ANISOTROPIC SURFACE MEASURES AS LIMITS OF VOLUME FRACTIONS

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Abstract. In this paper we consider the new characterization of the perimeter of a measurable set in $\mathbb{R}^n$ recently studied by Ambrosio, Bourgain, Brezis and Figalli. We modify their approach by using, instead of cubes, covering families made by translations of a given open bounded set with Lipschitz boundary. We show that the new functionals converge to an anisotropic surface measure, which is indeed a multiple of the perimeter if we allow for isotropic coverings (e.g. balls or arbitrary rotations of the given set). This result underlines that the particular geometry of the covering sets is not essential.

Contents

1. Introduction

The literature on approximation of Sobolev and BV norms, and on the characterizations of the corresponding spaces in terms of these approximations, is by now very wide, see in particular [4] for the case of Sobolev spaces, [16] and the more recent papers [6], [7]

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which deal with non-local approximations, in the sense of Γ-convergence of (a multiple of) the total variation norm, with intriguing connection to problems considered in image processing. Still in connection with non-local functionals, it is worth to mention the paper [8] which gave origin to the theory of nonlocal minimal surfaces.

Somehow in the same vein, motivated by [5], the first author, Bourgain, Brezis and Figalli recently studied in [2] and [1] a new characterization of the perimeter of a set in $\mathbb{R}^n$ by considering the following functionals originating from a BMO-type seminorm

$$\mathsf{I}_\varepsilon(f) = \varepsilon^{n-1} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} |f(x) - \fint_{Q'} f|\, dx, \tag{1.1}$$

where $\mathcal{G}_\varepsilon$ is any disjoint collection of ε-cubes Q' with arbitrary orientation and cardinality not exceeding ε^{1-n}.
In particular, they studied the case $f = \mathbf{1}_A$; that is, the characteristic function of a measurable set A, and proved that

$$\lim_{\varepsilon \to 0} \mathsf{I}_\varepsilon(\mathbf{1}_A) = \frac{1}{2} \min\{1, \mathsf{P}(A)\}. \tag{1.2}$$

This theme has been further investigated in [11], for BV functions, see also [12] for a variant of this construction leading to Sobolev norms and spaces.

In this paper we study more in detail the structure of the optimization problem in (1.1). We remove the upper bound on cardinality that seems to be very special of the case of cubes, at least if one is willing to get a precise formula as (1.2) and not only upper and lower bounds on I_ε. With this simplification, we prove that the existence of the limit and the emergence of a surface measure are general phenomena. In particular we prove that, for some dimensional constant $\xi = \xi(n)$, one has

$$\lim_{\varepsilon \to 0} H^B_\varepsilon(\mathbf{1}_A) = \xi \mathsf{P}(A), \tag{1.3}$$

where H^B_ε is defined as (1.1), without the bound on cardinality and using disjoint ε-balls. More generally, if C is a bounded open set with Lipschitz boundary and if we define

$$H^C_\varepsilon(A) := \varepsilon^{n-1} \sup_{\mathcal{H}_\varepsilon} \sum_{C' \in \mathcal{H}_\varepsilon} \fint_{C'} |\mathbf{1}_A(x) - \fint_{C'} \mathbf{1}_A|\, dx, \tag{1.4}$$

where $\mathcal{H}_\varepsilon$ is any disjoint family of translations C' of the set εC with no bounds on cardinality, we are able to prove the following result.

Theorem 1.1. *There exists $\varphi^C : \mathbb{S}^{n-1} \to (0, +\infty)$, bounded and lower semicontinuous, such that, for any set of finite perimeter A, one has*

$$\lim_{\varepsilon \to 0} H^C_\varepsilon(A) = \int_{\mathscr{F} A} \varphi^C(\nu_A(x))\, d\mathscr{H}^{n-1}(x), \tag{1.5}$$

where $\mathscr{F}A$ and ν_A are respectively the reduced boundary of A and the approximate unit normal to $\mathscr{F}A$. Moreover, if A is measurable and $\mathsf{P}(A) = \infty$, one has

$$\lim_{\varepsilon\to 0} H_\varepsilon^C(A) = +\infty. \tag{1.6}$$

The right hand side of (1.5) can be seen as an anisotropic version of the perimeter, $\mathsf{P}_\varphi(A)$. This result, while shows that the particular geometry of the covering sets is not essential, raises indeed some questions. The most important is maybe the following one:

$$\text{Is the function } \tilde{\varphi}^C(p) := \begin{cases} |p|\varphi^C\big(\frac{p}{|p|}\big) & \text{if } p \neq 0 \\ \\ 0 & \text{if } p = 0 \end{cases} \quad \text{convex?}$$

This question is natural, in view of the fact that the anisotropic perimeter $\int_{\mathscr{F}A} \varphi(\nu_A)\, d\mathscr{H}^{n-1}$ is lower semicontinuous w.r.t. the convergence in measure if and only if φ is the restriction to the unit sphere of a positively 1-homogeneous and convex function. The problem is nontrivial since we were able to prove that, if C is the unit square $(0,1)^2$ in $\mathbb{R}^2$, then $\tilde{\varphi}^C$ is not convex, as it is shown in Section 4. In particular, the convexity of C is not a sufficient condition to obtain $\tilde{\varphi}^C$ convex.

The paper is organized as follows: after a brief section containing preliminary results, in Section 3 we provide first the proof of (1.6), by a simple comparison argument based on the results of [1]. Then, we define suitable localized versions $H_\varepsilon(A,\Omega)$ and $H_\pm(A,\Omega)$ of our functionals; the latter arise by taking the lim sup and the lim inf w.r.t. the scale parameter ε. We can use symmetry and superadditivity arguments to show that $H_+ = H_-$ when both are evaluated in halfspaces A and in cubical domains Ω with faces parallel or orthogonal to the normal to the halfspace. Eventually, we use covering theorems as well as the fine properties of sets of finite perimeter to extend the result to general sets of finite perimeter and general domains.

In Section 4 we discuss examples and variants of our result. Finally, we provide in the appendix, for the reader's convenience, a proof of the existence of the "best volume fraction" in Kepler's packing problem (related to our problem when we choose as C a ball), whose value is presently known only in dimensions 2 and 3 ([19], [14]).

2. Notation and preliminary results

In this paper we assume $n \geq 2$. We denote by A^c the complement of A, by $|A|$ the Lebesgue measure of a Lebesgue measurable set $A \subset \mathbb{R}^n$, by $\mathscr{H}^{n-1}$ the Hausdorff $(n-1)$-dimensional measure and by $A\Delta B := (A\setminus B)\cup(B\setminus A)$ the symmetric difference of the sets A, B. If E is a set of finite perimeter, we denote by $\partial^* E$ its essential boundary, namely the complement of the sets of density and rarefaction points. We refer to Section 3.5 of [3] for the basic definitions and the general properties of sets of finite perimeter needed in this paper.

In order to avoid confusion, in the rest of the paper we will mostly work with the reduced boundary $\mathscr{F}E \subset \partial^* E$, using the larger essential boundary $\partial^* E$ only when strictly necessary, see for instance (3.18). The reason for this choice is that at every point of the reduced boundary the measure theoretic exterior normal is well defined, and we need it in order to prove results on densities of set functions. On the other hand, it is well-known that $\mathscr{H}^{n-1}(\partial^* E \setminus \mathscr{F}E) = 0$, hence every property which holds for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$ holds also for $\mathscr{H}^{n-1}$-a.e. $x \in \partial^* E$ and, when integrating w.r.t. $\mathscr{H}^{n-1}$, the difference is not seen.

An important result we need is the linear form of the relative relative isoperimetric inequality (see for instance Theorem 3.44 in [3]): for any open bounded set C with Lipschitz boundary there exists a constant $\gamma = \gamma(C)$ such that

$$\frac{|C \cap E||C \setminus E|}{|C|^2} \leq \gamma \mathsf{P}(E, C), \tag{2.1}$$

for any measurable set E. By scaling, it follows that if $C' = \varepsilon C$ we have

$$\frac{|C' \cap E||C' \setminus E|}{|C'|^2} \leq \varepsilon^{1-n} \gamma \mathsf{P}(E, C'),$$

for any measurable set E.

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we define a localized version of H_ε: for any measurable set A and any open set Ω we set

$$H_\varepsilon^C(A, \Omega) := \varepsilon^{n-1} \sup_{\mathcal{H}_\varepsilon} \sum_{C' \in \mathcal{H}_\varepsilon} \fint_{C'} |\mathbf{1}_A(x) - \fint_{C'} \mathbf{1}_A| \, dx, \tag{3.1}$$

where the supremum runs among all disjoint families $\mathcal{H}_\varepsilon$ made with translations of the set εC in Ω. Since

$$\fint_{C'} |\mathbf{1}_A(x) - \fint_{C'} \mathbf{1}_A| \, dx = \fint_{C'} \fint_{C'} |\mathbf{1}_A(x) - \mathbf{1}_A(y)| \, dx dy = 2 \frac{|C' \cap A||C' \setminus A|}{|C'|^2}, \tag{3.2}$$

we have the following equivalent definition

$$H_\varepsilon^C(A, \Omega) := \varepsilon^{n-1} \sup_{\mathcal{H}_\varepsilon} \sum_{C' \in \mathcal{H}_\varepsilon} 2 \frac{|C' \cap A||C' \setminus A|}{|C'|^2}, \tag{3.3}$$

which we are going to use mostly.
In addition, it is convenient to define the following set functions

$$H_+^C(A, \Omega) := \limsup_{\varepsilon \to 0} H_\varepsilon^C(A, \Omega), \tag{3.4}$$

$$H_-^C(A, \Omega) := \liminf_{\varepsilon \to 0} H_\varepsilon^C(A, \Omega). \tag{3.5}$$

Clearly, we have $H^C_-(A,\Omega) \le H^C_+(A,\Omega)$. In order to show the existence of the limit in the case of a set of finite perimeter A, we need to prove the converse inequality $H_-(A,\Omega) \ge H_+(A,\Omega)$.

