REGULARITY OF MINIMIZERS UNDER LIMIT GROWTH CONDITIONS

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This paper is dedicated to Nicola Fusco on the occasion of his 60th birthday.
Nicola is expert and master in regularity; we like here to give a small contribution to this field.

Abstract. It is well known that an integral of the Calculus of Variations satisfying anisotropic growth conditions may have unbounded minimizers if the growth exponents are too far apart. Under sharp assumptions on the exponents we prove the local boundedness of minimizers of functionals with anisotropic p, q-growth, via the De Giorgi method. As a by-product, regularity of minimizers of some non coercive functionals is obtained by reduction to coercive ones.

1. Introduction

An unusual point of view for the following integrals of the Calculus of Variations

$$\mathcal{F}(u) = \int_{B_1(0)} |x|^{\alpha} |Du|^r \, dx, \quad \mathcal{G}(u) = \int_{B_1(0)} |x|^{-\alpha} |Du|^r \, dx, \tag{1.1}$$

with $r > 1$ and $\alpha > 0$, is to include them in the class of functionals satisfying some p, q-growth conditions. In fact, for $\mathcal{F}(u)$ in (1.1) we have that for every exponent $p \in [1, r)$,

$$|Du|^p = (|x|^{\alpha}|Du|^r)^{\frac{p}{r}} (|x|^{-\alpha})^{\frac{p}{r}} \leq \frac{p}{r}|x|^{\alpha}|Du|^r + \frac{r-p}{r}|x|^{-\frac{\alpha p}{r-p}}$$

and $|x|^{\alpha}|Du|^r \leq |Du|^r$ for every $x \in B_1(0)$; so $q = r$. Hence $\mathcal{F}$, not coercive in $W^{1,r}_{\mathrm{loc}}(B_1(0))$, is coercive in $W^{1,p}_{\mathrm{loc}}(B_1(0))$. We claim that every local minimizer in $W^{1,p}_{\mathrm{loc}}(B_1(0))$ of the integral $\mathcal{F}$ is locally bounded whenever

$$\begin{cases} 0 < \alpha < r - 1 & \text{if } 1 < r \leq \frac{n}{n-1} \\ 0 < \alpha < \frac{r^2}{n+r} & \text{if } \frac{n}{n-1} < r \leq n. \end{cases} \tag{1.2}$$

This result is a particular case of our Theorem 2.5, that we now state not in its full generality.

Theorem 1.1. *Let $f(x, u, \xi)$ be a Carathéodory function convex with respect to $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and such that*

$$|\xi|^p - a(x) \leq f(x, u, \xi) \leq L \{|\xi|^q + |u|^q + a(x)\},$$

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for a.e. $x \in \Omega$, Ω *open bounded set in* $\mathbb{R}^n$, $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, *for some* $L > 0$ *and* $a \in L^s_{\mathrm{loc}}(\Omega)$. *Then, if* $1 \leq p \leq q \leq p^*$ *and* $s > \max\{\frac{n}{p}, 1\}$, *every local minimizer of* $\mathcal{F}(u) = \int_\Omega f(x, u, Du)\, dx$ *in the class* $W^{1,p}_{\mathrm{loc}}(\Omega)$ *is locally bounded in* Ω.

Indeed, if $0 < \alpha < r - p$, the function $a(x) := |x|^{-\frac{p\alpha}{r-p}}$ is in $L^s(B_1(0))$ for some $s > \frac{n}{p}$. Since we need $r \leq p^*$, if $r > \frac{n}{n-1}$ the largest upper bound on α is obtained for $p = \frac{rn}{n+r}$, so obtaining $\alpha < \frac{r^2}{n+r}$. When $r \leq \frac{n}{n-1}$, the largest upper bound on α is obtained for $p = 1$.

Similarly, we can deal with the integral $\mathcal{G}$ in (1.1). In fact, for $q > r$ we have

$$|x|^{-\alpha}|Du|^r \leq \frac{r}{q}|Du|^q + \frac{q-r}{q}|x|^{-\frac{\alpha q}{q-r}};$$

moreover, $|x|^{-\alpha}|Du|^r \geq |Du|^r$, for a.e. $x \in B_1(0)$. Again, by Theorem 1.1, applied with $p = r$ and $q \in (r, \frac{rn}{n-r}]$ (if $r < n$) or any $q > r$ (if $r = n$), we obtain that *every local minimizer of the integral* $\mathcal{G}(u)$ *in (1.1) is locally bounded* if

$$0 < \alpha < \frac{r^2}{n} \quad \text{if } r \leq n. \tag{1.3}$$

The functionals $\mathcal{F}$ and $\mathcal{G}$ described above are particular cases of the more general integral

$$\mathcal{F}(u) = \int_\Omega a\,(x)\,|Du|^r\, dx \tag{1.4}$$

with $r > 1$, $a(x) \geq 0$ a.e. in Ω, $a \in L^\sigma_{\mathrm{loc}}(\Omega)$ and $\frac{1}{a} \in L^\tau_{\mathrm{loc}}(\Omega)$, with $\sigma, \tau > 1$. In Theorem 6.1 we prove that, under suitable conditions on σ, τ related to n and r, see (6.2), there exist p and q, with $1 \leq p \leq r \leq q \leq p^*$, such that the integrand $f(x, Du) = a\,(x)\,|Du|^r$ satisfies the assumptions of Theorem 1.1 and therefore every local minimizer in $W^{1,p}_{\mathrm{loc}}(\Omega)$ is locally bounded.

Non-uniformly elliptic equations and integrals of the Calculus of Variations of the type (1.4) with $r = 2$ have been studied by Trudinger [29] in 1971; in particular Section 3 in [29] is devoted to the study of the local boundedness of weak solutions to the Euler's equation of integrals of the type in (1.1). Higher integrability has been considered in a similar context by [5]. See also [25], [26], [32], [8], [27], and recently [22].

In this paper we consider a more general framework. In Section 2 we state our main regularity results, in particular the *local boundedness* of *minimizers* (and of *quasi-minimizers* too) of general integrals of the Calculus of Variations of the type

$$\mathcal{F}(u; \Omega) := \int_\Omega f(x, u, Du)\, dx.$$

More precisely, let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be Carathéodory function, convex in $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$ for $|\xi|$ large enough and satisfying the following anisotropic growth condition

$$\sum_{i=1}^n [g(|\xi_i|)]^{p_i} \leq f(x, u, \xi) \leq L\left\{[g(|\xi|)]^q + [g(|u|)]^q + a(x)\right\}, \tag{1.5}$$

for a.e. $x \in \Omega$, every $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$; for $L > 0$ and $a \in L^s_{\mathrm{loc}}(\Omega)$ for some $s > 1$ and $1 \leq p_i \leq q$, $i = 1, \ldots, n$. Finally $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a function of class C^1, increasing and convex, $g(0) = 0$, $g \not\equiv 0$, satisfying $g\,(\lambda t) \leq \lambda^\mu g\,(t)$ for some $\mu \geq 1$ and every $\lambda > 1$.

Let $\overline{p}$ be the harmonic average of $\{p_i\}$; i.e., $\dfrac{1}{\overline{p}} := \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\dfrac{1}{p_i}$; and $\overline{p}^*$ is the usual Sobolev exponent of $\overline{p}$, that is $\overline{p}^* = \frac{n\overline{p}}{n-\overline{p}}$ if $\overline{p} < n$, otherwise $\overline{p}^*$ is any $t > \overline{p}$.

By assuming $q \leq \overline{p}^*$, every local minimizer or *quasi-minimizer* is locally bounded, see Theorem 2.2. Note that the equality $q = \overline{p}^*$ is a limit growth condition, due to the well known counterexamples in [18], [23], [24] and the results in [4], [16], [17].

We observe that anisotropic functionals as in (1.5) appear in several branches of applied analysis, in particular in models where the derivatives have different weights along distinct directions. Moreover, the presence of the convex function g permits to consider some particular variational model with logarithmic behavior, as it happens in the theory of plasticity.

In [14] De Giorgi developed an original geometric method for the boundedness and regularity of solutions to elliptic equation with discontinuous coefficients. The fundamental ideas of this technique have been successfully applied to get regularity for local minimizers of Calculus of Variations with standard p-growth (for an exhaustive overview on the subject, see [19]). The proofs of our results are based on this method. Although the strategy for establishing regularity goes as in the standard p-growth, we had to overcome some difficulties for the presence of p, q-growth and the anisotropy of the functionals. In particular, we obtain a special *unbalanced* Caccioppoli inequality, without the use of a p-growth coercivity from below, which allows us to carry out the De Giorgi procedure for the local boundedness of minimizers.

