Initial-boundary value problems for nearly incompressible vector fields, and applications to the Keyfitz and Kranzer system

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Abstract

We establish existence and uniqueness results for initial boundary value problems with nearly incompressible vector fields. We then apply our results to establish well-posedness of the initial-boundary value problem for the Keyfitz and Kranzer system of conservation laws in several space dimensions.

1 Introduction

The Keyfitz and Kranzer system is a system of conservation laws in several space dimensions that was introduced in [24] and takes the form

$$\partial_t U + \sum_{i=1}^{d} \partial_{x_i} (f^i(|U|)U) = 0.$$

The unknown is $U : \mathbb{R}^d \to \mathbb{R}^N$ and $|U|$ denotes its modulus. Also, for every $i = 1, \ldots, d$ the function $f^i : \mathbb{R} \to \mathbb{R}^N$ is smooth. In this work we establish existence and uniqueness results for the initial-boundary value problem associated to (1.1).

The well-posedness of the Cauchy problem associated to (1.1) was established by Ambrosio, Bouchut and De Lellis in [2, 6] by relying on a strategy suggested by Bressan in [12]. Note that the results in [2, 6] are one of the very few well-posedness results that apply to systems of conservation laws in several spaces dimensions. Indeed, establishing either existence or uniqueness for a general system of conservation laws in several space dimensions is presently a completely open problem, see [18, 27, 28] for an extended discussion on this topic.

The basic idea underpinning the argument in [2, 6] is that (1.1) can be (formally) written as the coupling between a *scalar* conservation law and a transport equation with very irregular coefficients. The scalar conservation law is solved by using the foundamental work by Kružkov [25], while the transport equation is handled by relying on Ambrosio's celebrated extension of the DiPerna-Lions' well-posendess theory, see [1] and [21], respectively, and [3, 20] for an overview. Note, however, that Ambrosio's theory [1] does not directly apply to (1.1) owing to a lack of control on the divergence of the vector fields. In order to tackle this issue, a theory of *nearly incompressible vector fields* was developed, see [19] for an extended discussion. Since we will need it in the following, we recall the definition here.

Definition 1.1. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set and $T > 0$. We say that a vector field $b \in L^\infty((0,T) \times \Omega; \mathbb{R}^d)$ is* ***nearly incompressible*** *if there are a density function $\rho \in L^\infty((0,T) \times \Omega)$ and a constant $C > 0$ such that*

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i. $0 \le \rho \le C$, $\mathcal{L}^{d+1} - a.e.$ *in* $(0,T) \times \Omega$, *and*

ii. the equation

$$\partial_t \rho + \operatorname{div}(\rho b) = 0 \tag{1.1}$$

holds in the sense of distributions in $(0,T) \times \Omega$.

The analysis in [2, 6, 19] ensures that, if $b \in L^\infty((0,T) \times \mathbb{R}^d; \mathbb{R}^d) \cap BV((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$ is a nearly incompressible vector field with density $\rho \in BV((0,T) \times \mathbb{R}^d)$, then the Cauchy problem

$$\begin{cases} \partial_t[\rho u] + \operatorname{div}[\rho b u] = 0 & \text{in } (0,T) \times \mathbb{R}^d \\ u = \overline{u} & \text{at } t = 0 \end{cases}$$

is well-posed for every initial datum $\overline{u} \in L^\infty(\mathbb{R}^d)$. This result is pivotal to the proof of the well-posedness of the Cauchy problem for the Keyfitz and Kranzer system (1.1). See also [4] for applications of nearly incompressible vector fields to the so-called chromatography system of conservation laws. Note, furthermore, that here and in the following we denote by BV the space of functions with *bounded variation*, see [7] for an extended introduction.

The present paper aims at extending the analysis in [2, 6, 19] to the case of initial-boundary value problems. First, we establish the well-posedness of initial-boundary value problems with BV, nearly incompressible vector fields, see Theorem 1.2 below for the precise statement. In doing so, we rely on well-posedness results for continuity and transport equations with weakly differentiable vector fields established in [16], see also [17] for related results. Next, we discuss the applications to the Keyfitz and Kranzer system (1.1).

We now provide a more precise description of our results concerning nearly incompressible vector fields. We fix an open, bounded set Ω and a nearly incompressible vector field b with density ρ and we consider the initial-boundary value problem

$$\begin{cases} \partial_t[\rho u] + \operatorname{div}[\rho u b] = 0 & \text{in } (0,T) \times \Omega \\ u = \overline{u} & \text{at } t = 0 \\ u = \overline{g} & \text{on } \Gamma^-, \end{cases} \tag{1.2}$$

where Γ^- is the part of the boundary $(0,T) \times \partial\Omega$ where the characteristic lines of the vector field ρb are *inward pointing*. Note that, in general, if b and ρ are only weakly differentiable, one cannot expect that the solution u is a regular function. Since Γ^- will in general be negligible, then assigning the value of u on Γ^- is in general not possible. In § 3 we provide the rigorous (distributional) formulation of the initial-boundary value problem (1.2) by relying on the theory of normal traces for low regularity vector fields, see [5, 8, 13, 14].

We can now state our well-posedness result concerning (1.2).

Theorem 1.2. *Let $T > 0$ and $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set with C^2 boundary. Also, let $b \in BV((0,T) \times \Omega; \mathbb{R}^d) \cap L^\infty((0,T) \times \Omega; \mathbb{R}^d)$ be a nearly incompressible vector field with density $\rho \in BV((0,T) \times \Omega) \cap L^\infty((0,T) \times \Omega)$, see Definition 1.1. Further, assume that $\overline{u} \in L^\infty(\Omega)$ and $\overline{g} \in L^\infty(\Gamma^-)$.*

Then there is a distributional solution $u \in L^\infty((0,T) \times \Omega)$ to (1.2) satisfying the maximum principle

$$\|u\|_{L^\infty} \le \max\{\|\overline{u}\|_{L^\infty}, \|\overline{g}\|_{L^\infty}\}. \tag{1.3}$$

Also, if u_1, $u_2 \in L^\infty((0,T) \times \Omega)$ are two different distributional solutions of the same initial-boundary value problem, then $\rho u_1 = \rho u_2$ a.e. in $(0,T) \times \Omega$.

Note that the reason why we do not exactly obtain uniqueness of the function u is because ρ can attain the value 0. If ρ is bounded away from 0, then the distributional solution u of (1.2) is unique. Also, we refer to [9, 11, 16, 17, 22, 26] for related results on the well-posedness of initial-boundary value problems for continuity and transport equation with weakly differentiable vector fields.

In § 7 we discuss the applications of Theorem 1.2 to the Keyfitz and Kranzer system and our main well-posedness result is Theorem 7.3. Note that the proof of Theorem 7.3 combines Theorem 1.2, the analysis in [19], and well-posedness results for the initial-boundary value problems for scalar conservation laws established in [10, 15, 28].

Paper outline

In § 2 we go over some preliminary results concerning normal traces of weakly differentiable vector fields. By relying on these results, in § 3 we provide the rigorous formulation of the initial-boundary value problem (1.2). In § 4 we establish the existence part of Theorem 1.2, and in § 5 the uniqueness. In § 6 we establish some stability and space continuity property results. Finally, in § 7 we discuss the applications to the Keyfitz and Kranzer system.

Notation

For the reader's convenience, we collect here the main notation used in the present paper.

- div: the divergence, computed with respect to the x variable only.
- Div: the complete divergence, i.e. the divergence computed with respect to the (t, x) variables.
- $\mathrm{Tr}(B, \partial\Lambda)$: the normal trace of the bounded, measure-divergence vector field B on the boundary of the set Λ, see § 2.
- $(\rho u)_0$, ρ_0: the initial datum of the functions ρu and ρ, see Lemma 3.1 and Remark 3.2 .
- $T(f)$: the trace of the BV function f, see Theorem 2.6.
- $\mathcal{H}^s$: the s-dimensional Hausdorff measure.
- $f_{|_E}$: the restriction of the function f to the set E.
- $\mu \llcorner E$: the restriction of the measure μ to the measurable set E.
- $a.e.$: almost everywhere.
- $|\mu|$: the total variation of the measure μ.
- $a \cdot b$: the (Euclidean) scalar product between a and b.
- $\mathbf{1}_E$: the characteristic function of the measurable set E.
- $\Gamma, \Gamma^-, \Gamma^+, \Gamma^0$: see (3.6).
- $\vec{n}$: the outward pointing, unit normal vector to Γ.

2 Preliminary results

In this section, we briefly recall some notions and results that shall be used in the sequel.

First, we discuss the notion of normal trace for weakly differentiable vector fields, see [5, 8, 13, 14]. Our presentation here closely follows that of [5]. Let $\Lambda \subseteq \mathbb{R}^N$ be an open set and let us denote by $\mathcal{M}_\infty(\Lambda)$, the family of bounded, measure-divergence vector fields. The space $\mathcal{M}_\infty(\Lambda)$, therefore, consists of bounded functions $B \in L^\infty(\Lambda; \mathbb{R}^N)$ such that the distributional divergence of B (denoted by $\mathrm{Div}B$) is a locally bounded Radon measure on Λ.

The normal trace of $B \in \mathcal{M}_\infty(\Lambda)$ on the boundary $\partial\Lambda$ can be defined as follows.

