \in (\lambda_u + Z_v) \cap \Xi^u_y \) satisfying \( |\lambda_u - \xi_v| \leq K \), proved in Proposition 27, implies that the intersection of \( B^u_v \) with \( \Xi^u_y \) is a compact submanifold \( T_v \) (with boundary); moreover, since the radius of \( B^u_v \) is strictly greater than \( |\lambda_u - \xi_v| \), it is also true that the intersection of \( \lambda_u + Z_u \) with \( \text{int}(T_v) \) is not empty.

Let us consider as before (cf. previous remark) the \( k_O \)-dimensional affine subspace \( \lambda_u + Z_u \), which is transversal to \( \Xi^u_x \); possibly increasing the radius \( C^u_0 \), the continuity of the map \( w \mapsto \lambda_u + Z_u \) ensures that \( \lambda_u + Z_u \) remains transversal to \( T_v \), and in particular that the intersection \( T_v \cap (\lambda_u + Z_u) \) is not empty (see Figure 2). Moreover, it is clear that this conclusion is local, that is with the same choice of \( C^u_0 \) it can be drawn on some full neighborhood \( \mathcal{W}_v \) of \( v \). Then, to find a ball \( B_u \) and a radius \( C_0 \) uniformly for the whole set \( \mathcal{V}_u \), it is sufficient to extract a finite sub-cover \( \mathcal{W}_{v_1}, \ldots, \mathcal{W}_{v_l} \) of \( \mathcal{V}_u \), and choose \( C_0 \) as the maximum between \( C^u_0, \ldots, C^v_0 \).

We conclude the proof setting \( A_u = B_u \cap (\lambda_u + Z_u) \); indeed \( A_u \) is a compact \( k_O \)-dimensional ball by construction, and moreover if we call \( \eta_v \) any element in the intersection \( T_v \cap (\lambda_u + Z_u) \), for \( v \in \mathcal{V}_u \), then it follows that:

\[
\eta \in \Xi^u_y \cap B_u \cap (\lambda_u + Z_u) = \Xi^u_y \cap A_u,
\]

that is, the intersection \( \Xi^u_y \cap A_u \) is not empty for every \( v \in \mathcal{V}_u \cap \mathcal{U}^\text{min}_y \). \( \square \)
Lemma 29. Let $O \subset \text{int}(A_{x_0}^T)$ be an open set, and let

$$k_O = \max_{x \in \Sigma_c \cap O} \text{class}(x).$$

Then there exists a neighborhood $O' \subset O$, such that $\text{class}(y) = k_O$, for every $y \in O'$.

Proof. Let $x \in \Sigma_c \cap O$ be a point of continuity for the value function $S_{x_0}^T$, having the property that $\text{class}(x) = k_O$. Assume by contradiction that we can find a sequence $\{x_n\}_{n \in \mathbb{N}}$ converging to $x$ and satisfying $\text{class}(x_n) \leq k_O - 1$ for every $n \in \mathbb{N}$. Accordingly, let $u_n \in \mathbb{U}^{\min}_{x_n}$ an associated sequence of optimal controls; in particular, for every $n \in \mathbb{N}$, we have by definition that

$$\text{class}(x_n) = \text{rank } d_{u_n} E_{x_0}^T.$$  

By Lemma [18] we can extract a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}} \subset \{u_n\}_{n \in \mathbb{N}}$ which converges to some optimal control $u$ steering $x_0$ to $x$, strongly in the $L^2$-topology, and write:

$$\text{class}(x) \leq \text{rank } d_u E_{x_0}^T \leq \liminf_{k \to \infty} \text{rank } d_{u_{n_k}} E_{x_0}^T = \liminf_{k \to \infty} \text{class}(x_{n_k}) \leq k_O - 1,$$

which is absurd by construction, and the claim follows. \hfill $\Box$

Collecting all the results we can now prove Theorem [25]

Proof of Theorem 25. Let $O$ be an open set in $\text{int}(A_{x_0}^T)$ and define

$$k_O = \max_{x \in \Sigma_c \cap O} \text{class}(x);$$

notice that this definition makes sense, since points of continuity are dense in $\text{int}(A_{x_0}^T)$ by Proposition [17]. Then we may suppose that $k_O$ is strictly less than $m$, for otherwise there would be nothing to prove. Moreover, by Lemma [29] it is not restrictive to assume that $\text{class}(y) = k_O$ for every $y \in O$.

