Multi-Scale Analysis by Γ-convergence of a Shell-Membrane Transition

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Abstract

We study the asymptotic behavior of functionals associated to the energy of a thin nonlinear elastic spherical shell in the limit of vanishing thickness (proportional to a small parameter) ε and under the assumption of radial deformations. The functionals are characterized by the presence of a nonlocal potential term and defined on suitable weighted functional spaces. The transition shell-membrane is studied at three relevant different scales. For each of them we give a compactness result and compute the Γ-limit. In particular, we show that if the energies on a sequence of configurations scale as $\varepsilon^{3/2}$ then the limit configuration describes a (locally) finite number of transitions between the undeformed and the everted configurations of the shell. We also highlight a kind of 'Gibbs' phenomenon' by showing that non-trivial optimal sequences restricted between the undeformed and the everted configurations must have energy scaling at least as $\varepsilon^{4/3}$.

1 Introduction and physical motivations

It is well known that a spherical shell under zero loads can assume at least two configurations, the trivial reference configuration and the everted configuration, both of them are stable solutions of the equilibrium problem. The existence of everted shapes was first proved by Antman [1] for thick spherical shells. Later the analysis concerning the eversion of thin shells has been carried out from the theoretical and numerical point of view by Podio-Guidugli *et al.* [22] and Geymonat *et al.* [15, 16].

The equations for the small finite axially symmetric deformations of a spherical cap, without applied loads, may be written as[1]

$$\theta^{-3}[\theta^3 S']' = -f(f+2), \qquad \varepsilon^2\theta^{-3}[\theta^3 f']' = S(1+f). \tag{1.1}$$

They have been introduced by several authors (see for example the pioneering papers by Reiss [23] and Bauer *et al.* [4]) for studying the buckling phenomena in spherical shells. The unknown f, function of the normalized polar angle θ, is related to the slope of the deformed middle surface of the cap with respect to the initial spherical shape and ε is the thinness parameter of the shell. Due to the symmetry of the deformations the boundary conditions at the origin are usually taken as $f(0) = S'(0) = 0$. Among the plausible conditions at the boundary of the cap one may assume that both the membrane and bending stresses satisfy the equilibrium conditions

$$S(1) = 0 \qquad f'(1) + (1+\nu)f(1) = 0. \tag{1.2}$$

Here $|\nu| < 1$ and the prime denotes the first derivative with respect to θ. A further homogeneous boundary condition for the displacement completes the description of the problem.

The problem has been studied in [22, 15, 16] in the weighted Sobolev spaces:
K: the space of square integrable functions on (0,1) with the weight θ^3 supplied with the scalar product $(f_1, f_2)_0 = \int_0^1 f_1\, f_2\, \theta^3 d\theta$;
K^1: the space of functions of K whose first derivatives also belong to K with scalar product $(f_1, f_2)_1 = \int_0^1 f_1'\, f_2'\, \theta^3 d\theta + (1+\nu) f_1(1) f_2(1)$;
K_0^1: the space of functions of K^1 vanishing at $\theta = 1$.

Following a method introduced by Berger [6], the boundary values problem can be formulated as a single operator equation in the Sobolev space K^1

[1] The unknowns f and S are related to the meridional and radial (oriented towards the center) displacements of the shell middle surface, denoted by z and w respectively, by the relations

$$f(\theta) = \frac{1}{R\psi}\frac{dw}{d\psi}, \quad S(\theta) = \frac{2}{\Lambda^2(1-\nu^2)R}\left[\left(\frac{dz}{d\psi} - w\right) + \frac{1}{2}R\psi^2 f^2(\theta) + \nu\left(\frac{z}{\psi} - w\right)\right]$$

where Λ is the given opening angle of the cap, R is the radius of cap middle surface and $\psi = \theta\Lambda$. Finally we recall that the parameter ε depends on the thickness h of the cap by the formula

$$\varepsilon^2 = \frac{2h^2}{3R^2\Lambda^4(1-\nu^2)}$$

whose solutions are the stationary points of the functional

$$J_\varepsilon(f) = \varepsilon^2 \|f\|_1^2 + \frac{1}{2}\|G_0[f^2 + 2f]\|_1^2,$$

where $G_0[g]$ is the Green operator defined by $(G_0[g], \psi)_1 = (g, \psi)_0$ for each $\psi \in K_0^1$; more precisely,

$$J_\varepsilon(f) = \varepsilon^2 \|f\|_1^2 \,+\, \frac{1}{2}\int_0^1 \frac{1}{\theta^3}\left(\int_0^\theta f(s)(f(s)+2)s^3 ds\right)^2 d\theta.$$

Beside the trivial stable solution $f = 0$ (absolute minimum for the above functionals), for ε small enough, there exists a second stable solution namely the everted stressed solution. The sequence of everted configurations tends to an unstressed configuration ($f = -2$) that can be described as the reflection of the cap reference configuration (see [22, 15, 16]).

More recently, the problem has been further investigated. The existence of infinitely many stable solutions for the limit problem has been predicted and several numerical experiments proposed by Geymonat ant Leger [14]. We also want to quote the papers by Antman and Srubshchik [2, 3], where the existence of everted states and their approximation by asymptotic expansions have been justified also for more sophisticated shell models. For control problem associated to linear and nonlinear thin spherical caps see Geymonat and Valente [17] and Lasiecka and Valente [18].

Although the problem has been carefully studied in [22, 15, 16], the asymptotic analysis of J_ε has remained an open problem. The aim of our paper is the study as $\varepsilon \to 0^+$ of the variational problem for $J_\varepsilon(f)$ introducing suitable functional spaces which allow to characterize the asymptotic solutions. The formal analogy with two-well potentials perturbed by a gradient term suggests to utilize the at Γ-convergence methods largely used in the study of phase transitions.

For the sake of symmetry we change variable by setting $u = f + 1$ and consider a slightly modified functional with two absolute minimum points $u = \pm 1$

$$F_\varepsilon(u) = \varepsilon^2 \int_0^1 \theta^3 (u'(\theta))^2 \, d\theta \,+ \int_0^1 \frac{1}{\theta^3}\Big(\int_0^\theta \varphi^3 (u^2(\varphi) - 1)\, d\varphi\Big)^2 d\theta\,.$$

Note that the difference with J_ε consist only in the absence of a boundary condition, whose presence however would not influence the asymptotic analysis.

To describe the asymptotic analysis of F_ε we first focus on sequences (u_ε) such that $F_\varepsilon(u_\varepsilon) = O(1)$. In that case we prove that (u_ε) is locally weakly compact in $L^1(0,1)$ and sequences giving the optimal lower bound may oscillate between the values -1 and 1. This behavior is described by the Γ-limit F^0 (see Theorem 3.6 for the precise form of the limit), that not only captures these oscillations but also shows that the non-local character of the functional is maintained in the limit.

Minimizing sequences are responsible of folding effects also observed for flat membranes. The analytical reconstruction of the shell surface texture could allow both to understand the material elastic properties and to study the interactions between two surfaces. It must be noted that the Γ-limit coincides with the lower-semicontinuous envelope of the functional

$$G(u) = \int_0^1 \theta^{-3}\Big(\int_0^\theta (u^2-1)\varphi^3\,d\varphi\Big)^2\,d\theta$$

with respect to the local weak L^1-convergence, characterized in Lemma 3.3, and that minimizers of this functionals are all functions with $|u| = 1$ a.e. In terms of recovery sequences, we note that they may develop oscillations, but the occurrence of these is due to a non-local effect (see the example in Remark 3.5).

The minimum value for the Γ-limit F^0 is 0 and is achieved exactly on all functions u with $|u| \le 1$. This large class of minimizers justifies the analysis at finer scales. We show that the next meaningful scale is when $F_\varepsilon(u_\varepsilon) = O(\varepsilon^{3/2})$. If this is the case then we show that such (u_ε) is strongly pre-compact in $L^1(0,1)$ and its limits u are locally piecewise constant in $(0,1)$, and $|u(\theta)| = 1$ a.e. We describe this behavior by showing that the Γ-limit of the scaled energies $\varepsilon^{-3/2}F_\varepsilon$ on those functions takes the form

$$F^{3/2}(u) = c_0 \sum_{\theta\in S(u)} \theta^3,$$

where we denote by $S(u)$ the set of discontinuities of u (see Theorem 4.3).

The formal analogies with the corresponding functionals of the gradient theory of phase transitions

$$H_\varepsilon(u) = \varepsilon^2\int_0^1 \theta^3(u'(\theta))^2\,d\theta + \int_0^1 \theta^3(u^2(\theta)-1)^2\,d\theta$$

must be noted. Upon a normalization factor, the functionals $\varepsilon^{-1}H_\varepsilon$ still Γ-converge to $F^{3/2}$. Apart from the different scaling ε^{-1}, this analogy does not carry to the details of the proof. Firstly, it must be noted that the compactness properties for functions with $F_\varepsilon(u_\varepsilon) = O(\varepsilon^{3/2})$ are much more difficult to prove by the cancellations that may occur in the integral $\int_0^\theta \varphi^3(u_\varepsilon^2-1)\,d\varphi$ due to the fact that $u_\varepsilon^2 - 1$ may change sign. Secondly, the way the constant c_0 is computed involves some optimal transitions that exhibit a sort of *Gibbs' phenomenon*: even though their limit takes only the value ± 1 these transitions must take values external to the interval $[-1,1]$.

In a final section, we show that this Gibbs' phenomenon is substantial: if we impose the constraint $|u_\varepsilon| \le 1$ to a sequence (u_ε) converging to u, then not only the values $\varepsilon^{-3/2}F_\varepsilon(u_\varepsilon)$ cannot converge to the value $F^{3/2}(u)$, but they must even diverge. We show that with this additional constraint the correct scaling is $\varepsilon^{-4/3}$. The scaled energies still converge to a phase-transition functional, but this time of a non-local form (see Theorem 5.3).

The Γ-convergence analysis presented here requires new and sophisticated strategies and allows to reconstruct and justify equilibrium everted shapes, as for example those predicted in the paper by Truesdell [24]. We believe that the techniques developed here can be adapted to other transition problems in nonlinear elasticity leading to the study of functionals with similar nonlocal terms. Moreover, our analysis can be generalized to different weighted spaces. For these reasons we report our results with reference to a more general weighted functional (details in the next section).

Finally, we point out that functionals J_ε are derived from the scaled energy

$$\frac{1}{\varepsilon}\int_{\mathcal{C}_\varepsilon}(\mu|\mathbf{D}|^2+\frac{\lambda}{2}(\operatorname{trace}\mathbf{D})^2)\,dx,$$

where $\mathcal{C}_\varepsilon$ parameterizes a thin spherical shell of thickness ε, $\mathbf{D}=\frac{\mathbf{1}}{\mathbf{2}}(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathbf{T}})+\frac{\mathbf{1}}{\mathbf{2}}(\nabla\mathbf{u}\nabla\mathbf{u}^{\mathbf{T}})$ is the *nonlinear deformation tensor* related to the deformation $\mathbf{u}$ of the shell, and λ,μ are the Lamé constants (we refer *e.g.* to [22] for the precise derivation). In this way, our paper may be partly related to recent works on dimension-reduction for thin structures by the use of Γ-convergence (see *e.g.* Le Dret and Raoult [19] for the limit analysis of thin shells, Friesecke *et al.* [13] for the analysis under various scaling, Ben Belgacem *et al.* [5] and Conti and Maggi [11] for complex patterns in recovery sequences, Braides *et al.* [10] for an example of application of the localization methods of Γ-convergence to thin structures, etc.).

2 Setting of the problem

We recall the definition of Γ-*convergence* of a sequence of functionals F_j defined on $L^p(0,1)$. We say that (F_j) Γ-converges to F (with respect to the L^p-convergence) on $L^p(0,1)$ if for all $u\in L^p(0,1)$ we have:

(i) (*liminf inequality*) for all sequences (u_j) converging to u we have

$$F(u)\le\liminf_{j\to+\infty}F_j(u_j);$$

(ii) (*limsup inequality*) there exists a sequence (u_j) converging to u such that

$$F(u)\ge\limsup_{j\to+\infty}F_j(u_j)\,.$$

If (i) and (ii) hold we write $F(u)=\Gamma(L^p)\text{-}\lim_{j\to+\infty}F_j(u)$ and F is the Γ-limit of F_j.

We also introduce the notation

$$\Gamma\text{-}\liminf_{j\to+\infty}F_j(u)=\inf\Big\{\liminf_{j\to+\infty}F_j(u_j):\,u_j\to u\Big\},$$

$$\Gamma\text{-}\limsup_{j\to+\infty}F_j(u)=\inf\Big\{\limsup_{j\to+\infty}F_j(u_j):\,u_j\to u\Big\},$$

so that the equality $\Gamma\text{-}\liminf_{j\to+\infty}F_j(u)=\Gamma\text{-}\limsup_{j\to+\infty}F_j(u)$ is equivalent to the existence of the $\Gamma(L^p)\text{-}\lim_{j\to+\infty}F_j(u)$.

We will say that a family (F_ε) Γ-converges to F if for all sequences (ε_j) of positive numbers converging to 0 (i) and (ii) above are satisfied with F_{ε_j} in place of F_j and we write $F(u)=\Gamma(L^p)\text{-}\lim_{\varepsilon\to0}F_\varepsilon(u)$.

Analogously, we define the Γ-convergence of (F_ε) with respect to the weak L^p-convergence, for which we write $F(u) = \Gamma(\text{w-}L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u)$.

For a comprehensive study of Γ-convergence we refer to [8] and [12] (see also [9] Part 2).

Our general weight function $\rho : [0,1] \mapsto \mathbf{R}$ will be a non-decreasing, continuous function, strictly positive on $(0,1]$. These conditions are quite general. In particular, we may take $\rho(\theta) = \theta^3$ to cover the case of the energies described in the Introduction or ρ a positive constant.

For all $\varepsilon > 0$ and $\alpha \geq 0$ we define

$$F_\varepsilon^\alpha(u) = \varepsilon^{2-\alpha} \int_0^1 \rho\,(u')^2\,d\theta \;+ \varepsilon^{-\alpha} \int_0^1 \frac{1}{\rho(\theta)} \Big(\int_0^\theta \rho\,(u^2-1)\,d\varphi\Big)^2 d\theta$$

for $u \in H^1(0,1)$. The functionals are understood to take the value $+\infty$ where not otherwise defined. We will isolate particular values of α for which the Γ-limit is meaningful.

We will separately consider the following cases:

(1) (Section 3) $\alpha = 0$. In this case minimizing sequences are weakly precompact in $L^2_{\rm loc}(0,1)$; hence, we compute the Γ-limit of F_ε^0 with respect to that convergence and for every $u \in L^2_{\rm loc}(0,1)$ we get

$$F^0(u) = \min\Big\{\int_0^1 \frac{1}{\rho(\theta)}\Big(\mu([0,\theta)) - \int_0^\theta \rho\;d\varphi\Big)^2 d\theta \,:\, \mu \geq \rho\,u^2\,d\varphi\Big\},$$

where the minimum is taken over all non-negative measures μ. The set of the minimum points of F^0 is $\{|u| \leq 1\}$.

