

Multi-scale analysis by Γ -convergence of a
one-dimensional non-local functional related to a
shell-membrane transition

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Abstract

We study the asymptotic behavior of one-dimensional functionals associated to the energy of a thin nonlinear elastic spherical shell in the limit of vanishing thickness (proportional to a small parameter) ε and under the assumption of radial deformations. The functionals are characterized by the presence of a nonlocal potential term and defined on suitable weighted functional spaces. The transition shell-membrane is studied at three relevant different scales. For each of them we give a compactness result and compute the Γ -limit. In particular, we show that if the energies on a sequence of configurations scale as $\varepsilon^{3/2}$ then the limit configuration describes a (locally) finite number of transitions between the undeformed and the everted configurations of the shell. We also highlight a kind of 'Gibbs' phenomenon' by showing that non-trivial optimal sequences restricted between the undeformed and the everted configurations must have energy scaling at least as $\varepsilon^{4/3}$.

1 Introduction

It is well known that a spherical shell under zero loads can assume at least two configurations, the trivial reference configuration and the everted configuration, both of them being stable solutions of the equilibrium problem. The existence of everted shapes was first proved by Antman [1] for thick spherical shells. Later, the analysis concerning the eversion of thin shells has been carried out from the theoretical and numerical point of view by Podio-Guidugli *et al.* [20] and Geymonat *et al.* [15, 16].

The energy functional associated to the axially symmetric deformations of a spherical cap, without applied loads, may be written as

$$F_\varepsilon(u) = \varepsilon^2 \int_0^1 \theta^3 (u'(\theta))^2 d\theta + \int_0^1 \frac{1}{\theta^3} \left(\int_0^\theta \varphi^3 (u^2(\varphi) - 1) d\varphi \right)^2 d\theta.$$

The unknown u , function of the normalized polar angle θ , is related to the slope of the deformed middle surface of the cap with respect to the initial spherical shape and ε is the thinness parameter of the shell (for the derivation of the model and physical interpretation of the variable see the pioneering papers by Reiss [21], Bauer *et al.* [2] and [20]).

In [20, 15, 16], under suitable boundary conditions, it has been proved that beside the trivial stable solution $u = 1$, for ε small enough there exists a second stable solution namely the everted stressed solution. The sequence of everted configurations tends to an unstressed configuration ($u = -1$) that can be described as the reflection of the cap reference configuration. More recently, the problem has been further investigated. The existence of infinitely many stable solutions for the limit problem has been predicted and several numerical experiments have been proposed by Geymonat and Leger [14].

Although the problem has been carefully studied in [20, 15, 16] for ε small, the asymptotic analysis of F_ε has remained an open problem, and is the object of the present work, through the notation and techniques of variational calculus.

To describe the asymptotic behavior of F_ε we first focus on sequences (u_ε) such that $F_\varepsilon(u_\varepsilon) = O(1)$. In that case we prove that (u_ε) is locally weakly compact in $L^1(0, 1)$ and sequences giving the optimal lower bound may oscillate between the values -1 and 1 . This behavior is described by the Γ -limit F^0 (see next section for the precise form of the limit), that not only captures these oscillations but also shows that the non-local character of the functional is maintained in the limit. Minimizing sequences are responsible of folding effects also observed for flat membranes. The analytical reconstruction of the shell surface texture could allow both to understand the material elastic properties and to study the interactions between two surfaces.

It must be noted that the Γ -limit F^0 coincides with the lower-semicontinuous envelope of the functional

$$G(u) = \int_0^1 \theta^{-3} \left(\int_0^\theta (u^2 - 1) \varphi^3 d\varphi \right)^2 d\theta$$

with respect to the local weak L^1 -convergence and that minimizers of this functionals are all functions with $|u| = 1$ a.e. In terms of recovery sequences, we note that they may develop oscillations, but the occurrence of these is due to a non-local effect (see the example in Remark 2.5).

The minimum value for the Γ -limit F^0 is 0 and is achieved exactly on all functions u with $|u| \leq 1$. This large class of minimizers justifies the analysis at finer scales. We show that the next meaningful scale is when $F_\varepsilon(u_\varepsilon) = O(\varepsilon^{3/2})$. If this is the case then we show that such (u_ε) is strongly pre-compact in $L^1(0,1)$ and its limits u are locally piecewise constant in $(0,1)$, and $|u(\theta)| = 1$ a.e. We describe this behavior by showing that the Γ -limit of the scaled energies $\varepsilon^{-3/2}F_\varepsilon$ on those functions takes the form

$$F^{3/2}(u) = c_0 \sum_{\theta \in S(u)} \theta^3,$$

where we denote by $S(u)$ the set of discontinuities of u (see Theorem 3.3).

The mechanical interpretation of this energetic asymptotic description is that minimal states of the energies F_ε subjected to boundary conditions will be approximately described for ε small by a mixture of undeformed and everted configurations, the transitions between whom are such as to minimize the energy $F^{3/2}$. A similar analysis can be performed for suitably scaling forcing terms (in which case the corresponding forcing term must be added to $F^{3/2}$).

The formal analogies with the corresponding functionals of the gradient theory of phase transitions

$$H_\varepsilon(u) = \varepsilon^2 \int_0^1 \theta^3 (u'(\theta))^2 d\theta + \int_0^1 \theta^3 (u^2(\theta) - 1)^2 d\theta$$

must be noted. Upon a normalization factor, the functionals $\varepsilon^{-1}H_\varepsilon$ also Γ -converge to $F^{3/2}$. Apart from the different scaling ε^{-1} , this analogy does not carry to the details of the proof. Firstly, it must be noted that the compactness properties for functions with $F_\varepsilon(u_\varepsilon) = O(\varepsilon^{3/2})$ are much more difficult to prove by the cancellations that may occur in the integral $\int_0^\theta \varphi^3 (u_\varepsilon^2 - 1) d\varphi$ owing to the fact that $u_\varepsilon^2 - 1$ may change sign. Secondly, the way the constant c_0 is computed involves some optimal transitions that exhibit a sort of *Gibbs' phenomenon*: even though their limit takes only the value ± 1 these transitions must take values external to the interval $[-1, 1]$.

In a final section, we show that this Gibbs' phenomenon is substantial: if we impose the constraint $|u_\varepsilon| \leq 1$ to a sequence (u_ε) converging to u , then not only the values $\varepsilon^{-3/2}F_\varepsilon(u_\varepsilon)$ cannot converge to the value $F^{3/2}(u)$, but they must even diverge. We show that with this additional constraint the correct scaling is $\varepsilon^{-4/3}$. The scaled energies still converge to a phase-transition functional, but this time of a non-local form (see Theorem 4.3).

We believe that the techniques developed here can be adapted to other transition problems in nonlinear elasticity leading to the study of functionals with similar nonlocal terms. Moreover, our analysis can be generalized

to different weighted spaces. For these reasons we report our results with reference to a more general weighted functional (details in the next section).

Finally, we point out that functionals F_ε are derived from the scaled energy

$$\frac{1}{\varepsilon} \int_{\mathcal{C}_\varepsilon} (\mu |\mathbf{D}|^2 + \frac{\lambda}{2} (\text{trace } \mathbf{D})^2) dx,$$

where \mathcal{C}_ε parameterizes a thin spherical shell of thickness ε , $\mathbf{D} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \frac{1}{2}(\nabla \mathbf{u} \nabla \mathbf{u}^T)$ is the *nonlinear deformation tensor* related to the deformation \mathbf{u} of the shell, and λ, μ are the Lamé constants (we refer *e.g.* to [20] for the precise derivation). In this way, our paper may be partly related to recent works on dimension-reduction for thin structures by the use of Γ -convergence (see *e.g.* Le Dret and Raoult [17] for the limit analysis of thin shells, Friesecke *et al.* [13] for the analysis under various scaling, Ben Belgacem *et al.* [3] for complex patterns in recovery sequences, Braides *et al.* [9] for an example of application of the localization methods of Γ -convergence to thin structures, etc.).

1.1 Analytical description of the results

We conclude this introduction with a brief analytical description of the results in a more general setting, where we replace the weight θ^3 with a more general weight $\rho(\theta)$. We use the techniques of Γ -convergence for the asymptotic analysis of our functionals, for whose definition, notation and a comprehensive study of Γ -convergence we refer to [6] and [11] (see also [7], [8] Part 2).

Let $\rho : [0, 1] \mapsto \mathbf{R}$ be a non-decreasing, continuous function, strictly positive on $(0, 1]$. For all $\varepsilon > 0$ and $\alpha \geq 0$ we define

$$F_\varepsilon^\alpha(u) = \varepsilon^{2-\alpha} \int_0^1 \rho(u')^2 d\theta + \varepsilon^{-\alpha} \int_0^1 \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u^2 - 1) d\varphi \right)^2 d\theta$$

for $u \in H^1(0, 1)$. When needed, the functionals are understood to take the value $+\infty$ where not otherwise defined. We will isolate particular values of α for which the Γ -limit is non-trivial.

We will separately consider the following cases:

(1) (Section 2) $\alpha = 0$. In this case minimizing sequences are weakly pre-compact in $L_{\text{loc}}^2(0, 1)$; hence, we compute the Γ -limit F^0 of F_ε^0 with respect to that convergence and for every $u \in L_{\text{loc}}^2(0, 1)$ we get

$$F^0(u) = \min \left\{ \int_0^1 \frac{1}{\rho(\theta)} \left(\mu([0, \theta]) - \int_0^\theta \rho d\varphi \right)^2 d\theta : \mu \geq \rho u^2 d\varphi \right\},$$

where the minimum is taken over all non-negative measures μ . The set of the minimum points of F^0 is $\{|u| \leq 1\}$.

(2) (Section 3) $\alpha = 3/2$. In this case we scale F_ε^0 further, and study the limit of $F_\varepsilon^{3/2} = \varepsilon^{-3/2} F_\varepsilon^0$. We prove that minimizing sequences are pre-compact with respect to the strong $L_{\text{loc}}^1(0, 1)$ -convergence, and their limits u

belong to $BV_{\text{loc}}((0, 1); \{-1, 1\})$; *i.e.*, u is locally piecewise constant on $(0, 1)$ and it only takes the values 1 and -1 . We compute the Γ -limit $F^{3/2}$ of $F_\varepsilon^{3/2}$ with respect to that convergence and we get

$$F^{3/2}(u) = c_0 \sum_{\theta \in S(u)} \rho(\theta)$$

for every $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$, where $S(u)$ is the set of points where u jumps between the points 1 and -1 , and

$$c_0 = \inf_{T>0} \inf \left\{ \int_{-T}^T (v')^2 ds + \int_{-T}^T \left(\int_{-T}^\sigma (v^2 - 1) ds \right)^2 d\sigma : \right. \\ \left. v \in H^1(-T, T), \quad v(\pm T) = \pm 1, \quad \int_{-T}^T (v^2 - 1) ds = 0 \right\}.$$

With this choice of the scaling we get a result of ‘Modica-Mortola’ type with a different characterization of the constant c_0 (see [18], [19] or [5]).

(3) (Section 4) In this section, we show that the characteristic scale changes if we impose the restriction that $u \in H^1((0, 1); [-1, 1])$, and that in that case the correct scaling power is $\alpha = 4/3$. We only treat the case $\rho = 1$ for the sake of simplicity. By (1) above, we have that the Γ -limit of the restriction of F_ε^0 to $H^1((0, 1); [-1, 1])$, G_ε^0 , is zero. We then rescale G_ε^0 to get a non trivial limit problem, considering the family of functionals $G_\varepsilon^{4/3} = \varepsilon^{-4/3} G_\varepsilon^0$ and we prove that the minimizing sequences are compact with respect to the strong L^1 -convergence and its Γ -limit is nonlocal

$$G^{4/3}(u) = \inf_{T>0} \inf \left\{ \sum_{i \in I} \left(\int_{-T}^T (v'_i)^2 ds + (\theta_{i+1} - \theta_i) \sum_{j=1}^i \int_{-T}^T (v_j^2 - 1) ds \right)^2 : \right. \\ \left. v_i \in H^1((-T, T); [-1, 1]), \quad v_i(\pm T) = \pm 1 \right\},$$

for every $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$, where we have labeled the points in $S(u)$ by a set of indices $I \subset \mathbf{N}$ in such a way that $\theta_i < \theta_{i+1}$.

2 The case $\alpha = 0$: oscillations

We consider the case $\alpha = 0$ first; *i.e.*,

$$F_\varepsilon^0(u) = \varepsilon^2 \int_0^1 \rho(u')^2 d\theta + \int_0^1 \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u^2 - 1) d\varphi \right)^2 d\theta$$

for $u \in H_{\text{loc}}^1(0, 1)$. In order to choose the topology in which to frame our limit problem we have to examine the compactness properties of sequences with bounded energy. Note that the presence of ε in the first term of $F_\varepsilon^0(u)$ only, suggests the use of the weak L^2 -convergence.

