Approximation and characterization of quasi-static H^1-evolutions for a cohesive interface with different loading-unloading regimes

Matteo Negri*, Enrico Vitali

Abstract. We consider the quasi-static evolution of a prescribed cohesive interface: dissipative under loading and elastic under unloading. We provide existence in terms of parametrized BV-evolutions, employing a discrete scheme based on local minimization, with respect to the H^1-norm, of a regularized energy. Technically, the evolution is fully characterized by: equilibrium, energy balance and Karush-Kuhn-Tucker conditions for the internal variable. Catastrophic regimes (discontinuities in time) are described by gradient flows of visco-elastic type.

1 Introduction

In this work we study a quasi-static evolution for an elastic material containing a cohesive crack. Models of this type have been studied under many different mechanical and mathematical hypotheses. Before presenting our setting and our results, we recall some recent works, covering different research directions.

First of all, we mention [9] and [7], a couple of results obtained in the framework of *energetic evolutions* [17]. From our perspective, and from the mechanical point of view in general, it is interesting that in these works the cohesive potential depends, at time t, both on the crack opening, say $[\![u]\!](t)$, and on an internal variable, say $\xi(t)$, given (roughly speaking) by the maximal crack opening $[\![u]\!]$ in the interval $[0, t]$. This feature allows to introduce irreversibility (by the monotonicity of ξ) and to distinguish between different loading-unloading regimes: [9] considers a constant unloading while [7] considers a more general convex unloading, introducing Young measures. These energetic evolutions are obtained, as usual, taking the limit of time-discrete evolutions in which the time-incremental problem is a *(global) energy minimization problem.* A similar approach is pursued also in [21] employing a "damage like" interface energy, in place of an internal variable. We finally mention [16] which studies global minimizers, for static problems, under very weak conditions on the adhesive (or cohesive) potential. In this context we would like to point out also the weaker notion of directional local minimizers proposed in [19], actually for a gradient damage model.

Let us turn to *BV-evolutions*, another class of quasi-static evolutions. In this framework, developed to overcome some issues of energetic evolutions, the system attains, at each time, an equilibrium configuration which is not necessarily an energy minimizer, as it is for energetic evolutions. Typically, BV-evolutions are obtained by *vanishing viscosity*, i.e., as the limit of auxiliary time-continuous parabolic systems (see [17] for abstract results and [6, 1] for cohesive models). Alternatively, see [18], they can be generated as the limit of time-discrete evolutions in which the time-incremental problem is a local energy minimization problem. In both cases, it is necessary to provide (or identify) a norm or a metric which, together with the energy, drives the evolution. Clearly, different choices of this norm or metric are interesting from the mathematical and mechanical point of view; for instance, in the frame of cohesive fracture, [6] employs the bulk L^2-norm while [1] employs a "metric" depending on the crack length (the surface energy actually has an activation threshold followed by a cohesive behaviour).

Let us briefly mention some results in the one dimensional setting, i.e., for elastic bars with cohesive cracks; this simplified setting is often useful to provide a representative picture of the complex behaviour of more realistic problems. For instance, [4] and [10] contain fine studies of (stable and unstable) equilibrium configurations, [15] studies a dynamic problem while [8] presents a quasi-static evolution generated by gradient flows, as incremental problems, along different loading-unloading paths.

We conclude this brief overview with some computational works, closely related to our work. We first mention [2] which makes use of a regularized cohesive potential, similar to the one employed

*Financial support was provided by INdAM-GNAMPA project "Flussi gradiente ed evoluzioni rate-independent: sviluppi dell'approccio variazionale ed applicazioni" and by the ERC Advanced Grant n. 290888 QuaDynEvoPro.

here, in order to obtain convenient (differentiable) energies for numerical simulations. The class of cohesive laws used here is inspired by [20] both for the loading-unloading regimes and for the regularization of the density (labelled "Smith-Ferrante" in [20]). We finally remember the recent [3] which contains an abstract approximation result (from discrete to continuum) applied to the viscosity approach of [6] and also [23] which employs an arc-length approach, similar to ours, to capture unstable regimes of propagation (see § 8).

Now, let us describe our setting and the main results, without going into technical details. We work within the anti-plane setting. We start with a traction-separation law $\tau(|[\![u]\!]|, \xi)$ (depending on the modulus of the opening $[\![u]\!]$ and on the internal variable ξ) which is linear in the unloading branch $0 < |[\![u]\!]| \le \xi$, decreasing and convex in the loading branch $|[\![u]\!]| > \xi$ (see Figure 2a). The cohesive potential $\psi(\cdot, \xi)$ is then obtained by integration of $\tau(\cdot, \xi)$. We remark that the cohesive density $[\![u]\!] \mapsto \psi(|[\![u]\!]|, \xi)$ is not differentiable in the origin, unless $\xi > 0$. Given a function $t \mapsto g(t)$ the potential energy is given by

$$\mathcal{F}(t, u, \xi) = \tfrac{1}{2} \int_{\Omega \setminus K} \mu |\nabla(u + g(t))|^2 \, dx + \int_K \psi(|[\![u]\!]|, \xi) \, d\mathcal{H}^1,$$

where Ω is the reference configuration, K is the cohesive interface (or crack), $u \in \mathcal{U} = \{u \in H^1(\Omega \setminus K) : u = 0 \text{ on } \partial_D \Omega\}$ where $\partial_D\Omega \subset \partial\Omega$. Since differentiability of the energy is a convenient property, both theoretically and numerically, we introduce a family of regularized (differentiable) potentials $\psi_\varepsilon(|[\![u]\!]|, \xi)$ approximating $\psi(|[\![u]\!]|, \xi)$. We denote by $\mathcal{F}_\varepsilon$ the corresponding energy.

We work within the framework of parametrized BV-evolutions [18]. Our strategy, to find an evolution for $\mathcal{F}$, is the following. First, we define a family of evolutions for the regularized energies $\mathcal{F}_\varepsilon$ and then, passing to the limit as $\varepsilon \to 0$, we find an evolution for $\mathcal{F}$. To find an evolution for $\mathcal{F}_\varepsilon$ we follow this approach. First, we employ a discrete (incremental) scheme, in which the updated configuration is given by a local minimization problem. More precisely, let $\Delta s_n \searrow 0$; for each $n \in \mathbb{N}$ define by induction a sequence $(t_{n,k}, u_{n,k}, \xi_{n,k})$, for $k \in \mathbb{N}$, as follows: if $\partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k}) \neq 0$ then

$$\begin{cases} t_{n,k+1} = t_{n,k} \,, \\ u_{n,k+1} \in \operatorname{argmin} \{ \mathcal{F}_\varepsilon(t_{n,k}, v, \xi_{n,k}) : \| v - u_{n,k} \|_{H^1} \le \Delta s_n \}, \\ \xi_{n,k+1} = \xi_{n,k} \vee |[\![u_{n,k+1}]\!]|, \end{cases}$$

while, if $\partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k}) = 0$ then

$$\begin{cases} t_{n,k+1} = t_{n,k} + \Delta s_n, \\ u_{n,k+1} = u_{n,k}, \\ \xi_{n,k+1} = \xi_{n,k}. \end{cases}$$

Note that in this scheme the time variable is updated only when an equilibrium configuration is attained (the approach is indeed inspired by minimizing movements for gradient flows). The piecewise-affine interpolation of the sequence $(t_{n,k}, u_{n,k}, \xi_{n,k})$, for $k \in \mathbb{N}$, in the discrete points $s_{n,k} = k \Delta s_n$ provides a parametrized "discrete" evolution $s \mapsto (t_n(s), u_n(s), \xi_n(s))$ for $s \in [0, +\infty)$. By construction, the evolutions (t_n, u_n, ξ_n), for $n \in \mathbb{N}$, are uniformly Lipschitz continuous and thus (upon extracting a subsequence) there exists a limit, say $s \mapsto (t_\varepsilon(s), u_\varepsilon(s), \xi_\varepsilon(s))$, which is indeed the parametrized BV-evolution for the energy $\mathcal{F}_\varepsilon$. Finally, passing to the limit for $\varepsilon \to 0$ yields a parametrized BV-evolution $s \mapsto (t(s), u(s), \xi(s))$ for the energy $\mathcal{F}$.

Note that in general s is not the physical time variable but an auxiliary "length" parameter in the (t, u) space. In this framework discontinuities in time are represented by intervals, say $[s^-, s^+]$, where $t' = 0$ while u (and possibly ξ) changes; on the contrary, continuity points in time correspond to parametrization points in which $t'(s) > 0$.

Now we describe in more detail the characterization of this evolution (for the precise statement see Definition 3.1):

(C) for almost every $s \in [0, +\infty)$ the following Karush-Kuhn-Tucker conditions hold,

$$\xi'(s) \ge 0, \qquad |[\![u(s)]\!]| \le \xi(s), \qquad \xi'(s)\big(|[\![u(s)]\!]| - \xi(s)\big) = 0\,, \qquad \mathcal{H}^1\text{-a.e. on } K;$$

(S) for every $s \in [0, +\infty)$ with $t'(s) > 0$ the following equilibrium condition holds,

$$|\partial_u^- \mathcal{F}(t(s), u(s), \xi(s))| = 0,$$

(E) for every $s \in [0, +\infty)$ the following energy balance holds,

$$\mathcal{F}(t(s), u(s), \xi(s)) = \mathcal{F}(t_0, w_0, \xi_0) + \int_0^s \partial_t \mathcal{F}(t(r), u(r), \xi(r))\, t'(r)\, dr + \\ - \int_0^s |\partial_u^- \mathcal{F}(t(r), u(r), \xi(r))|\, dr\,.$$

In (S) and (E) we have to consider the *slope* $|\partial_u^- \mathcal{F}(t, u, \xi)|$ (see § 2) since $\mathcal{F}$ is not everywhere differentiable with respect to u. We remark that in our characterization the energy balance (E) is an equality and, most important, that all the Karush-Kuhn-Tucker conditions (C) are provided in a strong form. Moreover, it is noteworthy that for every $T \in (0, +\infty)$ there exists $S \in (0, +\infty)$ such that $t(S) = T$; as a by-product we also prove that discrete evolutions in a finite time horizon $T > 0$ are parametrized in a single, finite length, interval, say $[0, S]$, and obtained by a finite number of induction steps.

Finally, considering in $\mathcal{U}$ the norm $\|u\| = (\int_{\Omega \setminus K} |\nabla u|^2\, dx)^{1/2}$, conditions (S) and (E) provide the following system of PDEs: for $v(s) = u(s) + g(t(s)) + \lambda(s) u'(s)$ and $\lambda(s) = |\partial_u^- \mathcal{F}(t(s), u(s), \xi(s))|$ it holds

$$\begin{cases} \Delta v = 0 & \text{in } H^{-1}(\Omega \setminus K), \\ v = g(t(s)) & \text{on } \partial_D \Omega, \\ \partial_\nu^+ v = \partial_\nu^- v = h & \text{on } K, \\ \partial_\nu v = 0 & \text{on } [(\partial\Omega) \setminus K] \setminus \partial_D \Omega, \end{cases}$$

where $h \in L^\infty(K)$ and

$$\begin{cases} h = \tau(|[\![u]\!]|, \xi)(\operatorname{sgn} [\![u]\!]) & \mathcal{H}^1\text{-a.e. on } \{([\![u]\!], \xi) \neq (0,0)\}, \\ |h| \le \tau(0,0) & \text{otherwise.} \end{cases}$$

As we will see in § 7 and § 8 this system gives both the equilibrium conditions in the continuity points, i.e. where $t'(s) > 0$, and the behaviour in the discontinuity intervals, i.e. where $t' = 0$. Note that, in the former case it turns out that $\lambda(s) = 0$, by condition (E), and thus $v(s)$ becomes simply the (total) displacement $u(s) + g(t(s))$; in the latter, when $\lambda(s) \neq 0$, we formally obtain a visco-elastic (Kelvin-Voigt) system; this is a consequence of the choice of the H^1-norm in the discrete scheme.

2 Preliminaries

L^p vector-valued functions. Let us recall the following result (see, e.g., [14], § 2.22)

Lemma 2.1 *Let X be a reflexive Banach space, and $T > 0$. Let Φ be a bounded linear functional on $L^p(0, T; X)$ $(1 \le p < +\infty)$. Then there exists $u \in L^{p'}(0, T; X')$ such that $\|\Phi\| = \|u\|_{L^{p'}}$ and*

$$\Phi(v) = \int_0^T \langle u(t), v(t) \rangle_{X', X}\, dt$$

for every $v \in L^p(0, T; X)$.

Remark 2.2 In particular, if X is a reflexive Banach space, then the space $L^\infty(0, T; X')$ can be identified with the dual of the space $L^1(0, T; X)$. The duality pair is given by

$$\langle u, v \rangle_{L^\infty(0,T;X'), L^1(0,T;X)} = \int_0^T \langle u(t), v(t) \rangle_{X', X}\, dt$$

Let us also recall that $L^p(0, T; X)$ $(1 \le p < +\infty)$ is separable if (and only if) X is separable (see, e.g., [14], § 2.20). Hence, if X is a separable reflexive Banach space, then bounded sets in $L^\infty(0, T; X')$ are sequentially relatively compact with respect to the weak* convergence.

Sobolev vector-valued functions. Let us recall (see, e.g., [5]) that if X is a Banach space and $g \in L^1(0, T; X)$ then the functions u defined by $u(t) = \int_0^t g(s)\, ds$ is a.e. differentiable in $(0, T)$ and $u' = g$ a.e. in $(0, T)$.

We define the space $W^{1,p}(0,T;X)$ (with $1 \le p \le \infty$) as the space of function $u\colon [0,T] \to X$ which can be represented as

$$u(t) = u(0) + \int_0^t g(s)\,ds \qquad (t \in [0,T])$$

for a suitable $g \in L^p(0,T;X)$. The function g equals the derivative u' of u a.e. in $(0,T)$. We set $\|u\|_{W^{1,p}(0,T;X)} = \|u\|_{L^p(0,T;X)} + \|u'\|_{L^p(0,T;X)}$.

Assume that X is reflexive and separable (hence X' is reflexive and separable). Then ([5], Cor. A.2) the space of Lipschitz functions $[0,T] \to X'$ coincides with $W^{1,\infty}(0,T;X')$. Moreover, the following proposition holds.

Proposition 2.3 *Let (u_n) be a bounded sequence in $W^{1,\infty}(0,T;X')$. Then there exists a function $u \in W^{1,\infty}(0,T;X')$ such that, up to a subsequence,*

$$\begin{aligned} &u_n(t) \rightharpoonup u(t) \qquad w\text{-}X' \quad \text{for every } t \in [0,T], \\ &u_n' \rightharpoonup u' \qquad w^*\text{-}L^\infty(0,T;X'). \end{aligned}$$

Moreover, the w-X' convergence of $u_n(t)$ is uniform with respect to $t \in [0,T]$, i.e.

$$\text{if } t_n \to t \text{ then } \quad u_n(t_n) \rightharpoonup u(t) \quad w\text{-}X'.$$

We will refer to the convergence properties just stated as *weak* convergence in* $W^{1,\infty}(0,T;X')$.

Slope of a functional. Directional derivatives. Let X be a Banach space, and F a functional $X \to \mathbb{R}$. We define the *slope* of F in $u_0 \in X$ as

$$|\partial^- F(u_0)| := \limsup_{u \to u_0} \frac{[F(u) - F(u_0)]_-}{\|u - u_0\|}\,,$$

where $[\cdot]_-$ denotes the negative part.

If F is Fréchet differentiable in u_0, then

$$|\partial^- F(u_0)| = \|dF(u_0)\|_{X'}\,.$$

Assume now that F admits only (unilateral) *directional derivatives*, i.e. for every $z \in X$ the following limit exists and is finite:

$$\partial F(u_0; z) := \lim_{h \to 0+} \frac{F(u_0 + hz) - F(u_0)}{h}. \tag{2.1}$$

The following result provides a relationship between the slope and the directional derivatives.

Proposition 2.4 *Let $u_0 \in X$, and assume that the limit* (2.1) *is uniform with respect to $\|z\| \le 1$. Then*

$$|\partial^- F(u_0)| = \sup\bigl\{[\partial F(u_0; z)]_- : \ \|z\| \le 1\bigr\}.$$

Proof. Let $z \in X$ with $\|z\| \le 1$ and $z \ne 0$. Then by continuity of $[\cdot]_-$

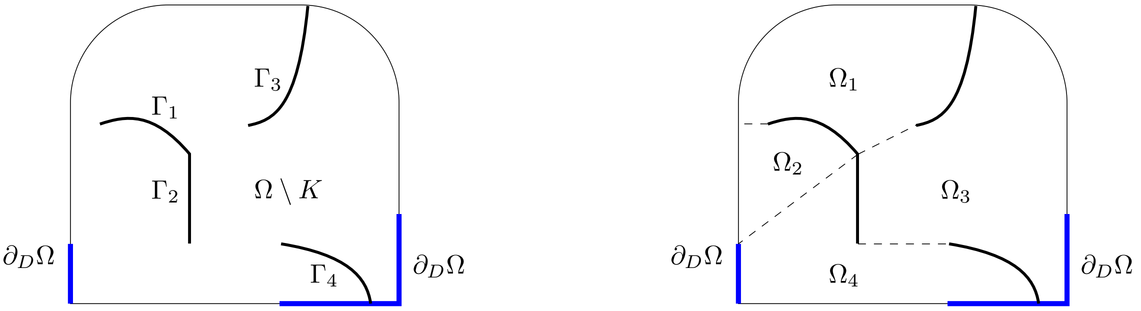
$$\begin{aligned} [\partial F(u_0; z)]_- &= \lim_{h \to 0^+} \frac{[F(u_0 + hz) - F(u_0)]_-}{h\|z\|}\,\|z\| \\ &\le \limsup_{u \to u_0} \frac{[F(u) - F(u_0)]_-}{\|u - u_0\|} = |\partial^- F(u_0)|. \end{aligned}$$

By the arbitrariness of z we have $|\partial^- F(u_0)| \ge \sup\bigl\{[\partial F(u_0; z)]_- : \ \|z\| \le 1\bigr\}$. Let us address the opposite inequality. Let $u_n \to u$ be a sequence satisfying

$$\lim_{n \to +\infty} \frac{[F(u_n) - F(u_0)]_-}{\|u_n - u_0\|} = |\partial^- F(u_0)|.$$

Let $h_n = \|u_n - u_0\|$ and $z_n = (u_n - u_0)/h_n$; thus $u_n = u_0 + h_n z_n$, with $\|z_n\| = 1$. Fix $\varepsilon > 0$. By assumption (see Remark 2.5 below, too) we can assume that for every n

$$\left| \frac{F(u_0 + h_n z_n) - F(u_0)}{h_n} - \partial F(u_0; z_n) \right| < \varepsilon,$$



(a) Crack path K and Dirichlet boundary $\partial_D\Omega$

(b) Decomposition of $\Omega \setminus K$ in piecewise-C^1 domains

Figure 1: Geometric setting

thus

$$\frac{F(u_0 + h_n z_n) - F(u_0)}{h_n} \geq \partial F(u_0; z_n) - \varepsilon.$$

Since $[\cdot]_-$ is monotone and $[x - \varepsilon]_- \leq [x]_- + \varepsilon$, we get

$$\frac{[F(u_0 + h_n z_n) - F(u_0)]_-}{h_n} \leq [\partial F(u_0; z_n)]_- + \varepsilon \leq \sup\bigl\{[\partial F(u_0; z)]_- : \ \|z\| \leq 1\bigr\} + \varepsilon.$$

The first item of these inequalities tends to $|\partial^- F(u_0)|$ and we conclude by the arbitrariness of ε. ■

Remark 2.5 *a*) The uniformity assumption in the preceding proposition can be expressed by requiring that for any positive infinitesimal sequence (h_n) and for every sequence (z_n) in X, with $\|z_n\| \leq 1$, we have

$$\lim_{n\to+\infty} \left| \frac{F(u_0 + h_n z_n) - F(u_0)}{h_n} - \partial F(u_0; z_n) \right| = 0.$$

b) It is easy to check that if F is Fréchet differentiable in u_0 then the limit (2.1) is uniform with respect to $\|z\| \leq 1$.

3 Setting

Let $\Omega \subseteq \mathbb{R}^2$ be an open, bounded and connected set with a piecewise-C^1 boundary (i.e. every point of $\partial\Omega$ has a neighbourhood which is the graph of a piecewise-C^1 function). Let $\alpha_1, \dots, \alpha_m$ be C^1 simple curves $[0, 1] \to \overline{\Omega}$ such that the sets $\Gamma_j = \alpha_j((0, 1))$ are pairwise disjoint, see Figure 1(a). Let $K := \bigcup_j \alpha_j([0, 1])$. We will assume that

i) $K \cap \partial\Omega$ is a subset of the set of endpoints of the arcs Γ_j; in particular, $(\partial\Omega) \setminus K$ consists of a finite number of arcs;

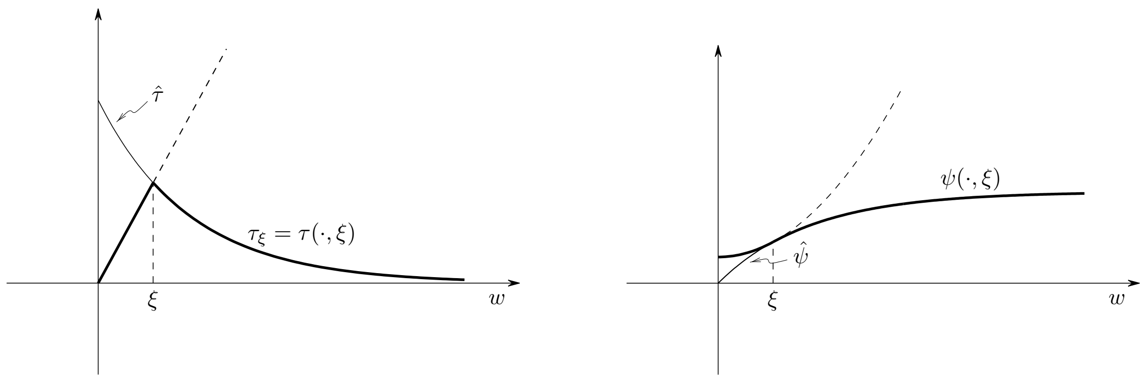
ii) up to a negligible set, $\Omega \setminus K$ is the disjoint union of finitely many connected piecewise-C^1 open sets Ω_i, see Figure 1(b); in particular, none of the curves Γ_j is tangent to $\partial\Omega$;

iii) each arc Γ_j is part of the boundaries of exactly two sets of the family $(\Omega_i)_i$.