(2) (Section 4) $\alpha = 3/2$. In this case we scale F_ε^0 further, and study the limit of $F_\varepsilon^{3/2} = \varepsilon^{-3/2} F_\varepsilon^0$. We prove that minimizing sequences are precompact with respect to the strong $L^1_{\rm loc}(0,1)$-convergence, and their limits u belong to $BV_{\rm loc}((0,1);\{-1,1\})$; *i.e.*, u is locally piecewise constant on $(0,1)$ and it only takes the values 1 and -1. We compute the Γ-limit $F^{3/2}$ of $F_\varepsilon^{3/2}$ with respect to that convergence and we get

$$F^{3/2}(u) = c_0 \sum_{\theta\in S(u)} \rho(\theta)$$

for every $u \in BV_{\rm loc}((0,1);\{-1,1\})$, where $S(u)$ is the set of points where u jumps between the points 1 and -1, and

$$c_0 = \inf_{T>0}\,\inf\Big\{\int_{-T}^T (v')^2\,ds \;+ \int_{-T}^T \Big(\int_{-T}^\sigma (v^2-1)\,ds\Big)^2 d\sigma \;: \\ v \in H^1(-T,T), \quad v(\pm T) = \pm 1, \quad \int_{-T}^T (v^2-1)\,ds = 0\Big\}.$$

With this choice of the scaling we get a result of 'Modica-Mortola' type with a different characterization of the constant c_0 (see [20], [21] or [7]).

(3) (Section 5) In this section, we show that the characteristic scale changes if we impose the restriction that $u \in H^1((0,1);[-1,1])$, and that in that case the correct scaling power is $\alpha = 4/3$. We only treat the case $\rho = 1$ for the case of simplicity. By (1) above, we have that the Γ-limit of the restriction of F^0_ε to $H^1((0,1);[-1,1])$, G^0_ε, is zero. We rescale then G^0_ε to get a non trivial limit problem. Hence, we consider the family of functionals $G^{4/3}_\varepsilon = \varepsilon^{-4/3} G^0_\varepsilon$ and we prove that the minimizing sequences are compact with respect to the strong L^1-convergence and its Γ-limit is nonlocal

$$G^{4/3}(u) = \inf_{T>0} \inf\Big\{ \sum_{i\in I} \Big(\int_{-T}^{T} (v_i')^2\, ds + (\theta_{i+1} - \theta_i) \sum_{j=1}^{i} \int_{-T}^{T} (v_j^2 - 1)\, ds \Big)^2 \Big) : \\ v_i \in H^1((-T,T);[-1,1]), \quad v_i(\pm T) = \pm 1 \Big\},$$

for every $u \in BV_{\mathrm{loc}}((0,1);\{-1,1\})$, where we have labeled the points in $S(u)$ by a set of indices $I \subset \mathbf{N}$ in such a way that $\theta_i < \theta_{i+1}$.

3 The case $\alpha = 0$: oscillations

We consider the case $\alpha = 0$ first; *i.e.*,

$$F^0_\varepsilon(u) = \varepsilon^2 \int_0^1 \rho\, (u')^2\, d\theta + \int_0^1 \frac{1}{\rho(\theta)} \Big(\int_0^\theta \rho\, (u^2 - 1)\, d\varphi \Big)^2\, d\theta$$

for $u \in H^1_{\mathrm{loc}}(0,1)$. In order to choose the topology in which to frame our limit problem we have to examine the compactness properties of sequences with bounded energy. Note that the presence of ε in the first term of $F^0_\varepsilon(u)$ only, suggests the use of the weak L^2-convergence.

Theorem 3.1 (Compactness) *Let (u_ε) be a sequence such that $\sup_\varepsilon F^0_\varepsilon(u_\varepsilon) < +\infty$, then, up to subsequences, (u_ε) converges weakly in $L^2_{\mathrm{loc}}(0,1)$.*

Proof. By assumption $\sup_\varepsilon \int_0^1 (1/\rho(\theta)) \Big(\int_0^\theta \rho\, (u_\varepsilon^2 - 1)\, d\varphi \Big)^2 d\theta < +\infty$, which implies that $\sup_\varepsilon \int_0^1 \Big(\int_0^\theta \rho\, u_\varepsilon^2\, d\varphi \Big)^2 d\theta < +\infty$; hence, by Hölder's inequality, there exists a constant c independent of ε such that

$$\int_0^1 \Big(\int_0^\theta \rho\, u_\varepsilon^2\, d\varphi \Big) d\theta \le c \tag{3.1}$$

for every $\varepsilon > 0$. By the monotonicity of $\theta \mapsto \int_0^\theta \rho\, u_\varepsilon^2\, d\varphi$ we have that for a fixed $\theta_0 \in (0,1)$

$$\int_0^{\theta_0} \rho\, u_\varepsilon^2\, d\varphi \le \int_0^\theta \rho\, u_\varepsilon^2\, d\varphi$$

for every $\theta \ge \theta_0$. Hence we get that

$$\int_{\theta_0}^1 \Big(\int_0^{\theta_0} \rho\, u_\varepsilon^2\, d\varphi \Big) d\theta \le \int_{\theta_0}^1 \Big(\int_0^\theta \rho\, u_\varepsilon^2\, d\varphi \Big) d\theta \le \int_0^1 \Big(\int_0^\theta \rho\, u_\varepsilon^2\, d\varphi \Big) d\theta$$

and, by (3.1), we can conclude that for every $\theta_0 \in (0,1)$ there exists a constant $c(\theta_0)$ depending only on θ_0 such that $\int_0^{\theta_0} \rho\, u_\varepsilon^2\, d\varphi \leq c(\theta_0)$ for every $\varepsilon > 0$, which gives the compactness of the sequence (u_ε) in $L^2_{\rm loc}$. □

Remark 3.2 The following example shows that we cannot expect weak compactness, but only local weak compactness, in $L^2(0,1)$ for a sequence (u_ε) with $\sup_\varepsilon F^0_\varepsilon(u_\varepsilon) < +\infty$. In fact, consider

$$u_\varepsilon(\theta) = \varepsilon^{-7/5}(\theta - \theta_\varepsilon)^+$$

with $\theta_\varepsilon = 1 - \varepsilon^{4/5}$, then $F^0_\varepsilon(u_\varepsilon) \leq c$ for every $\varepsilon > 0$ but

$$\int_0^1 u_\varepsilon^2\, d\theta = \frac{1}{3}\varepsilon^{-2/5}\,.$$

Lemma 3.3 *Let*

$$G(u) = \int_0^1 \frac{1}{\rho(\theta)} \Big(\int_0^\theta \rho\,(u^2 - 1)\, d\varphi \Big)^2 d\theta\,.$$

The lower semicontinuous envelope of G with respect to the weak $L^2_{\rm loc}$ topology is

$$\overline{G}(u) := \min\Big\{ \int_0^1 \frac{1}{\rho(\theta)} \Big(\mu([0,\theta)) - \int_0^\theta \rho\; d\varphi \Big)^2 d\theta \;:\; \mu \geq \rho u^2\, d\varphi \Big\}\,,$$

where the minimum is taken in the space $\mathcal{M}^+([0,1))$ of locally finite positive measures on $[0,1)$.

Proof. Let (u_N) be a sequence weakly converging to u in $L^2_{\rm loc}$ such that $\lim_{N\to+\infty} G(u_N) < +\infty$ and the sequence of positive measures $(\rho u_N^2\, d\varphi)$ converges $\mathrm{w}^*_{\rm loc}$ to μ in $\mathcal{M}^+([0,1))$. Then, by the lower semicontinuity of the L^2-norm, we have that

$$\frac{\mu(\theta-\delta,\theta+\delta)}{\delta} = \lim_{N\to+\infty} ⨍_{\theta-\delta}^{\theta+\delta} \rho u_N^2\, d\varphi \geq ⨍_{\theta-\delta}^{\theta+\delta} \rho u^2\, d\varphi\,;$$

by the Besicovitch Derivation Theorem, we conclude that for almost every θ

$$\frac{d\mu}{d\varphi}(\theta) \geq \rho(\theta) u^2(\theta)\,.$$

Since μ has at most countably many atoms, by the $\mathrm{w}^*_{\rm loc}$-convergence in $\mathcal{M}^+([0,1))$, we have that

$$\lim_{N\to+\infty} \int_0^\theta \rho u_N^2\, d\varphi = \mu([0,\theta)) \qquad \text{for a.e.}\, \theta \in (0,1)\,. \tag{3.2}$$

It follows that

$$\frac{1}{\sqrt{\rho(\theta)}} \Big(\int_0^\theta \rho\,(u_N^2 - 1)\, d\varphi \Big) \quad \overset{L^2}{\rightharpoonup} \quad \frac{1}{\sqrt{\rho(\theta)}} \Big(\mu([0,\theta)) - \int_0^\theta \rho\, d\varphi \Big)$$

and by the weak lower semicontinuity of the L^2-norm we get

$$\lim_{N\to+\infty} G(u_N) \geq \overline{G}(u)\,.$$

Note that the functional

$$G(\mu) = \int_0^1 \frac{1}{\rho(\theta)}\Big(\mu([0,\theta)) - \int_0^\theta \rho\; d\varphi\Big)^2\, d\theta$$

is weakly lower semicontinuous and coercive in $\mathcal{M}^+([0,1))$. Moreover, the set $\{\mu \geq \rho u^2\, d\varphi\}$ is convex; hence, the minimum is attained.

We now check the limsup inequality for every $u \in L^2_{\rm loc}(0,1)$ such that $\overline{G}(u) < +\infty$. Let $\mu \in \mathcal{M}^+([0,1))$ be such that

$$\overline{G}(u) = \int_0^1 \frac{1}{\rho(\theta)}\Big(\mu([0,\theta)) - \int_0^\theta \rho\; d\varphi\Big)^2\, d\theta\,.$$

For $0 < a < 1$ we define

$$u^a(\theta) = \begin{cases} u(\theta) & a \leq \theta \leq 1-a \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mu^a_N([0,\theta)) = \begin{cases} 0 & \text{if } \theta < a - 1/N \\ \mu([0,a)) & \text{if } a - 1/N \leq \theta < a \\ \mu([0,\theta)) & \text{if } a \leq \theta < 1-a \\ \mu([0,1-a)) & \text{if } 1-a \leq \theta < 1 \end{cases}$$

for $N \in \mathbf{N}$ and $N > 1$. Then, $u^a \in L^2(0,1)$, $u^a \to u$ in $L^2_{\rm loc}(0,1)$ as $a \to 0^+$, μ^a_N converges to the measure μ^a defined by

$$\mu^a([0,\theta)) = \begin{cases} 0 & \text{if } \theta < a \\ \mu([0,\theta)) & \text{if } a \leq \theta < 1-a \\ \mu([0,1-a)) & \text{if } 1-a \leq \theta < 1, \end{cases}$$

and

$$\overline{G}(u^a) \leq \int_a^1 \frac{1}{\rho(\theta)}\Big(\mu^a([0,\theta)) - \int_0^\theta \rho\; d\varphi\Big)^2\, d\theta + o(1) \leq \overline{G}(u) + o(1) \quad (3.3)$$

as $a \to 0^+$. In fact, since by assumption $\overline{G}(u) < +\infty$, then we have

$$\begin{aligned} & \int_a^1 \frac{1}{\rho(\theta)}\Big(\mu^a([0,\theta)) - \int_0^\theta \rho\; d\varphi\Big)^2\, d\theta \\ = \;& \int_a^{1-a} \frac{1}{\rho(\theta)}\Big(\mu([0,\theta)) - \int_0^\theta \rho\; d\varphi\Big)^2\, d\theta \\ & + \int_{1-a}^1 \frac{1}{\rho(\theta)}\Big(\mu([0,1-a)) - \int_0^\theta \rho\; d\varphi\Big)^2\, d\theta \\ \leq \;& \overline{G}(u) + 2\int_{1-a}^1 \frac{\mu([1-a,\theta))^2}{\rho(\theta)}\, d\theta \\ & + \int_{1-a}^1 \frac{1}{\rho(\theta)}\Big(\mu([0,\theta)) - \int_0^\theta \rho\; d\varphi\Big)^2\, d\theta \\ = \;& \overline{G}(u) + o(1)\,, \end{aligned}$$

as $a \to 0^+$.

We denote by $I_N = \{0, \cdots, N-1\}$ and by

$$\begin{cases} \overline{u}_N(\theta) = \fint_{i/N}^{(i+1)/N} u^a \, d\varphi \,, & i/N \le \theta \le (i+1)/N \,, \quad i \in I_N \\ v_N(\theta) = N \, \mu_N^a([i/N, (i+1)/N)) \,, & i/N \le \theta \le (i+1)/N \,, \quad i \in I_N \,. \end{cases}$$

Note that, in particular

$$\overline{u}_N(\theta) = 0 \,, \qquad 0 \le \theta \le [Na]/N$$

and

$$v_N(\theta) = \begin{cases} 0 & 0 \le \theta \le ([Na]-1)/N \\ N\mu([0,a)) & ([Na]-1)/N \le \theta \le [Na]/N \,. \end{cases}$$

Hence, we define

$$u_N(\theta) = \begin{cases} 0 & 0 \le \theta \le ([Na]-1)/N \\ \sqrt{N\mu([0,a))/\rho(\theta)} & ([Na]-1)/N \le \theta \le [Na]/N \\ \overline{u}_N + \sqrt{(v_N/\rho_N^i) - (\overline{u}_N)^2} & i/N \le \theta \le i/N + 1/2N, \\ & \quad i = [Na] \cdots, N-1 \\ c_N\Big(\overline{u}_N - \sqrt{(v_N/\rho_N^i) - (\overline{u}_N)^2}\Big) & i/N + 1/2N \le \theta \le (i+1)/N, \\ & \quad i = [Na] \cdots, N-1 \end{cases} \tag{3.4}$$

where

$$c_N(\theta) = \rho(\theta - 1/2N)/\rho(\theta) \,, \quad [Na]/N + 1/2N \le \theta \le 1$$

($[t]$ denotes the *integer part* of t). Finally, it remains to fix ρ_N^i for $i = [Na], \cdots, N-1$ such that

$$v_N(\theta) \ge \rho_N^i \, (\overline{u}_N(\theta))^2 \,, \quad i/N \le \theta \le (i+1)/N \,, \quad i = [Na] \cdots, N-1 \,.$$

Since $\mu_N^a \ge \rho \, (u^a)^2 \, d\varphi$, for every $i = [Na], \cdots, N-1$, there exists $\eta_N^i \in [i/N, (i+1)/N)$ such that

$$\begin{aligned} \mu_N^a[i/N, (i+1)/N) \quad &\ge \quad \int_{i/N}^{(i+1)/N} \rho \, (u^a)^2 \, d\varphi = \rho(\eta_N^i) \int_{i/N}^{(i+1)/N} (u^a)^2 \, d\varphi \\ &\ge \quad \rho(\eta_N^i) N \Big(\int_{i/N}^{(i+1)/N} u^a \, d\varphi \Big)^2 = \rho(\eta_N^i) \frac{1}{N} (\overline{u}_N)^2 \,. \end{aligned}$$

Hence, we choose $\rho_N^i = \rho(\eta_N^i)$ in (3.4).