It is noteworthy that Trudinger, in the quoted paper [29], pointed out that in this context of non-uniformly elliptic problems it is possible to give conditions to establish the local boundedness of weak solutions, but in general, due to the lack of the uniform ellipticity, it is not clear if they are Hölder continuous too.

In this paper we also study a class of variational integrals with linear growth from below; i.e., $\min\{p_i\} = 1$ in (1.5). Because of the lack of coercivity we consider the relaxed functional in the class of bounded variation functions $BV(\Omega)$, see Theorem 2.7; see also [3] for related results.

In recent years the study of integrals and equations with p, q-growth has undergone a remarkable development, also under the impulse of some applications, as in the study of strongly anisotropic materials, see [30] and [31]. The bibliography on the regularity under p, q-growth is large; we recall some recent papers on the subject: [2], [6], [7], [22] and, by the authors, [9], [12]; in the vector-valued case [10], [11]; we refer to [27] for a detailed survey on the subject.

The paper is organized as follows: in Section 2 we give the statement of the main regularity results; in Section 3 we collect some preliminary and technical properties; in Section 4 we establish an inequality of Caccioppoli type; Section 5 is devoted to the proofs of our main theorems; finally, Section 6 contains the applications of Theorem 2.5 to the functionals (1.4).

2. Assumptions and statement of the main results

Consider the integral functional of the type

$$\mathcal{F}(u;\Omega) := \int_\Omega f(x,u,Du)\,dx, \tag{2.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^n$, $n \geq 2$, and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function.

We recall the definition of quasi-minimizers of (2.1).

Definition 2.1. A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a *quasi-minimizer* of (2.1) if $f(x,u,Du) \in L^1_{\mathrm{loc}}(\Omega)$ and there exists $Q \geq 1$ such that

$$\mathcal{F}(u;\mathrm{supp}\ \varphi) \leq Q\mathcal{F}(u+\varphi;\mathrm{supp}\ \varphi),$$

for all $\varphi \in W^{1,1}(\Omega)$ with $\mathrm{supp}\,\varphi \Subset \Omega$. If $Q = 1$, then u is a *local minimizer* of (2.1).

We assume the following growth condition: there exist $L > 0$ and $1 \leq p_i \leq q$, $i = 1,\dots,n$, such that

$$\sum_{i=1}^n [g(|\xi_i|)]^{p_i} \leq f(x,u,\xi) \leq L\left\{[g(|\xi|)]^q + [g(|u|)]^q + a(x)\right\} \tag{2.2}$$

for a.e. x and for every $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Here $a \in L^s_{\mathrm{loc}}(\Omega)$, $s \in (1,\infty]$, and $g : [0,\infty) \to [0,\infty)$ is a N-function of Δ_2-class; precisely, we assume that g is of class C^1, convex, non-decreasing, $g(0) = 0$, $g \not\equiv 0$, satisfying, for some $\mu \geq 1$,

$$g(\lambda t) \leq \lambda^\mu g(t) \quad \text{for every} \quad \lambda > 1 \quad \text{and every} \quad t \geq t_0 \tag{2.3}$$

for some $t_0 > 0$.

We now require a convexity assumption *at infinity* on f. Precisely, Let us denote $f^{**} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ the convex envelope of $f(x,u,\xi)$ with respect to (u,ξ). We assume that

$$f(x,\cdot,\cdot) = f^{**}(x,\cdot,\cdot) \text{ in } \left(\mathbb{R} \times B_{t_0}(0)\right)^c. \tag{2.4}$$

From now on, without any loss of generality, we assume $t_0 = 1$ and $g(t) \geq 1$ for all $t \geq 1$. We observe that if $s = +\infty$ then $\frac{1}{s}$ has to be read as 0.

We denote by $\overline{p}$ the harmonic average of $\{p_i\}$; i.e., $\dfrac{1}{\overline{p}} := \dfrac{1}{n}\displaystyle\sum_{i=1}^n \dfrac{1}{p_i}$; moreover, $\overline{p}^*$ is the Sobolev exponent of $\overline{p}$:

$$\overline{p}^* := \begin{cases} \frac{n\overline{p}}{n-\overline{p}}, & \text{if } \overline{p} < n, \\ \text{any } t > \overline{p}, & \text{if } \overline{p} \geq n. \end{cases} \tag{2.5}$$

Let us now state our results. First, we deal with the case $q < \overline{p}^*$.

Theorem 2.2. *Assume (2.2)-(2.4) with* $1 \leq p_i \leq q$,

$$q < \overline{p}^* \quad \textit{and} \quad s > \frac{\overline{p}^*}{\overline{p}^* - \overline{p}}. \tag{2.6}$$

Then any quasi-minimizer u of (2.1) is locally bounded. Moreover, for every $B_R(x_0) \Subset \Omega$, there exists a positive constant c, depending on q, p_i, s, μ, Q, L, R, such that

$$\|g(|u|)\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \le c \left\{ 1 + \left(\int_{B_R(x_0)} g^q(|u|)\, dx \right)^\gamma \right\}, \tag{2.7}$$

where $\gamma = \frac{\overline{p}^*(1-1/s)-\overline{p}}{\overline{p}(\overline{p}^*-q)}$.

As far as the limit case $q = \overline{p}^*$ is concerned, we have the following result.

Theorem 2.3. *Assume (2.2)-(2.4) with $1 \le p_i \le q = \overline{p}^*$, and*

$$\text{either} \quad \max\{p_i\} < \overline{p}^* \quad \text{or} \quad g(|u|) \in L^{\overline{p}^*}_{\text{loc}}(\Omega).$$

If $s > \frac{\overline{p}^*}{\overline{p}^*-\overline{p}}$, then any quasi-minimizer u of (2.1) is locally bounded.

Remark 2.4. As far as the assumption on s is concerned, we notice that if $\overline{p} < n$ we have

$$s > \frac{\overline{p}^*}{\overline{p}^* - \overline{p}} \Leftrightarrow \overline{p}^* \left(1 - \frac{1}{s}\right) - \overline{p} > 0 \Leftrightarrow s > \frac{n}{\overline{p}}.$$

If, instead $\overline{p} \ge n$, due to the arbitrariness of $\overline{p}^*$, we can replace (2.6) with $s > 1$.

Note that, if the p_i's are equal, we obtain the straightforward consequence of the above results.

Theorem 2.5. *Assume (2.4) and that there exists $L > 0$, such that*

$$|\xi|^p \le f(x, u, \xi) \le L\left\{|\xi|^q + |u|^q + a(x)\right\}, \qquad 1 \le p < q \le p^*$$

for a.e. x, for every $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, with $a \in L^s_{\text{loc}}(\Omega)$, with $s > \max\{\frac{n}{p}, 1\}$. Then the quasi-minimizers of $\mathcal{F}$ are locally bounded.

Now, we deal with a minimization problem in a Dirichlet class.

We here consider $g(t) := t$; precisely, we assume that there exist $L > 0$ and $1 \le p_i \le q$, $i = 1, \dots, n$, $a \in L^s_{\text{loc}}(\Omega)$, $s > 1$, such that

$$\sum_{i=1}^n |\xi_i|^{p_i} \le f(x, u, \xi) \le L\left\{|\xi|^q + |u|^q + a(x)\right\}, \tag{2.8}$$

for a.e. x, for every $u \in \mathbb{R}$ and every $\xi \in \mathbb{R}^n$.

A first result, with $\min\{p_i\} > 1$, is the following.