Theorem 2.1 (Compactness) *Let (u_ε) be a sequence such that $\sup_\varepsilon F_\varepsilon^0(u_\varepsilon) < +\infty$, then, up to subsequences, the measures $\rho u_\varepsilon^2 d\varphi$ converge weakly* in $(-\infty, 1)$, and in particular (u_ε) converges weakly in $L_{\text{loc}}^2(0, 1)$.*

PROOF. By assumption $\sup_\varepsilon \int_0^1 (1/\rho(\theta)) \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta < +\infty$, which implies that $\sup_\varepsilon \int_0^1 \left(\int_0^\theta \rho u_\varepsilon^2 d\varphi \right)^2 d\theta < +\infty$; hence, by Hölder's inequality, there exists a constant c independent of ε such that

$$\int_0^1 \left(\int_0^\theta \rho u_\varepsilon^2 d\varphi \right) d\theta \leq c \quad (2.1)$$

for every $\varepsilon > 0$. By the monotonicity of $\theta \mapsto \int_0^\theta \rho u_\varepsilon^2 d\varphi$ we have that for a fixed $\theta_0 \in (0, 1)$

$$\int_0^{\theta_0} \rho u_\varepsilon^2 d\varphi \leq \int_0^\theta \rho u_\varepsilon^2 d\varphi$$

for every $\theta \geq \theta_0$. Hence we get that

$$\int_{\theta_0}^1 \left(\int_0^{\theta_0} \rho u_\varepsilon^2 d\varphi \right) d\theta \leq \int_{\theta_0}^1 \left(\int_0^\theta \rho u_\varepsilon^2 d\varphi \right) d\theta \leq \int_0^1 \left(\int_0^\theta \rho u_\varepsilon^2 d\varphi \right) d\theta$$

and, by (2.1), we can conclude that for every $\theta_0 \in (0, 1)$ there exists a constant $c(\theta_0)$ depending only on θ_0 such that $\int_0^{\theta_0} \rho u_\varepsilon^2 d\varphi \leq c(\theta_0)$ for every $\varepsilon > 0$, which gives the desired compactness properties. \square

Remark 2.2 The following example shows that we cannot expect weak compactness, but only local weak compactness, in $L^2(0, 1)$ for a sequence (u_ε) with $\sup_\varepsilon F_\varepsilon^0(u_\varepsilon) < +\infty$. In fact, consider

$$u_\varepsilon(\theta) = \varepsilon^{-7/5}(\theta - \theta_\varepsilon)^+$$

with $\theta_\varepsilon = 1 - \varepsilon^{4/5}$, then $F_\varepsilon^0(u_\varepsilon) \leq c$ for every $\varepsilon > 0$ but

$$\int_0^1 u_\varepsilon^2 d\theta = \frac{1}{3}\varepsilon^{-2/5}.$$

Lemma 2.3 *Let*

$$G(u) = \int_0^1 \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u^2 - 1) d\varphi \right)^2 d\theta.$$

The lower semicontinuous envelope of G with respect to the weak L_{loc}^2 topology is

$$\bar{G}(u) := \min \left\{ \int_0^1 \frac{1}{\rho(\theta)} \left(\mu([0, \theta]) - \int_0^\theta \rho d\varphi \right)^2 d\theta : \mu \geq \rho u^2 d\varphi \right\},$$

where the minimum is taken in the space $\mathcal{M}^+([0, 1])$ of locally finite positive measures on $[0, 1)$.

PROOF. Let (u_N) be a sequence weakly converging to u in L^2_{loc} such that $\lim_{N \rightarrow +\infty} G(u_N) < +\infty$ and the sequence of positive measures $(\rho u_N^2 d\varphi)$ converges weakly* to μ in $\mathcal{M}^+([0, 1])$. Then, by the lower semicontinuity of the L^2 -norm and the convergence of the measures of almost all intervals, we have that

$$\frac{\mu(\theta - \delta, \theta + \delta)}{\delta} = \lim_{N \rightarrow +\infty} \int_{\theta - \delta}^{\theta + \delta} \rho u_N^2 d\varphi \geq \int_{\theta - \delta}^{\theta + \delta} \rho u^2 d\varphi$$

for a.a. $\delta \in (0, \theta)$; by the Besicovitch Derivation Theorem, we conclude that for almost every θ

$$\frac{d\mu}{d\varphi}(\theta) \geq \rho(\theta)u^2(\theta).$$

Since μ has at most countably many atoms, by the weak*-convergence in $\mathcal{M}^+([0, 1])$, we have that

$$\lim_{N \rightarrow +\infty} \int_0^\theta \rho u_N^2 d\varphi = \mu([0, \theta)) \quad \text{for a.e. } \theta \in (0, 1). \quad (2.2)$$

It follows that

$$\frac{1}{\sqrt{\rho(\theta)}} \left(\int_0^\theta \rho (u_N^2 - 1) d\varphi \right) \xrightarrow{L^2} \frac{1}{\sqrt{\rho(\theta)}} \left(\mu([0, \theta)) - \int_0^\theta \rho d\varphi \right)$$

and by the weak lower semicontinuity of the L^2 -norm we get

$$\lim_{N \rightarrow +\infty} G(u_N) \geq \overline{G}(u).$$

Note that the functional

$$\mu \mapsto \int_0^1 \frac{1}{\rho(\theta)} \left(\mu([0, \theta)) - \int_0^\theta \rho d\varphi \right)^2 d\theta$$

is weakly lower semicontinuous and coercive in $\mathcal{M}^+([0, 1])$. Moreover, the set $\{\mu \geq \rho u^2 d\varphi\}$ is convex; hence, the minimum is attained.

We now check the limsup inequality for every $u \in L^2_{\text{loc}}(0, 1)$ such that $\overline{G}(u) < +\infty$. Let $\mu \in \mathcal{M}^+([0, 1])$ be such that

$$\overline{G}(u) = \int_0^1 \frac{1}{\rho(\theta)} \left(\mu([0, \theta)) - \int_0^\theta \rho d\varphi \right)^2 d\theta.$$

For $0 < a < 1$ we define

$$u^a(\theta) = \begin{cases} u(\theta) & a \leq \theta \leq 1 - a \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mu_N^a([0, \theta)) = \begin{cases} 0 & \text{if } \theta < a - 1/N \\ \mu([0, a)) & \text{if } a - 1/N \leq \theta < a \\ \mu([0, \theta)) & \text{if } a \leq \theta < 1 - a \\ \mu([0, 1 - a)) & \text{if } 1 - a \leq \theta < 1 \end{cases}$$

for $N \in \mathbf{N}$ and $N > 1$. Then, $u^a \in L^2(0, 1)$, $u^a \rightarrow u$ in $L^2_{\text{loc}}(0, 1)$ as $a \rightarrow 0^+$, μ_N^a converges to the measure μ^a defined by

$$\mu^a([0, \theta]) = \begin{cases} 0 & \text{if } \theta < a \\ \mu([0, \theta]) & \text{if } a \leq \theta < 1 - a \\ \mu([0, 1 - a]) & \text{if } 1 - a \leq \theta < 1, \end{cases}$$

(this can be checked, e.g., by using Theorem 1.16 in [12]) and

$$\overline{G}(u^a) \leq \int_a^1 \frac{1}{\rho(\theta)} \left(\mu^a([0, \theta]) - \int_0^\theta \rho \, d\varphi \right)^2 d\theta + o(1) \leq \overline{G}(u) + o(1) \quad (2.3)$$

as $a \rightarrow 0^+$. Indeed, since by assumption $\overline{G}(u) < +\infty$, then we have

$$\begin{aligned} & \int_a^1 \frac{1}{\rho(\theta)} \left(\mu^a([0, \theta]) - \int_0^\theta \rho \, d\varphi \right)^2 d\theta \\ &= \int_a^{1-a} \frac{1}{\rho(\theta)} \left(\mu([0, \theta]) - \int_0^\theta \rho \, d\varphi \right)^2 d\theta \\ & \quad + \int_{1-a}^1 \frac{1}{\rho(\theta)} \left(\mu([0, 1-a]) - \int_0^\theta \rho \, d\varphi \right)^2 d\theta \\ &\leq \overline{G}(u) + 2 \int_{1-a}^1 \frac{\mu([1-a, \theta])^2}{\rho(\theta)} d\theta \\ & \quad + 2 \int_{1-a}^1 \frac{1}{\rho(\theta)} \left(\mu([0, \theta]) - \int_0^\theta \rho \, d\varphi \right)^2 d\theta \\ &= \overline{G}(u) + o(1), \end{aligned}$$

as $a \rightarrow 0^+$.

We denote by $I_N = \{0, \dots, N-1\}$ and by

$$\begin{cases} \bar{u}_N(\theta) = \int_{i/N}^{(i+1)/N} u^a \, d\varphi, & i/N \leq \theta \leq (i+1)/N, \quad i \in I_N \\ v_N(\theta) = N \mu_N^a([i/N, (i+1)/N]), & i/N \leq \theta \leq (i+1)/N, \quad i \in I_N. \end{cases}$$

Note that, in particular

$$\bar{u}_N(\theta) = 0, \quad 0 \leq \theta \leq [Na]/N$$

and

$$v_N(\theta) = \begin{cases} 0 & 0 \leq \theta \leq ([Na] - 1)/N \\ N\mu([0, a]) & ([Na] - 1)/N \leq \theta \leq [Na]/N. \end{cases}$$

Hence, we define

$$u_N(\theta) = \begin{cases} 0 & 0 \leq \theta \leq ([Na] - 1)/N \\ \sqrt{N\mu([0, a])/\rho(\theta)} & ([Na] - 1)/N \leq \theta \leq [Na]/N \\ \bar{u}_N + \sqrt{(v_N/\rho_N^i) - (\bar{u}_N)^2} & i/N \leq \theta \leq i/N + 1/2N, \\ & i = [Na] \cdots, N - 1 \\ c_N \left(\bar{u}_N - \sqrt{(v_N/\rho_N^i) - (\bar{u}_N)^2} \right) & i/N + 1/2N \leq \theta \leq (i + 1)/N, \\ & i = [Na] \cdots, N - 1 \end{cases} \quad (2.4)$$

where

$$c_N(\theta) = \rho(\theta - 1/2N)/\rho(\theta), \quad [Na]/N + 1/2N \leq \theta \leq 1$$

($[t]$ denotes the *integer part* of t). Finally, it remains to fix ρ_N^i for $i = [Na], \dots, N - 1$ such that

$$v_N(\theta) \geq \rho_N^i (\bar{u}_N(\theta))^2, \quad i/N \leq \theta \leq (i + 1)/N, \quad i = [Na] \cdots, N - 1.$$

Since $\mu_N^\alpha \geq \rho(u^\alpha)^2 d\varphi$, for every $i = [Na], \dots, N - 1$, there exists $\eta_N^i \in [i/N, (i + 1)/N]$ such that

$$\begin{aligned} \mu_N^\alpha[i/N, (i + 1)/N] &\geq \int_{i/N}^{(i+1)/N} \rho(u^\alpha)^2 d\varphi = \rho(\eta_N^i) \int_{i/N}^{(i+1)/N} (u^\alpha)^2 d\varphi \\ &\geq \rho(\eta_N^i) N \left(\int_{i/N}^{(i+1)/N} u^\alpha d\varphi \right)^2 = \rho(\eta_N^i) \frac{1}{N} (\bar{u}_N)^2. \end{aligned}$$

Hence, we choose $\rho_N^i = \rho(\eta_N^i)$ in (2.4).

By definition (u_N) is bounded in L^2 ; hence, up to subsequence, it converges weakly in L^2 . To identify the weak limit function with u^α it is sufficient to check that $\lim_{N \rightarrow +\infty} \int_b^d u_N = \int_b^d u^\alpha$, for every $(b, d) \subseteq (0, 1)$. In fact,

$$\int_0^{[Na]/N} u_N d\theta = \sqrt{N\mu([0, a])} \int_{([Na]-1)/N}^{[Na]/N} \frac{d\theta}{\sqrt{\rho(\theta)}} \leq \frac{1}{\sqrt{N}} \sqrt{\frac{\mu([0, a])}{\rho(([Na] - 1)/N)}}; \quad (2.5)$$

while, for $i = [Na], \dots, N - 1$, we have that

$$\begin{aligned} \int_{i/N}^{(i+1)/N} u_N d\theta &= \int_{i/N}^{i/N+1/2N} \bar{u}_N + \sqrt{(v_N/\rho_N^i) - (\bar{u}_N)^2} d\theta \\ &\quad + \int_{i/N+1/2N}^{(i+1)/N} \frac{\rho(\theta - \frac{1}{2N})}{\rho(\theta)} \left(\bar{u}_N - \sqrt{(v_N/\rho_N^i) - (\bar{u}_N)^2} \right) d\theta \\ &= \int_{i/N}^{(i+1)/N} \bar{u}_N d\theta + \int_{i/N+1/2N}^{(i+1)/N} \bar{u}_N \left(\frac{\rho(\theta - \frac{1}{2N})}{\rho(\theta)} - 1 \right) d\theta \\ &\quad + \int_{i/N+1/2N}^{(i+1)/N} \sqrt{(v_N/\rho_N^i) - (\bar{u}_N)^2} \left(1 - \frac{\rho(\theta - \frac{1}{2N})}{\rho(\theta)} \right) d\theta \end{aligned} \quad (2.6)$$

$$\begin{aligned}
&= \left(\int_{i/N}^{(i+1)/N} u^a d\theta \right) + \int_{i/N+1/2N}^{(i+1)/N} \bar{u}_N \left(\frac{\rho(\theta - \frac{1}{2N})}{\rho(\theta)} - 1 \right) d\theta \\
&\quad + \int_{i/N+1/2N}^{(i+1)/N} \sqrt{(v_N/\rho_N^i) - (\bar{u}_N)^2} \left(1 - \frac{\rho(\theta - \frac{1}{2N})}{\rho(\theta)} \right) d\theta.
\end{aligned}$$

If we sum up on i , by Hölder inequality, we get that

$$\begin{aligned}
&\left| \sum_{i \geq [Na]} \int_{i/N+1/2N}^{(i+1)/N} \bar{u}_N \left(\frac{\rho(\theta - \frac{1}{2N})}{\rho(\theta)} - 1 \right) d\theta \right| \\
&\leq \left(\int_0^1 (\bar{u}_N)^2 d\theta \right)^{1/2} \left(\int_{a/2}^1 \left(1 - \frac{\rho(\theta - \frac{1}{2N})}{\rho(\theta)} \right)^2 d\theta \right)^{1/2} \quad (2.7)
\end{aligned}$$

and

$$\begin{aligned}
&\left| \sum_{i \geq [Na]} \int_{i/N+1/2N}^{(i+1)/N} \sqrt{(v_N/\rho_N^i) - (\bar{u}_N)^2} \left(1 - \frac{\rho(\theta - \frac{1}{2N})}{\rho(\theta)} \right) d\theta \right| \\
&\leq \left(\sum_{i \geq [Na]} \int_{i/N+1/2N}^{(i+1)/N} (v_N/\rho_N^i) + (\bar{u}_N)^2 d\theta \right)^{1/2} \left(\int_{a/2}^1 \left(1 - \frac{\rho(\theta - \frac{1}{2N})}{\rho(\theta)} \right)^2 d\theta \right)^{1/2} \\
&\leq \left(\frac{\mu([0, 1-a])}{\rho(a/2)} + \int_0^1 (\bar{u}_N)^2 d\theta \right)^{1/2} \left(\int_{a/2}^1 \left(1 - \frac{\rho(\theta - \frac{1}{2N})}{\rho(\theta)} \right)^2 d\theta \right)^{1/2}. \quad (2.8)
\end{aligned}$$

Note that the sequence (\bar{u}_N) is bounded in the L^2 since it converges to u^a strongly; moreover,

$$\lim_{N \rightarrow +\infty} \int_{a/2}^1 \left(1 - \frac{\rho(\theta - \frac{1}{2N})}{\rho(\theta)} \right)^2 d\theta = 0.$$

Hence, by (2.5), (2.6), (2.7) and (2.8), we can easily conclude that $\lim_{N \rightarrow +\infty} \int_b^d u_N = \int_b^d u^a$, for every $(b, d) \subseteq (0, 1)$ and, therefore, the weak convergence of (u_N) to u^a in $L^2(0, 1)$.