By definition (u_N) is bounded in L^2; hence, up to subsequence, it converges weakly in L^2. To identify the weak limit function with u^a it is sufficient to check that $\lim_{N\to+\infty}\int_b^d u_N = \int_b^d u^a$, for every $(b,d)\subseteq(0,1)$. In fact,

$$\int_0^{[Na]/N} u_N\,d\theta = \sqrt{N\mu([0,a))}\int_{([Na]-1)/N}^{[Na]/N}\frac{d\theta}{\sqrt{\rho(\theta)}} \le \frac{1}{\sqrt{N}}\sqrt{\frac{\mu([0,a))}{\rho(([Na]-1)/N)}}\,; \tag{3.5}$$

while, for $i=[Na],\cdots,N-1$, we have that

$$\begin{aligned}
\int_{i/N}^{(i+1)/N} u_N\,d\theta &= \int_{i/N}^{i/N+1/2N} \overline{u}_N + \sqrt{(v_N/\rho_N^i)-(\overline{u}_N)^2}\,d\theta \qquad (3.6)\\
&\quad + \int_{i/N+1/2N}^{(i+1)/N} \frac{\rho(\theta-\frac{1}{2N})}{\rho(\theta)}\Big(\overline{u}_N - \sqrt{(v_N/\rho_N^i)-(\overline{u}_N)^2}\Big)\,d\theta\\
&= \int_{i/N}^{(i+1)/N} \overline{u}_N\,d\theta + \int_{i/N+1/2N}^{(i+1)/N} \overline{u}_N\Big(\frac{\rho(\theta-\frac{1}{2N})}{\rho(\theta)}-1\Big)\,d\theta\\
&\quad + \int_{i/N+1/2N}^{(i+1)/N} \sqrt{(v_N/\rho_N^i)-(\overline{u}_N)^2}\Big(1-\frac{\rho(\theta-\frac{1}{2N})}{\rho(\theta)}\Big)\,d\theta\\
&= \Big(\int_{i/N}^{(i+1)/N} u^a\,d\theta\Big) + \int_{i/N+1/2N}^{(i+1)/N} \overline{u}_N\Big(\frac{\rho(\theta-\frac{1}{2N})}{\rho(\theta)}-1\Big)\,d\theta\\
&\quad + \int_{i/N+1/2N}^{(i+1)/N} \sqrt{(v_N/\rho_N^i)-(\overline{u}_N)^2}\Big(1-\frac{\rho(\theta-\frac{1}{2N})}{\rho(\theta)}\Big)\,d\theta\,.
\end{aligned}$$

If we sum up on i, by Hölder inequality, we get that

$$\begin{aligned}
&\Big|\sum_{i\ge[Na]}\int_{i/N+1/2N}^{(i+1)/N} \overline{u}_N\Big(\frac{\rho(\theta-\frac{1}{2N})}{\rho(\theta)}-1\Big)\,d\theta\Big|\\
&\le \Big(\int_0^1 (\overline{u}_N)^2\,d\theta\Big)^{1/2}\Big(\int_{a/2}^1\Big(1-\frac{\rho(\theta-\frac{1}{2N})}{\rho(\theta)}\Big)^2\,d\theta\Big)^{1/2}
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&\Big|\sum_{i\ge[Na]}\int_{i/N+1/2N}^{(i+1)/N} \sqrt{(v_N/\rho_N^i)-(\overline{u}_N)^2}\Big(1-\frac{\rho(\theta-\frac{1}{2N})}{\rho(\theta)}\Big)\,d\theta\Big|\\
&\le \Big(\sum_{i\ge[Na]}\int_{i/N+1/2N}^{(i+1)/N} (v_N/\rho_N^i)+(\overline{u}_N)^2\,d\theta\Big)^{1/2}\Big(\int_{a/2}^1\Big(1-\frac{\rho(\theta-\frac{1}{2N})}{\rho(\theta)}\Big)^2\,d\theta\Big)^{1/2}\\
&\le \Big(\frac{\mu([0,1-a))}{\rho(a/2)}+\int_0^1(\overline{u}_N)^2\,d\theta\Big)^{1/2}\Big(\int_{a/2}^1\Big(1-\frac{\rho(\theta-\frac{1}{2N})}{\rho(\theta)}\Big)^2\,d\theta\Big)^{1/2}.
\end{aligned} \tag{3.8}$$

Note that the sequence $(\overline{u}_N)$ is bounded in the L^2 since it converges to u^a strongly; moreover,

$$\lim_{N\to+\infty}\int_{a/2}^1\Big(1-\frac{\rho(\theta-\frac{1}{2N})}{\rho(\theta)}\Big)^2\,d\theta = 0\,.$$

Hence, by (3.5), (3.6), (3.7) and (3.8), we can easily conclude that $\lim_{N\to+\infty}\int_b^d u_N = \int_b^d u^a$, for every $(b,d)\subseteq(0,1)$ and, therefore, the weak convergence of (u_N) to u^a in $L^2(0,1)$.

We now examine

$$\begin{aligned} G(u_N) &= \int_0^a \frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\, u_N^2 - \int_0^\theta \rho\ d\varphi\Big)^2 d\theta \\ &\quad + \int_a^1 \frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\, u_N^2 - \int_0^\theta \rho\ d\varphi\Big)^2 d\theta\,; \end{aligned} \tag{3.9}$$

in particular, we have that

$$\begin{aligned} &\int_0^{[Na]/N} \frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\, u_N^2\, d\varphi\Big)^2 d\theta \\ = \quad & N^2\mu([0,a))^2 \int_{([Na]-1)/N}^{[Na]/N} \frac{1}{\rho(\theta)}\Big(\theta - \frac{[Na]-1}{N}\Big)^2 d\theta \\ \leq \quad & \frac{1}{3N}\,\frac{\mu([0,a))^2}{\rho(([Na]-1)/N)}\,. \end{aligned} \tag{3.10}$$

Then, passing to the limit as $N\to+\infty$ in (3.10), we get that

$$\limsup_{N\to+\infty}\int_0^a \frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\, u_N^2 - \int_0^\theta \rho\ d\varphi\Big)^2 d\theta = o(1)\,, \tag{3.11}$$

as $a\to 0^+$. It remains to study the second term in (3.9). For $i=[Na],\cdots,N-1$ we have that

$$\begin{aligned} &\int_{i/N}^{(i+1)/N} \rho\, u_N^2\, d\theta \\ = \quad & \int_{i/N}^{i/N+1/2N} \rho\Big(\frac{v_N}{\rho_N^i} + 2\overline{u}_N\,\sqrt{(v_N/\rho_N^i)-(\overline{u}_N)^2}\Big)\, d\theta \\ &+ \int_{i/N+1/2N}^{(i+1)/N} \rho\Big(\theta-\frac{1}{2N}\Big)\Big(\frac{v_N}{\rho_N^i} - 2\overline{u}_N\,\sqrt{(v_N/\rho_N^i)-(\overline{u}_N)^2}\Big)\, d\theta \\ = \quad & 2\int_{i/N}^{i/N+1/2N} \frac{\rho(\theta)}{\rho(\eta_N^i)}\, v_N(\theta)\, d\theta\,, \end{aligned} \tag{3.12}$$

where $\eta_N^i\in[i/N,(i+1)/N)$, $i=[Na],\cdots,N-1$. We recall that $\rho u_N^2=0$ in $[0,([Na]-1)/N)\cup[([N(1-a)]+1)/N,1]$ while $\rho u_N^2 = N\mu([0,a))$ in $[([Na]-1)/N,[Na]/N)$; hence, by (3.12), we have that

$$\lim_{N\to+\infty}\int_0^\theta \rho\, u_N^2\, d\theta \quad = \quad \mu^a([0,\theta)) \quad \text{a.e.}\,\theta\in(a,1)\,.$$

Moreover, as already observed, (u_N) is bounded in L^2; hence,

$$\frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\, u_N^2\, d\varphi - \int_0^\theta \rho\ d\varphi\Big)^2 \leq c$$

for every $\theta \in (a,1)$ and for every N. Therefore, we can apply the Lebesgue's Theorem in the second term of (3.9) and by (3.11), (3.3) we have

$$\begin{aligned}\limsup_{N\to+\infty} G(u_N) &\leq \int_a^1 \frac{1}{\rho(\theta)}\Big(\mu^a([0,\theta)) - \int_0^\theta \rho\, d\varphi\Big)^2 d\theta + o(1)\\ &\leq \overline{G}(u) + o(1)\,, \quad \text{as} \quad a \to 0^+\,. \end{aligned} \tag{3.13}$$

Since $u^a \to u$ in $L^2_{\rm loc}(0,1)$, as $a \to 0^+$, we can conclude that, passing to a further subsequence, $u_N \rightharpoonup u$ in $L^2_{\rm loc}(0,1)$ as $N \to +\infty$ and

$$\limsup_{N\to+\infty} G(u_N) \leq \overline{G}(u)$$

as desired. □

Remark 3.4 Note that, $\{|u| \leq 1\}$ is the set of minimizers for $\overline{G}(u)$. In fact, if $\overline{G}(u) = 0$ then

$$\mu([0,\theta)) = \int_0^\theta \rho\, d\varphi \qquad \text{for a.e. } \theta \in (0,1)\,.$$

Hence, $\mu = \rho\, d\varphi$; the constraint becomes $\rho u^2 \leq \rho$, so that $u^2 \leq 1$. Conversely, if u is a function such that $u^2 \leq 1$ then $\mu = \rho\, d\varphi$ satisfies the constraint and $\overline{G}(u) = 0$.

Remark 3.5 The functional $\overline{G}$ can be estimated from above and from below as follows

$$G^-(u) \leq \overline{G}(u) \leq G^+(u) \tag{3.14}$$

where

$$G^+(u) = \int_0^1 \frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\,(u^2-1)^+\, d\varphi\Big)^2 d\theta\,,$$

and

$$G^-(u) = \int_0^1 \frac{1}{\rho(\theta)}\Big(\Big(\int_0^\theta \rho\,(u^2-1)\, d\varphi\Big)^+\Big)^2 d\theta\,.$$

The second inequality in (3.14) easily follows by testing the definition of $\overline{G}(u)$ with the measure $\mu = \rho(u^2 \vee 1)\, d\varphi$. To check the first inequality from below we preliminary note that $G(u) \geq G^-(u)$ and, by Fatou's lemma and the weak $L^2_{\rm loc}$-lower semicontinuity of the functional $\int_0^\theta \rho\, u^2\, d\varphi$ for every $\theta \in (0,1)$, we get that G^- is weakly $L^2_{\rm loc}$ lower semicontinuous. Hence, by Lemma 3.3 and the lower semicontinuity of G^- we have that for every $u \in L^2_{\rm loc}(0,1)$ there exists a sequence u_N weakly $L^2_{\rm loc}$ converging to u such that

$$\overline{G}(u) = \lim_{N\to+\infty} G(u_N) \geq \liminf_{N\to+\infty} G^-(u_N) \geq G^-(u)\,.$$

We show now an example of function u such that $\overline{G}(u) = G^-(u)$. Let us consider for simplicity $\rho = 1$. Let

$$u(\varphi) = \begin{cases} \sqrt{2} & \varphi \in (0,1/4) \\ 0 & \varphi \in (1/4,1)\,. \end{cases} \tag{3.15}$$

Note that

$$\int_0^\theta (u^2 - 1)\, d\varphi < 0 \qquad \text{for all } \theta > 1/2\,,$$

so that

$$G^-(u) = \int_0^{1/4} \theta^2\, d\theta + \int_{1/4}^{1/2} \Big(\frac{1}{2} - \theta\Big)^2 d\theta = \int_0^1 (\mu([0,\theta)) - \theta)^2 d\theta,$$

where the measure $\mu = v\, d\varphi$ is defined by

$$v(\varphi) = \begin{cases} 2 & \varphi \in (0, 1/4) \\ 0 & \varphi \in (1/4, 1/2) \\ 1 & (1/2, 1)\,. \end{cases}$$

Since $\mu \geq u^2\, d\varphi$ we can test the definition of $\overline{G}(u)$ with μ getting $G^-(u) \geq \overline{G}(u)$, and then we have that $G^-(u) = \overline{G}(u)$ by (3.14). A recovery sequence (corresponding to (3.4) with $a = 0$ and $\rho = 1$) for $\overline{G}(u)$ is shown in Fig. 1. It highlights the non-local nature of the oscillations that start at $\varphi = 1/2$ while the target function is 0 on the whole $(1/4, 1)$.

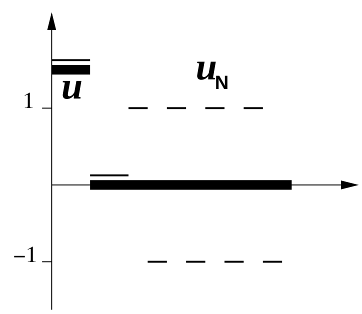

Figure 1: non-local oscillations

Note that u as in (3.15) is also an example of function such that $\overline{G}(u) < G^+(u)$.

Finally, we note that also the inequality $G^- \leq \overline{G}$ is sharp: if we consider

$$u(\varphi) = \begin{cases} 0 & \varphi \in (0, 1/2) \\ \sqrt{2} & \varphi \in (1/2, 1) \end{cases}$$

we have that $G^-(u) = 0$ while $\overline{G}(u) > 0$ by Remark 3.4.

Theorem 3.6 (Γ-convergence result) *We have*

$$\Gamma(\text{w-}L^2_{\text{loc}})\text{-}\lim_{\varepsilon \to 0} F^0_\varepsilon(u) = \overline{G}(u)$$

for every $u \in L^2_{\text{loc}}(0, 1)$.

Proof. Let $u \in L^2_{\rm loc}(0,1)$. By definition of $\overline{G}$ for every sequence (u_ε) $L^2_{\rm loc}$-weakly converging to u we have

$$F^0_\varepsilon(u_\varepsilon) \geq G(u_\varepsilon) \geq \overline{G}(u_\varepsilon)\,;$$

hence by the weak lower semicontinuity of $\overline{G}$ we get the liminf inequality.

Conversely, let $u \in L^2_{\rm loc}(0,1)$ and let

$$u^a(\theta) = \begin{cases} u(\theta) & \text{if } a \leq \theta \leq 1-a \\ 0 & \text{otherwise} \end{cases}$$

with $0 < a < 1$; hence, $u^a \in L^2(0,1)$ and $u^a \to u$ in $L^2_{\rm loc}(0,1)$ as $a \to 0^+$. By Lemma 3.3, there exists a sequence $(u_N) \in L^2(\mathbf{R})$ weakly converging to u^a in $L^2(0,1)$ such that

$$\overline{G}(u^a) = \lim_{N\to+\infty} G(u_N)\,.$$

Let $\eta : \mathbf{R} \mapsto [0,+\infty)$ be a mollifier, we define $\eta_\varepsilon(\theta) = \frac{1}{\sqrt{\varepsilon}}\,\eta(\frac{\theta}{\sqrt{\varepsilon}})$ then $u^N_\varepsilon = u_N * \eta_\varepsilon \in C^\infty_c(\mathbf{R})$ and $u^N_\varepsilon \to u_N$ in $L^2(0,1)$ as $\varepsilon \to 0$ for every N. Hence,

$$\begin{aligned} \lim_{\varepsilon\to0} F^0_\varepsilon(u^N_\varepsilon) &= \lim_{\varepsilon\to0}\Big(\varepsilon^2\int_0^1 \rho\,(u_N * \eta'_\varepsilon)^2\,d\theta + G(u^N_\varepsilon)\Big) \\ &= \lim_{\varepsilon\to0} G(u^N_\varepsilon) = G(u_N) \end{aligned}$$

and, by the lower semicontinuity of the Γ-limsup and (3.3), we have that

$$\begin{aligned} \Gamma\text{-}\limsup_{\varepsilon\to0} F^0_\varepsilon(u^a) &\leq \liminf_{N\to+\infty}\Big(\Gamma\text{-}\limsup_{\varepsilon\to0} F^0_\varepsilon(u_N)\Big) \\ &\leq \liminf_{N\to+\infty} G(u_N) = \overline{G}(u^a) \leq \overline{G}(u) + o(1) \end{aligned}$$

as $a \to 0^+$. We again use the lower semicontinuity of the Γ-limsup to get, as $a \to 0^+$, that

$$\Gamma\text{-}\limsup_{\varepsilon\to0} F^0_\varepsilon(u) \leq \overline{G}(u)$$

which concludes the proof of the limsup inequality. (See [8] Remark 1.29). □

4 The case $\alpha = 3/2$: phase transitions

In Section 3 we have shown that the set of the minimum points of the Γ-limit F^0 is $\{|u| \leq 1\}$ and $\min F^0 = 0$. To reduce the choice in the minimizers of the limit problem we may further rescale F^0_ε; the next meaningful scaling is $\alpha = 3/2$. We then consider the following family of functionals

$$F^{3/2}_\varepsilon(u) = \sqrt{\varepsilon}\int_0^1 \rho\,(u')^2\,d\theta \,+ \varepsilon^{-3/2}\int_0^1 \frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\,(u^2-1)\,d\varphi\Big)^2\,d\theta$$

for $u \in H^1(0,1)$.