Theorem 2.6. *Assume (2.4) and (2.8), with $1 < p_i \le q \le \overline{p}^*$, $i = 1, \dots, n$. Let $u_0 \in W^{1,1}(\Omega) \cap L^{\overline{p}^*}_{\text{loc}}(\Omega)$ be such that $\mathcal{F}(u_0; \Omega) < +\infty$. If u is a minimizer of $\mathcal{F}(\cdot; \Omega)$ in $u_0 + W^{1,(p_1,\dots,p_n)}_0(\Omega)$, and $s > \frac{n}{\overline{p}}$ (if $\overline{p} < n$) or $s > \frac{\overline{p}^*}{\overline{p}^*-\overline{p}}$ (if $\overline{p} \ge n$) then u is locally bounded.*

Let us consider the case $\min\{p_i\} = 1$. Fix $u_0 \in W^{1,1}(\Omega)$, such that $\mathcal{F}(u_0; \Omega) < +\infty$. Since $\min\{p_i\} = 1$, then $W^{1,(p_1,\dots,p_n)}(\Omega)$ is a non-reflexive space and the direct method generally fails. So, minimizers of $\mathcal{F}$ in $u_0 + W^{1,(p_1,\dots,p_n)}_0(\Omega)$ may not exist. Consider

$$\overline{\mathcal{F}}(u) := \inf\left\{ \liminf_{k \to +\infty} \mathcal{F}(u_k) \ : \ u_k \to u \text{ in } L^1(\Omega),\ u_k \in u_0 + W^{1,(p_1,\dots,p_n)}_0(\Omega) \right\}, \tag{2.9}$$

the relaxed functional of $\mathcal{F}(\cdot;\Omega)$ in $BV(\Omega)$. We prove that minimizers of $\overline{\mathcal{F}}$ exist in $BV(\Omega)$ and are locally bounded.

Theorem 2.7. *Assume (2.4) and (2.8), with* $1 \le p_i \le q < \overline{p}^*$, $\min\{p_i\} = 1$.
Fixed $u_0 \in W^{1,1}(\Omega)$, *such that* $\mathcal{F}(u_0;\Omega) < +\infty$, *there exists a minimizer* $\bar{u} \in BV(\Omega)$ *of* $\overline{\mathcal{F}}$, *such that* $\bar{u} \in L^\infty_{\rm loc}(\Omega)$ *and, for all* $B_R(x_0) \Subset \Omega$,

$$\|\bar{u}\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \le c\left\{1 + \left(\overline{\mathcal{F}}(\bar{u}) + \|u_0\|_{W^{1,(p_1,\dots,p_n)}(\Omega)}\right)^{q\gamma}\right\},$$

where $\gamma := \frac{\overline{p}^*(1-1/s)-\overline{p}}{\overline{p}(\overline{p}^*-q)}$ *and* c *is depending on* q, p_i, s, L, R.

3. Preliminary results

We consider the following anisotropic Sobolev space:

$$W^{1,(p_1,\dots,p_n)}(\Omega) := \left\{u \in W^{1,1}(\Omega) \,:\, u_{x_i} \in L^{p_i}(\Omega), \text{ for all } i = 1,\dots,n\right\},$$

endowed with the norm

$$\|u\|_{W^{1,(p_1,\dots,p_n)}(\Omega)} := \|u\|_{L^1(\Omega)} + \sum_{i=1}^n \|u_{x_i}\|_{L^{p_i}(\Omega)}.$$

Let us denote $W_0^{1,(p_1,\dots,p_n)}(\Omega) = W_0^{1,1}(\Omega) \cap W^{1,(p_1,\dots,p_n)}(\Omega)$.

We recall the following embedding results for anisotropic Sobolev spaces. We refer to [28].

Theorem 3.1. *Let* $p_i \ge 1$, $i = 1,\dots,n$, *and* $\overline{p}^*$ *be as in (2.5). Let* $u \in W_0^{1,(p_1,\dots,p_n)}(\Omega)$ *and* Ω *be an open bounded set in* $\mathbb{R}^n$. *Then there exists* c, *depending on* n, p_i *and, only in the case* $\overline{p} \ge n$, *also on* $\overline{p}^*$ *and on the measure of the support of* u, *such that*

$$\|u\|_{L^{\overline{p}^*}(\Omega)} \le c\left(\Pi_{i=1}^n \|u_{x_i}\|_{L^{p_i}(\Omega)}\right)^{\frac{1}{n}}. \tag{3.1}$$

Remark 3.2. In general if $n \ge 2$, the inclusion $W^{1,(p_1,\dots,p_n)}(\Omega) \subset L^{\overline{p}^*}(\Omega)$ may not hold, even if Ω is a rectangular domain. See [20] and [21].

We also need the following result; see Proposition 1 in [9] for the proof.

Proposition 3.3. *Let* g *be a* Δ_2 *and* N*-function of* C^1 *class (see Section 2) and* $u \in W^{1,1}_{\rm loc}(\Omega)$. *Suppose that* $g(|u_{x_i}|) \in L^{p_i}_{\rm loc}(\Omega)$, *with* $1 \le p_i < \overline{p}^*$ *for every* $i = 1,\dots,n$. *Then* $g(|u|) \in L^{\overline{p}^*}_{\rm loc}(\Omega)$.

Moreover we need of some properties of the Δ_2-functions; see [9] for the proof.

Lemma 3.4. *Consider* $g : [0,\infty) \to [0,\infty)$ *of class* C^1, *convex, non-decreasing and satisfying (2.3), with* $t_0 = 1$. *Then*

$$g(\lambda t) \le \lambda^\mu (g(t) + g(1)) \quad \text{and} \quad g'(t)t \le \mu(g(t) + g(1)),$$

for all $t \ge 0$ *and all* $\lambda > 1$. *Moreover, for every* $(t_1,\dots,t_k) \in [0,\infty)^k$, *we have:*

$$k^{-1}\sum_{i=1}^k g(t_i) \le g\left(\sum_{i=1}^k t_i\right) \le k^\mu\left\{g(1) + \sum_{i=1}^k g(t_i)\right\}.$$

The following is a well known classical result; see, e.g., [19].

Lemma 3.5. *Let $\phi(t)$ be a non-negative and bounded function, defined in $[\tau_0, \tau_1]$. Suppose that, for all s, t, such that $\tau_0 \le s < t \le \tau_1$, ϕ satisfies*

$$\phi(s) \le \theta\phi(t) + \frac{A}{(t-s)^\alpha} + B,$$

where A, B, α are non-negative constants and $0 < \theta < 1$. Then, for all ρ and R, such that $\tau_0 \le \rho \le R \le \tau_1$, we have

$$\phi(\rho) \le C\left\{\frac{A}{(R-\rho)^\alpha} + B\right\}.$$

4. Caccioppoli inequality

For $u \in W^{1,1}_{\text{loc}}(\Omega)$ and $B_R(x_0) \subseteq \Omega$, we define the super-level sets:

$$A_{k,R} := \{x \in B_R(x_0) \,:\, u(x) > k\}, \qquad k \in \mathbb{R}.$$

For a quasi-minimizer of $\mathcal{F}$ the following Caccioppoli inequality holds.

Proposition 4.1. *Assume (2.4) and*

$$0 \le f(x, u, \xi) \le L\left\{[g(|\xi|)]^q + [g(|u|)]^q + a(x)\right\}, \tag{4.1}$$

with $q > 1$ and $a \in L^s(\Omega)$, $s > 1$. Let $u \in W^{1,1}_{\text{loc}}(\Omega)$ be a quasi-minimizer of $\mathcal{F}$, such that $g(|u|) \in L^q_{\text{loc}}(\Omega)$. Then for any $B_R(x_0) \Subset \Omega$, $0 < \rho < R$, and for any $k, d \in \mathbb{R}$, $d \ge k \ge 1$,

$$\int_{A_{k,\rho}} f(x, u, Du)\,dx \le \frac{c}{(R-\rho)^{\mu q}} \int_{A_{k,R}} \{g^q(u-k) + g^q(d)\}\,dx + c\|a\|_{L^s(B_R)}|A_{k,R}|^{1-\frac{1}{s}}, \tag{4.2}$$

with c depending on n, q, μ, Q, L.

