# CONTINUITY OF AN OPTIMAL TRANSPORT IN MONGE PROBLEM

ILARIA FRAGALÀ, MARIA STELLA GELLI, AND ALDO PRATELLI



Abstract. Given two absolutely continuous probability measures $f^{\pm}$ in $\mathbb{R}^2$, we consider the classical Monge transport problem, with the Euclidean distance as cost function. We prove the existence of a *continuous* optimal transport, under the assumptions that (the densities of) $f^{\pm}$ are continuous and strictly positive in the interior part of their supports, and that such supports are convex, compact, and disjoint. We show through several examples that our statement is nearly optimal. Moreover, under the same hypotheses, we also obtain the continuity of the transport density.

**Keywords.** *Monge problem, optimal transport, continuity, maximal transport rays.*
**MSC2000.** 49N60*,* 28A50*,* 90B06*,* 74P20*.*

## 1. Introduction

Monge transport problem dates back to 1781. It can be formulated as follows: given two probability measures $f^+$ and $f^-$ on $\mathbb{R}^N$, find a solution to the variational problem

$$\inf\left\{ \int_{\mathbb{R}^N} |x-\tau(x)|\,df^+(x) \ :\ \tau \text{ is a transport of } f^+ \text{ into } f^- \right\}. \tag{1.1}$$

Here admissible transports of $f^+$ into $f^-$ are understood as Borel functions $\tau : \mathbb{R}^N \to \mathbb{R}^N$ such that $\tau_\# f^+ = f^-$, being $\tau_\# f^+$ the push-forward measure defined as usual by $\tau_\# f^+(B) := f^+(\tau^{-1}(B))$ for every Borel set $B$ of $\mathbb{R}^N$ (notice that the class of such functions may happen to be empty). If we look at $|x-\tau(x)|$ as the cost needed to move a unit mass from $x$ into $\tau(x)$, it is clear that finding a solution to (1.1) corresponds to determining the optimal way to "transport" the mass $f^+$ into the mass $f^-$. More general versions of Monge problem can be obtained for instance replacing $\mathbb{R}^N$ by a metric space $X$, or the Euclidean norm $|x-\tau(x)|$ by a lower semicontinuous cost $c = c(x,\tau(x))$.

Since the original work by Monge [17], the transport problem has a long story. It has been reformulated almost two centuries later by Kantorovich [15, 16], and in recent years it has been investigated with a renewed interest by several authors. In fact, it does have many applications in different areas, such as economic sciences, shape optimization, fluid mechanics, evolution equations. Here we do not pretend to introduce the uninitiated reader to this wide field: the lecture notes [1], the book [23] and the PhD thesis [19] may be indicated as general references containing a detailed account on transport theory, as well as a large amount of bibliography.

The problem of establishing the existence of optimal transports in problem (1.1) is quite delicate to solve, as it cannot be attacked by the classical methods of the Calculus of Variations.

The first existence result was given by Evans and Gangbo in 1999 via a PDE approach [13], by making suitable assumptions on $f^{\pm}$. Later on, these assumptions have been weakened by Caffarelli, Feldmann and McCann [9], Trudinger and Wang [22], and Ambrosio [1]. In all these existence results, the optimal transports built by the various authors turn out to be the same, even if they are constructed in different ways. We will refer to such map $t$ as the *non-decreasing optimal transport*, where the meaning of the word "non-decreasing" will be clarified in Section 2.2.

At present, the most general result available in the literature is the one obtained in 2003 by Ambrosio [1], by following a clever proof scheme suggested by Sudakov in 1979 [21]. Such proof is based on the crucial idea that an optimal transport must move $f^+$ onto $f^-$ along a family of pairwise disjoint line segments, called *transport rays*. Thus, one is reduced to study a family of one-dimensional transport problems, which are immediate to solve; the delicate point is then to show that the corresponding optimal transports can be suitably "glued" to form an optimal transport for the $N$-dimensional problem. We recall the outcoming result, where $\mathscr{L}^N$ denotes the Lebesgue measure on $\mathbb{R}^N$.

**Theorem 1.1.** *If $f^\pm$ are measures on $\mathbb{R}^N$ with finite first order moments, and $f^+$ is absolutely continuous w.r.t. $\mathscr{L}^N$, then problem (1.1) admits a solution.*

Being this one the state of the art in existence theory, we turn our attention to regularity; in this paper, we are concerned with the following question:

"when does problem (1.1) admit a *continuous* solution $\tau$?"

Our main result reads as follows. We define $T$ the so-called *transport set*, which is obtained as the union of all transport rays (see Section 2.1 for the precise definitions); moreover for any set $A$ we denote by $A^0$ its interior.