Definition 2.1. *Let $\Lambda \subseteq \mathbb{R}^N$ be an open and bounded set with Lipschitz continuous boundary and let $B \in \mathcal{M}_\infty(\Lambda)$. The normal trace of B on $\partial\Lambda$ is a distribution defined by the identity*

$$\Big\langle \mathrm{Tr}(B, \partial\Lambda), \psi \Big\rangle = \int_\Lambda \nabla\psi \cdot B \, dy + \int_\Lambda \psi \, d(\mathrm{Div}B), \qquad \forall \ \psi \in C_c^\infty(\mathbb{R}^N). \tag{2.1}$$

Here $\mathrm{Div}B$ *denotes the distributional divergence of* B *and is a bounded Radon measure on* Λ.

Note that, owing to the Gauss-Green formula, if B is a smooth vector field, then $\mathrm{Tr}(B, \partial\Lambda) = B \cdot \vec{n}$, where $\vec{n}$ denotes the outward pointing, unit normal vector to $\partial\Lambda$.

Note, furthermore, that the analysis in [5] shows that the normal trace distribution satisfies the following properties.

(a) The normal trace distribution is induced by an L^∞ function on $\partial\Lambda$, which we shall continue to refer to as $\mathrm{Tr}(B, \partial\Lambda)$. The bounded function $\mathrm{Tr}(B, \partial\Lambda)$ satisfies

$$\|\mathrm{Tr}(B, \partial\Lambda)\|_{L^\infty(\partial\Lambda)} \leq \|B\|_{L^\infty(\Lambda)}.$$

(b) Let Σ be a Borel set contained in $\partial\Lambda_1 \cap \partial\Lambda_2$, and let $\vec{n}_1 = \vec{n}_2$ on Σ (here $\vec{n}_1, \vec{n}_2$ denote the outward pointing, unit normal vectors to $\partial\Lambda_1, \partial\Lambda_2$ respectively). Then

$$\mathrm{Tr}(B, \partial\Lambda_1) = \mathrm{Tr}(B, \partial\Lambda_2) \qquad \mathcal{H}^{N-1}\text{-a.e. on } \Sigma. \tag{2.2}$$

In the following we will use several times the following renormalization result, which was established in [5].

Theorem 2.2. *Let $B \in BV(\Lambda; \mathbb{R}^N) \cap L^\infty(\Lambda; \mathbb{R}^N)$ and $w \in L^\infty(\Lambda)$ be such that $\mathrm{Div}(wB)$ is a Radon measure. If $\Lambda' \subset\subset \Lambda$ is an open set with bounded and Lipschitz continuous boundary and $h \in C^1(\mathbb{R})$, then*

$$\mathrm{Tr}(h(w)B, \partial\Lambda') = h\left(\frac{\mathrm{Tr}(wB, \partial\Lambda')}{\mathrm{Tr}(B, \partial\Lambda')}\right)\mathrm{Tr}(B, \partial\Lambda') \qquad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Lambda',$$

where the ratio $\dfrac{\mathrm{Tr}(wB, \partial\Lambda')}{\mathrm{Tr}(B, \partial\Lambda')}$ is arbitrarily defined at points where the trace $\mathrm{Tr}(B, \partial\Lambda')$ vanishes.

We can now introduce the notion of normal trace on a general bounded, Lipschitz continuous, oriented hypersurface $\Sigma \subseteq \mathbb{R}^N$ in the following manner. Since Σ is oriented, an orientation of the normal vector $\vec{n}_\Sigma$ is given. We can then find a domain $\Lambda_1 \subseteq \mathbb{R}^N$ such that $\Sigma \subseteq \partial\Lambda_1$ and the normal vectors $\vec{n}_\Sigma, \vec{n}_1$ coincide. Using (2.2), we can then define

$$\mathrm{Tr}^-(B, \Sigma) := \mathrm{Tr}(B, \partial\Lambda_1).$$

Similarly, if $\Lambda_2 \subseteq \mathbb{R}^N$ is an open set satisfying $\Sigma \subseteq \partial\Lambda_2$, and $\vec{n}_2 = -\vec{n}_\Sigma$, we can define

$$\mathrm{Tr}^+(B, \Sigma) := -\mathrm{Tr}(B, \partial\Lambda_2).$$

Furthermore we have the formula

$$(\mathrm{Div}B)\llcorner\Sigma = \Big(\mathrm{Tr}^+(B, \Sigma) - \mathrm{Tr}^-(B, \Sigma)\Big)\mathcal{H}^{N-1}\llcorner\Sigma.$$

Thus Tr^+ and Tr^- coincide $\mathcal{H}^{N-1}$−a.e. on Σ if and only if Σ is a $(\mathrm{Div}B)$-negligible set.

We next recall some results from [5] concerning space continuity.

Definition 2.3. *A family of oriented surfaces $\{\Sigma_r\}_{r\in I} \subseteq \mathbb{R}^N$ (where $I \subseteq \mathbb{R}$ is an open interval) is called a family of graphs if there exist*

- *a bounded open set $D \subseteq \mathbb{R}^{N-1}$,*
- *a Lipschitz function $f : D \to \mathbb{R}$,*
- *a system of coordinates $(x_1, \cdots, x_N)$*

such that the following holds true: For each $r \in I$, we can write

$$\Sigma_r = \Big\{(x_1, \cdots, x_N) : f(x_1, \cdots, x_{N-1}) - x_N = r\Big\}, \tag{2.3}$$

and the orientation of Σ_r is determined by the normal $\dfrac{(-\nabla f, 1)}{\sqrt{1 + |\nabla f|^2}}$.

We now quote a space continuity result.

Theorem 2.4 (see [5]). *Let $B \in \mathcal{M}_\infty(\mathbb{R}^N)$ and let $\{\Sigma_r\}_{r\in I}$ be a family of graphs as above. For a fixed $r_0 \in I$, let us define the functions $\alpha_0, \alpha_r : D \to \mathbb{R}$ as*

$$\begin{aligned} &\alpha_0(x_1, \cdots, x_{N-1}) := \mathrm{Tr}^-(B, \Sigma_{r_0})(x_1, \cdots, x_{N-1}, f(x_1, \cdots, x_{N-1}) - r_0), \text{ and} \\ &\alpha_r(x_1, \cdots, x_{N-1}) := \mathrm{Tr}^+(B, \Sigma_r)(x_1, \cdots, x_{N-1}, f(x_1, \cdots, x_{N-1}) - r). \end{aligned} \tag{2.4}$$

Then $\alpha_r \overset{*}{\rightharpoonup} \alpha_0$ weakly in $L^\infty(D, \mathcal{L}^{N-1} \llcorner D)$ as $r \to r_0^+$.*

We will also need the following result, which was originally established in [16].

Lemma 2.5. *Let $\Lambda \subseteq \mathbb{R}^N$ be an open and bounded set with bounded and Lipschitz continuous boundary and let B belong to $\mathcal{M}_\infty(\Lambda)$. Then the vector field*

$$\tilde{B}(z) := \begin{cases} B(z) & z \in \Lambda \\ 0 & \text{otherwise} \end{cases}$$

belongs to $\mathcal{M}_\infty(\mathbb{R}^N)$.

We conclude by recalling some results concerning traces of BV functions and we refer to [7, §3] for a more extended discussion.

Theorem 2.6. *Let $\Lambda \subseteq \mathbb{R}^N$ be an open and bounded set with bounded and Lipschitz continuous boundary. There exists a bounded linear mapping*

$$T : BV(\Lambda) \to L^1(\partial\Lambda; \mathcal{H}^{N-1}) \tag{2.5}$$

such that $T(f) = f_{|\partial\Lambda}$ if f is continuous up to the boundary. Also,

$$\int_\Lambda \nabla\psi \cdot f \, dy = -\int_\Lambda \psi \, d(\mathrm{Div} f) + \int_{\partial\Lambda} \psi \, Tf \cdot \vec{n} \, d\mathcal{H}^{N-1}, \tag{2.6}$$

for all $f \in BV(\Lambda)$ and $\psi \in C_c^\infty(\mathbb{R}^N)$. In the above expression, $\vec{n}$ denotes the outward pointing, unit normal vector to $\partial\Lambda$.

By comparing (2.1) and (2.6) we conclude that

$$\mathrm{Tr}(f, \partial\Lambda) = T(f) \cdot \vec{n}, \quad \text{for every } f \in BV(\Lambda). \tag{2.7}$$

By combining Theorems 3.9 and 3.88 in [7] we get the following result.

Theorem 2.7 ([7]). *Assume $\Lambda \subseteq \mathbb{R}^N$ is an open set with bounded and Lipschitz continuous boundary. If $f \in BV(\Lambda; \mathbb{R}^m)$, then there is a sequence $\{\tilde{f}_m\} \subseteq C^\infty(\Lambda)$ such that*

$$\tilde{f}_m \to f \text{ strongly in } L^1(\Lambda; \mathbb{R}^m), \qquad T(\tilde{f}_m) \to T(f) \text{ strongly in } L^1(\partial\Lambda; \mathbb{R}^m). \tag{2.8}$$

Also, we can choose $\tilde{f}_m$ in such a way that

- *$\tilde{f}_m \geq 0$ if $f \geq 0$,*
- *if $f \in L^\infty(\Lambda; \mathbb{R}^m)$, then*

$$\|\tilde{f}_m\|_{L^\infty} \leq 4\|f\|_{L^\infty}. \tag{2.9}$$

A sketch of the proof of Theorem 2.7 is provided in § 4.3.

3 Distributional formulation of the problem

In this section, we follow [11, 16] and we provide the distributional formulation of the problem (1.2). We first establish a preliminary result.