Fix then a point $x \in \Sigma_c \cap O$; since the hypotheses of Proposition [27] are satisfied, for every $u \in \mathbb{U}^{\min}_{x}$ we can find a neighborhood $\mathcal{V}_u \subset \Omega_{x_0}^T$ of $u$, fix $\lambda_u \in \Xi_x$, and construct accordingly a compact $k_O$-dimensional ball $A_u$, centered at $\lambda_u$ and transversal to $\Xi_x$, such that (Corollary [28])

$$A_u \cap \Xi_v = \emptyset \quad \text{for every } v \in \mathcal{V}_u,$$

and $y = E_{x_0}^T(v)$.

Since $\mathbb{U}^{\min}_{x}$ is compact (Definition [24]), we can choose finitely many elements $u_1, \ldots, u_l$ in $\mathbb{U}^{\min}_{x}$ such that

$$\mathbb{U}^{\min}_{x} \subset \bigcup_{i=1}^l \mathcal{V}_{u_i}.$$  

The union $A_{u_1} \cup \ldots \cup A_{u_l}$ is again of positive codimension. Moreover, for every sequence $x_n$ of fair points converging to $x$, and whose associated sequence of optimal controls (by uniqueness of the optimal control, necessarily $u_n \in \mathbb{U}^{\min}_{x_n}$) the sequence $u_n$ converges to some $v \in \mathcal{V}_{u_i} \subset \mathbb{U}^{\min}_{x}$, we have that $A_{u_i}$ is also transversal to $\Xi_{x_n}$. In particular, possibly enlarging the ball $A_{u_i}$, we can assume that

$$A_{u_i} \cap \Xi_{x_n} = \emptyset, \quad \text{for every } n \in \mathbb{N}.$$  

For any fair point $z \in \Sigma_f \cap O$, the optimal control admits a normal lift, and we have the equality

$$E_{x_0}^T(\Xi_z) = z,$$

where $E_{x_0}^T$ is the exponential map with base point $x_0$ at time $T$ of Definition [7] so that we eventually deduce the inclusion:

$$\Sigma_f \cap O \subset E_{x_0}^T(A_{u_1} \cup \ldots \cup A_{u_l}).$$

(3.5)
The set on the right-hand side is closed, being the image of a compact set; moreover, it is of measure zero by the classical Sard Lemma [18], as it is the image of a set of positive codimension by construction. Since the set $\Sigma_f \cap O$ is dense in $O$ by Corollary [15] passing to the closures in [3.5] we conclude that $\text{meas}(O) = 0$, which is impossible. 

Combining now Lemma [20] and Theorem [25] we obtain the following (cf. Theorem [2]).

**Corollary 30.** The set $\Sigma_k$ of tame points is open and dense in $\text{int} \,(A^T(x_0))$.

4. **Smooth Points**

In this section we investigate the regularity properties of the value function $S^T_{x_0}$ in the presence of tame points. Since tame points are in particular points of continuity for $S^T_{x_0}$, the arguments of Lemma [18] with minor changes, prove the following result.

**Lemma 31.** Let $K \subset \Sigma_k$ be a compact subset of tame points. Then the set of optimal controls reaching points of $K$

$$\mathcal{M}_K = \{u \in \Omega^T_{x_0} \mid E^T_{x_0}(u) \in K \text{ and } C_T(u) = S^T_{x_0}(E^T_{x_0}(u))\}$$

is strongly compact in the $L^2$-topology.

The first result of this section, which is an adaptation of an argument of [15] [1], is as follows:

**Proposition 32.** Let $K \subset \Sigma_k$ be a compact subset of tame points. Then $S^T_{x_0}$ is Lipschitz continuous on $K$.

**Proof.** By compactness, it is sufficient to show that $S^T_{x_0}$ is locally Lipschitz continuous on $K$.

Fix a point $x \in K$ and let $u$ be associated with an optimal trajectory joining $x_0$ and $x$. By assumption, $d_uE^T_{x_0}$ is surjective, so that there are neighborhoods $\mathcal{V}_u \subset \Omega^T_{x_0}$ of $u$ and $O_x \subset \text{int} \,(A^T_{x_0})$ of $x$ such that

$$E^T_{x_0}|_{\mathcal{V}_u} : \mathcal{V}_u \to O_x$$

is surjective, and there exists a smooth right inverse $\Phi : O_x \to \mathcal{V}_u$ such that $E^T_{x_0}(\Phi(y)) = y$ for every $y \in O_x$.