The following scaling properties will be useful:

$$H^{\lambda C}_\varepsilon(A,\Omega) = \lambda^{1-n} H^C_{\varepsilon\lambda}(A,\Omega), \qquad H^{\lambda C}_\pm(A,\Omega) = \lambda^{1-n} H^C_\pm(A,\Omega). \tag{3.6}$$

In the sequel, we also often assume with no loss of generality that $\mathrm{diam}(C) = 1$. Indeed, if we set $\tilde{C} := C/\mathrm{diam}(C)$, then (3.6) with $\lambda = \mathrm{diam}(C)$, so that $C = \lambda\tilde{C}$, implies

$$H^C_\varepsilon(A,\Omega) = \mathrm{diam}(C)^{1-n} H^{\tilde{C}}_{\varepsilon\mathrm{diam}(C)}(A,\Omega), \qquad H^C_\pm(A,\Omega) = \mathrm{diam}(C)^{1-n} H^{\tilde{C}}_\pm(A,\Omega).$$

It is also not difficult to compare H^C_ε to H^D_ε when $D \subset C$ and D is an open set containing the origin[1]. Indeed, it is clear that that for any measurable set A one has

$$\frac{|D \cap A||D \setminus A|}{|D|^2} \le \frac{|C|^2}{|D|^2} \frac{|C \cap A||C \setminus A|}{|C|^2}, \tag{3.7}$$

and that the same holds for any translated and dilated copies of C and D. Now, for any disjoint family $\mathcal{H}_{\varepsilon,D}$ of translations of εD we can find a family $\mathcal{H}_{\varepsilon,C}$ of translations of εC such that for any $D_j \in \mathcal{H}_{\varepsilon,D}$ there exists $C_j \in \mathcal{H}_{\varepsilon,C}$ with $D_j \subset C_j$. Even though the family $\mathcal{H}_{\varepsilon,C}$ is not disjoint in general, it is easily seen, using the inclusions

$$B(x_j, \lambda\varepsilon) \subset D_j \subset C_j \subset B(x_j, \varepsilon) \qquad \text{for some } x_j \in \mathbb{R}^n$$

(where $\lambda > 0$ satisfies $B(0,\lambda) \subset D$), that it has bounded overlap. More precisely, there exists $\theta = \theta(n,\lambda) > 0$ such that for any fixed j we have $\#\{k : \ B(x_k,\varepsilon) \cap B(x_j,\varepsilon) \neq \emptyset\} \le \theta$ and so the same property holds if we replace the balls by the corresponding sets C_j. Therefore, the family $\mathcal{H}_{\varepsilon,C}$ can be seen as the union of at most θ disjoint subfamilies $\mathcal{H}_{\varepsilon,C,i}$. This argument yields

$$\varepsilon^{n-1} \sum_{D' \in \mathcal{H}_{\varepsilon,D}} 2\frac{|D' \cap A||D' \setminus A|}{|D'|^2} \le \frac{|C|^2}{|D|^2} \sum_{i=1}^{\theta} \varepsilon^{n-1} \sum_{C' \in \mathcal{H}_{\varepsilon,C,i}} 2\frac{|C' \cap A||C' \setminus A|}{|C'|^2} \le \frac{|C|^2}{|D|^2} \theta H^C_\varepsilon(A,\Omega),$$

and, taking the supremum over the families $\mathcal{H}_{\varepsilon,D}$, we obtain

$$H^D_\varepsilon(A,\Omega) \le \frac{|C|^2}{|D|^2} \theta H^C_\varepsilon(A,\Omega), \qquad H^D_\pm(A,\Omega) \le \frac{|C|^2}{|D|^2} \theta H^C_\pm(A,\Omega). \tag{3.8}$$

In addition, we notice that, for any rotation R we have $H^{R(C)}_\varepsilon(R(A), R(\Omega)) = H^C_\varepsilon(A,\Omega)$ and $H^{R(C)}_\pm(R(A), R(\Omega)) = H^C_\pm(A,\Omega)$.

Since in the following the set C will be mostly fixed, we drop the superscript C from H^C_ε, $H^C_\pm$.

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[1]Without loss of generality, we can always assume $0 \in D \subset C$.

3.1. **Proof of** (1.6)**.** In this section we prove (1.6), which follows easily from the results of [1] and from comparison arguments.

Let

$$\mathsf{I}_\varepsilon(\mathbf{1}_A,\Omega) := \varepsilon^{n-1}\sup_{\mathcal{F}_\varepsilon}\sum_{Q'\in\mathcal{F}_\varepsilon} 2\frac{|Q'\cap A||Q'\setminus A|}{|Q'|^2},$$

where $\mathcal{F}_\varepsilon$ denotes a collection of disjoint open cubes $Q' \subset \Omega$ with side length ε and arbitrary orientation. In [1] it was shown that, for any Borel set A, one has

$$\lim_{\varepsilon\to 0}\mathsf{I}_\varepsilon(\mathbf{1}_A,\mathbb{R}^n) = \frac{1}{2}\mathsf{P}(A). \tag{3.9}$$

For later purposes, we recall also a local version of (3.9) which is proved in [1] in order to get the global version, namely

$$\liminf_{\varepsilon\to 0}\mathsf{I}_\varepsilon(\mathbf{1}_A,\Omega) \geq \frac{1}{2}\mathsf{P}(A,\Omega) \qquad \text{for any open set } \Omega\subset\mathbb{R}^n. \tag{3.10}$$

Arguing as in the proof of (3.8), we observe that for any cube Q' with arbitrary orientation and side length $2\varepsilon/\sqrt{n}$, we can find an open ε-ball $B' \supset Q'$. Hence, for any collection $\mathcal{F}_{2\varepsilon/\sqrt{n}}$ of disjoint cubes Q' with arbitrary orientation and side length $2\varepsilon/\sqrt{n}$, we find a family $\mathcal{G}_{\varepsilon,B}$ of ε-balls with bounded overlap; that is, there exists $\theta_n > 0$ such that for any fixed $B' \in \mathcal{G}_{\varepsilon,B}$ we have $\#\{B'' \in \mathcal{G}_{\varepsilon,B} : B''\cap B' \neq \emptyset\} \leq \theta_n$.
Then, if we denote by H^B_ε the functional where we take a covering with ε-balls, we get

$$\theta_n H^B_\varepsilon(A,\Omega) \geq \frac{4^n}{n^n\omega_n^2}\mathsf{I}_{2\varepsilon/\sqrt{n}}(\mathbf{1}_A,\Omega) \tag{3.11}$$

If $\mathsf{P}(A) = +\infty$, inequalities (3.9) and (3.11) clearly give

$$\liminf_{\varepsilon\to 0} H^B_\varepsilon(A) \geq \liminf_{\varepsilon\to 0}\frac{4^n}{n^n\omega_n^2\theta_n}\mathsf{I}_{\frac{2}{\sqrt{n}}\varepsilon}(\mathbf{1}_A) = +\infty.$$

The case of a general bounded open set C containing the origin follows immediately by (3.8), since $C \supset B(0,\lambda)$ for some $\lambda = \lambda(C) > 0$.

3.2. **First properties of H_ε and $H_\pm$.** We show now some elementary properties of the functionals H_ε and $H_\pm$, omitting the proof of the simplest ones and assuming the normalization $\mathrm{diam}(C) = 1$.

(1) Translation invariance: for any $\tau\in\mathbb{R}^n$, we have $H_\varepsilon(A+\tau,\Omega+\tau) = H_\varepsilon(A,\Omega)$; taking limits, one has also $H_\pm(A+\tau,\Omega+\tau) = H_\pm(A,\Omega)$;
(2) Monotonicity: $H_\varepsilon(A,\cdot)$ and $H_\pm(A,\cdot)$ are increasing set functions on the class of open sets in $\mathbb{R}^n$;
(3) Homogeneity: for any $t>0$, $H_{t\varepsilon}(tA,t\Omega) = t^{n-1}H_\varepsilon(A,\Omega)$. Indeed, $tC'\subset t\Omega$ if and only if $C'\subset\Omega$, and

$$\frac{|tC'\cap tA||tC'\setminus tA|}{|tC'|^2} = \frac{|C'\cap A||C'\setminus A|}{|C'|^2}.$$

It follows immediately that

$$H_\pm(tA, t\Omega) = t^{n-1} H_\pm(A, \Omega). \tag{3.12}$$

(4) Superadditivity of H_-: it is easy to see that

$$H_\varepsilon(A, \Omega_1 \cup \Omega_2) = H_\varepsilon(A, \Omega_1) + H_\varepsilon(A, \Omega_2) \tag{3.13}$$

whenever $\Omega_1 \cap \Omega_2 = \emptyset$. From (3.13) we get

$$H_-(A, \Omega_1 \cup \Omega_2) \geq H_-(A, \Omega_1) + H_-(A, \Omega_2). \tag{3.14}$$

(5) Almost subadditivity of H_+:

$$H_\varepsilon(A, \Omega_1 \cup \Omega_2) \leq H_\varepsilon(A, I_\varepsilon(\Omega_1)) + H_\varepsilon(A, I_\varepsilon(\Omega_2)), \tag{3.15}$$

for any open set Ω_1, Ω_2, where $I_t(\Omega) := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) < t\}$. Indeed, if $C' \subset \Omega_1 \cup \Omega_2$, then it must be contained in the ε-neighbourhood of one of the two open sets, since $\operatorname{diam}(C') = \varepsilon \operatorname{diam}(C) = \varepsilon$. From (3.15) we get

$$H_+(A, \Omega_1 \cup \Omega_2) \leq H_+(A, W_1) + H_+(A, W_2), \tag{3.16}$$

for any open sets $W_i \supset I_\delta(\Omega_i)$, $i = 1, 2$, for some $\delta > 0$.

(6) Upper bound for H_+: using (2.1), we see that

$$H_\varepsilon(A, \Omega) \leq 2\gamma \mathsf{P}(A, \Omega)$$

and so

$$H_+(A, \Omega) \leq 2\gamma \mathsf{P}(A, \Omega). \tag{3.17}$$

3.3. **Lower and upper density of $H_\pm$.** We set

$$\varphi_+(\nu) := H_+(S_\nu, Q_\nu),$$
$$\varphi_-(\nu) := H_-(S_\nu, Q_\nu),$$

where $\nu \in \mathbb{S}^{n-1}$, $S_\nu := \{x \in \mathbb{R}^n : \ x \cdot \nu \geq 0\}$ and Q_ν a unit cube centered in the origin having one face orthogonal to ν and bisected by the hyperplane ∂S_ν.
Due to the translation invariance, this definition does not actually depend on the choice of the origin, since we could take any hyperplane $\{(x - x_0) \cdot \nu \geq 0\}$ and cubes centered in x_0.

It is obvious that $\varphi_-(\nu) \leq \varphi_+(\nu)$. We collect in the next proposition a few elementary properties of $\varphi_\pm$ (more refined estimates in some special cases will be given in Section 4) and then we prove that these two functions coincide.

Proposition 3.1. *We have the following upper and lower bounds for $\varphi_\pm$:*

(1) $\varphi_+ \leq 2\gamma$, where γ is the same constant in (3.17)*;*

(2) $\varphi_- \geq \lambda^{n+1} \dfrac{2^{2n-1}}{|C|^2 \theta n^n \theta_n}$, where $\lambda = \lambda(C) := \sup\{r > 0 : B(0, r) \subset C\}$, $\theta = \theta(n, \lambda)$ and θ_n are defined in the proofs of (3.8) *and* (1.6)*, respectively.*

In addition, $\varphi_- = \varphi_+$ and φ_- is lower semicontinuous.