Proof. Let $B_R(x_0) \Subset \Omega$. Let ρ, s, t be such that $0 < \rho \le s < t \le R$. Let $\eta \in C^\infty_0(B_t)$ be a cut-off function, satisfying the following assumptions:

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ in } B_s(x_0), \quad |D\eta| \le \frac{2}{t-s}. \tag{4.3}$$

Fixed $k \ge 1$, define

$$w := \max(u-k, 0) \quad \text{and} \quad \varphi := -\eta^{\mu q} w.$$

Consider a number d, such that $d \ge k$. By the quasi-minimality of u, we get

$$\int_{A_{k,s}} f(x,u,Du)\,dx \le \int_{A_{k,t}\cap \operatorname{supp}\eta} f(x,u,Du)\,dx \le Q\int_{A_{k,t}\cap \operatorname{supp}\eta} f(x, u+\varphi, Du + D\varphi)\,dx$$

$$= Q\int_{A_{k,t}\cap \operatorname{supp}\eta} f\left(x, (1-\eta^{\mu q})u + \eta^{\mu q}k, (1-\eta^{\mu q})Du + \mu q\eta^{\mu q-1}(k-u)D\eta\right)dx.$$

Denote

$$\Omega_k := \{x \in \Omega \,:\, \left((1-\eta^{\mu q})u + \eta^{\mu q}k, (1-\eta^{\mu q})Du + \mu q\eta^{\mu q-1}(k-u)D\eta\right) \in \mathbb{R}\times B_1(0)\}.$$

By (2.4), for a.e. $x \in (A_{k,t} \cap \operatorname{supp} \eta) \setminus \Omega_k$,

$$\begin{aligned} & f\left(x, (1-\eta^{\mu q})u + \eta^{\mu q} k, (1-\eta^{\mu q})Du + \mu q \eta^{\mu q - 1}(k-u)D\eta\right) \\ & = f^{**}\left(x, (1-\eta^{\mu q})u + \eta^{\mu q} k, (1-\eta^{\mu q})Du + \mu q \eta^{\mu q - 1}(k-u)D\eta\right) \\ & \leq (1-\eta^{\mu q}) f^{**}(x, u, Du) + \eta^{\mu q} f^{**}\left(x, k, \mu q \eta^{-1}(k-u)D\eta\right) \\ & \leq (1-\eta^{\mu q}) f(x, u, Du) + \eta^{\mu q} f\left(x, k, \mu q \eta^{-1}(k-u)D\eta\right). \end{aligned} \tag{4.4}$$

By the growth assumption (4.1)

$$f\left(x, k, \mu q \eta^{-1}(k-u)D\eta\right) \leq L\left\{a(x) + g^q(\mu q|\frac{u-k}{\eta} D\eta|) + g^q(d)\right\}. \tag{4.5}$$

Lemma 3.4 and (4.3) imply

$$g^q(|\mu q \frac{u-k}{\eta} D\eta|) \leq \frac{(2\mu q)^{\mu q}}{(t-s)^{\mu q} \eta^{\mu q}} \left\{g^q(|u-k|) + g^q(1)\right\}. \tag{4.6}$$

Therefore, by (4.4), (4.5), (4.6) and $1 \leq d$, for a.e. $x \in (A_{k,t} \cap \operatorname{supp} \eta) \setminus \Omega_k$

$$\begin{aligned} & f\left(x, (1-\eta^{\mu q})u + \eta^{\mu q} k, (1-\eta^{\mu q})Du + \mu q \eta^{\mu q - 1}(k-u)D\eta\right) \\ & \leq (1-\eta^{\mu q}) f(x, u, Du) + L\eta^{\mu q} a(x) + L\eta^{\mu q} \frac{(2\mu q)^{\mu q}}{(t-s)^{\mu q} \eta^{\mu q}} \left\{g^q(|u-k|) + g^q(d)\right\}. \end{aligned} \tag{4.7}$$

Let us now consider the case $x \in A_{k,t} \cap \operatorname{supp} \eta \cap \Omega_k$. Since g is increasing and convex, by Lemma 3.4 we have

$$g((1-\eta^{\mu q})u + \eta^{\mu q} k) \leq (1-\eta^{\mu q}) g(u) + \eta^{\mu q} g(k) \leq c\,(g(u-k) + g(d)).$$

Therefore, again by (4.1), for a.e $x \in A_{k,t} \cap \operatorname{supp} \eta \cap \Omega_k$ and for (4.6)

$$f\left(x, (1-\eta^{\mu q})u + \eta^{\mu q} k, (1-\eta^{\mu q})Du + \mu q \eta^{\mu q - 1}(k-u)D\eta\right) \leq c\,(a(x) + g^q(u-k) + g^q(d)). \tag{4.8}$$

Taking into account that $\operatorname{supp}(1-\eta^{\mu q}) \subset B_t \setminus B_s$, $\eta^{\mu(q-1)} \leq 1$, we have

$$\int_{A_{k,t} \cap \operatorname{supp} \eta \setminus \Omega_k} (1-\eta^{\mu q}) f(x, u, Du)\, dx \leq \int_{A_{k,t} \setminus A_{k,s}} f(x, u, Du)\, dx.$$

By (4.7) and (4.8) we obtain

$$\begin{aligned} & \int_{A_{k,t} \cap \operatorname{supp} \eta} f\left(x, (1-\eta^{\mu q})u + \eta^{\mu q} k, (1-\eta^{\mu q})Du + \mu q \eta^{\mu q - 1}(k-u)D\eta\right)\, dx \\ & \leq \int_{A_{k,t} \setminus A_{k,s}} f(x, u, Du)\, dx + c \int_{A_{k,R}} \left(a(x) + g^q(u-k) + g^q(d)\right)\, dx \\ & + c \frac{(2\mu q)^{\mu q}}{(t-s)^{\mu q}} \int_{A_{k,R}} \left\{g^q(u-k) + g^q(d)\right\}\, dx. \end{aligned} \tag{4.9}$$

Therefore

$$\int_{A_{k,s}} f(x,u,Du)\,dx \le Q\int_{A_{k,t}\setminus A_{k,s}} f(x,u,Du)\,dx$$
$$+ Qc\|a\|_{L^s(B_R)}|A_{k,R}|^{1-\frac{1}{s}} + \frac{c}{(t-s)^{\mu q}}\int_{A_{k,R}} \{g^q\,(u-k)+g^q(d)\}\,dx \tag{4.10}$$

with $c = c(n,\mu,q,Q,L)$.

Conclusion.

By (4.10), adding to both sides Q times the left hand side, we get:

$$\int_{A_{k,s}} f(x,u,Du)\,dx \le \frac{Q}{Q+1}\int_{A_{k,t}} f(x,u,Du)\,dx + \frac{Qc}{Q+1}\|a\|_{L^s(B_R)}|A_{k,R}|^{1-\frac{1}{s}}$$
$$+ \frac{c}{(Q+1)(t-s)^{\mu q}}\int_{A_{k,R}} \{g^q(u-k)+g^q(d)\}\,dx.$$

Thus, by Lemma 3.5, with $\tau_0 := \rho$, $\tau_1 := R$, $\phi(t) := \int_{A_{k,t}} f(x,u,Du)\,dx$, and

$$A := \int_{A_{k,R}} \{g^q(u-k)+g^q(d)\}\,dx, \qquad B := \frac{Qc}{Q+1}\|a\|_{L^s(B_R)}|A_{k,R}|^{1-\frac{1}{s}},$$

we get (4.2). $\square$

5. Proof of Theorems 2.2, 2.3 and 2.6

Assume that g satisfies the assumptions in Section 2. Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be such that $g(|u|) \in L^q_{\mathrm{loc}}(\Omega)$. Consider $B_{R_0}(x_0) \Subset \Omega$, such that

$$\int_{B_{R_0}(x_0)} g^q(|u|)\,dx \le 1. \tag{5.1}$$

For any $0 < R \le R_0$, define the decreasing sequences

$$\rho_h := \frac{R}{2} + \frac{R}{2^{h+1}} = \frac{R}{2}(1+\frac{1}{2^h}), \qquad \bar\rho_h := \frac{\rho_h+\rho_{h+1}}{2} = \frac{R}{2}(1+\frac{3}{4\cdot 2^h}).$$

Fixed a positive constant $d \ge 2$, to be chosen later, define the increasing sequence of positive real numbers

$$k_h := d\left(1-\frac{1}{2^{h+1}}\right), \; h \in \mathbb{N}\cup\{0\}. \tag{5.2}$$

Define the sequence (J_h),

$$J_h := \int_{A_{k_h,\rho_h}} g^q(u-k_h)\,dx. \tag{5.3}$$

Notice that, by (5.1), $J_h \le 1$ for every h and that J_h is a decreasing sequence, because

$$J_{h+1} \le \int_{A_{k_{h+1},\rho_h}} g^q(u-k_{h+1})\,dx \le \int_{A_{k_{h+1},\rho_h}} g^q(u-k_h)\,dx \le J_h. \tag{5.4}$$

The following lemma is the common root to prove Theorems 2.2 and 2.3.