**Theorem 1.2.** *Let $N = 2$ and let $f^\pm$ be absolutely continuous w.r.t. $\mathscr{L}^2$, with densities $\underline{f}^\pm$. Assume that*

(i) $\operatorname{spt} f^\pm$ *are compact, convex, and disjoint subsets of* $\mathbb{R}^2$*;*
(ii) $\underline{f}^\pm$ *are continuous functions on* $\operatorname{spt} f^\pm$*;*
(iii) $\underline{f}^\pm$ *are strictly positive on* $\big(\operatorname{spt} f^\pm\big)^0$*.*

*Then the non-decreasing optimal transport $t$ is continuous on $\big(T \cap \operatorname{spt} f^+\big)^0$.*

Notice that in (ii) the continuity of $\underline{f}^\pm$ is required just on $\operatorname{spt} f^\pm$ and not on the whole $\mathbb{R}^2$, so that one can consider for instance characteristic functions; notice also that (ii) must be formally read as "there exist representatives of $\underline{f}^\pm$ which are continuous functions on $\operatorname{spt} f^\pm$".

At first glance, some of the assumptions in the above result, as well as the choice of the set where the continuity is stated, might look unnatural (even though they almost coincide with the ones of Evans and Gangbo in [13], who only strengthen (ii) asking $\underline{f}^\pm$ to be Lipschitz functions). On the contrary, they are fairly optimal, as it can be deduced through a deeper analysis accomplished by several (counter)examples, that we postpone to Section 4.

As far as we are aware, Theorem 1.2 is the first regularity result for optimal transports appearing in the literature. In fact, all previous regularity results concern the *transport density*, a measure which is relevant in the study of Monge problem from many different points of view (see *e.g.* [5, 4, 13, 1]). In this paper, as a by-product of Theorem 1.2, we obtain a first continuity result for the transport density (which is far from being optimal), see Section 8.

We like to conclude this introduction by giving a quick idea of the proof of Theorem 1.2. By a disintegration argument, there exist probabilities $f^\pm_C$ defined on the closure $C$ of each maximal transport ray such that, as mentioned above, the non-decreasing optimal transport $t$ is built by gluing suitable one-dimensional optimal transports of $f^+_C$ into $f^-_C$. Through the coarea formula, we are able to write down an explicit expression of $f^\pm_C$. From such expression, it emerges that $t$ is continuous on $(\operatorname{spt} f^+ \cap T)^0$ provided both the closure $C(p)$ of the unique maximal transport ray containing $p$ and the gradient of a suitable function $\varphi$ involved in the coarea factor, depend continuously on $p$. While our use of the coarea formula is quite close to the one in [1, 2], the proofs of the continuity of the maps $p \mapsto C(p)$ and $p \mapsto \nabla\varphi(p)$

under assumptions (i)-(iii) are the most challenging part and actually the basic contribution of this work. These proofs are obtained by means of a thorough geometrical analysis of the behavior of transport rays, a special role being played by the *doubling points*, namely the possible common extremes of two or more maximal transport rays.

**Plan of the paper.** The contents are organized as follows. In Section 2 we first introduce some preliminary notation and definitions of transport theory, and then we sketch how the optimal non-decreasing transport $t$ is constructed. In Section 3 we explain in detail how the continuity of $t$ can be derived if both the maps $p \mapsto C(p)$ and $p \mapsto \nabla\varphi(p)$ are continuous. In Section 4 we show through several concrete examples that neither the assumptions of Theorem 1.2 can be substantially weakened, nor its thesis can be substantially strengthened. Section 5 contains the main geometrical lemmas, which are used in Sections 6 and 7 to prove the needed regularity of the maps $p \mapsto C(p)$ and $p \mapsto \varphi(p)$. Finally, in Section 8 we establish and discuss our continuity result for the transport density.

## 2. Notation and known facts on transport theory.

2.1. **Some preliminaries.** For convenience, we list here some notation and definitions which will be used throughout the paper. We adopt the symbol $|\cdot|$ for the Euclidean norm; convergence w.r.t. the Euclidean norm will be denoted by an arrow $\to$. We denote by $\mathrm{dist}(A, B)$ the Euclidean distance between sets $A$, $B \subseteq \mathbb{R}^2$, and by $\mathrm{diam}(A)$ the Euclidean diameter of $A$. When dealing with an absolutely continuous measure $\eta \ll \mathscr{L}^2$, in absence of ambiguity we use the same symbol $\eta$ also for its density; so, by saying that a measure $\eta$ is continuous (or in $L^p$), we mean that $\eta \ll \mathscr{L}^2$ and its density is a continuous (or $L^p$) function.

Given points $x, y \in \mathbb{R}^2$, we set:

- $xy$ the open oriented segment from $x$ to $y$;
- $[xy]$ the closed oriented segment from $x$ to $y$;
- $\overrightarrow{xy}$ the half-line with vertex in $x$ and containing $y$;
- $\overleftrightarrow{xy}$ the straight line containing both $x$ and $y$;
- $\overline{xy} = |y - x|$ the Euclidean length of $xy$.