We now examine

$$\begin{aligned}
G(u_N) &= \int_0^a \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho u_N^2 - \int_0^\theta \rho d\varphi \right)^2 d\theta \\
&\quad + \int_a^1 \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho u_N^2 - \int_0^\theta \rho d\varphi \right)^2 d\theta; \quad (2.9)
\end{aligned}$$

in particular, we have that

$$\begin{aligned}
&\int_0^{[Na]/N} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho u_N^2 d\varphi \right)^2 d\theta \\
&= N^2 \mu([0, a])^2 \int_{([Na]-1)/N}^{[Na]/N} \frac{1}{\rho(\theta)} \left(\theta - \frac{[Na]-1}{N} \right)^2 d\theta \\
&\leq \frac{1}{3N} \frac{\mu([0, a])^2}{\rho(([Na]-1)/N)}. \quad (2.10)
\end{aligned}$$

Then, passing to the limit as $N \rightarrow +\infty$ in (2.10), we get that

$$\limsup_{N \rightarrow +\infty} \int_0^a \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho u_N^2 - \int_0^\theta \rho d\varphi \right)^2 d\theta = o(1), \quad (2.11)$$

as $a \rightarrow 0^+$. It remains to study the second term in (2.9). For $i = [Na], \dots, N-1$ we have that

$$\begin{aligned} & \int_{i/N}^{(i+1)/N} \rho u_N^2 d\theta \\ &= \int_{i/N}^{i/N+1/2N} \rho \left(\frac{v_N}{\rho_N^i} + 2\bar{u}_N \sqrt{(v_N/\rho_N^i) - (\bar{u}_N)^2} \right) d\theta \\ & \quad + \int_{i/N+1/2N}^{(i+1)/N} \rho \left(\theta - \frac{1}{2N} \right) \left(\frac{v_N}{\rho_N^i} - 2\bar{u}_N \sqrt{(v_N/\rho_N^i) - (\bar{u}_N)^2} \right) d\theta \\ &= 2 \int_{i/N}^{i/N+1/2N} \frac{\rho(\theta)}{\rho(\eta_N^i)} v_N(\theta) d\theta, \end{aligned} \quad (2.12)$$

where $\eta_N^i \in [i/N, (i+1)/N]$, $i = [Na], \dots, N-1$. We recall that $\rho u_N^2 = 0$ in $[0, ([Na]-1)/N] \cup [([N(1-a)]+1)/N, 1]$ while $\rho u_N^2 = N\mu([0, a])$ in $[([Na]-1)/N, [Na]/N]$; hence, by (2.12), we have that

$$\lim_{N \rightarrow +\infty} \int_0^\theta \rho u_N^2 d\theta = \mu^a([0, \theta]) \quad \text{a.e. } \theta \in (a, 1).$$

Moreover, as already observed, (u_N) is bounded in L^2 ; hence,

$$\frac{1}{\rho(\theta)} \left(\int_0^\theta \rho u_N^2 d\varphi - \int_0^\theta \rho d\varphi \right)^2 \leq c$$

for every $\theta \in (a, 1)$ and for every N . Therefore, we can apply the Lebesgue's Theorem in the second term of (2.9) and by (2.11), (2.3) we have

$$\begin{aligned} \limsup_{N \rightarrow +\infty} G(u_N) &\leq \int_a^1 \frac{1}{\rho(\theta)} \left(\mu^a([0, \theta]) - \int_0^\theta \rho d\varphi \right)^2 d\theta + o(1) \\ &\leq \overline{G}(u) + o(1), \quad \text{as } a \rightarrow 0^+. \end{aligned} \quad (2.13)$$

Since $u^a \rightarrow u$ in $L_{\text{loc}}^2(0, 1)$, as $a \rightarrow 0^+$, we can conclude that, passing to a further subsequence, $u_N \rightarrow u$ in $L_{\text{loc}}^2(0, 1)$ as $N \rightarrow +\infty$ and

$$\limsup_{N \rightarrow +\infty} G(u_N) \leq \overline{G}(u)$$

as desired. \square

Remark 2.4 Note that the set of minimizers for $\overline{G}(u)$ is $\{|u| \leq 1\}$.

Remark 2.5 The functional \overline{G} can be estimated from above and from below as follows

$$G^-(u) \leq \overline{G}(u) \leq G^+(u) \quad (2.14)$$

where

$$G^+(u) = \int_0^1 \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u^2 - 1)^+ d\varphi \right)^2 d\theta,$$

and

$$G^-(u) = \int_0^1 \frac{1}{\rho(\theta)} \left(\left(\int_0^\theta \rho(u^2 - 1) d\varphi \right)^+ \right)^2 d\theta.$$

The second inequality in (2.14) easily follows by testing the definition of $\overline{G}(u)$ with the measure $\mu = \rho(u^2 \vee 1) d\varphi$. To check the first inequality from below we preliminary note that $G(u) \geq G^-(u)$ and, by Fatou's lemma and the weak L^2_{loc} -lower semicontinuity of the functional $\int_0^\theta \rho u^2 d\varphi$ for every $\theta \in (0, 1)$, we get that G^- is weakly L^2_{loc} lower semicontinuous. Hence, by Lemma 2.3 and the lower semicontinuity of G^- we have that for every $u \in L^2_{\text{loc}}(0, 1)$ there exists a sequence u_N weakly L^2_{loc} converging to u such that

$$\overline{G}(u) = \lim_{N \rightarrow +\infty} G(u_N) \geq \liminf_{N \rightarrow +\infty} G^-(u_N) \geq G^-(u).$$

We show now an example of function u such that $\overline{G}(u) = G^-(u)$. Let us consider for simplicity $\rho = 1$. Let

$$u(\varphi) = \begin{cases} \sqrt{2} & \varphi \in (0, 1/4) \\ 0 & \varphi \in (1/4, 1). \end{cases} \quad (2.15)$$

Note that

$$\int_0^\theta (u^2 - 1) d\varphi < 0 \quad \text{for all } \theta > 1/2,$$

so that

$$G^-(u) = \int_0^{1/4} \theta^2 d\theta + \int_{1/4}^{1/2} \left(\frac{1}{2} - \theta \right)^2 d\theta = \int_0^1 (\mu([0, \theta]) - \theta)^2 d\theta,$$

where the measure $\mu = v d\varphi$ is defined by

$$v(\varphi) = \begin{cases} 2 & \varphi \in (0, 1/4) \\ 0 & \varphi \in (1/4, 1/2) \\ 1 & \varphi \in (1/2, 1). \end{cases}$$

Since $\mu \geq u^2 d\varphi$ we can test the definition of $\overline{G}(u)$ with μ getting $G^-(u) \geq \overline{G}(u)$, and then we have that $G^-(u) = \overline{G}(u)$ by (2.14). A recovery sequence (corresponding to (2.4) with $a = 0$ and $\rho = 1$) for $\overline{G}(u)$ is shown in Fig. 1, it highlights the non-local nature of the oscillations that start at $\varphi = 1/2$ while the target function is 0 on the whole $(1/4, 1)$.

Note that u as in (2.15) is also an example of function such that $\overline{G}(u) < G^+(u)$.

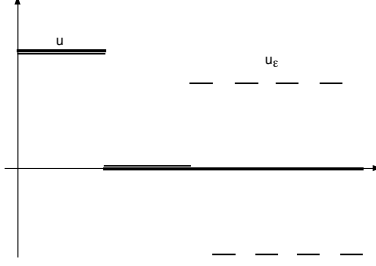


Figure 1: non-local oscillations

Finally, we note that also the inequality $G^- \leq \overline{G}$ is sharp: if we consider

$$u(\varphi) = \begin{cases} 0 & \varphi \in (0, 1/2) \\ \sqrt{2} & \varphi \in (1/2, 1) \end{cases}$$

we have that $G^-(u) = 0$ while $\overline{G}(u) > 0$ by Remark 2.4.

Theorem 2.6 (Γ -convergence result) *We have*

$$\Gamma(\text{w-}L^2_{\text{loc}})\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon^0(u) = \overline{G}(u)$$

for every $u \in L^2_{\text{loc}}(0, 1)$.

PROOF. Let $u \in L^2_{\text{loc}}(0, 1)$. By definition of \overline{G} for every sequence (u_ε) L^2_{loc} -weakly converging to u we have

$$F_\varepsilon^0(u_\varepsilon) \geq G(u_\varepsilon) \geq \overline{G}(u_\varepsilon);$$

hence by the weak lower semicontinuity of \overline{G} we get the liminf inequality.

Conversely, let $u \in L^2_{\text{loc}}(0, 1)$ and let

$$u^a(\theta) = \begin{cases} u(\theta) & \text{if } a \leq \theta \leq 1 - a \\ 0 & \text{otherwise} \end{cases}$$

with $0 < a < 1$; hence, $u^a \in L^2(0, 1)$ and $u^a \rightarrow u$ in $L^2_{\text{loc}}(0, 1)$ as $a \rightarrow 0^+$. By Lemma 2.3, there exists a sequence $(u_N) \in L^2(\mathbf{R})$ weakly converging to u^a in $L^2(0, 1)$ such that

$$\overline{G}(u^a) = \lim_{N \rightarrow +\infty} G(u_N).$$

Let $\eta : \mathbf{R} \mapsto [0, +\infty)$ be a mollifier, we define $\eta_\varepsilon(\theta) = \frac{1}{\sqrt{\varepsilon}} \eta(\frac{\theta}{\sqrt{\varepsilon}})$ then $u_\varepsilon^N = u_N * \eta_\varepsilon \in C_c^\infty(\mathbf{R})$ and $u_\varepsilon^N \rightarrow u_N$ in $L^2(0, 1)$ as $\varepsilon \rightarrow 0$ for every N . Hence,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} F_\varepsilon^0(u_\varepsilon^N) &= \lim_{\varepsilon \rightarrow 0} \left(\varepsilon^2 \int_0^1 \rho(u_N * \eta'_\varepsilon)^2 d\theta + G(u_\varepsilon^N) \right) \\ &= \lim_{\varepsilon \rightarrow 0} G(u_\varepsilon^N) = G(u_N) \end{aligned}$$

and, by the lower semicontinuity of the Γ -limsup and (2.3), we have that

$$\begin{aligned} \Gamma\text{-lim sup}_{\varepsilon \rightarrow 0} F_\varepsilon^0(u^a) &\leq \liminf_{N \rightarrow +\infty} \left(\Gamma\text{-lim sup}_{\varepsilon \rightarrow 0} F_\varepsilon^0(u_N) \right) \\ &\leq \liminf_{N \rightarrow +\infty} G(u_N) = \overline{G}(u^a) \leq \overline{G}(u) + o(1) \end{aligned}$$

as $a \rightarrow 0^+$. We again use the lower semicontinuity of the Γ -limsup to get, as $a \rightarrow 0^+$, that

$$\Gamma\text{-lim sup}_{\varepsilon \rightarrow 0} F_\varepsilon^0(u) \leq \overline{G}(u)$$

which concludes the proof of the limsup inequality. (See [6] Remark 1.29). \square

3 The case $\alpha = 3/2$: phase transitions

In Section 2 we have shown that the set of the minimum points of the Γ -limit F^0 is $\{|u| \leq 1\}$ and $\min F^0 = 0$. To reduce the choice in the minimizers of the limit problem we may further rescale F_ε^0 ; the next meaningful scaling is $\alpha = 3/2$. We then consider the following family of functionals

$$F_\varepsilon^{3/2}(u) = \sqrt{\varepsilon} \int_0^1 \rho(u')^2 d\theta + \varepsilon^{-3/2} \int_0^1 \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u^2 - 1) d\varphi \right)^2 d\theta$$

for $u \in H^1(0, 1)$.

Theorem 3.1 (Compactness) *Let (u_ε) be a sequence of equi-bounded energy; i.e., $\sup_\varepsilon F_\varepsilon^{3/2}(u_\varepsilon) < +\infty$, then (u_ε) is equi-bounded in $L_{\text{loc}}^\infty(0, 1)$ and, up to subsequences, (u_ε) converges to $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$ strongly in L_{loc}^1 .*

PROOF. Let $\eta_\varepsilon^\pm \neq \pm 1$ such that $-1 < \eta_\varepsilon^+, \eta_\varepsilon^- < 1$ or $\eta_\varepsilon^+, \eta_\varepsilon^- > 1$ or $\eta_\varepsilon^+, \eta_\varepsilon^- < -1$. We denote by $(\delta_\varepsilon^-, \delta_\varepsilon^+)$ an interval such that $u_\varepsilon(\delta_\varepsilon^-) = \eta_\varepsilon^-$, $u_\varepsilon(\delta_\varepsilon^+) = \eta_\varepsilon^+$ and u_ε takes values between η_ε^- and η_ε^+ . We use in the sequel the notation $\eta^\pm = \eta_\varepsilon^\pm$, $\delta^\pm = \delta_\varepsilon^\pm$ not to overburden notation. By assumption,

$$\sqrt{\varepsilon} \int_{\delta^-}^{\delta^+} \rho(u'_\varepsilon)^2 d\theta + \varepsilon^{-3/2} \int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \leq c. \quad (3.1)$$

For any fixed $0 < a < 1$, we assume $\delta^- \geq a$.