In the setting of anti-plane elasticity, the displacement is a scalar function on $\Omega\setminus K$. On a portion $\partial_D\Omega$ of the boundary $\partial\Omega$ with $\mathcal{H}^1(\partial_D\Omega) > 0$ we impose boundary conditions, parametrized over the positive 'time' axis $\mathbb{R}^+ = [0, +\infty)$: if g is a given function $[0, +\infty) \to H^1(\Omega)$, we require that the displacement equals $g(t)$ on $\partial_D\Omega$ at any t. More precisely, we assume that $g \in C^1\bigl([0, +\infty); H^1(\Omega)\bigr)$ with $\|g\|_{C^1([0,+\infty);H^1)} < +\infty$.[1] In particular $g \in W^{1,\infty}\bigl([0, T]; H^1(\Omega)\bigr)$ for every $T > 0$. Note that, directly from the definition of g', the map $\nabla g \colon [0, +\infty) \to L^2(\Omega, \mathbb{R}^2)$ is a.e. differentiable, and

$$\frac{d}{dt}\nabla g(t) = \nabla g'(t) \qquad \text{a.e. in } \Omega.$$

[1] The case of a datum g which is assigned on a bounded interval $[0, T]$ can be managed in a similar way.



(a) Traction-separation law: function $\tau(\cdot, \xi)$ (b) Potential $\psi(\cdot, \xi)$

Figure 2: Traction-separation law and potential for a cohesive failure model

For convenience, for every t, we will write the admissible displacements as $v = g(t) + u$ where $u \in \mathcal{U}$, with

$$\mathcal{U} = \{u \in H^1(\Omega \setminus K) : u = 0 \text{ on } \partial_D \Omega\}.$$

This space will be equipped with the usual H^1 norm: we simply write $\|u\|$ if $u \in \mathcal{U}$.

A natural assumption on $\partial_D \Omega$ is that each connected component A of $\Omega \setminus K$ shares a part of the boundary where the datum g is placed (for instance, this guarantees that we can control the H^1 norm on A by the L^2 norm of the gradient). Thus, we require that

$$\mathcal{H}^1(\partial_D \Omega \cap \partial A) > 0, \qquad \text{for every connected component } A \text{ of } \Omega \setminus K.$$

Moreover, we require that $\partial_D \Omega$ consists of a finite number of C^1 arcs.

If $u \in \mathcal{U}$, we denote by

$$[\![u]\!] = u^+ - u^-$$

the jump of u on K, with respect to a fixed orientation (however, the relevant results involve only the absolute value of $[\![u]\!]$; see Remark 3.2, too).

We consider an elastic energy with the simple form:

$$\mathcal{E}(t, u) = \tfrac{1}{2} \int_{\Omega \setminus K} \mu |\nabla (u + g(t))|^2 \, dx \qquad (t \geq 0, \ u \in \mathcal{U}),$$

where $\mu > 0$ is the shear modulus. For the sake of simplicity, we will assume, without loss of generality, that $\mu = 1$. We match this energy with a cohesive potential energy, which we define starting from the traction-separation law, as follows.

Let $\hat{\tau} \colon [0, +\infty) \to [0, +\infty)$ be a C^1, non-increasing, summable, convex function: $\hat{\tau}$ can be interpreted as the *traction-separation law* for the originally unfractured configuration in a cohesive failure model. Denote by w the crack opening, defined (pointwise) on the crack path K; consider a configuration where the maximum opening previously experienced by the material is given pointwise by the non-negative function ξ. If $\xi = 0$ we define $\tau(w, \xi) = \hat{\tau}(w)$. If $\xi > 0$ we assume a linear loading-unloading regime followed by a softening loading regime; thus we get a traction-separation law of the form (see Figure 2(a)):

$$\tau(w, \xi) = \begin{cases} (\hat{\tau}(\xi)/\xi) w & \text{if } w \leq \xi, \\ \hat{\tau}(w) & \text{if } w \geq \xi. \end{cases}$$

Next, we define the cohesive energy density ψ as a function of both w and the maximum opening ξ through the traction-separation law $\tau(w, \xi)$ as (see Figure 2(b)):

$$\psi(w, \xi) = \int_0^\xi \hat{\tau}(r) \, dr - \int_w^\xi \tau(r, \xi) \, dr. \tag{3.1}$$

The first term in (3.1) corresponds to the energy of the opening crack ξ, while the second term gives the released energy when the opening is reduced to $w \leq \xi$. Note that the underlying physical model will naturally force the condition $w \leq \xi$ in the definition of an evolution path, however (3.1) defines ψ for every $(w, \xi) \in \mathbb{R}^+ \times \mathbb{R}^+$ (and not only for $w \leq \xi$). Clearly $\partial\psi/\partial w = \tau$.

We point out that ψ can be equivalently expressed as

$$\psi(w, \xi) = \int_0^w \hat{\tau}(r)\, dr + \int_w^\xi [\hat{\tau}(r) - \tau(r, \xi)]\, dr \tag{3.2}$$

and also as

$$\psi(w, \xi) = \psi^s(w, \xi) + \psi^d(\xi), \tag{3.3}$$

where

$$\psi^s(w, \xi) = \int_0^w \tau(r, \xi)\, dr, \qquad \psi^d(\xi) = \int_0^\xi [\hat{\tau}(r) - \tau(r, \xi)]\, dr$$

denote the *stored* and the *dissipated* energy, respectively. In the sequel, we will work with the density ψ without making the distinction between stored and dissipated energy. In Proposition 3.3 we gather the main properties of ψ.

On the crack path K we consider the energy

$$\mathcal{K}(u, \xi) = \int_K \psi(|[\![u]\!]|, \xi)\, d\mathcal{H}^1 \,, \tag{3.4}$$

defined in $\mathcal{U} \times L^2_+(K)$ (here $L^2_+(K)$ denotes the space of positive functions in $L^2(K)$). For ease of notation, we extend $\psi(\cdot, \xi)$ all over $\mathbb{R}$ as an even function; thus, we can also write

$$\mathcal{K}(u, \xi) = \int_K \psi([\![u]\!], \xi)\, d\mathcal{H}^1 \,.$$

The two terms previously set forth give the energy functional $\mathcal{F}\colon \mathbb{R}^+ \times \mathcal{U} \times L^2_+(K) \to \mathbb{R}^+$ defined by

$$\mathcal{F}(t, u, \xi) = \mathcal{E}(t, u) + \mathcal{K}(u, \xi) = \tfrac{1}{2} \int_{\Omega \setminus K} |\nabla(u + g(t))|^2\, dx + \int_K \psi(|[\![u]\!]|, \xi)\, d\mathcal{H}^1.$$

Let us now introduce the notion of *quasi-static evolution* we deal with in this paper; as in [18] we express it in terms of *parametrized BV evolutions*.

Definition 3.1 *Let $u_0 \in \mathcal{U}$ and $\xi_0 \in L^2_+(K)$, with $|[\![u_0]\!]| \leq \xi_0$ a.e. on K. Let $(t, u, \xi)\colon [0, +\infty) \to \mathbb{R}^+ \times \mathcal{U} \times L^2_+(K)$ be a Lipschitz map such that*

$$(t(0), u(0), \xi(0)) = (0, u_0, \xi_0), \qquad \lim_{s \to +\infty} t(s) = +\infty,$$

with $t(\cdot)$ a non-decreasing function.

The map (t, u, ξ) is a parametrized BV evolution *for $\mathcal{F}$ if*

(C) for almost every $s \in [0, +\infty)$ we have

$$\xi'(s) \geq 0, \qquad |[\![u(s)]\!]| \leq \xi(s), \qquad \xi'(s)\big(|[\![u(s)]\!]| - \xi(s)\big) = 0\,, \qquad \mathcal{H}^1\text{-a.e. on } K; \tag{3.5}$$

(S) for every $s \in [0, +\infty)$ with $t'(s) > 0$ we have

$$|\partial_u^- \mathcal{F}(t(s), u(s), \xi(s))| = 0\,; \tag{3.6}$$

(E) for every $s \in [0, +\infty)$ we have

$$\begin{aligned} \mathcal{F}(t(s), u(s), \xi(s)) = \mathcal{F}(t_0, w_0, \xi_0) &+ \int_0^s \partial_t \mathcal{F}(t(r), u(r), \xi(r))\, t'(r)\, dr + \\ &- \int_0^s |\partial_u^- \mathcal{F}(t(r), u(r), \xi(r))|\, dr\,. \end{aligned} \tag{3.7}$$

When the boundary condition is defined in a finite time interval $[0,T]$ the parametrization (t,u,ξ) is defined in a finite interval $[0,S]$ and correspondingly the conditions (C), (S) and (E) hold in $[0,S]$.

In Theorem 6.2 we prove the existence of a parametrized BV evolution for the energy $\mathcal{F}$.

Finally, we collect here a few properties which we will need in the sequel.

Remark 3.2 Let ν be a unit, normal vector field on $\partial\Omega$ and $K\cap\Omega$; assume that $\nu_{|\partial\Omega}$ is the exterior normal vector, and that $\nu_{|\Gamma_j}$ is continuous for every j. Let $u \in H^1(\Omega\setminus K)$. For every i, the trace of u on $\partial\Omega_i$ is well defined as a function in $H^{1/2}(\partial\Omega_i)$. This yields a trace u° of u on $(\partial\Omega)\setminus K$. As to the trace on K, let Γ be any of the arcs Γ_j which decompose K. Let i_+ and i_- be such that $\Gamma \subseteq \partial\Omega_{i_+} \cap \partial\Omega_{i_-}$ and that the orientation ν on Γ agrees with the outer unit normal of Ω_{i_+} on Γ. We denote by u^+ on Γ the trace of $u_{|\Omega_{i_+}}$ on Γ, and by u^- on Γ the trace of $u_{|\Omega_{i_-}}$ on Γ.

Let us now point out some properties of these traces.

a) By the continuity of the trace operator on each Ω_i, if $u \in H^1(\Omega\setminus K)$ then

$$\|[\![u]\!]\|_{L^2(K)} \le \|u^+\|_{L^2(K)} + \|u^-\|_{L^2(K)} \le C\|u\|$$

for a suitable constant C depending only on Ω and K.

b) Let Γ, i_+ and i_- be as above. Then the trace operators map continuously $H^1(\Omega_{i_+}), H^1(\Omega_{i_-})$ to the space $H^{1/2}(\Gamma)$, which is continuously and compactly embedded in $L^q(\Gamma)$ for every $q \in [2,+\infty)$ (see, e.g., [11], §§ 6 and 7).

c) If (u_n) is a sequence in $H^1(\Omega\setminus K)$ which converges weakly to an element u, then the continuity of the trace operator implies that $u_n^\pm \rightharpoonup u^\pm$ weakly in $H^{1/2}(\Gamma)$ for every Γ as above, hence $u_n^\pm \to u^\pm$ in $L^2(K)$ and in particular $[\![u_n]\!] \to [\![u]\!]$ in $L^2(K)$.

Proposition 3.3 *The following properties hold.*

a) *ψ is continuous and bounded in $\mathbb{R}^+ \times \mathbb{R}^+$.*

b) *Define (see Fig. 2(b))*

$$\hat\psi(w) = \int_0^w \hat\tau(r)\,dr, \qquad \textit{for every } w \ge 0.$$

Then, if $\xi > 0$

$$\psi(w,\xi) = \begin{cases} \hat\psi(\xi) - \frac{1}{2}\frac{\hat\tau(\xi)}{\xi}(\xi^2 - w^2) & \textit{if } 0 \le w \le \xi, \\ \\ \hat\psi(w) & \textit{if } w \ge \xi \end{cases}$$

while $\psi(w,0) = \hat\psi(w)$ for every $w \ge 0$.

c) *$\psi(\cdot,\xi) \in C^1\big([0,+\infty)\big)$ for every $\xi \ge 0$, and $\partial_w\psi = \tau$. In particular, $0 \le \partial_w\psi \le \hat\tau(0)$ on $[0,+\infty)$, and $\partial_w\psi(0,\xi) = 0$ if $\xi > 0$; remembering that $\psi(\cdot,\xi)$ is extended from $\mathbb{R}^+$ to $\mathbb{R}$ by even symmetry, it follows that $\psi(\cdot,\xi) \in C^1(\mathbb{R})$ for every $\xi > 0$.*

d) *$\psi(w,\cdot)$ is non-decreasing on $[0,+\infty)$ for every $w \ge 0$; moreover, it is continuously differentiable on $[w,+\infty)$, and $0 \le \partial_\xi\psi \le \frac{1}{2}\hat\tau(0)$.*

e) *ψ is Lipschitz continuous on $\mathbb{R}^+ \times \mathbb{R}^+$ (hence on $\mathbb{R}\times\mathbb{R}^+$).*

Proof. *a*) The continuity is immediate; the boundedness follows from the fact that $\hat\tau$ is summable, and both the integrals in (3.1) are bounded by $\int_0^{+\infty} \hat\tau$.

b) This follows from (3.1) making use of the explicit form of τ.

c) The property can be immediately deduced from (3.1) since $\tau(\cdot,\xi)$ is continuous on $[0,+\infty)$ for every $\xi \ge 0$.

d) Let $w \ge 0$. Since $\psi(w,\xi) = \hat\psi(w)$ if $\xi \le w$, to prove that $\psi(w,\cdot)$ is non-decreasing we have only to show that $\psi(w,\xi_1) \le \psi(w,\xi_2)$ if $w \le \xi_1 \le \xi_2$: this follows immediately from equation (3.1) since $\tau(\cdot,\xi_1) \ge \tau(\cdot,\xi_2)$.

Let us now prove the C^1-differentiability of $\psi(w,\cdot)$ on $[w,+\infty)$. If $w=0$ then

$$\psi(0,\xi)=\hat{\psi}(\xi)-\tfrac{1}{2}\hat{\tau}(\xi)\xi, \qquad \text{for every } \xi\geq 0,$$

and

$$\frac{d}{d\xi}\psi(0,\xi)=\hat{\tau}(\xi)-\tfrac{1}{2}\hat{\tau}'(\xi)\xi-\tfrac{1}{2}\hat{\tau}(\xi)=\tfrac{1}{2}\big(\hat{\tau}(\xi)-\hat{\tau}'(\xi)\xi\big);$$

now, taking the convexity of $\hat{\tau}$ into account, we have $0\leq\hat{\tau}(\xi)-\hat{\tau}'(\xi)\xi\leq\hat{\tau}(0)$.

If $w>0$ and $\xi\geq w$, then $\psi(w,\xi)=\hat{\psi}(\xi)-\frac{1}{2}\frac{\hat{\tau}(\xi)}{\xi}(\xi^2-w^2)$, so that

$$\frac{\partial}{\partial\xi}\psi(w,\xi)=\tfrac{1}{2}\frac{\hat{\tau}(\xi)-\hat{\tau}'(\xi)\xi}{\xi^2}(\xi^2-w^2).$$

Since $0\leq(\xi^2-w^2)/\xi^2\leq 1$, we conclude again by the convexity of $\hat{\tau}$.

e) By (*c*) and (*d*) the functions $\psi(\cdot,\xi)$ and $\psi(w,\cdot)$ are Lipschitz continuous on $\mathbb{R}^+$ with Lipschitz constants independent of ξ and w (recall that $\psi(w,\cdot)$ is constant on $[0,w]$): the global Lipschitz continuity of ψ on $\mathbb{R}^+\times\mathbb{R}^+$ then follows. ■

Corollary 3.4 *The functional* $\mathcal{K}\colon\mathcal{U}\times L^2_+(K)\to\mathbb{R}^+$ *is Lipschitz continuous.*

Proof. Take Proposition 3.3 (e) into account together with Remark 3.2 (*a*). ■

Lemma 3.5 *The functional* $\mathcal{E}$ *is of class* C^1 *on* $\mathbb{R}^+\times\mathcal{U}$ *(i.e., it is Fréchet differentiable on* $\mathbb{R}^+\times\mathcal{U}$ *with continuous derivative), and*

$$\partial_t\mathcal{E}(t,u)=\int_{\Omega\setminus K}\nabla\big(u+g(t)\big)\nabla g'(t)\,dx,$$
$$\partial_u\mathcal{E}(t,u)[z]=\int_{\Omega\setminus K}\nabla\big(u+g(t)\big)\nabla z\,dx \qquad (z\in\mathcal{U}).$$

Proof. It is enough to show that the partial Fréchet derivatives exist and are continuous. The result about $\partial_u\mathcal{E}(t,u)$ is standard; that about $\partial_t\mathcal{E}(t,u)$, can be obtained by composition. ■

Since the partial derivative $\partial_w\psi(w,\xi)$ does not exist in the origin, i.e. for $w=\xi=0$, it will be useful to have the directional derivative $\partial_w\psi(w,\xi;z)$ of ψ in $\mathbb{R}\times\mathbb{R}^+$ (according to (2.1) with $X=\mathbb{R}$): for every $(w,\xi)\in\mathbb{R}\times\mathbb{R}^+$ and $z\in\mathbb{R}$ it turns out that

$$\partial_w\psi(w,\xi;z)=\begin{cases}\partial_w\psi(w,\xi)z=\tau(|w|,\xi)(\operatorname{sgn}w)z & \text{if } (w,\xi)\neq(0,0)\\ \hat{\tau}(0)|z|=\tau(0,0)|z| & \text{if } (w,\xi)=(0,0),\end{cases} \tag{3.8}$$

where we have set $\operatorname{sgn}0=0$ (however, note that if $w=0$ then $\tau(|w|,\xi)=0$ if $\xi\neq 0$). In the following result we study the directional differentiability of $\mathcal{F}$ according to (2.1),

Lemma 3.6 *The functional* $\mathcal{K}$ *admits (unilateral) directional derivative* $\partial_u\mathcal{K}(u,\xi;z)$ *for any* $z\in\mathcal{U}$*, and*

$$\partial_u\mathcal{K}(u,\xi;z)=\int_K\partial_w\psi([\![u]\!],\xi;[\![z]\!])\,d\mathcal{H}^1 \tag{3.9}$$

(where $\partial_w\psi$ *is defined in* (3.8)*). Moreover, the limit defining* $\partial_u\mathcal{K}(u,\xi;z)$ *is uniform with respect to* $z\in\mathcal{U}$*, with* $\|z\|\leq 1$*. In particular, by Remark 2.5 if* $z_n\to z$ *and* h_n *is positive and infinitesimal then*

$$\lim_{n\to+\infty}\left|\frac{\mathcal{K}(u+h_nz_n,\xi)-\mathcal{K}(u,\xi)}{h_n}-\partial_u\mathcal{K}(u,\xi;z_n)\right|=0.$$

Proof. Let $u\in\mathcal{U}$ and $\xi\in L^2_+(K)$ be fixed. Let (h_n) be a positive infinitesimal sequence and (z_n) a sequence in $\mathcal{U}$, with $\|z_n\|\leq 1$. Denote $[\![u]\!]$ and $[\![z_n]\!]$ by w and w_n, respectively. According to Remark 2.5 (*a*), consider

$$\begin{aligned}&\left|\frac{\mathcal{K}(u+h_nz_n,\xi)-\mathcal{K}(u,\xi)}{h_n}-\int_K\partial_w\psi(w,\xi;w_n)\,d\mathcal{H}^1\right|\\ &\leq\int_K\left|\frac{\psi(w+h_nw_n,\xi)-\psi(w,\xi)}{h_n}-\partial_w\psi(w,\xi;w_n)\right|d\mathcal{H}^1=\int_K\sigma_n|w_n|d\mathcal{H}^1,\end{aligned}$$

where

$$\sigma_n = 1_{\{w_n \neq 0\}} \left| \frac{\psi(w + h_n w_n, \xi) - \psi(w,\xi)}{h_n |w_n|} - \partial_w \psi(w, \xi; \operatorname{sgn} w_n) \right|$$

(note that $\partial_w \psi(w, \xi; \lambda w_n) = \lambda \partial_w \psi(w, \xi; w_n)$ for $\lambda \geq 0$). Since $h_n z_n \to 0$ in $H^1(\Omega \setminus K)$, we can assume (up to a subsequence) that $h_n w_n \to 0$ in $L^2(K)$ and pointwise a.e. on K. Setting $\eta_n = h_n |w_n|$ if $w_n > 0$ we have

$$\left| \frac{\psi(w + h_n w_n, \xi) - \psi(w,\xi)}{h_n |w_n|} - \partial_w \psi(w, \xi; \operatorname{sgn} w_n) \right| = \left| \frac{\psi(w + \eta_n, \xi) - \psi(w,\xi)}{\eta_n} - \partial_w \psi(w, \xi; 1) \right|$$

while for $w_n < 0$

$$\left| \frac{\psi(w + h_n w_n, \xi) - \psi(w,\xi)}{h_n |w_n|} - \partial_w \psi(w, \xi; \operatorname{sgn} w_n) \right| = \left| \frac{\psi(w - \eta_n, \xi) - \psi(w,\xi)}{\eta_n} - \partial_w \psi(w, \xi; -1) \right|.$$

Considering the subsequences where $w_n > 0$ or $w_n < 0$ we have in both the cases that the difference quotient converge to the directional derivative and hence $\sigma_n \to 0$ a.e. on K.