Fix now an optimal transport $\tau$ for problem (1.1). We list some standard definitions and well-known related properties which can be found for instance in [13, 2, 19].

- $xy$ is a *transport ray* if $x \in \mathrm{spt}\, f^+$ and $\tau(x) = y$;
- $xy$ is a *maximal transport ray* if any $z \in xy$ belongs to the closure of some transport ray contained into $xy$, and $xy$ is not strictly contained into any segment with the same property;
- $xy$ is a *transport segment* if it is contained into a maximal transport ray and it has the same orientation;
- $T := \big\{z \in \mathbb{R}^2 : \ z \text{ is contained into some maximal transport ray}\big\}$ is the *transport set*.

**Remark 2.1.** The definition of transport ray depends on the choice of the optimal transport $\tau$; nevertheless, maximal transport rays (and consequently transport segments and the transport set $T$) are independent of $\tau$.

**Remark 2.2.** If $\{x_n y_n\}$ is a sequence of transport segments with $x_n \to x$ and $y_n \to y$, then $xy$ is also a transport segment. Indeed, there exists some continuous function $u$, usually called *Kantorovich potential*, such that (provided $\mathrm{spt}\, f^{\pm}$ are convex) $u(z) - u(w) = |z - w|$ if and only if $zw$ is a transport segment.

**Remark 2.3** (Cyclical monotonicity)**.** The following inequality holds for any pair $xy$, $x'y'$ of transport rays:

$$\overline{xy} + \overline{x'y'} \leq \overline{xy'} + \overline{x'y}. \tag{2.1}$$

*A fortiori*, (2.1) holds whenever $xy$ and $x'y'$ are transport segments.

**Remark 2.4** (Non-intersection property)**.** As an immediate consequence of Remark 2.3, two different maximal transport rays cannot intersect.

In view of Remark 2.4, we can define:

- $R(z)$ as the unique maximal transport ray containing a given $z \in T$;
- $\mathrm{lx}(z) \in \mathrm{spt}\, f^+$ and $\mathrm{rx}(z) \in \mathrm{spt}\, f^-$ as the endpoints of $R(z)$, that we will call respectively *left extreme* and *right extreme* of $z \in T$;
- *doubling point* as a point $z \in \mathrm{spt}\, f^+$ (or $z \in \mathrm{spt}\, f^-$) which is the left (or the right) extreme of more than one maximal transport ray.

Note that left and right extremes, and in particular doubling points, do not belong to $T$ but to its closure.

2.2. **Construction of the map $t$.** In this subsection, we introduce the non-decreasing optimal transport $t$ for problem (1.1), which is the natural candidate to be continuous. It is precisely the one guessed by Sudakov and built in all the existence proofs available in the literature.

• *Starting from this section, $t$ will always denote this special map.*

We begin with the one-dimensional case.

**Lemma 2.5** (One-dimensional case)**.** *Let $f^\pm$ be probability measures on a closed oriented segment $C$, with $f^+$ non-atomic (*i.e. *$f^+(\{x\}) = 0\ \forall x \in C$). Then the unique admissible transport of $f^+$ into $f^-$ which is non-decreasing as a function from $C$ to itself is the map $t_C(f^+, f^-)$ defined by*

$$t_C(f^+, f^-)(x) := \sup\Big\{ y \in C \ : \ f^-\big([\inf C, y]\big) \leq f^+\big([\inf C, x]\big) \Big\}. \tag{2.2}$$

*Moreover, $t_C(f^+, f^-)$ is an optimal transport for problem (1.1), and it is continuous provided* $\mathrm{spt}\, f^-$ *is connected.*

*Proof.* The uniqueness of $t_C(f^+, f^-)$ among non-decreasing admissible transports and its optimality in problem (1.1) are well-known, see *e.g.* [1, Theorem 3.1]. The last part of the statement concerning the continuity of $t_C(f^+, f^-)$ is easily checked and we leave it to the reader. □

**Remark 2.6.** Under the assumptions of Lemma 2.5, there is also a unique non-increasing admissible transport $\tilde{t}_C(f^+, f^-)$ of $f^+$ into $f^-$. We point out that, if both $f^\pm$ are non-atomic and with connected support, the maps $t_C(f^+, f^-)$ and $\tilde{t}_C(f^+, f^-)$ are the only invertible continuous transports of $f^+$ into $f^-$.