Proof. The inequality $\varphi_+ \leq 2\gamma$ is easy, since by (2.1) we have

$$H_+(S_\nu, Q_\nu) \leq 2\gamma \mathsf{P}(S_\nu, Q_\nu)$$

and $\mathsf{P}(S_\nu, Q_\nu) = 1$, by the definition of S_ν and Q_ν.
As for the lower bound on $\varphi_-(\nu)$, it can be obtained as follows: first we take $r > 0$ such that $B(0,r) \subset C$, then we apply (3.8), (3.6) and eventually (3.11) to get

$$\begin{aligned} H_\varepsilon^C(S_\nu, Q_\nu) &\geq \frac{|B(0,r)|^2}{|C|^2\theta} H_\varepsilon^{B(0,r)}(S_\nu, Q_\nu) \\ &= r^{1-n}\frac{|B(0,r)|^2}{|C|^2\theta} H_{\varepsilon r}^{B}(S_\nu, Q_\nu) \\ &\geq r^{n+1}\frac{|B|^2}{|C|^2\theta}\frac{4^n}{n^n\omega_n^2\theta_n}\mathsf{I}_{2\varepsilon\lambda/\sqrt{n}}(S_\nu, Q_\nu) = r^{n+1}\frac{1}{|C|^2\theta}\frac{4^n}{n^n\theta_n}\mathsf{I}_{2\varepsilon\lambda/\sqrt{n}}(S_\nu, Q_\nu). \end{aligned}$$

Now we let $\varepsilon \to 0$, using (3.10) with $A = S_\nu$ and $\Omega = Q_\nu$, and finally we take the supremum over $r > 0$ such that $B(0,r) \subset C$.
Finally, homogeneity implies that

$$\varphi_-(\nu) = H_-(S_\nu, Q_\nu) = \liminf_{\varepsilon \to 0} \varepsilon^{n-1} H_1(S_\nu, (1/\varepsilon)Q_\nu),$$

since $(1/\varepsilon)S_\nu = S_\nu$ for any $\varepsilon > 0$.
We observe that $(1/\varepsilon)Q_\nu$ contains the union of at least $\lfloor (t/\varepsilon) \rfloor^{n-1}$ open disjoint cubes of side length $1/t$, for any $t > \varepsilon$, which are translations of $(1/t)Q_\nu$ centered in points of ∂S_ν. Clearly, $H_\varepsilon(S_\nu, x + Q_\nu) = H_\varepsilon(S_\nu, Q_\nu)$ for any $x \in \partial S_\nu$. Hence, the monotonicity in the second argument, the additivity of H_ε and the homogeneity imply

$$\varphi_-(\nu) \geq \liminf_{\varepsilon \to 0} \varepsilon^{n-1} \lfloor (t/\varepsilon) \rfloor^{n-1} H_1(S_\nu, (1/t)Q_\nu) = t^{n-1} H_{t(1/t)}((1/t)S_\nu, (1/t)Q_\nu) = H_t(S_\nu, Q_\nu),$$

which implies $\varphi_-(\nu) \geq \sup_{t>0} H_t(S_\nu, Q_\nu)$.
On the other hand, it is clear that

$$\varphi_-(\nu) \leq \varphi_+(\nu) = \limsup_{\varepsilon \to 0} H_\varepsilon(S_\nu, Q_\nu) = \lim_{\varepsilon \to 0} \sup_{0<s<\varepsilon} H_s(S_\nu, Q_\nu) \leq \sup_{s>0} H_s(S_\nu, Q_\nu)$$

from which we deduce that $\varphi_-(\nu) = \varphi_+(\nu) = \sup_{t>0} H_t(S_\nu, Q_\nu)$.
As a byproduct, we also obtain that φ_- is lower semicontinuous in ν, being the supremum with respect to the parameter t of the supremum over the families $\mathcal{H}_t$ of translations of tC of the quantities

$$t^{n-1} \sum_{C' \in \mathcal{H}_t} 2\frac{|C' \cap S_\nu||C' \setminus S_\nu|}{|C'|^2},$$

which are continuous functions of ν.

□

We define $\varphi(\nu) := \lim_{\varepsilon \to 0} H_\varepsilon(S_\nu, Q_\nu)$, since Proposition 3.1 showed the existence of the limit.
To prove that the upper and lower densities of $H_\pm$ coincide with φ, we need a modulus of

continuity for $E \to H_\varepsilon(E,\Omega)$ similar to the one shown in [1], Lemma 3.6. We recall that for any E, F sets of finite perimeter in Ω we have

$$\mathscr{H}^{n-1}(\partial^*(E\Delta F)\cap\Omega) = \mathscr{H}^{n-1}((\partial^* E\Delta\partial^* F)\cap\Omega), \tag{3.18}$$

see for instance [1], Section 2.

Lemma 3.2. *For any E, F sets of finite perimeter in Ω and any $\varepsilon > 0$ we have*

$$H_\varepsilon(F,\Omega) \le H_\varepsilon(E,\Omega) + 4\gamma\mathscr{H}^{n-1}((\mathscr{F}E\Delta\mathscr{F}F)\cap\Omega). \tag{3.19}$$

In particular one has

$$H_\pm(F,\Omega) \le H_\pm(E,\Omega) + 4\gamma\mathscr{H}^{n-1}((\mathscr{F}E\Delta\mathscr{F}F)\cap\Omega). \tag{3.20}$$

Proof. For any C' and any measurable set $L \subset C'$ we have the relative isoperimetric inequality (2.1) and, combining it with the inequality $\min\{t, 1-t\} \le 2t(1-t)$ for any $t \in [0,1]$, we obtain also

$$\min\{|L|, |C'\setminus L|\} \le 2\gamma|C|\varepsilon\mathsf{P}(L,C'). \tag{3.21}$$

Let now $\mathcal{H}_\varepsilon$ be a disjoint family of translations of εC in Ω.
For any $C' \in \mathcal{H}_\varepsilon$, we have

$$\fint_{C'}\fint_{C'} |\mathbf{1}_F(x)-\mathbf{1}_F(y)|\,dx\,dy \le \fint_{C'}\fint_{C'} |\mathbf{1}_E(x)-\mathbf{1}_E(y)|\,dx\,dy + \frac{2}{|C|}\varepsilon^{-n}|C'\cap(F\Delta E)|. \tag{3.22}$$

Indeed,

$$\begin{aligned}
&\fint_{C'}\fint_{C'} |\mathbf{1}_F(x)-\mathbf{1}_E(x)-\mathbf{1}_F(y)+\mathbf{1}_E(y)|\,dx\,dy \\
&= \frac{2}{|C'|^2}(2|C'\cap(F\setminus E)||C'\cap(E\setminus F)| + |C'\setminus(F\Delta E)||C'\cap(F\Delta E)|) \\
&= \frac{2}{|C'|^2}(2|C'\cap(F\setminus E)||C'\cap(E\setminus F)| + |C'||C'\cap(F\Delta E)| - (|C'\cap(E\setminus F)| + |C'\cap(F\setminus E)|)^2) \\
&\le \frac{2}{|C|}\varepsilon^{-n}|C'\cap(F\Delta E)|.
\end{aligned}$$

Since $\mathbf{1}_{E^c}(x) - \mathbf{1}_{E^c}(y) = \mathbf{1}_E(y) - \mathbf{1}_E(x)$, then we have also

$$\fint_{C'}\fint_{C'} |\mathbf{1}_F(x)-\mathbf{1}_F(y)|\,dx\,dy \le \fint_{C'}\fint_{C'} |\mathbf{1}_E(x)-\mathbf{1}_E(y)|\,dx\,dy + \frac{2}{|C|}\varepsilon^{-n}|C'\cap(F\Delta E^c)|. \tag{3.23}$$

It is clear that $F\Delta E = \Omega\setminus(F\Delta E^c)$, hence we can apply (3.21) to $L = C'\cap(F\Delta E)$. Therefore, by (3.2), we obtain

$$\varepsilon^{n-1}\sum_{C'\in\mathcal{H}_\varepsilon}\fint_{C'}|\mathbf{1}_F(x) - \fint_{C'}\mathbf{1}_F|\,dx \le H_\varepsilon(E,\Omega) + 4\gamma\sum_{C'\in\mathcal{H}_\varepsilon}\mathsf{P}(F\Delta E,C').$$

Since $\sum_{C'\in\mathcal{H}_\varepsilon}\mathsf{P}(F\Delta E,C') \le \mathsf{P}(F\Delta E,\Omega) = \mathscr{H}^{n-1}((\mathscr{F}E\Delta\mathscr{F}F)\cap\Omega)$ by (3.18), we can pass to the supremum at the left hand side and we get (3.19). □

Let now $x \in \partial S_\nu$. If we denote by $Q_\nu(x,r)$ the cube of side length r centered in $x \in \mathbb{R}^n$ and with one face orthogonal to ν, by homogeneity we have

$$\lim_{r\to 0} \frac{H_\pm(S_\nu, Q_\nu(x,r))}{r^{n-1}} = H_\pm(S_\nu, Q_\nu(x,1)) = \varphi(\nu). \tag{3.24}$$

Theorem 3.3. *Let E be a set of finite perimeter and ν_E be its measure theoretic interior normal. Then, for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$, we have*

$$\liminf_{r\to 0} \frac{H_-(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} \geq \varphi(\nu_E(x)). \tag{3.25}$$

Proof. By our previous remarks, the result holds if E is the half-space $\{y : (y-x)\cdot\nu_E(x) \geq 0\}$. Indeed, $\mathsf{P}(E, Q_{\nu_E(x)}(x,r)) = r^{n-1}$, so that (3.24) implies (3.25).

If E is a set of finite perimeter, for any $x \in \mathscr{F}E$ there exists the measure theoretic interior normal $\nu_E(x)$ and the approximate tangent space to the measure $|D\chi_E|$ is $\nu_E^\perp(x)$, namely

$$\mathrm{Tan}^{n-1}(|D\chi_E|, x) = \mathscr{H}^{n-1} \llcorner \nu_E^\perp(x).$$

This implies that

$$\lim_{r\to 0} \frac{\mathscr{H}^{n-1}(\mathscr{F}E \cap Q_{\nu_E(x)}(x,r))}{r^{n-1}} = \mathscr{H}^{n-1}((\nu_E^\perp(x)) \cap Q_{\nu_E(x)}(x,1)) = 1.$$

Therefore, since $\mathsf{P}(E,\cdot) = \mathscr{H}^{n-1} \llcorner \mathscr{F}E$, we deduce that for all $x \in \mathscr{F}E$ one has

$$\mathsf{P}(E, Q_{\nu_E(x)}(x,r)) = r^{n-1} + o(r^{n-1}). \tag{3.26}$$

If F is the subgraph of a C^1 function in a neighbourhood of x, then $(F-x)/\rho$ is bi-Lipschitz equivalent to the half-space $S_{\nu_F(x)}$ in $Q_{\nu_F(x)}(0,1)$, with bi-Lipschitz constants converging to 1 as $\rho \to 0$. Hence, we can use a C^1 deformation map Φ with bi-Lipschitz constant close to 1 near to x to transform any disjoint family C_i' admissible for F into a disjoint family $D_i = \Phi(C_i)$; we can then find $C_i'' \subset D_i \subset C_i'''$ translated and scaled copies of C_i' whose diameters satisfy $\mathrm{diam}(C_i')/\mathrm{diam}(C_i) \sim 1$, $\mathrm{diam}(C_i''')/\mathrm{diam}(C_i) \sim 1$. Summing up, for $r > 0$ small enough there exists a nonnegative modulus of continuity $\omega(r)$ satisfying

$$(1-\omega(r))|C'' \cap S_{\nu_F(x)}| \leq |C' \cap F| \leq (1+\omega(r))|C''' \cap S_{\nu_F(x)}|$$

for $0 < \rho < r$ and any translated copy C' of ρC contained in $Q_{\nu_F(x)}(x,r)$. We can choose the modulus of continuity in such a way that similar inequalities hold with the roles of F and $S_{\nu_F(x)}$ reversed. Hence, we have

$$\frac{|C' \cap F||C' \setminus F|}{|C'|^2} \leq (1+\omega(r))^2 \frac{|C''' \cap S_{\nu_F(x)}||C''' \setminus S_{\nu_F(x)}|}{|C'''|^2}, \tag{3.27}$$

and

$$\frac{|C' \cap F||C' \setminus F|}{|C'|^2} \geq (1-\omega(r))^2 \frac{|C'' \cap S_{\nu_F(x)}||C'' \setminus S_{\nu_F(x)}|}{|C''|^2}. \tag{3.28}$$

In particular, (3.28) and (3.24) imply

$$H_-(F, Q_{\nu_E(x)}(x,r)) \geq \varphi(\nu_F(x))r^{n-1} + o(r^{n-1}). \tag{3.29}$$