By Hölder's inequality and Remark 3.2

$$\int_K \sigma_n |w_n| d\mathcal{H}^1 \leq C \|\sigma_n\|_{L^2(K)} \|z_n\| \leq C \|\sigma_n\|_{L^2(K)}.$$

By the Lipschitz continuity of $\psi(\cdot, \xi)$ (see Proposition 3.3 (e)) the incremental quotients in the definition of σ_n are bounded by $\max |\partial_w \psi(\cdot, \xi; \pm 1)| = \hat{\tau}(0)$. Therefore, $|\sigma_n| \leq 2\hat{\tau}(0)$, and, by the dominated convergence theorem $\|\sigma_n\|_{L^2(K)} \to 0$.

The convergence now proved yields both (3.9) (take a constant sequence (z_n)) and the uniform condition for the limit (2.1) for $\mathcal{K}$. ∎

Since the elastic energy $\mathcal{E}$ is Fréchet differentiable we can introduce the directional derivative

$$\partial_u \mathcal{F}(t, u, \xi; z) = \partial_u \mathcal{E}(t, u)[z] + \partial_u \mathcal{K}(u, \xi; z). \tag{3.10}$$

By the previous lemma and by Proposition 2.4 we can represent the slope as

$$|\partial_u^- \mathcal{F}(t, u, \xi)| = \sup\big\{ [\partial_u \mathcal{F}(t, u, \xi; z)]_- : \ \|z\| \leq 1 \big\}. \tag{3.11}$$

4 Regularized energy

The main result of this paper will be first proved for a modified energy $\mathcal{F}_\varepsilon$ where an additional regularity is required for the energy density on the crack. Thus, a modified traction-separation law is considered, to overcome the lack of differentiability in zero of the function $\psi(|\cdot|, \xi)$ which enters the line energy (3.4).

For every $\varepsilon > 0$ and $w \in \mathbb{R}^+$ let (see Figure 3)

$$\hat{\tau}_\varepsilon(w) = \min[w/\varepsilon, \hat{\tau}(w)]. \tag{4.1}$$

Let $\xi_\varepsilon > 0$ such that $\xi_\varepsilon / \varepsilon = \hat{\tau}(\xi_\varepsilon)$, then the regularized function τ_ε takes the form $\hat{\tau}_\varepsilon(w) = \tau(w, \xi_\varepsilon)$. For $(w, \xi) \in \mathbb{R}^+ \times \mathbb{R}^+$ we define

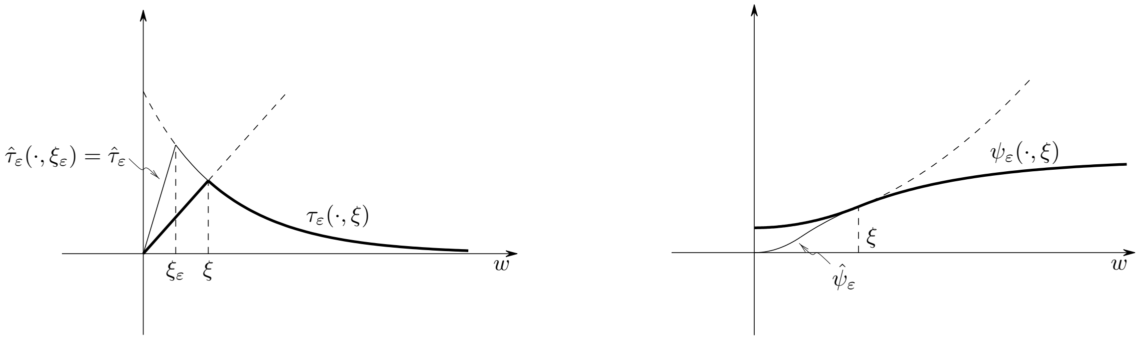
$$\tau_\varepsilon(w, \xi) = \begin{cases} \tau(w, \xi_\varepsilon) = \hat{\tau}_\varepsilon(w) & \text{if } \xi \leq \xi_\varepsilon, \\ \tau(w, \xi) & \text{if } \xi \geq \xi_\varepsilon. \end{cases} \tag{4.2}$$

Thus, $\tau_\varepsilon(0, \xi) = 0$, and $\tau_\varepsilon(\cdot, \xi)$ is Lipschitz continuous on $\mathbb{R}^+$, uniformly with respect to $\xi \in \mathbb{R}^+$. Moreover, it is worthwhile to note that $\xi_\varepsilon \to 0$ as $\varepsilon \to 0$ and that

$$\tau_\varepsilon(w, \xi) = \tau(w, \xi) \qquad \text{if } \xi \geq \xi_\varepsilon \text{ or } w \geq \xi_\varepsilon. \tag{4.3}$$

Next, we define the regularized potential $\psi_\varepsilon(w, \xi)$ by analogy with the definition of ψ:

$$\begin{aligned} \psi_\varepsilon(w, \xi) &= \int_0^\xi \hat{\tau}_\varepsilon(r)\, dr - \int_w^\xi \tau_\varepsilon(r, \xi)\, dr \\ &= \int_0^w \hat{\tau}_\varepsilon(r)\, dr + \int_w^\xi [\hat{\tau}_\varepsilon(r) - \tau_\varepsilon(r, \xi)]\, dr \\ &= \psi_\varepsilon^s(w, \xi) + \psi_\varepsilon^d(\xi), \end{aligned} \tag{4.4}$$



(a) Traction-separation law: function $\tau_\varepsilon(\cdot,\xi)$.

(b) Potential $\psi_\varepsilon(\cdot,\xi)$.

Figure 3: A regularized model

where

$$\psi^s_\varepsilon(w,\xi)=\int_0^w \tau_\varepsilon(r,\xi)\,dr, \qquad \psi^d_\varepsilon(\xi)=\int_0^\xi [\hat\tau_\varepsilon(r)-\tau_\varepsilon(r,\xi)]\,dr.$$

We extend $\psi_\varepsilon(\cdot,\xi)$ to the whole $\mathbb{R}$ as an even function. In the following proposition we collect some of the main properties of the function ψ_ε (see Figure 3).

Let us now introduce the regularized energy as the energy corresponding to the potential in (4.4). Let

$$\mathcal{K}_\varepsilon(u,\xi)=\int_K \psi_\varepsilon([\![u]\!],\xi)\,d\mathcal{H}^1\,,$$

and

$$\mathcal{F}_\varepsilon(t,u,\xi)=\mathcal{E}(t,u)+\mathcal{K}_\varepsilon(u,\xi)=\tfrac12\int_{\Omega\setminus K}|\nabla\big(u+g(t)\big)|^2\,dx+\int_K \psi_\varepsilon([\![u]\!],\xi)\,d\mathcal{H}^1\,.$$

In analogy with Definition 4.1 a *parametrized BV-evolution* for $\mathcal{F}_eps$ can be defined as follows.

Definition 4.1 *Let $u_0\in\mathcal{U}$ and $\xi_0\in L^2_+(K)$, with $|[\![u_0]\!]|\le\xi_0$ a.e. on K. Let $(t,u,\xi)\colon[0,+\infty)\to\mathbb{R}^+\times\mathcal{U}\times L^2_+(K)$ be a Lipschitz map such that*

$$(t(0),u(0),\xi(0))=(0,u_0,\xi_0), \qquad \lim_{s\to+\infty}t(s)=+\infty,$$

with $t(\cdot)$ a non-decreasing function.

The map (t,u,ξ) is a parametrized BV evolution *for $\mathcal{F}_\varepsilon$ if*

(C) *for almost every $s\in[0,+\infty)$ we have*

$$\xi'(s)\ge0,\qquad |[\![u(s)]\!]|\le\xi(s),\qquad \xi'(s)\big(|[\![u(s)]\!]|-\xi(s)\big)=0\,,\qquad \mathcal{H}^1\text{-a.e. on } K; \tag{4.5}$$

(S) *for every $s\in[0,+\infty)$ with $t'(s)>0$ we have*

$$\|\partial_u\mathcal{F}_\varepsilon(t(s),u(s),\xi(s))\|_{\mathcal{U}'}=0\,; \tag{4.6}$$

(E) *for every $s\in[0,+\infty)$ we have*

$$\begin{aligned}\mathcal{F}_\varepsilon(t(s),u(s),\xi(s))=\mathcal{F}_\varepsilon(t_0,w_0,\xi_0)+\int_0^s \partial_t\mathcal{F}_\varepsilon(t(r),u(r),\xi(r))\,t'(r)\,dr+\\ -\int_0^s\|\partial_u\mathcal{F}_\varepsilon(t(r),u(r),\xi(r))\|_{\mathcal{U}'}\,dr\,.\end{aligned} \tag{4.7}$$

Let us see the properties of the regularized energies and their convergence as ε vanishes.

Proposition 4.2 *The following properties hold.*

a) ψ_ε is continuous and bounded, uniformly with respect to $\varepsilon>0$, in $\mathbb{R}\times\mathbb{R}^+$;

b) *Define*

$$\hat{\psi}_\varepsilon(w) = \int_0^w \hat{\tau}_\varepsilon(r)\,dr, \qquad \text{for every } w \geq 0.$$

Then $\psi_\varepsilon(\cdot,\xi) = \hat{\psi}_\varepsilon$ *on* $\mathbb{R}^+$ *if* $0 \leq \xi \leq \xi_\varepsilon$. *Moreover, if* $\xi \geq \xi_\varepsilon$ *then*

$$\psi_\varepsilon(w,\xi) = \begin{cases} \hat{\psi}_\varepsilon(\xi) - \frac{1}{2}\frac{\hat{\tau}(\xi)}{\xi}(\xi^2 - w^2) & \text{if } 0 \leq w \leq \xi, \\ \\ \hat{\psi}_\varepsilon(w) & \text{if } w \geq \xi. \end{cases}$$

c) $\psi_\varepsilon(\cdot,\xi) \in C^1(\mathbb{R})$ *for every* $\xi \geq 0$, *and* $\partial_w \psi_\varepsilon = \tau_\varepsilon$.

d) $\psi_\varepsilon(w,\cdot)$ *is non-decreasing on* $[0,+\infty)$ *for every* $w \in \mathbb{R}$.

Proof. (a), (b) and (d) can be proved as the analogous properties in Proposition 3.3. Property (c) follows from the fact that, as pointed out above, $\tau_\varepsilon(0,\xi) = 0$. ■

Proposition 4.3 *As* $\varepsilon \to 0$ *we have:*

a) $\psi_\varepsilon \to \psi$ *in* $\mathbb{R} \times \mathbb{R}^+$, *uniformly;*

b) $\partial_w \psi_\varepsilon \to \partial_w \psi$ *uniformly on compact subsets of* $\big(\mathbb{R} \times \mathbb{R}^+\big) \setminus \{(0,0)\}$. *Moreover,*

$$\limsup_{n\to+\infty} \partial_w \psi_{\varepsilon_n}(w_n,\xi_n) z \leq \partial_w \psi(w,\xi;z)$$

whenever $\varepsilon_n \to 0$, $w_n \to w$ *and* $\xi_n \to \xi$.

Proof. $a)$ It is enough to consider $\mathbb{R}^+ \times \mathbb{R}^+$. Let $w, \xi \in \mathbb{R}^+$ be fixed. From the definition of ψ and ψ_ε, we have

$$\psi(w,\xi) - \psi_\varepsilon(w,\xi) = \int_0^\xi \big(\hat{\tau}(r) - \hat{\tau}_\varepsilon(r)\big)\,dr - \int_w^\xi \big(\tau(r,\xi) - \tau_\varepsilon(r,\xi)\big)\,dr.$$

If $\xi \geq \xi_\varepsilon$ then the second integral vanishes by (4.2). Otherwise, its absolute value is not greater than $\int_0^{\xi_\varepsilon} \big(\hat{\tau}(r) - \hat{\tau}_\varepsilon(r)\big)\,dr$, which tends to zero as $\varepsilon \to 0$, uniformly with respect to w and ξ, since $\xi_\varepsilon \to 0$. The first integral is bounded by $\int_0^{\xi_\varepsilon} \big(\hat{\tau}(r) - \hat{\tau}_\varepsilon(r)\big)\,dr$, too.

(b) On compact subsets of $\big(\mathbb{R}^+ \times \mathbb{R}^+\big) \setminus \{(0,0)\}$ we have $\partial_w \psi_\varepsilon = \tau_\varepsilon$ and $\partial_w \psi = \tau$, hence the uniform convergence is an immediate consequence of (4.3). This implies also the uniform convergence on the whole $\big(\mathbb{R} \times \mathbb{R}^+\big) \setminus \{(0,0)\}$ since the extensions of $\psi_\varepsilon(\cdot,\xi)$ and $\psi(\cdot,\xi)$ to $\mathbb{R}$ is even.

Let now $\varepsilon_n \to 0$, $w_n \to w$ and $\xi_n \to \xi$. If $(w,\xi) = (0,0)$ then $\partial_w \psi(w,\xi;z) = \hat{\tau}(0)|z|$ for every $z \in \mathbb{R}$ and

$$|\partial_w \psi_{\varepsilon_n}(w_n,\xi_n) z| = \tau_\varepsilon(|w_n|,\xi_n)|z| \leq \tau(|w_n|,\xi_n)|z| \leq \hat{\tau}(0)|z|;$$

therefore the lim sup inequality is trivial. If $(w,\xi) \neq (0,0)$ then (w_n,ξ_n) is bounded away from $(0,0)$ for n large enough, so that $\partial_w \psi_{\varepsilon_n}(w_n,\xi_n) z \to \partial_w \psi(w,\xi) z = \partial_w \psi(w,\xi;z)$ for every $z \in \mathbb{R}$. ■

Lemma 4.4 *For every* $\xi \in L^2_+(K)$ *the functional* $\mathcal{F}_\varepsilon(\cdot,\cdot,\xi)$ *is of class* C^1 *on* $\mathbb{R}^+ \times \mathcal{U}$, *with*

$$\partial_t \mathcal{F}_\varepsilon(t,u,\xi) = \partial_t \mathcal{E}(t,u),$$
$$\partial_u \mathcal{F}_\varepsilon(t,u,\xi)[z] = \partial_u \mathcal{E}(t,u)[z] + \partial_u \mathcal{K}_\varepsilon(u,\xi)[z],$$

where $\partial_t \mathcal{E}$ *and* $\partial_u \mathcal{E}$ *are given in Lemma 3.5, and, for every* $z \in \mathcal{U}$,

$$\partial_u \mathcal{K}_\varepsilon(u,\xi)[z] = \int_K \partial_w \psi_\varepsilon([\![u]\!],\xi)[\![z]\!]\,d\mathcal{H}^1 = \int_K \tau_\varepsilon(|[\![u]\!]|,\xi)\,\mathrm{sgn}([\![u]\!])[\![z]\!]\,d\mathcal{H}^1.$$

Moreover, the map $u \mapsto \partial_u \mathcal{F}_\varepsilon(t,u,\xi)$ *is Lipschitz continuous from* $\mathcal{U}$ *to* $\mathcal{U}'$, *uniformly with respect to* $(t,\xi) \in \mathbb{R}^+ \times L^2_+(K)$.

Proof. Let $\xi \in L^2_+$ be fixed. For every $z \in \mathcal{U}$ we have

$$\lim_{h\to 0} \frac{\mathcal{K}_\varepsilon(u+hz,\xi) - \mathcal{K}_\varepsilon(u,\xi)}{h} = \int_K \partial_w \psi_\varepsilon(\llbracket u \rrbracket, \xi)\llbracket z \rrbracket \, d\mathcal{H}^1$$

since $\psi_\varepsilon(\cdot,\xi)$ is C^1 with bounded derivative. The continuity of the right-hand side with respect to $z \in \mathcal{U}$ follows from Remark 3.2(a). Thus $\mathcal{K}_\varepsilon(\cdot,\xi)$ is Gâteaux differentiable, with derivative

$$\partial_u \mathcal{K}_\varepsilon(u,\xi)[z] = \int_K \partial_w \psi_\varepsilon(\llbracket u \rrbracket, \xi)\llbracket z \rrbracket \, d\mathcal{H}^1 = \int_K \tau_\varepsilon(|\llbracket u \rrbracket|, \xi)\,\mathrm{sgn}(\llbracket u \rrbracket)\llbracket z \rrbracket \, d\mathcal{H}^1 .$$

Let $u_1, u_2 \in \mathcal{U}$. For every $z \in \mathcal{U}$ we have

$$|\partial_u \mathcal{K}_\varepsilon(u_1,\xi)[z] - \partial_u \mathcal{K}_\varepsilon(u_2,\xi)[z]| \le L_\varepsilon \int_K |\llbracket u_1 \rrbracket - \llbracket u_2 \rrbracket| |\llbracket z \rrbracket| \, d\mathcal{H}^1 \le L'_\varepsilon \|u_1 - u_2\| \, \|z\| ,$$

where L_ε denotes a Lipschitz constant for the function $r \mapsto \tau_\varepsilon(|r|,\xi)\,\mathrm{sgn}(r)$. It follows that

$$\|\partial_u \mathcal{K}_\varepsilon(u_1,\xi) - \partial_u \mathcal{K}_\varepsilon(u_2,\xi)\|_{\mathcal{U}'} \le L'_\varepsilon \|u_1 - u_2\|$$

and hence $\mathcal{K}_\varepsilon(\cdot,\xi)$ is Fréchet differentiable, with Lipschitz derivative. The same Lipschitz property is shared by $\partial_u \mathcal{E}(t,\cdot)$, indeed

$$|\partial_u \mathcal{E}(t,u_1)[z] - \partial_u \mathcal{E}(t,u_2)[z]| \le \int_{\Omega\setminus K} |\nabla(u_1 - u_2)|\,|\nabla z| \, dx.$$

We conclude that the map $u \mapsto \partial_u \mathcal{F}(t,u,\xi)$ is Lipschitz continuous from $\mathcal{U}$ to $\mathcal{U}'$, uniformly with respect to $(t,\xi) \in \mathbb{R}^+ \times L^2_+(K)$. ■

Lemma 4.5 *Let (t_n), (u_n) and (ξ_n) be such that*

$$t_n \to t; \quad u_n \rightharpoonup u \text{ in } H^1(\Omega\setminus K); \quad \xi_n \to \xi \text{ in } L^2(K).$$

Then

$$(a) \quad \mathcal{F}_\varepsilon(t,u,\xi) \le \liminf_{n\to+\infty} \mathcal{F}_\varepsilon(t_n,u_n,\xi_n);$$

$$(b) \quad \|\partial_u \mathcal{F}_\varepsilon(t,u,\xi)\|_{\mathcal{U}'} \le \liminf_{n\to+\infty} \|\partial_u \mathcal{F}_\varepsilon(t_n,u_n,\xi_n)\|_{\mathcal{U}'};$$

$$(c) \quad \lim_{n\to+\infty} \partial_t \mathcal{F}_\varepsilon(t_n,u_n,\xi_n) = \partial_t \mathcal{F}_\varepsilon(t,u,\xi) .$$

Proof. (a) By the weak-L^2 convergence of (∇u_n) and the convergence of $\big(\nabla g(t_n)\big)$ in L^2, we have the lower semicontinuity inequality for $\mathcal{E}$. As to $\mathcal{K}_\varepsilon$, consider a subsequence (not relabeled) such that $\liminf_{n\to+\infty} \int_K \psi_\varepsilon(\llbracket u_n \rrbracket, \xi_n) d\mathcal{H}^1$ is a limit. By the strong convergence of (ξ_n) and by Remark 3.2 (c), we can assume that (ξ_n) and $(\llbracket u_n \rrbracket)$ converge a.e. Hence, by Fatou's Lemma $\int_K \psi_\varepsilon(\llbracket u \rrbracket, \xi)\, d\mathcal{H}^1 \le \liminf_{n\to+\infty} \int_K \psi_\varepsilon(\llbracket u_n \rrbracket, \xi_n)\, d\mathcal{H}^1$.

(b) Let $z \in \mathcal{U}$ with $\|z\| \le 1$. Then

$$\begin{aligned}\|\partial_u \mathcal{F}_\varepsilon(t_n,u_n,\xi_n)\|_{\mathcal{U}'} &\ge \partial_u \mathcal{F}_\varepsilon(t_n,u_n,\xi_n)[z] \\ &= \int_{\Omega\setminus K} \nabla(u_n + g(t_n))\nabla z \, dx + \int_K \tau_\varepsilon(|\llbracket u_n \rrbracket|, \xi_n)\,\mathrm{sgn}(\llbracket u_n \rrbracket)\llbracket z \rrbracket \, d\mathcal{H}^1 .\end{aligned}$$

Remembering that $\tau_\varepsilon(\cdot,\xi)$ is continuous and that $\tau_\varepsilon(0,\xi) = 0$, a similar argument as in (a) yields the lower semicontinuity for both these integral terms; hence

$$\begin{aligned}\liminf_{n\to+\infty} \|\partial_u \mathcal{F}_\varepsilon(t_n,u_n,\xi_n)\|_{\mathcal{U}'} &\ge \int_{\Omega\setminus K} \nabla(u + g(t))\nabla z \, dx + \int_K \tau_\varepsilon(|\llbracket u \rrbracket|, \xi)\,\mathrm{sgn}(\llbracket u \rrbracket)\llbracket z \rrbracket \, d\mathcal{H}^1 \\ &= \partial_u \mathcal{F}_\varepsilon(t,u,\xi)[z] .\end{aligned}$$

By the arbitrariness of z we get

$$\liminf_{n\to+\infty} \|\partial_u \mathcal{F}_\varepsilon(t_n,u_n,\xi_n)\|_{\mathcal{U}'} \ge \|\partial_u \mathcal{F}_\varepsilon(t,u,\xi)\|_{\mathcal{U}'}.$$

(c) This property is an immediate consequence of the expression of $\partial_t \mathcal{F}_\varepsilon$ and the continuity of the map $t \mapsto \nabla g'(t)$ in $L^2(\Omega,\mathbb{R}^2)$. ■

Lemma 4.6 *Let (t_n), (u_n) and (ξ_n) be sequences such that*

$$t_n \to t; \quad u_n \rightharpoonup u \text{ in } H^1(\Omega \setminus K); \quad \xi_n \to \xi \text{ in } L^2(K).$$

Then, for $\varepsilon_n \to 0$ we have

$$\begin{aligned} &(a) \quad \mathcal{F}(t,u,\xi) \le \liminf_{n\to+\infty} \mathcal{F}_{\varepsilon_n}(t_n,u_n,\xi_n)\,, \\ &(b) \quad |\partial_u^- \mathcal{F}(t,u,\xi)| \le \liminf_{n\to+\infty} \|\partial_u \mathcal{F}_{\varepsilon_n}(t_n,u_n,\xi_n)\|_{\mathcal{U}'}\,, \\ &(c) \quad \partial_t \mathcal{F}(t,u,\xi) = \lim_{n\to+\infty} \partial_t \mathcal{F}_{\varepsilon_n}(t_n,u_n,\xi_n)\,. \end{aligned}$$

Proof. We can assume that $(\llbracket u_n \rrbracket)$ and (ξ_n) converge a.e. (recall Remark 3.2 (c)).