Lemma 5.1. *Let $u \in W^{1,1}_{\rm loc}(\Omega)$ be a quasi-minimizer of $\mathcal{F}$. Assume (2.2)-(2.4), with $q \le \overline{p}^*$, and $a \in L^s_{\rm loc}(\Omega)$, $s > 1$. Moreover assume that $g(|u|) \in L^q_{\rm loc}(\Omega)$ and let $B_{R_0}(x_0) \Subset \Omega$ be such that $\int_{B_{R_0}(x_0)} g^q(|u|)\,dx \le 1$. Let J_h be as in (5.3).*

Then there exists a constant $C > 0$, depending on $\|a\|_{L^s(B_{R_0})}$, such that, for all $h \in \mathbb{N} \cup \{0\}$,

$$J_{h+1} \le \frac{C}{(g(d))^{q-\frac{q^2}{\overline{p}^*}}} \left(\frac{1}{R}\right)^{\mu\frac{q^2}{\overline{p}}} \lambda^h J_h^{1+\alpha},$$

where $\lambda = 4^{\mu\frac{q^2}{\overline{p}}}$ and $\alpha = \frac{q}{\overline{p}}\left(1 - \frac{1}{s}\right) - \frac{q}{\overline{p}^*}$.

Proof. Since u is a quasi-minimizer of $\mathcal{F}$ and (2.2) holds, then $g(|u_{x_i}|) \in L^{p_i}_{\rm loc}(\Omega)$.
If $q < \overline{p}^*$, then $\max\{p_i\} < \overline{p}^*$ and, by Proposition 3.3, $g(|u|) \in L^{\overline{p}^*}_{\rm loc}(\Omega)$.
If $q = \overline{p}^*$, we have, by assumption, that $g(|u|) \in L^{\overline{p}^*}(B_{R_0})$. Thus, J_h is finite for all h.
Let, now, define a sequence (ζ_h) of cut-off functions, satisfying the following properties:

$$\zeta_h \in C^\infty_c(B_{\bar\rho_h}(x_0)), \quad \zeta_h \equiv 1 \quad \text{in} \quad B_{\rho_{h+1}}, \quad \text{and} \quad |D\zeta_h| \le \frac{2^{h+4}}{R}.$$

Denoting $(u - k_{h+1})_+ := \max\{u - k_{h+1}, 0\}$, by the Hölder inequality we get

$$\begin{aligned} J_{h+1} &\le |A_{k_{h+1},\bar\rho_h}|^{1-\frac{q}{\overline{p}^*}} \left(\int_{A_{k_{h+1},\bar\rho_h}} (g(u-k_{h+1})\zeta_h)^{\overline{p}^*}\,dx\right)^{\frac{q}{\overline{p}^*}} \\ &= |A_{k_{h+1},\bar\rho_h}|^{1-\frac{q}{\overline{p}^*}} \left(\int_{B_{\bar\rho_h}} (\zeta_h g((u-k_{h+1})_+))^{\overline{p}^*}\,dx\right)^{\frac{q}{\overline{p}^*}}. \end{aligned} \tag{5.5}$$

To apply the Sobolev embedding Theorem 3.1 to the function $g((u-k_{h+1})_+)\zeta_h$, we need to prove that $g((u-k_{h+1})_+)\zeta_h \in W^{1,(p_1,\dots,p_n)}_0(B_{\bar\rho_h}(x_0))$ i.e. that $(\zeta_h g((u-k_{h+1})_+))_{x_i} \in L^{p_i}(B_{\bar\rho_h}(x_0))$.

Taking into account that

$$(g((u(x)-k_{h+1})_+)))_{x_i} = g'(u(x)-k_{h+1})u_{x_i}(x)\chi_{A_{k_{h+1},\bar\rho_h}}(x) \qquad \text{for a.e. } x \in B_{\bar\rho_h}(x_0),$$

(here $\chi_{A_{k_{h+1},\bar\rho_h}}$ is, as usual, the characteristic function of the set $A_{k_{h+1},\bar\rho_h}$), noting that, by the monotonicity of g and g', $g'(t_1)t_2 \le g'(t_1)t_1 + g'(t_2)t_2$, and using Lemma 3.4 we get

$$\begin{aligned} |(\zeta_h g((u-k_{h+1})_+))_{x_i}| &\le g((u-k_{h+1})_+)|(\zeta_h)_{x_i}| + \zeta_h g'(u-k_{h+1})|u_{x_i}|\chi_{A_{k_{h+1},\bar\rho_h}} \\ &\le g((u-k_{h+1})_+)|D\zeta_h| + \zeta_h\left\{g'(u-k_{h+1})(u-k_{h+1}) + g'(|u_{x_i}|)|u_{x_i}|\right\}\chi_{A_{k_{h+1},\bar\rho_h}} \\ &\le g(u-k_{h+1})|D\zeta_h|\chi_{A_{k_{h+1},\bar\rho_h}} + \zeta_h\mu\left(g(u-k_{h+1}) + g(|u_{x_i}|) + 2g(1)\right)\chi_{A_{k_{h+1},\bar\rho_h}}. \end{aligned}$$

Since $g(d) \ge g(1)$, we have, for a.e. $x \in B_{\bar\rho_h}(x_0)$,

$$\begin{aligned} &|(\zeta_h g((u-k_{h+1})_+))_{x_i}| \\ &\le c(\mu)\frac{2^h}{R}\left(g(u-k_{h+1}) + g(d)\right)\chi_{A_{k_{h+1},\bar\rho_h}} + \mu g(|u_{x_i}|)\chi_{A_{k_{h+1},\bar\rho_h}}. \end{aligned} \tag{5.6}$$

Since $g(|u|)$ and $g(|u_{x_i}|)$ are in $L^{p_i}_{\mathrm{loc}}(\Omega)$, we have $(\zeta_h g((u-k_{h+1})_+))_{x_i} \in L^{p_i}(B_{\bar\rho_h}(x_0))$. Thus, by (5.5) and the embedding Theorem 3.1,

$$J_{h+1} \le c|A_{k_{h+1},\bar\rho_h}|^{1-\frac{q}{\bar p^*}} \left\{ \Pi_{i=1}^n \left(\int_{B_{\bar\rho_h}} |(\zeta_h g((u-k_{h+1})_+))_{x_i}|^{p_i}\, dx \right)^{\frac{1}{p_i}} \right\}^{\frac{q}{n}}. \tag{5.7}$$

Let us estimate the integrals in the right hand side. By (5.6), since $(a+b)^{\frac{1}{p_i}} \le a^{\frac{1}{p_i}} + b^{\frac{1}{p_i}}$, $a^{p_i} \le a^q + 1$ for every $a \ge 0$, $\bar\rho_h \le \rho_h$, and since (5.4) holds, then

$$\begin{aligned} \left(\int_{B_{\bar\rho_h}} |(\zeta_h g((u-k_{h+1})_+)_{x_i}|^{p_i}\, dx \right)^{\frac{1}{p_i}} &\le \mu \left(\int_{A_{k_{h+1},\bar\rho_h}} [g(|u_{x_i}|)]^{p_i}\, dx \right)^{\frac{1}{p_i}} \\ &\quad + \frac{c2^h}{R} \left(\int_{A_{k_{h+1},\rho_h}} \{g^q(u-k_{h+1}) + g^q(d)\}\, dx \right)^{\frac{1}{p_i}} \\ \le \mu \left(\int_{A_{k_{h+1},\bar\rho_h}} [g(|u_{x_i}|)]^{p_i}\, dx \right)^{\frac{1}{p_i}} &+ \frac{c2^h}{R} \left(J_h + g^q(d)|A_{k_{h+1},\rho_h}| \right)^{\frac{1}{p_i}}, \end{aligned} \tag{5.8}$$

where we used that $g(d) \ge g(1) \ge 1$. By (2.2), the Caccioppoli inequality (4.2) and (5.4), we obtain, for any $i = 1, \dots, n$,

$$\begin{aligned} &\int_{A_{k_{h+1},\bar\rho_h}} g^{p_i}(|u_{x_i}|)\, dx \le \int_{A_{k_{h+1},\bar\rho_h}} f(x,u,Du)\, dx \\ &\le c \left(\frac{2^h}{R} \right)^{\mu q} \int_{A_{k_{h+1},\rho_h}} \{g^q(u-k_{h+1}) + g^q(d)\}\, dx + c\|a\|_{L^s(B_R)} |A_{k_{h+1},\rho_h}|^{1-\frac{1}{s}} \\ &\le c \left(\|a\|_{L^s(B_R)} + 1 \right) \left(\frac{2^h}{R} \right)^{\mu q} \left(J_h + g^q(d)|A_{k_{h+1},\rho_h}| + |A_{k_{h+1},\rho_h}|^{1-\frac{1}{s}} \right). \end{aligned} \tag{5.9}$$