Now, in order to obtain (3.29) also for E, we are going to use the rectifiability of $\mathscr{F}E$ and apply Lemma 3.2 to E and to the subgraph of one of the C^1 hypersurfaces Γ_i whose union covers $\mathscr{H}^{n-1}$-almost all of $\mathscr{F}E$ and such that $\nu_E|_{\Gamma_i}$ is the interior normal of the subgraph of Γ_i. Indeed, we fix i and observe that for $\mathscr{H}^{n-1}$-a.e. $x \in \Gamma_i \cap \mathscr{F}E$ one has

$$\mathscr{H}^{n-1}((\Gamma_i \Delta \mathscr{F}E) \cap B(x,r)) = o(r^{n-1}),$$

arguing as in the proof of Lemma 3.7 of [1] and using the density properties of the Hausdorff measure (see [3], Theorem 2.56 and Eq. (2.41)). It follows easily that we have also

$$\mathscr{H}^{n-1}((\Gamma_i \Delta \mathscr{F}E) \cap Q_{\nu_E(x)}(x,r)) = o(r^{n-1})$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \Gamma_i \cap \mathscr{F}E$. Now we use (3.20) choosing $\Omega = Q_{\nu_E(x)}(x,r)$ and F to be the subgraph of Γ_i inside $Q_{\nu_E(x)}(x,r)$, obtaining

$$H_-(F, Q_{\nu_E(x)}(x,r)) \leq H_-(E, Q_{\nu_E(x)}(x,r)) + 4\gamma \mathscr{H}^{n-1}((\Gamma_i \Delta \mathscr{F}E) \cap Q_{\nu_E(x)}(x,r)).$$

Since Γ_i is a C^1 hypersurface, we have (3.29) for F, with $\nu_F(x) = \nu_E(x)$. Since i is arbitrary this implies (3.29) for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$.

Combining (3.26) and (3.29), we get the desired resut. □

Theorem 3.4. *For any Borel set $B \subset \mathscr{F}E$ and $t > 0$, we have that*

$$\liminf_{r \to 0} \frac{H_-(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} \geq t \tag{3.30}$$

for all $x \in B$ implies $H_-(E,U) \geq t\mathscr{H}^{n-1}(B)$ for any open set $U \supset B$.

Proof. Without loss of generality, let $U \supset B$ be a bounded open set, since $\mathscr{H}^{n-1} \llcorner B$ is inner regular. For a given $\delta \in (0,1)$, we consider the family $\mathcal{F}$ of all the closed cubes inside U centered in the points $x \in B$ with one face oriented as $\nu_E(x)$, such that, if we denote their interior by $Q_{\nu_E(x)}(x,r)$, we have $H_-(E, Q_{\nu_E(x)}(x,r)) \geq t(1-\delta)\mathsf{P}(E, Q_{\nu_E(x)}(x,r))$ and $|D\chi_E|(\partial Q_{\nu_E(x)}(x,r)) = 0$.

In this way, we can apply the version of Vitali theorem for cubes (see Theorem 5.13 of [15]) and find a disjoint countable subfamily $\{\overline{Q_j}\}$ which covers $\mathscr{H}^{n-1}$-almost all of B. It is also clear that $\mathsf{P}(E, \overline{Q_j}) = \mathsf{P}(E, Q_j)$, hence we can use an open covering. Therefore, the superadditivity of $H_-(E, \cdot)$ implies

$$\begin{aligned} t\mathscr{H}^{n-1}(B) &\leq t\mathsf{P}(E, \bigcup_j Q_j) = t\sum_j \mathsf{P}(E, Q_j) \leq (1-\delta)^{-1} \sum_j H_-(E, Q_j) \\ &\leq (1-\delta)^{-1} H_-(E, \bigcup_j Q_j) \leq (1-\delta)^{-1} H_-(E, U). \end{aligned}$$

Letting $\delta \to 0$, we prove the theorem. □

We can now extend the result of Theorem 3.3 to $H_+(E,\cdot)$ using similar techniques.

Theorem 3.5. *Let E be a set of finite perimeter and ν_E be its measure theoretic interior normal. Then, for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$, we have*

$$\limsup_{r\to 0} \frac{H_+(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} \le \varphi(\nu_E(x)). \tag{3.31}$$

Proof. In the beginning of the proof of Theorem 3.3 we showed that $\mathsf{P}(E, Q_{\nu_E(x)}(x,r)) = r^{n-1} + o(r^{n-1})$ for all $x \in \mathscr{F}E$.
By (3.24), (3.31) holds if E is a half space S_ν. Then we need to use estimate (3.27) in order to prove the inequality in the case that E is a subgraph of a C^1 function in a neighbourhood of x.
Finally, we switch the roles of F and E in (3.20) and we repeat the steps of the last part of the proof of Theorem 3.3 to obtain

$$H_+(E, Q_{\nu_E(x)}(x,r)) \le \varphi(\nu_E(x)) r^{n-1} + o(r^{n-1})$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$.
Combining these results, we obtain (3.31). □

In order to prove the upper estimate for H_+, we need to consider the inner regularization of the nondecreasing set functions $H_+(E,\cdot)$ defined on the open sets of $\mathbb{R}^n$.

Definition 3.6. *Let $\mathcal{A}$ be the family of open sets in $\mathbb{R}^n$ and let $\alpha : \mathcal{A} \to [0,+\infty]$ be a nondecreasing set function. The inner regular envelope of α is the function $\alpha^* : \mathcal{A} \to \overline{\mathbb{R}}$ defined by*

$$\alpha^*(A) := \sup\{\alpha(A') : \ A' \Subset A\}.$$

It is not hard to show (see for instance [10]) that α^* is the largest inner regular function smaller than α (namely $\alpha^*(A) = \sup\{\alpha^*(A') : \ A' \Subset A\}$). Recall also that any inner regular and subadditive function α is σ-subadditive, namely

$$\alpha(A) \le \sum_{i=0}^{\infty} \alpha(A_i) \qquad \text{whenever } A \subset \bigcup_{i=0}^{\infty} A_i.$$

The proof of this statement can be adapted also to the case when α is weakly subadditive as our set function H_+, this leads to the following result.

Proposition 3.7. *$H_+^*(E,\cdot)$ is σ-subadditive and*

$$H_+^*(E,\Omega) \le 2\gamma \mathsf{P}(E,\Omega). \tag{3.32}$$

Proof. Given open sets Ω_i, $i = 1, 2$, let $0 < t < H_+^*(E, \Omega_1 \cup \Omega_2)$. Then there exists $W \Subset \Omega_1 \cup \Omega_2$ such that $H_+(E,W) \ge t$. By Lemma 14.20 of [10], there exist open sets Ω_i', $i = 1, 2$, such that $W \Subset \Omega_1' \cup \Omega_2'$ and $\Omega_i' \Subset \Omega_i$, $i = 1, 2$. Hence, we can find open sets W_i such that $\Omega_i' \Subset W_i \Subset \Omega_i$, $i = 1, 2$, and, by (3.16), we obtain

$$t \le H_+(E,W) \le H_+(E,W_1) + H_+(E,W_2) \le H_+^*(E,\Omega_1) + H_+^*(E,\Omega_2).$$

Since $t < H_+^*(E, \Omega_1 \cup \Omega_2)$ is arbitrary, this proves the subadditivity.

Since $H_+^*(E, \cdot)$ is inner regular and subadditive the σ-subadditivity follows. The last statement follows by (3.17) and the inner regularity of $\mathscr{H}^{n-1} \llcorner \mathscr{F}E$. □

We are now able to show the same result of Theorem 3.4 for H_+.

Theorem 3.8. *For any Borel set $B \subset \mathscr{F}E$ and $t > 0$, we have that*

$$\limsup_{r\to 0} \frac{H_+(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} \le t \tag{3.33}$$

for all $x \in B$ implies $H_+^*(E,U) \le t\mathsf{P}(E,U) + 2\gamma \mathsf{P}(E, U \setminus B)$ for any open set $U \supset B$.

Proof. Since H_+^* is inner regular, we may assume $U \supset B$ to be a bounded open set without loss of generality. We fix $\delta \in (0,1)$ and we consider the family $\mathcal{F}$ of all the closed cubes inside U centered in the points $x \in B$ with one face oriented as $\nu_E(x)$, such that, if we denote their interior by $Q_{\nu_E(x)}(x,r)$, we have

$$H_+^*(E, Q_{\nu_E(x)}(x,r)) \le H_+(E, Q_{\nu_E(x)}(x,r)) \le (1+\delta)t\mathsf{P}(E, Q_{\nu_E(x)}(x,r)),$$

and $|D\chi_E|(\partial Q_{\nu_E(x)}(x,r)) = 0$.

As in the proof of Theorem 3.4, we can apply the version of Vitali theorem for cubes (see Theorem 5.13 of [15]) and find a disjoint countable subfamily $\{\overline{Q_j}\}$ which covers $\mathscr{H}^{n-1}$-almost all B. It is also clear that, since $\mathsf{P}(E, \overline{Q_j}) = \mathsf{P}(E, Q_j)$, then we have

$$\mathscr{H}^{n-1}(B \setminus \bigcup_j Q_j) = 0. \tag{3.34}$$

Therefore the subadditivity of $H_+^*(E, \cdot)$ and (3.32) imply

$$\begin{aligned} H_+^*(E,U) &\le H_+^*(E, \bigcup_{j=1}^N Q_j) + H_+^*(E, U \setminus \bigcup_{j=1}^N (1-\delta)\overline{Q_j}) \\ &\le (1+\delta)t \sum_{j=1}^N \mathsf{P}(E, Q_j) + 2\gamma \mathsf{P}(E, U \setminus \bigcup_{j=1}^N (1-\delta)\overline{Q_j}) \\ &\le (1+\delta)t\mathsf{P}(E,U) + 2\gamma\mathsf{P}(E, U \setminus \bigcup_{j=1}^N (1-\delta)\overline{Q_j}). \end{aligned}$$

Letting first $\delta \to 0$ and then $N \to +\infty$, we obtain

$$\begin{aligned} H_+^*(E,U) &\le t\mathsf{P}(E,U) + 2\gamma\mathsf{P}(E, U \setminus \bigcup_j Q_j) \\ &= t\mathsf{P}(E,U) + 2\gamma\mathsf{P}(E, U \setminus (B \cup \bigcup_j Q_j)) + 2\gamma\mathsf{P}(E, B \setminus \bigcup_j Q_j) \\ &\le t\mathsf{P}(E,U) + 2\gamma\mathsf{P}(E, U\setminus B), \end{aligned}$$

because of (3.34). □

Remark 3.9. *We notice that, by combining Theorems 3.3 and 3.5, for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$ we obtain*

$$\varphi(\nu_E(x)) \le \liminf_{r\to 0} \frac{H_-(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} \le \limsup_{r\to 0} \frac{H_+(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} \le \varphi(\nu_E(x)),$$

which yields the following equalities:

$$\liminf_{r\to 0} \frac{H_-(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} = \limsup_{r\to 0} \frac{H_+(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} = \varphi(\nu_E(x)). \tag{3.35}$$

3.4. **Final estimates.** Now we use the results of the previous section to adapt the classical results concerning differentiation of Radon measures to the nondecreasing set functions $H_\pm(E,\cdot)$.