Theorem 4.1 (Compactness) *Let (u_ε) be a sequence of equi-bounded energy; i.e.,* $\sup_\varepsilon F_\varepsilon^{3/2}(u_\varepsilon) < +\infty$*, then (u_ε) is equi-bounded in $L^\infty_{\rm loc}(0,1)$ and, up to subsequences, (u_ε) converges to $u \in BV_{\rm loc}((0,1);\{-1,1\})$ strongly in $L^1_{\rm loc}$.*

Proof. Let $\eta_\varepsilon^\pm \neq \pm 1$ such that $-1 < \eta_\varepsilon^+, \eta_\varepsilon^- < 1$ or $\eta_\varepsilon^+, \eta_\varepsilon^- > 1$ or $\eta_\varepsilon^+, \eta_\varepsilon^- < -1$. We denote by $(\delta_\varepsilon^-, \delta_\varepsilon^+)$ an interval such that $u_\varepsilon(\delta_\varepsilon^-) = \eta_\varepsilon^-$, $u_\varepsilon(\delta_\varepsilon^+) = \eta_\varepsilon^+$ and u_ε takes values between η_ε^- and η_ε^+. We use in the sequel the notation $\eta^\pm = \eta_\varepsilon^\pm$, $\delta^\pm = \delta_\varepsilon^\pm$ not to overburden notation. By assumption,

$$\sqrt{\varepsilon}\int_{\delta^-}^{\delta^+} \rho\,(u_\varepsilon')^2\,d\theta \,+ \varepsilon^{-3/2}\int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2 d\theta \le c\,. \qquad (4.1)$$

For any fixed $0 < a < 1$, we assume $\delta^- \ge a$.

Step 1. We define the set

$$A_\varepsilon = \Big\{\theta \in (a,1]: \quad \frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2 \le \varepsilon\Big\}\,,$$

and we denote by A_ε^c its complementary set. Since $\sup_\varepsilon F_\varepsilon^{3/2}(u_\varepsilon) < +\infty$ we have that

$$|A_\varepsilon^c|\varepsilon < \int_{A_\varepsilon^c} \frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2 \le c\varepsilon^{3/2}$$

which implies that there exists $\overline{c}(\varepsilon) \le c$ such that

$$|A_\varepsilon^c| = \overline{c}(\varepsilon)\,\sqrt{\varepsilon}\,. \qquad (4.2)$$

Let $\theta_\varepsilon \in A_\varepsilon$ be such that $\theta_\varepsilon = \max\{\theta \in A_\varepsilon : \theta \le \delta^-\}$; by definition

$$\varepsilon \ge \frac{1}{\rho(\theta_\varepsilon)}\Big(\int_0^{\theta_\varepsilon} \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2 \ge \frac{1}{\rho(1)}\Big(\int_0^{\theta_\varepsilon} \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2.$$

It follows that there exists $c_1(\varepsilon)$ such that $|c_1(\varepsilon)| \le \sqrt{\rho(1)}$ and

$$\int_0^{\theta_\varepsilon} \rho\,(u_\varepsilon^2-1)\,d\varphi = c_1(\varepsilon)\,\sqrt{\varepsilon}\,. \qquad (4.3)$$

Moreover, for every $\theta \in (\theta_\varepsilon, \delta^-)$ we have that

$$\begin{aligned} |u_\varepsilon(\theta) - \eta^-| &= |u_\varepsilon(\theta) - u_\varepsilon(\delta^-)| \le \Big(|A_\varepsilon^c|\int_{\theta_\varepsilon}^{\delta^-}(u_\varepsilon')^2\,d\theta\Big)^{1/2} \\ &\le \Big(\frac{|A_\varepsilon^c|}{\rho(a)}\int_{\theta_\varepsilon}^{\delta^-}\rho\,(u_\varepsilon')^2\,d\theta\Big)^{1/2}. \end{aligned}$$

Since $\sup_\varepsilon F_\varepsilon^{3/2}(u_\varepsilon) < +\infty$, by (4.2) we have that

$$|u_\varepsilon(\theta)| \le |\eta^-| + \tilde{c} \qquad \forall\,\theta \in [\theta_\varepsilon, \delta^-] \qquad (4.4)$$

and

$$\Big|\int_{\theta_\varepsilon}^{\delta^-} \rho\,(u_\varepsilon^2-1)\,d\theta\Big| \le c(\eta^-)\,\sqrt{\varepsilon}\,.$$

where $c(\eta^-) = c\,\rho(1)(1+(|\eta^-|+\tilde{c})^2)$. Hence, there exists $c_2(\varepsilon,\eta^-)$ such that $|c_2(\varepsilon,\eta^-)| \le c(\eta^-)$ and

$$\int_{\theta_\varepsilon}^{\delta^-} \rho\,(u_\varepsilon^2-1)\,d\theta = c_2(\varepsilon,\eta^-)\,\sqrt{\varepsilon}\,. \tag{4.5}$$

By (4.3) and (4.5) we get that

$$\begin{aligned}
&\int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2\,d\theta\\
=\;&\int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)}\Big(\int_0^{\theta_\varepsilon} \rho\,(u_\varepsilon^2-1)\,d\varphi + \int_{\theta_\varepsilon}^{\delta^-} \rho\,(u_\varepsilon^2-1)\,d\varphi + \int_{\delta^-}^{\theta} \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2\,d\theta\\
=\;&c_5(c_1(\varepsilon)+c_2(\varepsilon,\eta^-))^2\,\varepsilon\,\delta + \int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)}\Big(\int_{\delta^-}^{\theta} \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2\,d\theta\\
&+2(c_1(\varepsilon)+c_2(\varepsilon,\eta^-))\,\sqrt{\varepsilon}\int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)}\Big(\int_{\delta^-}^{\theta} \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)\,d\theta\,,
\end{aligned} \tag{4.6}$$

where $c_5 = \fint_{\delta^-}^{\delta^+}(1/\rho)\,d\theta$.

By assumption u_ε takes values between η^- and η^+ for every $\theta\in[\delta^-,\delta^+]$ (note that either $(u_\varepsilon^2-1)>0$ or $(u_\varepsilon^2-1)<0$ for every $\theta\in[\delta^-,\delta^+]$); hence,

$$|u_\varepsilon^2-1| \ge \lambda := ||\eta^-|^2-1| \wedge ||\eta^+|^2-1|$$

where we use the notation λ without an explicit dependence on $\eta^\pm$ since we want to emphasize that λ does not depend on the values of $\eta^\pm$ but on the minimum distance of $|\eta^\pm|^2$ from 1.

Moreover,

$$|u_\varepsilon^2-1| \le c(\eta^\pm) := ||\eta^-|^2-1| \vee ||\eta^+|^2-1|\,,$$

then there exist $c_3(\varepsilon,\eta^\pm)$ and $c_4(\varepsilon,\lambda)$ such that

$$0 < |c_3(\varepsilon,\eta^\pm)| \le \frac{c(\eta^\pm)}{2}\,,\qquad c_4(\varepsilon,\lambda) \ge \frac{\lambda^2}{3}\,\frac{\rho(a)^2}{\rho(1)} \tag{4.7}$$

and

$$\int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)}\Big(\int_{\delta^-}^{\theta} \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)\,d\theta = c_3(\varepsilon,\eta^\pm)\,\delta^2\,, \tag{4.8}$$

$$\int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)}\Big(\int_{\delta^-}^{\theta} \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2\,d\theta = c_4(\varepsilon,\lambda)\,\delta^3\,. \tag{4.9}$$

By (4.6), (4.8) and (4.9) we get that

$$\begin{aligned}
&\int_{\delta^-}^{\delta^+} \frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2\,d\theta \\
= \quad & \delta^3\Big\{c_4(\varepsilon,\lambda)+c_5(c_1(\varepsilon)+c_2(\varepsilon,\eta^-))^2\Big(\frac{\sqrt{\varepsilon}}{\delta}\Big)^2 \\
&\quad +2c_3(\varepsilon,\eta^\pm)(c_1(\varepsilon)+c_2(\varepsilon,\eta^-))\Big(\frac{\sqrt{\varepsilon}}{\delta}\Big)\Big\}. \qquad (4.10)
\end{aligned}$$

Note that c_4 and c_5 are always strictly positive while $c_3 \neq 0$ with sign $(c_3) =$ sign $(u_\varepsilon^2 - 1)$. Moreover,

$$c_4 - (c_3^2/c_5) \geq c(\lambda) \qquad (4.11)$$

where $c(\lambda) := \lambda^2\, c(\delta^\pm)\rho(a)^2/\rho(1)$ with $1/12 \leq c(\delta^\pm) \leq 1/3$ for every $\delta^\pm$. Also in this case we prefer to use the notation $c(\lambda)$ omitting the dependence on $\delta^\pm$ cause of the bound of $c(\delta^\pm)$. We check now (4.11). Let us denote

$$f(\theta) = \int_{\delta^-}^{\theta} \rho\,|u_\varepsilon^2 - 1|\,d\varphi\,;$$

hence, $f'(\theta) = \rho(\theta)|u_\varepsilon^2(\theta) - 1| \geq \lambda\rho(a)$. Then, there exists $\delta_0 \in (\delta^-, \delta^+)$ such that

$$|c_3| = \frac{1}{\delta^2}\int_{\delta^-}^{\delta^+} \frac{f(\theta)}{\rho(\theta)}\,d\theta = \frac{f(\delta_0)}{\delta^2}\int_{\delta^-}^{\delta^+}\frac{d\theta}{\rho(\theta)} = \frac{f(\delta_0)}{\delta}c_5 \qquad (4.12)$$

and

$$|f(\theta) - f(\delta_0)| \geq \lambda\rho(a)\,|\theta - \delta_0|\,.$$

It follows that

$$\begin{aligned}
\int_{\delta^-}^{\delta^+} \frac{(f(\theta)-f(\delta_0))^2}{\rho(\theta)}\,d\theta \quad &\geq \quad (\lambda\rho(a))^2\int_{\delta^-}^{\delta^+}\frac{(\theta-\delta_0)^2}{\rho(\theta)}\,d\theta \\
&\geq \quad (\lambda\rho(a))^2\,\frac{(\delta^+-\delta_0)^3-(\delta^--\delta_0)^3}{3\rho(1)} \\
&= \quad \delta^3\left(\lambda^2\,c(\delta^\pm)\,\frac{\rho(a)^2}{\rho(1)}\right) \qquad (4.13)
\end{aligned}$$

where $1/12 \leq c(\delta^\pm) \leq 1/3$ for every $\delta^\pm$. On the other hand, by (4.12) we have that

$$\begin{aligned}
&\int_{\delta^-}^{\delta^+} \frac{(f(\theta)-f(\delta_0))^2}{\rho(\theta)}\,d\theta \\
= \quad & \int_{\delta^-}^{\delta^+}\frac{f(\theta)^2}{\rho(\theta)}\,d\theta + f(\delta_0)^2\int_{\delta^-}^{\delta^+}\frac{d\theta}{\rho(\theta)} - 2f(\delta_0)\int_{\delta^-}^{\delta^+}\frac{f(\theta)}{\rho(\theta)}\,d\theta \\
= \quad & \delta^3\Big(c_4 - c_5\Big(\frac{f(\delta_0)}{\delta}\Big)^2\Big) \\
= \quad & \delta^3\Big(c_4 - \frac{c_3^2}{c_5}\Big)\,; \qquad (4.14)
\end{aligned}$$

hence, by (4.13) we get (4.11).

We now estimate the term with the derivative in (4.1); by Hölder's inequality we get that

$$\sqrt{\varepsilon}\int_{\delta^-}^{\delta^+} \rho\,(u_\varepsilon')^2\,d\theta \geq \rho(a)\,|\eta^+ - \eta^-|^2\Big(\frac{\sqrt{\varepsilon}}{\delta}\Big)\,. \tag{4.15}$$

By (4.10) and (4.15) we have then

$$\begin{aligned} &\sqrt{\varepsilon}\int_{\delta^-}^{\delta^+} \rho\,(u_\varepsilon')^2\,d\theta\,+\varepsilon^{-3/2}\int_{\delta^-}^{\delta^+}\frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2 d\theta \\ \geq\;& \rho(a)\,|\eta^+ - \eta^-|^2\Big(\frac{\sqrt{\varepsilon}}{\delta}\Big) + \Big(\frac{\delta}{\sqrt{\varepsilon}}\Big)^3\Big(\gamma\Big(\frac{\sqrt{\varepsilon}}{\delta}\Big)^2 \pm \beta\Big(\frac{\sqrt{\varepsilon}}{\delta}\Big) + \alpha\Big)\,, \end{aligned} \tag{4.16}$$

where

$$\alpha = c_4\,,\quad \gamma = c_5(c_1+c_2)^2\,,\quad \beta = 2|c_3(c_1+c_2)|\,.$$

Note that, as already observed, $\alpha > 0$, $\gamma \geq 0$ and $c_3(c_1+c_2)$ may be ≥ 0 or ≤ 0 ($\gamma = 0$ if and only if $\beta = 0$). By (4.11), if we minimize $\gamma(\sqrt{\varepsilon}/\delta)^2 \pm \beta(\sqrt{\varepsilon}/\delta) + \alpha$ in $(\sqrt{\varepsilon}/\delta)$, we have

$$\begin{aligned} &\rho(a)\,|\eta^+ - \eta^-|^2\,\Big(\frac{\sqrt{\varepsilon}}{\delta}\Big) + \Big(\gamma\Big(\frac{\sqrt{\varepsilon}}{\delta}\Big)^2 \pm \beta\Big(\frac{\sqrt{\varepsilon}}{\delta}\Big) + \alpha\Big)\Big(\frac{\delta}{\sqrt{\varepsilon}}\Big)^3 \\ \geq\;& \rho(a)|\eta^+ - \eta^-|^2\,\Big(\frac{\sqrt{\varepsilon}}{\delta}\Big) + \Big(\alpha - \frac{\beta^2}{4\gamma}\Big)\Big(\frac{\delta}{\sqrt{\varepsilon}}\Big)^3 \\ =\;& \rho(a)|\eta^+ - \eta^-|^2\,\Big(\frac{\sqrt{\varepsilon}}{\delta}\Big) + (c_4 - c_3^2/c_5)\Big(\frac{\delta}{\sqrt{\varepsilon}}\Big)^3 \\ \geq\;& \rho(a)|\eta^+ - \eta^-|^2\,\Big(\frac{\sqrt{\varepsilon}}{\delta}\Big) + c(\lambda)\Big(\frac{\delta}{\sqrt{\varepsilon}}\Big)^3\,. \end{aligned} \tag{4.17}$$

Step 2. If

$$|\eta^+ - \eta^-| \geq \zeta > 0\,,$$

with ζ independent of ε, studying the function $x \mapsto \rho(a)|\eta^+ - \eta^-|^2 x + c(\lambda)/x^3$ for $x > 0$, by (4.1), (4.16) and (4.17) we have that $\delta/\sqrt{\varepsilon}$ is bounded; *i.e.,* there exist two positive constants α_1, α_2 such that $\alpha_1\sqrt{\varepsilon} \leq \delta \leq \alpha_2\sqrt{\varepsilon}$.