We sketch now the construction of $t$, under the assumptions of Theorem 1.1, via a "gluing" of one-dimensional transports along the family of maximal transport rays (see [1] for all the details here omitted). Let $\mathcal{S}$ be the set of bounded oriented closed segments in $\mathbb{R}^2$, endowed with the natural metric $d\big([xy], [x'y']\big) = \overline{xx'} + \overline{yy'}$. Define $r : T \to \mathcal{S}$ as the map which associates with every $x \in T$ the closure of the maximal transport ray $R(x)$; this map turns out to be Borel, see [2]. Set now $\lambda := r_\# f^+$; then $\lambda$ is a measure on $\mathcal{S}$, which also coincides with $r_\# f^-$. Disintegrating $f^\pm$ w.r.t. $r$ (see [3] for the definition and the main properties of disintegration), we can write $f^\pm = f^\pm_C \otimes \lambda$ for a family of probability measures $\{f^\pm_C\}_{C \in \mathcal{S}}$ on $\mathbb{R}^2$; this identity has to be understood in the usual sense of measure theory, that is

$\langle f^{\pm}, g\rangle = \int_{\mathcal{S}} \langle f^{\pm}_C, g\rangle \, d\lambda(C)$ for any bounded continuous function $g$ on $\mathcal{S}$. Under the assumptions of Theorem 1.1, it turns out that $f^+_C \ll \mathscr{H}^1 \llcorner C$ for $\lambda-$a.e. $C \in \mathcal{S}$ and that left extremes are $f^+-$negligible, so $f^+(\mathbb{R}^2 \setminus T) = 0$. Finally, the map $t$ is given by

$$t(x) = t_{C(x)}\big(f^+_{C(x)}, f^-_{C(x)}\big)(x) \quad \text{for } f^+\text{−a.e. } x, \tag{2.3}$$

where $C(x)$ is the closure of $R(x)$, and $t_{C(x)}\big(f^+_{C(x)}, f^-_{C(x)}\big)$ is defined according to (2.2). Note that the identity (2.3) is meaningful because $T$ is of full $f^+-$measure.

## 3. Sufficient conditions for the continuity of the optimal map in Theorem 1.1

• *In this section, in order to avoid confusion in some formulas, we use the different symbol* $\underline{\eta}$ *for the density of any absolutely continuous measure* $\eta$.

In view of (2.3), the regularity of $t$ is related to the explicit expression of the probabilities $f^{\pm}_C$. We will prove that their densities $\underline{f}^{\pm}_C$ are given by (3.2) in Lemma 3.3; thanks to such characterization, we will be able to determine sufficient conditions for the continuity of $t$. Actually, the proof of Theorem 1.2 will be obtained through the crucial Proposition 3.4 below.
A major role in the sequel will be played by the hyperplane $H$ and the function $\varphi$ defined as follows.

**Definition 3.1.** Under assumption (i) of Theorem 1.2, let $H \subseteq \mathbb{R}^2$ be an hyperplane (that is, a straight line) separating by Hahn-Banach Theorem the disjoint convex sets $\operatorname{spt} f^+$ and $\operatorname{spt} f^-$. Then set

$$\varphi : T \to H, \quad \varphi(x) := H \cap R(x)\,. \tag{3.1}$$

Notice that the fibers of $\varphi$ are precisely the maximal transport rays, that is, $\varphi^{-1}(p) = R(p)$ for any $p \in \varphi(T) = H \cap T$.

• *In the sequel, we fix an orthonormal basis* $\{e_1, e_2\}$ *such that* $H = \{x_1 = 0\}$ *and* $\operatorname{spt} f^+ \subseteq \{x_1 < 0\}$. *Moreover, given a segment $xy$ such that the line $\overleftrightarrow{xy}$ is not parallel to $H$, a point $z$ will be said to be* above *(resp.* below*) the segment $xy$ (or the half-line $\overrightarrow{xy}$, or the line $\overleftrightarrow{xy}$) if $z$ lies in the closed upper (resp. lower) half-plane delimited by $\overleftrightarrow{xy}$.*

**Lemma 3.2** (Continuity of the map $\varphi$)**.** *The function $\varphi$ defined by* (3.1) *is continuous on $T$.*

*Proof.* Let $\{x_n\} \subseteq T$ be a sequence with $x_n \to x \in T$, and assume by compactness that $\varphi(x_n) \to p$. Since $x_n\varphi(x_n)$ is a transport segment for any $n$, by Remark 2.2 $xp$ is a transport segment too; moreover, since $H$ is closed, $p \in H$. Hence $p = \varphi(x)$. □

With the help of the function $\varphi$, we can now characterize the measures $f^{\pm}_C = \underline{f}^{\pm}_C \mathscr{H}^1 \llcorner C$ (in fact, both $f^{\pm}_C \ll \mathscr{H}^1 \llcorner C$ when both $f^{\pm} \ll \mathscr{L}^2$, see Subsection 2.2).