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded set with C^2 boundary and let $T > 0$. We assume that $b \in L^\infty((0,T)\times\Omega;\mathbb{R}^d)$ is a nearly incompressible vector field with density $\rho \in L^\infty((0,T)\times\Omega)$, see Definition 1.1. If $u \in L^\infty((0,T)\times\Omega)$ satisfies*

$$\int_0^T \int_\Omega \rho u(\partial_t\phi + b\cdot\nabla\phi)\, dxdt = 0, \quad \forall\, \phi \in \mathcal{C}_c^\infty((0,T)\times\Omega), \tag{3.1}$$

then there are two unique functions, which we henceforth denote by $\mathrm{Tr}(\rho u b) \in L^\infty((0,T)\times\partial\Omega)$ and $(\rho u)_0 \in L^\infty(\Omega)$, that satisfy

$$\int_0^T\int_\Omega \rho u(\partial_t\psi + b\cdot\nabla\psi)\, dxdt = \int_0^T\int_{\partial\Omega} \mathrm{Tr}(\rho u b)\psi\, d\mathcal{H}^{d-1}\, dt - \int_\Omega \psi(0,\cdot)(\rho u)_0\, dx, \quad \forall \psi \in \mathcal{C}_c^\infty([0,T)\times\mathbb{R}^d). \tag{3.2}$$

Also, we have the bounds

$$\|\mathrm{Tr}(\rho u b)\|_{L^\infty((0,T)\times\partial\Omega)},\ \|(\rho u)_0\|_{L^\infty(\Omega)} \le \max\{\|\rho u\|_{L^\infty((0,T)\times\Omega)}; \|\rho u b\|_{L^\infty((0,T)\times\Omega)}\}. \tag{3.3}$$

Proof. First of all, let us note that the uniqueness of such functions follow from the liberty in choosing the test functions ψ. Therefore it is enough to discuss the existence of the functions with the above properties. Let us define

$$B(t,x) := \begin{cases} (u\rho, u\rho b) & (t,x) \in (0,T)\times\Omega \\ 0 & \text{elsewhere in } \mathbb{R}^{d+1}. \end{cases} \tag{3.4}$$

Then $B \in L^\infty(\mathbb{R}^{d+1})$ and from (3.1), it also follows that $\left[\mathrm{Div} B \llcorner (0,T)\times\Omega\right] = 0$. We can now apply Lemma 2.5 with $\Lambda = (0,T)\times\Omega$ to conclude that $B \in \mathcal{M}_\infty(\mathbb{R}^{d+1})$. Hence B induces the existence of normal trace on $\partial\Lambda$. Let

$$\mathrm{Tr}(\rho u b) := \mathrm{Tr}(B,\partial\Lambda)\Big|_{(0,T)\times\partial\Omega}, \quad (\rho u)_0 := -\mathrm{Tr}(B,\partial\Lambda)\Big|_{\{0\}\times\Omega}.$$

The identity (3.2) then follows from (2.1) by virtue of the fact that $\mathrm{Div} B = 0$ in $(0,T)\times\Omega$. □

Remark 3.2. *We define the vector field $P := (\rho, \rho b)$ and we point out that $P \in L^\infty((0,T)\times\Omega;\mathbb{R}^{d+1})$ since ρ and b are both bounded functions. By introducing the same extension as in (3.4) and using the fact that*

$$\int_0^T \int_\Omega \rho(\partial_t\phi + b\cdot\nabla\phi)\, dxdt = 0, \quad \forall\, \phi \in \mathcal{C}_c^\infty((0,T)\times\Omega),$$

we can argue as in the proof of the above lemma to establish the existence of unique functions $\mathrm{Tr}(\rho b) \in L^\infty((0,T)\times\partial\Omega)$ and $\rho_0 \in L^\infty(\Omega)$ defined as

$$\mathrm{Tr}(\rho b) := \mathrm{Tr}(P,\partial\Lambda)\Big|_{(0,T)\times\partial\Omega}, \quad \rho_0 := -\mathrm{Tr}(P,\partial\Lambda)\Big|_{\{0\}\times\Omega}.$$

In this way, we can give a meaning to the normal trace $\mathrm{Tr}(\rho b)$ and to the initial datum ρ_0. Also, we have the bounds

$$\|\mathrm{Tr}(\rho b)\|_{L^\infty((0,T)\times\partial\Omega)},\ \|\rho_0\|_{L^\infty(\Omega)} \le \max\{\|\rho\|_{L^\infty((0,T)\times\Omega)}; \|\rho b\|_{L^\infty((0,T)\times\Omega)}\}. \tag{3.5}$$

We can now introduce the distributional formulation to the problem (1.2) by using Lemma 3.1. We introduce the following notation:

$$\begin{aligned} &\Gamma := (0,T)\times\partial\Omega, && \Gamma^- := \{(t,x)\in\Gamma:\ \mathrm{Tr}(\rho b)(t,x) < 0\}, \\ &\Gamma^+ := \{(t,x)\in\Gamma:\ \mathrm{Tr}(\rho b)(t,x) > 0\}, && \Gamma^0 := \{(t,x)\in\Gamma:\ \mathrm{Tr}(\rho b)(t,x) = 0\}. \end{aligned} \tag{3.6}$$

Definition 3.3. *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded set with C^2 boundary and let $T > 0$. Let $b \in L^\infty((0,T) \times \Omega; \mathbb{R}^d)$ be a nearly incompressible vector field with density ρ, see Definition 1.1. Fix $\overline{u} \in L^\infty(\Omega)$ and $\overline{g} \in L^\infty(\Gamma^-)$. We say that a function $u \in L^\infty((0,T) \times \Omega)$ is a distributional solution of* (1.2) *if the following conditions are satisfied:*

i. u *satisfies* (3.1)*;*

ii. $(\rho u)_0 = \overline{u}\rho_0$*;*

iii. $\mathrm{Tr}(\rho u b) = \overline{g}\mathrm{Tr}(\rho b)$ *on the set* Γ^-*.*

4 Proof of Theorem 1.2: existence of solution

In this section we establish the existence part of Theorem 1.2, namely we prove the existence of functions $u \in L^\infty((0,T) \times \Omega)$ and $w \in L^\infty(\Gamma^0 \cup \Gamma^+)$ such that for every $\psi \in C^\infty_c([0,T) \times \mathbb{R}^d)$,

$$\int_0^T \int_\Omega \rho u(\partial_t \psi + b \cdot \nabla \psi)\, dxdt = \int_{\Gamma^-} \overline{g}\mathrm{Tr}(\rho b)\psi \, d\mathcal{H}^{d-1}dt + \int_{\Gamma^+ \cup \Gamma^0} \mathrm{Tr}(\rho b)\psi w \, d\mathcal{H}^{d-1}dt - \int_\Omega \rho_0 \, \overline{u}\, \psi(0,\cdot)\, dx. \tag{4.1}$$

We proceed as follows: first, in § 4.1 we introduce an approximation scheme. Next, in § 4.2 we pass to the limit and establish existence.

4.1 Approximation scheme

In this section we rely on the analysis in [19, § 3.3], but we employ a more refined approximation scheme which guarantees strong convergence of the traces.

We set $\Lambda := (0,T) \times \Omega$ and we recall that by assumption $\rho \in BV(\Lambda) \cap L^\infty(\Lambda)$. We apply Theorem 2.7 and we select a sequence $\{\tilde{\rho}_m\} \subseteq C^\infty(\Lambda)$ satisfying (2.8) and (2.9). Next, we set

$$\rho_m := \frac{1}{m} + \tilde{\rho}_m \geq \frac{1}{m}. \tag{4.2}$$

We then apply Theorem 2.7 to the function $b\rho$ and we set

$$b_m := \frac{\widetilde{(b\rho)}_m}{\rho_m}. \tag{4.3}$$

Owing to Theorem 2.7 we have

$$\rho_m \to \rho \text{ strongly in } L^1((0,T) \times \Omega), \quad b_m \rho_m \to b\rho \text{ strongly in } L^1((0,T) \times \Omega; \mathbb{R}^d). \tag{4.4}$$

and, by using the identity (2.7),

$$\begin{gathered} \mathrm{Tr}(\rho_m) \to \mathrm{Tr}(\rho) \text{ strongly in } L^1(\Gamma), \quad \mathrm{Tr}(\rho_m b_m) \to \mathrm{Tr}(\rho b) \text{ strongly in } L^1(\Gamma), \\ \rho_{m0} \to \rho_0 \text{ strongly in } L^1(\Omega). \end{gathered} \tag{4.5}$$

Note, furthermore, that

$$\|\mathrm{Tr}(b_m \rho_m)\|_{L^\infty} \overset{(3.5)}{\leq} \|b_m \rho_m\|_{L^\infty} \overset{(2.9)}{\leq} 4\|b\rho\|_{L^\infty}. \tag{4.6}$$

In the following, we will use the notation

$$\Gamma^-_m := \big\{(t,x) \in \Gamma : \ \mathrm{Tr}(\rho_m b_m) < 0\big\}, \qquad \Gamma^+_m := \big\{(t,x) \in \Gamma : \ \mathrm{Tr}(\rho_m b_m) > 0\big\} \tag{4.7}$$

Finally, we extend the function $\overline{g}$ to the whole Γ by setting it equal to 0 outside Γ^- and we construct two sequences $\{\overline{g}_m\} \subseteq C^1(\Gamma)$ and $\{\overline{u}_m\} \subseteq C^\infty(\Omega)$ such that

$$\overline{g}_m \to \overline{g} \text{ strongly in } L^1(\Gamma), \quad \overline{u}_m \to \overline{u} \text{ strongly in } L^1(\Omega) \tag{4.8}$$

and

$$\|\overline{g}_m\|_{L^\infty} \leq \|\overline{g}\|_{L^\infty}, \quad \|\overline{u}_m\|_{L^\infty} \leq \|\overline{u}\|_{L^\infty}. \tag{4.9}$$