Step 3. The minimum point of $x \mapsto \rho(a)|\eta^+ - \eta^-|^2 x + c(\lambda)/x^3$ for $x > 0$ is $x_m = c_m/|\eta^+ - \eta^-|^{1/2}$ then by (4.16) and (4.17) we have that

$$\begin{aligned} &\sqrt{\varepsilon}\int_{\delta^-}^{\delta^+} \rho\,(u_\varepsilon')^2\,d\theta\,+\varepsilon^{-3/2}\int_{\delta^-}^{\delta^+}\frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2 d\theta \\ \geq\;& \tilde{c}(\lambda)\,|\eta^+ - \eta^-|^{3/2} \geq \tilde{c}(\lambda)\,\zeta^{3/2}\,; \end{aligned} \tag{4.18}$$

where $\tilde{c}(\lambda) = (\rho(a)c_m + c(\lambda)/c_m^3)$. We recall that λ is the minimum distance of $|\eta^\pm|^2$ from 1. Since (u_ε) is a sequence with bounded energy, and estimate (4.18) depends on λ only, we deduce that the number of transitions of u_ε from η^- to η^+ is equibounded independently of ε.

Conclusions. By (4.1) and (4.18) we conclude that (u_ε) is equi-bounded in $L^\infty_{\rm loc}(0,1)$. Moreover, by Steps 2 and 3, we have that for every fixed $0 < a < 1$ (u_ε) converges in measure in $(a,1)$ to $u \in BV_{\rm loc}((0,1);\{-1,1\})$; hence, up to subsequences, $u_\varepsilon \to u$ a.e. $\theta \in (a,1)$. Since (u_ε) is equi-bounded in $L^\infty_{\rm loc}(0,1)$ we can conclude that, up to subsequences, (u_ε) converges strongly in $L^1_{\rm loc}(0,1)$ to $u \in BV_{\rm loc}((0,1);\{-1,1\})$. □

The following proposition gives an estimate of the measure of the set where a sequence of bounded energy is not close to ± 1.

Proposition 4.2 *Let (u_ε) be a sequence converging to u in $L^1_{\rm loc}(0,1)$ such that $\sup_\varepsilon F^{3/2}_\varepsilon(u_\varepsilon) < +\infty$. Then for every fixed $0 < a < 1$ and $\eta > 0$ there exists $\overline{c} = \overline{c}(\eta)$ such that*

$$|\{\theta \in [a, 1-a] : \, |u^2_\varepsilon(\theta) - 1| > \eta\}| \le \overline{c}\,\sqrt{\varepsilon}\,. \tag{4.19}$$

Proof. Let $0 < a < 1$ and $\eta > 0$ be fixed.

$$\{\theta \in [a, 1-a] : \, |u^2_\varepsilon(\theta) - 1| > \eta\} \subseteq \bigcup_i [\delta_i^-, \delta_i^+]$$

such that $\delta_i^- < \delta_i^+$, $\delta_i^+ \le \delta_j^-$ for every $i < j$ and

$$\begin{cases} |(u_\varepsilon(\delta_i^\pm))^2 - 1| = \eta/2 & \text{execept if } \delta_i^+ = 1 \\ |u^2_\varepsilon(\theta) - 1| > \eta/2 & \text{if } \theta \in (\delta_i^-, \delta_i^+) \\ \exists \delta_i \in (\delta_i^-, \delta_i^+) & \text{such that } |(u_\varepsilon(\delta_i))^2 - 1| = \eta\,. \end{cases}$$

Hence, we may have two cases:

$$u_\varepsilon(\delta_i^-) \neq u_\varepsilon(\delta_i^+) \quad \text{ and hence } \quad u_\varepsilon(\delta_i^\pm) \in \left\{\pm\sqrt{1-\eta/2}\right\} \tag{4.20}$$

or

$$u_\varepsilon(\delta_i^-) = u_\varepsilon(\delta_i^+) \in \left\{\pm\sqrt{1-\eta/2}\,,\, \pm\sqrt{1+\eta/2}\right\}. \tag{4.21}$$

In case (4.20) we may apply Steps 1–3 in Theorem 4.1 with $\eta^\pm \in \left\{\pm\sqrt{1-\eta/2}\right\}$, $\lambda = \eta/2$ and $\zeta = 2\,\sqrt{1-\eta/2}$. While if we are in case (4.21) we consider

$$\eta^- \in \left\{\pm\sqrt{1-\eta/2}\,,\, \pm\sqrt{1+\eta/2}\right\} \quad , \quad \eta^+ \in \left\{\max_{[\delta_i^-, \delta_i^+]} u_\varepsilon\,,\, \min_{[\delta_i^-, \delta_i^+]} u_\varepsilon\right\},$$

or conversely. For example, if $u_\varepsilon(\delta_i^-) = u_\varepsilon(\delta_i^+) = \sqrt{1+\eta/2}$ then we apply Steps 1–3 in Theorem 4.1 two times: the first one to

$$\eta^- = \sqrt{1+\eta/2}\,, \quad \eta^+ = \max_{[\delta_i^-, \delta_i^+]} u_\varepsilon$$

and the second one to

$$\eta^- = \max_{[\delta_i^-, \delta_i^+]} u_\varepsilon\,, \quad \eta^+ = \sqrt{1+\eta/2}\,.$$

Hence, also in this case we may apply Steps 1–3 in Theorem 4.1 with $\zeta > 0$.

We can then conclude that the number of intervals $[\delta_i^-, \delta_i^+]$ is finite (and independent of ε) and there exist $\alpha_i^1, \alpha_i^2 > 0$ such that $\alpha_i^1\sqrt{\varepsilon} \le (\delta_i^+ - \delta_i^-) \le \alpha_i^2\sqrt{\varepsilon}$ which proves (4.19). □

Theorem 4.3 (Γ-convergence result) *We have*

$$\Gamma(L^1_{\rm loc})\text{-}\lim_{\varepsilon\to 0} F_\varepsilon^{3/2}(u) = c_0 \sum_{\theta\in S(u)} \rho(\theta)$$

for every $u \in BV_{\rm loc}((0,1);\{-1,1\})$*, where*

$$\begin{aligned} c_0 \;=\; \inf_{T>0} \inf\Big\{ & \int_{-T}^{T} (v')^2\,ds + \int_{-T}^{T}\Big(\int_{-T}^{\sigma}(v^2-1)\,ds\Big)^2 d\sigma \;: \\ & v \in H^1(-T,T), \quad v(\pm T) = \pm 1, \quad \int_{-T}^{T}(v^2-1)\,ds = 0\Big\}. \end{aligned}$$

PROOF. *Liminf inequality.* Let $u \in BV_{\rm loc}((0,1);\{-1,1\})$ and let (u_ε) be a sequence converging to u in $L^1_{\rm loc}(0,1)$ such that $\sup_\varepsilon F_\varepsilon^{3/2}(u_\varepsilon) < +\infty$.

Step 1. We fix $0 < a < 1$ and consider $\theta_i \in S(u)$ such that $a < \theta_i < 1-a$. Without loss of generality, we may assume that $u(\theta_i\pm) = \mp 1$. Let $\theta_\varepsilon \to \theta_i$, as $\varepsilon \to 0$, and $M > 0$ be such that

$$u_\varepsilon > \frac{1}{2} \quad \text{on } I_\varepsilon^- := (\theta_\varepsilon - 2M\sqrt{\varepsilon}, \theta_\varepsilon - M\sqrt{\varepsilon})$$

and

$$u_\varepsilon < \frac{1}{2} \quad \text{on } I_\varepsilon^+ := (\theta_\varepsilon + M\sqrt{\varepsilon}, \theta_\varepsilon + 2M\sqrt{\varepsilon})\,.$$

By (4.19) there exists a constant $c > 0$ and $\theta_\varepsilon^\pm \in I_\varepsilon^\pm$ of the form $\theta_\varepsilon^\pm = \theta_\varepsilon \pm M_\varepsilon\sqrt{\varepsilon}$, with $M \le M_\varepsilon \le 2M$, such that

$$\Big|\int_0^{\theta_\varepsilon^\pm} \rho\,(u_\varepsilon^2 - 1)\,d\varphi\Big| \le \frac{c}{\sqrt{M}}\sqrt{\varepsilon} \tag{4.22}$$

and

$$|u_\varepsilon^2(\theta_\varepsilon^\pm) - 1| \le \eta\,. \tag{4.23}$$

In fact, with η fixed, reasoning by contradiction, assume that for every constant $c > 0$ we cannot find two points $\theta_\varepsilon^\pm \in I_\varepsilon^\pm$ such that (4.22) and (4.23) are satisfied at the same time; if we denote

$$B_c = \Big\{\theta \in (0,1) : \Big|\int_0^{\theta} \rho\,(u_\varepsilon^2 - 1)\,d\varphi\Big| > \frac{c}{\sqrt{M}}\sqrt{\varepsilon}\Big\}\,,$$

we have that

$$\begin{aligned} F_\varepsilon^{3/2}(u_\varepsilon) &\ge \varepsilon^{-3/2}\int_{B_c} \frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\,(u_\varepsilon^2 - 1)\,d\varphi\Big)^2 d\theta \\ &\ge \varepsilon^{-1/2}\frac{c^2}{\rho(1)}\frac{|B_c|}{M} \ge \frac{c^2}{\rho(1)}\Big(2 - \frac{\bar{c}}{M}\Big)\,, \end{aligned}$$

Note that we can choose M large enough such that $\overline{c} < 2M$. Since $\sup_\varepsilon F_\varepsilon^{3/2}(u_\varepsilon) < +\infty$ we get a contradiction by the arbitrariness of $c > 0$.

Step 2. We give an estimate on the contribution between θ_ε^- and θ_ε^+. By (4.22), there exists a sequence (c_-^ε), bounded independently of ε, such that

$$\begin{aligned} & \sqrt{\varepsilon}\int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \rho\,(u_\varepsilon')^2\,d\theta + \varepsilon^{-3/2}\int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \frac{1}{\rho(\theta)}\Big(\int_0^\theta \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2 d\theta \\ = & \sqrt{\varepsilon}\int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \rho\,(u_\varepsilon')^2\,d\theta + \varepsilon^{-3/2}\int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \frac{1}{\rho(\theta)}\Big(\frac{c_-^\varepsilon}{\sqrt{M}}\sqrt{\varepsilon} + \int_{\theta_\varepsilon^-}^\theta \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2 d\theta \\ = & \int_{-M_\varepsilon}^{M_\varepsilon} \rho_\varepsilon\,(v')^2\,ds + \int_{-M_\varepsilon}^{M_\varepsilon} \frac{1}{\rho_\varepsilon(\sigma)}\Big(\frac{c_-^\varepsilon}{\sqrt{M}} + \int_{-M_\varepsilon}^\sigma \rho_\varepsilon\,(v^2-1)\,ds\Big)^2 d\sigma, \end{aligned} \tag{4.24}$$

where $s = (\varphi-\theta_\varepsilon)/\sqrt{\varepsilon}$, $\sigma = (\theta-\theta_\varepsilon)/\sqrt{\varepsilon}$, $\rho_\varepsilon(t) = \rho(\theta_\varepsilon + t\sqrt{\varepsilon})$ and $v(s) = u_\varepsilon(\theta_\varepsilon + \sqrt{\varepsilon}s)$. By (4.22) there exists also a sequence (c_+^ε), bounded independently of ε, such that

$$\int_0^{\theta_\varepsilon^+} \rho\,(u_\varepsilon^2-1)\,d\varphi = \frac{c_+^\varepsilon}{\sqrt{M}}\sqrt{\varepsilon}\,;$$

hence,

$$\frac{1}{\sqrt{\varepsilon}}\int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \rho\,(u_\varepsilon-1)^2\,d\varphi = \int_{-M_\varepsilon}^{M_\varepsilon} \rho_\varepsilon\,(v^2-1)\,ds = \frac{c_+^\varepsilon - c_-^\varepsilon}{\sqrt{M}}\,. \tag{4.25}$$

By (4.23) we get that $|v^2(\pm M_\varepsilon) - 1| \le \eta$. We consider $v(s) = u_\varepsilon(\theta_\varepsilon + \sqrt{\varepsilon}s)$ for $s \in [-M_\varepsilon, M_\varepsilon]$ and we extend it to $[-3M, 3M]$ in such a way that

$$v(\pm 3M) = \mp 1, \qquad \int_{-3M}^{3M} \rho_\varepsilon\,(v^2-1)\,ds = 0, \qquad \int_{-3M}^{-M_\varepsilon} \rho_\varepsilon\,(v^2-1)\,ds = \frac{c_-^\varepsilon}{\sqrt{M}}\,. \tag{4.26}$$

Note that by(4.25) and (4.26) we have that

$$\int_{M_\varepsilon}^{3M} \rho_\varepsilon\,(v^2-1)\,ds = -\frac{c_+^\varepsilon}{\sqrt{M}}\,.$$

We explicitly construct v on $[-3M, -M_\varepsilon]$, the construction on $[M_\varepsilon, 3M]$ being analogous. We also suppose that $c_-^\varepsilon \le 0$ and $v(-M_\varepsilon) \ge 1$ (the construction being the same or simpler in the other cases). We define v on $[-3M, -M_\varepsilon]$ as follows

$$v(s) = \begin{cases} 1 & \text{if } s \in [-3M, -h_\varepsilon) \\ -s + \sqrt{1-\eta} - k_\varepsilon^- & \text{if } s \in [-h_\varepsilon, -k_\varepsilon^-) \\ \sqrt{1-\eta} & \text{if } s \in [-k_\varepsilon^-, -k_\varepsilon^+) \\ s + \sqrt{1-\eta} + k_\varepsilon^+ & \text{if } s \in [-k_\varepsilon^+, -M_\varepsilon], \end{cases} \tag{4.27}$$

where

$$-k_\varepsilon^- = -h_\varepsilon + 1 - \sqrt{1-\eta}, \qquad -k_\varepsilon^+ = -M_\varepsilon - v(-M_\varepsilon) + \sqrt{1-\eta}\,.$$