**Lemma 3.3** (Characterization of $\underline{f}^{\pm}_C$)**.** *Under the assumptions of Theorem 1.2, if $\varphi$ is of class* $\mathrm{C}^1$ *on $T^0$ and* $\inf_{x\in T^0} |\nabla\varphi(x)| > 0$, *then*

$$\underline{f}^{\pm}_{C(x)}(x) = \Big\{ \int_{C(x)} \frac{\underline{f}^{\pm}(z)}{|\nabla\varphi|(z)}\, d\mathscr{H}^1(z) \Big\}^{-1} \frac{\underline{f}^{\pm}(x)}{|\nabla\varphi|(x)} \qquad \textit{for } f^+\textit{−a.e. } x \in \mathbb{R}^2\,. \tag{3.2}$$

*Proof.* By the regularity assumption on $\varphi$, the coarea formula (see *e.g.* [3]) gives the decomposition

$$|\nabla\varphi| \mathscr{L}^2 = \mathscr{H}^1 \llcorner \varphi^{-1}(p) \otimes \mathscr{H}^1 \llcorner H\,. \tag{3.3}$$

On the other hand, there is a natural identification between maximal transport rays and points of $H\cap T$, *i.e.* there exists a Borel function $i:\mathcal{S}\to H$ such that $i(C)=H\cap C$ whenever $C$ is the closure of a maximal transport ray. Then the disintegration $f^\pm = f^\pm_C\otimes\lambda$ may be rewritten as

$$\underline{f}^\pm\mathscr{L}^2 = f^\pm = f^\pm_{i^{-1}(p)}\otimes i_\#\lambda = \underline{f}^\pm_{C(p)}\mathscr{H}^1\llcorner C(p)\otimes\eta\,, \tag{3.4}$$

where $\eta := i_\#\lambda(=\varphi_\# f^+ = \varphi_\# f^-)$ is a measure on $H$.
Now, multiplying (3.3) by $\underline{f}^\pm$ and (3.4) by $|\nabla\varphi|$, we get

$$\underline{f}^\pm\mathscr{H}^1\llcorner C(p)\otimes\mathscr{H}^1\llcorner H = |\nabla\varphi|\underline{f}^\pm_{C(p)}\mathscr{H}^1\llcorner C(p)\otimes\eta\,.$$

From this equality, since the fibers $C(p)$ are essentially disjoint, one may argue as in [19, Lemma 6.14] to deduce that the two disintegrations appearing in the above equation agree, in the sense that their left (resp. right) terms coincide up to the multiplication (resp. division) by a factor $h:H\to\mathbb{R}^+$. Formally,

$$\underline{f}^\pm(x) = h\big(\varphi(x)\big)|\nabla\varphi(x)|\underline{f}^\pm_{C(x)}(x)\quad\text{for } f^\pm\text{−a.e. } x\qquad\text{and}\qquad \eta = h\mathscr{H}^1\llcorner H\,.$$

Since by hypothesis $\inf_{x\in T^0}|\nabla\varphi(x)|>0$, and $\underline{f}^\pm_C$ are probability measures, we deduce that $h(p) = \int_{C(p)}\big(\underline{f}^\pm/|\nabla\varphi|\big)\,d\mathscr{H}^1$. Finally, (3.2) follows by noticing that $h(p)>0$ for $\eta-$a.e. $p$ (so that $h(\varphi(x))>0$ for $f^\pm-$a.e. $x$). □

Now, we can give two sufficient conditions for the continuity of $t$ which arise in a natural way from (3.2).

**Proposition 3.4** (Two sufficient conditions for the continuity of $t$)**.** *Under the assumptions of Theorem 1.2, the optimal transport $t$ is continuous on $\big(T\cap\operatorname{spt} f^+\big)^0$ provided the following two conditions are satisfied:*

(a) *the maps* lx *and* rx *are continuous on $T^0$;*
(b) *the map $\varphi$ is of class* $\mathrm{C}^1$ *on $T^0$, and* $\inf_{x\in T^0}|\nabla\varphi|>0$.

*Proof.* In view of (2.3) and (2.2), the transport $t$ is continuous on $\big(T\cap\operatorname{spt} f^+\big)^0$ if the maps

$$T^0\ni x\mapsto \underline{f}^\pm_{C(x)}(x)\in\mathbb{R}^+,\qquad\qquad T^0\ni x\mapsto C(x)\in\mathcal{S}$$

are continuous. This condition is immediately satisfied. Indeed, thanks to (b) the characterization (3.2) holds; moreover, all the functions involved in the right-hand side of (3.2) are continuous: in particular, the integrals in (3.2) are made over $C(x)\cap\operatorname{spt} f^\pm = [\mathrm{lx}(x)\mathrm{rx}(x)]\cap\operatorname{spt} f^\pm$, and these segments vary continuously by (a) and taking into account the regularity of $\partial\operatorname{spt} f^\pm$. □

Let us conclude by remarking that the results of this section hold more in general (with the same proofs) in the $N-$dimensional case. However, in this paper the validity of the sufficient conditions of Proposition 3.4 is proved in the special case $N=2$.
We also point out that if –instead of an hyperplane– there exists a $\mathrm{C}^1$ curve which separates $\operatorname{spt} f^+$ and $\operatorname{spt} f^-$ and intersects precisely once each maximal transport ray, then a similar version of (3.2) can be written in terms of the corresponding function $\varphi$. Therefore, Proposition 3.4 remains true.