Collecting (5.8) and (5.9), we have

$$\begin{aligned} &\left(\int_{A_{k_{h+1},\bar\rho_h}} |(\zeta_h g(u-k_{h+1}))_{x_i}|^{p_i}\, dx \right)^{\frac{1}{p_i}} \\ &\le \left\{ c \left(\|a\|_{L^s(B_R)} + 1 \right) \left(\frac{2^h}{R} \right)^{\mu q} \left(J_h + g^q(d)|A_{k_{h+1},\rho_h}| + |A_{k_{h+1},\rho_h}|^{1-\frac{1}{s}} \right) \right\}^{\frac{1}{p_i}}. \end{aligned}$$

Using the above inequality to estimate (5.7), it follows that

$$J_{h+1} \le c|A_{k_{h+1},\bar\rho_h}|^{1-\frac{q}{\bar p^*}} \left\{ \Pi_{i=1}^n \left(\frac{2^h}{R} \right)^{\mu \frac{q}{p_i}} \left(J_h + g^q(d)|A_{k_{h+1},\rho_h}| + |A_{k_{h+1},\rho_h}|^{1-\frac{1}{s}} \right)^{\frac{1}{p_i}} \right\}^{\frac{q}{n}}, \tag{5.10}$$

with c depending on $\|a\|_{L^s(B_{R_0})}$. Note that

$$\begin{aligned} J_h &\geq \int_{A_{k_{h+1},\rho_h}} g^q(u-k_h)\,dx \geq g^q(k_{h+1}-k_h)|A_{k_{h+1},\rho_h}| \\ &= g^q(\frac{d}{2^{h+2}})|A_{k_{h+1},\rho_h}| \geq \frac{g^q(d)}{2^{(h+2)\mu q}}|A_{k_{h+1},\rho_h}|, \end{aligned}$$

therefore

$$|A_{k_{h+1},\bar{\rho}_h}| \leq |A_{k_{h+1},\rho_h}| \leq \frac{2^{(h+2)\mu q}}{g^q(d)} J_h \leq c\frac{2^{h\mu q}}{g^q(d)} J_h. \tag{5.11}$$

By (5.10) and (5.11), recalling that $J_h \leq 1$ for every h, so $J_h \leq J_h^{1-\frac{1}{s}}$, we obtain

$$\begin{aligned} J_{h+1} &\leq c\left(\frac{2^{h\mu q}}{g^q(d)}J_h\right)^{1-\frac{q}{\overline{p}^*}} \left\{\Pi_{i=1}^n \left(\frac{2^h}{R}\right)^{\mu\frac{q}{p_i}} \left(J_h\,+\,2^{h\mu q}J_h + \left(\frac{2^{(h+2)\mu q}}{g^q(d)}J_h\right)^{1-\frac{1}{s}}\right)^{\frac{1}{p_i}}\right\}^{\frac{q}{n}} \\ &\leq c\left(\frac{2^{h\mu q}}{g^q(d)}J_h\right)^{1-\frac{q}{\overline{p}^*}} \left\{\Pi_{i=1}^n \left(\frac{2^h}{R}\right)^{\mu\frac{q}{p_i}} \left(2^{h\mu q}J_h^{1-\frac{1}{s}}\right)^{\frac{1}{p_i}}\right\}^{\frac{q}{n}} \\ &= c\left(\frac{2^{h\mu q}}{g^q(d)}J_h\right)^{1-\frac{q}{\overline{p}^*}} \left\{\left[\left(\frac{2^h}{R}\right)^{\mu q} 2^{h\mu q}J_h^{1-\frac{1}{s}}\right]^{\sum_i\frac{1}{p_i}}\right\}^{\frac{q}{n}} \\ &= c\left(\frac{2^{h\mu q}}{g^q(d)}J_h\right)^{1-\frac{q}{\overline{p}^*}} \left(\frac{2^h}{R}\right)^{\mu\frac{q^2}{\overline{p}}} \left(2^{h\mu q}J_h^{1-\frac{1}{s}}\right)^{\frac{q}{\overline{p}}} \\ &\leq \frac{C}{R^{\mu\frac{q^2}{\overline{p}}}\left(g^q(d)\right)^{1-\frac{q}{\overline{p}^*}}} \left(4^{\mu\frac{q^2}{\overline{p}}}\right)^h J_h^{1+\frac{q}{\overline{p}}(1-\frac{1}{s})-\frac{q}{\overline{p}^*}} \end{aligned}$$

with C depending on $\|a\|_{L^s(B_{R_0})}$. The conclusion follows. $\square$

To prove Theorems 2.2 and 2.3 we will use the following classical result; see, e.g., [19].

Lemma 5.2. *Let $\alpha > 0$ and (J_h) a sequence of real positive numbers, such that*

$$J_{h+1} \leq A\,\lambda^h J_h^{1+\alpha},$$

with $A > 0$ and $\lambda > 1$. If $J_0 \leq A^{-\frac{1}{\alpha}}\lambda^{-\frac{1}{\alpha^2}}$, then $J_h \leq \lambda^{-\frac{h}{\alpha}}J_0$ and $\lim_{h\to\infty} J_h = 0$.

We are now ready to prove the regularity result under the assumption $q < \overline{p}^*$.

Proof of Theorem 2.2. Let d be a positive constant, $d \geq 2$, to be chosen later.
We notice that by (2.2) and since $q < \overline{p}^*$, it follows that $g(|u|) \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$ (see Proposition 3.3). Therefore, fixed $x_0 \in \Omega$, there exists $R_0 > 0$ small enough, such that $B_{R_0}(x_0) \Subset \Omega$ and $\int_{B_{R_0}(x_0)} g^q(|u|)\,dx \leq 1$. By Lemma 5.1, we have that, for all h,

$$J_{h+1} \leq \frac{C}{(g(d))^{q-\frac{q^2}{\overline{p}^*}}}\left(\frac{1}{R}\right)^{\mu\frac{q^2}{\overline{p}}}\lambda^h J_h^{1+\alpha},$$

with $\lambda := 4^{\mu\frac{q^2}{\overline{p}}}$ and $\alpha := \frac{q}{\overline{p}}\left(1-\frac{1}{s}\right)-\frac{q}{\overline{p}^*} > 0$. Using Lemma 5.2, with $A := \dfrac{C}{R^{\mu\frac{q^2}{\overline{p}}}\,(g(d))^{q-\frac{q^2}{\overline{p}^*}}}$, we have that, if

$$J_0 \le K[g(d)]^{\frac{\overline{p}(\overline{p}^*-q)}{\overline{p}^*(1-1/s)-\overline{p}}}, \qquad \text{with } K := \left\{\frac{C}{R^{\mu\frac{q^2}{\overline{p}}}}\right\}^{-\frac{1}{\alpha}} \lambda^{-\frac{1}{\alpha^2}}, \tag{5.12}$$

then $\lim_{h\to+\infty} J_h = 0$.
Since

$$J_0 := \int_{A_{\frac{d}{2},R}} g^q(u-\frac{d}{2})\,dx \le \int_{B_R} g^q(|u|)\,dx,$$

it is easy to check that (5.12) is satisfied, if we choose d such that

$$g(d) = g(2) + \left\{\frac{1}{K}\int_{B_R} g^q(|u|)\,dx\right\}^{\frac{\overline{p}^*(1-1/s)-\overline{p}}{\overline{p}(\overline{p}^*-q)}}. \tag{5.13}$$

Indeed we get $d \ge 2$ and

$$g(d) \ge \left\{\frac{1}{K}J_0\right\}^{\frac{\overline{p}^*(1-1/s)-\overline{p}}{\overline{p}(\overline{p}^*-q)}},$$

so (5.12) follows. Therefore, we have $\lim\limits_{h\to+\infty} J_h = \displaystyle\int_{A_{d,\frac{R}{2}}} g^q(u-d)\,dx = 0$. This implies $|A_{d,\frac{R}{2}}| = 0$ and we conclude that $B_{\frac{R}{2}} \subseteq \{u \le d\}$.