(a) The l.s.c. inequality for $\mathcal{E}$, which is independent of ε, has already been checked in Lemma 4.5 (it follows from the weak-L^2 convergence of (∇u_n) and the convergence of $\big(\nabla g(t_n)\big)$ in L^2). As to $\mathcal{K}$, it is enough to apply Lemma 4.3 (a) and Fatou's Lemma, indeed

$$\mathcal{K}(u,\xi) = \int_K \psi(|\llbracket u \rrbracket|,\xi)\, d\mathcal{H}^1 \le \liminf_{n\to+\infty} \int_K \psi_{\varepsilon_n}(|\llbracket u_n \rrbracket|,\xi_n)\, d\mathcal{H}^1 = \liminf_{n\to+\infty} \mathcal{K}_{\varepsilon_n}(u_n,\xi_n).$$

(b) Let $z \in \mathcal{U}$ be fixed, with $\|z\| \le 1$. Recall that, by Lemma 3.5 and Lemma 4.4

$$\partial_u \mathcal{F}_{\varepsilon_n}(t_n,u_n,\xi_n)[z] = \int_{\Omega\setminus K} \nabla\big(u_n + g(t_n)\big)\nabla z\, dx + \int_K \partial_w \psi_{\varepsilon_n}(\llbracket u_n \rrbracket,\xi_n)\llbracket z \rrbracket\, d\mathcal{H}^1.$$

The first integral in the right-hand side converges to $\int_{\Omega\setminus K} \nabla\big(u+g(t)\big)\nabla z\, dx$. Moreover, by Proposition 4.3 (b), a.e. on K we have

$$\limsup_{n\to+\infty} \partial_w \psi_{\varepsilon_n}(\llbracket u_n \rrbracket,\xi_n)\llbracket z \rrbracket \le \partial_w \psi(\llbracket u \rrbracket,\xi;\llbracket z \rrbracket)$$

since $(\llbracket u_n \rrbracket,\xi_n) \to (\llbracket u \rrbracket,\xi)$ a.e. on K. Therefore Fatous's Lemma yields

$$\limsup_{n\to+\infty} \partial_u \mathcal{F}_{\varepsilon_n}(t_n,u_n,\xi_n)[z] \le \int_{\Omega\setminus K} \nabla\big(u+g(t)\big)\nabla z\, dx + \int_K \partial_w \psi(\llbracket u \rrbracket,\xi;\llbracket z \rrbracket)\, d\mathcal{H}^1 = \partial_u \mathcal{F}(t,u,\xi;z),$$

where the directional derivative $\partial_w \mathcal{F}(t,u,\xi;z)$ has been defined in (3.10). Note now that for any real sequence (a_n) it holds $(\limsup a_n)_- = (\liminf(-a_n))_+$; then by the monotonicity of $(\cdot)_-$

$$\begin{aligned} \big(\partial_u \mathcal{F}(t,u,\xi;z)\big)_- &\le \Big(\limsup_{n\to+\infty} \partial_u \mathcal{F}_{\varepsilon_n}(t_n,u_n,\xi_n)[z]\Big)_- \\ &= \Big(\liminf_{n\to+\infty} \partial_u \mathcal{F}_{\varepsilon_n}(t_n,u_n,\xi_n)[-z]\Big)_+ \\ &\le \liminf_{n\to+\infty} \|\partial_u \mathcal{F}_{\varepsilon_n}(t_n,u_n,\xi_n)\|_{\mathcal{U}'}. \end{aligned}$$

We can now conclude by taking the supremum with respect to z, thank to (3.11).

(c) Since $\partial_t \mathcal{F} = \partial_t \mathcal{E} = \partial_t \mathcal{F}_{\varepsilon_n}$, this item is as in Lemma 4.5. ■

5 Quasi-static evolution for the regularized energy $\mathcal{F}_\varepsilon$

In the space $\mathbb{R}^+ \times \mathcal{U} \times L^2_+(K)$ of the variables t, u and ξ we first introduce (Subsection 5.1) a discrete evolution (from an initial point $(0,u_0,\xi_0)$), depending on an incremental parameter Δs which acts both as a time increment and as a range for the local minimality of the displacement (see below). This sequence of points is read as a piecewise-affine function on the space of the parameter s. Actually, the increment Δs varies along a sequence $\Delta s_n \to 0$; thus we get a sequence (t_n,u_n,ξ_n) of piecewise-affine approximating evolutions. We prove (Subsection 5.2) its convergence (up to a subsequence) to a *parametrized BV evolution* for $\mathcal{F}_\varepsilon$ according to Definition 3.1.

In Subsection 5.1 (Theorem 5.3) we prove that the functions t_n satisfy a coercivity condition, uniform with respect to n; this guarantees that the discrete evolution is globally defined in the time interval $[0,+\infty)$. Moreover, as a by-product, we get that the polygonal path in $\mathcal{U}$ given by (u_n) has locally-finite length, uniformly bounded with respect to n.

5.1 Discrete (in time) evolution

Fix $\varepsilon > 0$. Let $\Delta s_n \searrow 0$ (we assume $\Delta s_n \leq 1$). Let $u_0 \in \mathcal{U}$ and $\xi_0 \in L^2_+(K)$ be given, with $|[\![u_0]\!]| \leq \xi_0$ a.e. on K. Let

$$t_{n,0} = 0,\ u_{n,0} = u_0,\ \xi_{n,0} = \xi_0,$$

and define $(t_{n,k}, u_{n,k}, \xi_{n,k})$, for every $k \in \mathbb{N}$, by applying the following *recursive rule:*

(r_1) If $\partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k}) = 0$ then

$$\begin{cases} t_{n,k+1} = t_{n,k} + \Delta s_n\ , \\ u_{n,k+1} = u_{n,k}\ , \\ \xi_{n,k+1} = \xi_{n,k}\ . \end{cases}$$

(r_2) If $\partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k}) \neq 0$ then

$$\begin{cases} t_{n,k+1} = t_{n,k}\ , \\ u_{n,k+1} \in \operatorname{argmin}\{\mathcal{F}_\varepsilon(t_{n,k}, v, \xi_{n,k}) :\ v \in \mathcal{U},\ \|v - u_{n,k}\| \leq \Delta s_n\}, \\ \xi_{n,k+1} = \xi_{n,k} \vee |[\![u_{n,k+1}]\!]|. \end{cases}$$

In the recursive rule (r_2) the internal variable is updated a posteriori, i.e. after the minimization of $\mathcal{F}_\varepsilon(t_{n,k}, \cdot, \xi_{n,k})$. In particular it may happen that $[\![u_{n,k+1}]\!] \geq \xi_{n,k}$. This is not an issue, since, a posteriori, the minimization of $\mathcal{F}_\varepsilon(t_{n,k}, \cdot, \xi_{n,k+1})$ provides the same minimizer, as stated in the next proposition.

Lemma 5.1 *If $\partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k}) \neq 0$ then*

$$u_{n,k+1} \in \operatorname{argmin}\{\mathcal{F}_\varepsilon(t_{n,k+1}, v, \xi_{n,k+1}) :\ v \in \mathcal{U},\ \|v - u_{n,k}\| \leq \Delta s_n\}.$$

Proof. Clearly $t_{n,k+1} = t_{n,k}$ by definition; hence $\mathcal{F}_\varepsilon(t_{n,k+1}, \cdot, \xi_{n,k+1}) = \mathcal{F}_\varepsilon(t_{n,k}, \cdot, \xi_{n,k+1})$. Next, we show that for every $v \in \mathcal{U}$ with $\|v - u_{n,k}\| \leq \Delta s_n$

$$\mathcal{F}_\varepsilon(t_{n,k}, v, \xi_{n,k+1}) \geq \mathcal{F}_\varepsilon(t_{n,k}, v, \xi_{n,k}) \geq \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k+1}, \xi_{n,k}) = \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k+1}, \xi_{n,k+1}). \tag{5.1}$$

from which the thesis follows.

The first inequality is a direct consequence of the increasing monotonicity of $\psi_\varepsilon(w, \cdot)$ (see Proposition 4.2 (d)). The second follows by minimality. As to the last equality, it is enough to consider the points on K where $|[\![u_{n,k+1}]\!]| > \xi_{n,k}$; in this case: $\xi_{n,k} < \xi_{n,k+1} = |[\![u_{n,k+1}]\!]|$, which implies that $\psi_\varepsilon(|[\![u_{n,k+1}]\!]|, \xi_{n,k+1}) = \hat{\psi}_\varepsilon(|[\![u_{n,k+1}]\!]|) = \psi_\varepsilon(|[\![u_{n,k+1}]\!]|, \xi_{n,k})$ (recall Proposition 4.2 (b)). Thus, the line integrals in the definition of both sides of the second inequality in (5.1) are the same. ■

At this point we define the map

$$(t_n, u_n, \xi_n)\colon [0, +\infty) \to [0, +\infty) \times H^1(\Omega \setminus K) \times L^2_+(K) \tag{5.2}$$

as a piecewise-affine function taking the values $(t_{n,k}, u_{n,k}, \xi_{n,k})$ at the points $s_{n,k} = k\Delta s_n$.

The following proposition points out that the local minimization appearing in the recursive rule behaves as a normalized gradient flow.

Proposition 5.2 *Assume that $\partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k}) \neq 0$ and $\partial_u \mathcal{F}_\varepsilon(t_{n,k+1}, u_{n,k+1}, \xi_{n,k+1}) \neq 0$. Then $\|u_{n,k+1} - u_{n,k}\| = \Delta s_n$ and there exists $\lambda > 0$ such that*

$$\partial_u \mathcal{F}_\varepsilon(t_{n,k+1}, u_{n,k+1}, \xi_{n,k+1})[v] = \langle \lambda \frac{u_{n,k} - u_{n,k+1}}{\|u_{n,k} - u_{n,k+1}\|}, v \rangle_{H^1(\Omega \setminus K)} \tag{5.3}$$

for every $v \in \mathcal{U}$. In particular $\lambda = \|\partial_u \mathcal{F}_\varepsilon(t_{n,k+1}, u_{n,k+1}, \xi_{n,k+1})\|_{\mathcal{U}'}$ and

$$\partial_u \mathcal{F}_\varepsilon(t_{n,k+1}, u_{n,k+1}, \xi_{n,k+1})[u_{n,k+1} - u_{n,k}] = -\|\partial_u \mathcal{F}_\varepsilon(t_{n,k+1}, u_{n,k+1}, \xi_{n,k+1})\|_{\mathcal{U}'} \|u_{n,k+1} - u_{n,k}\|.$$

Proof. Let $\mathcal{G} = \mathcal{F}_\varepsilon(t_{n,k+1}, \cdot, \xi_{n,k+1})$. Since $\partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k}) \neq 0$, by the previous lemma $u_{n,k+1}$ minimizes $\mathcal{G}$ on the closed ball in $\mathcal{U}$ with centre $u_{n,k}$ and radius Δs_n. Since $\partial\mathcal{G}(u_{n,k+1}) \neq 0$, we have $\|u_{n,k+1} - u_{n,k}\| = \Delta s_n$ (otherwise, the minimality condition would require the vanishing of the derivative). Let $z_0 = u_{n,k} - u_{n,k+1}$. It is easy to check that $\partial\mathcal{G}(u_{n,k+1})$ vanishes on $z_0^\perp$, the orthogonal complement of the span of z_0 in the subspace $\mathcal{U}$ of the Hilbert space $H^1(\Omega \setminus K)$; indeed, fix $\alpha > 0$ and $v \in z_0^\perp$, and let $z = \alpha z_0 + v$. Then $\|(u_{n,k+1} + hz) - u_{n,k}\| < \Delta s_n$ if $0 < h < \delta$, with $\delta > 0$ sufficiently small; thus

$$0 \leq \frac{d}{dh}\mathcal{G}(u_{n,k+1} + hz)_{\big|h=0+} = \partial\mathcal{G}(u_{n,k+1})[z] = \alpha\partial\mathcal{G}(u_{n,k+1})[z_0] + \partial\mathcal{G}(u_{n,k+1})[v].$$

Therefore $\partial\mathcal{G}(u_{n,k+1})[v] = 0$ by the arbitrariness of α and v.

Hence, we can represent $\partial\mathcal{G}(u_{n,k+1})$ through an element of the span of z_0, i.e. (5.3) holds (λ is positive since $u_{n,k+1}$ is a minimum). ■

The following theorem proves a *uniform coercivity condition* for the time parametrization; it implies that the whole time interval $[0, +\infty)$ is parametrized.

Theorem 5.3 *There exist $c_0, c_1 > 0$, independent of n, Δs_n and ε, such that*

$$t_n(S) \geq c_0 S - c_1\,,$$

for every $S \geq 0$ and $n \in \mathbb{N}$.

For the proof we need a technical lemma.

Lemma 5.4 *Let $(t_{n,k}, u_{n,k}, \xi_{n,k})$ be as above. Define*

$$w_{n,k} = [\![u_{n,k}]\!], \quad w_{n,k+1} = [\![u_{n,k+1}]\!].$$

Then

$$\begin{aligned}\Big[\tau_\varepsilon\big(|w_{n,k}|, \xi_{n,k}\big)\operatorname{sgn}(w_{n,k}) - \tau_\varepsilon\big(|w_{n,k+1}|, \xi_{n,k+1}\big)\operatorname{sgn}(w_{n,k+1})\Big]\,(w_{n,k+1} - w_{n,k}) \leq \\ \leq \big|\hat{\tau}(\xi_{n,k+1}) - \hat{\tau}(\xi_{n,k})\big||w_{n,k+1} - w_{n,k}|\end{aligned} \tag{5.4}$$

a.e. on K.

Proof. If $w_{n,k+1} > 0 > w_{n,k}$ or $w_{n,k+1} < 0 < w_{n,k}$, then the left-hand side in (5.4) is non-positive, and the inequality holds. Therefore, we assume that $w_{n,k}$ and $w_{n,k+1}$ have the same sign. Let $w_{n,k}, w_{n,k+1} \geq 0$.

If $\xi_{n,k+1} = \xi_{n,k}$ then the left-hand side of (5.4) is non-positive by the monotonicity of $\tau_\varepsilon(\cdot, \xi_{n,k})$ on $[0, \xi_{n,k}]$. If $\xi_{n,k+1} > \xi_{n,k}$ then $w_{n,k+1} = \xi_{n,k+1} > \xi_{n,k} \geq w_{n,k}$; thus $w_{n,k+1} > w_{n,k}$ and

$$\tau_\varepsilon(w_{n,k}, \xi_{n,k}) - \tau_\varepsilon(w_{n,k+1}, \xi_{n,k+1}) \leq \tau_\varepsilon(\xi_{n,k}, \xi_{n,k}) - \tau_\varepsilon(\xi_{n,k+1}, \xi_{n,k+1}).$$

Now we have to consider two subcases. If $\xi_{n,k+1} \leq \xi_\varepsilon$ (see (4.2)) then $\tau_\varepsilon(\cdot, \xi_{n,k})$ and $\tau_\varepsilon(\cdot, \xi_{n,k+1})$ are the same linear function with slope $1/\varepsilon$ on the interval $[0, \xi_\varepsilon]$; therefore we have $\tau_\varepsilon(\xi_{n,k}, \xi_{n,k}) - \tau_\varepsilon(\xi_{n,k+1}, \xi_{n,k+1}) \leq 0$, and (5.4) holds. If, on the contrary, $\xi_{n,k+1} > \xi_\varepsilon$, then $\tau_\varepsilon(\xi_{n,k+1}, \xi_{n,k+1}) = \hat{\tau}(\xi_{n,k+1})$; thus

$$\tau_\varepsilon(\xi_{n,k}, \xi_{n,k}) - \tau_\varepsilon(\xi_{n,k+1}, \xi_{n,k+1}) \leq \hat{\tau}(\xi_{n,k}) - \hat{\tau}(\xi_{n,k+1})\,,$$

and (5.4) follows again.

The proof in the case $w_{n,k}, w_{n,k+1} \leq 0$ is analogous. ■

Proof of Theorem 5.3. We need to consider separately the cases (r_1) and (r_2) in the recursive rule. In particular, for the second, we will provide first an estimate for a pair of consecutive indices and then an estimate for a "maximal" interval of indices where (r_2) holds.

First step. For every $k \in \mathbb{N}$ let $\gamma_k = \|\partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k}))\|_{\mathcal{U}'}$. Assume that $\gamma_k \neq 0$ and $\gamma_{k+1} \neq 0$. Then, by Proposition 5.2:

$$\partial_u \mathcal{F}_\varepsilon(t_{n,k+1}, u_{n,k+1}, \xi_{n,k+1})[u_{n,k+1} - u_{n,k}] = -\gamma_{k+1}\|u_{n,k+1} - u_{n,k}\|.$$

Since $\partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k})[u_{n,k+1} - u_{n,k}] \geq -\gamma_k \|u_{n,k+1} - u_{n,k}\|$, we deduce that

$$\partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k})[u_{n,k+1} - u_{n,k}] - \partial_u \mathcal{F}_\varepsilon(t_{n,k+1}, u_{n,k+1}, \xi_{n,k+1})[u_{n,k+1} - u_{n,k}] \geq \\ \geq (\gamma_{k+1} - \gamma_k)\|u_{n,k+1} - u_{n,k}\|.$$

Let us estimate the left-hand side; since $t_{n,k+1} = t_{n,k}$, this term reads as

$$\int_{\Omega \setminus K} \nabla(u_{n,k} - u_{n,k+1}) \nabla(u_{n,k+1} - u_{n,k})\, dx + \\ + \int_K \Big[\tau_\varepsilon\big(|[\![u_{n,k}]\!]|, \xi_{n,k}\big) \operatorname{sgn}([\![u_{n,k}]\!]) + \\ - \tau_\varepsilon\big(|[\![u_{n,k+1}]\!]|, \xi_{n,k+1}\big) \operatorname{sgn}([\![u_{n,k+1}]\!]) \Big] [\![u_{n,k+1} - u_{n,k}]\!]\, d\mathcal{H}^1 .$$

By Lemma 5.4 a bound from above is given by

$$-\|\nabla(u_{n,k+1} - u_{n,k})\|^2_{L^2(\Omega \setminus K)} + \int_K \big|\hat\tau(\xi_{n,k+1}) - \hat\tau(\xi_{n,k})\big|\, |[\![u_{n,k+1} - u_{n,k}]\!]|\, dx.$$

Therefore

$$(\gamma_{k+1} - \gamma_k)\|u_{n,k+1} - u_{n,k}\| \leq -\|\nabla(u_{n,k+1} - u_{n,k})\|^2_{L^2(\Omega \setminus K)} + \\ + \|\hat\tau(\xi_{n,k+1}) - \hat\tau(\xi_{n,k})\|_{L^2(K)}\, \|[\![u_{n,k+1} - u_{n,k}]\!]\|_{L^2(K)}.$$

Let $c, C > 0$ be such that $\|\nabla u\|^2_{L^2} \geq c\|u\|^2$ and $\|[\![u]\!]\|_{L^2(K)} \leq C\|u\|$ for every $u \in \mathcal{U}$ (recall Remark 3.2). Then

$$(\gamma_{k+1} - \gamma_k)\|u_{n,k+1} - u_{n,k}\| \leq -c\|u_{n,k+1} - u_{n,k}\|^2 + \\ + C\|\hat\tau(\xi_{n,k+1}) - \hat\tau(\xi_{n,k})\|_{L^2(K)}\, \|u_{n,k+1} - u_{n,k}\|.$$

Since $\gamma_{n,k} \neq 0$ and $\gamma_{n,k+1} \neq 0$ then $\|u_{n,k+1} - u_{n,k}\| \neq 0$ (by Proposition 5.2) and thus

$$\gamma_{k+1} - \gamma_k \leq -c\|u_{n,k+1} - u_{n,k}\| + C\|\hat\tau(\xi_{n,k+1}) - \hat\tau(\xi_{n,k})\|_{L^2(K)} ,$$

i.e.

$$c\|u_{n,k+1} - u_{n,k}\| \leq \gamma_k - \gamma_{k+1} + C\|\hat\tau(\xi_{n,k+1}) - \hat\tau(\xi_{n,k})\|_{L^2(K)} . \tag{5.5}$$

In order to get a telescopic sum, we need to replace the L^2 norm by an L^1 term. By interpolation inequality

$$\|\hat\tau(\xi_{n,k+1}) - \hat\tau(\xi_{n,k})\|_{L^2(K)} \leq \|\hat\tau(\xi_{n,k+1}) - \hat\tau(\xi_{n,k})\|^\alpha_{L^1(K)} \|\hat\tau(\xi_{n,k+1}) - \hat\tau(\xi_{n,k})\|^{1-\alpha}_{L^q(K)} ,$$

where α and q satisfy $2 < q$ and $1/2 = \alpha + (1-\alpha)/q$. Apply now Young's inequality to the right-hand side: for every $\delta > 0$ there exists a constant C_δ such that

$$\|\hat\tau(\xi_{n,k+1}) - \hat\tau(\xi_{n,k})\|_{L^2(K)} \leq C_\delta \|\hat\tau(\xi_{n,k+1}) - \hat\tau(\xi_{n,k})\|_{L^1(K)} + \delta\|\hat\tau(\xi_{n,k+1}) - \hat\tau(\xi_{n,k})\|_{L^q(K)} .$$