The crucial argument in the proof will be to estimate the length $\delta^+ - \delta^-$ of any such interval. We will show that

$$\begin{aligned} &\sqrt{\varepsilon} \int_{\delta^-}^{\delta^+} \rho(u'_\varepsilon)^2 d\theta + \varepsilon^{-3/2} \int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \\ &\geq \rho(a) |\eta^+ - \eta^-|^2 \left(\frac{\sqrt{\varepsilon}}{\delta} \right) + c(\lambda) \left(\frac{\delta}{\sqrt{\varepsilon}} \right)^3, \end{aligned}$$

where λ is the minimum between $|\eta_-^2 - 1|$ and $|\eta_+^2 - 1|$, and $c(\lambda)$ is a suitable positive constant depending only on λ . From this estimate we will deduce that if $|\eta^+ - \eta^-| \geq \zeta > 0$ then $\delta^+ - \delta^-$ is of the order $\sqrt{\varepsilon}$, and that

$$\sqrt{\varepsilon} \int_{\delta^-}^{\delta^+} \rho(u'_\varepsilon)^2 d\theta + \varepsilon^{-3/2} \int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \geq c\zeta^{3/2}.$$

From this, we will finally obtain that frequent large oscillations are forbidden, and in particular that the number of transitions from values close to 1 to values close to -1 , and conversely, is equi-bounded, and then deduce the L^1_{loc} -convergence in $(0, 1)$.

Step 1. We define the set

$$A_\varepsilon = \left\{ \theta \in (a, 1] : \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 \leq \varepsilon \right\},$$

and we denote by A_ε^c its complementary set. Since $\sup_\varepsilon F_\varepsilon^{3/2}(u_\varepsilon) < +\infty$ we have that

$$|A_\varepsilon^c| \varepsilon < \int_{A_\varepsilon^c} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 \leq c\varepsilon^{3/2}$$

which implies that there exists $\bar{c}(\varepsilon) \leq c$ such that

$$|A_\varepsilon^c| = \bar{c}(\varepsilon) \sqrt{\varepsilon}. \quad (3.2)$$

Let $\theta_\varepsilon \in A_\varepsilon$ be such that $\theta_\varepsilon = \max\{\theta \in A_\varepsilon : \theta \leq \delta^-\}$; by definition

$$\varepsilon \geq \frac{1}{\rho(\theta_\varepsilon)} \left(\int_0^{\theta_\varepsilon} \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 \geq \frac{1}{\rho(1)} \left(\int_0^{\theta_\varepsilon} \rho(u_\varepsilon^2 - 1) d\varphi \right)^2.$$

It follows that there exists $c_1(\varepsilon)$ such that $|c_1(\varepsilon)| \leq \sqrt{\rho(1)}$ and

$$\int_0^{\theta_\varepsilon} \rho(u_\varepsilon^2 - 1) d\varphi = c_1(\varepsilon) \sqrt{\varepsilon}. \quad (3.3)$$

Moreover, for every $\theta \in (\theta_\varepsilon, \delta^-) \subset A_\varepsilon^c$ we have that

$$\begin{aligned} |u_\varepsilon(\theta) - \eta^-| &= |u_\varepsilon(\theta) - u_\varepsilon(\delta^-)| \leq \left(|A_\varepsilon^c| \int_{\theta_\varepsilon}^{\delta^-} (u'_\varepsilon)^2 d\theta \right)^{1/2} \\ &\leq \left(\frac{|A_\varepsilon^c|}{\rho(a)} \int_{\theta_\varepsilon}^{\delta^-} \rho(u'_\varepsilon)^2 d\theta \right)^{1/2}. \end{aligned}$$

Since $\sup_\varepsilon F_\varepsilon^{3/2}(u_\varepsilon) < +\infty$, by (3.2) we have that

$$|u_\varepsilon(\theta)| \leq |\eta^-| + \tilde{c} \quad \forall \theta \in [\theta_\varepsilon, \delta^-] \quad (3.4)$$

that in turn implies

$$\left| \int_{\theta_\varepsilon}^{\delta^-} \rho(u_\varepsilon^2 - 1) d\theta \right| \leq c(\eta^-) \sqrt{\varepsilon}.$$

where $c(\eta^-) = c\rho(1)(1 + (|\eta^-| + \tilde{c})^2)$. Hence, there exists $c_2(\varepsilon, \eta^-)$ such that $|c_2(\varepsilon, \eta^-)| \leq c(\eta^-)$ and

$$\int_{\theta_\varepsilon}^{\delta^-} \rho(u_\varepsilon^2 - 1) d\theta = c_2(\varepsilon, \eta^-) \sqrt{\varepsilon}. \quad (3.5)$$

By (3.3) and (3.5) we get that

$$\begin{aligned} & \int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \\ &= \int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)} \left(\int_0^{\theta_\varepsilon} \rho(u_\varepsilon^2 - 1) d\varphi + \int_{\theta_\varepsilon}^{\delta^-} \rho(u_\varepsilon^2 - 1) d\varphi + \int_{\delta^-}^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \\ &= c_5(c_1(\varepsilon) + c_2(\varepsilon, \eta^-))^2 \varepsilon \delta + \int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)} \left(\int_{\delta^-}^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \\ & \quad + 2(c_1(\varepsilon) + c_2(\varepsilon, \eta^-)) \sqrt{\varepsilon} \int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)} \left(\int_{\delta^-}^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right) d\theta, \end{aligned} \quad (3.6)$$

where $c_5 = \int_{\delta^-}^{\delta^+} (1/\rho) d\theta$ and $\delta = \delta^+ - \delta^-$.

By assumption u_ε takes values between η^- and η^+ for every $\theta \in [\delta^-, \delta^+]$ (note that either $(u_\varepsilon^2 - 1) > 0$ or $(u_\varepsilon^2 - 1) < 0$ for every $\theta \in [\delta^-, \delta^+]$); hence,

$$|u_\varepsilon^2 - 1| \geq \lambda := \|\eta^-\|^2 - 1 \wedge \|\eta^+\|^2 - 1$$

where we use the notation λ without an explicit dependence on η^\pm since we want to emphasize that λ does not depend on the values of η^\pm but on the minimum distance of $|\eta^\pm|^2$ from 1.

Moreover, on $[\delta^-, \delta^+]$

$$|u_\varepsilon^2 - 1| \leq c(\eta^\pm) := \|\eta^-\|^2 - 1 \vee \|\eta^+\|^2 - 1,$$

then there exist $c_3(\varepsilon, \eta^\pm)$ and $c_4(\varepsilon, \lambda)$ such that

$$0 < |c_3(\varepsilon, \eta^\pm)| \leq \frac{c(\eta^\pm)}{2}, \quad c_4(\varepsilon, \lambda) \geq \frac{\lambda^2}{3} \frac{\rho(a)^2}{\rho(1)} \quad (3.7)$$

and

$$\int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)} \left(\int_{\delta^-}^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right) d\theta = c_3(\varepsilon, \eta^\pm) \delta^2, \quad (3.8)$$

$$\int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)} \left(\int_{\delta^-}^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta = c_4(\varepsilon, \lambda) \delta^3. \quad (3.9)$$

By (3.6), (3.8) and (3.9) we get that

$$\begin{aligned} & \int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \\ &= \delta^3 \left\{ c_4(\varepsilon, \lambda) + c_5(c_1(\varepsilon) + c_2(\varepsilon, \eta^-))^2 \left(\frac{\sqrt{\varepsilon}}{\delta} \right)^2 \right. \\ & \quad \left. + 2c_3(\varepsilon, \eta^\pm)(c_1(\varepsilon) + c_2(\varepsilon, \eta^-)) \left(\frac{\sqrt{\varepsilon}}{\delta} \right) \right\}. \end{aligned} \quad (3.10)$$

Note that c_4 and c_5 are always strictly positive while $c_3 \neq 0$ with $\text{sign}(c_3) = \text{sign}(u_\varepsilon^2 - 1)$. Moreover, we claim that

$$c_4 - (c_3^2/c_5) \geq c(\lambda) \quad (3.11)$$

where $c(\lambda) := \lambda^2 c(\delta^\pm) \rho(a)^2 / \rho(1)$ with $1/12 \leq c(\delta^\pm) \leq 1/3$ for every δ^\pm . Also in this case we prefer to use the notation $c(\lambda)$ omitting the dependence on δ^\pm because of the bound on $c(\delta^\pm)$. We check now (3.11). Let us denote

$$f(\theta) = \int_{\delta^-}^{\theta} \rho |u_\varepsilon^2 - 1| d\varphi;$$

hence, $f'(\theta) = \rho(\theta) |u_\varepsilon^2(\theta) - 1| \geq \lambda \rho(a)$. Then, there exists $\delta_0 \in (\delta^-, \delta^+)$ such that

$$|c_3| = \frac{1}{\delta^2} \int_{\delta^-}^{\delta^+} \frac{f(\theta)}{\rho(\theta)} d\theta = \frac{f(\delta_0)}{\delta^2} \int_{\delta^-}^{\delta^+} \frac{d\theta}{\rho(\theta)} = \frac{f(\delta_0)}{\delta} c_5 \quad (3.12)$$

and

$$|f(\theta) - f(\delta_0)| \geq \lambda \rho(a) |\theta - \delta_0|.$$

It follows that

$$\begin{aligned} \int_{\delta^-}^{\delta^+} \frac{(f(\theta) - f(\delta_0))^2}{\rho(\theta)} d\theta &\geq (\lambda \rho(a))^2 \int_{\delta^-}^{\delta^+} \frac{(\theta - \delta_0)^2}{\rho(\theta)} d\theta \\ &\geq (\lambda \rho(a))^2 \frac{(\delta^+ - \delta_0)^3 - (\delta^- - \delta_0)^3}{3\rho(1)} \\ &= \delta^3 \left(\lambda^2 c(\delta^\pm) \frac{\rho(a)^2}{\rho(1)} \right) \end{aligned} \quad (3.13)$$

where $1/12 \leq c(\delta^\pm) \leq 1/3$ for every δ^\pm . On the other hand, by (3.12) we have that

$$\begin{aligned} &\int_{\delta^-}^{\delta^+} \frac{(f(\theta) - f(\delta_0))^2}{\rho(\theta)} d\theta \\ &= \int_{\delta^-}^{\delta^+} \frac{f(\theta)^2}{\rho(\theta)} d\theta + f(\delta_0)^2 \int_{\delta^-}^{\delta^+} \frac{d\theta}{\rho(\theta)} - 2f(\delta_0) \int_{\delta^-}^{\delta^+} \frac{f(\theta)}{\rho(\theta)} d\theta \\ &= \delta^3 \left(c_4 - c_5 \left(\frac{f(\delta_0)}{\delta} \right)^2 \right) \\ &= \delta^3 \left(c_4 - \frac{c_3^2}{c_5} \right); \end{aligned} \quad (3.14)$$

hence, by (3.13) we get (3.11).

We now estimate the term with the derivative in (3.1); by Hölder's inequality we get that

$$\sqrt{\varepsilon} \int_{\delta^-}^{\delta^+} \rho (u'_\varepsilon)^2 d\theta \geq \rho(a) |\eta^+ - \eta^-|^2 \left(\frac{\sqrt{\varepsilon}}{\delta} \right). \quad (3.15)$$

By (3.10) and (3.15) we have then

$$\begin{aligned} & \sqrt{\varepsilon} \int_{\delta^-}^{\delta^+} \rho(u'_\varepsilon)^2 d\theta + \varepsilon^{-3/2} \int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \\ & \geq \rho(a) |\eta^+ - \eta^-|^2 \left(\frac{\sqrt{\varepsilon}}{\delta} \right) + \left(\frac{\delta}{\sqrt{\varepsilon}} \right)^3 \left(\gamma \left(\frac{\sqrt{\varepsilon}}{\delta} \right)^2 \pm \beta \left(\frac{\sqrt{\varepsilon}}{\delta} \right) + \alpha \right), \end{aligned} \quad (3.16)$$

where

$$\alpha = c_4, \quad \gamma = c_5(c_1 + c_2)^2, \quad \beta = 2|c_3(c_1 + c_2)|.$$

Note that, as already observed, $\alpha > 0$, $\gamma \geq 0$ and $c_3(c_1 + c_2)$ may be ≥ 0 or ≤ 0 ($\gamma = 0$ if and only if $\beta = 0$). By (3.11), if we minimize $\gamma(\sqrt{\varepsilon}/\delta)^2 \pm \beta(\sqrt{\varepsilon}/\delta) + \alpha$ in $(\sqrt{\varepsilon}/\delta)$, we have

$$\begin{aligned} & \rho(a) |\eta^+ - \eta^-|^2 \left(\frac{\sqrt{\varepsilon}}{\delta} \right) + \left(\gamma \left(\frac{\sqrt{\varepsilon}}{\delta} \right)^2 \pm \beta \left(\frac{\sqrt{\varepsilon}}{\delta} \right) + \alpha \right) \left(\frac{\delta}{\sqrt{\varepsilon}} \right)^3 \\ & \geq \rho(a) |\eta^+ - \eta^-|^2 \left(\frac{\sqrt{\varepsilon}}{\delta} \right) + \left(\alpha - \frac{\beta^2}{4\gamma} \right) \left(\frac{\delta}{\sqrt{\varepsilon}} \right)^3 \\ & = \rho(a) |\eta^+ - \eta^-|^2 \left(\frac{\sqrt{\varepsilon}}{\delta} \right) + (c_4 - c_3^2/c_5) \left(\frac{\delta}{\sqrt{\varepsilon}} \right)^3 \\ & \geq \rho(a) |\eta^+ - \eta^-|^2 \left(\frac{\sqrt{\varepsilon}}{\delta} \right) + c(\lambda) \left(\frac{\delta}{\sqrt{\varepsilon}} \right)^3. \end{aligned} \quad (3.17)$$

Step 2. If

$$|\eta^+ - \eta^-| \geq \zeta > 0,$$

with ζ independent of ε , studying the function $x \mapsto \rho(a) |\eta^+ - \eta^-|^2 x + c(\lambda)/x^3$ for $x > 0$, by (3.1), (3.16) and (3.17) we have that $\delta/\sqrt{\varepsilon}$ is bounded; *i.e.*, there exist two positive constants α_1, α_2 such that $\alpha_1 \sqrt{\varepsilon} \leq \delta \leq \alpha_2 \sqrt{\varepsilon}$.