Theorem 3.10. *For any set of finite perimeter E in $\mathbb{R}^n$ one has*

$$H_+(E) = H_-(E) = \int_{\mathscr{F}E} \varphi(\nu_E(x))\, d\mathscr{H}^{n-1}(x). \tag{3.36}$$

Proof. We consider first the case of H_-. Then, fixed $t > 1$, we define the Borel sets

$$D_k := \{x \in \mathscr{F}E : \varphi(\nu_E(x)) \in (t^k, t^{k+1}]\}$$

for $k \in \mathbb{Z}$.

For any $\varepsilon_k > 0$ we can find compact sets $K_k \subset D_k$ such that

$$\mathscr{H}^{n-1}(D_k \setminus K_k) < \varepsilon_k. \tag{3.37}$$

Since this family of compact sets is disjoint, it is then clear that

$$\min_{-J\le k\ne k'\le J} \operatorname{dist}(K_k, K_{k'}) > 0. \qquad \forall J \in \mathbb{N}.$$

Hence, for any J, we can find a disjoint family of open sets $U_k \supset K_k$, for $-J \le k \le J$. By the superadditivity of H_-, Theorem 3.4 and (3.35), we get

$$\begin{aligned} H_-(E) &\ge H_-(E, \bigcup_{-J\le k\le J} U_k) \ge \sum_{-J\le k\le J} H_-(E, U_k) \\ &\ge \sum_{-J\le k\le J} t^k \mathscr{H}^{n-1}(K_k) \\ &\ge \sum_{-J\le k\le J} t^{-1} \int_{K_k} \varphi(\nu_E)\, d\mathscr{H}^{n-1} \\ &= t^{-1} \int_{\bigcup_{-J\le k\le J} K_k} \varphi(\nu_E)\, d\mathscr{H}^{n-1} \end{aligned} \tag{3.38}$$

for any $J \in \mathbb{N}$. Since the measure $\mathscr{H}^{n-1} \llcorner \mathscr{F}E$ is regular and ε_k are arbitrary, we can pass to the supremum to get

$$H_-(E) \ge t^{-1} \int_{\bigcup_{-J\le k\le J} D_k} \varphi(\nu_E)\, d\mathscr{H}^{n-1}$$

Finally, we pass to the supremum over J and then send $t \to 1$ to get

$$H_-(E) \geq \int_{\mathscr{F}E} \varphi(\nu_E)\, d\mathscr{H}^{n-1}. \tag{3.39}$$

Now we deal with H_+. Fixed $t > 1$, we define the Borel sets D_k as above. For $\varepsilon > 0$, we can therefore find open sets $U_k \supset D_k$ with

$$\sum_k t^{k+1} \mathsf{P}(E, U_k \setminus D_k) < \varepsilon, \quad \sum_k 2\gamma \mathsf{P}(E, U_k \setminus D_k) < \varepsilon.$$

Since $\bigcup_k U_k$ covers $\mathscr{H}^{n-1}$-almost all of $\mathscr{F}E$ we can cover $\mathbb{R}^n \setminus \bigcup_k U_k$ with an open set U_0 with $\mathsf{P}(E, U_0)$ arbitrarily small and use the σ-subadditivity of H_+^* and (3.32) to get

$$H_+^*(E, \mathbb{R}^n) \leq \sum_k H_+^*(E, U_k).$$

Now, using Theorem 3.8, we estimate

$$\begin{aligned} H_+^*(E, \mathbb{R}^n) &\leq \sum_k H_+^*(E, U_k) \\ &\leq \sum_k t^{k+1} \mathsf{P}(E, U_k) + 2\gamma \mathsf{P}(E, U_k \setminus D_k) \\ &\leq \sum_k t^{k+1} \mathsf{P}(E, D_k) + 2\varepsilon \\ &\leq \sum_k t \int_{(\mathscr{F}E)\cap D_k} \varphi(\nu_E)\, d\mathscr{H}^{n-1} + 2\varepsilon \\ &\leq t \int_{\mathscr{F}E} \varphi(\nu_E)\, d\mathscr{H}^{n-1} + 2\varepsilon. \end{aligned} \tag{3.40}$$

Now we let $\varepsilon \downarrow 0$ and $t \downarrow 1$ to get

$$H_+^*(E, \mathbb{R}^n) \leq \int_{\mathscr{F}E} \varphi(\nu_E)\, d\mathscr{H}^{n-1}.$$

We show now that $H_+^*(E, \mathbb{R}^n) = H_+(E, \mathbb{R}^n)$. Indeed, we need only to show $H_+(E, \mathbb{R}^n) \leq H_+^*(E, \mathbb{R}^n)$. Fix $W \Subset \mathbb{R}^n$ open and let Ω such that $W \Subset \Omega$; by (3.16) we have

$$H_+(E, \mathbb{R}^n) \leq H_+(E, \Omega) + H_+(E, \mathbb{R}^n \setminus \overline{W}),$$

since we can take $\tilde{\Omega}$ and $\tilde{W}$ such that $W \Subset \tilde{W} \Subset \tilde{\Omega} \Subset \Omega$ and write $\mathbb{R}^n = \tilde{\Omega} \cup (\mathbb{R}^n \setminus \overline{\tilde{W}})$. By (3.17), we have

$$H_+(E, \mathbb{R}^n) \leq H_+^*(E, \mathbb{R}^n) + 2\gamma \mathsf{P}(E, \mathbb{R}^n \setminus \overline{W}),$$

which implies $H_+(E, \mathbb{R}^n) \leq H_+^*(E, \mathbb{R}^n)$, since W is arbitrary. In this way we obtain the inequality

$$H_+(E) \leq \int_{\mathscr{F}E} \varphi(\nu_E)\, d\mathscr{H}^{n-1}. \tag{3.41}$$

Combining (3.39) and (3.41), we prove the theorem. □

Remark 3.11 (A local version of Theorem 3.10)**.** By similar arguments one can prove that $\mathsf{P}(E,\mathbb{R}^n)<\infty$ implies that the family

$$\mathcal{R}:=\left\{A\subset\mathbb{R}^n:\ A \text{ open},\ H_\pm(E,A)=\int_{A\cap\mathscr{F}E}\varphi(\nu_E)\,d\mathscr{H}^{n-1}\right\}$$

is rich, namely the set $\{i\in[0,1]:\ A_i\notin\mathcal{R}\}$ is at most countable whenever the family $\{A_i\}_{i\in[0,1]}$ satisfies $A_i\Subset A_j$ for $i<j$.
Indeed, since the density arguments are local, one need just to start with $H_-(E,A)$ in (3.38) and with $H_+^*(E,A)$ in (3.40) and to estimate in a finer way. Then, we recall that $H_+^*(E,\cdot)=H_+(E,\cdot)$ on a rich family of open sets. More specifically, one can use (3.32) and an argument similar to the last part of the proof of Theorem 3.10 to prove that any open set $A\subset\mathbb{R}^n$ such that $|D\chi_E|(\partial A)=0$ belongs to this family.

4. Examples and variants

In this section we discuss a few examples and estimates of the function φ. We also introduce a variant of the functionals H_ε in which we allow for dilations ηC, for any $\eta\in(0,\varepsilon]$ (i.e. the size of the sets in the family need not be the same).

4.1. **Covering with balls.** If we choose the set C to be the unit ball $B(0,1)$, it is easy to see that the function φ is a constant ξ_n depending only on the space dimension. Indeed, in this case the functionals H_ε and $H_\pm$ are rotationally invariant.
We are also able to estimate ξ_n, see (4.4) below. A result due to Cianchi ([9]) shows that we have the following sharp form of the relative linear isoperimetric inequality in the unit ball B:

$$\frac{|E\cap B||B\setminus E|}{|B|^2}\le\frac{1}{4\omega_{n-1}}\mathsf{P}(E,B)\qquad\text{for any measurable set } E.$$

This inequality clearly gives us the upper bound

$$\xi_n=H_+^B(S_\nu,Q_\nu)\le\frac{1}{2\omega_{n-1}}\mathsf{P}(S_\nu,Q_\nu)=\frac{1}{2\omega_{n-1}}.\tag{4.1}$$

On the other hand, the derivation of a lower bound is related to the well-known Kepler's problem (see for instance [13], [17]). This problem, also called "packing problem", consists in looking for the best way to place finite unions of disjoint open balls with the same (small) radius inside a unit cube in $\mathbb{R}^n$ in order to cover as much volume as possible. As the radius tends to 0, this problem identifies the best fraction $\rho_n\in(0,1]$ of volume covered. Kepler's problem is highly non trivial, since only in 1998 Hales ([14]) was able to prove that in three dimensions the best packing is the face centered cubic lattice (which is the one used to pack oranges and cannon balls), and that $\rho_3=\frac{\pi}{3\sqrt{2}}$, as Kepler conjectured. In two dimensions the best packing is the exagonal lattice and therefore $\rho_2=\frac{\pi}{2\sqrt{3}}$, as it was proved by Thue in 1890 (see [18] and [19]), while in dimensions higher than 3 the problem is still essentially open. Nevertheless, it is not difficult to prove the existence

of the constant ρ_n by standard subadditivity arguments; for the reader's convenience, we include a proof of this fact in the appendix.
Our aim is to give a lower estimate of the number of disjoint ε-balls which can stay inside Q_ν and are bisected by ∂S_ν. Thus, it is clear that this problem is related to the one of looking for the optimal fraction $\rho_n \in (0,1]$ of the volume of the n-dimensional unit cube covered by finite unions of disjoint balls with the same radii. We claim that we have

$$\xi_n \geq \frac{\rho_{n-1}}{2\omega_{n-1}}. \tag{4.2}$$

Indeed, we can cover $\partial S_\nu \cap Q_\nu$ with a number N_ε of $(n-1)$-dimensional ε-balls satisfying

$$N_\varepsilon \sim \rho_{n-1}\frac{1}{\omega_{n-1}\varepsilon^{n-1}}. \tag{4.3}$$

Such $(n-1)$-dimensional ε-balls can be seen as the sections $\partial S_\nu \cap B'$ for some disjoint n-dimensional ε-balls B' which are bisected by the hyperplane ∂S_ν and lie inside the cube Q_ν. Therefore, we get

$$\xi_n = H_-^B(S_\nu, Q_\nu) \geq \liminf_{\varepsilon\to 0} \varepsilon^{n-1}\frac{1}{2}N_\varepsilon = \frac{\rho_{n-1}}{2\omega_{n-1}}.$$

Combining (4.1) and (4.2), we obtain

$$\frac{\rho_{n-1}}{2\omega_{n-1}} \leq \xi_n \leq \frac{1}{2\omega_{n-1}}. \tag{4.4}$$

In particular, it is easy to see that $\rho_1 = 1$, since the ball centered in the origin of radius r coincides with the cube, being the interval $(-r,r)$. Therefore, we conclude that $\xi_2 = 1/(2\omega_1) = 1/4$.