## 4. Discussion on the statement of Theorem 1.2: examples

This section is entirely devoted to show that the statement of Theorem 1.2 is fairly optimal in dimension $N=2$. Concerning the assumptions, we provide several examples in which, due to the fact that one of them is removed, the transport $t$ fails to be continuous. In all the cases presented, the same procedure can be adopted to show that $\tilde t$ is not continuous, being $\tilde t$ the optimal transport obtained in a similar way as in (2.3), by gluing the non-increasing one-dimensional transports $\tilde t_C(f^+_C, f^-_C)$ mentioned in Remark 2.6;

more generally, one can easily convince himself that there does not exist *any* continuous optimal transport. Concerning the thesis we show, again through examples, that the continuity of $t$ in $(T \cap \operatorname{spt} f^+)^0$ is satisfactory, as in general there is no continuity on $T \cap \operatorname{spt} f^+$.
In many of our examples we make use of Lemma 4.1 below as a crucial tool to determine, in a formally correct way, the optimal transport $t$ (which in many cases can be intuitively guessed). This result is a special case of the result in [20], which extends Lemma 8.1 in [2].

**Lemma 4.1.** *Let $Q^+ := [-3,-1] \times [-1,1]$ and $Q^- := [1,3] \times [-1,1]$, and assume that the supports of $f^\pm$ are contained into $Q^\pm$ respectively. Let $\mathscr{R} := \{R_i\}_{i \in I}$ be a covering of $Q^+ \cup Q^-$ made by pairwise disjoint open segments with left (resp. right) extremes in $\partial Q^+ \setminus \big(\{-1\} \times [-1,1]\big)$ (resp. $\partial Q^- \setminus \big(\{1\} \times [-1,1]\big)$). Let $E_i$ be the upper half-plane above $R_i$. If*

$$f^+(E_i) = f^-(E_i) \qquad \forall i \in I\,, \tag{4.1}$$

*then each maximal transport ray is contained into some $R_i \in \mathscr{R}$ .*

4.1. **About assumption (i).** The hypothesis that the supports of the sources are compact and disjoint is quite standard in the literature on transport theory, and it is not so strong. In fact, the key point in assumption (i) of Theorem 1.2 is the convexity request. Clearly, the map $t$ can lose continuity if the intersection of some maximal transport ray with $\operatorname{spt} f^-$ is not connected (recall Lemma 2.5 about the one-dimensional case and see also Example 4.2).
It is less immediate why, instead of simply asking $R \cap \operatorname{spt} f^-$ to be connected for any maximal transport ray $R$, we have to require the stronger hypothesis of convexity for both $\operatorname{spt} f^-$ and $\operatorname{spt} f^+$. The point is that a necessary ingredient for the continuity of $t$ is the continuity of lx and rx, which does not remain true without a convexity assumption for both $\operatorname{spt} f^+$ and $\operatorname{spt} f^-$ (see Example 4.3). Moreover, such assumption leads to some regularity of $\partial \operatorname{spt} f^\pm$, whose absence may prevent the continuity of $t$ (see Example 4.4); the regularity of $\partial \operatorname{spt} f^\pm$ has been already invoked in the proof of Proposition 3.4 and will be widely used in the sequel.

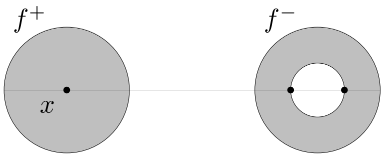


FIGURE 1. Example 4.2

**Example 4.2.** If $\operatorname{spt} f^+ \subseteq Q^+$ and $\operatorname{spt} f^- \subseteq Q^-$ are respectively a disk and an annulus as in Figure 1, one can choose densities $f^\pm$ such that, by Lemma 4.1, maximal transport rays are horizontal. Hence $t$ is discontinuous, for instance at the center $x$ of the disk.

**Example 4.3.** Let $f^+$ and $f^-$ be the characteristic functions respectively of the union of two half-disks and of a disk as in Figure 2.a. By Lemma 4.1, maximal transport rays are horizontal, so that $t$ is obtained by gluing two different horizontal translations, and clearly it is not continuous. In particular, for $x$, $y$ and $z$ as in the figure, if $\{x_n\}$ is a sequence of points in the upper (resp. lower) half of $\operatorname{spt} f^+$ converging to $x$, then the limit of $t(x_n)$ is $z$ (resp. $y$). The same example, exchanging the role of $f^+$ and $f^-$, enlightens the necessity of the convexity assumption also for $\operatorname{spt} f^-$.