On the other hand, since $-u$ is a quasi-minimizer of the functional

$$\mathcal{I}(v) := \int \overline{f}(x,u,Du)\,dx,$$

where $\overline{f}(x,u,\xi) := f(x,-u,-\xi)$, which satisfies the same assumptions of f, we obtain that $B_{\frac{R}{2}} \subseteq \{u \ge -d\}$. Therefore, by (5.13) and the monotonicity of g,

$$g(|u|) \le g(2) + \left\{\left(\frac{C}{R^{\mu\frac{q^2}{\overline{p}}}}\right)^{\frac{1}{\alpha}} \lambda^{\frac{1}{\alpha^2}} \int_{B_R} g^q(|u|)\,dx\right\}^{\frac{\overline{p}^*(1-1/s)-\overline{p}}{\overline{p}(\overline{p}^*-q)}} \qquad \text{a.e. in} \quad B_{\frac{R}{2}},$$

that is

$$\|g(|u|)\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \le g(2) + \frac{c}{R^{\mu\frac{q\overline{p}^*}{\overline{p}(\overline{p}^*-q)}}}\left(\int_{B_R} g^q(|u|)\,dx\right)^{\frac{\overline{p}^*(1-1/s)-\overline{p}}{\overline{p}(\overline{p}^*-q)}}.$$

By a covering argument, we can obtain estimate (2.7).

□

We now turn to the proof of our boundedness result, under the assumption $q = \overline{p}^*$.

Proof of Theorem 2.3. If $\max\{p_i\} = \overline{p}^*$, we know, by assumption, that $g(|u|) \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$. The same conclusion holds if $\max\{p_i\} < \overline{p}^*$. Indeed, (2.2) implies $g(|u_{x_i}|) \in L^{p_i}_{\mathrm{loc}}(\Omega)$; so, by Proposition 3.3, $g(|u|) \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$. Consider $B_{R_0}(x_0) \Subset \Omega$ such that $\int_{B_{R_0}(x_0)} g^{\overline{p}^*}(|u|)\, dx \leq 1$. With J_h defined as at the beginning of this section and using Lemma 5.1, with $q = \overline{p}^*$, we get

$$J_{h+1} \leq C \left(\frac{1}{R}\right)^{\mu \frac{(\overline{p}^*)^2}{\overline{p}}} \lambda^h J_h^{1+\alpha}$$

where $\lambda = 4^{\mu \frac{(\overline{p}^*)^2}{\overline{p}}}$ and $\alpha = \frac{\overline{p}^*}{\overline{p}} \left(1 - \frac{1}{s}\right) - 1$. Therefore, by Lemma 5.2 we have that $\lim_{h \to +\infty} J_h = 0$, if

$$J_0 \leq \left(C \left(\frac{1}{R}\right)^{\mu \frac{(\overline{p}^*)^2}{p}} \right)^{-\frac{1}{\alpha}} \left(4^{\mu \frac{(\overline{p}^*)^2}{p}} \right)^{-\frac{1}{\alpha^2}}. \tag{5.14}$$

By definition, $J_0 = \int_{A_{\frac{d}{2},R}} g^{\overline{p}^*}(u - \frac{d}{2})\, dx$. Since $g^{\overline{p}^*}(|u|) \in L^1(B_R)$ we can choose d large, such that (5.14) holds; in fact

$$J_0 = \int_{A_{\frac{d}{2},R}} g^{\overline{p}^*} \left(u - \frac{d}{2}\right) dx \leq \int_{A_{\frac{d}{2},R}} g^{\overline{p}^*}(|u|)\, dx \to_{d \to +\infty} 0.$$

With this choice of d, we get

$$\lim_{h \to \infty} J_h = \int_{A_{d,\frac{R}{2}}} g^{\overline{p}^*}(u - d)\, dx = 0.$$

Therefore, $u \leq d$ a.e. in $B_{\frac{R}{2}}(x_0)$. To get a bound from below, we proceed as in the previous Theorem 2.2. $\square$

We conclude the section with the proof of Theorems 2.6 and 2.7.

Proof of Theorem 2.6. If $q < \overline{p}^*$, then we get the thesis by Theorem 2.2. Assume $q = \overline{p}^*$. By $\mathcal{F}(u_0) < +\infty$ and (2.8), we get $u_0 \in W^{1,(p_1,\ldots,p_n)}(\Omega)$. Theorem 3.1 implies $u - u_0 \in L^{\overline{p}^*}(\Omega)$. Thus, $u \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$. The conclusion follows by Theorem 2.3. $\square$

Proof of Theorem 2.7. We proceed similarly to Theorem 2.5 in [12]. However, for the sake of completeness we give a sketch of the proof.

Assume (2.8) with $\min\{p_i\} = 1$ and define $\overline{\mathcal{F}}$ as in (2.9). By Rellich's Theorem in BV, every minimizing sequence for $\mathcal{F}$ in $u_0 + W_0^{1,(p_1,\ldots,p_n)}(\Omega)$ has a L^1-convergent subsequence. The lower semicontinuity of $\overline{\mathcal{F}}$ gives the existence of a minimizer $\bar{u}$ in BV, such that

$$\overline{\mathcal{F}}(\bar{u}) = \min_{u \in BV} \overline{\mathcal{F}}(u) = \inf_{u \in u_0 + W_0^{1,(p_1,\ldots,p_n)}(\Omega)} \mathcal{F}(u). \tag{5.15}$$

By the minimality of $\bar{u}$ and (5.15), there exists a sequence (u_k) in $u_0 + W_0^{1,(p_1,\ldots,p_n)}(\Omega)$, such that, for all k,

$$\mathcal{F}(u_k) \leq \inf_{u_0 + W_0^{1,(p_1,\ldots,p_n)}(\Omega)} \mathcal{F} + \frac{1}{k}, \qquad \text{and} \qquad u_k \to_{k \to +\infty} \bar{u} \text{ in } L^1(\Omega). \tag{5.16}$$

By the Ekeland's variational principle, see [15], for every k there exists a function $v_k \in u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega)$, such that

$$\mathcal{F}(v_k) \le \mathcal{F}(u) + \frac{1}{\sqrt{k}} \sum_{i=1}^{n} \left(\int_\Omega |(v_k - u)_{x_i}|^{p_i}\, dx \right)^{\frac{1}{p_i}} \quad \forall u \in u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega), \tag{5.17}$$

and

$$\sum_{i=1}^{n} \left(\int_\Omega |(v_k - u_k)_{x_i}|^{p_i}\, dx \right)^{\frac{1}{p_i}} \le \frac{1}{\sqrt{k}} \quad \forall k. \tag{5.18}$$

Since $u_k - v_k \in W_0^{1,(p_1,\dots,p_n)}(\Omega)$, then (5.16) and (5.18) imply that $v_k \to \bar{u}$ in L^1.
Note that $a^{1/p_i} \le a + 1$ for every $a > 0$ and every $i = 1, \dots, n$. Thus, using (5.17) and (2.8), we get that, for all $u \in u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega)$,

$$\begin{aligned}\mathcal{F}(v_k) &\le \mathcal{F}(u) + \frac{1}{\sqrt{k}} \left\{ \sum_{i=1}^{n} \left(\int_\Omega |(v_k)_{x_i}|^{p_i}\, dx \right)^{1/p_i} + \sum_{i=1}^{n} \left(\int_\Omega |u_{x_i}|^{p_i}\, dx \right)^{1/p_i} \right\} \\ &\le \left(1 + \frac{1}{\sqrt{k}}\right) \mathcal{F}(u) + \frac{1}{\sqrt{k}} \mathcal{F}(v_k) + \frac{2}{\sqrt{k}},\end{aligned}$$

that implies

$$\left(1 - \frac{1}{\sqrt{k}}\right) \mathcal{F}(v_k) \le \left(1 + \frac{1}{\sqrt{k}}\right) \mathcal{F}(u) + \frac{2}{\sqrt{k}}.$$