Note that k_ε^+ is fixed by Step 1 while k_ε^- is fixed by (4.26). In fact, since $|v^2-1|\le\eta$ on $[-h_\varepsilon,-k_\varepsilon^-)$ and $[-k_\varepsilon^+,-M_\varepsilon)$ we have that

$$\Big|\int_{-h_\varepsilon}^{-k_\varepsilon^-}\rho_\varepsilon\,(v^2-1)\,ds+\int_{-k_\varepsilon^+}^{-M_\varepsilon}\rho_\varepsilon\,(v^2-1)\,ds\Big|\le\eta\,\rho(1)(1+\sqrt{1+\eta}-2\sqrt{1-\eta})\,;$$

hence, there exists $r(\eta,\varepsilon)$ such that $|r(\eta,\varepsilon)|\le\rho(1)(1+\sqrt{1+\eta}-2\sqrt{1-\eta})$ uniformly in ε and

$$\int_{-3M}^{-M_\varepsilon}\rho_\varepsilon\,(v^2-1)\,ds=\eta\,(-k_\varepsilon+r(\eta,\varepsilon))=\frac{c_-^\varepsilon}{\sqrt{M}}\,,$$

where $k_\varepsilon:=-k_\varepsilon^++k_\varepsilon^-$. It follows that

$$k_\varepsilon=\frac{|c_-^\varepsilon|}{\eta\sqrt{M}}+r(\eta,\varepsilon)\,,\tag{4.28}$$

with $\lim_{\eta\to0^+}|r(\eta,\varepsilon)|=0$. Reasoning as above, by (4.28), we can also observe that

$$\begin{aligned}&\int_{-3M}^{-M_\varepsilon}\frac{1}{\rho_\varepsilon(\sigma)}\Big(\int_{-3M}^{\sigma}\rho_\varepsilon\,(v^2-1)\,ds\Big)^2\,d\sigma\\
=\;&\int_{-h_\varepsilon}^{-M_\varepsilon}\frac{1}{\rho_\varepsilon(\sigma)}\Big(\int_{-h_\varepsilon}^{\sigma}\rho_\varepsilon\,(v^2-1)\,ds\Big)^2\,d\sigma\\
\le\;&\eta^2\,\frac{\rho(1)}{3}\Big(k_\varepsilon+1+\sqrt{1+\eta}-2\sqrt{1-\eta}\Big)^3\\
=\;&\eta^2\,\frac{\rho(1)}{3}\Big(\frac{|c_-^\varepsilon|}{\eta\sqrt{M}}+r(\eta,\varepsilon)+1+\sqrt{1+\eta}-2\sqrt{1-\eta}\Big)^3\,.\end{aligned}$$

We can prove a similar estimate also for the contribution corresponding to the interval $[M_\varepsilon,3M]$; hence, we can conclude that there exists $R(M,\eta,\varepsilon)>0$, bounded independently of ε, such that

$$\begin{aligned}R(M,\eta,\varepsilon)\;=\;&\int_{-3M}^{-M_\varepsilon}\frac{1}{\rho_\varepsilon(\sigma)}\Big(\int_{-3M}^{\sigma}\rho_\varepsilon\,(v^2-1)\,ds\Big)^2\,d\sigma\\
&+\int_{M_\varepsilon}^{3M}\frac{1}{\rho_\varepsilon(\sigma)}\Big(\frac{c_+^\varepsilon}{\sqrt{M}}+\int_{M_\varepsilon}^{\sigma}\rho_\varepsilon\,(v^2-1)\,ds\Big)^2\,d\sigma\end{aligned}\tag{4.29}$$

and

$$\limsup_{\eta\to0^+}\limsup_{M\to\infty}\limsup_{\varepsilon\to0}R(M,\eta,\varepsilon)=0\,.\tag{4.30}$$

By (4.29) we have that

$$\begin{aligned}&\int_{-M_\varepsilon}^{M_\varepsilon}\frac{1}{\rho_\varepsilon(\sigma)}\Big(\frac{c_-^\varepsilon}{\sqrt{M}}+\int_{-M_\varepsilon}^{\sigma}\rho_\varepsilon\,(v^2-1)\,ds\Big)^2\,d\sigma\\
=\;&\int_{-3M}^{3M}\frac{1}{\rho_\varepsilon(\sigma)}\Big(\int_{-3M}^{\sigma}\rho_\varepsilon\,(v^2-1)\,ds\Big)^2\,d\sigma-R(M,\eta,\varepsilon)\,.\end{aligned}\tag{4.31}$$

The function v, constructed as in (4.27), gives the following contribution of the term with the derivative

$$\int_{-3M}^{-M_\varepsilon} \rho_\varepsilon \, (v')^2 \, ds \le \rho(1)(1 + \sqrt{1+\eta} - 2\sqrt{1-\eta}) \, ;$$

hence, reasoning similarly on $[M_\varepsilon, 3M]$, there exists $R_1(\eta,\varepsilon) > 0$, bounded independently of ε, such that

$$\int_{-M_\varepsilon}^{M_\varepsilon} \rho_\varepsilon \, (v')^2 \, ds = \int_{-3M}^{3M} \rho_\varepsilon \, (v')^2 \, ds - R_1(\eta,\varepsilon) \tag{4.32}$$

and

$$\limsup_{\eta\to 0^+} \limsup_{\varepsilon\to 0} R_1(\eta,\varepsilon) = 0 \, . \tag{4.33}$$

By (4.24), (4.32) and (4.31) we get that

$$\begin{aligned}
& \sqrt{\varepsilon} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \rho(\theta)(u_\varepsilon')^2 \, d\theta + \varepsilon^{-3/2} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \frac{1}{\rho(\theta)} \Big(\int_0^\theta \rho \, (u_\varepsilon^2 - 1) \, d\varphi \Big)^2 \, d\theta \\
= \; & \int_{-3M}^{3M} \rho_\varepsilon \, (v')^2 \, ds + \int_{-3M}^{3M} \frac{1}{\rho_\varepsilon(\sigma)} \Big(\int_{-3M}^{\sigma} \rho_\varepsilon \, (v^2 - 1) \, ds \Big)^2 \, d\sigma \\
& - R_1(\eta,\varepsilon) - R(M,\eta,\varepsilon) \\
\ge \; & \inf \Big\{ \int_{-T}^{T} \rho_\varepsilon \, (v')^2 \, ds \; + \; \int_{-T}^{T} \frac{1}{\rho_\varepsilon(\sigma)} \Big(\int_{-T}^{\sigma} \rho_\varepsilon \, (v^2 - 1) \, ds \Big)^2 \, d\sigma \; : \\
& \qquad v \in H^1(-T,T), \quad v(\pm T) = \pm 1, \quad \int_{-T}^{T} \rho_\varepsilon \, (v^2 - 1) \, ds = 0 \Big\} \\
& + \delta(T,\eta,\varepsilon),
\end{aligned} \tag{4.34}$$

where $T = 3M$ and $\delta(T,\eta,\varepsilon) = -R_1(\eta,\varepsilon) - \tilde{R}(T,\eta,\varepsilon)$ with $\tilde{R}(T,\eta,\varepsilon) = R(M,\eta,\varepsilon)$. Note that in the last infimum problem we can take the boundary values indifferently as $v(\pm T) = \pm 1$ or $v(\pm T) = \mp 1$, by the symmetry of the problem, so that both types of transitions are taken into account.

By (4.30) and (4.33), we have that

$$\limsup_{\eta\to 0^+} \limsup_{T\to\infty} \limsup_{\varepsilon\to 0} |\delta(T,\eta,\varepsilon)| = 0 \, ;$$

hence, it remains to study the behavior of the minimum problems as ε tends to 0. By the uniform convergence of ρ_ε to $\rho(\theta_i)$, as ε tends to 0, we have immediately the Γ-convergence of the functionals with respect to the strong L^2 convergence; *i.e.,*

$$\begin{aligned}
& \Gamma(L^2)\text{-} \lim_{\varepsilon\to 0} \Big(\int_{-T}^{T} \rho_\varepsilon \, (v')^2 \, ds \; + \; \int_{-T}^{T} \frac{1}{\rho_\varepsilon(\sigma)} \Big(\int_{-T}^{\sigma} \rho_\varepsilon \, (v^2 - 1) \, ds \Big)^2 \, d\sigma \Big) \\
= \; & \rho(\theta_i) \Big(\int_{-T}^{T} (v')^2 \, ds \; + \; \int_{-T}^{T} \Big(\int_{-T}^{\sigma} (v^2 - 1) \, ds \Big)^2 \, d\sigma \Big) \, .
\end{aligned}$$

This Γ-convergence result is stable by adding the constraint; in fact, since the constraint $\int_{-T}^{T} \rho_\varepsilon\,(v^2-1)\,ds = 0$ is close for the strong L^2 convergence the liminf inequality is trivial. To check the limsup inequality, let $v \in H^1(-T,T)$ such that $v(\pm T) = \pm 1$ and $\int_{-T}^{T}(v^2-1)\,ds = 0$. To obtain a recovery sequence we consider

$$v_\varepsilon(s) = v(s) + t_\varepsilon \phi(s) \qquad s \in [-T,T] \tag{4.35}$$

where

$$\phi(s) = \begin{cases} (s+T)/T & s \in [-T,0) \\ \\ (T-s)/T & s \in [0,T] \end{cases} \tag{4.36}$$

(note that $v_\varepsilon \in H^1(-T,T)$ and $v_\varepsilon(\pm T) = \pm 1$) and $t_\varepsilon \in \mathbf{R}$ is chosen such that (v_ε) satisfies the constraint and converges to v in $L^2(-T,T)$; *i.e.,*

$$\int_{-T}^{T} \rho_\varepsilon\,((v + t_\varepsilon\phi)^2 - 1)\,ds = 0 \quad \text{and} \quad t_\varepsilon \to 0 \quad \text{as}\, \varepsilon \to 0\,. \tag{4.37}$$

More precisely, (v_ε) satisfies the constraint in (4.37) if $t_\varepsilon \in \mathbf{R}$ is solution of the following second order equation

$$\Big(\int_{-T}^{T} \rho_\varepsilon\, \phi^2\, ds\Big)\, t_\varepsilon^2 + 2\Big(\int_{-T}^{T} \rho_\varepsilon\, v\phi\, ds\Big)\, t_\varepsilon + \Big(\int_{-T}^{T} \rho_\varepsilon\,(v^2-1)\,ds\Big) = 0\,. \tag{4.38}$$

Since $\lim_{\varepsilon\to 0}\int_{-T}^{T} \rho_\varepsilon (v^2-1)\,ds = 0$, for ε small enough, equation (4.38) has two solutions, real and distinct, such that one of two tends to 0 as ε tends to 0.

To conclude the limsup inequality, we have to note that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\Big(\int_{-T}^{T} \rho_\varepsilon\,(v_\varepsilon')^2\,ds \;+\; \int_{-T}^{T} \frac{1}{\rho_\varepsilon(\sigma)}\Big(\int_{-T}^{\sigma} \rho_\varepsilon\,(v_\varepsilon^2-1)\,ds\Big)^2 d\sigma\Big) \\ =\;\; & \rho(\theta_i)\Big(\int_{-T}^{T} (v')^2\,ds \;+\; \int_{-T}^{T}\Big(\int_{-T}^{\sigma} (v^2-1)\,ds\Big)^2 d\sigma\Big)\,. \end{aligned} \tag{4.39}$$

We go back now to (4.34), by the property of convergence of minima (see [8] Theorem 1.21) we have that

$$\begin{aligned} &\liminf_{\varepsilon\to 0}\Big(\sqrt{\varepsilon}\int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \rho\,(u_\varepsilon')^2\,d\theta + \varepsilon^{-3/2}\int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \frac{1}{\rho(\theta)}\Big(\int_0^{\theta} \rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2 d\theta\Big) \\ \geq\;\; & \rho(\theta_i)\inf\Big\{\int_{-T}^{T} (v')^2\,ds \;+\; \int_{-T}^{T}\Big(\int_{-T}^{\sigma}(v^2-1)\,ds\Big)^2 d\sigma \,:\, v \in H^1(-T,T), \\ & \qquad\qquad v(\pm T) = \pm 1, \quad \int_{-T}^{T}(v^2-1)\,ds = 0\Big\} \\ & -\limsup_{\varepsilon\to 0}|\delta(T,\eta,\varepsilon)| \\ \geq\;\; & \rho(\theta_i)\inf_{T>0}\,\inf\Big\{\int_{-T}^{T} (v')^2\,ds \;+\; \int_{-T}^{T}\Big(\int_{-T}^{\sigma}(v^2-1)\,ds\Big)^2 d\sigma \,: \end{aligned}$$

$$v \in H^1(-T,T), \quad v(\pm T) = \pm 1, \quad \int_{-T}^{T} (v^2-1)\,ds = 0 \Big\}$$

$$\begin{aligned} & - \limsup_{T\to+\infty} \limsup_{\varepsilon\to 0} |\delta(T,\eta,\varepsilon)| \\ = \quad & \rho(\theta_i)\, c_0 - \limsup_{T\to+\infty} \limsup_{\varepsilon\to 0} |\delta(T,\eta,\varepsilon)| \,, \end{aligned}$$

where

$$\begin{aligned} c_0 \quad := \quad & \inf_{T>0} \inf \Big\{ \int_{-T}^{T} (v')^2\,ds \; + \; \int_{-T}^{T} \Big(\int_{-T}^{\sigma} (v^2-1)\,ds \Big)^2 d\sigma \; : \; v \in H^1(-T,T) \\ & \qquad v(\pm T) = \pm 1, \quad \text{and} \quad \int_{-T}^{T} (v^2-1)\,ds = 0 \Big\} \,. \end{aligned}$$

Passing to the limit as $\eta \to 0^+$, we get

$$\begin{aligned} & \liminf_{\varepsilon \to 0} \Big(\sqrt{\varepsilon} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \rho\,(u_\varepsilon')^2\,d\theta + \varepsilon^{-3/2} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \frac{1}{\rho(\theta)} \Big(\int_0^\theta \rho\,(u_\varepsilon^2-1)\,d\varphi \Big)^2 d\theta \Big) \\ \geq \quad & \rho(\theta_i)\, c_0 \,. \end{aligned}$$

Step 3. If we repeat Steps 1-2 for every $\theta \in S(u) \cap (a, 1-a)$ we immediately get that

$$\liminf_{\varepsilon\to 0} F_\varepsilon^{3/2}(u_\varepsilon) \geq c_0 \sum_{\theta \in S(u)\cap(a,1-a)} \rho(\theta)$$

and then the liminf inequality taking the supremum in a.

Limsup inequality. Let $u \in BV((0,1);\{-1,1\})$ we denote $S(u) = \{\theta_1, \cdots, \theta_N\}$ with $\theta_i < \theta_{i+1}$. Fixed $\eta > 0$ there exist $T > 0$ and $v \in H^1(-T,T)$ such that $v(\pm T) = \pm 1$, $\int_{-T}^{T} (v^2-1)\,ds = 0$ and

$$\int_{-T}^{T} (v')^2\,ds \; + \; \int_{-T}^{T} \Big(\int_{-T}^{\sigma} (v^2-1)\,ds \Big)^2 d\sigma \leq c_0 + \eta \,. \tag{4.40}$$

We denote $\delta = T\sqrt{\varepsilon}$, $K_i = [\theta_i + \delta, \theta_{i+1} - \delta]$ for $i = 1, \cdots, N-1$, $K_N = [\theta_N + \delta, 1]$. We construct a sequence u_ε by setting

$$u_\varepsilon(\theta) = \begin{cases} v_\varepsilon^i(\pm \varepsilon^{-1/2}(\theta - \theta_i)) & \text{if } \theta \in [\theta_i - \delta, \theta_i + \delta], \quad i = 1, \cdots, N \\ \\ u(\theta) & \text{if } \theta \in (0, \theta_1 - \delta) \cup_{i=1}^N K_i \end{cases} \tag{4.41}$$

where $v_\varepsilon^i = v + t_\varepsilon^i \phi$ is defined as in Step 2 with ϕ given by (4.36) and $t_\varepsilon^i \in \mathbf{R}$ such that

$$\int_{-T}^{T} \rho_\varepsilon^i\,((v + t_\varepsilon^i \phi)^2 - 1)\,ds = 0 \quad \text{and} \quad t_\varepsilon^i \to 0 \quad \text{as } \varepsilon \to 0 \tag{4.42}$$

for every $i = 1, \cdots, N$ where $\rho_\varepsilon^i(s) = \rho(\theta_i + s\sqrt{\varepsilon})$. Note that the choice between the plus and minus sign, in (4.41), is made in such a way that the resulting function is continuous. The construction of u_ε is illustrated in Fig. 2.