We can now define the function u_m as the solution of the initial-boundary value problem

$$\begin{cases} \partial_t u_m + b_m \cdot \nabla u_m = 0 & \text{on } (0,T)\times\Omega \\ u_m = \overline{u}_m & \text{at } t=0 \\ u_m = \overline{g}_m & \text{on } \tilde{\Gamma}_m^-, \end{cases} \tag{4.10}$$

where $\tilde{\Gamma}_m^-$ is the subset of Γ such that the characteristic lines of b_m starting at a point in $\tilde{\Gamma}_m^-$ are entering $(0,T)\times\Omega$. We recall (4.7) and we point out that

$$\Gamma_m^- \subseteq \tilde{\Gamma}_m^- \subseteq \big\{(t,x)\in\Gamma : \; b_m\cdot\vec{n} \leq 0\big\}.$$

In the previous expression, $\vec{n}$ denotes as the outward pointing, unit normal vector to $\partial\Omega$. By using the classical method of characteristics (see also [9]) we establish the existence of a solution u_m satisfying

$$\|u_m\|_\infty \leq \max\{\|\overline{u}_m\|_\infty, \|\overline{g}_m\|_\infty\} \overset{(4.9)}{\leq} \max\{\|\overline{u}\|_\infty, \|\overline{g}\|_\infty\}. \tag{4.11}$$

We now introduce the function h_m by setting

$$h_m := \partial_t \rho_m + \operatorname{div}(b_m\rho_m) \tag{4.12}$$

and by using the equation at the first line of (4.10) we get that

$$\partial_t(\rho_m u_m) + \operatorname{div}(b_m\rho_m u_m) = h_m u_m.$$

Owing to the Gauss-Green formula, this implies that, for every $\psi \in C^\infty_c([0,T)\times\mathbb{R}^d)$,

$$\begin{aligned} &\int_0^T \int_\Omega \rho_m u_m[\partial_t\psi + b_m\cdot\nabla\psi]\, dxdt + \int_0^T\int_\Omega h_m u_m \psi\, dxdt \\ &\quad = -\int_\Omega \psi(0,x)\overline{\rho}_{m0}\overline{u}_m\, dx - \int_0^T\int_{\partial\Omega} \psi u_m \rho_m b_m\cdot\vec{n}\, d\mathcal{H}^{d-1}dt \\ &\quad = -\int_\Omega \psi(0,x)\overline{\rho}_{m0}\overline{u}_m\, dx - \int_0^T\int_{\partial\Omega} \mathbf{1}_{\Gamma_m^-}\overline{g}_m\psi \operatorname{Tr}(\rho_m b_m) d\mathcal{H}^{d-1}dt - \int_0^T\int_{\partial\Omega} \mathbf{1}_{\Gamma_m^+} u_m\psi\operatorname{Tr}(\rho_m b_m) d\mathcal{H}^{d-1}dt. \end{aligned} \tag{4.13}$$

In the above expression, we have used the notation introduced in (4.7) and the fact that $\operatorname{Tr}(\rho_m b_m) = 0$ on $\Gamma \setminus (\Gamma_m^- \cup \Gamma_m^+)$.

4.2 Passage to the limit

Owing to the uniform bound (4.11), there are a subsequence of $\{u_m\}$ (which, to simplify notation, we do not relabel) and a function $u \in L^\infty((0,T)\times\Omega$ such that

$$u_m \overset{*}{\rightharpoonup} u \text{ weakly}^* \text{ in } L^\infty((0,T)\times\Omega). \tag{4.14}$$

The goal of this paragraph is to show that the function u in (4.14) is a distributional solution of (1.2) by passing to the limit in (4.13). We first introduce a technical lemma.

Lemma 4.1. *We can construct the approximating sequences $\{\rho_m\}$ and $\{b_m\}$ in such a way that the sequence $\{h_m\}$ defined as in (4.12) satisfies*

$$h_m \to 0 \text{ strongly in } L^1((0,T)\times\Omega). \tag{4.15}$$

The proof of Lemma 4.1 is deferred to § 4.3 . For future reference, we state the next simple convergence result as a lemma.

Lemma 4.2. *Assume that*

$$\mathrm{Tr}(\rho_m b_m) \to \mathrm{Tr}(\rho b) \text{ strongly in } L^1(\Gamma). \tag{4.16}$$

Let Γ_m^- *and* Γ_m^+ *as in* (4.7) *and* Γ^- *and* Γ^+ *as in* (3.6)*, respectively. Then, up to subsequences,*

$$\mathbf{1}_{\Gamma_m^-} \to \mathbf{1}_{\Gamma^-} + \mathbf{1}_{\Gamma'} \text{ strongly in } L^1(\Gamma) \tag{4.17}$$

and

$$\mathbf{1}_{\Gamma_m^+} \to \mathbf{1}_{\Gamma^+} + \mathbf{1}_{\Gamma''} \text{ strongly in } L^1(\Gamma), \tag{4.18}$$

where Γ' *and* Γ'' *are (possibly empty) measurable sets satisfying*

$$\Gamma', \Gamma'' \subseteq \Gamma^0. \tag{4.19}$$

Proof of Lemma 4.2. Owing to (4.16) we have that, up to subsequences, the sequence $\{\mathrm{Tr}(\rho_m b_m)\}$ satisfies

$$\mathrm{Tr}(\rho_m b_m)(t,x) \to \mathrm{Tr}(\rho b)(t,x), \quad \text{for } \mathcal{L}^1 \otimes \mathcal{H}^{d-1}\text{-almost every } (t,x) \in \Gamma.$$

Owing to the Lebesgue Dominated Convergence Theorem, this implies (4.17) and (4.18). □

We can now pass to the limit in all the terms in (4.13). First, by combining (4.4), (4.11), (4.14) and (4.15) we get that

$$\int_0^T \int_\Omega \rho_m u_m [\partial_t \psi + b_m \cdot \nabla \psi] \, dxdt + \int_0^T \int_\Omega h_m u_m \psi \, dxdt \to \int_0^T \int_\Omega \rho u [\partial_t \psi + b \cdot \nabla \psi] \, dxdt, \tag{4.20}$$

for every $\psi \in C_c^\infty([0,T) \times \mathbb{R}^d)$. Also, by combining the second line of (4.5) with (4.8) and (4.9) we arrive at

$$\int_\Omega \psi(0,x) \rho_{m0} \overline{u}_m \, dx \to \int_\Omega \psi(0,x) \rho_0 \overline{u} \, dx, \tag{4.21}$$

for every $\psi \in C_c^\infty([0,T) \times \mathbb{R}^d)$. Next, we combine (4.5), (4.8), (4.9), (4.17), (4.19) and the fact that $\mathrm{Tr}(\rho b) = 0$ on Γ^0 to get that

$$\begin{aligned} \int_0^T \int_{\partial\Omega} \mathbf{1}_{\Gamma_m^-} \overline{g}_m \psi \mathrm{Tr}(\rho_m b_m) d\mathcal{H}^{d-1} dt &\to \int_0^T \int_{\partial\Omega} \mathbf{1}_{\Gamma^-} \overline{g} \psi \mathrm{Tr}(\rho b) d\mathcal{H}^{d-1} dt \\ &= \int_0^T \int_{\Gamma^-} \overline{g} \psi \mathrm{Tr}(\rho b) d\mathcal{H}^{d-1} dt, \end{aligned} \tag{4.22}$$

for every $\psi \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^d)$. We are left with the last term in (4.13): first, we denote by $u_{m|\Gamma}$ the restriction of u_m to Γ. Since u_m is a smooth function, then

$$\|u_{m|\Gamma}\|_{L^\infty(\Gamma)} \leq \|u_m\|_{L^\infty((0,T)\times\Omega)} \overset{(4.11)}{\leq} \max\left\{\|\overline{u}\|_{L^\infty}, \|\overline{g}\|_{L^\infty}\right\}$$

and hence there is a function $w \in L^\infty(\Gamma)$ such that, up to subsequences,

$$u_{m|\Gamma} \overset{*}{\rightharpoonup} w \text{ weakly}^* \text{ in } L^\infty(\Gamma). \tag{4.23}$$

By combining (4.5), (4.18), (4.23) and the fact that $\mathrm{Tr}(\rho b) = 0$ on Γ^0 we get that

$$\begin{aligned} \int_0^T \int_{\partial\Omega} \mathbf{1}_{\Gamma_m^+} u_m \psi \mathrm{Tr}(\rho_m b_m) d\mathcal{H}^{d-1} dt &\to \int_0^T \int_{\partial\Omega} \mathbf{1}_{\Gamma^+} w \psi \mathrm{Tr}(\rho b) d\mathcal{H}^{d-1} dt \\ &= \int_{\Gamma^+ \cup \Gamma^0} w \psi \mathrm{Tr}(\rho b) d\mathcal{H}^{d-1} dt. \end{aligned} \tag{4.24}$$

By combining (4.20), (4.21), (4.22) and (4.24) we get that u satisfies (4.1) and this establishes existence of a distributional solution of (1.2).

4.3 Proof of Lemma 4.1

To ensure that (4.15) holds we use the same approximation *à la* Meyers-Serrin as in [7, pp.122-123]. We now recall some details of the construction. First, we fix a countable family of open sets $\{\Lambda_h\}$ such that

i. Λ_h is compactly contained in Λ, for every h;

ii. $\{\Lambda_h\}$ is a covering of Λ, namely
$$\bigcup_{h=1}^{\infty} \Lambda_h = \Lambda;$$

iii. every point in Λ is contained in at most 4 sets Λ_h.