Fix local coordinates around $x$, and let $B_x(r) \subset M$ and $B_u(r) \subset \Omega^T_{x_0}$ denote some balls of radius $r > 0$ centered at $x$ and $u$ respectively. As $\Phi$ is smooth, there exists $R > 0$ and $C_0 > 0$ such that:

\[
B_x(C_0r) \subset E^T_{x_0}(B_u(r)), \quad \text{for every } 0 \leq r \leq R.
\]

Observe that there also exists $C_1 > 0$ such that, for every $v, w \in B_u(R)$ we have

\[
|C_T(v) - C_T(w)| \leq C_1\|v - w\|_{L^2}.
\]

Indeed our main assumption implies that the subset $\{x_v(t) \mid t \in [0,T], \, v \in B_u(R)\}$ is contained in a compact set $K$ of $M$, on which the smooth function $Q$, together with its derivative $Q'$, attains both a maximum and a minimum. Then, using the mean value theorem and [19] Proposition 3.5], we deduce that

\[
\int_0^T |Q(x_v(t)) - Q(x_w(t))|dt \leq \sup_{y \in K} |Q'(y)| \int_0^T |x_v(t) - x_w(t)|dt \leq C\|v - w\|_{L^2},
\]

and by means of the triangular inequality, (4.2) is proved.
Pick any point \( y \in K \) such that \( |y - x| = C_0 r \), with \( 0 \leq r \leq R \). Then by (4.1) there exists \( v \in B_u(r) \) satisfying \( \|u - v\|_{L^2} \leq r \) and such that \( E_{x_0}^T(v) = y \); since \( C_T(u) = S_{x_0}^T(x) \) and \( S_{x_0}^T(y) \leq C_T(v) \), we have

\[
S_{x_0}^T(y) - S_{x_0}^T(x) \leq C_T(v) - C_T(u) \leq C_1 \|v - u\|_{L^2} \leq C_1 C_0 |y - x|.
\]

Using the compactness of both \( K \) and \( M_K \) (cf. Lemma 31), all the constants can be made uniform, and the role of \( x \) and \( y \) can be exchanged, so that we have indeed

\[
|S_{x_0}^T(x) - S_{x_0}^T(y)| \leq \frac{C_1}{C_0} |x - y|,
\]

for every pair of points \( x \) and \( y \) such that \( |x - y| \leq C_0 R \). \( \square \)

**Definition 33.** We define the set of \( \Sigma \subset \text{int}(A_{x_0}^T) \) as the set of points \( x \) such that

(a) there exists a unique optimal trajectory \( t \mapsto x_u(t) \) steering \( x_0 \) to \( x \) in time \( T \), which is strictly normal,

(b) \( x \) is not conjugate to \( x_0 \) along \( x_u(\cdot) \) (cf. Definition 9).

Item (a) in the Definition 33 is equivalent to require that \( x \) is in fact a point that is at the same time fair and tame. Notice that as a consequence of the results of Section 3, and in particular of Corollary 30, the set \( \Sigma_f \cap \Sigma_t \) is dense in \( \text{int}(A_{x_0}^T) \).

The following result finally proves Theorem 1.

**Theorem 34** (Density of smooth points). \( \Sigma \) is open and dense in \( \text{int}(A_{x_0}^T) \). Moreover \( S_{x_0}^T \) is smooth on \( \Sigma \).

**Proof.** (i.a) Let us show that \( \Sigma \) is dense. First we prove that, for any open set \( O \), we have \( \Sigma \cap O \neq \emptyset \). Since the set points satisfying (a) (i.e. the set \( \Sigma_f \cap \Sigma_t \)) is dense in \( \text{int}(A_{x_0}^T) \), by Proposition 32 we can choose a point \( x \in \Sigma_f \cap \Sigma_t \cap O \) and \( O' \subset O \cap \Sigma_t \) relatively compact, such that \( S_{x_0}^T \) is Lipschitz on \( O' \). Thanks to the classical Rademacher theorem we know that \( S_{x_0}^T \) is differentiable almost everywhere on \( O' \), and therefore, since any point of differentiability is a fair point by Corollary 16, \( \text{meas}(\Sigma_f \cap O') = \text{meas}(O') \). Moreover, any point in \( \Sigma_f \cap O' \) is also contained in the image of the exponential map \( E_{x_0}^T \); and Sard Lemma implies that the set of regular points is of full measure in \( \Sigma_f \cap O' \). By definition any such point is in \( \Sigma \), that is we have \( \text{meas}(\Sigma \cap O') = \text{meas}(\Sigma_f \cap O') = \text{meas}(O') \), which implies that \( \Sigma \cap O' \neq \emptyset \), and this concludes the proof.