Step 3. The minimum point of $x \mapsto \rho(a) |\eta^+ - \eta^-|^2 x + c(\lambda)/x^3$ for $x > 0$ is $x_m = c_m/|\eta^+ - \eta^-|^{1/2}$ then by (3.16) and (3.17) we have that

$$\begin{aligned} & \sqrt{\varepsilon} \int_{\delta^-}^{\delta^+} \rho(u'_\varepsilon)^2 d\theta + \varepsilon^{-3/2} \int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \\ & \geq \tilde{c}(\lambda) |\eta^+ - \eta^-|^{3/2} \geq \tilde{c}(\lambda) \zeta^{3/2}; \end{aligned} \quad (3.18)$$

where $\tilde{c}(\lambda) = (\rho(a)c_m + c(\lambda)/c_m^3)$. We recall that λ is the minimum distance of $|\eta^\pm|^2$ from 1. Since (u_ε) is a sequence with bounded energy, and estimate (3.18) depends on λ only, we deduce that the number of transitions of u_ε from η^- to η^+ is equibounded independently of ε .

Conclusions. By (3.1) and (3.18) we conclude that (u_ε) is equi-bounded in $L_{\text{loc}}^\infty(0, 1)$. Moreover, by Steps 2 and 3, we have that for every fixed $0 < a < 1$ (u_ε) converges in measure in $(a, 1)$ to $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$; hence, up to subsequences, $u_\varepsilon \rightarrow u$ a.e. $\theta \in (a, 1)$. Since (u_ε) is equi-bounded in $L_{\text{loc}}^\infty(0, 1)$ we can conclude that, up to subsequences, (u_ε) converges strongly in $L_{\text{loc}}^1(0, 1)$ to $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$. \square

The following proposition gives an estimate of the measure of the set where a sequence of bounded energy is not close to ± 1 .

Proposition 3.2 *Let (u_ε) be a sequence converging to u in $L^1_{\text{loc}}(0, 1)$ such that $\sup_\varepsilon F_\varepsilon^{3/2}(u_\varepsilon) < +\infty$. Then for every fixed $0 < a < 1$ and $\eta > 0$ there exists $\bar{c} = \bar{c}(\eta)$ such that*

$$|\{\theta \in [a, 1-a] : |u_\varepsilon^2(\theta) - 1| > \eta\}| \leq \bar{c}\sqrt{\varepsilon}. \quad (3.19)$$

PROOF. Let $0 < a < 1$ and $\eta > 0$ be fixed. We define δ_i^\pm such that

$$\{\theta \in [a, 1-a] : |u_\varepsilon^2(\theta) - 1| > \eta\} \subseteq \bigcup_i [\delta_i^-, \delta_i^+],$$

with $\delta_i^- < \delta_i^+$, $\delta_i^+ \leq \delta_j^-$ for every $i < j$ and

$$\begin{cases} |(u_\varepsilon(\delta_i^\pm))^2 - 1| = \eta/2 & \text{except if } \delta_i^+ = 1 \\ |u_\varepsilon^2(\theta) - 1| > \eta/2 & \text{if } \theta \in (\delta_i^-, \delta_i^+) \\ \exists \delta_i \in (\delta_i^-, \delta_i^+) & \text{such that } |(u_\varepsilon(\delta_i))^2 - 1| = \eta. \end{cases}$$

Hence, we may have two cases:

$$u_\varepsilon(\delta_i^-) \neq u_\varepsilon(\delta_i^+) \quad \text{and hence} \quad u_\varepsilon(\delta_i^\pm) \in \left\{ \pm\sqrt{1-\eta/2} \right\} \quad (3.20)$$

or

$$u_\varepsilon(\delta_i^-) = u_\varepsilon(\delta_i^+) \in \left\{ \pm\sqrt{1-\eta/2}, \pm\sqrt{1+\eta/2} \right\}. \quad (3.21)$$

In case (3.20) we may apply Steps 1–3 in Theorem 3.1 with $\eta^\pm \in \left\{ \pm\sqrt{1-\eta/2} \right\}$, $\lambda = \eta/2$ and $\zeta = 2\sqrt{1-\eta/2}$. While if we are in case (3.21) we consider

$$\eta^- \in \left\{ \pm\sqrt{1-\eta/2}, \pm\sqrt{1+\eta/2} \right\}, \quad \eta^+ \in \left\{ \max_{[\delta_i^-, \delta_i^+]} u_\varepsilon, \min_{[\delta_i^-, \delta_i^+]} u_\varepsilon \right\},$$

or conversely. For example, if $u_\varepsilon(\delta_i^-) = u_\varepsilon(\delta_i^+) = \sqrt{1+\eta/2}$ then we apply Steps 1–3 in Theorem 3.1 two times: the first one to

$$\eta^- = \sqrt{1+\eta/2}, \quad \eta^+ = \max_{[\delta_i^-, \delta_i^+]} u_\varepsilon$$

and the second one to

$$\eta^- = \max_{[\delta_i^-, \delta_i^+]} u_\varepsilon, \quad \eta^+ = \sqrt{1+\eta/2}.$$

Hence, also in this case we may apply Steps 1, 2, 3 in Theorem 3.1 with $\zeta > 0$.

We can then conclude that the number of intervals $[\delta_i^-, \delta_i^+]$ is finite (and independent of ε) and there exist $\alpha_i^1, \alpha_i^2 > 0$ such that $\alpha_i^1\sqrt{\varepsilon} \leq (\delta_i^+ - \delta_i^-) \leq \alpha_i^2\sqrt{\varepsilon}$ which proves (3.19). \square

Theorem 3.3 (Γ -convergence result) *We have*

$$\Gamma(L^1_{\text{loc}})\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon^{3/2}(u) = c_0 \sum_{\theta \in S(u)} \rho(\theta)$$

for every $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$, where

$$c_0 = \inf_{T > 0} \inf \left\{ \int_{-T}^T (v')^2 ds + \int_{-T}^T \left(\int_{-T}^\sigma (v^2 - 1) ds \right)^2 d\sigma : \right. \\ \left. v \in H^1(-T, T), \quad v(\pm T) = \pm 1, \quad \int_{-T}^T (v^2 - 1) ds = 0 \right\}.$$

PROOF. Liminf inequality. Let $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$ and let (u_ε) be a sequence converging to u in $L^1_{\text{loc}}(0, 1)$ such that $\sup_\varepsilon F_\varepsilon^{3/2}(u_\varepsilon) < +\infty$.

Step 1. We fix $0 < a < 1$ and consider $\theta_i \in S(u)$ such that $a < \theta_i < 1 - a$. Without loss of generality, we may assume that $u(\theta_i \pm) = \mp 1$. Let $\theta_\varepsilon \rightarrow \theta_i$, as $\varepsilon \rightarrow 0$, and $M > 0$ be such that

$$u_\varepsilon > \frac{1}{2} \quad \text{on } I_\varepsilon^- := (\theta_\varepsilon - 2M\sqrt{\varepsilon}, \theta_\varepsilon - M\sqrt{\varepsilon})$$

and

$$u_\varepsilon < \frac{1}{2} \quad \text{on } I_\varepsilon^+ := (\theta_\varepsilon + M\sqrt{\varepsilon}, \theta_\varepsilon + 2M\sqrt{\varepsilon}).$$

We claim that for every fixed $\eta > 0$ there exist a constant $c > 0$ and $\theta_\varepsilon^\pm \in I_\varepsilon^\pm$ of the form $\theta_\varepsilon^\pm = \theta_\varepsilon \pm M_\varepsilon\sqrt{\varepsilon}$, with $M \leq M_\varepsilon \leq 2M$, such that

$$\left| \int_0^{\theta_\varepsilon^\pm} \rho(u_\varepsilon^2 - 1) d\varphi \right| \leq \frac{c}{\sqrt{M}} \sqrt{\varepsilon} \quad (3.22)$$

and

$$|u_\varepsilon^2(\theta_\varepsilon^\pm) - 1| \leq \eta. \quad (3.23)$$

In fact, reasoning by contradiction, assume that for every constant $c > 0$ we cannot find two points $\theta_\varepsilon^\pm \in I_\varepsilon^\pm$ such that (3.22) and (3.23) are satisfied at the same time. If we denote

$$B_c = \left\{ \theta : M\sqrt{\varepsilon} \leq |\theta - \theta_\varepsilon| \leq 2M\sqrt{\varepsilon}, \left| \int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right| > \frac{c}{\sqrt{M}} \sqrt{\varepsilon} \right\},$$

note that by (3.19) we then have $|B_c| \geq \sqrt{\varepsilon}(2M - \bar{c})$, and hence

$$F_\varepsilon^{3/2}(u_\varepsilon) \geq \varepsilon^{-3/2} \int_{B_c} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \\ \geq \varepsilon^{-1/2} \frac{c^2}{\rho(1)} \frac{|B_c|}{M} \geq \frac{c^2}{\rho(1)} \left(2 - \frac{\bar{c}}{M} \right),$$

Note that we can choose M large enough such that $\bar{c} < 2M$. Since $\sup_\varepsilon F_\varepsilon^{3/2}(u_\varepsilon) < +\infty$ we get a contradiction by the arbitrariness of $c > 0$.

Step 2. We give an estimate on the contribution between θ_ε^- and θ_ε^+ . By (3.22), there exists a sequence (c_-^ε) , bounded independently of ε , such that

$$\begin{aligned}
& \sqrt{\varepsilon} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \rho(u'_\varepsilon)^2 d\theta + \varepsilon^{-3/2} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \\
&= \sqrt{\varepsilon} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \rho(u'_\varepsilon)^2 d\theta + \varepsilon^{-3/2} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \frac{1}{\rho(\theta)} \left(\frac{c_-^\varepsilon}{\sqrt{M}} \sqrt{\varepsilon} + \int_{\theta_\varepsilon^-}^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \\
&= \int_{-M_\varepsilon}^{M_\varepsilon} \rho_\varepsilon(v')^2 ds + \int_{-M_\varepsilon}^{M_\varepsilon} \frac{1}{\rho_\varepsilon(\sigma)} \left(\frac{c_-^\varepsilon}{\sqrt{M}} + \int_{-M_\varepsilon}^\sigma \rho_\varepsilon(v^2 - 1) ds \right)^2 d\sigma, \quad (3.24)
\end{aligned}$$

where $s = (\varphi - \theta_\varepsilon)/\sqrt{\varepsilon}$, $\sigma = (\theta - \theta_\varepsilon)/\sqrt{\varepsilon}$, $\rho_\varepsilon(t) = \rho(\theta_\varepsilon + t\sqrt{\varepsilon})$ and $v(s) = u_\varepsilon(\theta_\varepsilon + \sqrt{\varepsilon}s)$. By (3.22) there exists also a sequence (c_+^ε) , bounded independently of ε , such that

$$\int_0^{\theta_\varepsilon^+} \rho(u_\varepsilon^2 - 1) d\varphi = \frac{c_+^\varepsilon}{\sqrt{M}} \sqrt{\varepsilon};$$

hence,

$$\frac{1}{\sqrt{\varepsilon}} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \rho(u_\varepsilon - 1)^2 d\varphi = \int_{-M_\varepsilon}^{M_\varepsilon} \rho_\varepsilon(v^2 - 1) ds = \frac{c_+^\varepsilon - c_-^\varepsilon}{\sqrt{M}}. \quad (3.25)$$

By (3.23) we get that $|v^2(\pm M_\varepsilon) - 1| \leq \eta$. We consider $v(s) = u_\varepsilon(\theta_\varepsilon + \sqrt{\varepsilon}s)$ for $s \in [-M_\varepsilon, M_\varepsilon]$ and we extend it to $[-3M, 3M]$ in such a way that

$$v(\pm 3M) = \mp 1, \quad \int_{-3M}^{3M} \rho_\varepsilon(v^2 - 1) ds = 0, \quad \int_{-3M}^{-M_\varepsilon} \rho_\varepsilon(v^2 - 1) ds = \frac{c_-^\varepsilon}{\sqrt{M}}. \quad (3.26)$$

Note that by (3.25) and (3.26) we have that

$$\int_{M_\varepsilon}^{3M} \rho_\varepsilon(v^2 - 1) ds = -\frac{c_+^\varepsilon}{\sqrt{M}}.$$

We explicitly construct v on $[-3M, -M_\varepsilon]$, the construction on $[M_\varepsilon, 3M]$ being analogous. We also suppose that $c_-^\varepsilon \leq 0$ and $v(-M_\varepsilon) \geq 1$ (the construction being the same or simpler in the other cases). We define v on $[-3M, -M_\varepsilon]$ as follows

$$v(s) = \begin{cases} 1 & \text{if } s \in [-3M, -h_\varepsilon) \\ -s + \sqrt{1 - \eta} - k_\varepsilon^- & \text{if } s \in [-h_\varepsilon, -k_\varepsilon^-) \\ \sqrt{1 - \eta} & \text{if } s \in [-k_\varepsilon^-, -k_\varepsilon^+) \\ s + \sqrt{1 - \eta} + k_\varepsilon^+ & \text{if } s \in [-k_\varepsilon^+, -M_\varepsilon], \end{cases} \quad (3.27)$$

where

$$-k_\varepsilon^- = -h_\varepsilon + 1 - \sqrt{1 - \eta}, \quad -k_\varepsilon^+ = -M_\varepsilon - v(-M_\varepsilon) + \sqrt{1 - \eta}.$$