We notice that we can use the above arguments to estimate φ also in the case when C is the spherical shell $B(0,1)\setminus\overline{B(0,r)}$, for some $r\in(0,1)$.
Indeed, it is clear that φ is a constant $\xi_{n,r}$ depending only on the interior radius and the space dimension, due to the rotational invariance. If we choose the arrangement of disjoint copies of εC which are bisected by ∂S_ν and cover the maximum fraction of surface area, then their number will be the same N_ε as in (4.3): in fact, $C\cap(\partial S_\nu)$ occupies the same surface area as $B(0,1)\cap(\partial S_\nu)$. Hence, we have

$$\xi_{n,r} \geq \frac{\rho_{n-1}}{2\omega_{n-1}}.$$

On the other hand, it is clear $C\subset B(0,1)$ and that any disjoint family of translations of εC generates a disjoint family of full ε-balls. Hence, the inequalities (3.7) and (4.1) imply

$$\xi_{n,r} \leq \frac{|B(0,1)|^2}{|B(0,1)\setminus\overline{B(0,r)}|^2}\frac{1}{2\omega_{n-1}} = \frac{1}{(1-r^n)^2 2\omega_{n-1}}.$$

4.2. **Isotropic coverings.** If we redefine H_ε in an isotropic way; that is, allowing for any orientation of the sets C' in the covering, we clearly get the rotational invariance for the modified functionals $H_\varepsilon^{\rm iso}$ and so the associated function $\varphi^{\rm iso}$ is a constant $\xi(C)$. This was done in [1] with C equal to the unit cube Q and it is not difficult to show that $\xi(Q) = 1/2$, as Ambrosio, Brezis, Bourgain and Figalli proved. Indeed, by the relative isoperimetric inequality in the unit cube

$$|E|(1-|E|) \leq \frac{1}{4}\mathsf{P}(E,Q) \tag{4.5}$$

for any measurable set $E \subset Q$ (see [1], (2.2)), we have that

$$\frac{|Q' \cap S_\nu||Q' \setminus S_\nu|}{|Q'|^2} \leq \frac{1}{4}\varepsilon^{1-n}\mathsf{P}(S_\nu, Q')$$

for any ε-cube Q'. This gives $H_+^{{\rm iso},Q}(S_\nu, Q_\nu) \leq \frac{1}{2}\mathsf{P}(S_\nu, Q_\nu) = \frac{1}{2}$.
On the other hand, we can take the ε-cubes with one face oriented as ν, bisected by ∂S_ν and whose intersection with it gives the canonical partition of $\partial S_\nu \cap Q_\nu$ in order to obtain $H_\varepsilon^{{\rm iso},Q}(S_\nu, Q_\nu) \geq \frac{1}{2}\varepsilon^{n-1}\lfloor \varepsilon^{1-n} \rfloor$. This gives the result, coherently with [1].
Actually, using (3.10) and (4.5) we have immediately

$$\lim_{\varepsilon \to 0} H_\varepsilon^{{\rm iso},Q}(A, \Omega) = \frac{1}{2}\mathsf{P}(A, \Omega) \tag{4.6}$$

for any measurable set A and any open sets Ω.
It is also possible to show that we obtain a similar result if C is the pluri-rectangle $R = \prod_{j=1}^n (-a_j/2, a_j/2)$, for $a_j > 0$.
Indeed, we can take the copies of εR having one face oriented as ν, bisected by ∂S_ν and whose intersection with it gives a partition of $\partial S_\nu \cap Q_\nu$ with the largest cardinality; that is, at least $\lfloor \varepsilon^{1-n}/m \rfloor$, where

$$m := \min_{i=1,\dots,n} \prod_{j \neq i} a_j.$$

Thus, we obtain the lower bound $H_\varepsilon^{{\rm iso},R}(S_\nu, Q_\nu) \geq \frac{1}{2}\varepsilon^{n-1}\lfloor \varepsilon^{1-n}/m \rfloor$ and so $\xi(R) \geq \frac{1}{2m}$.
As for the upper bound, by (4.17) in the following subsection, we have $\xi(R) = 1/(2m) = 1/(2\min\{a_1, a_2\})$ if $n = 2$.

We notice that in these isotropic cases the result of Theorem 1.1 for sets of finite perimeter follows directly from Theorems 3.4 and 3.8 with $B = \mathscr{F}E$. Indeed, these theorems still hold true since $H_\varepsilon^{\rm iso}$ has the same properties of H_ε.
Then, if we take $t = \xi(C)$, Theorem 3.4 implies $H_-^{\rm iso}(E, U) \geq \xi(C)\mathscr{H}^{n-1}(\mathscr{F}E)$ for any open set $U \supset \mathscr{F}E$: it follows immediately that $H_-^{\rm iso}(E) \geq \xi(C)\mathsf{P}(E)$.
On the other hand, if we take an open set U containing $\mathscr{F}E$ and an open set $W \Subset U$, the subadditivity of $H_+^{{\rm iso},*}$ gives

$$H_+^{{\rm iso},*}(E) \leq H_+^{{\rm iso},*}(E, U) + H_+^{{\rm iso},*}(E, \mathbb{R}^n \setminus \overline{W}) \leq H_+^{{\rm iso},*}(E, U) + 2\gamma\mathsf{P}(E, \mathbb{R}^n \setminus \overline{W})$$

and clearly $\mathsf{P}(E, \mathbb{R}^n \setminus \overline{W}) \downarrow 0$ as $\overline{W} \uparrow U$.
Now, Theorem 3.8 yields $H_+^{{\rm iso},*}(E) \leq \xi(C)\mathsf{P}(E, U) = \xi(C)\mathsf{P}(E)$. It suffices now to repeat

the same argument at the end of the proof of Theorem 3.10 in order to obtain $H^{\mathrm{iso}}_+(E) \le \xi(C)\mathsf{P}(E)$, which concludes the proof.

4.3. **Examples of anisotropic coverings.** We present now three examples of estimates of φ in some special cases in $\mathbb{R}^2$.

Let us consider at first C be the unit cube $Q = (0,1)^n$ in $\mathbb{R}^n$.
In order to evaluate $\varphi^Q(\nu)$, we want to maximize in $x \in \mathbb{R}^n$ for any fixed unit vector ν the function

$$f(x,\nu) := \begin{cases} \frac{|(x+Q)\cap S_\nu||(x+Q)\setminus S_\nu|}{\mathsf{P}(S_\nu,(x+Q))} & \text{if } \mathsf{P}(S_\nu,(x+Q))>0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

We define then

$$g(\nu) := \sup_{x\in\mathbb{R}^n} f(x,\nu). \tag{4.8}$$

We observe that g is well defined and that the supremum is a maximum.
Indeed, for any fixed $\nu \in \mathbb{S}^{n-1}$, $f(x,\nu)$ is continuous in x. Clearly, $f(x,\nu) = 0$ if $x \notin \{y : -\sqrt{n} \le y\cdot\nu \le \sqrt{n}\}$, and, if $v\cdot\nu = 0$, then $f(x+v,\nu) = f(x,\nu)$. Thus, by simmetry, we can restrict ourselves to a compact set K_ν containing the origin inside the stripe $\{y : -\sqrt{n} \le y\cdot\nu \le \sqrt{n}\}$ such that $f(K_\nu,\nu) = f(\mathbb{R}^n,\nu)$. Next, we notice that if we have a sequence $y_k \to x$ with $\mathsf{P}(S_\nu,(y_k+Q)) > 0$ for any k and $\mathsf{P}(S_\nu,(y_k+Q)) \to 0$ as $y_k \to x$, then $\min\{|(y_k+Q)\cap S_\nu|, |(y_k+Q)\setminus S_\nu|\} = o(\mathsf{P}(S_\nu,(y_k+Q)))$, because one of the two parts of (y_k+Q) reduces to a simplex, for k large enough, and so its volume is proportional to the product of the basis area, $\mathsf{P}(S_\nu,(y_k+Q))$, and the relative height, which is going to zero.
Hence, $\sup_{x\in\mathbb{R}^n} f(x,\nu) = \max_{x\in K_\nu} f(x,\nu)$.

By the definition of φ^Q, it follows that $\varphi^Q(\nu) \le 2g(\nu)$, since

$$\begin{aligned}\varphi^Q(\nu) &= \lim_{\varepsilon\to0}\varepsilon^{n-1}\sup_{\mathcal{H}_\varepsilon}\sum_{Q'\in\mathcal{H}_\varepsilon} 2\frac{|Q'\cap S_\nu||Q'\setminus S_\nu|}{|Q'|^2} \\ &\le \lim_{\varepsilon\to0}\sup_{\mathcal{H}_\varepsilon} 2g(\nu)\mathsf{P}\left(S_\nu, \bigcup_{Q'\in\mathcal{H}_\varepsilon} Q'\right) \le 2g(\nu)\mathsf{P}(S_\nu,Q_\nu) = 2g(\nu).\end{aligned}$$

On the other hand, by symmetry, there exists $\tau \ge 0$ such that $g(\nu) = f(\pm\tau\nu + tv,\nu)$, for any $v \in \mathbb{S}^{n-1}$ orthogonal to ν and $t > 0$. Then, for any ε, we can choose the disjoint family $\mathcal{G}_\varepsilon$ of translations of εQ inside Q_ν which corresponds to a subset of $\{\pm\tau\nu + tv : v \in \mathbb{S}^{n-1}, v\cdot\nu = 0, t > 0\}$ and which covers ∂S_ν up to a set of $\mathscr{H}^{n-1}$-measure going to zero as $\varepsilon \to 0$. The existence of such a family of translation for any fixed $\varepsilon > 0$ follows easily from the fact that one can cover $\mathbb{R}^n$ with a tessellation of open disjoint cubes, up to a Lebesgue negligible set. For such a sequence of families we obtain

$$\varphi^Q(\nu) \ge \lim_{\varepsilon\to0} 2g(\nu)\mathsf{P}\left(S_\nu, \bigcup_{Q'\in\mathcal{G}_\varepsilon} Q'\right) = 2g(\nu).$$

Thus, we conclude that $\varphi^Q(\nu) = 2g(\nu)$.

We consider now the case $n = 2$. By the symmetries of the problem, we can redefine the function f as

$$f(q,m) := \frac{|Q \cap S_m^q||Q \setminus S_m^q|}{\mathsf{P}(S_m^q, Q)}, \tag{4.9}$$

where $Q = (0,1) \times (0,1)$, $S_m^q := \{(x,y) \in \mathbb{R}^2 : y \geq mx + q\}$, $q \in [-m, 1]$, $m = -(\nu_1/\nu_2) \in [0, +\infty)$. It is enough now to distinguish between the cases $0 \leq m \leq 1$ and $m \geq 1$.