If L denotes a Lipschitz constant for $\hat\tau$, then

$$\|\hat\tau(\xi_{n,k+1}) - \hat\tau(\xi_{n,k})\|_{L^2(K)} \leq C_\delta \|\hat\tau(\xi_{n,k+1}) - \hat\tau(\xi_{n,k})\|_{L^1(K)} + \delta L\|\xi_{n,k+1} - \xi_{n,k}\|_{L^q(K)} . \tag{5.6}$$

Note that

$$|\xi_{n,k+1} - \xi_{n,k}| \leq \big||[\![u_{n,k+1}]\!]| - |[\![u_{n,k}]\!]|\big| \leq \big|[\![u_{n,k+1}]\!] - [\![u_{n,k}]\!]\big| = |[\![u_{n,k+1} - u_{n,k}]\!]|,$$

so that

$$\|\xi_{n,k+1} - \xi_{n,k}\|_{L^q(K)} \leq C'\|u_{n,k+1} - u_{n,k}\|$$

for a suitable constant C'. From (5.5) and (5.6) we can choose δ sufficiently small in such a way that

$$c\|u_{n,k+1} - u_{n,k}\| \leq \gamma_k - \gamma_{k+1} + C\|\hat\tau(\xi_{n,k+1}) - \hat\tau(\xi_{n,k})\|_{L^1(K)} \tag{5.7}$$

(possibly with a new value for c and C). Note that, by monotonicity of the sequence $(\xi_{n,k})_k$ and of the function $\hat{\tau}$, we have

$$\|\hat{\tau}(\xi_{n,k+1}) - \hat{\tau}(\xi_{n,k})\|_{L^1(K)} = \int_K \big(\hat{\tau}(\xi_{n,k}) - \hat{\tau}(\xi_{n,k+1})\big)\, dx.$$

Second step. Given $k_2 \in \mathbb{N}$ with $k_2 > 0$ and $\partial_u \mathcal{F}_\varepsilon(t_{n,k_2}, u_{n,k_2}, \xi_{n,k_2}) \neq 0$ let us denote

$$k_1 = \min\{0 \le k \le k_2 : \partial_u \mathcal{F}_\varepsilon(t_{n,m}, u_{n,m}, \xi_{n,m}) \neq 0 \text{ for every } k \le m \le k_2\}.$$

Note that the interval of indices $[k_1, k_2]$ is "maximal on the left-side" and that

$$\text{either} \quad k_1 = 0 \quad \text{or} \quad \partial_u \mathcal{F}_\varepsilon(t_{n,k_1-1}, u_{n,k_1-1}, \xi_{n,k_1-1}) = 0.$$

Consider the case $k_1 < k_2$. We will prove that there exists $C_0 > 0$, independent of n, Δs_n and ε, such that,

$$(k_2 - k_1)(\Delta s_n) \le \begin{cases} C_0\Big(\Delta s_n + \displaystyle\int_K \big(\hat{\tau}(\xi_{n,k_1}) - \hat{\tau}(\xi_{n,k_2})\big)\, d\mathcal{H}^1 + 1\Big) & \text{if } k_1 = 0, \\[2ex] C_0\Big(\Delta s_n + \displaystyle\int_K \big(\hat{\tau}(\xi_{n,k_1}) - \hat{\tau}(\xi_{n,k_2})\big)\, d\mathcal{H}^1\Big) & \text{otherwise.} \end{cases} \tag{5.8}$$

By (5.7)

$$\begin{aligned} c \sum_{k=k_1}^{k_2-1} \|u_{n,k+1} - u_{n,k}\| &\le \gamma_{k_1} - \gamma_{k_2} + C \int_K \big(\hat{\tau}(\xi_{n,k_1}) - \hat{\tau}(\xi_{n,k_2})\big)\, d\mathcal{H}^1 \\ &\le \gamma_{k_1} + C \int_K \big(\hat{\tau}(\xi_{n,k_1}) - \hat{\tau}(\xi_{n,k_2})\big)\, d\mathcal{H}^1. \end{aligned}$$

Consider now the case $k_1 = 0$; then γ_{k_1} is bounded by a constant depending only on u_0 and ξ_0 (recall the form of $\partial_u \mathcal{F}_\varepsilon$ given in Lemma 4.4). Otherwise, by assumption, the index $k_0 := k_1 - 1$ satisfies $\partial_u \mathcal{F}_\varepsilon(t_{n,k_0}, u_{n,k_0}, \xi_{n,k_0}) = 0$ and thus $t_{n,k_1} = t_{n,k_0} + \Delta s_n$, $u_{n,k_1} = u_{n,k_0}$ and $\xi_{n,k_1} = \xi_{n,k_0}$ by (r_1). Therefore, by the Lipschitz continuity of $\partial_u \mathcal{E}$ (hence of $\partial_u \mathcal{F}_\varepsilon$) with respect to t, we have

$$\gamma_{k_1} = \|\partial_u \mathcal{F}_\varepsilon(t_{n,k_0} + \Delta s_n, u_{n,k_0}, \xi_{n,k_0}) - \partial_u \mathcal{F}_\varepsilon(t_{n,k_0}, u_{n,k_0}, \xi_{n,k_0})\| \le C'' \Delta s_n$$

for a suitable constant C'' depending on $\|g\|_{C^1([0,+\infty);H^1)}$ This concludes the proof of (5.8), since, as remarked in Proposition 5.2, it turns out that $\|u_{n,k+1} - u_{n,k}\| = \Delta s_n$ for every $k_1 \le k < k_2$.

Third step. Let now $S > 0$ be fixed, and denote by $N_n(S) = \lfloor S/(\Delta s_n) \rfloor$ the integer part of $S/(\Delta s_n)$. Following the recursive rule, we set

$$\begin{aligned} A_n(S) &= \{k \in [1, N_n(S)] : \ \partial_u \mathcal{F}(t_{n,k}, u_{n,k}, \xi_{n,k}) = 0\}, \\ Z_n(S) &= \{k \in [1, N_n(S)] : \ \partial_u \mathcal{F}(t_{n,k}, u_{n,k}, \xi_{n,k}) \neq 0\}. \end{aligned}$$

For technical reasons it is useful to distinguish between isolated points and interval of indices in $Z_n(S)$. Therefore we further split $Z_n(S)$ into the two subsets

$$\begin{aligned} &Z_n^0(S) = \{k \in Z_n(S) : \ \partial_u \mathcal{F}(t_{n,k-1}, u_{n,k-1}, \xi_{n,k-1}) = 0 \text{ and } \partial_u \mathcal{F}(t_{n,k+1}, u_{n,k+1}, \xi_{n,k+1}) = 0\}, \\ &Z_n^1(S) = Z_n(S) \setminus Z_n^0(S). \end{aligned}$$

Let $I^i = [k_1^i, k_2^i]$ with $k_1^i < k_2^i$ $(i = 1, \dots, l_n)$ denote the maximal intervals of indices in $Z_n^1(S)$. By the recursive rule (r_2) we have $\#A_n(S) \le (t_n(S)/\Delta s_n) + 1$, moreover

$$\#Z_n^0(S) \le (N_n(S) + 1)/2, \qquad \#Z_n^1(S) \le \sum_{i=1}^{l_n} \big((k_2^i - k_1^i) + 1\big), \qquad l_n \le \#A_n(S) + 1.$$

Note that for every $I^i = [k_1^i, k_2^i]$ $(i = 1, \dots, l_n)$ we have

$$k_1^i = \min\{0 \le k \le k_2^i : \partial_u \mathcal{F}_\varepsilon(t_{n,m}, u_{n,m}, \xi_{n,m}) \neq 0 \text{ for every } k \le m \le k_2^i\}.$$

Thus we can apply (5.8) to each interval I^i. At most one interval has $k_1^i = 0$ and thus

$$\sum_{i=1}^{l_n}(k_2^i - k_1^i)\Delta s_n \le C_0\Big(l_n\Delta s_n + 1 + \int_K \sum_{i=1}^{l_n}\Big(\hat\tau(\xi_{n,k_1^i}) - \hat\tau(\xi_{n,k_2^i})\Big)\,d\mathcal{H}^1\Big).$$

Since $\hat\tau$ is monotone decreasing we deduce that

$$\sum_{i=1}^{l_n}(k_2^i - k_1^i)\Delta s_n \le C_0\Big(l_n\Delta s_n + 1 + \int_K \hat\tau(\xi_{n,k_1^1})\,d\mathcal{H}^1\Big) \le C_0\big(l_n\Delta s_n + 1 + \hat\tau(0)\mathcal{H}^1(K)\big).$$

Up to a suitable change in the definition of C_0,

$$\sum_{i=1}^{l_n}(k_2^i - k_1^i)\Delta s_n \le C_0(l_n\Delta s_n + 1).$$

It follows that

$$\#Z_n^1(S)\Delta s_n \le l_n\Delta s_n + \sum_{i=1}^{l_n}(k_2^i - k_1^i)\Delta s_n \le C_0(l_n\Delta s_n + 1) \le C_0'(\#A_n(S)\Delta s_n + 1).$$

Since

$$N_n(S) \le \#A_n(S) + \#Z_n^0(S) + \#Z_n^1(S) \le \#A_n(S) + \tfrac{1}{2}N_n(S) + 1 + \#Z_n^1(S)$$

we get

$$\tfrac{1}{2}N_n(S)\Delta s_n \le C_1(\#A_n(S)\Delta s_n + 1) \le C_1(t_n(S) + 2)$$

for a suitable $C_1 > 0$. We conclude since $N_n(S)\Delta s_n \ge (S-1)$. ■

Corollary 5.5 *Let $T > 0$ and $k_n(T) = \min\{k : \ t_{n,k} \ge T\}$ (note that $k_n(T)$ is finite by Theorem 5.3). Then*

$$\sum_{k=0}^{k_n(T)-1} \|u_{n,k+1} - u_{n,k}\| \le (T + c_1)/c_0,$$

where c_0 and c_1 are as in Theorem 5.3. Hence, the length of the polygonal path $(u_{n,k})_{0\le k\le k_n(T)}$ in $\mathcal{U}$ is bounded independently of n and $\varepsilon > 0$.

Proof. By Theorem 5.3, $c_0\big(k_n(T)\Delta s_n\big) - c_1 \le T$; since $\|u_{n,k+1} - u_{n,k}\| \le \Delta s_n$ for every k, we deduce that

$$\sum_{k=0}^{k_n(T)-1} \|u_{n,k+1} - u_{n,k}\| \le k_n(T)\Delta s_n \le (T + c_1)/c_0\,.$$

■

The following energy estimate for the discrete evolution $(t_{n,k}, u_{n,k}, \xi_{n,k})_k$ will be used in the next subsection to prove the energy balance for the limit evolution.

Proposition 5.6 *Let $T > 0$ be fixed. For every $k \in \mathbb{N}$ with $t_{n,k+1} < T$ we have*

$$\begin{aligned}\mathcal{F}_\varepsilon(t_{n,k+1}, u_{n,k+1}, \xi_{n,k+1}) \le \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k}) + \int_{t_{n,k}}^{t_{n,k+1}} \partial_t\mathcal{F}_\varepsilon(t, u_{n,k}, \xi_{n,k})\,dt\, + \\ - \|\partial_u\mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k})\|_{\mathcal{U}'}\Delta s_n + C_\varepsilon(\Delta s_n)^2,\end{aligned} \tag{5.9}$$

where C_ε depends on ε and $\|g\|_{C^1([0,+\infty);H^1(\Omega))}$.

Proof. Let $w_{n,k} = |[\![u_{n,k}]\!]|$ and $w_{n,k+1} = |[\![u_{n,k+1}]\!]|$. First of all, note that $\psi_\varepsilon(w_{n,k+1}, \xi_{n,k+1}) = \psi_\varepsilon(w_{n,k+1}, \xi_{n,k})$. Clearly, the equality has to be checked only if k falls within recursive rule (r_2) and $\xi_{n,k} < \xi_{n,k+1}$; in this case, $\xi_{n,k} < w_{n,k+1} = \xi_{n,k+1}$ and, by Proposition 4.2 (b),

$$\psi_\varepsilon(w_{n,k+1}, \xi_{n,k}) = \hat\psi_\varepsilon(w_{n,k+1}) = \psi_\varepsilon(w_{n,k+1}, \xi_{n,k+1}).$$

Therefore

$$\mathcal{F}_\varepsilon(t_{n,k+1}, u_{n,k+1}, \xi_{n,k+1}) = \mathcal{F}_\varepsilon(t_{n,k+1}, u_{n,k+1}, \xi_{n,k}). \tag{5.10}$$

Second,

$$\mathcal{F}_\varepsilon(t_{n,k+1}, u_{n,k+1}, \xi_{n,k}) = \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k+1}, \xi_{n,k}) + \int_{t_{n,k}}^{t_{n,k+1}} \partial_t \mathcal{F}_\varepsilon(t, u_{n,k+1}, \xi_{n,k})\, dt.$$

By Lemma 4.4 and Lemma 3.5

$$\big|\partial_t \mathcal{F}_\varepsilon(t, u_{n,k+1}, \xi_{n,k}) - \partial_t \mathcal{F}_\varepsilon(t, u_{n,k}, \xi_{n,k})\big| \le C \|u_{n,k+1} - u_{n,k}\|_{H^1(\Omega\setminus K)} \le C\Delta s_n\,,$$

where $C = \|g\|_{C^1([0,+\infty);H^1(\Omega))}$. Then

$$\mathcal{F}_\varepsilon(t_{n,k+1}, u_{n,k+1}, \xi_{n,k}) \le \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k+1}, \xi_{n,k}) + \int_{t_{n,k}}^{t_{n,k+1}} \partial_t \mathcal{F}_\varepsilon(t, u_{n,k}, \xi_{n,k})\, dt + C_T(\Delta s_n)^2\,. \tag{5.11}$$

Third, it is not restrictive to assume that $u_{n,k+1} \neq u_{n,k}$ (otherwise $\|\partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k})\|_{\mathcal{U}'} = 0$ and there is nothing else to prove). Let $z \in \mathcal{U}$ with $\|z\| \le 1$, by the minimality property of $u_{n,k+1}$ we have

$$\mathcal{F}_\varepsilon(t_{n,k}, u_{n,k+1}, \xi_{n,k}) \le \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k} + \Delta s_n z, \xi_{n,k}). \tag{5.12}$$

Moreover

$$\begin{aligned}\mathcal{F}_\varepsilon(t_{n,k}, u_{n,k} + \Delta s_n z, \xi_{n,k}) &= \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k}) + \int_0^{\Delta s_n} \frac{d}{dh}\mathcal{F}_\varepsilon(t_{n,k}, u_{n,k} + hz, \xi_{n,k})\, dh \\ &= \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k}) + \int_0^{\Delta s_n} \partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k} + hz, \xi_{n,k})[z]\, dh.\end{aligned}$$

By Lemma 4.4, for every $h \in [0, \Delta s_n]$

$$\|\partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k} + hz, \xi_{n,k}) - \partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k})\|_{\mathcal{U}'} \le C_\varepsilon \|hz\|_{H^1} \le C_\varepsilon \Delta s_n\,,$$

for a suitable constant C_ε depending on ε. Therefore

$$\mathcal{F}_\varepsilon(t_{n,k}, u_{n,k} + \Delta s_n z, \xi_{n,k}) \le \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k}) + \Delta s_n \partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k})[z] + C_\varepsilon(\Delta s_n)^2;$$

by (5.12) we get

$$\mathcal{F}_\varepsilon(t_{n,k}, u_{n,k+1}, \xi_{n,k}) \le \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k}) + \Delta s_n \partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k})[z] + C_\varepsilon(\Delta s_n)^2.$$

By the arbitrariness of z we conclude that

$$\mathcal{F}_\varepsilon(t_{n,k}, u_{n,k+1}, \xi_{n,k}) \le \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k}) - \Delta s_n \|\partial_u \mathcal{F}_\varepsilon(t_{n,k}, u_{n,k}, \xi_{n,k})\|_{\mathcal{U}'} + C_\varepsilon(\Delta s_n)^2.$$

This, together with (5.10) and (5.11), gives the stated inequality. ■

5.2 Quasi-static evolution for the regularized energy

Let (t_n, u_n, ξ_n) be the map defined in (5.2). Let $S > 0$ be fixed. From the definition it is easy to see that:

$$(t_n) \text{ is bounded in } W^{1,\infty}(0,S), \qquad (u_n) \text{ is bounded in } W^{1,\infty}(0,S; H^1(\Omega\setminus K)).$$

Moreover, these sequences are bounded independently of ε. Remember that

$$\|\xi_{n,k+1} - \xi_{n,k}\|_{L^p(K)} \le C\|u_{n,k+1} - u_{n,k}\| \le C\Delta s_n \tag{5.13}$$

for every $1 \le p < +\infty$ and for a suitable constant $C > 0$, independent of n, k and ε. We conclude that (ξ_n) is bounded in $W^{1,\infty}(0,S; L^p(K))$ for every $1 \le p < +\infty$.

By recalling Proposition 2.3, and by applying a standard diagonal argument, we deduce the following result.

Proposition 5.7 *Let $(t_n), (u_n)$ and (ξ_n) be defined as above. Then, up to a subsequence (not relabeled) $t_n \overset{*}{\rightharpoonup} t$ in $W^{1,\infty}(0,S)$, $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(0,S;H^1(\Omega\setminus K))$, $\xi_n \overset{*}{\rightharpoonup} \xi$ in $W^{1,\infty}(0,S;L^p(K))$ for $1<p<+\infty$ and for any finite interval $(0,S)$.*

Theorem 5.8 *Let (t_n,u_n,ξ_n) and (t,u,ξ) be as in Proposition 5.7. Then (t,u,ξ) is a (parametrized) BV evolution for the energy $\mathcal{F}_\varepsilon$ according to Definition 4.1.*

Proof. By Theorem 5.3 it turns out that $t(s)\to+\infty$ as $s\to+\infty$. The sequences (t_n), (u_n) and ξ_n are uniformly Lipschitz continuous in $(0,+\infty)$, by the recursive rule and by (5.13), hence their limits are Lipschitz continuous as well. Let $S>0$ be fixed.

Proof of (C) in Definition 3.1: for almost every $s\in[0,S]$

$$\xi'(s)\geq 0,\qquad |[\![u(s)]\!]|\leq\xi(s),\qquad \xi'(s)\big(|[\![u(s)]\!]|-\xi(s)\big)=0\quad \mathcal{H}^1\text{-a.e. on } K.$$

By definition $\xi_{n,k+1}\geq\xi_{n,k}$ pointwise on K for every $k\in\mathbb{N}$; then $\xi_n(s_2)-\xi_n(s_1)\geq 0$ pointwise on K if $0\leq s_1\leq s_2\leq S$. Passing to the limit (with respect to the weak convergence in $L^2(K)$) we get $\xi(s_2)-\xi(s_1)\geq 0$ and thus

$$\xi'(s)\geq 0\quad \mathcal{H}^1\text{-a.e. on } K \text{ for a.e. } s\in[0,S].$$

Let $s=\lambda s_{n,k}+(1-\lambda)s_{n,k+1}$, for some $k\in\mathbb{N}$ and $\lambda\in[0,1]$. Then $u_n(s)=\lambda u_{n,k}+(1-\lambda)u_{n,k+1}$ and $\xi_n(s)=\lambda\xi_{n,k}+(1-\lambda)\xi_{n,k+1}$. By linearity of the trace operator

$$|[\![u_n(s)]\!]|\leq\lambda|[\![u_{n,k}]\!]|+(1-\lambda)|[\![u_{n,k+1}]\!]|.$$

Since $|[\![u_{n,k}]\!]|\leq\xi_{n,k}$ and $|[\![u_{n,k+1}]\!]|\leq\xi_{n,k+1}$ we deduce that

$$|[\![u_n(s)]\!]|\leq\xi_n(s)\quad \mathcal{H}^1\text{-a.e. on } K \text{ for every } s\in[0,S].$$

Since $u_n(s)\rightharpoonup u(s)$ in $H^1(\Omega\setminus K)$, by Remark 3.2 (c) we have that $|[\![u_n(s)]\!]|\to|[\![u(s)]\!]|$ in $L^2(K)$ for every $s\in[0,S]$. Then, the $w-L^2(K)$ convergence of $\big(\xi_n(s)\big)$ implies that:

$$|[\![u(s)]\!]|\leq\xi(s)\quad \mathcal{H}^1\text{-a.e. on } K \text{ for every } s\in[0,S].$$

Let us now address the equation

$$\xi'(s)\big([\![u(s)]\!]|-\xi(s)\big)=0\quad \mathcal{H}^1\text{-a.e. on } K \text{ for a.e. } s\in[0,S],$$

which is equivalent to

$$\int_0^\sigma ds\int_B\xi'(s)\big(|[\![u(s)]\!]|-\xi(s)\big)d\mathcal{H}^1=0\quad\text{for every }\sigma\in[0,S]\text{ and for every Borel subset } B \text{ of } K.$$

From the definition, we deduce that (pointwise on K) either $\xi_{n,k+1}-\xi_{n,k}=0$ or $\xi_{n,k+1}-|[\![u_{n,k+1}]\!]|=0$, i.e.

$$(\xi_{n,k+1}-\xi_{n,k})(|[\![u_{n,k+1}]\!]|-\xi_{n,k+1})=0\quad\text{pointwise on } K.$$

Then, for a.e. $s\in[0,S]$

$$\xi_n'(s)\big(|[\![u_{n,k+1}]\!]|-\xi_{n,k+1}\big)=0, \tag{5.14}$$

where $k=k(n,s)$ satisfies $s_{n,k}<s<s_{n,k+1}$.