Note that k_ε^+ is fixed by Step 1 while k_ε^- is fixed by (3.26). In fact, since $|v^2 - 1| \leq \eta$ on $[-h_\varepsilon, -k_\varepsilon^-)$ and $[-k_\varepsilon^+, -M_\varepsilon)$ we have that

$$\left| \int_{-h_\varepsilon}^{-k_\varepsilon^-} \rho_\varepsilon(v^2 - 1) ds + \int_{-k_\varepsilon^+}^{-M_\varepsilon} \rho_\varepsilon(v^2 - 1) ds \right| \leq \eta \rho(1)(1 + \sqrt{1 + \eta} - 2\sqrt{1 - \eta});$$

hence, there exists $r(\eta, \varepsilon)$ such that $|r(\eta, \varepsilon)| \leq \rho(1)(1 + \sqrt{1 + \eta} - 2\sqrt{1 - \eta})$ uniformly in ε and

$$\int_{-3M}^{-M_\varepsilon} \rho_\varepsilon(v^2 - 1) ds = \eta(-k_\varepsilon + r(\eta, \varepsilon)) = \frac{c_-^\varepsilon}{\sqrt{M}},$$

where $k_\varepsilon := -k_\varepsilon^+ + k_\varepsilon^-$. It follows that

$$k_\varepsilon = \frac{|c_-^\varepsilon|}{\eta\sqrt{M}} + r(\eta, \varepsilon), \quad (3.28)$$

with $\lim_{\eta \rightarrow 0^+} |r(\eta, \varepsilon)| = 0$. Reasoning as above, by (3.28), we can also observe that

$$\begin{aligned} & \int_{-3M}^{-M_\varepsilon} \frac{1}{\rho_\varepsilon(\sigma)} \left(\int_{-3M}^\sigma \rho_\varepsilon(v^2 - 1) ds \right)^2 d\sigma \\ &= \int_{-h_\varepsilon}^{-M_\varepsilon} \frac{1}{\rho_\varepsilon(\sigma)} \left(\int_{-h_\varepsilon}^\sigma \rho_\varepsilon(v^2 - 1) ds \right)^2 d\sigma \\ &\leq \eta^2 \frac{\rho(1)}{3} \left(k_\varepsilon + 1 + \sqrt{1 + \eta} - 2\sqrt{1 - \eta} \right)^3 \\ &= \eta^2 \frac{\rho(1)}{3} \left(\frac{|c_-^\varepsilon|}{\eta\sqrt{M}} + r(\eta, \varepsilon) + 1 + \sqrt{1 + \eta} - 2\sqrt{1 - \eta} \right)^3. \end{aligned}$$

We can prove a similar estimate also for the contribution corresponding to the interval $[M_\varepsilon, 3M]$; hence, we can conclude that there exists $R(M, \eta, \varepsilon) > 0$, bounded independently of ε , such that

$$\begin{aligned} R(M, \eta, \varepsilon) &= \int_{-3M}^{-M_\varepsilon} \frac{1}{\rho_\varepsilon(\sigma)} \left(\int_{-3M}^\sigma \rho_\varepsilon(v^2 - 1) ds \right)^2 d\sigma \\ &\quad + \int_{M_\varepsilon}^{3M} \frac{1}{\rho_\varepsilon(\sigma)} \left(\frac{c_+^\varepsilon}{\sqrt{M}} + \int_{M_\varepsilon}^\sigma \rho_\varepsilon(v^2 - 1) ds \right)^2 d\sigma \quad (3.29) \end{aligned}$$

and

$$\limsup_{\eta \rightarrow 0^+} \limsup_{M \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} R(M, \eta, \varepsilon) = 0. \quad (3.30)$$

By (3.29) we have that

$$\begin{aligned} & \int_{-M_\varepsilon}^{M_\varepsilon} \frac{1}{\rho_\varepsilon(\sigma)} \left(\frac{c_-^\varepsilon}{\sqrt{M}} + \int_{-M_\varepsilon}^\sigma \rho_\varepsilon(v^2 - 1) ds \right)^2 d\sigma \\ &= \int_{-3M}^{3M} \frac{1}{\rho_\varepsilon(\sigma)} \left(\int_{-3M}^\sigma \rho_\varepsilon(v^2 - 1) ds \right)^2 d\sigma - R(M, \eta, \varepsilon). \quad (3.31) \end{aligned}$$

The function v , constructed as in (3.27), gives the following contribution of the term with the derivative

$$\int_{-3M}^{-M_\varepsilon} \rho_\varepsilon (v')^2 ds \leq \rho(1)(1 + \sqrt{1 + \eta} - 2\sqrt{1 - \eta});$$

hence, reasoning similarly on $[M_\varepsilon, 3M]$, there exists $R_1(\eta, \varepsilon) > 0$, bounded independently of ε , such that

$$\int_{-M_\varepsilon}^{M_\varepsilon} \rho_\varepsilon (v')^2 ds = \int_{-3M}^{3M} \rho_\varepsilon (v')^2 ds - R_1(\eta, \varepsilon) \quad (3.32)$$

and

$$\limsup_{\eta \rightarrow 0^+} \limsup_{\varepsilon \rightarrow 0} R_1(\eta, \varepsilon) = 0. \quad (3.33)$$

By (3.24), (3.32) and (3.31) we get that

$$\begin{aligned} & \sqrt{\varepsilon} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \rho(\theta)(u_\varepsilon')^2 d\theta + \varepsilon^{-3/2} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \\ &= \int_{-3M}^{3M} \rho_\varepsilon (v')^2 ds + \int_{-3M}^{3M} \frac{1}{\rho_\varepsilon(\sigma)} \left(\int_{-3M}^\sigma \rho_\varepsilon (v^2 - 1) ds \right)^2 d\sigma \\ & \quad - R_1(\eta, \varepsilon) - R(M, \eta, \varepsilon) \\ &\geq \inf \left\{ \int_{-T}^T \rho_\varepsilon (v')^2 ds + \int_{-T}^T \frac{1}{\rho_\varepsilon(\sigma)} \left(\int_{-T}^\sigma \rho_\varepsilon (v^2 - 1) ds \right)^2 d\sigma : \right. \\ & \quad \left. v \in H^1(-T, T), \quad v(\pm T) = \pm 1, \quad \int_{-T}^T \rho_\varepsilon (v^2 - 1) ds = 0 \right\} \\ & \quad + \delta(T, \eta, \varepsilon), \end{aligned} \quad (3.34)$$

where $T = 3M$ and $\delta(T, \eta, \varepsilon) = -R_1(\eta, \varepsilon) - \tilde{R}(T, \eta, \varepsilon)$ with $\tilde{R}(T, \eta, \varepsilon) = R(M, \eta, \varepsilon)$. Note that in the last infimum problem we can take the boundary values indifferently as $v(\pm T) = \pm 1$ or $v(\pm T) = \mp 1$, by the symmetry of the problem, so that both types of transitions are taken into account.

By (3.30) and (3.33), we have that

$$\limsup_{\eta \rightarrow 0^+} \limsup_{T \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} |\delta(T, \eta, \varepsilon)| = 0;$$

hence, it remains to study the behavior of the minimum problems as ε tends to 0. By the uniform convergence of ρ_ε to $\rho(\theta_i)$, as ε tends to 0, we have immediately the Γ -convergence of the functionals with respect to the strong L^2 convergence; *i.e.*,

$$\begin{aligned} & \Gamma(L^2)\text{-}\lim_{\varepsilon \rightarrow 0} \left(\int_{-T}^T \rho_\varepsilon (v')^2 ds + \int_{-T}^T \frac{1}{\rho_\varepsilon(\sigma)} \left(\int_{-T}^\sigma \rho_\varepsilon (v^2 - 1) ds \right)^2 d\sigma \right) \\ &= \rho(\theta_i) \left(\int_{-T}^T (v')^2 ds + \int_{-T}^T \left(\int_{-T}^\sigma (v^2 - 1) ds \right)^2 d\sigma \right). \end{aligned}$$

This Γ -convergence result is stable by adding the constraint; indeed, since the constraint $\int_{-T}^T \rho_\varepsilon (v^2 - 1) ds = 0$ is close for the strong L^2 convergence the liminf inequality is trivial. To check the limsup inequality, let $v \in H^1(-T, T)$ such that $v(\pm T) = \pm 1$ and $\int_{-T}^T (v^2 - 1) ds = 0$. To obtain a recovery sequence we consider

$$v_\varepsilon(s) = v(s) + t_\varepsilon \phi(s) \quad s \in [-T, T] \quad (3.35)$$

where

$$\phi(s) = \begin{cases} (s+T)/T & s \in [-T, 0) \\ (T-s)/T & s \in [0, T] \end{cases} \quad (3.36)$$

(note that $v_\varepsilon \in H^1(-T, T)$ and $v_\varepsilon(\pm T) = \pm 1$) and $t_\varepsilon \in \mathbf{R}$ is chosen such that (v_ε) satisfies the constraint and converges to v in $L^2(-T, T)$; *i.e.*,

$$\int_{-T}^T \rho_\varepsilon ((v + t_\varepsilon \phi)^2 - 1) ds = 0 \quad \text{and} \quad t_\varepsilon \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0. \quad (3.37)$$

More precisely, (v_ε) satisfies the constraint in (3.37) if $t_\varepsilon \in \mathbf{R}$ is solution of the following second order equation

$$\left(\int_{-T}^T \rho_\varepsilon \phi^2 ds \right) t_\varepsilon^2 + 2 \left(\int_{-T}^T \rho_\varepsilon v \phi ds \right) t_\varepsilon + \left(\int_{-T}^T \rho_\varepsilon (v^2 - 1) ds \right) = 0. \quad (3.38)$$

Since $\lim_{\varepsilon \rightarrow 0} \int_{-T}^T \rho_\varepsilon (v^2 - 1) ds = 0$, for ε small enough, equation (3.38) has two solutions, real and distinct, such that one of two tends to 0 as ε tends to 0.

To conclude the proof of the limsup inequality, we have to note that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \left(\int_{-T}^T \rho_\varepsilon (v'_\varepsilon)^2 ds + \int_{-T}^T \frac{1}{\rho_\varepsilon(\sigma)} \left(\int_{-T}^\sigma \rho_\varepsilon (v_\varepsilon^2 - 1) ds \right)^2 d\sigma \right) \\ &= \rho(\theta_i) \left(\int_{-T}^T (v')^2 ds + \int_{-T}^T \left(\int_{-T}^\sigma (v^2 - 1) ds \right)^2 d\sigma \right). \end{aligned} \quad (3.39)$$

We go back now to (3.34), by the property of convergence of minima (see [6] Theorem 1.21) we have that

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \left(\sqrt{\varepsilon} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \rho(u'_\varepsilon)^2 d\theta + \varepsilon^{-3/2} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \right) \\ & \geq \rho(\theta_i) \inf \left\{ \int_{-T}^T (v')^2 ds + \int_{-T}^T \left(\int_{-T}^\sigma (v^2 - 1) ds \right)^2 d\sigma : v \in H^1(-T, T), \right. \\ & \quad \left. v(\pm T) = \pm 1, \quad \int_{-T}^T (v^2 - 1) ds = 0 \right\} \\ & \quad - \limsup_{\varepsilon \rightarrow 0} |\delta(T, \eta, \varepsilon)| \\ & \geq \rho(\theta_i) \inf_{T>0} \inf \left\{ \int_{-T}^T (v')^2 ds + \int_{-T}^T \left(\int_{-T}^\sigma (v^2 - 1) ds \right)^2 d\sigma : \right. \end{aligned}$$

$$\begin{aligned}
& v \in H^1(-T, T), \quad v(\pm T) = \pm 1, \quad \int_{-T}^T (v^2 - 1) ds = 0 \Big\} \\
& - \limsup_{T \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} |\delta(T, \eta, \varepsilon)| \\
& = \rho(\theta_i) c_0 - \limsup_{T \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} |\delta(T, \eta, \varepsilon)|,
\end{aligned}$$

where

$$\begin{aligned}
c_0 := \inf_{T > 0} \inf \Big\{ & \int_{-T}^T (v')^2 ds + \int_{-T}^T \left(\int_{-T}^\sigma (v^2 - 1) ds \right)^2 d\sigma : v \in H^1(-T, T) \\
& v(\pm T) = \pm 1, \quad \text{and} \quad \int_{-T}^T (v^2 - 1) ds = 0 \Big\}.
\end{aligned}$$

Passing to the limit as $\eta \rightarrow 0^+$, we get

$$\begin{aligned}
& \liminf_{\varepsilon \rightarrow 0} \left(\sqrt{\varepsilon} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \rho(u_\varepsilon')^2 d\theta + \varepsilon^{-3/2} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \right) \\
& \geq \rho(\theta_i) c_0.
\end{aligned}$$

Step 3. If we repeat Steps 1-2 for every $\theta \in S(u) \cap (a, 1-a)$ we immediately get that

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon^{3/2}(u_\varepsilon) \geq c_0 \sum_{\theta \in S(u) \cap (a, 1-a)} \rho(\theta)$$

and then the liminf inequality taking the supremum in a .

Limsup inequality. Let $u \in BV((0, 1); \{-1, 1\})$ we denote $S(u) = \{\theta_1, \dots, \theta_N\}$ with $\theta_i < \theta_{i+1}$. Fixed $\eta > 0$ there exist $T > 0$ and $v \in H^1(-T, T)$ such that $v(\pm T) = \pm 1$, $\int_{-T}^T (v^2 - 1) ds = 0$ and

$$\int_{-T}^T (v')^2 ds + \int_{-T}^T \left(\int_{-T}^\sigma (v^2 - 1) ds \right)^2 d\sigma \leq c_0 + \eta. \quad (3.40)$$

We denote $\delta = T\sqrt{\varepsilon}$, $K_i = [\theta_i + \delta, \theta_{i+1} - \delta]$ for $i = 1, \dots, N-1$, $K_N = [\theta_N + \delta, 1]$. We construct a sequence u_ε by setting

$$u_\varepsilon(\theta) = \begin{cases} v_\varepsilon^i(\pm \varepsilon^{-1/2}(\theta - \theta_i)) & \text{if } \theta \in [\theta_i - \delta, \theta_i + \delta], \quad i = 1, \dots, N \\ u(\theta) & \text{if } \theta \in (0, \theta_1 - \delta) \cup_{i=1}^N K_i \end{cases} \quad (3.41)$$

where $v_\varepsilon^i = v + t_\varepsilon^i \phi$ is defined as in Step 2 with ϕ given by (3.36) and $t_\varepsilon^i \in \mathbf{R}$ such that

$$\int_{-T}^T \rho_\varepsilon^i((v + t_\varepsilon^i \phi)^2 - 1) ds = 0 \quad \text{and} \quad t_\varepsilon^i \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad (3.42)$$

for every $i = 1, \dots, N$ where $\rho_\varepsilon^i(s) = \rho(\theta_i + s\sqrt{\varepsilon})$. Note that the choice between the plus and minus sign, in (3.41), is made in such a way that

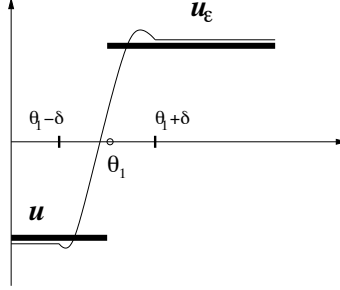


Figure 2: recovery sequence with a Gibbs' phenomenon

the resulting function is continuous. The construction of u_ϵ is illustrated in Fig. 2.