If $0 \leq m \leq 1$, then we need only to consider $q \in [0,1]$. The line $\{y = mx + q\}$ intersects the edges of Q in the points $A = (0,q)$ and

$$B = \begin{cases} (1, m+q) & \text{if } 0 \leq q \leq 1-m, \\ \left(\frac{1-q}{m}, 1\right) & \text{if } 1-m \leq q \leq 1. \end{cases}$$

Hence, we have

$$f(q,m) = \frac{1}{\sqrt{1+m^2}} \begin{cases} \left(q + \frac{m}{2}\right) - \left(q + \frac{m}{2}\right)^2 & \text{if } 0 \leq q \leq 1-m, \\ \frac{1-q}{2} - \frac{(1-q)^3}{4m} & \text{if } 1-m \leq q \leq 1. \end{cases} \tag{4.10}$$

It is easy to see that, for any fixed m, the partial derivative in q is

$$\frac{\partial f}{\partial q}(q,m) = \frac{1}{\sqrt{1+m^2}} \begin{cases} 1 - 2q - m & \text{if } 0 < q < 1-m, \\ -\frac{1}{2} + \frac{3}{4m}(1-q)^2 & \text{if } 1-m < q < 1. \end{cases}$$

Hence, $\frac{\partial f}{\partial q} \geq 0$ if and only if

$$\begin{cases} q \leq \frac{1-m}{2} & \text{if } 0 < q < 1-m, \\ q \leq 1 - \sqrt{\frac{2m}{3}} & \text{if } 1-m < q < 1, \end{cases}$$

which means that

$$\max_{q \in [0, 1-m]} f(q,m) = \frac{1}{4\sqrt{1+m^2}}$$

and

$$\max_{q \in [1-m, 1]} f(q,m) = \frac{1}{\sqrt{1+m^2}} \begin{cases} \frac{1}{3}\sqrt{\frac{2m}{3}} & \text{if } \frac{2}{3} < m \leq 1, \\ \frac{m}{2}\left(1 - \frac{m}{2}\right) & \text{if } 0 \leq m \leq \frac{2}{3}. \end{cases}$$

Since

$$\frac{m}{2}\left(1 - \frac{m}{2}\right) \leq \frac{1}{4}$$

for any $m \in [0,1]$, and

$$\frac{1}{3}\sqrt{\frac{2m}{3}} \leq \frac{1}{4}$$

only for $0 \le m \le (27/32)$, it follows that

$$\max_{q\in[0,1]} f(q,m) = \frac{1}{\sqrt{1+m^2}} \begin{cases} \frac{1}{3}\sqrt{\frac{2m}{3}} & \text{if } \frac{27}{32} \le m \le 1, \\ \frac{1}{4} & \text{if } 0 \le m \le \frac{27}{32}. \end{cases} \tag{4.11}$$

If $m > 1$, we need only to consider $q \in [1-m, 1]$ and the intersections are

$$A = \begin{cases} (0,q) & \text{if } 0 \le q \le 1, \\ \left(-\frac{q}{m}, 0\right) & \text{if } 1-m \le q \le 0, \end{cases}$$

and $B = \left(\dfrac{1-q}{m}, 1\right)$. Hence, we have

$$f(q,m) = \frac{1}{\sqrt{1+m^2}} \begin{cases} \frac{1-q}{2} - \frac{(1-q)^3}{4m} & \text{if } 0 \le q \le 1, \\ \left(\frac{1}{2} - q\right) - \frac{1}{m}\left(\frac{1}{2} - q\right)^2 & \text{if } 1-m \le q \le 0. \end{cases} \tag{4.12}$$

We have that, for any fixed m, the partial derivative in q is

$$\frac{\partial f}{\partial q}(q,m) = \frac{1}{\sqrt{1+m^2}} \begin{cases} -\frac{1}{2} + \frac{3}{4m}(1-q)^2 & \text{if } 0 < q < 1, \\ -1 + \frac{2}{m}\left(\frac{1}{2} - q\right) & \text{if } 1-m < q < 0. \end{cases}$$

Hence, $\frac{\partial f}{\partial q} \ge 0$ if and only if

$$\begin{cases} q \le 1 - \sqrt{\frac{2m}{3}} & \text{if } 0 < q < 1, \\ q \le \frac{1-m}{2} & \text{if } 1-m < q < 0, \end{cases}$$

which means that

$$\max_{q\in[1-m,0]} f(q,m) = \frac{m}{4\sqrt{1+m^2}}$$

and

$$\max_{q\in[0,1]} f(q,m) = \frac{1}{\sqrt{1+m^2}} \begin{cases} \frac{1}{3}\sqrt{\frac{2m}{3}} & \text{if } 1 < m < \frac{3}{2}, \\ \frac{2m-1}{4m} & \text{if } m \ge \frac{3}{2}. \end{cases}$$

Since

$$\frac{2m-1}{m} \le m$$

for any $m > 1$, and

$$\frac{1}{3}\sqrt{\frac{2m}{3}} \le \frac{m}{4}$$

only for $1 < m \leq (32/27)$, it follows that

$$\max_{q\in[1-m,1]} f(q,m) = \frac{1}{\sqrt{1+m^2}} \begin{cases} \frac{1}{3}\sqrt{\frac{2m}{3}} & \text{if } 1 < m \leq \frac{32}{27}, \\ \frac{m}{4} & \text{if } m \geq \frac{32}{27}. \end{cases} \tag{4.13}$$

Because of the symmetry of the cube, we can conclude that

$$g(\nu) = \begin{cases} \frac{|\nu_2|}{4} & \text{if } |\nu_1| \leq \frac{27}{32}|\nu_2|, \\ \frac{1}{3}\sqrt{\frac{2}{3}|\nu_1||\nu_2|} & \text{if } \frac{27}{32}|\nu_2| \leq |\nu_1| \leq \frac{32}{27}|\nu_2|, \\ \frac{|\nu_1|}{4} & \text{if } |\nu_1| \geq \frac{32}{27}|\nu_2|, \end{cases}$$

which means

$$g(\nu) = \begin{cases} \frac{1}{3}\sqrt{\frac{2}{3}|\nu_1||\nu_2|} & \text{if } \frac{27}{32}|\nu_2| \leq |\nu_1| \leq \frac{32}{27}|\nu_2|, \\ \frac{\|\nu\|_\infty}{4} & \text{if } |\nu_1| \leq \frac{27}{32}|\nu_2| \text{ or } |\nu_1| \geq \frac{32}{27}|\nu_2|, \end{cases} \tag{4.14}$$

and so

$$\varphi^Q(\nu) = \begin{cases} \frac{2}{3}\sqrt{\frac{2}{3}|\nu_1||\nu_2|} & \text{if } \frac{27}{32}|\nu_2| \leq |\nu_1| \leq \frac{32}{27}|\nu_2|, \\ \frac{\|\nu\|_\infty}{2} & \text{if } |\nu_1| \leq \frac{27}{32}|\nu_2| \text{ or } |\nu_1| \geq \frac{32}{27}|\nu_2|. \end{cases}$$

It is clear that its 1-homogeneous extension $\Phi^Q(x,y) := \sqrt{x^2+y^2}\varphi^Q\left(\frac{(x,y)}{\sqrt{x^2+y^2}}\right)$ is indeed $\varphi^Q(x)$ and that it is not convex in the region $\{\frac{27}{32}|y| \leq |x| \leq \frac{32}{27}|y|\}$.

We also notice that $\max_{\nu\in\mathbb{S}^1}\varphi^Q(\nu) = (1/2)$, coherently with the results of [1] in the isotropic case.

Let us now consider anisotropic coverings with rectangles. Let $R = \prod_{i=1}^n (0,a_i)$, $a_i > 0$, then we can argue as before to show that

$$\varphi^R(\nu) = 2\sup_{x\in\mathbb{R}^n} \frac{|(x+R)\cap S_\nu||(x+R)\setminus S_\nu|}{|R|^2\mathsf{P}(R_\nu,(x+R))}.$$

If $n = 2$, we can work with $R_\lambda = (0,1)\times(0,\lambda)$, $\lambda > 0$, since for a generic rectangle $R = (0,a)\times(0,b)$, we have $R = aR_\lambda$ if $\lambda = (b/a)$, and so $\varphi^R(\nu) = (1/a)\varphi^{R_\lambda}(\nu)$.
In order to deal with the explicit calculation, we can proceed in a similar way as before, by considering the function

$$f_\lambda(q,m) := \frac{|R_\lambda\cap S^q_m||R_\lambda\setminus S^q_m|}{|R_\lambda|^2\mathsf{P}(S^q_m,R_\lambda)}, \tag{4.15}$$

where $S^q_m := \{(x,y)\in\mathbb{R}^2 : y \geq mx+q\}$, $q\in[-m,\lambda]$, $m = -(\nu_1/\nu_2)\in[0,+\infty)$, and dividing in the two cases $0 \leq m \leq \lambda$ and $m \geq \lambda$. Then, it is not difficult to show that we have

$$g^{R_\lambda}(\nu) = \begin{cases} \frac{|\nu_2|}{4} & \text{if } |\nu_1| \le \frac{27}{32}\lambda|\nu_2|, \\ \frac{1}{3}\sqrt{\frac{2}{3\lambda}|\nu_1||\nu_2|} & \text{if } \frac{27}{32}\lambda|\nu_2| \le |\nu_1| \le \frac{32}{27}\lambda|\nu_2|, \\ \frac{|\nu_1|}{4\lambda} & \text{if } |\nu_1| \ge \frac{32}{27}\lambda|\nu_2|. \end{cases} \tag{4.16}$$

Since $\varphi^{R_\lambda}(\nu) = 2g^{R_\lambda}(\nu)$, then neither the 1-homogeneous extension of this function is convex.

In conclusion, for the rectangle $R = (0,a) \times (0,b)$ we have

$$\varphi^{R}(\nu) = \begin{cases} \frac{|\nu_2|}{2a} & \text{if } a|\nu_1| \le \frac{27}{32}b|\nu_2|, \\ \frac{2}{3}\sqrt{\frac{2}{3ab}|\nu_1||\nu_2|} & \text{if } \frac{27}{32}b|\nu_2| \le a|\nu_1| \le \frac{32}{27}b|\nu_2|, \\ \frac{|\nu_1|}{2b} & \text{if } a|\nu_1| \ge \frac{32}{27}b|\nu_2|. \end{cases}$$

It is also easy to see that

$$\max_{\nu\in\mathbb{S}^1} \varphi^R(\nu) = \frac{1}{2\min\{a,b\}}, \tag{4.17}$$

which gives the value of the constant $\xi(R)$ in the case of isotropic coverings with rectangles, when arbitrary rotations are allowed.

As a last example, let now C be the ellipse $E = \{(x,y) : (x/a)^2 + (y/b)^2 < 1\}$, for some a, $b > 0$.

In order to estimate φ from below, we choose the arrangement of copies of εE such that each one is bisected by the line ∂S_ν and the contiguous copies are tangent in the intersection between their boundaries and ∂S_ν. Hence, we need to evaluate the length of the segment intersected by a copy of εE on the line $\partial S_\nu = \{(x,y)\cdot\nu = 0\}$.

If $m = -\nu_1/\nu_2$, then $\partial S_\nu = \{y = mx\}$ and the intesections with ∂E are $\pm\dfrac{ab}{\sqrt{b^2+m^2a^2}}(1,m)$.

Therefore[2], the length of the segment is $2\varepsilon ab\sqrt{(1+m^2)/(b^2+m^2a^2)}$. Since the copies of εE need to cover the unitary segment $(\partial S_\nu)\cap Q_\nu$, we obtain

$$\varphi^E(\nu) \ge \lim_{\varepsilon\to 0} \frac{1}{2}\varepsilon \left\lfloor \frac{\sqrt{b^2+m^2a^2}}{\varepsilon 2ab\sqrt{1+m^2}} \right\rfloor = \frac{1}{4ab}\sqrt{b^2\nu_2^2 + a^2\nu_1^2}.$$

In particular, if $a = b = 1$, we get $\varphi^B(\nu) \ge (1/4)$, coherently with (4.4).

4.4. **A variant.** One may define a family of functionals similar to H_ε allowing for different dilations of the set C under a fixed level $\varepsilon > 0$. More specifically, we set

$$\tilde H_\varepsilon(A,\Omega) := \sup_{\mathcal{G}_\varepsilon} \sum_{C'\in\mathcal{G}_\varepsilon} 2(\varepsilon(C'))^{n-1}\frac{|C'\cap A||C'\setminus A|}{|C'|^2}, \tag{4.18}$$

[2]If $\nu_2 = 0$, the length is $2\varepsilon b$.

where $C' = \varepsilon(C')(C + a)$, for some translation vector a, and $\mathcal{G}_\varepsilon$ is a disjoint family inside Ω of translations of the set ηC, for any $\eta \in (0, \varepsilon]$.