Fix σ and B as above. Since $[\![u_n]\!]\to[\![u]\!]$ in $L^1(0,S;L^2(K))$ and since $\xi_n\overset{*}{\rightharpoonup}\xi$ in $L^\infty(0,S;L^2(K))$, we have

$$\lim_{n\to+\infty}\int_0^\sigma ds\int_B\xi_n'(s)|[\![u_n(s)]\!]|d\mathcal{H}^1=\int_0^\sigma ds\int_B\xi'(s)|[\![u(s)]\!]|d\mathcal{H}^1\,. \tag{5.15}$$

Note now that, if $k=k(n,s)$ is as in (5.14), and $u_n(s)=\lambda u_{n,k}+(1-\lambda)u_{n,k+1}$, then we have

$$\|[\![u_{n,k+1}]\!]-[\![u_n(s)]\!]\|_{L^2(K)}=\lambda\|[\![u_{n,k+1}-u_{n,k}]\!]\|_{L^2(K)}\leq C\|u_{n,k+1}-u_{n,k}\|\leq C\Delta s_n\,.$$

Therefore

$$\lim_{n\to+\infty}\int_0^\sigma ds\int_B \xi_n'(s)|[\![u_{n,k(n,s)+1}]\!]|d\mathcal{H}^1=\int_0^\sigma ds\int_B \xi'(s)|[\![u(s)]\!]|d\mathcal{H}^1\,. \tag{5.16}$$

Let us now consider the term $\xi_n'(s)\xi_{n,k+1}$ in (5.14). By monotonicity of ξ_n and by (5.14) we have

$$\int_0^\sigma ds\int_B \xi_n'(s)\xi_n(s)d\mathcal{H}^1\le\int_0^\sigma ds\int_B \xi_n'(s)\xi_{n,k+1}d\mathcal{H}^1=\int_0^\sigma ds\int_B \xi_n'(s)|[\![u_{n,k+1}]\!]|d\mathcal{H}^1$$

so that by (5.16)

$$\limsup_{n\to+\infty}\int_0^\sigma ds\int_B \xi_n'(s)\xi_n(s)d\mathcal{H}^1\le\int_0^\sigma ds\int_B \xi'(s)|[\![u(s)]\!]|d\mathcal{H}^1\,. \tag{5.17}$$

Since $\xi_n^2\in W^{1,\infty}(0,S;L^r(K))$ for some $r>1$ and $(\xi_n^2)'=2\xi_n'\xi_n$ (see Remark 5.9 below) we can apply the fundamental theorem of calculus, see §2, to write

$$\int_0^\sigma ds\int_B \xi_n'(s)\xi_n(s)d\mathcal{H}^1=\tfrac12\int_B\big(\xi_n^2(\sigma)-\xi^2(0)\big)\,d\mathcal{H}^1\,. \tag{5.18}$$

By the weak $L^2(K)$-convergence of $(\xi_n(\sigma))$ we deduce that

$$\liminf_{n\to+\infty}\int_0^\sigma ds\int_B \xi_n'(s)\xi_n(s)d\mathcal{H}^1\ge\tfrac12\int_B\big(\xi(\sigma)^2-\xi(0)^2\big)d\mathcal{H}^1=\int_0^\sigma ds\int_B \xi'(s)\xi(s)d\mathcal{H}^1\,, \tag{5.19}$$

where the last equality follows again by Remark 5.9. Recalling (5.17) we get

$$\begin{aligned}\int_0^\sigma ds\int_B \xi'(s)\xi(s)d\mathcal{H}^1 &\le \liminf_{n\to+\infty}\int_0^\sigma ds\int_B \xi_n'(s)\xi_n(s)d\mathcal{H}^1\\ &\le \limsup_{n\to+\infty}\int_0^\sigma ds\int_B \xi_n'(s)\xi_n(s)d\mathcal{H}^1\\ &\le \int_0^\sigma ds\int_B \xi'(s)|[\![u(s)]\!]|d\mathcal{H}^1\le\int_0^\sigma ds\int_B \xi'(s)\xi(s)d\mathcal{H}^1,\end{aligned}$$

where the last inequality follows from $\xi'(s)\ge0$ and $|[\![u(s)]\!]|-\xi(s)\le0$.

In addition, we have proved that

$$\lim_{n\to+\infty}\int_0^\sigma ds\int_B \xi_n'(s)\xi_n(s)d\mathcal{H}^1=\int_0^\sigma ds\int_B \xi'(s)\xi(s)d\mathcal{H}^1=\tfrac12\int_B\big(\xi(\sigma)^2-\xi(0)^2\big)d\mathcal{H}^1.$$

This allows to get an improvement of the convergence properties of the sequence (ξ_n). Indeed, the limit in the left-hand side equals $\lim_{n\to+\infty}\frac12\int_B\big(\xi_n(\sigma)^2-\xi(0)^2\big)d\mathcal{H}^1$ by (5.18); thus

$$\lim_{n\to+\infty}\int_K \xi_n(\sigma)^2d\mathcal{H}^1=\int_K\xi(\sigma)^2d\mathcal{H}^1\,.$$

Since $\xi_n(\sigma)\rightharpoonup\xi(\sigma)$ weakly in $L^2(K)$, we deduce that

$$\xi_n(\sigma)\to\xi(\sigma)\quad\text{strongly in }L^2(K)\text{ for every }\sigma\in[0,S].$$

By the uniform Lipschitz continuity of ξ_n it is easy to check that for $\sigma_n\to\sigma$

$$\xi_n(\sigma_n)\to\xi(\sigma)\quad\text{strongly in }L^2(K). \tag{5.20}$$

Proof of (S): for every $s\in[0,S]$ with $t'(s)>0$ we have

$$\|\partial_u\mathcal{F}_\varepsilon(t(s),u(s),\xi(s))\|_{\mathcal{U}'}=0\,.$$

Let $s\in(0,S)$ be such that $t'(s)>0$. Let $\delta>0$ be fixed; we note that there exists $\overline{n}\in\mathbb{N}$ such that for every $n\ge\overline{n}$ we can find $k\in\mathbb{N}$ with the property that

$$|s_{n,k}-s|<\delta\quad\text{and}\quad t_{n,k}<t_{n,k+1}\,.$$

Indeed, assume, by contradiction, that there exists an increasing sequence (n_j) of integers such that for every k satisfying $|s_{n,k}-s| < \delta$ we have $t_{n,k} = t_{n,k+1}$. Then $t_{n_j}(\cdot)$ is constant in a neighbourhood of s, thus $t'(s) = 0$.

The arbitrariness of δ implies that there exists a sequence $s_{n,k_n} \to s$ such that $t_{n,k_n} < t_{n,k_n+1}$. By Proposition 5.7 we know that

$$t_{n,k_n} = t_n(s_{n,k_n}) \to t(s), \qquad u_{n,k_n} = u_n(s_{n,k_n}) \rightharpoonup u(s) \quad w\text{-}H^1(\Omega \setminus K)$$

while, by (5.20) and the equi-boundedness of the Lipschitz constants of $(\xi_n)_n$ (see (5.13)):

$$\xi_{n,k_n} = \xi_n(s_{n,k_n}) \to \xi(s) \qquad \text{strongly in } L^2(K).$$

These convergences allow to apply Lemma 4.5 and get

$$\|\partial_u \mathcal{F}_\varepsilon(t(s), u(s), \xi(s))\|_{\mathcal{U}'} \leq \liminf_{n\to+\infty} \|\partial_u \mathcal{F}_\varepsilon(t_{n,k_n}, u_{n,k_n}, \xi_{n,k_n})\|_{\mathcal{U}'} \,.$$

Now, we conclude since $\partial_u \mathcal{F}_\varepsilon(t_{n,k_n}, u_{n,k_n}, \xi_{n,k_n}) = 0$: this is a direct consequence of the recursive rule, otherwise $t_{n,k_n} = t_{n,k_n+1}$.

Proof of (E): for every $s \in [0, S]$ we have

$$\begin{aligned}\mathcal{F}_\varepsilon(t(s), u(s), \xi(s)) = \mathcal{F}_\varepsilon(t_0, w_0, \xi_0) &+ \int_0^s \partial_t \mathcal{F}_\varepsilon(t(r), u(r), \xi(r))\, t'(r)\, dr + \\ &- \int_0^s \|\partial_u \mathcal{F}_\varepsilon(t(r), u(r), \xi(r))\|_{\mathcal{U}'}\, dr \,.\end{aligned}$$

It is useful to introduce the function $(\hat{t}_n, \hat{u}_n, \hat{\xi}_n)$ as the right-continuous piecewise-constant function on $[0, +\infty)$ taking the value $(t_{n,k}, u_{n,k}, \xi_{n,k})$ on $[s_{n,k}, s_{n,k+1})$. In particular, the integral on the right-hand side in (5.9) can be written as:

$$\int_{s_{n,k}}^{s_{n,k+1}} \partial_t \mathcal{F}_\varepsilon\big(t_n(s), \hat{u}_n(r), \hat{\xi}_n(r)\big) t_n'(r)\, dr.$$

Let $s \in [0, S)$ be fixed, and n sufficiently large so that $s + \Delta s_n < S$. Let k_n be such that $s_{n,k_n} \leq s < s_{n,k_n+1}$ (i.e. $k_n \Delta s_n \leq s < (k_n + 1)\Delta s_n$). Since $t_{n,k} \leq k\Delta s_n \leq S$ for every $k = 0, \ldots, k_n + 1$, we can apply the energy estimate (5.9) with the constant $C_{\varepsilon,S}$ (depending on $\varepsilon > 0$ and on $\|g\|_{W^{1,\infty}(0,S;H^1)}$). Summing up for every $k = 0, \ldots, k_n$ yields the energy estimates

$$\begin{aligned}\mathcal{F}_\varepsilon(t_{n,k_n+1}, u_{n,k_n+1}, \xi_{n,k_n+1}) \leq \mathcal{F}_\varepsilon(0, u_0, \xi_0) &+ \int_0^{s_{n,k_n+1}} \partial_t \mathcal{F}_\varepsilon\big(t_n(r), \hat{u}_n(r), \hat{\xi}_n(r)\big) t_n'(r)\, dr \\ &- \int_0^{s_{n,k_n+1}} \|\partial_u \mathcal{F}_\varepsilon\big(\hat{t}_n(r), \hat{u}_n(r), \hat{\xi}_n(r)\big)\|_{\mathcal{U}'}\, dr + C_{\varepsilon,S} \Delta s_n S.\end{aligned} \tag{5.21}$$

As above, we have:

$$t_{n,k_n+1} \to t(s), \qquad u_{n,k_n+1} \rightharpoonup u(s),\ w\text{-}H^1(\Omega \setminus K) \qquad \xi_{n,k_n+1} \to \xi(s) \ \ L^2(K).$$

Therefore, Lemma 4.5 (a) implies that

$$\mathcal{F}_\varepsilon\big(t(s), u(s), \xi(s)\big) \leq \liminf_{n\to+\infty} \mathcal{F}_\varepsilon(t_{n,k_n+1}, u_{n,k_n+1}, \xi_{n,k_n+1}).$$

Consider now the right-hand side in (5.21). Note that, for every $r \in [0, S)$,

$$t_n(r) \to t(r); \quad \hat{t}_n(r) \to t(r); \quad \hat{u}_n(r) \rightharpoonup u(r) \quad w\text{-}H^1(\Omega \setminus K); \quad \hat{\xi}_n(r) \to \xi(r) \ \ L^2(K).$$

Denote now the second term in the right-hand side of (5.21) by I_n^1; then

$$I_n^1 = \int_0^S 1_n(r)\ \partial_t \mathcal{F}_\varepsilon\big(t_n(r), \hat{u}_n(r), \hat{\xi}_n(r)\big)\, t_n'(r)\, dr,$$

where 1_n is the characteristic function of the interval $(0, s_{n,k_n+1})$. Denote by $h_n(r)$ the function $\partial_t \mathcal{F}_\varepsilon\big(t_n(r), \hat{u}_n(r), \hat{\xi}_n(r)\big)$; because of the convergence properties of $(t_n, \hat{u}_n, \hat{\xi}_n)$, by Lemma 4.5 (c) we have

$$h_n(r)\chi_n(r) \to \partial_t \mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big)\, 1_{(0,S)}(r) \qquad \text{for a.e. } r \in (0, S),$$

where $1_{(0,S)}$ denotes the characteristic function of the interval $(0, S)$. Moreover (recall Lemma 4.4)

$$\begin{aligned} |h_n(r)| &\le \int_{\Omega\setminus K} |\nabla\big(u_n(r) + g(t_n(r))\big)\nabla g'\big(t_n(r)\big)|\, dx \\ &\le \big(\|u_n\|_{W^{1,\infty}(0,S;H^1)} + \|g\|_{W^{1,\infty}(0,S;H^1)}\big)\|g\|_{W^{1,\infty}(0,S;H^1)}\,. \end{aligned}$$

The equi-boundedness of (h_n) on $(0, S)$ follows. Hence $h_n 1_n$ converge in $L^1(0, S)$; since $t'_n \overset{*}{\rightharpoonup} t'$ in $L^\infty(0, S)$, we conclude that

$$\lim_{n\to+\infty} I_n^1 = \int_0^s \partial_t \mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big)\, t'(r)\, dr.$$

Let now I_n^2 be the third term in the right-hand side of (5.21). It can be written as

$$I_n^2 = -\int_0^S 1_n(r)\|\partial_u \mathcal{F}_\varepsilon\big(\hat{t}_n(r), \hat{u}_n(r), \hat{\xi}_n(r)\big)\|_{\mathcal{U}'}\, dr.$$

Thus, by Lemma 4.5 and Fatou's Lemma,

$$\limsup_{n\to+\infty} I_n^2 \le -\int_0^s \|\partial_u \mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big)\|_{\mathcal{U}'}\, dr.$$

By collecting the estimates for the terms I_n^1 and I_n^2, we conclude that

$$\begin{aligned} \mathcal{F}_\varepsilon\big(t(s), u(s), \xi(s)\big) \le \mathcal{F}_\varepsilon(0, u_0, \xi_0) &+ \int_0^s \partial_t \mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big) t'(r)\, dr \\ &- \int_0^s \|\partial_u \mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big)\|_{\mathcal{U}'}\, dr. \end{aligned} \tag{5.22}$$

We have now to prove the opposite inequality. To this aim we compute the derivative of the map $r \mapsto \mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big)$ which is Lipschitz continuous and, hence, a.e. differentiable. Fix a differentiability point $r \in (0, S)$; by the monotonicity of $\mathcal{F}_\varepsilon$ with respect to ξ, it turns out that

$$\begin{aligned} \frac{d}{dr}\mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big) &= \lim_{h\to 0+} \frac{1}{h}\big[\mathcal{F}_\varepsilon\big(t(r+h), u(r+h), \xi(r+h)\big) - \mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big)\big] \\ &\ge \liminf_{h\to 0+} \frac{1}{h}\big[\mathcal{F}_\varepsilon\big(t(r+h), u(r+h), \xi(r)\big) - \mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big)\big]. \end{aligned}$$

Since $\mathcal{F}_\varepsilon(\cdot, \cdot, \xi)$ is Fréchet differentiable for every ξ, it turns out that for a.e. $r \in (0, S)$ the last term in the previous inequality can be computed by the usual chain rule. Thus, for a.e. $r \in (0, S)$, this term equals

$$\begin{aligned} \partial_t \mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big) t'(r) &+ \partial_u \mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big)[u'(r)] \\ &\ge \partial_t \mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big) t'(r) - \|\partial_u \mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big)\|_{\mathcal{U}'} \end{aligned}$$

where we used that $\|u'(r)\| \le 1$.

Therefore, we can estimate the right-hand side of (5.22):

$$\begin{aligned} \mathcal{F}_\varepsilon(0, u_0, \xi_0) &+ \int_0^s \partial_t \mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big) t'(r)\, dr - \int_0^s \|\partial_u \mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big)\|_{\mathcal{U}'}\, dr \\ &\le \mathcal{F}_\varepsilon(0, u_0, \xi_0) + \int_0^s \frac{d}{dr}\mathcal{F}_\varepsilon\big(t(r), u(r), \xi(r)\big)\, dr = \mathcal{F}_\varepsilon\big(t(s), u(s), \xi(s)\big). \end{aligned}$$

We conclude that in (5.22) the equality holds. ■

Remark 5.9 If $z \in W^{1,\infty}(0,S;L^p(K))$ for every $1 \le p < \infty$ then $z^2 \in W^{1,\infty}(0,S;L^r(K))$ for every $1 < r < +\infty$ and $(z^2)' = 2z'z$. Since $L^r(K)$ is reflexive and separable, to prove that $z^2 \in W^{1,\infty}$ it is enough to show (see §2) that z^2 is a Lipschitz map in $L^r(K)$. Let $p_1, p_2 \in (2,\infty)$, and let $r > 1$ be such that $1/r = (1/p_1) + (1/p_2)$; then

$$\begin{aligned}\left(\int_K |z^2(s_2) - z^2(s_1)|^r\,dx\right)^{1/r} &= \left(\int_K |z(s_2) - z(s_1)|^r\,|z(s_2) + z(s_1)|^r\,dx\right)^{1/r}\\ &\le \|z(s_2) - z(s_1)\|_{L^{p_2}}\,\|z(s_2) + z(s_1)\|_{L^{p_1}} \le C|s_2 - s_1|.\end{aligned}$$

For the chain rule, let us write

$$\frac{z^2(s+h) - z^2(s)}{h} = \frac{z(s+h) - z(s)}{h}(z(s+h) + z(s))$$

Then for a.e. $s \in (0,S)$ the left-hand side converges strongly in $L^r(K)$, and thus in $L^1(K)$, to $(z^2)'(s)$. Moreover $(z(s+h) - z(s))/h \to z'(s)$ and $z(s+h) \to z(s)$ again strongly in $L^2(K)$.

6 Quasi-static evolution for the energy $\mathcal{F}$

For every $\varepsilon > 0$, Proposition 5.7 and Theorem 5.8 provide a triple $(t_\varepsilon, u_\varepsilon, \xi_\varepsilon)$ which is a parametrized BV evolution for the energy $\mathcal{F}_\varepsilon$. By the estimates shown in introducing Proposition 5.7, for every $S > 0$ the functions t_ε, u_ε, and ξ_ε turn out to be bounded, uniformly with respect to $\varepsilon > 0$, in $W^{1,\infty}(0,S)$, $W^{1,\infty}(0,S;H^1(\Omega\setminus K))$ and $W^{1,\infty}(0,S;L^q(K))$ (for any $1 \le q < +\infty$) respectively and the map

$$s \mapsto \big(t_\varepsilon(s), u_\varepsilon(s), \xi_\varepsilon(s)\big) \colon [0,+\infty) \to [0,+\infty) \times H^1(\Omega \setminus K) \times L^q(K)$$

has a Lipschitz constant independent of ε (see §5.2). Therefore, Proposition 2.3 and a standard diagonal argument yield the following compactness result.

Proposition 6.1 *Let (ε_n) be a positive infinitesimal sequence. There exists a map $(t,u,\xi)\colon [0,+\infty) \to [0,+\infty) \times H^1(\Omega \setminus K) \times L^q(K)$ such that (up to a subsequence) $t_{\varepsilon_n} \overset{*}{\rightharpoonup} t$ in $W^{1,\infty}(0,S)$, $u_{\varepsilon_n} \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(0,S;H^1(\Omega \setminus K))$ and $\xi_{\varepsilon_n} \overset{*}{\rightharpoonup} \xi$ in $W^{1,\infty}(0,S;L^q(K))$ for $1 < q < +\infty$ and for any finite interval $(0,S)$. Moreover, for any $1 \le q < +\infty$, the map (t,u,ξ) is Lipschitz continuous.*

Theorem 6.2 *The triple (t,u,ξ) in Proposition 6.1 is a (parametrized) BV evolution for the energy $\mathcal{F}$ according to Definition 3.1.*

Proof. The Lipschitz continuity has been checked in the previous Proposition. Let $S > 0$ be fixed. If (ε_n) is as above, we denote $\mathcal{F}_{\varepsilon_n}$ simply by $\mathcal{F}_n$ and similarly for (t_n), (u_n) and (ξ_n).

Let us retrace the proof of Theorem 5.8. Note that the convergence properties of the sequence (t_n, u_n, ξ_n) are the same in both cases.

First, let us prove condition (C) in Definition 3.1, i.e., for almost every $s \in [0,S]$

$$\xi'(s) \ge 0, \qquad |[\![u(s)]\!]| \le \xi(s), \qquad \xi'(s)\big(|[\![u(s)]\!]| - \xi(s)\big) = 0\,, \qquad \mathcal{H}^1\text{-a.e. on } K. \tag{6.1}$$

The first two items of (6.1) follow by passing to the limit in the corresponding inequalities for ξ_n and u_n.