Next, we consider a partition of unity associated to $\{\Lambda_h\}$, namely a countably family of smooth, nonnegative functions $\{\zeta_h\}$ such that

iv. we have
$$\sum_{h=1}^{\infty} \zeta_h \equiv 1 \quad \text{in } \Omega; \tag{4.25}$$

v. for every $h > 0$, the support of ζ_h is contained in Λ_h.

Finally, we fix a convolution kernel $\eta : \mathbb{R}^{d+1} \to \mathbb{R}^+$ and we define η_ε by setting
$$\eta_\varepsilon(z) := \frac{1}{\varepsilon^{d+1}} \eta\left(\frac{z}{\varepsilon}\right)$$

For every $m > 0$ and $h > 0$ we can choose ε_{mh} in such a way that $(\rho\zeta_h) * \eta_{\varepsilon_{mh}}$ is supported in Λ_h and furthermore
$$\int_0^T \int_\Omega |\rho\zeta_h - (\rho\zeta_h) * \eta_{\varepsilon_{mh}}| + |\rho\, \partial_t \zeta_h - (\rho\, \partial_t \zeta_h) * \eta_{\varepsilon_{mh}}| + |\rho b \cdot \nabla \zeta_h - (\rho b \cdot \nabla \zeta_h) * \eta_{\varepsilon_{mh}}| dxdt \le \frac{2^{-h}}{m}. \tag{4.26}$$

We then define $\tilde{\rho}_m$ by setting
$$\tilde{\rho}_m := \sum_{h=1}^{\infty} (\rho\zeta_h) * \eta_{\varepsilon_{mh}}. \tag{4.27}$$

The function $(\widetilde{\rho b})_m$ is defined analogously. Next, we proceed as in [7, p.123] and we point out that
$$\begin{aligned} h_m &\overset{(4.12)}{=} \partial_t \rho_m + \operatorname{div}(\rho_m b_m) \overset{(4.12)}{=} \underbrace{\sum_{h=1}^{\infty} (\partial_t \rho\zeta_h) * \eta_{\varepsilon_{mh}} + \sum_{h=1}^{\infty} (\operatorname{div}(\rho b)\zeta_h) * \eta_{\varepsilon_{mh}}}_{=0 \text{ by } (1.1)} \\ &\quad + \sum_{h=1}^{\infty} (\rho\, \partial_t \zeta_h) * \eta_{\varepsilon_{mh}} + \sum_{h=1}^{\infty} (\rho b \cdot \nabla \zeta_h) * \eta_{\varepsilon_{mh}} \\ &\overset{(4.25)}{=} \sum_{h=1}^{\infty} (\rho\, \partial_t \zeta_h) * \eta_{\varepsilon_{mh}} - \rho \sum_{h=1}^{\infty} \partial_t \zeta_h \quad + \sum_{h=1}^{\infty} (\rho b \cdot \nabla \zeta_h) * \eta_{\varepsilon_{mh}} - \rho b \cdot \sum_{h=1}^{\infty} \nabla \zeta_h \end{aligned}$$

By using (4.26) we then get that
$$\int_0^T \int_\Omega |h_m| dxdt \le \sum_{h=1}^{\infty} \frac{2^{-h}}{m} = \frac{1}{m}$$

and this establishes (4.15).

5 Proof Theorem 1.2: comparison principle and uniqueness

In this section we complete the proof of Theorem 1.2. More precisely, we establish the following comparison principle.

Lemma 5.1. *Let Ω, b and ρ as in the statement of Theorem 1.2. Assume u_1 and $u_2 \in L^\infty((0,T)\times\Omega)$ are distributional solutions (in the sense of Definition 3.3) of the initial-boundary value problem (1.2) corresponding to the initial and boundary data $\overline{u}_1 \in L^\infty(\Omega)$, $\overline{g}_1 \in L^\infty(\Gamma^-)$ and $\overline{u}_2 \in L^\infty(\Omega)$, $\overline{g}_2 \in L^\infty(\Gamma^-)$, respectively. If $\overline{u}_1 \geq \overline{u}_2$ and $\overline{g}_1 \geq \overline{g}_2$, then*

$$\rho u_1 \geq \rho u_2 \quad \text{a.e. in } (0,T)\times\Omega. \tag{5.1}$$

Note that the uniqueness of ρu, where u is a distributional solution of the initial-boundary value problem (1.2), immediately follows from the above result.

Proof of Lemma 5.1. Let us define the function

$$\tilde\beta(u) = \begin{cases} u^2 & u \geq 0 \\ 0 & u < 0. \end{cases}$$

In what follows, we shall prove that the identity $\rho\ \tilde\beta(u_2 - u_1) = 0$ holds almost everywhere, whence the comparison principle follows. To see this, we proceed as described below. First, we point out that, since the equation at the first line of (1.2) is linear, then $u_2 - u_1$ is a distributional solution of the initial boundary value problem with data $\overline{u}_2 - \overline{u}_1$, $\overline{g}_2 - \overline{g}_1$. In particular, for every $\psi \in C^\infty_c([0,T)\times\mathbb{R}^d)$ we have

$$\int_0^T \int_\Omega \rho(u_2-u_1)(\partial_t\psi + b\cdot\nabla\psi)\, dxdt = \int_0^T \int_{\partial\Omega} [\mathrm{Tr}(\rho u_2 b) - \mathrm{Tr}(\rho u_1 b)]\,\psi\, d\mathcal{H}^{d-1}dt - \int_\Omega \psi(0,\cdot)\rho_0(\overline{u}_2 - \overline{u}_1)\, dx \tag{5.2}$$

and

$$\mathrm{Tr}(\rho u_2 b) = \overline{g}_2 \mathrm{Tr}(\rho b), \quad \mathrm{Tr}(\rho u_1 b) = \overline{g}_1 \mathrm{Tr}(\rho b) \quad \text{on } \Gamma^-. \tag{5.3}$$

Note that (5.2) implies that

$$\int_0^T \int_\Omega \rho(u_2 - u_1)(\partial_t\phi + b\cdot\nabla\phi)\, dxdt = 0, \quad \forall \phi \in C^\infty_c((0,T)\times\Omega). \tag{5.4}$$

By using [19, Lemma 5.10] (renormalization property inside the domain), we get

$$\int_0^T \int_\Omega \rho\ \tilde\beta(u_2 - u_1)(\partial_t\phi + b\cdot\nabla\phi)\, dxdt = 0, \qquad \forall \phi \in C^\infty_c((0,T)\times\Omega). \tag{5.5}$$

We next apply Lemma 3.1 to the function $\tilde\beta(u_2 - u_1)$ to infer that there are bounded functions $\mathrm{Tr}(\rho\tilde\beta(u_2 - u_1)b)$ and $(\rho\tilde\beta(u_2 - u_1))_0$ such that, for every $\psi \in C^\infty_c([0,T)\times\mathbb{R}^d)$, we have

$$\int_0^T \int_\Omega \rho\ \tilde\beta(u_2-u_1)(\partial_t\psi + b\cdot\nabla\psi)\, dxdt = \int_0^T \int_{\partial\Omega} \mathrm{Tr}(\rho\ \tilde\beta(u_2-u_1)b)\,\psi\, d\mathcal{H}^{d-1}dt - \int_\Omega \psi(0,\cdot)(\rho\ \tilde\beta(u_2-u_1))_0\, dx. \tag{5.6}$$

We recall (5.2) and we apply Lemma 2.2 (trace renormalization property) with $w = u_2 - u_1$, $h = \tilde\beta$, $B = (\rho, \rho b)$, $\Lambda = \mathbb{R}^{d+1}$ and $\Lambda' = (0,T)\times\Omega$. We recall that the vector field P is defined by setting $P := (\rho, \rho b)$ and we get

$$\begin{aligned} (\rho\ \tilde\beta(u_2 - u_1))_0 = -\mathrm{Tr}(\tilde\beta(u_2 - u_1)P, \partial\Lambda')\Big|_{\{0\}\times\Omega} &= -\tilde\beta\left(\frac{(\rho(u_2 - u_1))_0}{\mathrm{Tr}(P,\partial\Lambda')\Big|_{\{0\}\times\Omega}}\right)\mathrm{Tr}(P,\partial\Lambda')\Big|_{\{0\}\times\Omega} \\ &= -\tilde\beta\left(\frac{\rho_0(\overline{u}_2 - \overline{u}_1)}{\overline{\rho}}\right)\rho_0 \\ &= 0, \text{ since } \overline{u}_1 \geq \overline{u}_2 \end{aligned} \tag{5.7}$$

and

$$\mathrm{Tr}(\rho\ \tilde\beta(u_2-u_1)b) = \mathrm{Tr}(\tilde\beta(u_2-u_1)\rho, \partial\Lambda')\Big|_{(0,T)\times\partial\Omega} = \tilde\beta\left(\frac{\mathrm{Tr}((u_2-u_1)\rho, \partial\Lambda')\Big|_{(0,T)\times\partial\Omega}}{\mathrm{Tr}(P, \partial\Lambda')\Big|_{(0,T)\times\partial\Omega}}\right)\mathrm{Tr}(P, \partial\Lambda')\Big|_{(0,T)\times\partial\Omega}$$
$$= \tilde\beta\left(\frac{\mathrm{Tr}(\rho(u_2-u_1)b)}{\mathrm{Tr}(\rho b)}\right)\mathrm{Tr}(\rho b).$$

By recalling (5.3) and the inequality $\bar g_1 \geq \bar g_2$, we conclude that

$$\mathrm{Tr}(\rho\ \tilde\beta(u_2-u_1)b) = 0 \quad \text{on } \Gamma^-$$

and, since $\tilde\beta \geq 0$, that

$$\mathrm{Tr}(\rho\ \tilde\beta(u_2-u_1)b) \geq 0 \quad \text{on } \Gamma. \tag{5.8}$$