It is clear that $H_\varepsilon(A, \Omega) \le \tilde{H}_\varepsilon(A, \Omega)$, and so (1.6) follows for $\tilde{H}_\varepsilon$ in the case $\mathsf{P}(A) = +\infty$. We can define the functionals $\tilde{H}_\pm$ as liminf and limsup of $\tilde{H}_\varepsilon$. It is also not difficult to see that $\tilde{H}_\varepsilon$ and $\tilde{H}_\pm$ satisfy the same elementary properties of H_ε shown in Section 3.2. For instance, the homogeneity $\tilde{H}_{t\varepsilon}(tA, t\Omega) = t^{n-1}\tilde{H}_\varepsilon(A, \Omega)$ follows from the fact that each set $C'' \in \mathcal{G}_{t\varepsilon}$ can be seen as $C'' = tC'$, with $C' = \varepsilon(C')C$, for $\varepsilon(C') \le \varepsilon$, and so $C' \in \mathcal{G}_\varepsilon$.

Since these functionals satisfy the same properties of H_ε and $H_\pm$, we can define the functions $\tilde{\varphi}_\pm(\nu) := \tilde{H}_\pm(S_\nu, Q_\nu)$ and show an analogous version of Proposition 3.1 for them. Then, one may follow the same steps in order to prove Theorem 1.1 for $\tilde{H}_\varepsilon$ in the rectifiable case. Thus, we obtain that for any set of finite perimeter A

$$\lim_{\varepsilon \to 0} \tilde{H}_\varepsilon(A) = \int_{\mathscr{F}A} \tilde{\varphi}(\nu_A)\, d\mathscr{H}^{n-1}.$$

Let us now consider the case in which the set C is the unit ball. Then $\tilde{\varphi}$ is a constant, since $\tilde{H}_\varepsilon$ is rotation invariant, and $\tilde{\varphi} \equiv 1/(2\omega_{n-1})$, since arbitrarily small radii are allowed. Indeed, the upper estimate is given by (4.1).

On the other hand, we notice that we can find a lower bound by considering only the family of balls which are bisected by the hyperplane ∂S_ν. For any fixed $\varepsilon > 0$, we can apply Vitali-Besicovitch Theorem (Theorem 2.19 in [3]) to the measure $\mu = \mathscr{H}^{n-1} \llcorner Q''$, where Q'' is a unit cube in $\mathbb{R}^{n-1}$ and to a fine cover of balls with radii smaller than ε, in order to find a disjoint family $\mathcal{F}_{\varepsilon,(n-1)}$ of $(n-1)$-dimensional balls with radii smaller than ε such that

$$\mathscr{H}^{n-1}\left(Q'' \setminus \bigcup_{B'' \in \mathcal{F}_{\varepsilon,(n-1)}} B''\right) = 0. \tag{4.19}$$

Hence, we can take the family $\mathcal{F}_\varepsilon$ of n-dimensional balls bisected by $(\partial S_\nu) \cap Q_\nu$ and whose intersections with it generate the family $\mathcal{F}_{\varepsilon,(n-1)}$. Then, we use (4.19) to obtain

$$\begin{aligned}\tilde{\varphi}(\nu) &\ge \lim_{\varepsilon \to 0} \frac{1}{2} \sum_{B' \in \mathcal{F}_\varepsilon} \varepsilon^{n-1} \\ &= \lim_{\varepsilon \to 0} \frac{1}{2\omega_{n-1}} \sum_{B'' \in \mathcal{F}_{\varepsilon,(n-1)}} \omega_{n-1}\varepsilon^{n-1} \\ &= \frac{1}{2\omega_{n-1}} \mathscr{H}^{n-1}((\partial S_\nu) \cap Q_\nu) = \frac{1}{2\omega_{n-1}}.\end{aligned}$$

Finally, we observe that if we redefine $\tilde{H}_\varepsilon$ allowing for the possibility to rotate the sets C' in the covering, we obtain that $\tilde{\varphi}$ is a constant, as it happens for H_ε.

In particular, if we take C to be the unit cube Q as in [1], then, by (4.6) and (4.5), we have

$$\frac{1}{2}\mathsf{P}(A, \Omega) = \lim_{\varepsilon \to 0} H_\varepsilon^{\mathrm{iso},Q}(A, \Omega) \le \lim_{\varepsilon \to 0} \tilde{H}_\varepsilon^{\mathrm{iso},Q}(A, \Omega) \le \frac{1}{2}\mathsf{P}(A, \Omega),$$

which gives $\lim_{\varepsilon\to 0} \tilde{H}_\varepsilon^{\mathrm{iso},Q}(A,\Omega) = \frac{1}{2}\mathsf{P}(A,\Omega)$ for any measurable set A and open set Ω.

5. Appendix: Proof of the existence of ρ_n

For any $\varepsilon > 0$ and for any open set $U \subset \mathbb{R}^n$ we denote by $\Lambda_\varepsilon(U)$ the supremum of the Lebesgue measure of the finite unions of disjoint ε-balls inside U. In a similar way as we did in the beginning of Section 3, we list now some elementary properties of the set functions Λ_ε:

(1) translation invariance and monotonicity;
(2) homogeneity:
$$\Lambda_{t\varepsilon}(tU) = t^n \Lambda_\varepsilon(U) \qquad t > 0; \tag{5.1}$$
(3) superadditivity: for any open sets U, V such that $U \cap V = \emptyset$, one has
$$\Lambda_\varepsilon(U \cup V) \geq \Lambda_\varepsilon(U) + \Lambda_\varepsilon(V);$$
(4) almost subadditivity: for any open sets U, V,
$$\Lambda_\varepsilon(U \cup V) \leq \Lambda_\varepsilon(I_{2\varepsilon}(U)) + \Lambda_\varepsilon(V); \tag{5.2}$$
(5) upper and lower bounds:
$$0 \leq \Lambda_\varepsilon(U) \leq |U|. \tag{5.3}$$

For $U \subset \mathbb{R}^n$ open, we then set
$$\Lambda_-(U) := \liminf_{\varepsilon\downarrow 0} \Lambda_\varepsilon(U), \qquad \Lambda_+(U) := \limsup_{\varepsilon\downarrow 0} \Lambda_\varepsilon(U).$$
These set functions inherit from Λ_ε the translation invariance, the monotonicity, the homogeneity. In addition Λ_- is superadditive and subadditivity for Λ_+ holds in this form: for any bounded open sets U, V,
$$\Lambda_+(U \cup V) \leq \Lambda_+(W) + \Lambda_+(V), \text{whenever } U \Subset W. \tag{5.4}$$
We have also the upper and lower bounds
$$0 \leq \Lambda_-(U) \leq \Lambda_+(U) \leq |U|.$$

Theorem 5.1. *There exists a constant $\rho_n \in [\omega_n/2^n, 1]$ such that*
$$\Lambda_+(U) = \Lambda_-(U) = \rho_n|U|.$$

Proof. We set $\rho_\pm := \Lambda_\pm(Q)$, where $Q = [0,1]^n$. Because of the translation invariance, we see that $\rho_\pm = \Lambda_\pm(Q)$ for any unit cube Q in $\mathbb{R}^n$ and, using the homogeneity, we have $\Lambda_\pm(Q(x,r)) = \rho_\pm r^n$ for any $x \in \mathbb{R}^n$ and any $r > 0$. If we fix an open set U with finite measure and $\delta \in (0,1)$, there exists a finite disjoint family of open cubes $\{Q_i\}_{1\leq i\leq N}$ inside U which covers a fraction larger or equal to $(1-\delta)$ of the volume of U; that is, $|U \setminus \bigcup_{i=1}^N Q_i| \leq \delta|U|$. Then, monotonicity and superadditivity yield
$$\Lambda_-(U) \geq \Lambda_-(\bigcup_{i=1}^N Q_i) \geq \sum_{i=1}^N \Lambda_-(Q_i) = \rho_- \sum_{i=1}^N |Q_i| \geq \rho_-(1-\delta)|U|.$$

Therefore, since U and δ are arbitrary, by monotonicity and inner approximation we see that $\Lambda_-(U) \geq \rho_-|U|$.
We take now U open with finite measure, $\delta > 0$ and $\{Q_i\}_{i=1,\dots,N}$ as above. In addition, let Q_i' be cubes concentric with Q_i, satisfying $Q_i \Subset Q_i'$. Since we have $I_r(\bigcup_{i=1}^N Q_i) \subset \bigcup_{i=1}^N I_r(Q_i)$, applying (5.2), for ε small enough we obtain

$$\begin{aligned}\Lambda_\varepsilon(U) &\leq \Lambda_\varepsilon(U \setminus \bigcup_{i=1}^N (1-\varepsilon)\overline{Q_i}) + \Lambda_\varepsilon\big(I_{2\varepsilon}(\bigcup_{i=1}^N Q_i)\big) \\ &\leq \beta(\varepsilon)\delta|U| + \sum_{i=1}^N \Lambda_\varepsilon(I_{2\varepsilon}(Q_i)) \\ &\leq \beta(\varepsilon)\delta|U| + \sum_{i=1}^N \Lambda_\varepsilon(Q_i'),\end{aligned}$$

where $\beta(\varepsilon) \downarrow 1$ as $\varepsilon \downarrow 0$. Taking the limsup yields

$$\Lambda_+(U) \leq \delta|U| + \rho_+ \sum_{i=1}^N |Q_i'|$$

and if we let first Q_i' tend to Q_i and then $\delta \downarrow 0$, we deduce $\Lambda_+(U) \leq \rho_+|U|$.
Therefore, to complete the proof it is enough to show that $\rho_+ \leq \rho_-$. Let r_i and ε_i be infinitesimal sequences for which

$$\lim_{i\to\infty} \Lambda_{\varepsilon_i}(Q) = \rho_+, \qquad \lim_{i\to\infty} \Lambda_{r_i}(Q) = \rho_-,$$

where $Q = [0,1]^n$. Up to subsequences, we may assume that $\varepsilon_i = o(r_i)$ and we set $\lambda_i = \varepsilon_i/r_i \to 0$. We can cover Q with $k_i = [\lambda_i^{-n}]$ disjoint subcubes Q_j with side length λ_i, up to a set of measure less or equal to $\eta_i := 1 - k_i\lambda_i^n$. Since $I_s(tQ) = tI_{s/t}(Q)$, properties (5.2), (5.3) and homogeneity imply

$$\Lambda_{\varepsilon_i}(Q) \leq \sum_{j=1}^{k_i} \Lambda_{\varepsilon_i}(I_{2\varepsilon_i}(Q_j)) + \eta_i \leq \lambda_i^{-n}\Lambda_{r_i\lambda_i}(\lambda_i I_{2r_i}(Q)) + \eta_i = \Lambda_{r_i}(I_{2r_i}(Q)) + \eta_i.$$

We take now a cube Q' such that $Q \Subset Q'$. Then, for i large enough we have

$$\Lambda_{\varepsilon_i}(Q) \leq \Lambda_{r_i}(Q') + \eta_i$$

and passing to the limit we obtain $\rho_+ \leq \rho_-|Q'|$ for any such cube Q'. Hence, it follows that $\rho_+ \leq \rho_-$; that is, $\rho_+ = \rho_- =: \rho_n \in [0,1]$. This means that $\Lambda_\varepsilon(U) \to \rho_n|U|$ for any open set $U \subset \mathbb{R}^n$.

If $\rho_\pm$ were defined using cubes instead of balls, we would obviously get $\rho_n = 1$. Comparing cubes with balls we get $\rho_n \geq \omega_n/2^n$. □

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