(i.b) Let us prove that \( \Sigma \) is open. Fix as before an open set \( O \) having compact closure in \( \text{int}(A_{x_0}^T) \). Assume by contradiction that there exists a sequence of points \( x_n \in O \) converging to \( x \in \Sigma \) and such that there are (at least) two optimal trajectories connecting them with \( x_0 \). Call \( \{u_n\}_{n \in \mathbb{N}} \) and \( \{v_n\}_{n \in \mathbb{N}} \) the corresponding sequences of optimal controls associated with such trajectories. Lemma 18 then guarantees that, up to considering subsequences, it is not restrictive to assume the existence of both \( u = \lim_{n \to \infty} u_n \) and \( v = \lim_{n \to \infty} v_n \) in \( L^2([0, T], \mathbb{R}^d) \); however, the uniqueness of the minimizer steering \( x_0 \) to \( x \) implies that \( u = v \).

Then both \( d_{u_n}E_{x_0}^T \) and \( d_{v_n}E_{x_0}^T \) have maximal rank for \( n \) large enough (\( u \) is strictly normal because \( x \) is a smooth point), and we can define the families of covectors \( \lambda_n \) and \( \xi_n \), as elements of \( T_{x_0}^* M \), satisfying the identities

\[
\lambda_n d_{u_n}E_{x_0}^T = d_{u_n}C_T, \quad \xi_n d_{v_n}E_{x_0}^T = d_{v_n}C_T.
\]

Taking the limit on these two equations we see that \( \lim_{n \to \infty} \lambda_n = \lim_{n \to \infty} \xi_n = \lambda \), where \( \lambda \) is the covector associated with the unique optimal control \( u \) steering \( x_0 \) to \( x \). If, for any \( s \in [0, T] \),
we let \( \lambda_n^* = (P_{n,T})^* \lambda_n \) and \( \xi_n^* = (P_{n,T})^* \xi_n \), then we see that even the “initial covectors” \( \lambda_0 \) and \( \xi_0 \) converge to the same element \( \lambda \).

On the other hand, since by the point (b) of Definition 33 \( x \) is not conjugate to \( x_0 \) along the unique optimal trajectory \( x_u(\cdot) \), we have that \( \lambda^0 \) is a regular point for the exponential map \( \exp_x^\lambda \). Then there exist full neighborhoods \( V \subset T_{x_0}^* M \) of \( \lambda^0 \) and \( O_x \subset \text{int} (A_{x_0}^\lambda) \) of \( x \) such that the exponential map \( \exp_x^\lambda|_V : V \to O_x \) is a diffeomorphism. In particular, if we pick some point \( y \in O_x \), there is a unique optimal trajectory \( x_u(\cdot) \) steering \( x_0 \) to \( y \); moreover the covector \( \lambda_y \) associated with \( x_u(\cdot) \) is a regular point for \( \exp_x^\lambda \), and from the equality \( \exp_x^\lambda(u) = \exp_x^\lambda(\lambda_y) \), we see that \( u \) has to be strictly normal. This shows that \( O_x \subset \Sigma \), which in the end is an open set.

(ii). Next we prove the smoothness of \( S_T^{x_0} \) on \( \Sigma \). Let us consider a covector \( \lambda \in T_{x_0}^* M \) associated with the unique optimal trajectory connecting \( x_0 \) and \( x \). By the arguments of Theorem 34 there are neighborhoods \( V_\lambda \subset T_{x_0}^* M \) of \( \lambda \) and \( O_x \subset \text{int} (A_{x_0}^\lambda) \) of \( x \) such that \( \exp_x^\lambda|_{V_\lambda} : V_\lambda \to O_x \) is a diffeomorphism.

It is then possible to define a smooth inverse \( \Phi : O_x \to V_\lambda \) sending \( y \) to the corresponding “initial” covector \( \lambda_y \). Along (strictly normal) trajectories associated with covectors \( \lambda_y \) in \( V_\lambda \) we have therefore (compare with (2.3))

\[
u^y_i(t) = \langle \Phi(y), X_i(x^y_u(t)) \rangle,
\]

which means that the control \( u^y \in \Omega_{x_0}^T \) and, in turn, the cost \( C_T(u) \) itself, are smooth on \( O_x \). \( \Box \)

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