We now choose a test function $\nu \in C_c^\infty(\mathbb{R}^d)$ in such a way that $\nu \equiv 1$ on the bounded set Ω. Note that

$$\partial_t \nu + b\cdot\nabla\nu = 0 \quad \text{on } (0,T)\times\Omega. \tag{5.9}$$

Next we choose a sequence of functions $\chi_n \in C_c^\infty([0,+\infty))$ that satisfy

$$\chi_n \equiv 1 \text{ on } [0,\bar t],\ \chi_n \equiv 0 \text{ on } [\bar t + \frac{1}{n}, +\infty),\ \chi_n' \leq 0,$$

and we define

$$\psi_n(t,x) := \chi_n(t)\nu(x),\ (t,x)\in[0,T)\times\mathbb{R}^d.$$

Note that ψ is smooth, non-negative and compactly supported in $[0,T)\times\mathbb{R}^d$. By combining the identities (5.6), (5.7) and the inequality (5.8), we get

$$\begin{aligned}
0 &\leq \int_0^T\int_\Omega \rho\ \tilde\beta(u_2-u_1)[\partial_t(\chi_n\nu) + b\cdot\nabla(\chi_n\nu)]\ dxdt \\
&= \int_0^T\int_\Omega \nu\rho\ \tilde\beta(u_2-u_1)\chi_n'\ dxdt + \int_0^T\int_\Omega \chi_n\rho\ \tilde\beta(u_2-u_1)(\partial_t\nu + b\cdot\nabla\nu)\ dxdt \\
&\overset{(5.9)}{=} \int_0^T\int_\Omega \nu\rho\ \chi_n'\ \tilde\beta(u_2-u_1)\ dxdt.
\end{aligned}$$

Passing to the limit as $n\to+\infty$ and noting that $\chi_n' \to -\delta_{\bar t}$ as $n\to\infty$ in the sense of distributions and recalling that $\nu\equiv 1$ on Ω we obtain

$$\int_\Omega \rho(\bar t,\cdot)\tilde\beta(u_2-u_1)(\bar t,\cdot) \leq 0.$$

Since the above inequality is true for arbitrary $\bar t\in[0,T]$, we can conclude that

$$\rho\ \tilde\beta(u_2-u_1) = 0, \text{ for almost every } (t,x) \Rightarrow \rho u_1 \geq \rho u_2, \text{ for almost every } (t,x). \tag{5.10}$$

This concludes the proof of Lemma 5.1. □

6 Stability and space continuity properties

In this section, we discuss some qualitative properties of solutions of the initial-boundary value problem (1.2). First, we establish Theorem 6.1, which establishes (weak) stability of solutions with respect to perturbations in the vector fields and the data. Theorem 6.2 implies that, under stronger hypotheses, we can establish strong stability. Finally, Theorem 6.3 establishes space continuity properties.

Theorem 6.1. *Let $T > 0$ and let $\Omega \subseteq \mathbb{R}^d$ be an open and bounded set with C^2 boundary. Assume that*

$$b_n, b \in BV((0,T) \times \Omega; \mathbb{R}^d) \cap L^\infty((0,T) \times \Omega; \mathbb{R}^d), \qquad \rho_n, \rho \in BV((0,T) \times \Omega) \cap L^\infty([0,T) \times \Omega)$$

satisfy

$$\begin{aligned} \partial_t \rho_n + \operatorname{div}(b_n \rho_n) = 0, \\ \partial_t \rho + \operatorname{div}(b\rho) = 0, \end{aligned} \tag{6.1}$$

in the sense of distributions on $(0,T) \times \Omega$. Assume furthermore that

$$0 \le \rho_n, \rho \le C \text{ and } \|b_n\|_\infty \text{ is uniformly bounded,} \tag{6.2}$$

$$(b_n, \rho_n) \xrightarrow[n\to\infty]{} (b, \rho) \text{ strongly in } L^1((0,T) \times \Omega; \mathbb{R}^{d+1}), \tag{6.3}$$

$$\rho_{n0} \xrightarrow[n\to\infty]{} \rho_0 \text{ strongly in } L^1(\Omega), \tag{6.4}$$

$$\operatorname{Tr}(\rho_n b_n) \xrightarrow[n\to\infty]{} \operatorname{Tr}(\rho b) \text{ strongly in } L^1(\Gamma), \tag{6.5}$$

Let $u_n \in L^\infty((0,T) \times \Omega)$ be a distributional solution (in the sense of Definition 3.3) of the initial-boundary value problem

$$\begin{cases} \partial_t(\rho_n u_n) + \operatorname{div}(\rho_n u_n b_n) = 0 & \text{in } (0,T) \times \Omega \\ u_n = \overline{u}_n & \text{at } t = 0 \\ u_n = \overline{g}_n & \text{on } \Gamma_n^- \end{cases} \tag{6.6}$$

and $u \in L^\infty((0,T) \times \Omega)$ be a distributional solution of the equation

$$\begin{cases} \partial_t(\rho u) + \operatorname{div}(\rho u b) = 0 & \text{in } (0,T) \times \Omega \\ u = \overline{u} & \text{at } t = 0 \\ u = \overline{g} & \text{on } \Gamma^-. \end{cases} \tag{6.7}$$

If $u_m, \bar{u} \in L^\infty(\Omega)$ and $\overline{g}_n, \overline{g} \in L^\infty(\Gamma)$ satisfy

$$\overline{u}_n \overset{*}{\rightharpoonup} \overline{u} \text{ weak-* in } L^\infty(\Omega), \tag{6.8}$$

$$\overline{g}_n \overset{*}{\rightharpoonup} \overline{g} \text{ weak-* in } L^\infty(\Gamma), \tag{6.9}$$

then

$$\rho_n u_n \overset{*}{\rightharpoonup} \rho u \text{ weak-* in } L^\infty((0,T) \times \Omega) \tag{6.10}$$

and

$$\operatorname{Tr}(\rho_n u_n b_n) \overset{*}{\rightharpoonup} \operatorname{Tr}(\rho u b) \text{ weak-* in } L^\infty(\Gamma). \tag{6.11}$$

Note that in the statement of the above theorem $\overline{g}_m$ and $\overline{g}$ are functions defined on the whole Γ, although the values of $\rho_m u_m$ and ρu are only determined by their values on Γ_m^- and Γ^-, respectively.

Proof. We proceed according to the following steps.
STEP 1: we apply Theorem 1.2 and we infer that the function $\rho_n u_n$ satisfying (6.6) is unique. Also, without loss of generality, we can redefine the function u_n on the set $\{\rho_n = 0\}$ in such a way that u_n satisfies the maximum principle (1.3). Owing to (6.11), the sequences $\|\overline{u}_m\|_{L^\infty}$ and $\|\overline{g}_m\|_{L^\infty}$ are both uniformly bounded and by the maximum principle so is $\|u_m\|_{L^\infty}$. Also, by combining (3.3) and (6.2) we infer that the sequence $\|\operatorname{Tr}(\rho_n b_n u_n)\|_\infty$ is also uniformly bounded. We conclude that, up to subsequences (which we do not label to simplify the notation), we have

$$\begin{aligned} & u_n \overset{*}{\rightharpoonup} r_1 \text{ weak-* in } L^\infty((0,T) \times \Omega), \\ & \operatorname{Tr}(\rho_n u_n b_n) \overset{*}{\rightharpoonup} r_2 \text{ weak-* in } L^\infty(\Gamma) \end{aligned} \tag{6.12}$$

for some $r_1 \in L^\infty((0,T)\times\Omega)$ and $r_2 \in L^\infty(\Gamma)$. By using (3.1) and (3.2), we get that

$$\int_0^T \int_\Omega \rho r_1 (\partial_t \phi + b\cdot\nabla\phi)\, dxdt = 0, \quad \forall \phi \in C_c^\infty((0,T)\times\Omega), \tag{6.13}$$

and

$$\int_0^T \int_\Omega \rho r_1 (\partial_t \psi + b\nabla\psi)\, dxdt = \int_0^T \int_{\partial\Omega} r_2 \psi \, d\mathcal{H}^{d-1} dt - \int_\Omega \psi(0,\cdot)\rho_0 \, \overline{u}\, dx, \ \forall \psi \in C_c^\infty([0,T)\times\mathbb{R}^d). \tag{6.14}$$

From Lemma 3.1, it also follows that

$$r_2 = \mathrm{Tr}(\rho r_1 b). \tag{6.15}$$

Assume for the time being that we have established the equality

$$r_2 = \overline{g}\mathrm{Tr}(\rho b), \quad \text{on } \Gamma^-, \tag{6.16}$$

then by recalling (6.15) and the uniqueness part in Theorem 1.2 we conclude that $r_1 = \rho u$ and $r_2 = \mathrm{Tr}(\rho b u)$. Owing to (6.12), this concludes the proof of the theorem.
Step 2: we establish (6.16). First, we decompose $\mathrm{Tr}(\rho_m u_m b_m)$ as

$$\begin{aligned} \mathrm{Tr}(\rho_n u_n b_n) &= \mathrm{Tr}(\rho_n u_n b_n)\mathbf{1}_{\Gamma_n^-} + \mathrm{Tr}(\rho_n u_n b_n)\mathbf{1}_{\Gamma_n^+} + \mathrm{Tr}(\rho_n u_n b_n)\mathbf{1}_{\Gamma_n^0} \\ &= \overline{g}_n \mathrm{Tr}(\rho_n b_n)\mathbf{1}_{\Gamma_n^-} + \mathrm{Tr}(\rho_n u_n b_n)\mathbf{1}_{\Gamma_n^+} + \mathrm{Tr}(\rho_n u_n b_n)\mathbf{1}_{\Gamma_n^0}, \end{aligned} \tag{6.17}$$

where Γ_n^-, Γ_n^+ and Γ_n^0 are defined as in (3.6). By using Lemma 2.2 (trace renormalization), one could actually prove that the last term in the above expression is actually 0. This is actually not needed here. Indeed, it suffices to recall (6.5) and Lemma 4.2 and point out that by combining (4.17) and (4.18) we get

$$\mathbf{1}_{\Gamma_n^0} \to \mathbf{1}_{\Gamma^0} - \mathbf{1}_{\Gamma'} - \mathbf{1}_{\Gamma''}. \tag{6.18}$$