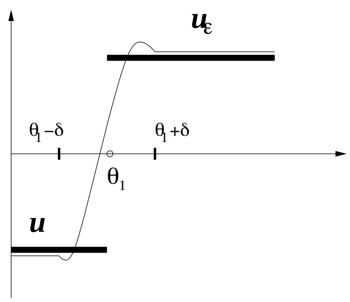

Figure 2: recovery sequence with a Gibbs' phenomenon

Note that, reasoning as in Step 2, we get

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\Big(\int_{-T}^{T}\rho_\varepsilon^i\,(v_\varepsilon^i)'^2\,ds\;+\;\int_{-T}^{T}\frac{1}{\rho_\varepsilon^i(\sigma)}\Big(\int_{-T}^{\sigma}\rho_\varepsilon^i\,((v_\varepsilon^i)^2-1)\,ds\Big)^2\,d\sigma\Big)\\
=\;&\rho(\theta_i)\Big(\int_{-T}^{T}(v')^2\,ds\;+\;\int_{-T}^{T}\Big(\int_{-T}^{\sigma}(v^2-1)\,ds\Big)^2\,d\sigma\Big)\,. \qquad (4.43)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
F_\varepsilon^{3/2}(u_\varepsilon)\;=\;&\sqrt{\varepsilon}\sum_{i=1}^{N}\int_{\theta_i-\delta}^{\theta_i+\delta}\rho\,(u_\varepsilon')^2\,d\theta\\
&+\varepsilon^{-3/2}\sum_{i=1}^{N}\Big(\int_{K_i}\frac{1}{\rho(\theta)}\Big(\int_0^{\theta}\rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2\,d\theta\\
&\qquad+\int_{\theta_i-\delta}^{\theta_i+\delta}\frac{1}{\rho(\theta)}\Big(\int_0^{\theta}\rho\,(u_\varepsilon^2-1)\,d\varphi\Big)^2\,d\theta\Big)\,.
\end{aligned}
$$

By the changes of variable $s=\varepsilon^{-1/2}(\varphi-\theta_i)$, $\sigma=\varepsilon^{-1/2}(\theta-\theta_i)$ and (4.42), we get that

$$
\begin{aligned}
F_\varepsilon^{3/2}(u_\varepsilon)\;=\;&\sum_{i=1}^{N}\int_{-T}^{T}\rho_\varepsilon^i\,(v_\varepsilon^i)'^2\,ds\\
&+\varepsilon^{-1/2}\sum_{i=1}^{N}\Big(\int_{K_i}\frac{d\theta}{\rho(\theta)}\Big)\Big(\sum_{j=1}^{i}\int_{-T}^{T}\rho_\varepsilon^j\,((v_\varepsilon^j)^2-1)\,ds\Big)^2\\
&+\varepsilon^{-1/2}\sum_{i=1}^{N}\int_{\theta_i-\delta}^{\theta_i+\delta}\frac{1}{\rho(\theta)}\Big(\sum_{j=1}^{i-1}\int_{-T}^{T}\rho_\varepsilon^j\,((v_\varepsilon^j)^2-1)\,ds\\
&\qquad+\int_{-T}^{\varepsilon^{-1/2}(\theta-\theta_i)}\rho_\varepsilon^i\,((v_\varepsilon^i)^2-1)\,ds\Big)^2\,d\theta
\end{aligned}
$$

$$= \sum_{i=1}^{N} \int_{-T}^{T} \rho_\varepsilon^i \, (v_\varepsilon^i)'^2 \, ds + \int_{-T}^{T} \frac{1}{\rho_\varepsilon^i(\sigma)} \Big(\int_{-T}^{\sigma} \rho_\varepsilon^i \, ((v_\varepsilon^i)^2 - 1) \, ds \Big)^2 \, d\sigma \, .$$

By (4.40) and (4.43), we have

$$\begin{aligned} \limsup_{\varepsilon \to 0} F_\varepsilon^{3/2}(u_\varepsilon) &= \Big(\sum_{i=1}^{N} \rho(\theta_i) \Big) \Big(\int_{-T}^{T} (v')^2 \, ds + \int_{-T}^{T} \Big(\int_{-T}^{\sigma} (v^2 - 1) \, ds \Big)^2 \, d\sigma \Big) \\ &\leq \Big(\sum_{i=1}^{N} \rho(\theta_i) \Big) (c_0 + \eta) \, . \end{aligned}$$

By the arbitrariness of η we get the limsup inequality for every $u \in BV((0,1); \{-1,1\})$.

We now consider $u \in BV_{\rm loc}((0,1); \{-1,1\})$; let $u^a \in BV((0,1); \{-1,1\})$ be such that u^a converges to u strongly in $L^1(0,1)$ as $a \to 0^+$; *e.g.*,

$$u^a(\theta) = \begin{cases} u(a) & \theta \in [0, a) \\ u(\theta) & \theta \in [a, 1-a] \\ u(1-a) & \theta \in (1-a, 1] \end{cases} ,$$

with $0 < a < 1$ and $a, 1-a \notin S(u)$. By the lower semicontinuity of the Γ-limsup we have that

$$\begin{aligned} \Gamma\text{-}\limsup_{\varepsilon \to 0} F_\varepsilon^{3/2}(u) &\leq \liminf_{a \to 0^+} \Big(\Gamma\text{-}\limsup_{\varepsilon \to 0} F_\varepsilon^{3/2}(u^a) \Big) \\ &\leq c_0 \liminf_{a \to 0^+} \sum_{\theta \in S(u^a)} \rho(\theta) \leq c_0 \sum_{\theta \in S(u)} \rho(\theta) \end{aligned}$$

for every $u \in BV_{\rm loc}((0,1); \{-1,1\})$ (see [8] Remark 1.29). □

5 The case $\alpha = 4/3$ with constrained phase transitions

If we define G_ε^0 as the restriction of F_ε^0 to the space of functions $u : (0,1) \to [-1,1]$ then, by Remark 3.4, the Γ-limit of G_ε^0 is identically 0. Still one may find another scaling, $\alpha = 4/3$, such that the Γ-limit of $G_\varepsilon^{4/3} = \varepsilon^{-4/3} G_\varepsilon^0$ is not trivial. We consider for simplicity $\rho \equiv 1$, so that

$$G_\varepsilon^{4/3}(u) = \varepsilon^{2/3} \int_0^1 (u')^2 \, d\theta + \varepsilon^{-4/3} \int_0^1 \Big(\int_0^\theta (u^2 - 1) \, d\varphi \Big)^2 \, d\theta \, ,$$

for every $u \in H^1((0,1); [-1,1])$. Note that since $\varepsilon^{-4/3} F_\varepsilon^0 = \varepsilon^{1/6} F_\varepsilon^{3/2}$, by Section 4, the Γ-limit of $\varepsilon^{-4/3} F_\varepsilon^0$ with respect to the strong L^1-convergence is zero. Hence, the constraint $|u| \leq 1$ completely changes the characteristic scaling of the energy.

Theorem 5.1 (Compactness) *Let $(u_\varepsilon) \in H^1((0,1); [-1,1])$ be a sequence such that $\sup_\varepsilon G_\varepsilon^{4/3}(u_\varepsilon) < +\infty$ then, up to subsequences, (u_ε) converges strongly in $L^1(0,1)$ to $u \in BV_{\rm loc}((0,1); \{-1,1\})$.*

Proof. Let $\eta_\varepsilon^\pm \neq \pm 1$ such that $-1 < \eta_\varepsilon^+, \eta_\varepsilon^- < 1$. We denote by $(\delta_\varepsilon^-, \delta_\varepsilon^+)$ an interval such that $u_\varepsilon(\delta_\varepsilon^-) = \eta_\varepsilon^-$, $u_\varepsilon(\delta_\varepsilon^+) = \eta_\varepsilon^+$ and u_ε takes values between η_ε^- and η_ε^+. We use in the sequel the notation $\eta^\pm = \eta_\varepsilon^\pm$, $\delta^\pm = \delta_\varepsilon^\pm$ not to overburden notation. By the constraint, $(u_\varepsilon^2 - 1)$ never changes sign; hence, we can select in the energy $G_\varepsilon^{4/3}(u_\varepsilon)$ the most significative contribution which permits to easily estimate $(\delta^+ - \delta^-)$. More precisely, for every fixed $0 < a < 1$ we consider $\delta^+ \leq 1 - a$; then,

$$\begin{aligned} G_\varepsilon^{4/3}(u_\varepsilon) &\geq \varepsilon^{2/3} \int_{\delta^-}^{\delta^+} (u')^2 \, d\theta + \varepsilon^{-4/3} \int_{\delta^+}^{1} \Big(\int_{\delta^-}^{\delta^+} (u_\varepsilon^2 - 1) \, d\varphi \Big)^2 d\theta \\ &\geq |\eta^+ - \eta^-|^2 \Big(\frac{\varepsilon^{2/3}}{\delta} \Big) + a \, \lambda^2 \Big(\frac{\delta}{\varepsilon^{2/3}} \Big)^2, \end{aligned}$$

where $\lambda := ||\eta^-|^2 - 1| \wedge ||\eta^+|^2 - 1|$; *i.e.,* the minimum distance of $|\eta^\pm|^2$ from 1. Hence, if $|\eta^+ - \eta^-| \geq \zeta > 0$, with ζ independent of ε, we have that $\delta/\varepsilon^{2/3}$ is bounded; *i.e.,* there exist two positive constant α_1, α_2 such that $\alpha_1 \varepsilon^{2/3} \leq \delta \leq \alpha_2 \varepsilon^{2/3}$. Moreover, the number of intervals (δ^-, δ^+) is finite in $[0, 1-a]$ for every $0 < a < 1$. Then (u_ε) converges in measure to $u \in BV_{\rm loc}((0,1); \{-1,1\})$ and since $|u_\varepsilon| \leq 1$ we can conclude that, up to subsequences, (u_ε) converges strongly to u in L^1. □

Remark 5.2 Note that in general we cannot expect $u \in BV((0,1); \{-1,1\})$. To show this we construct a sequence (u_ε) with $\sup_\varepsilon G_\varepsilon^{4/3}(u_\varepsilon) < +\infty$ and strongly converging in L^1 to $u \in BV_{\rm loc}((0,1); \{-1,1\})$ with infinitely many jump points. To this end, consider a strictly increasing sequence $(\theta_i) \in (0,1)$ such that $\sup_{i \in \mathbf{N}} \theta_i = 1$ and let $v_i(s) = (s + T_i/T_i) - 1$ for $s \in [-T_i, T_i]$. Fixed $k \in \mathbf{N}$ and ε small enough, we define the sequence (u_ε^k) as follows

$$u_\varepsilon^k(\theta) = \begin{cases} v_i(\pm \varepsilon^{-2/3}(\theta - \theta_i)) & \theta \in [\theta_i - T_i \varepsilon^{2/3}, \theta_i + T_i \varepsilon^{2/3}], \; i = 1, \cdots, k \\ \pm 1 & \text{otherwise in } [0,1] \end{cases}$$

where the choice between the plus and minus sign is made in such a way that the resulting function u_ε^k is continuous. For every $k \in \mathbf{N}$, we have that

$$\begin{aligned} G_\varepsilon^{4/3}(u_\varepsilon^k) &= \sum_{i=1}^{k} \Big(\frac{2}{T_i} + (\theta_{i+1} - \theta_i) \Big(c \sum_{j \leq i} T_j \Big)^2 \Big) + O(\varepsilon^{2/3}) \\ &\leq \sum_{i \in \mathbf{N}} \Big(\frac{2}{T_i} + (\theta_{i+1} - \theta_i) \Big(c \sum_{j \leq i} T_j \Big)^2 \Big) + O(\varepsilon^{2/3}) \end{aligned}$$

(see a similar computation in the next Theorem 5.3 for the proof of the limsup inequality). If we fix $T_i = i^\beta$, with $\beta > 1$, and (θ_i) such that $(\theta_{i+1} - \theta_i) = \gamma \, i^{(-3\beta - 2)}$, with γ satisfing the condition $\gamma \sum_{i \in \mathbf{N}} i^{(-3\beta-2)} = (1 - \theta_1)$, then $\sup_\varepsilon G_\varepsilon^{4/3}(u_\varepsilon^k) \leq c$, with c independent on k. Therefore, if $(\theta_i)_{i \in \mathbf{N}}$ is an increasing sequence of points distributed in $(0,1)$ as above, for every fixed $k \in \mathbf{N}$, we can construct a suitable sequence (u_ε^k) strongly converging in

L^1 to $u^k \in BV((0,1);\{-1,1\})$, as $\varepsilon \to 0$, with $\sup_\varepsilon G_\varepsilon^{4/3}(u_\varepsilon^k) \le c$ and $S(u^k) = \{\theta_1, \cdots, \theta_k\}$. We now consider $u \in BV_{\rm loc}((0,1);\{-1,1\})$ such that $S(u) = (\theta_i)_{i\in\mathbf{N}}$ and $u = u^k$ in $[0, \theta_{k+1})$ then u^k converges strongly in L^1 to u as k tends to $+\infty$. By a diagonal procedure we may extract from (u_ε^k) a subsequence with bounded energy and strongly converging to u in L^1.

Theorem 5.3 (Nonlocal Γ-limit) *We have*

$$\begin{aligned}
&\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon^{4/3}(u) \\
= \;& \inf_{T>0} \inf\Big\{ \sum_{i\in I}\Big(\int_{-T}^{T} (v_i')^2\, ds \,+ (\theta_{i+1}-\theta_i)\Big(\sum_{j=1}^{i} \int_{-T}^{T} (v_j^2 - 1)\, ds\Big)^2\Big) \; : \\
&\qquad\qquad v_i \in H^1((-T,T);[-1,1]), \quad v_i(\pm T) = \pm 1 \Big\},
\end{aligned}$$

for every $u \in BV_{\rm loc}((0,1);\{-1,1\})$, where $I = \{i \in \mathbf{N} : \theta_i \in S(u)\,, \theta_i < \theta_{i+1}\}$.