Note that, reasoning as in Step 2, we get

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \left(\int_{-T}^T \rho_\epsilon^i (v_\epsilon^i)'{}^2 ds + \int_{-T}^T \frac{1}{\rho_\epsilon^i(\sigma)} \left(\int_{-T}^\sigma \rho_\epsilon^i ((v_\epsilon^i)^2 - 1) ds \right)^2 d\sigma \right) \\ &= \rho(\theta_i) \left(\int_{-T}^T (v')^2 ds + \int_{-T}^T \left(\int_{-T}^\sigma (v^2 - 1) ds \right)^2 d\sigma \right). \end{aligned} \quad (3.43)$$

Hence,

$$\begin{aligned} F_\epsilon^{3/2}(u_\epsilon) &= \sqrt{\epsilon} \sum_{i=1}^N \int_{\theta_i - \delta}^{\theta_i + \delta} \rho(u_\epsilon')^2 d\theta \\ &+ \epsilon^{-3/2} \sum_{i=1}^N \left(\int_{K_i} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\epsilon^2 - 1) d\varphi \right)^2 d\theta \right. \\ &\quad \left. + \int_{\theta_i - \delta}^{\theta_i + \delta} \frac{1}{\rho(\theta)} \left(\int_0^\theta \rho(u_\epsilon^2 - 1) d\varphi \right)^2 d\theta \right). \end{aligned}$$

By the changes of variable $s = \epsilon^{-1/2}(\varphi - \theta_i)$, $\sigma = \epsilon^{-1/2}(\theta - \theta_i)$ and (3.42), we get that

$$\begin{aligned} F_\epsilon^{3/2}(u_\epsilon) &= \sum_{i=1}^N \int_{-T}^T \rho_\epsilon^i (v_\epsilon^i)'{}^2 ds \\ &+ \epsilon^{-1/2} \sum_{i=1}^N \left(\int_{K_i} \frac{d\theta}{\rho(\theta)} \right) \left(\sum_{j=1}^i \int_{-T}^T \rho_\epsilon^j ((v_\epsilon^j)^2 - 1) ds \right)^2 \\ &+ \epsilon^{-1/2} \sum_{i=1}^N \int_{\theta_i - \delta}^{\theta_i + \delta} \frac{1}{\rho(\theta)} \left(\sum_{j=1}^{i-1} \int_{-T}^T \rho_\epsilon^j ((v_\epsilon^j)^2 - 1) ds \right. \\ &\quad \left. + \int_{-T}^{\epsilon^{-1/2}(\theta - \theta_i)} \rho_\epsilon^i ((v_\epsilon^i)^2 - 1) ds \right)^2 d\theta \end{aligned}$$

$$= \sum_{i=1}^N \int_{-T}^T \rho_\varepsilon^i (v_\varepsilon^i)'{}^2 ds + \int_{-T}^T \frac{1}{\rho_\varepsilon^i(\sigma)} \left(\int_{-T}^\sigma \rho_\varepsilon^i ((v_\varepsilon^i)^2 - 1) ds \right)^2 d\sigma.$$

By (3.40) and (3.43), we have

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} F_\varepsilon^{3/2}(u_\varepsilon) &= \left(\sum_{i=1}^N \rho(\theta_i) \right) \left(\int_{-T}^T (v')^2 ds + \int_{-T}^T \left(\int_{-T}^\sigma (v^2 - 1) ds \right)^2 d\sigma \right) \\ &\leq \left(\sum_{i=1}^N \rho(\theta_i) \right) (c_0 + \eta). \end{aligned}$$

By the arbitrariness of η we get the limsup inequality for every $u \in BV((0, 1); \{-1, 1\})$.

We now consider $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$; let $u^a \in BV((0, 1); \{-1, 1\})$ be such that u^a converges to u strongly in $L^1(0, 1)$ as $a \rightarrow 0^+$; *e.g.*,

$$u^a(\theta) = \begin{cases} u(a) & \theta \in [0, a] \\ u(\theta) & \theta \in [a, 1-a] \\ u(1-a) & \theta \in (1-a, 1] \end{cases},$$

with $0 < a < 1$ and $a, 1-a \notin S(u)$. By the lower semicontinuity of the Γ -limsup we have that

$$\begin{aligned} \Gamma\text{-lim sup}_{\varepsilon \rightarrow 0} F_\varepsilon^{3/2}(u) &\leq \liminf_{a \rightarrow 0^+} \left(\Gamma\text{-lim sup}_{\varepsilon \rightarrow 0} F_\varepsilon^{3/2}(u^a) \right) \\ &\leq c_0 \liminf_{a \rightarrow 0^+} \sum_{\theta \in S(u^a)} \rho(\theta) \leq c_0 \sum_{\theta \in S(u)} \rho(\theta) \end{aligned}$$

for every $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$ (see [6] Remark 1.29). \square

4 The case $\alpha = 4/3$ with constrained phase transitions

If we define G_ε^0 as the restriction of F_ε^0 to the space of functions $u : (0, 1) \rightarrow [-1, 1]$ then, by Remark 2.4, the Γ -limit of G_ε^0 is identically 0. Still one may find another scaling, $\alpha = 4/3$, such that the Γ -limit of $G_\varepsilon^{4/3} = \varepsilon^{-4/3} G_\varepsilon^0$ is not trivial. We consider for simplicity $\rho \equiv 1$, so that

$$G_\varepsilon^{4/3}(u) = \varepsilon^{2/3} \int_0^1 (u')^2 d\theta + \varepsilon^{-4/3} \int_0^1 \left(\int_0^\theta (u^2 - 1) d\varphi \right)^2 d\theta,$$

for every $u \in H^1((0, 1); [-1, 1])$. Note that since $\varepsilon^{-4/3} F_\varepsilon^0 = \varepsilon^{1/6} F_\varepsilon^{3/2}$, by Section 3, the Γ -limit of $\varepsilon^{-4/3} F_\varepsilon^0$ with respect to the strong L^1 -convergence is zero. Hence, the constraint $|u| \leq 1$ completely changes the characteristic scaling of the energy.

Theorem 4.1 (Compactness) *Let $(u_\varepsilon) \in H^1((0, 1); [-1, 1])$ be a sequence such that $\sup_\varepsilon G_\varepsilon^{4/3}(u_\varepsilon) < +\infty$ then, up to subsequences, (u_ε) converges strongly in $L^1(0, 1)$ to $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$.*

PROOF. Let $\eta_\varepsilon^\pm \neq \pm 1$ such that $-1 < \eta_\varepsilon^+, \eta_\varepsilon^- < 1$. We denote by $(\delta_\varepsilon^-, \delta_\varepsilon^+)$ an interval such that $u_\varepsilon(\delta_\varepsilon^-) = \eta_\varepsilon^-, u_\varepsilon(\delta_\varepsilon^+) = \eta_\varepsilon^+$ and u_ε takes values between η_ε^- and η_ε^+ . We use in the sequel the notation $\eta^\pm = \eta_\varepsilon^\pm$, $\delta^\pm = \delta_\varepsilon^\pm$ not to overburden notation. By the constraint, $(u_\varepsilon^2 - 1)$ never changes sign; hence, we can select in the energy $G_\varepsilon^{4/3}(u_\varepsilon)$ the most significative contribution which permits to easily estimate $(\delta^+ - \delta^-)$. More precisely, for every fixed $0 < a < 1$ we consider $\delta^+ \leq 1 - a$; then,

$$\begin{aligned} G_\varepsilon^{4/3}(u_\varepsilon) &\geq \varepsilon^{2/3} \int_{\delta^-}^{\delta^+} (u')^2 d\theta + \varepsilon^{-4/3} \int_{\delta^+}^1 \left(\int_{\delta^-}^{\delta^+} (u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \\ &\geq |\eta^+ - \eta^-|^2 \left(\frac{\varepsilon^{2/3}}{\delta} \right) + a \lambda^2 \left(\frac{\delta}{\varepsilon^{2/3}} \right)^2, \end{aligned}$$

where $\lambda := \|\eta^+ - 1\| \wedge \|\eta^- - 1\|$; *i.e.*, the minimum distance of $|\eta^\pm|$ from 1. Hence, if $|\eta^+ - \eta^-| \geq \zeta > 0$, with ζ independent of ε , we have that $\delta/\varepsilon^{2/3}$ is bounded; *i.e.*, there exist two positive constant α_1, α_2 such that $\alpha_1 \varepsilon^{2/3} \leq \delta \leq \alpha_2 \varepsilon^{2/3}$. Moreover, the number of intervals (δ^-, δ^+) is finite in $[0, 1 - a]$ for every $0 < a < 1$. Then (u_ε) converges in measure to $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$ and since $|u_\varepsilon| \leq 1$ we can conclude that, up to subsequences, (u_ε) converges strongly to u in L^1 . \square

Remark 4.2 Note that in general we cannot expect $u \in BV((0, 1); \{-1, 1\})$. To show this we construct a sequence (u_ε) with $\sup_\varepsilon G_\varepsilon^{4/3}(u_\varepsilon) < +\infty$ and strongly converging in L^1 to $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$ with infinitely many jump points. To this end, consider a strictly increasing sequence $(\theta_i) \in (0, 1)$ such that $\sup_{i \in \mathbf{N}} \theta_i = 1$ and let $v_i(s) = (s + T_i/T_i) - 1$ for $s \in [-T_i, T_i]$. Fixed $k \in \mathbf{N}$ and ε small enough, we define the sequence (u_ε^k) as follows

$$u_\varepsilon^k(\theta) = \begin{cases} v_i(\pm \varepsilon^{-2/3}(\theta - \theta_i)) & \theta \in [\theta_i - T_i \varepsilon^{2/3}, \theta_i + T_i \varepsilon^{2/3}], i = 1, \dots, k \\ \pm 1 & \text{otherwise in } [0, 1] \end{cases}$$

where the choice between the plus and minus sign is made in such a way that the resulting function u_ε^k is continuous. For every $k \in \mathbf{N}$, we have that

$$\begin{aligned} G_\varepsilon^{4/3}(u_\varepsilon^k) &= \sum_{i=1}^k \left(\frac{2}{T_i} + (\theta_{i+1} - \theta_i) \left(c \sum_{j \leq i} T_j \right)^2 \right) + O(\varepsilon^{2/3}) \\ &\leq \sum_{i \in \mathbf{N}} \left(\frac{2}{T_i} + (\theta_{i+1} - \theta_i) \left(c \sum_{j \leq i} T_j \right)^2 \right) + O(\varepsilon^{2/3}) \end{aligned}$$

(see a similar computation in the next Theorem 4.3 for the proof of the limsup inequality). If we fix $T_i = i^\beta$, with $\beta > 1$, and (θ_i) such that $(\theta_{i+1} - \theta_i) = \gamma i^{(-3\beta-2)}$, with γ satisfying the condition $\gamma \sum_{i \in \mathbf{N}} i^{(-3\beta-2)} = (1 - \theta_1)$, then $\sup_\varepsilon G_\varepsilon^{4/3}(u_\varepsilon^k) \leq c$, with c independent on k . Therefore, if $(\theta_i)_{i \in \mathbf{N}}$ is an increasing sequence of points distributed in $(0, 1)$ as above, for every fixed $k \in \mathbf{N}$, we can construct a suitable sequence (u_ε^k) strongly converging in

L^1 to $u^k \in BV((0, 1); \{-1, 1\})$, as $\varepsilon \rightarrow 0$, with $\sup_\varepsilon G_\varepsilon^{4/3}(u_\varepsilon^k) \leq c$ and $S(u^k) = \{\theta_1, \dots, \theta_k\}$. We now consider $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$ such that $S(u) = (\theta_i)_{i \in \mathbb{N}}$ and $u = u^k$ in $[0, \theta_{k+1})$ then u^k converges strongly in L^1 to u as k tends to $+\infty$. By a diagonal procedure we may extract from (u_ε^k) a subsequence with bounded energy and strongly converging to u in L^1 .

Theorem 4.3 (Nonlocal Γ -limit) *We have*

$$\begin{aligned} & \Gamma(L^1)\text{-}\lim_{\varepsilon \rightarrow 0} G_\varepsilon^{4/3}(u) \\ &= \inf_{T > 0} \inf \left\{ \sum_{i \in I} \left(\int_{-T}^T (v'_i)^2 ds + (\theta_{i+1} - \theta_i) \left(\sum_{j=1}^i \int_{-T}^T (v_j^2 - 1) ds \right)^2 \right) : \right. \\ & \quad \left. v_i \in H^1((-T, T); [-1, 1]), \quad v_i(\pm T) = \pm 1 \right\}, \end{aligned}$$

for every $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$, where $I = \{i \in \mathbb{N} : \theta_i \in S(u), \theta_i < \theta_{i+1}\}$.