Next, we recall that the sequence $\|\mathrm{Tr}(\rho_n u_n b_n)\|_{L^\infty}$ is uniformly bounded owing to the uniform bounds on $\|\rho_n\|_{L^\infty}$ and $\|u_n\|_{L^\infty}$. By recalling (6.9), we conclude that

$$\overline{g}_n \mathrm{Tr}(\rho_n b_n)\mathbf{1}_{\Gamma_n^-} \overset{*}{\rightharpoonup} \overline{g}\,\mathrm{Tr}(\rho b)\Big(\mathbf{1}_{\Gamma^-} + \mathbf{1}_{\Gamma'}\Big) \qquad \text{weak-* in } L^\infty(\Gamma). \tag{6.19}$$

By recalling that $\Gamma' \subseteq \Gamma^0$ we get that $\mathrm{Tr}(\rho b)\mathbf{1}_{\Gamma'} = 0$. We now pass to the weak star limit in (6.17) and using (4.17), (4.18), (6.12), (6.9) and (6.19) we get

$$r_2 = \overline{g}\mathrm{Tr}(\rho b)\mathbf{1}_{\Gamma^-} + r_2\Big(\mathbf{1}_{\Gamma^+} + \mathbf{1}_{\Gamma'}\Big) + r_2\Big(\mathbf{1}_{\Gamma^0} - \mathbf{1}_{\Gamma'} - \mathbf{1}_{\Gamma''}\Big), \tag{6.20}$$

which owing to the properties

$$\Gamma^- \cap \Gamma^0 = \emptyset, \quad \Gamma^- \cap \Gamma' = \emptyset, \quad \Gamma^- \cap \Gamma'' = \emptyset$$

implies (6.16). This concludes the proof Theorem 6.1. □

Theorem 6.2. *Under the same assumptions as in Theorem 6.1, if we furthermore assume that*

$$\overline{u}_n \xrightarrow[n\to\infty]{} \overline{u} \text{ strongly in } L^1(\Omega), \tag{6.21}$$

$$\overline{g}_n \xrightarrow[n\to\infty]{} \overline{g} \text{ strongly in } L^1(\Gamma), \tag{6.22}$$

then we get

$$\begin{aligned} &\rho_n u_n \xrightarrow[n\to\infty]{} \rho u \text{ strongly in } L^1((0,T)\times\Omega), \\ &\mathrm{Tr}(\rho_n u_n b_n) \xrightarrow[n\to\infty]{} \mathrm{Tr}(\rho u b) \text{ strongly in } L^1(\Gamma). \end{aligned} \tag{6.23}$$

Proof. First, we point out that the first equation in (6.11) implies that

$$\rho_n u_m \rightharpoonup \rho u \text{ weakly in } L^2((0,T)\times\Omega). \tag{6.24}$$

Next, by using Lemma 2.2 (trace-renormalization property), we get that $\rho_m u_n^2$ and ρu^2 satisfy (in the sense of distributions)

$$\begin{cases} \partial_t(\rho_n u_n^2) + \operatorname{div}(\rho_n u_n^2 b_n) = 0 & \text{in } (0,T)\times\Omega \\ u_n^2 = \overline{u}_n^2 & \text{at } t=0 \\ u_n^2 = \overline{g}_n^2 & \text{on } \Gamma_n^-, \end{cases}$$

and

$$\begin{cases} \partial_t(\rho u^2) + \operatorname{div}(\rho u^2 b) = 0 & \text{in } (0,T)\times\Omega \\ u^2 = \overline{u}^2 & \text{at } t=0 \\ u^2 = \overline{g}^2 & \text{on } \Gamma^-, \end{cases}$$

respectively. Also, by combinig (6.8),(6.9), (6.21) and (6.22), we get that

$$\overline{u}_n^2 \overset{*}{\rightharpoonup} \overline{u}^2 \text{ weak-* in } L^\infty(\Omega), \qquad \overline{g}_n^2 \overset{*}{\rightharpoonup} \overline{g}^2 \text{ weak-* in } L^\infty(\Gamma)$$

and by applying Theorem 6.1 to $\rho_m u_m^2$ we conclude that

$$\rho_m u_m^2 \overset{*}{\rightharpoonup} \rho u^2 \text{ weak-* in } L^\infty((0,T)\times\Omega)$$

and that

$$\operatorname{Tr}(\rho_n u_n^2 b_n) \overset{*}{\rightharpoonup} \operatorname{Tr}(\rho u^2 b) \text{ weak-* in } L^\infty(\Gamma). \tag{6.25}$$

Since the sequence $\|\rho_m\|_{L^\infty}$ is uniformly bounded, then by recalling (6.3) we get

$$\rho_m^2 u_m^2 \overset{*}{\rightharpoonup} \rho^2 u^2 \text{ weak-* in } L^\infty((0,T)\times\Omega)$$

and hence

$$\rho_m^2 u_m^2 \rightharpoonup \rho^2 u^2 \text{ weakly in } L^2((0,T)\times\Omega). \tag{6.26}$$

By combining (6.24) and (6.26) we get that $\rho_m^2 u_m^2 \longrightarrow \rho u$ strongly in $L^2((0,T)\times\Omega)$ and this implies the first convergence in (6.23).

Next, we establish the second convergence in $L^2((0,T)\times\Omega)$. Since Γ is a set of finite measure, from (6.11) and (6.25) we can infer that

$$\begin{aligned} &\operatorname{Tr}(\rho_n u_n b_n) \rightharpoonup \operatorname{Tr}(\rho u b) \text{ weakly in } L^2(\Gamma), \\ &\operatorname{Tr}(\rho_n u_n^2 b_n) \rightharpoonup \operatorname{Tr}(\rho u^2 b) \text{ weakly in } L^2(\Gamma). \end{aligned} \tag{6.27}$$

By using the uniform bounds for $\|\operatorname{Tr}(\rho_n b_n)\|_\infty$, we infer from the L^1 convergence of $\operatorname{Tr}(\rho_n b_n)$ to $\operatorname{Tr}(\rho b)$ that

$$\operatorname{Tr}(\rho_n b_n) \xrightarrow[n\to\infty]{} \operatorname{Tr}(\rho b) \text{ strongly in } L^2(\Gamma). \tag{6.28}$$

Next, we apply Lemma 2.2 (trace renormalization property) and we get that

$$[\operatorname{Tr}(\rho_n u_n b_n)]^2 = \left[\frac{\operatorname{Tr}(\rho_n u_n b_n)}{\operatorname{Tr}(\rho_n b_n)}\right]^2 [\operatorname{Tr}(\rho_n b_n)]^2 = \operatorname{Tr}(\rho_n u_n^2 b_n)\operatorname{Tr}(\rho_n b_n)$$

and

$$[\operatorname{Tr}(\rho u b)]^2 = \left[\frac{\operatorname{Tr}(\rho u b)}{\operatorname{Tr}(\rho b)}\right]^2 [\operatorname{Tr}(\rho b)]^2 = \operatorname{Tr}(\rho u^2 b)\operatorname{Tr}(\rho b).$$

From (6.27) and (6.28), we can then conclude that

$$[\operatorname{Tr}(\rho_n u_n b_n)]^2 \rightharpoonup [\operatorname{Tr}(\rho u b)]^2 \text{ weakly in } L^2(\Gamma), \tag{6.29}$$

and by recalling (6.27) the second convergence in (6.23) follows. □

Finally, we establish space-continuity properties of the vector field $(\rho u, \rho u b)$ similar to those established in [11, 16].

Theorem 6.3. *Under the same assumptions as in Theorem 1.2, let P be the vector field $P := (\rho, \rho b)$, u be a distributional solution of (1.2) and $\{\Sigma_r\}_{r\in I} \subseteq \mathbb{R}^d$ be a family of graphs as in Definition 2.3. Also, fix $r_0 \in I$ and let $\gamma_0, \gamma_r : (0,T)\times D \to \mathbb{R}$ be defined by*

$$\begin{aligned} \gamma_0(t, x_1, \cdots, x_{d-1}) &:= \mathrm{Tr}^-(uP, (0,T)\times\Sigma_{r_0})(t, x_1, \cdots, x_{d-1}, f(x_1,\cdots,x_{d-1}) - r_0), \\ \gamma_r(t, x_1, \cdots, x_{d-1}) &:= \mathrm{Tr}^+(uP, (0,T)\times\Sigma_{r})(t, x_1, \cdots, x_{d-1}, f(x_1,\cdots,x_{d-1}) - r). \end{aligned} \tag{6.30}$$

Then $\gamma_r \to \gamma_0$ strongly in $L^1((0,T)\times D)$ as $r \to r_0^+$.

The proof of the above result follows the same strategy as the proof of [16, Proposition 3.5] and is therefore omitted.