Therefore, we have that v_k is a quasi-minimizer of the functional

$$\mathcal{I}(u) := \int_\Omega (f(x, u, Du) + 1)\, dx,$$

with Q independent of k. Since $(x, s, \xi) \mapsto f(x, s, \xi) + 1$ satisfies properties analogous to f, we can apply Theorem 2.2 and then $v_k \in L^\infty_{\mathrm{loc}}(\Omega)$. Fixed $x_0 \in \Omega$, consider $Q_R(x_0) \Subset \Omega$, cube centered at x_0, with edges, of length $2R$, parallel to the coordinate axes, by the estimate (2.7) on the cubes, there exist $\gamma > 0$ and $c > 0$, independent of k, but depending on R, such that

$$\|v_k\|_{L^\infty(Q_{\frac{R}{2}}(x_0))} \le c(R) \left\{ 1 + \left(\int_{Q_R(x_0)} |v_k|^q\, dx \right)^{\gamma} \right\}. \tag{5.19}$$

Since $\mathcal{F}(u_0) < \infty$, then $u_0 \in W^{1,(p_1,\dots,p_n)}(\Omega)$. By the embedding theorem for anisotropic Sobolev spaces on rectangular sets (see for example Lemma 2.1 in [1]), we have that $u_0 \in L^{\overline{p}^*}(Q_R(x_0))$ and

$$\|u_0\|_{L^{\overline{p}^*}(Q_R)} \le c \left\{ \|u_0\|_{L^1(Q_R)} + \sum_{i=1}^{n} \|(u_0)_{x_i}\|_{L^{p_i}(Q_R)} \right\} \tag{5.20}$$

for some $c > 0$. On the other hand, by applying the inequality (3.1) of Theorem 3.1 to the function $v_k - u_0 \in W_0^{1,(p_1,\dots,p_n)}(\Omega)$ and by taking into account (5.20), we get

$$\left\{ \int_{Q_R(x_0)} |v_k|^q\, dx \right\}^{\frac{1}{q}} \le c \sum_{i=1}^{n} \left(\int_\Omega |(v_k - u_0)_{x_i}|^{p_i}\, dx \right)^{\frac{1}{p_i}} + c\|u_0\|_{W^{1,(p_1,\dots,p_n)}(\Omega)}.$$

Using the growth assumption (2.8), we have that

$$\sum_{i=1}^{n}\left(\int_{\Omega}|(v_k-u_0)_{x_i}|^{p_i}\,dx\right)^{\frac{1}{p_i}} \le c\,\{\mathcal{F}(v_k)+1\}+\sum_{i=1}^{n}\left\{\int_{\Omega}|(u_0)_{x_i}|^{p_i}\,dx\right\}^{\frac{1}{p_i}}$$
$$\le c\left\{\mathcal{F}(v_k)+1+\|u_0\|_{W^{1,(p_1,\dots,p_n)}(\Omega)}\right\}. \tag{5.21}$$

Collecting (5.19)-(5.21), we obtain

$$\|v_k\|_{L^\infty(Q_{\frac{R}{2}}(x_0))} \le c\left\{1+\left(\mathcal{F}(v_k)+\|u_0\|_{W^{1,(p_1,\dots,p_n)}(\Omega)}\right)^{q\gamma}\right\},$$

for some positive c independent of k and u_0. By (5.17), (5.18) and (5.16), we have

$$\|v_k\|_{L^\infty(Q_{\frac{R}{2}}(x_0))} \le c\left\{1+\left(\inf_{u_0+W_0^{1,(p_1,\dots,p_n)}(\Omega)}\mathcal{F}+\frac{2}{k}+\|u_0\|_{W^{1,(p_1,\dots,p_n)}(\Omega)}\right)^{q\gamma}\right\}.$$

So, up to subsequences, v_k converges to to $\bar{u}$ in the $*$-weak topology of L^∞ and by the lower semicontinuity of the L^∞-norm, we conclude. □

6. Applications

In this section we discuss some applications of the local boundedness result Theorem 2.5. Let us consider

$$\mathcal{I}(u)=\int_{\Omega}a(x)|Du|^r\,dx, \qquad 1<r\le n, \tag{6.1}$$

$a(x)\ge 0$, $a\in L^\sigma_{\mathrm{loc}}(\Omega)$ and $a^{-1}\in L^\tau_{\mathrm{loc}}(\Omega)$, $\sigma,\tau>1$. An application of Theorem 2.5 gives the following result.

Theorem 6.1. *If σ,τ satisfy*

$$\max\{1,\frac{n}{\sigma}\cdot\frac{n+\sigma r}{n+r}\}<r-\frac{n}{\tau}, \tag{6.2}$$

then the local minimizers of $\mathcal{I}$ belongs to $W^{1,p}_{\mathrm{loc}}(\Omega)$ for some $p>1$ and they are locally bounded.

The idea of the proof is to observe that for every $1<p<r<q\le p^*$ we have, by the Young inequality, that there exist $c_1,c_2>0$ such that

$$c_1|Du|^p\le a(x)|Du|^r+a(x)^{-\frac{p}{r-p}}\le c_2\left\{|Du|^q+b(x)\right\},$$

with

$$b:=a^{\frac{q}{q-r}}+a^{-\frac{p}{r-p}}.$$

Taking into account that $\mathcal{I}$ and

$$\mathcal{J}(u):=\int_{\Omega}\left(a(x)|Du|^r+a(x)^{-\frac{p}{r-p}}\right)\,dx, \tag{6.3}$$

have the same local minimizers, if we show that $b\in L^s_{\mathrm{loc}}(\Omega)$ for some $s>\frac{n}{p}$, than we can conclude by applying Theorem 2.5 to $\mathcal{J}$.

Proof of Theorem 6.1. As remarked above, it is enough to show that it is possible to choose p, q in such a way that

$$1 < p < r < q \leq p^* \qquad \text{and} \qquad b := a^{\frac{q}{q-r}} + a^{-\frac{p}{r-p}} \in L^s_{\rm loc}(\Omega)$$

for some $s > \frac{n}{p}$.

By (6.2) there exists p such that

$$\max\{1, \frac{n}{\sigma} \cdot \frac{n+\sigma r}{n+r}\} < p < r - \frac{n}{\tau}.$$

Then, in particular, $p\sigma > n$. We note that

$$\frac{n}{\sigma} \cdot \frac{n+\sigma r}{n+r} < p \Leftrightarrow r < \frac{rp\sigma}{p\sigma - n} < p^*.$$

Thus, there exists $q \in \left(\frac{rp\sigma}{p\sigma-n}, p^*\right]$. Since

$$q > \frac{rp\sigma}{p\sigma - n} \Leftrightarrow \frac{q}{q-r} \cdot \frac{n}{p} < \sigma$$

and

$$p < r - \frac{n}{\tau} \Leftrightarrow \frac{p}{r-p} \cdot \frac{n}{p} < \tau,$$

there exists $s > \frac{n}{p}$ such that

$$\frac{q}{q-r} s < \sigma \qquad \frac{p}{r-p} s < \tau.$$

This implies that $b \in L^s_{\rm loc}(\Omega)$ for $s > \frac{n}{p}$. This allows to apply Theorem 2.5 to $\mathcal{J}$, with our choice of p and q. $\square$

We observe that the functionals (1.1) in the Introduction are particular cases of (6.1). Let

$$\mathcal{F}(u) = \int_{B_1(0)} |x|^\alpha |Du|^r \, dx,$$

with $\alpha > 0$ and $1 < r \leq n$. Assume that (1.2) holds. Then the local minimizers of $\mathcal{F}$ are locally bounded. In fact $a(x) := |x|^\alpha \in L^\infty$ and $a^{-1} \in L^\tau$ for every $\tau < \frac{n}{\alpha}$. Since σ in (6.2) can be arbitrarily chosen it is easy to check that (6.2) can be formulated as

$$\max\{1, \frac{nr}{n+r}\} < r - \alpha,$$

which is equivalent to (1.2).

Let us consider

$$\mathcal{G}(u) = \int_{B_1(0)} |x|^{-\alpha} |Du|^r \, dx,$$

with $\alpha > 0$ and $1 < r \leq n$. If $0 < \alpha < \frac{r^2}{n}$, then the local minimizers of $\mathcal{G}$ are locally bounded. In fact, $a(x) := |x|^{-\alpha} \in L^\sigma$ for every $\sigma < \frac{n}{\alpha}$ and $a^{-1} \in L^\tau$ for every $\tau > 1$. Since τ in (6.2) can be arbitrarily chosen it is easy to check that (6.2) becomes

$$\max\{1, (\alpha + r)\frac{n}{n+r}\} < r,$$

which is equivalent to $\alpha < \frac{r^2}{n}$.

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