Consider now the third item in (6.1). This, as in the proof of Theorem 5.8, is equivalent to

$$\int_0^\sigma ds \int_B \xi'(s)\big(|[\![u(s)]\!]| - \xi(s)\big)d\mathcal{H}^1 = 0 \tag{6.2}$$

for every $\sigma \in [0,S]$ and for every Borel subset B of K.We know that

$$\xi_n'(s)\big(|[\![u_n(s)]\!]| - \xi_n(s)\big) = 0\,, \qquad \mathcal{H}^1\text{-a.e. on } K.$$

By the same argument applied in Theorem 5.8, equation (5.15) continues to hold, i.e.

$$\lim_{n\to+\infty} \int_0^\sigma ds \int_B \xi_n'(s)|[\![u_n(s)]\!]|d\mathcal{H}^1 = \int_0^\sigma ds \int_B \xi'(s)|[\![u(s)]\!]|d\mathcal{H}^1\,.$$

Since $\xi_n'(s)|[\![u_n(s)]\!]| = \xi_n'(s)\xi_n(s)$, this implies that

$$\lim_{n\to+\infty}\int_0^\sigma ds\int_B \xi_n'(s)\xi_n(s)\,d\mathcal{H}^1 = \int_0^\sigma ds\int_B \xi'(s)|[\![u(s)]\!]|\,d\mathcal{H}^1\,.$$

Since $\xi,\xi_n \in W^{1,\infty}(0,S;L^q(K))$ (for any $1<q<+\infty$), we can apply Remark 5.9; then:

$$\begin{aligned}\int_0^\sigma ds\int_B \xi'(s)|[\![u(s)]\!]|\,d\mathcal{H}^1 &= \lim_{n\to+\infty}\int_0^\sigma ds\int_B \xi_n'(s)\xi_n(s)\,d\mathcal{H}^1 = \lim_{n\to+\infty}\frac{1}{2}\int_B\big(\xi_n(\sigma)^2-\xi(0)^2\big)\,d\mathcal{H}^1\\ &\geq \frac{1}{2}\int_B\big(\xi(\sigma)^2-\xi(0)^2\big)\,d\mathcal{H}^1 = \int_0^\sigma ds\int_B \xi'(s)\xi(s)\,d\mathcal{H}^1\,.\end{aligned}$$

Since we know that $|[\![u(s)]\!]| \leq \xi(s)$, the first and the last term term in the above inequalities must coincide, i.e. (6.2) holds. Moreover, we deduce that $\int_K \xi_n(\sigma)^2\,d\mathcal{H}^1 \to \int_K \xi(\sigma)^2\,d\mathcal{H}^1$; thus, the weak L^2-convergence implies

$$\xi_n(\sigma)\to\xi(\sigma)\quad\text{strongly in } L^2(K) \text{ for every } \sigma\in[0,S].$$

By the uniform Lipschitz continuity of ξ_n we deduce that for $\sigma_n\to\sigma$

$$\xi_n(\sigma_n)\to\xi(\sigma)\quad\text{strongly in } L^2(K).$$

Now, let us address condition (S) of Definition 3.1, i.e.: for every $s\in[0,S]$ with $t'(s)>0$

$$|\partial_u^-\mathcal{F}(t(s),u(s),\xi(s))| = 0. \tag{6.3}$$

Let $s\in[0,S]$ be such that $t'(s)>0$. Let us note that there exists a sequence $s_n\to s$ such that $t_n'(s_n)>0$ for n sufficiently large. Assume by contradiction that there exists $\delta>0$ with the property that, for every $k\in\mathbb{N}$ we can find $n>k$ such that $t_n'\equiv 0$ in $I_\delta=(s-\delta,s+\delta)$. Then, an increasing sequence $(n_k)_k$ would exist with $t_{n_k}'\equiv 0$ in I_δ; this implies, in particular, that $t'(s)=0$: a contradiction. Then, from

$$t_n(s_n)\to t(s),\qquad u_n(s_n)\rightharpoonup u(s),\quad\text{in } H^1(\Omega\setminus K),\qquad \xi_n(s_n)\to\xi(s)\quad\text{in } L^2(K),$$

by Lemma 4.6 we have

$$|\partial_u^-\mathcal{F}(t(s),u(s),\xi(s))| \leq \liminf_{n\to+\infty}\|\partial_u\mathcal{F}_n(t_n(s_n),u_n(s_n),\xi_n(s_n))\|_{\mathcal{U}'}.$$

Being condition (S) satisfied by (t_n,u_n,ξ_n), the right-hand side of this inequality is zero, thus the left-hand side is zero, too.

Let us now address condition (E) of Definition 3.1 i.e.: for every $s\in[0,S]$

$$\begin{aligned}\mathcal{F}(t(s),u(s),\xi(s)) = \mathcal{F}(0,w_0,\xi_0) &+ \int_0^s \partial_t\mathcal{F}(t(r),u(r),\xi(r))\,t'(r)\,dr\,+\\ &-\int_0^s |\partial_u^-\mathcal{F}(t(r),u(r),\xi(r))|\,dr\,.\end{aligned} \tag{6.4}$$

By Theorem 5.8 this holds for the energy $\mathcal{F}_n=\mathcal{F}_{\varepsilon_n}$ and the triple (t_n,u_n,ξ_n), i.e. $(t_{\varepsilon_n},u_{\varepsilon_n},\xi_{\varepsilon_n})$. Passing to the limit we get

$$\begin{aligned}\liminf_{n\to+\infty}\mathcal{F}_n(t_n(s),u_n(s),\xi_n(s)) \leq \limsup_{n\to+\infty}\mathcal{F}_n(0,u_0,\xi_0) &+ \limsup_{n\to+\infty}\int_0^s \partial_t\mathcal{F}_n(t_n(r),u_n(r),\xi_n(r))\,t_n'(r)\,dr\\ &-\liminf_{n\to+\infty}\int_0^s \|\partial_u\mathcal{F}_n(t_n(r),u_n(r),\xi_n(r))\|_{\mathcal{U}'}\,dr\,.\end{aligned}$$

The pointwise convergence of ψ_ε as $\varepsilon\to0$ (Proposition 4.3) together with the uniform boundedness of ψ_ε (Proposition 4.2) yields $\mathcal{F}_n(0,u_0,\xi_0)\to\mathcal{F}(0,u_0,\xi_0)$ as $n\to+\infty$. Moreover, $\partial_t\mathcal{F}_n(t_n(\cdot),u_n(\cdot),\xi_n(\cdot))$ converge to $\partial_t\mathcal{F}(t(\cdot),u(\cdot),\xi(\cdot))$ pointwise, by Lemma 4.6 (c), and then in $L^1(0,s)$ by dominated convergence. Taking into account Lemma 4.6 (b) we manage the last term. Summing up

$$\begin{aligned}\mathcal{F}(t(s),u(s),\xi(s)) \leq \mathcal{F}(0,&u_0,\xi_0)\,+\\ &+\int_0^s \partial_t\mathcal{F}(t(r),u(r),\xi(r))\,t'(r)\,dr - \int_0^s |\partial_u^-\mathcal{F}(t(r),u(r),\xi(r))|\,dr\,.\end{aligned} \tag{6.5}$$

As in the proof of Theorem 5.8, to get the opposite inequality note that $r \mapsto \mathcal{F}\big(t(r), u(r), \xi(r)\big)$ is Lipschitz continuous as a composition of Lipschitz functions; indeed, $\mathcal{E}$ is locally Lipschitz continuous on $\mathbb{R}^+ \times \mathcal{U}$ since it is quadratic in $u+g$ and both u and g are Lipschitz function from $[0,+\infty)$ to $H^1(\Omega \setminus K)$; moreover, $\mathcal{K}$ is Lipschitz by Corollary 3.4.

Let now r be a point of differentiability for the functions t, u and $\mathcal{F}\big(t(\cdot), u(\cdot), \xi(\cdot)\big)$. Let (h_n) be a positive infinitesimal sequence. By the monotonicity of ψ with respect to ξ, it turns out that

$$\begin{aligned}\frac{d}{dr}\mathcal{F}\big(t(r), u(r), \xi(r)\big) &= \frac{d}{dr}\mathcal{E}\big(t(r), u(r)\big) + \lim_{n\to+\infty} \frac{1}{h_n}\big[\mathcal{K}\big(u(r+h_n), \xi(r+h_n)\big) - \mathcal{K}\big(u(r), \xi(r)\big)\big] \\ &\geq \frac{d}{dr}\mathcal{E}\big(t(r), u(r)\big) + \liminf_{n\to+\infty} \frac{1}{h_n}\big[\mathcal{K}\big(u(r+h_n), \xi(r)\big) - \mathcal{K}\big(u(r), \xi(r)\big)\big].\end{aligned}$$

Since $\mathcal{E}$ is Fréchet differentiable, the usual chain rule yields:

$$\frac{d}{dr}\mathcal{E}\big(t(r), u(r)\big) = \partial_t \mathcal{E}\big(t(r), u(r)\big) t'(r) + \partial_u \mathcal{E}\big(t(r), u(r)\big)[u'(r)].$$

As to the other term, write $u(r+h_n) - u(r)$ as $h_n\big(u'(r) + Z(h_n)\big)$, where $Z(h) \to 0$ in $\mathcal{U}$ as $h \to 0$. Let $z_n = u'(r) + Z(h_n)$. Then, by Lemma 3.6

$$\left| \frac{1}{h_n}\big[\mathcal{K}\big(u(r) + h_n z_n, \xi(r)\big) - \mathcal{K}\big(u(r), \xi(r)\big)\big] - \partial_u \mathcal{K}\big(u(r), \xi(r); z_n\big) \right| \to 0 \qquad \text{as } n \to +\infty.$$

It follows that

$$\liminf_{n\to+\infty} \frac{1}{h_n}\big[\mathcal{K}\big(u(r+h_n), \xi(r)\big) - \mathcal{K}\big(u(r), \xi(r)\big)\big] \geq \liminf_{n\to+\infty} \partial_u \mathcal{K}\big(u(r), \xi(r); z_n\big).$$

From the convergence $z_n \to u'(r)$ in $\mathcal{U}$ and the explicit form of $\partial_u \mathcal{K}(u, \xi; z)$ given in Lemma 3.6 we deduce that $\partial_u \mathcal{K}\big(u(r), \xi(r); z_n\big) \to \partial_u \mathcal{K}\big(u(r), \xi(r); u'(r)\big)$, so that

$$\frac{d}{dr}\mathcal{F}\big(t(r), u(r), \xi(r)\big) \geq \partial_t \mathcal{E}\big(t(r), u(r)\big) t'(r) + \partial_u \mathcal{E}\big(t(r), u(r)\big)[u'(r)] + \partial_u \mathcal{K}\big(u(r), \xi(r); u'(r)\big),$$

i.e.

$$\frac{d}{dr}\mathcal{F}\big(t(r), u(r), \xi(r)\big) \geq \partial_t \mathcal{E}\big(t(r), u(r)\big) t'(r) + \partial_u \mathcal{F}\big(u(r), \xi(r); u'(r)\big). \tag{6.6}$$

Now, recall that $\|u'(r)\| \leq 1$:

$$\begin{aligned}\partial_u \mathcal{F}\big(t(r), u(r), \xi(r); u'(r)\big) &\geq -\Big(\partial_u \mathcal{F}\big(t(r), u(r), \xi(r); u'(r)\big)\Big)_- \\ &\geq -\sup\Big\{\Big(\partial_u \mathcal{F}\big(t(r), u(r), \xi(r); z\big)\Big)_- : \ \ \|z\| \leq 1\Big\} \\ &= -\big|\partial^- \mathcal{F}\big(t(r), u(r), \xi(r)\big)\big|,\end{aligned}$$

where in the last line we have used the representation (3.11) of the slope. We conclude that

$$\frac{d}{dr}\mathcal{F}\big(t(r), u(r), \xi(r)\big) \geq \partial_t \mathcal{E}\big(t(r), u(r)\big) t'(r) - \big|\partial_u^- \mathcal{F}\big(t(r), u(r), \xi(r)\big)\big|.$$

This inequality, together with (6.5), implies

$$\begin{aligned}\mathcal{F}(t(s), u(s), \xi(s)) &\leq \mathcal{F}(0, u_0, \xi_0) + \int_0^s \partial_t \mathcal{F}(t(r), u(r), \xi(r))\, t'(r)\, dr - \int_0^s |\partial_u^- \mathcal{F}(t(r), u(r), \xi(r))|\, dr \\ &\leq \mathcal{F}(0, u_0, \xi_0) + \int_0^s \frac{d}{dr}\mathcal{F}\big(t(r), u(r), \xi(r)\big)\, dr = \mathcal{F}(t(s), u(s), \xi(s)).\end{aligned}$$

Therefore, inequality (6.5) must actually be an equality. ■

7 Equilibrium condition in PDE form

In this section we express the equilibrium condition $|\partial_u^- \mathcal{F}(t(s), u(s), \xi(s))| = 0$ of equation (3.6) in a more explicit form. We need some preliminary remarks.

Let $v \in H^1(\Omega \setminus K)$ be such that the distributional Laplacian Δv is in $L^2(\Omega \setminus K)$. For every $z \in H^1(\Omega \setminus K)$ define

$$L_v z = \int_{\Omega \setminus K} \nabla v \nabla z \, dx + \int_{\Omega \setminus K} (\Delta v) z \, dx. \tag{7.1}$$

Then L_v is linear and continuous on $H^1(\Omega \setminus K)$.

Remark 7.1 Let us first consider the case of a regular v. Assume, e.g., that

$$v \in C^\infty(\Omega \setminus K), \quad \text{and} \quad v \in C^\infty(\overline{\Omega}_i) \quad \text{for every } i$$

(where the sets Ω_i are introduced in § 3). In the same way we followed introducing the trace $u^\pm$ for a function $u \in H^1(\Omega \setminus K)$ according to the chosen orientation ν on K (Remark 3.2), we can define the traces $(\nabla v)^\pm$ on K (actually, these simply are the restrictions of ∇v to a suitable boundary). For every $z \in H^1(\Omega \setminus K)$, and for every i, we have:

$$\int_{\Omega_i} \nabla v \nabla z \, dx + \int_{\Omega_i} (\Delta v) z \, dx = \int_{\partial \Omega_i} z (\nabla v) \cdot n \, d\mathcal{H}^1,$$

where n is the outer unit normal. Summing up over i we get

$$L_v z = \int_{(\partial\Omega) \setminus K} z^\circ (\nabla v) \cdot \nu \, d\mathcal{H}^1 + \int_K z^+ (\nabla v)^+ \cdot \nu \, d\mathcal{H}^1 - \int_K z^- (\nabla v)^- \cdot \nu \, d\mathcal{H}^1;$$

by introducing the integral operators $\partial_\nu v$, $\partial_\nu^\pm v$ with density $(\nabla v) \cdot \nu$ or $(\nabla v)^\pm \cdot \nu$ on $\partial\Omega$ and K, respectively, this equation can be written as:

$$L_v z = \langle \partial_\nu v, z^\circ \rangle + \langle \partial_\nu^+ v, z^+ \rangle - \langle \partial_\nu^- v, z^- \rangle \tag{7.2}$$

Hence, the value $L_v z$ depends on z only through the trace $\operatorname{tr} z := (z^\circ, z^+, z^-)$. This is true even in the general case, stated in the following result.

Proposition 7.2 *Let $v \in H^1(\Omega \setminus K)$ be such that the distributional Laplacian Δv is in $L^2(\Omega \setminus K)$. Let $z \in H^1(\Omega \setminus K)$ and let L_v be as in* (7.1). *If $\operatorname{tr} z = 0$ (i.e., $z \in H_0^1(\Omega \setminus K)$), then $L_v z = 0$.*

Proof. It is enough to prove the statement assuming that $z = \varphi \in C_c^\infty(\Omega \setminus K)$. Let $\Omega' \subset\subset (\Omega \setminus K)$ be a regular open set containing the support of φ. Let $V = \nabla v$; thus $V \in L^2(\Omega'; \mathbb{R}^2)$ and the distributional divergence $\operatorname{div} V$ is in $L^2(\Omega')$. Then (see, e.g., [22], Theorem 1.1) there exists a sequence (V_k) in $C^\infty(\overline{\Omega}'; \mathbb{R}^2)$ such that

$$V_k \to V \quad \text{in } L^2(\Omega'; \mathbb{R}^2), \qquad \operatorname{div} V_k \to \operatorname{div} V \quad \text{in } L^2(\Omega').$$

An integration by parts gives

$$\int_{\Omega'} V_k \nabla \varphi \, dx + \int_{\Omega'} (\operatorname{div} V_k) \, \varphi \, dx = \int_{\partial\Omega'} \varphi V_k \cdot n \, d\mathcal{H}^1,$$

where n is the outer unit normal. The right-hand side vanishes since $\varphi = 0$ on $\partial\Omega'$. Now it is enough to pass to the limit as $k \to \infty$. ■

Therefore, L_v defines a linear operator on the quotient space $H^1(\Omega \setminus K)/H_0^1(\Omega \setminus K)$; this can be identified with the space $T(\Omega \setminus K)$ of the traces $\boldsymbol{\zeta} = (z^\circ, z^+, z^-)$ when z varies in $H^1(\Omega \setminus K)$:

$$L_v \boldsymbol{\zeta} = \int_{\Omega \setminus K} \nabla v \nabla z \, dx + \int_{\Omega \setminus K} (\Delta v) z \, dx, \qquad \boldsymbol{\zeta} = \operatorname{tr} z, \ z \in H^1(\Omega \setminus K). \tag{7.3}$$

It is standard that the operator L_v is linear and continuous with respect to the quotient norm

$$\|\boldsymbol{\zeta}\|_{H^1/H_0^1} = \inf\{\|z\|_{H^1(\Omega \setminus K)} : \ \operatorname{tr} z = \boldsymbol{\zeta}\}.$$

Let us now turn to the equilibrium condition (3.6). Let us denote $(t(s), u(s), \xi(s))$ simply by (t, u, ξ). By (3.11) this is equivalent to

$$[\partial_u \mathcal{F}(t, u, \xi; z)]_- = 0 \qquad \text{for every } z \in \mathcal{U},$$

or

$$\partial_u \mathcal{F}(t, u, \xi; z) \geq 0 \qquad \text{for every } z \in \mathcal{U},$$

i.e.

$$\int_{\Omega\setminus K} \nabla\big(u + g(t)\big)\nabla z \, dx + \int_K \partial_w \psi([\![u]\!], \xi; [\![z]\!]) \, d\mathcal{H}^1 \geq 0 \qquad \text{for every } z \in \mathcal{U}. \tag{7.4}$$

If $z \in H^1(\Omega) \cap \mathcal{U}$, then $[\![z]\!] = 0$ on K and the second integral vanishes. By linearity:

$$\int_{\Omega\setminus K} \nabla\big(u + g(t)\big)\nabla z \, dx = 0 \qquad \text{for every } z \in H^1(\Omega) \cap \mathcal{U}.$$

This implies, in particular, that $\Delta v = 0$ in $H^{-1}(\Omega \setminus K)$, where $v = u + g(t)$. Therefore $L_v \boldsymbol{\zeta} = \int_{\Omega\setminus K} \nabla v \nabla z \, dx$ for $\boldsymbol{\zeta} \in T(\Omega \setminus K)$ with $\boldsymbol{\zeta} = \operatorname{tr} z$; moreover

$$L_v \boldsymbol{\zeta} + \int_K \partial_w \psi([\![u]\!], \xi; [\![z]\!]) \, d\mathcal{H}^1 \geq 0, \qquad \text{for every } z \in \mathcal{U}, \text{ with } \boldsymbol{\zeta} = \operatorname{tr} z, \tag{7.5}$$

$$L_v \boldsymbol{\zeta} = 0 \qquad \text{for every } \boldsymbol{\zeta} \in \mathcal{T}^1 := \{\operatorname{tr} z : \ z \in H^1(\Omega) \cap \mathcal{U}\}. \tag{7.6}$$

Remark 7.3 Condition (7.6) can be "splitted" into

$$\begin{aligned} &L_v \boldsymbol{\zeta} = 0 \quad \text{for every } \boldsymbol{\zeta} \in \mathcal{T}^1_0 := \{\operatorname{tr} z : \ z \in H^1_0(\Omega)\} \subseteq \mathcal{T}^1, \\ &L_v \boldsymbol{\zeta} = 0 \quad \text{for every } \boldsymbol{\zeta} \in \mathcal{T}^1_K := \{\operatorname{tr} z : \ z \in H^1(\Omega) \cap \mathcal{U}, \ z^+ = z^- = 0\} \subseteq \mathcal{T}^1. \end{aligned}$$

Recalling the meaning of L_v in case of a regular v (see (7.2)) we can understand these equations respectively as a weak form of

$$\partial^+_\nu v - \partial^-_\nu v = 0 \quad \text{on } K, \qquad \partial_\nu v = 0 \quad \text{on } [(\partial\Omega) \setminus K] \setminus \partial_D \Omega.$$

In the sequel we go further towards a more precise definition of these normal-derivative trace operators. In order to do this we will define a "localization" of the functional L_v to K and $[(\partial\Omega) \setminus K] \setminus \partial_D \Omega$.

Let γ be a curve which is part of the boundary of a piecewise-C^1 open set $A \subseteq \mathbb{R}^2$. Let ζ be a function on ∂A such that $\zeta_{|\gamma} \in H^{1/2}(\gamma)$ and $\zeta_{|\gamma^c} \in H^{1/2}(\gamma^c)$, where $\gamma^c = (\partial A) \setminus \gamma$. Theorem 1.5.2.3 in [13] gives necessary and sufficient (integrability) conditions that guarantee that ζ has a lifting to a function in $H^1(A)$. These conditions motivate the following definition.

Definition 7.4 *Let $x \colon [0, l_\gamma] \to \gamma$ be the length distance along γ. We denote by $W_0(\gamma)$ the subspace of $H^{1/2}(\gamma)$ consisting of the functions ζ such that*

$$\sigma \mapsto \frac{\zeta\big(x(\sigma)\big)^2}{\sigma}, \qquad \sigma \mapsto \frac{\zeta\big(x(\sigma)\big)^2}{l_\gamma - \sigma}$$

are integrable in a neighbourhood of 0 *and l_γ, respectively.*

For instance, $W_0(\gamma)$ contains all piecewise-C^1 functions with compact support.
From Theorem 1.5.2.3 in [13] we deduce the following result.