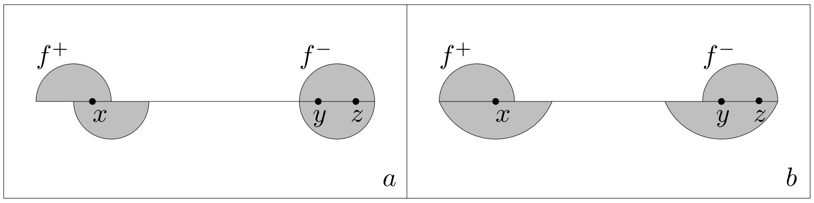


FIGURE 2. Examples 4.3 and 4.4

**Example 4.4.** Let $f^+$ and $f^-$ be the characteristic functions of two sets as in Figure 2.b . Then the very same facts as in Example 4.3 are true, but the non-continuity of $t$ is due now only to the non-regularity of $\partial \mathrm{spt}\, f^\pm$ (while lx and rx are continuous and $\varphi$ is $\mathrm{C}^1$).

4.2. **About assumption (ii).** Clearly, this assumption is not "necessary", since also highly non-regular sources may happen to admit a continuous transport. On the other hand, it is easy to imagine how the lack of continuity of $f^\pm$ may produce a discontinuity of the map $t$: a simple example is constructed below.

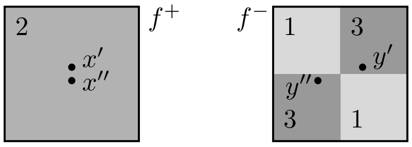


FIGURE 3. Example 4.5

**Example 4.5.** Let $f^+ \equiv 2$ on $Q^+$ and let $f^-$ be the function on $Q^-$ whose value is 1 or 3 as indicated in Figure 3. Then the covering $\mathscr{R}$ of $Q^+ \cup Q^-$ made by horizontal segments satisfies the assumptions of Lemma 4.1. Hence, $t$ is easily written as

$$t(x) = \begin{cases} (2x_1 + 7, x_2) & \text{if } x_1 < -5/2 \text{ and } x_2 > 0, \\ (2/3\, x_1 + 11/3, x_2) & \text{if } x_1 > -5/2 \text{ and } x_2 > 0, \\ (2/3\, x_1 + 3, x_2) & \text{if } x_1 < -3/2 \text{ and } x_2 < 0, \\ (2x_1 + 5, x_2) & \text{if } x_1 > -3/2 \text{ and } x_2 < 0. \end{cases}$$

The map $t$ is discontinuous on the line $\{x_2 = 0\}$ (notice that $t(x') = y'$ and $t(x'') = y''$ in Figure 3).

**Remark 4.6.** We remark that, in the above example, the lack of continuity for $t$ is essentially due to the discontinuity of $f^-$ on the line $\{x_2 = 0\}$: indeed, the discontinuity of $f^-$ on the line $\{x_1 = 2\}$ can be easily removed, still maintaining the discontinuity of $t$. More in general, since in the one-dimensional case (Lemma 2.5) the continuity of the sources is not needed, assumption (ii) of Theorem 1.2 could be weakened allowing discontinuities of $f^\pm$ *just along* transport rays.

4.3. **About assumption (iii).** It is easy to produce cases where the lack of assumption (iii) leads to some discontinuity of $t$, as in the next example.

**Example 4.7.** Let $f_1^\pm$ and $f_2^\pm$ be probabilities respectively on $Q^\pm \cap \{x_2 \geq 0\}$ and $Q^\pm \cap \{x_2 \leq 0\}$, with continuous densities $\underline{f}_{1,2}^\pm$ vanishing on $Q^\pm \cap \{x_2 = 0\}$; a suitable choice of $\underline{f}_{1,2}^\pm$ can produce transports $t_{1,2}$ of $f_{1,2}^+$ into $f_{1,2}^-$ which do not agree on $Q^\pm \cap \{x_2 = 0\}$. Then, if $f^\pm$ are obtaining by pasting $f_1^\pm$ and $f_2^\pm$, assumption (iii) is violated and $t$ is not continuous on $Q^+ \cap \{x_2 = 0\}$.

We like now to present a delicate example: it enlightens how the lack of assumption (iii) can prevent the $C^1$−smoothness of the map $\varphi$, which is an irremissible ingredient for the continuity of $t$.