7 Applications to the Keyfitz and Kranzer system

In this section, we consider the initial-boundary value problem for the Keyfitz and Kranzer system [24] of conservation laws in several space dimensions, namely

$$\begin{cases} \partial_t U + \displaystyle\sum_{i=1}^d \partial_{x_i}(f^i(|U|)U) = 0 & \text{in } (0,T)\times\Omega \\ U = U_0 & \text{at } t = 0 \\ U = U_b & \text{on } \Gamma. \end{cases} \tag{7.1}$$

Note that, in general, we cannot expect that the boundary datum is pointwise attained on the whole boundary Γ. We come back to this point in the following.

We follow the same approach as in [2, 6, 12, 20] and we formally split the equation at the first line of (7.1) as the coupling between a scalar conservation law and a linear transport equation. More precisely, we set $F := (f^1, \cdots, f^d)$ and we point out that the modulus $\rho := |U|$ formally solves the initial-boundary value problem

$$\begin{cases} \partial_t \rho + \mathrm{div}(F(\rho)\rho) = 0 & \text{in } (0,T)\times\Omega \\ \rho = |U_0| & \text{at } t = 0 \\ \rho = |U_b| & \text{on } \Gamma. \end{cases} \tag{7.2}$$

We follow [10, 15, 28] and we extend notion of *entropy admissible* solution (see [25]) to initial boundary value problems.

Definition 7.1. *A function $\rho \in L^\infty((0,T)\times\Omega) \cap BV((0,T)\times\Omega)$ is an entropy solution of (7.2) if for all $k \in \mathbb{R}$,*

$$\begin{aligned} &\int_0^T \int_\Omega \Big\{ |\rho(t,x) - k|\ \partial_t\psi + \mathrm{sgn}(\rho - k)[F(\rho) - F(k)]\cdot\nabla\psi \Big\}\, dxdt \\ &+ \int_\Omega |\rho_0 - k|\ \psi(0,\cdot)\ dx - \int_0^T \int_{\partial\Omega} \mathrm{sgn}(|U_b|(t,x) - k)\ [F(T(\rho)) - F(k)]\cdot\vec{n}\ \psi\ dxdt \geq 0, \end{aligned}$$

for any positive test function $\psi \in C_c^\infty([0,T)\times\mathbb{R}^d; \mathbb{R}^+)$. In the above expression $T(\rho)$ denotes the trace of the function ρ on the boundary Γ and $\vec{n}$ is the outward pointing, unit normal vector to Γ.

Existence and uniqueness results for entropy admissible solutions of the above systems were obtained by Bardos, le Roux and Nédélec [10] by extending the analysis by Kružkov to initial-boundary value problems (see also [15, 28] for a more recent discussion). Note, however, that one cannot expect that the boundary value $|U_b|$ is pointwise attained on the whole boundary Γ, see again [10, 15, 28] for a more extended discussion.

Next, we introduce the equation for the *angular part* of the solution of (7.1). We recall that, if $|U_b|$ and $|U_0|$ are of bounded variation, then so is ρ and hence the trace of $F(\rho)\rho$ on Γ is well defined. As usual, we denote it by $T(F(\rho)\rho)$. In particular, we can introduce the set

$$\Gamma^- := \big\{(t,x) \in \Gamma : \ T(F(\rho)\rho) \cdot \vec{n} < 0\big\},$$

where as usual $\vec{n}$ denotes the outward pointing, unit normal vector to Γ. We consider the vector $\theta = (\theta_1, \cdots, \theta_N)$ and we impose

$$\begin{cases} \partial_t(\rho\theta) + \operatorname{div}(F(\rho)\rho\theta) = 0 & \text{in } (0,T) \times \Omega \\ \theta = \dfrac{U_0}{|U_0|} & \text{at } t = 0 \\ \theta = \dfrac{U_b}{|U_b|} & \text{on } \Gamma^-, \end{cases} \tag{7.3}$$

where the ratios $U_0/|U_0|$ and $U_b/|U_b|$ are defined to be an arbitrary unit vector when $|U_0| = 0$ and $|U_b| = 0$, respectively. Note that the product $U = \theta\rho$ formally satisfies the equation at the first line of (7.1). We now extend the notion of *renormalized entropy solution* given in [2, 6, 20] to initial-boundary value problems.

Definition 7.2. *A renormalized entropy solution of* (7.1) *is a function* $U \in L^\infty((0,T) \times \Omega; \mathbb{R}^N)$ *such that* $U = \rho\theta$, *where*

- $\rho = |U|$ *and* ρ *is an entropy admissible solution of* (7.2).
- $\theta = (\theta_1, \dots, \theta_N)$ *is a distributional solution, in the sense of Definition 3.3, of* (7.3).

Some remarks are here in order. First, we can repeat the proof of [19, Proposition 5.7] and conclude that, under fairly general assumptions, any renormalized entropy solution is an entropy solution. More precisely, let us fix a renormalized entropy solution U and an *entropy-entropy flux pair* (η, Q), namely a couple of functions $\eta : \mathbb{R}^N \to \mathbb{R}$, $Q : \mathbb{R}^N \to \mathbb{R}^d$ such that

$$\nabla\eta Df^i = \nabla Q^i, \quad \text{for every } i = 1, \dots, d.$$

Assume that

$$\mathcal{L}^1\big\{r \in \mathbb{R} : \ (f^1)'(r) = \cdots = (f^d)'(r) = 0\big\} = 0.$$

By arguing as in [19] we conclude that, if η is convex, then

$$\int_0^T \int_\Omega \eta(U)\partial_t\phi + Q(U) \cdot \nabla\phi \, dxdt \geq 0$$

for every *entropy-entropy flux pair* (η, Q) and for every nonnegative test function $\phi \in C_c^\infty((0,T) \times \Omega)$.

Second, we point out that, as the Bardos, le Roux and Nédélec [10] solutions of scalar initial-boundary value problems, renormalized entropy solutions of the Keyfitz and Kranzer system do not, in general pointwise attain the boundary datum U_0 on the whole boundary Γ.

We now state our well-posedness result.

Theorem 7.3. *Assume* Ω *is a bounded open set with* C^2 *boundary. Also, assume that* $U_0 \in L^\infty(\Omega; \mathbb{R}^N)$ *and* $U_b \in L^\infty(\Gamma; \mathbb{R}^N)$ *satisfy* $|U_0| \in BV(\Omega)$, $|U_b| \in BV(\Gamma)$. *Then there is a unique renormalized entropy solution of* (7.1) *that satisfies* $U \in L^\infty((0,T) \times \Omega; \mathbb{R}^N)$.

Proof. We first establish existence, next uniqueness.
Existence: first, we point out that the results in [10, 15, 28] imply that there is an entropy

admissible solution of (7.2) satisfying $\rho \in L^\infty((0,T)\times\Omega)\cap BV((0,T)\times\Omega)$. Also, ρ satisfies the maximum principle, namely

$$0 \le \rho \le \max\big\{\|U_0\|_{L^\infty}, \|U_b\|_{L^\infty}\big\}. \tag{7.4}$$

For every $j = 1,\dots,N$ we consider the initial-boundary value problem

$$\begin{cases} \partial_t(\rho\theta_j) + \operatorname{div}(F(\rho)\rho\theta_j) = 0 & \text{in } (0,T)\times\Omega \\ \theta_j = \dfrac{U_{0j}}{|U_0|} & \text{at } t = 0 \\ \theta_j = \dfrac{U_{bj}}{|U_b|} & \text{on } \Gamma^-, \end{cases} \tag{7.5}$$

where U_{0j} and U_{bj} is the j-th component of U_0 and U_b, respectively. The existence of a distributional solution θ_j follows from the existence part in Theorem 1.2.

We now set $U := \rho\theta$, where $\theta = (\theta_1,\dots,\theta_N)$. To conclude the existence part we are left to show that $|U| = \rho$. To this end, we point out that, by combining [19, Lemma 5.10] (renormalization property inside the domain) with Theorem 2.2 (trace renormalization property) and by arguing as in § 5, we conclude that, for every $j = 1,\dots,N$, θ_j^2 is a distributional solution, in the sense of Definition 3.3, of the initial-boundary value problem

$$\begin{cases} \partial_t(\rho\theta_j^2) + \operatorname{div}(F(\rho)\rho\theta_j^2) = 0 & \text{in } (0,T)\times\Omega \\ \theta_j = \dfrac{U_{0j}^2}{|U_0|^2} & \text{at } t = 0 \\ \theta = \dfrac{U_{bj}^2}{|U_b|^2} & \text{on } \Gamma^-. \end{cases}$$

By adding from 1 to N, we conclude that $|\theta|^2$ is a distributional solution of

$$\begin{cases} \partial_t(\rho|\theta|^2) + \operatorname{div}(F(\rho)\rho|\theta|^2) = 0 & \text{in } (0,T)\times\Omega \\ \theta_j = 1 & \text{at } t = 0 \\ \theta = 1 & \text{on } \Gamma^-. \end{cases}$$

By recalling the equation at the first line of (7.2) we infer that $|\theta|^2 = 1$ is a solution of the above initial-boundary value problem. By the uniqueness part of Theorem 1.2, we then deduce that $\rho|\theta|^2 = \rho$ and this concludes the proof of the existence part.
Uniqueness: assume U_1 and U_2 are two renormalized entropy solutions, in the sense of Definition 7.2, of the initial-boundary value problem (7.1). Then $\rho_1 := |U_1|$ and $\rho_2 := |U_2|$ are two entropy admissible solutions of the initial-boundary value problem (7.2) and hence $\rho_1 = \rho_2$. By applying the uniqueness part of Theorem 1.2 to the initial-boundary value problem (7.5), for every $j = 1,\dots,N$, we can then conclude that $U_1 = U_2$. □

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