Theorem 7.5 *Let A and γ be as above. Let $\zeta \in H^{1/2}(\gamma)$; extend ζ to the whole of ∂A with value 0. Then ζ is the trace on ∂A of a function in $H^1(A)$ if and only if $\zeta \in W_0(\gamma)$.*

Let now Γ be any of the arcs Γ_j which decompose K. Let i_+ and i_- be such that $\Gamma \subseteq \partial\Omega_{i_+} \cap \partial\Omega_{i_-}$ and that the orientation ν on Γ agrees with the outer unit normal of Ω_{i_+} on Γ. Apply the previous remarks with $\gamma = \Gamma$. Let $\zeta \in W_0(\Gamma)$. By the previous theorem there exists a function $z \in H^1(\Omega_{i_+})$

whose trace on Γ is ζ, and whose trace on $(\partial\Omega_{i_+})\setminus\Gamma$ is 0. The function z can be extended (with value 0) to a function in $H^1(\Omega\setminus K)$; therefore

$$z \in H^1(\Omega\setminus K)\cap\mathcal{U}, \quad \text{and} \quad \boldsymbol{\zeta}_+ := (z^\circ, z^+, z^-) = (0, 1_\Gamma\zeta, 0). \tag{7.7}$$

In the same way we get the existence of a function z such that

$$z \in H^1(\Omega\setminus K)\cap\mathcal{U}, \quad \text{and} \quad \boldsymbol{\zeta}_- := (z^\circ, z^+, z^-) = (0, 0, 1_\Gamma\zeta). \tag{7.8}$$

This suggests the following definition.

Definition 7.6 *Let $\Gamma \subseteq K$ be as above, and $\zeta \in W_0(\Gamma)$. We set*

$$\langle (\partial_\nu^+ v)_{|\Gamma}, \zeta\rangle = L_v\boldsymbol{\zeta}_+, \qquad \langle (\partial_\nu^- v)_{|\Gamma}, \zeta\rangle = -L_v\boldsymbol{\zeta}_-,$$

where $\boldsymbol{\zeta}_\pm$ are defined in (7.7) *and* (7.8) *(and L_v in* (7.3)*).*

We explicitly note the slight abuse in using a pointwise-restriction notation in denoting this operator.

Let us now address the problem of the trace of the normal derivative on $\partial\Omega$.

By assumption, $(\partial\Omega)\setminus K$ consists of a finite number of piecewise-C^1 curves γ_i. Let $z \in H^1(\Omega)\cap\mathcal{U}$ with $z^+ = z^- = 0$; then, for each i, we can apply Theorem 7.5 (to a suitable neihbourhood of γ_i) and deduce that for every maximal arc γ in $\gamma_i\setminus\partial_D\Omega$, the trace of z on γ belongs to $W_0(\gamma)$. On the other hand, if ζ is a function on $[(\partial\Omega)\setminus K]\setminus\partial_D\Omega$ such that $\zeta_{|\gamma} \in W_0(\gamma)$ for every maximal arc γ in $[(\partial\Omega)\setminus K]\setminus\partial_D\Omega$, then it is the trace of a function $z \in H^1(\Omega)\cap\mathcal{U}$ with $z^+ = z^- = 0$ on K. Therefore, the space of traces $\mathcal{T}_K^1$ in Remark 7.3 is the natural domain for the normal-derivative operator on $[(\partial\Omega)\setminus K]\setminus\partial_D\Omega$ along the line of Definition 7.6.

Definition 7.7 *Let $\boldsymbol{\zeta} = (\zeta, 0, 0) \in \mathcal{T}_K^1$ (i.e. ζ is the trace of z on $(\partial\Omega)\setminus K$ for some $z \in H^1(\Omega)\cap\mathcal{U}$ with $z^+ = z^- = 0$ on K). We set*

$$\langle (\partial_\nu v)_{[(\partial\Omega)\setminus K]\setminus\partial_D\Omega}, \zeta\rangle = L_v\boldsymbol{\zeta}.$$

We are now in a position to prove the next result, following the analysis in [6].

Theorem 7.8 *Let $(t, u, \xi) := \big(t(s), u(s), \xi(s)\big)$ satisfy the equilibrium condition* (3.6). *Let $v = u(s) + g(t(s))$. Then*

$$\begin{cases} \Delta v = 0 & \text{in } H^{-1}(\Omega\setminus K), \\ v = g(t) & \text{on } \partial_D\Omega, \\ \partial_\nu^+ v = \partial_\nu^- v & \text{on every } \Gamma_j \subset K, \\ \partial_\nu v = 0 & \text{on } [(\partial\Omega)\setminus K]\setminus\partial_D\Omega, \end{cases} \tag{7.9}$$

where the boundary operators are introduced in Definitions 7.6 and 7.7 and the sets Γ_j are defined in § 3.

In addition, there exists $h \in L^\infty(K)$ such that the following properties hold.

a) Let Γ be any of the arcs Γ_j which decompose K. Let $\zeta \in W_0(\Gamma)$. Then

$$\langle \partial_\nu^+ v, \zeta\rangle = \langle \partial_\nu^- v, \zeta\rangle = \int_\Gamma h\zeta\, d\mathcal{H}^1. \tag{7.10}$$

and thus $\partial_\nu^+ v = \partial_\nu^- v = h$ in Γ (in the sense of Definition 7.6).

b) Further

$$\begin{cases} h = \partial_w\psi([\![u]\!], \xi) & \mathcal{H}^1\text{-a.e. on } \{x \in \Gamma : ([\![u]\!](x), \xi(x)) \neq (0,0)\} \\ |h| \le \hat\tau(0) & \text{otherwise.} \end{cases}$$

Proof. To prove (7.9) only the statement about the normal-derivative boundary conditions have to be addressed.

Remark 7.3 immediately implies that the operator introduced in Definition 7.7 vanishes: this condition is summed up in the equation $\partial_\nu v = 0$ on $[(\partial\Omega) \setminus K] \setminus \partial_D\Omega$.

Let Γ be any of the arcs Γ_j which decompose K. Let $\zeta \in W_0(\Gamma)$ and let $\boldsymbol{\zeta}_\pm$ be as in (7.7) and (7.8). Then $\boldsymbol{\zeta}_+ + \boldsymbol{\zeta}_- \in \mathcal{T}_0^1$ and $L_v(\boldsymbol{\zeta}_+ + \boldsymbol{\zeta}_-) = 0$ according to Remark 7.3. By linearity and the definition of $\partial_\nu^\pm v$ we conclude that $\partial_\nu^+ v = \partial_\nu^- v$ on Γ.

Let us now address the integral representation of $\partial_\nu^\pm v$. Let Γ, ζ, and $\boldsymbol{\zeta}_+$ be as above.

By (7.5) applied to $\boldsymbol{\zeta}_+$ and $-\boldsymbol{\zeta}_+$ we have

$$|L_v \boldsymbol{\zeta}_+| \le \int_\Gamma |\partial_w \psi([\![u]\!], \xi; \zeta)| \, d\mathcal{H}^1 \le \hat{\tau}(0) \|\zeta\|_{L^1(\Gamma)}.$$

It follows that the functional $\zeta \mapsto L_v \boldsymbol{\zeta}_+$ is linear and continuous on $W_0(\Gamma)$ with respect to the $L^1(\Gamma)$-norm. Therefore, it can be extended to a bounded linear functional on $L^1(\Gamma)$ which admits an integral representation through a function $h \in L^\infty(\Gamma)$, i.e. we have (7.10). Moreover, $|h| \le \hat{\tau}(0)$ a.e. on Γ.

As to property (b), by (7.5) and the definition of h, we get

$$\int_\Gamma h\zeta \, d\mathcal{H}^1 + \int_\Gamma \partial_w \psi([\![u]\!], \xi; \zeta) \, d\mathcal{H}^1 \ge 0, \qquad \text{for every } \zeta \in W_0(\Gamma).$$

By density, and recalling the definition (3.8) of $\partial_w \psi$, this inequality holds for every $\zeta \in L^1(\Gamma)$. Let now $J := \{x \in \Gamma : ([\![u]\!](x), \xi(x)) \neq (0,0)\}$. Note that $\partial_w \psi([\![u]\!], \xi; \zeta) = 1_J \partial_w \psi([\![u]\!], \xi)\zeta + 1_{J^c} \hat{\tau}(0)|\zeta|$. Then

$$\int_\Gamma \big[h - 1_J \partial_w \psi([\![u]\!], \xi)\big] \zeta \, d\mathcal{H}^1 + \int_\Gamma 1_{J^c} \hat{\tau}(0) |\zeta| \, d\mathcal{H}^1 \ge 0, \qquad \text{for every } \zeta \in L^1(\Gamma).$$

By choosing $\zeta > 0$ and $\zeta < 0$, this implies that

$$|h - 1_J \partial_w \psi([\![u]\!], \xi)| \le 1_{J^c} \hat{\tau}(0) \qquad \mathcal{H}^1\text{-a.e. on } \Gamma.$$

In particular, $\mathcal{H}^1$-a.e. on J we have $|h - \partial_w \psi([\![u]\!], \xi)| = 0$. ∎

8 Jump transition in PDE form

Let $t^* \in [0, T]$. Let us assume that $t^{-1}(t^*) = [s^-, s^+]$ with $s^- < s^+$. Clearly $t(s) = t^*$ for every $s \in [s^-, s^+]$. Denote $u(s^\pm) = u^\pm$ and $\xi(s^\pm) = \xi^\pm$. Under these assumptions, the map $s \mapsto (u(s), \xi(s))$ for $s \in [s^-, s^+]$ describes (in the parametric setting) the instantaneous transition from (u^-, ξ^-) to (u^+, ξ^+) at time t^*. The following theorem provides a characterization of the evolution in PDE form; it is formally that of Theorem 7.8 for a different function v.

Theorem 8.1 *Assume the space $\mathcal{U}$ is equipped with the equivalent norm $\|u\| = (\int_{\Omega\setminus K} |\nabla u|^2 \, dx)^{1/2}$. Under the above assumptions, for $t^* = t(s)$, let $\lambda(s) = |\partial_u^- \mathcal{F}(t^*, u(s), \xi(s))|$. Let $v(s) = (u(s) + g(t^*)) + \lambda(s) u'(s)$. Then, a.e. in $[s^-, s^+]$, v solves the following system*

$$\begin{cases} \Delta v = 0 & \text{in } H^{-1}(\Omega \setminus K), \\ v = g(t^*) & \text{on } \partial_D\Omega, \\ \partial_\nu^+ v = \partial_\nu^- v & \text{on every } \Gamma_j \subset K, \\ \partial_\nu v = 0 & \text{on } [(\partial\Omega) \setminus K] \setminus \partial_D\Omega, \end{cases} \tag{8.1}$$

where the boundary operators are introduced in Definitions 7.6 and 7.7 and the sets Γ_j are defined in § 3. In addition, a.e. in $[s^-, s^+]$, there exists $h \in L^\infty(K)$ such that the following properties hold.

a) Let Γ be any of the arcs Γ_j which decompose K. Let $\zeta \in W_0(\Gamma)$. Then

$$\langle \partial_\nu^+ v, \zeta \rangle = \langle \partial_\nu^- v, \zeta \rangle = \int_\Gamma h\zeta \, d\mathcal{H}^1 \tag{8.2}$$

and thus $\partial_\nu^+ v = \partial_\nu^- v = h$ in Γ (in the sense of Definition 7.6).

b) Further

$$\begin{cases} h = \partial_w \psi([\![u]\!], \xi) & \mathcal{H}^1\text{-}a.e.\ on\ \{x \in \Gamma : ([\![u]\!](x), \xi(x)) \neq (0,0)\} \\ |h| \leq \hat{\tau}(0) & otherwise. \end{cases}$$

Note that, being $t(s) = t^*$ constant in $[s^-, s^+]$, we can write $v(s) = (u(s) + g(t^*)) + \lambda(s)(u(s) + g(t^*))'$. In this way (8.1) becomes formally a visco-elastic (Kelvin-Voigt) system with stress $\nabla v(s) = \nabla(u(s) + g(t^*)) + \lambda(s)\nabla(u(s) + g(t^*))'$.

Proof. From the proof of Theorem 6.2 (see (6.6)) we know that $r \mapsto \mathcal{F}(t(r), u(r), \xi(r))$ is a.e. differentiable and

$$\frac{d}{dr}\mathcal{F}\big(t(r), u(r), \xi(r)\big) \geq \partial_t \mathcal{E}\big(t(r), u(r)\big) t'(r) + \partial_u \mathcal{F}\big(t(r), u(r), \xi(r); u'(r)\big)$$

for a.e. $r \in [0, +\infty)$. On the other hand, the energy balance (6.4) yields

$$\begin{aligned} \frac{d}{dr}\mathcal{F}\big(t(r), u(r), \xi(r)\big) &= \partial_t \mathcal{F}(t(r), u(r), \xi(r))\, t'(r) - |\partial_u^- \mathcal{F}(t(r), u(r), \xi(r))| \\ &= \partial_t \mathcal{E}(u(r), \xi(r))\, t'(r) - |\partial_u^- \mathcal{F}(t(r), u(r), \xi(r))|. \end{aligned}$$

Therefore, we deduce that

$$\partial_u \mathcal{F}\big(t(r), u(r), \xi(r); u'(r)\big) \leq -|\partial_u^- \mathcal{F}(t(r), u(r), \xi(r))|.$$

Let us now express the right-hand side as a supremum according to (3.11); then (being $-(a)_- \leq a$) we have

$$\begin{aligned} \partial_u \mathcal{F}\big(t(r), u(r), \xi(r); u'(r)\big) &\leq -\sup\Big\{\big[\partial_u \mathcal{F}\big(t(r), u(r), \xi(r); z\big)\big]_- : \ \ \|z\| \leq 1\Big\} \\ &= \inf\Big\{-\big[\partial_u \mathcal{F}\big(t(r), u(r), \xi(r); z\big)\big]_- : \ \ \|z\| \leq 1\Big\} \\ &\leq \inf\Big\{\partial_u \mathcal{F}\big(t(r), u(r), \xi(r); z\big) : \ \ \|z\| \leq 1\Big\}. \end{aligned}$$

Hence $u'(r) \in \operatorname{argmin}\{\partial_u \mathcal{F}(t(r), u(r), \xi(r); z) : \|z\| \leq 1\}$ and

$$\partial_u \mathcal{F}\big(t(r), u(r), \xi(r); u'(r)\big) = -|\partial_u^- \mathcal{F}(t(r), u(r), \xi(r))|.$$

Let $\mathcal{G}$ be the functional $\partial_u \mathcal{F}\big(t(r), u(r), \xi(r);\ \cdot\big)$ on $\mathcal{U}$; now we use the fact that the space $\mathcal{U}$ is equipped with the norm $\|u\| = (\int_{\Omega \setminus K} |\nabla u|^2\, dx)^{1/2}$ and denote by $\langle\, , \rangle$ the corresponding scalar product. $\mathcal{G}$ is convex, continuous and positively 1-homogeneous. Denote by B the closed unit ball in $\mathcal{U}$ and by I_B the indicator function of B. Since $u'(r)$ minimizes $\mathcal{G} + I_B$ on $\mathcal{U}$ we have

$$0 \in \partial\big(\mathcal{G} + I_B\big)(u'(r)), \tag{8.3}$$

where the right-hand side denotes the subdifferential of $\mathcal{G} + I_B$ in $u'(r)$. We know that (see, e.g., [12, Proposition 5.6]) that

$$\partial\big(\mathcal{G} + I_B\big)(u'(r)) = \partial\mathcal{G}(u'(r)) + \partial I_B(u'(r)), \qquad \partial I_B(z) = \begin{cases} \{0\} & \text{if } \|z\| < 1, \\ \{\lambda z : \ \lambda \geq 0\} & \text{if } \|z\| = 1. \end{cases}$$

Then, by (8.3) we deduce the existence of $\lambda(r) \geq 0$ such that

$$-\lambda(r) u'(r) \in \partial\mathcal{G}(u'(r))$$

(note that $\lambda(r) = 0$ if $\|u'(r)\| < 1$). Therefore, by the definition of subdifferential we have:

$$\mathcal{G}(z) \geq \mathcal{G}(u'(r)) - \lambda(r)\langle u'(r), z - u'(r)\rangle \qquad \text{for every } z \in \mathcal{U}.$$

If $\|u'(r)\| = 1$ then, by taking $z = 2u'(r)$ and $z = 0$, we get $\mathcal{G}(u'(r)) + \lambda(r) = 0$, so that

$$\lambda(r) = -\mathcal{G}(u'(r)) = |\partial_u^- \mathcal{F}(t(r), u(r), \xi(r))|,$$

and the previous inequality yields

$$\mathcal{G}(z) + \lambda(r)\langle u'(r), z\rangle \geq \mathcal{G}(u'(r)) + \lambda(r) = 0 \qquad \text{for every } z \in \mathcal{U}.$$

If $\|u'(r)\| < 1$ then $\lambda(r) = 0$ and, by the positive 1-homogeneity of $\mathcal{G}$, we have that the minimum value $\mathcal{G}(u'(r))$ of $\mathcal{G}$ is 0, too. In any case, we have proved that $\lambda(r) = |\partial_u^- \mathcal{F}(t(r), u(r), \xi(r))| \geq 0$ satisfies

$$\mathcal{G}(z) + \lambda(r)\langle u'(r), z\rangle \geq 0 \qquad \text{for every } z \in \mathcal{U}.$$

At this point, remembering that the duality above is in $H^1(\Omega \setminus K)$ endowed with the norm $\|u\| = (\int_{\Omega\setminus K} |\nabla u|^2\, dx)^{1/2}$, we can write the previous variational inequality as

$$\int_{\Omega\setminus K} \nabla\big(u(r) + g(t) + \lambda(r)u'(r)\big)\nabla z\, dx + \int_K \partial_w \psi([\![u]\!](r), \xi(r); [\![z]\!])\, d\mathcal{H}^1 \geq 0 \qquad \text{for every } z \in \mathcal{U}.$$

Defining $v(r) = (u(r) + g(t)) + \lambda(r)u'(r)$ and following step by step the proof of Theorem 7.8 we get the thesis. ■

Remark 8.2 The PDE characterizations of Theorem 7.8 and Theorem 8.1 distinguish between equilibrium configurations (in continuity points) and jump transitions (in discontinuity points) because the mechanical behaviour is different. However, it is possible to provide a unified mathematical characterization: the system of PDEs is indeed the same and the function $v(s) = (u(s) + g(t)) + \lambda(s)u'(s)$, appearing in Theorem 8.1, boils down to $v(s) = u(s) + g(t)$ when, under the assumptions of Theorem 7.8, $\lambda(s) = |\partial_u^- \mathcal{F}(t(s), u(s), \xi(s))| = 0$.

References

[1] S. Almi. Energy release rate and quasi-static evolution via vanishing viscosity in a fracture model depending on the crack opening. *ESAIM Control Optim. Calc. Var.*, to appear.

[2] M. Angelillo, E. Babilio, and A. Fortunato. Numerical solutions for crack growth based on the variational theory of fracture. *Comp. Mech.*, 50(3):285–301, 2012.

[3] M. Artina, F. Cagnetti, M. Fornasier, and F. Solombrino. Linearly constrained evolutions of critical points and an application to cohesive fractures. Technical report, 2015.

[4] A. Braides, G. Dal Maso, and A. Garroni. Variational formulation of softening phenomena in fracture mechanics: the one-dimensional case. *Arch. Ration. Mech. Anal.*, 146(1):23–58, 1999.

[5] H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[6] F. Cagnetti. A vanishing viscosity approach to fracture growth in a cohesive zone model with prescribed crack path. *Math. Models Methods Appl. Sci.*, 18(7):1027–1071, 2008.

[7] F. Cagnetti and R. Toader. Quasistatic crack evolution for a cohesive zone model with different response to loading and unloading: a Young measures approach. *ESAIM Control Optim. Calc. Var.*, 17(1):1–27, 2011.

[8] C. Comi, S. Mariani, M. Negri, and U. Perego. A 1d variational formulation for quasi-brittle fracture. *Journal of Mechanics of Materials and Structures*, 1(8):1323–1343, 2006.

[9] G. Dal Maso and C. Zanini. Quasi-static crack growth for a cohesive zone model with prescribed crack path. *Proc. Roy. Soc. Edinburgh Sect. A*, 137(2):253–279, 2007.

[10] G. Del Piero and L. Truskinovsky. Elastic bars with cohesive energy. *Contin. Mech. Thermodyn.*, 21(2):141–171, 2009.

[11] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[12] I. Ekeland and R. Temam. *Convex analysis and variational problems.* North-Holland Publishing Co., Amsterdam, 1976.

[13] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 69 of *Classics in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Reprint of the 1985 original [MR0775683], With a foreword by Susanne C. Brenner.

[14] A. Kufner, O. John, and S. Fučík. *Function spaces.* Noordhoff International Publishing, Leyden; Academia, Prague, 1977. Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis.

[15] C.J. Larsen and V. Slastikov. Dynamic cohesive fracture: models and analysis. *Math. Models Methods Appl. Sci.*, 24(9):1857–1875, 2014.

[16] F. Maddalena, D. Percivale, and F. Tomarelli. Debonding and collapse of glued thin structures, 2013.

[17] A. Mielke. Differential, energetic, and metric formulations for rate-independent processes. In *Nonlinear PDE's and applications*, volume 2028 of *Lecture Notes in Math.*, pages 87–170. Springer, Heidelberg, 2011.

[18] M. Negri. Quasi-static rate-independent evolutions: characterization, existence, approximation and application to fracture mechanics. *ESAIM Control Optim. Calc. Var.*, 20(4):983–1008, 2014.

[19] K. Pham and J.-J. Marigo. Stability of homogeneous states with gradient damage models: size effects and shape effects in the three-dimensional setting. *J. Elasticity*, 110(1):63–93, 2013.

[20] M. Ortiz and A. Pandolfi. A class of cohesive elements for the simulation of three-dimensional crack propagation. *Int. J. Numer. Methods Eng.*, 44:1267–1282, 1999.

[21] T. Roubiček, V. Mantič, and C. G. Panagiotopoulos. A quasistatic mixed-mode delamination model. *Discrete Contin. Dyn. Syst. Ser. S*, 6(2):591–610, 2013.

[22] R. Temam. *Navier-Stokes equations. Theory and numerical analysis.* North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Studies in Mathematics and its Applications, Vol. 2.

[23] C.V. Verhoosel, J.J.C. Remmers, and M.A. Gutiérrez. A dissipation-based arc-length method for robust simulation of brittle and ductile failure . *Int. J. Numer. Meth. Engng.*, 77(9):1290–1321, 2009.