Proof. Let $u \in BV_{\rm loc}((0,1);\{-1,1\})$ and let $(u_\varepsilon) \in H^1((0,1);[-1,1])$ be a sequence strongly converging to u in L^1 such that

$$\liminf_{\varepsilon\to 0} G_\varepsilon^{4/3}(u_\varepsilon) < +\infty\,.$$

For every fixed $0 < a < 1$, by Theorem 5.1, the limit function u has a finite number of discontinuity points in the interval $(0, 1-a]$; *i.e.,* $S(u)\cap(0,1-a] = \{\theta_1, \cdots, \theta_{N(a)}\}$ with $\theta_i < \theta_{i+1}$. Up to subsequences, $u_\varepsilon \to u$ for a.e. $\theta \in (0,1)$, as ε tends to 0; hence, fixed $\eta \in (0,1)$, we consider δ_i^1, δ_i^2 such that

$$u_\varepsilon(\theta_i - \delta_i^1) = -(1-\eta)\,, \qquad\qquad u_\varepsilon(\theta_i + \delta_i^2) = 1-\eta$$

or

$$u_\varepsilon(\theta_i - \delta_i^1) = 1-\eta\,, \qquad\qquad u_\varepsilon(\theta_i + \delta_i^2) = -(1-\eta)$$

for $i = 1, \cdots, N(a)$. The following estimate consists in eliminating all the contributions of u_ε on the intervals where the sequence takes values 'close' to $\{-1,1\}$; this choice is justified by the construction of the optimal sequence, in the limsup inequality, that will be equal to $\{-1,1\}$ on such intervals (see (5.2)). We have then

$$\begin{aligned}
G_\varepsilon^{4/3}(u_\varepsilon) \;\;\ge\;\; & \varepsilon^{2/3} \sum_{i=1}^{N(a)} \int_{I_i} (u_\varepsilon')^2 d\theta + \varepsilon^{-4/3}\Big(\sum_{i=1}^{N(a)} \int_{K_i} \Big(\sum_{j=1}^{i} \int_{I_j} (u_\varepsilon^2 - 1)\, d\varphi\Big)^2 d\theta \\
& + \sum_{i=1}^{N(a)} \int_{I_i} \Big(\sum_{j=1}^{i-1} \int_{I_j} (u_\varepsilon^2 - 1)\, d\varphi + \int_{\theta_i - \delta_i^1}^{\theta} (u_\varepsilon^2 - 1)\, d\varphi\Big)^2 d\theta\Big)\,,
\end{aligned}$$

where $I_i = [\theta_i - \delta_i^1, \theta_i + \delta_i^2]$ for $i = 1, \cdots, N(a)$ and $K_i = (\theta_i + \delta_i^2, \theta_{i+1} - \delta_{i+1}^1)$ for $i = 1, \cdots, N(a)-1$, $K_{N(a)} = (\theta_{N(a)} + \delta_{N(a)}^2, 1-a)$. We make the following change of variable

$$w_j(s) = u_\varepsilon\Big(\varepsilon^{2/3} s + \theta_j + \frac{\delta_j^2 - \delta_j^1}{2}\Big)\,;$$

hence, setting $T_j=\varepsilon^{-2/3}\delta_j$ with $\delta_j=\Big(\frac{\delta_j^2+\delta_j^1}{2}\Big)$ and

$$T_i(\theta)=\varepsilon^{-2/3}\Big(\theta-\theta_i-\frac{(\delta_i^2-\delta_i^1)}{2}\Big)\,,$$

we get

$$\begin{aligned}G_\varepsilon^{4/3}(u_\varepsilon) \;&\geq\; \sum_{i=1}^{N(a)}\Big(\int_{-T_i}^{T_i}(w_i')^2\,ds+\int_{K_i}\Big(\sum_{j=1}^{i}\int_{-T_j}^{T_j}(w_j^2-1)\,ds\Big)^2\,d\theta\\ &\qquad+\int_{I_i}\Big(\sum_{j=1}^{i-1}\int_{-T_j}^{T_j}(w_j^2-1)\,ds+\int_{-T_i}^{T_i(\theta)}(w_i^2-1)\,ds\Big)^2\,d\theta\Big)\,.\end{aligned}$$

We denote now

$$B_i=\int_{-T_i}^{T_i}(w_i')^2\,ds\,,\qquad\qquad A_j=\int_{-T_j}^{T_j}(w_j^2-1)\,ds\,,$$

by the change of variable $\sigma=T_i(\theta)$ we get

$$\begin{aligned}G_\varepsilon^{4/3}(u_\varepsilon) \;&\geq\; \sum_{i=1}^{N(a)}B_i+\sum_{i=1}^{N(a)}\Big((2\delta_i+\theta_{i+1}-\theta_i-\delta_{i+1}^1-\delta_i^2)\Big(\sum_{j=1}^{i}A_j\Big)^2\Big)\\ &\qquad+\varepsilon^{2/3}\sum_{i=1}^{N(a)}\Big(\int_{-T_i}^{T_i}\Big(\int_{\sigma}^{T_i}(w_i^2-1)\,ds\Big)^2\,d\sigma\\ &\qquad\qquad-2\Big(\sum_{j=1}^{i}A_j\Big)\int_{-T_i}^{T_i}\Big(\int_{\sigma}^{T_i}(w_i^2-1)\,ds\Big)\,d\sigma\Big)\\ &=\; \sum_{i=1}^{N(a)}\Big(B_i+(\theta_{i+1}-\theta_i)\Big(\sum_{j=1}^{i}A_j\Big)^2\Big)+\sum_{i=1}^{N(a)}(\delta_i^1-\delta_{i+1}^1)\Big(\sum_{j=1}^{i}A_j\Big)^2\\ &\qquad+\varepsilon^{2/3}\Big(\sum_{i=1}^{N(a)}\int_{-T_i}^{T_i}\Big(\int_{\sigma}^{T_i}(w_i^2-1)\,ds\Big)^2\,d\sigma\\ &\qquad\qquad-2\Big(\sum_{j=1}^{i}A_j\Big)\int_{-T_i}^{T_i}\Big(\int_{\sigma}^{T_i}(w_i^2-1)\,ds\Big)\,d\sigma\Big)\end{aligned}$$

where $\theta_{N(a)+1}=1-a$ and $\delta^1_{N(a)+1}=0$. Hence,

$$G_\varepsilon^{4/3}(u_\varepsilon)\geq\sum_{i=1}^{N(a)}\Big(\int_{-T_i}^{T_i}(w_i')^2\,ds+(\theta_{i+1}-\theta_i)\Big(\sum_{j=1}^{i}\int_{-T_j}^{T_j}(w_j^2-1)\,ds\Big)^2\Big)+O(\varepsilon^{2/3})\,,$$

where $w_i(\pm T_i)$ is equal to $\pm(1-\eta)$ or $\mp(1-\eta)$, and

$$\liminf_{\varepsilon\to 0}G_\varepsilon^{4/3}(u_\varepsilon)$$

$$\geq \inf_{T>0} \inf\Big\{ \sum_{i=1}^{N(a)} \Big(\int_{-T}^{T} (v_i')^2\, ds \,+ (\theta_{i+1}-\theta_i)\Big(\sum_{j=1}^{i} \int_{-T}^{T} (v_j^2-1)\, ds\Big)^2 \Big) \;:\; v_i \in H^1((-T,T);[-1,1]), \quad v_i(\pm T) = \pm(1-\eta) \Big\}$$

where, by symmetry, we may fix the boundary conditions as $v_i(\pm T) = \pm(1-\eta)$. We may now first pass to the limit as $\eta \to 0$ and then take the supremum on a; *i.e.,*

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} G_\varepsilon^{4/3}(u_\varepsilon) \\
&\geq \sup_{0<a<1} \inf_{T>0} \inf\Big\{ \sum_{i=1}^{N(a)} \Big(\int_{-T}^{T} (v_i')^2\, ds \,+ (\theta_{i+1}-\theta_i)\Big(\sum_{j=1}^{i} \int_{-T}^{T} (v_j^2-1)\, ds\Big)^2 \Big) \;: \\
&\qquad\qquad v_i \in H^1((-T,T);[-1,1]), \quad v_i(\pm T) = \pm 1 \Big\} \\
&= \inf_{T>0} \inf\Big\{ \sum_{i\in I} \Big(\int_{-T}^{T} (v_i')^2\, ds \,+ (\theta_{i+1}-\theta_i)\Big(\sum_{j=1}^{i} \int_{-T}^{T} (v_j^2-1)\, ds\Big)^2 \Big) \;: \\
&\qquad\qquad v_i \in H^1((-T,T);[-1,1]), \quad v_i(\pm T) = \pm 1 \Big\}
\end{aligned}$$

where we have labeled the points in $S(u)$ by a set of indices $I \subset \mathbf{N}$ in such a way that $\theta_i < \theta_{i+1}$.

We now check the limsup inequality. Let $u \in BV((0,1);\{-1,1\})$ with $S(u) = \{\theta_1,\cdots,\theta_N\}$ and $\theta_i < \theta_{i+1}$. We denote by

$$G^{4/3}(u) = \inf_{T>0} \inf\Big\{ \sum_{i=1}^{N} \Big(\int_{-T}^{T} (v_i')^2\, ds \,+ (\theta_{i+1}-\theta_i)\Big(\sum_{j=1}^{i} \int_{-T}^{T} (v_j^2-1)\, ds\Big)^2 \Big) \;: \; v_i \in H^1((-T,T);[-1,1]), \quad v_i(\pm T) = \pm 1 \Big\}.$$

Fixed $\eta > 0$, there exist $T > 0$ and $(v_1,\cdots,v_N) \in H^1((-T,T);[-1,1])$ such that $v_i(\pm T) = \pm 1$ and

$$\sum_{i=1}^{N} \Big(\int_{-T}^{T} (v_i')^2\, ds + (\theta_{i+1}-\theta_i)\Big(\sum_{j=1}^{i} \int_{-T}^{T} (v_j^2-1)\, ds\Big)^2 \Big) \leq G^{4/3}(u) + \eta\,. \tag{5.1}$$

We denote $\delta = T\varepsilon^{2/3}$, $I_i = [\theta_i-\delta, \theta_i+\delta]$ for $i = 1,\cdots,N$, $K_i = (\theta_i+\delta, \theta_{i+1}-\delta)$ for $i = 1,\cdots,N-1$ and $K_N = (\theta_N+\delta, 1]$. We construct a sequence u_ε by setting

$$u_\varepsilon(\theta) = \begin{cases} v_i(\pm\varepsilon^{-2/3}(\theta-\theta_i)) & \text{if } \theta \in I_i \quad i = 1,\cdots,N \\ u(\theta) & \text{if } \theta \in (0,\theta_1-\delta) \cup \Big(\cup_{i=1}^{N} K_i\Big) \end{cases} \tag{5.2}$$

where the choice between the plus and minus sign is made in such a way that the resulting function is continuous. Hence, by the change of variables

$s = \varepsilon^{-2/3}(\theta - \theta_i)$, we get

$$
\begin{aligned}
G^{4/3}_\varepsilon(u_\varepsilon) &= \varepsilon^{2/3}\sum_{i=1}^N \int_{I_i} (u'_\varepsilon)^2\, d\theta + \varepsilon^{-4/3}\sum_{i=1}^N \Big(\int_{K_i}\Big(\int_0^\theta (u_\varepsilon^2-1)\, d\varphi\Big)^2 d\theta\Big) \\
&\quad + \varepsilon^{-4/3}\sum_{i=1}^N \Big(\int_{I_i}\Big(\int_0^\theta (u_\varepsilon^2-1)\, d\varphi\Big)^2 d\theta\Big) \\
&= \sum_{i=1}^N \Big(\int_{-T}^T (v'_i)^2\, ds + \int_{K_i}\Big(\sum_{j=1}^i \int_{-T}^T (v_j^2-1)\, ds\Big)^2 d\theta \\
&\quad + \int_{I_i}\Big(\sum_{j=1}^{i-1}\int_{-T}^T (v_j^2-1)\, ds + \int_{-T}^{T_i(\theta)} (v_i^2-1)\, ds\Big)^2 d\theta\Big)
\end{aligned}
$$

where $T_i(\theta) = \varepsilon^{-2/3}(\theta-\theta_i)$. Setting

$$
B_i = \int_{-T}^T (v'_i)^2\, ds\,, \qquad A_j = \int_{-T}^T (v_j^2-1)\, ds
$$

we then have

$$
\begin{aligned}
G^{4/3}_\varepsilon(u_\varepsilon) &= \sum_{i=1}^N \Big(B_i + (\theta_{i+1} - \theta_i - 2\delta)\Big(\sum_{j=1}^i A_j\Big)^2\Big) \\
&\quad + \sum_{i=1}^N \int_{I_i}\Big(\sum_{j=1}^i A_j - \int_{T_i(\theta)}^T (v_i^2-1)\, ds\Big)^2 d\theta \\
&= \sum_{i=1}^N \Big(B_i + (\theta_{i+1}-\theta_i)\Big(\sum_{j=1}^i A_j\Big)^2\Big) + \sum_{i=1}^N \int_{I_i}\Big(\int_{T_i(\theta)}^T (v_i^2-1)\, ds\Big)^2 d\theta \\
&\quad - 2\sum_{i=1}^N\Big(\sum_{j=1}^i A_j\Big)\int_{I_i}\Big(\int_{T_i(\theta)}^T (v_i^2-1)\, ds\Big) d\theta\,.
\end{aligned}
$$

By (5.1) and the change of variable $\sigma = T_i(\theta)$, we get that

$$
\begin{aligned}
G^{4/3}_\varepsilon(u_\varepsilon) &= \sum_{i=1}^N \Big(B_i + (\theta_{i+1}-\theta_i)\Big(\sum_{j=1}^i A_j\Big)^2\Big) + O(\varepsilon^{2/3}) \\
&\leq G^{4/3}(u) + \eta + O(\varepsilon^{2/3})\,.
\end{aligned}
$$

Passing to the limit as ε tends to 0, by the arbitrariness of η, we get the limsup inequality for every $u \in BV((0,1);\{-1,1\})$.

We now consider $u \in BV_{\rm loc}((0,1);\{-1,1\})$. There exists $u^a \in BV((0,1);\{-1,1\})$ such that u^a converges to u strongly in $L^1(0,1)$ as $a \to 0^+$; *i.e.*,

$$
u^a(\theta) = \begin{cases} u(a) & \theta \in [0,a) \\ u(\theta) & \theta \in [a, 1-a] \\ u(1-a) & \theta \in (1-a, 1]\ , \end{cases}
$$

with $0 < a < 1$ and $a, 1-a \notin S(u)$. Hence, by the lower semicontinuity of the Γ-limsup, we have that

$$
\begin{aligned}
& \Gamma\text{-}\limsup_{\varepsilon\to 0} G_\varepsilon^{4/3}(u) \\
\le\ & \liminf_{a\to 0^+} \Big(\Gamma\text{-}\limsup_{\varepsilon\to 0} G_\varepsilon^{4/3}(u^a)\Big) \\
\le\ & \liminf_{a\to 0^+} \inf_{T>0} \inf\Big\{ \sum_{i\in I(a)} \Big(\int_{-T}^{T} (v_i')^2\, ds \ + (\theta_{i+1}-\theta_i)\Big(\sum_{j=1}^{i} \int_{-T}^{T} (v_j^2-1)\, ds \Big)^2 \Big) : \\
& \qquad\qquad v_i \in H^1((-T,T);[-1,1]), \quad v_i(\pm T) = \pm 1 \Big\} \\
\le\ & \inf_{T>0} \inf\Big\{ \sum_{i\in I} \Big(\int_{-T}^{T} (v_i')^2\, ds \ + (\theta_{i+1}-\theta_i)\Big(\sum_{j=1}^{i} \int_{-T}^{T} (v_j^2-1)\, ds \Big)^2 \Big) : \\
& \qquad\qquad v_i \in H^1((-T,T);[-1,1]), \quad v_i(\pm T) = \pm 1 \Big\},
\end{aligned}
$$

where $I(a) = \{i \in \mathbf{N} : \theta_i \in S(u^a), \theta_i < \theta_{i+1}\}$ and $I = \{i \in \mathbf{N} : \theta_i \in S(u), \theta_i < \theta_{i+1}\}$. □

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