PROOF. Let $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$ and let $(u_\varepsilon) \in H^1((0, 1); [-1, 1])$ be a sequence strongly converging to u in L^1 such that

$$\liminf_{\varepsilon \rightarrow 0} G_\varepsilon^{4/3}(u_\varepsilon) < +\infty.$$

For every fixed $0 < a < 1$, by Theorem 4.1, the limit function u has a finite number of discontinuity points in the interval $(0, 1-a]$; i.e., $S(u) \cap (0, 1-a] = \{\theta_1, \dots, \theta_{N(a)}\}$ with $\theta_i < \theta_{i+1}$. Up to subsequences, $u_\varepsilon \rightarrow u$ for a.e. $\theta \in (0, 1)$, as ε tends to 0; hence, fixed $\eta \in (0, 1)$, we consider δ_i^1, δ_i^2 such that

$$u_\varepsilon(\theta_i - \delta_i^1) = -(1 - \eta), \quad u_\varepsilon(\theta_i + \delta_i^2) = 1 - \eta$$

or

$$u_\varepsilon(\theta_i - \delta_i^1) = 1 - \eta, \quad u_\varepsilon(\theta_i + \delta_i^2) = -(1 - \eta)$$

for $i = 1, \dots, N(a)$. The following estimate consists in eliminating all the contributions of u_ε on the intervals where the sequence takes values ‘close’ to $\{-1, 1\}$; this choice is justified by the construction of the optimal sequence, in the limsup inequality, that will be equal to $\{-1, 1\}$ on such intervals (see (4.2)). We have then

$$\begin{aligned} G_\varepsilon^{4/3}(u_\varepsilon) &\geq \varepsilon^{2/3} \sum_{i=1}^{N(a)} \int_{I_i} (u'_\varepsilon)^2 d\theta + \varepsilon^{-4/3} \left(\sum_{i=1}^{N(a)} \int_{K_i} \left(\sum_{j=1}^i \int_{I_j} (u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \right. \\ & \quad \left. + \sum_{i=1}^{N(a)} \int_{I_i} \left(\sum_{j=1}^{i-1} \int_{I_j} (u_\varepsilon^2 - 1) d\varphi + \int_{\theta_i - \delta_i^1}^\theta (u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \right), \end{aligned}$$

where $I_i = [\theta_i - \delta_i^1, \theta_i + \delta_i^2]$ for $i = 1, \dots, N(a)$ and $K_i = (\theta_i + \delta_i^2, \theta_{i+1} - \delta_{i+1}^1)$ for $i = 1, \dots, N(a) - 1$, $K_{N(a)} = (\theta_{N(a)} + \delta_{N(a)}^2, 1 - a)$. We make the following change of variable

$$w_j(s) = u_\varepsilon \left(\varepsilon^{2/3} s + \theta_j + \frac{\delta_j^2 - \delta_j^1}{2} \right);$$

hence, setting $T_j = \varepsilon^{-2/3} \delta_j$ with $\delta_j = \left(\frac{\delta_j^2 + \delta_j^1}{2}\right)$ and

$$T_i(\theta) = \varepsilon^{-2/3} \left(\theta - \theta_i - \frac{(\delta_i^2 - \delta_i^1)}{2} \right),$$

we get

$$\begin{aligned} G_\varepsilon^{4/3}(u_\varepsilon) &\geq \sum_{i=1}^{N(a)} \left(\int_{-T_i}^{T_i} (w'_i)^2 ds + \int_{K_i} \left(\sum_{j=1}^i \int_{-T_j}^{T_j} (w_j^2 - 1) ds \right)^2 d\theta \right. \\ &\quad \left. + \int_{I_i} \left(\sum_{j=1}^{i-1} \int_{-T_j}^{T_j} (w_j^2 - 1) ds + \int_{-T_i}^{T_i(\theta)} (w_i^2 - 1) ds \right)^2 d\theta \right). \end{aligned}$$

We denote now

$$B_i = \int_{-T_i}^{T_i} (w'_i)^2 ds, \quad A_j = \int_{-T_j}^{T_j} (w_j^2 - 1) ds,$$

by the change of variable $\sigma = T_i(\theta)$ we get

$$\begin{aligned} G_\varepsilon^{4/3}(u_\varepsilon) &\geq \sum_{i=1}^{N(a)} B_i + \sum_{i=1}^{N(a)} \left((2\delta_i + \theta_{i+1} - \theta_i - \delta_{i+1}^1 - \delta_i^2) \left(\sum_{j=1}^i A_j \right)^2 \right) \\ &\quad + \varepsilon^{2/3} \sum_{i=1}^{N(a)} \left(\int_{-T_i}^{T_i} \left(\int_{\sigma}^{T_i} (w_i^2 - 1) ds \right)^2 d\sigma \right. \\ &\quad \left. - 2 \left(\sum_{j=1}^i A_j \right) \int_{-T_i}^{T_i} \left(\int_{\sigma}^{T_i} (w_i^2 - 1) ds \right) d\sigma \right) \\ &= \sum_{i=1}^{N(a)} \left(B_i + (\theta_{i+1} - \theta_i) \left(\sum_{j=1}^i A_j \right)^2 \right) + \sum_{i=1}^{N(a)} (\delta_i^1 - \delta_{i+1}^1) \left(\sum_{j=1}^i A_j \right)^2 \\ &\quad + \varepsilon^{2/3} \left(\sum_{i=1}^{N(a)} \int_{-T_i}^{T_i} \left(\int_{\sigma}^{T_i} (w_i^2 - 1) ds \right)^2 d\sigma \right. \\ &\quad \left. - 2 \left(\sum_{j=1}^i A_j \right) \int_{-T_i}^{T_i} \left(\int_{\sigma}^{T_i} (w_i^2 - 1) ds \right) d\sigma \right) \end{aligned}$$

where $\theta_{N(a)+1} = 1 - a$ and $\delta_{N(a)+1}^1 = 0$. Hence,

$$G_\varepsilon^{4/3}(u_\varepsilon) \geq \sum_{i=1}^{N(a)} \left(\int_{-T_i}^{T_i} (w'_i)^2 ds + (\theta_{i+1} - \theta_i) \left(\sum_{j=1}^i \int_{-T_j}^{T_j} (w_j^2 - 1) ds \right)^2 \right) + O(\varepsilon^{2/3}),$$

where $w_i(\pm T_i)$ is equal to $\pm(1 - \eta)$ or $\mp(1 - \eta)$, and

$$\liminf_{\varepsilon \rightarrow 0} G_\varepsilon^{4/3}(u_\varepsilon)$$

$$\geq \inf_{T>0} \inf \left\{ \sum_{i=1}^{N(a)} \left(\int_{-T}^T (v'_i)^2 ds + (\theta_{i+1} - \theta_i) \left(\sum_{j=1}^i \int_{-T}^T (v_j^2 - 1) ds \right)^2 \right) : \right. \\ \left. v_i \in H^1((-T, T); [-1, 1]), \quad v_i(\pm T) = \pm(1 - \eta) \right\}$$

where, by symmetry, we may fix the boundary conditions as $v_i(\pm T) = \pm(1 - \eta)$. We may now first pass to the limit as $\eta \rightarrow 0$ and then take the supremum on a ; *i.e.*,

$$\liminf_{\varepsilon \rightarrow 0} G_\varepsilon^{4/3}(u_\varepsilon) \\ \geq \sup_{0 < a < 1} \inf_{T > 0} \inf \left\{ \sum_{i=1}^{N(a)} \left(\int_{-T}^T (v'_i)^2 ds + (\theta_{i+1} - \theta_i) \left(\sum_{j=1}^i \int_{-T}^T (v_j^2 - 1) ds \right)^2 \right) : \right. \\ \left. v_i \in H^1((-T, T); [-1, 1]), \quad v_i(\pm T) = \pm 1 \right\} \\ = \inf_{T > 0} \inf \left\{ \sum_{i \in I} \left(\int_{-T}^T (v'_i)^2 ds + (\theta_{i+1} - \theta_i) \left(\sum_{j=1}^i \int_{-T}^T (v_j^2 - 1) ds \right)^2 \right) : \right. \\ \left. v_i \in H^1((-T, T); [-1, 1]), \quad v_i(\pm T) = \pm 1 \right\}$$

where we have labeled the points in $S(u)$ by a set of indices $I \subset \mathbf{N}$ in such a way that $\theta_i < \theta_{i+1}$.

We now check the limsup inequality. Let $u \in BV((0, 1); \{-1, 1\})$ with $S(u) = \{\theta_1, \dots, \theta_N\}$ and $\theta_i < \theta_{i+1}$. We denote by

$$G^{4/3}(u) = \inf_{T > 0} \inf \left\{ \sum_{i=1}^N \left(\int_{-T}^T (v'_i)^2 ds + (\theta_{i+1} - \theta_i) \left(\sum_{j=1}^i \int_{-T}^T (v_j^2 - 1) ds \right)^2 \right) : \right. \\ \left. v_i \in H^1((-T, T); [-1, 1]), \quad v_i(\pm T) = \pm 1 \right\}.$$

Fixed $\eta > 0$, there exist $T > 0$ and $(v_1, \dots, v_N) \in H^1((-T, T); [-1, 1])$ such that $v_i(\pm T) = \pm 1$ and

$$\sum_{i=1}^N \left(\int_{-T}^T (v'_i)^2 ds + (\theta_{i+1} - \theta_i) \left(\sum_{j=1}^i \int_{-T}^T (v_j^2 - 1) ds \right)^2 \right) \leq G^{4/3}(u) + \eta. \quad (4.1)$$

We denote $\delta = T\varepsilon^{2/3}$, $I_i = [\theta_i - \delta, \theta_i + \delta]$ for $i = 1, \dots, N$, $K_i = (\theta_i + \delta, \theta_{i+1} - \delta)$ for $i = 1, \dots, N - 1$ and $K_N = (\theta_N + \delta, 1]$. We construct a sequence u_ε by setting

$$u_\varepsilon(\theta) = \begin{cases} v_i(\pm\varepsilon^{-2/3}(\theta - \theta_i)) & \text{if } \theta \in I_i \quad i = 1, \dots, N \\ u(\theta) & \text{if } \theta \in (0, \theta_1 - \delta) \cup \left(\bigcup_{i=1}^N K_i \right) \end{cases} \quad (4.2)$$

where the choice between the plus and minus sign is made in such a way that the resulting function is continuous. Hence, by the change of variables

$s = \varepsilon^{-2/3}(\theta - \theta_i)$, we get

$$\begin{aligned}
G_\varepsilon^{4/3}(u_\varepsilon) &= \varepsilon^{2/3} \sum_{i=1}^N \int_{I_i} (u'_\varepsilon)^2 d\theta + \varepsilon^{-4/3} \sum_{i=1}^N \left(\int_{K_i} \left(\int_0^\theta (u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \right) \\
&\quad + \varepsilon^{-4/3} \sum_{i=1}^N \left(\int_{I_i} \left(\int_0^\theta (u_\varepsilon^2 - 1) d\varphi \right)^2 d\theta \right) \\
&= \sum_{i=1}^N \left(\int_{-T}^T (v'_i)^2 ds + \int_{K_i} \left(\sum_{j=1}^i \int_{-T}^T (v_j^2 - 1) ds \right)^2 d\theta \right) \\
&\quad + \int_{I_i} \left(\sum_{j=1}^{i-1} \int_{-T}^T (v_j^2 - 1) ds + \int_{-T}^{T_i(\theta)} (v_i^2 - 1) ds \right)^2 d\theta
\end{aligned}$$

where $T_i(\theta) = \varepsilon^{-2/3}(\theta - \theta_i)$. Setting

$$B_i = \int_{-T}^T (v'_i)^2 ds, \quad A_j = \int_{-T}^T (v_j^2 - 1) ds$$

we then have

$$\begin{aligned}
G_\varepsilon^{4/3}(u_\varepsilon) &= \sum_{i=1}^N \left(B_i + (\theta_{i+1} - \theta_i - 2\delta) \left(\sum_{j=1}^i A_j \right)^2 \right) \\
&\quad + \sum_{i=1}^N \int_{I_i} \left(\sum_{j=1}^i A_j - \int_{T_i(\theta)}^T (v_i^2 - 1) ds \right)^2 d\theta \\
&= \sum_{i=1}^N \left(B_i + (\theta_{i+1} - \theta_i) \left(\sum_{j=1}^i A_j \right)^2 \right) + \sum_{i=1}^N \int_{I_i} \left(\int_{T_i(\theta)}^T (v_i^2 - 1) ds \right)^2 d\theta \\
&\quad - 2 \sum_{i=1}^N \left(\sum_{j=1}^i A_j \right) \int_{I_i} \left(\int_{T_i(\theta)}^T (v_i^2 - 1) ds \right) d\theta.
\end{aligned}$$

By (4.1) and the change of variable $\sigma = T_i(\theta)$, we get that

$$\begin{aligned}
G_\varepsilon^{4/3}(u_\varepsilon) &= \sum_{i=1}^N \left(B_i + (\theta_{i+1} - \theta_i) \left(\sum_{j=1}^i A_j \right)^2 \right) + O(\varepsilon^{2/3}) \\
&\leq G^{4/3}(u) + \eta + O(\varepsilon^{2/3}).
\end{aligned}$$

Passing to the limit as ε tends to 0, by the arbitrariness of η , we get the limsup inequality for every $u \in BV((0, 1); \{-1, 1\})$.

We now consider $u \in BV_{\text{loc}}((0, 1); \{-1, 1\})$. There exists $u^a \in BV((0, 1); \{-1, 1\})$ such that u^a converges to u strongly in $L^1(0, 1)$ as $a \rightarrow 0^+$; *i.e.*,

$$u^a(\theta) = \begin{cases} u(a) & \theta \in [0, a] \\ u(\theta) & \theta \in [a, 1-a] \\ u(1-a) & \theta \in (1-a, 1] \end{cases},$$

with $0 < a < 1$ and $a, 1 - a \notin S(u)$. Hence, by the lower semicontinuity of the Γ -limsup, we have that

$$\begin{aligned}
& \Gamma\text{-lim sup}_{\varepsilon \rightarrow 0} G_\varepsilon^{4/3}(u) \\
& \leq \liminf_{a \rightarrow 0^+} \left(\Gamma\text{-lim sup}_{\varepsilon \rightarrow 0} G_\varepsilon^{4/3}(u^a) \right) \\
& \leq \liminf_{a \rightarrow 0^+} \inf_{T > 0} \inf \left\{ \sum_{i \in I(a)} \left(\int_{-T}^T (v'_i)^2 ds + (\theta_{i+1} - \theta_i) \left(\sum_{j=1}^i \int_{-T}^T (v_j^2 - 1) ds \right)^2 \right) : \right. \\
& \quad \left. v_i \in H^1((-T, T); [-1, 1]), \quad v_i(\pm T) = \pm 1 \right\} \\
& \leq \inf_{T > 0} \inf \left\{ \sum_{i \in I} \left(\int_{-T}^T (v'_i)^2 ds + (\theta_{i+1} - \theta_i) \left(\sum_{j=1}^i \int_{-T}^T (v_j^2 - 1) ds \right)^2 \right) : \right. \\
& \quad \left. v_i \in H^1((-T, T); [-1, 1]), \quad v_i(\pm T) = \pm 1 \right\},
\end{aligned}$$

where $I(a) = \{i \in \mathbf{N} : \theta_i \in S(u^a), \theta_i < \theta_{i+1}\}$ and $I = \{i \in \mathbf{N} : \theta_i \in S(u), \theta_i < \theta_{i+1}\}$. \square

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