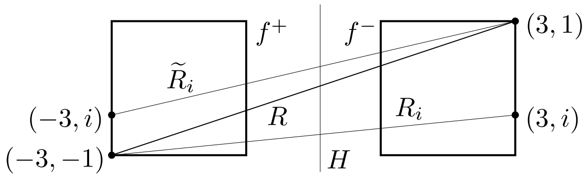


Figure 4. Example 4.8

**Example 4.8.** For any $i \in [-1,1]$, let $R_i$ and $\widetilde{R}_i$ be the open segments joining $(-3,-1)$ with $(3,i)$, and $(-3,i)$ with $(3,1)$ respectively (see Figure 4). We want to find suitable functions $f^\pm$ in order to apply Lemma 4.1 with $\mathscr{R} := \{R_i, \widetilde{R}_i\}_{i\in[-1,1]}$, since in this case the function $\varphi$ will be not differentiable on $R := R_1 \equiv \widetilde{R}_{-1}$. Indeed,

$$\varphi(x) = \begin{cases} \left(0, \dfrac{3x_2 - x_1}{3 + x_1}\right) & \text{if } x_1 \geq 3x_2, \\ \left(0, \dfrac{3x_2 - x_1}{3 - x_1}\right) & \text{if } x_1 < 3x_2. \end{cases}$$

For the sake of simplicity, we look for functions $f^\pm$ constant on each $Q^\pm \cap R_i$ and $Q^\pm \cap \widetilde{R}_i$; then, we denote by $g^\pm(i)$ (resp. $\tilde{g}^\pm(i)$) the constant value of $f^\pm$ on $Q^\pm \cap R_i$ (resp. $Q^\pm \cap \widetilde{R}_i$). One can easily check that condition (4.1) is satisfied if and only if, for all $i \in [-1,1]$,

$$g^+(i) = 5g^-(i) \qquad \text{and} \qquad \tilde{g}^-(i) = 5\tilde{g}^+(i)\,. \tag{4.2}$$

If (4.2) holds, $t$ does not depend on the particular choice of $g^\pm$, $\tilde{g}^\pm$. Indeed, by Lemma 4.1, it can be explicitly determined as $t(x) = y$, where

$$\begin{cases} y_1 = \sqrt{16 + 5(x_1+3)^2} - 3 \quad \text{and} \quad y_2 = \dfrac{(y_1+3)(1+x_2)}{3+x_1} - 1\,, & \text{if } x_1 > 3x_2, \\ y_1 = 3 - \sqrt{\dfrac{(3-x_1)^2 - 16}{5}} \quad \text{and} \quad y_2 = \dfrac{(y_1-3)(1-x_2)}{3-x_1} + 1\,, & \text{if } x_1 < 3x_2. \end{cases}$$

Thus, $t$ is not continuous on $R \cap Q^+$. We remark that, if (4.2) holds and $f^\pm$ are continuous, then $g^\pm$ and $\tilde{g}^\pm$ must be continuous on $[-1,1]$, with $g^\pm(1) = \tilde{g}^\pm(-1)$. So, again by (4.2), $f^+$ and $f^-$ are forced to vanish on $R$, and assumption (iii) of Theorem 1.2 is violated.

4.4. **About the thesis.** Here we briefly discuss why the continuity of $t$ in $(T \cap \operatorname{spt}(f^+))^0$ is a nearly optimal assert. First, one should not be satisfied with an almost-everywhere continuity statement; for instance, it would not be reasonable to consider "continuous" the transport $t$ of Example 4.7, although it lacks continuity just in a negligible set.
Now, one is led to investigate the pointwise continuity on the set $T \cap \operatorname{spt} f^+$, where $t$ is naturally defined. Actually, the existence of a continuous extension of $t$ to a larger set should not be expected, since doubling points (where continuity is always lost) arise also in "very good" situations, as in the next example.

**Example 4.9.** If $f^\pm$ are constant densities respectively on a circular sector and on an annulus sector as in Figure 5, Lemma 4.1 allows to prove that, as one expects, optimal transports move the mass radially. Consequently, $t$ is continuous on $T \cap \operatorname{spt} f^+$, but (as well as any other optimal transport) it cannot be continuously extended at the doubling point $x$ indicated in the figure.

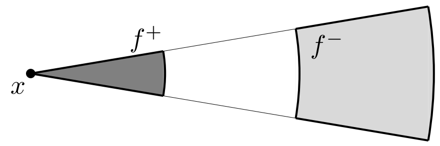


Figure 5. Example 4.9

We finally explain why the continuity of $t$ is stated on the *interior part* of $T \cap \operatorname{spt} f^+$. First notice that, since the map lx is continuous (see Section 6), the set $T$ is open within $\operatorname{spt} f^+$, thus taking the interior part has the only effect to rule out $\partial \operatorname{spt} f^+$:

$$\big(T \cap \operatorname{spt} f^+\big)^0 = T \cap \big(\operatorname{spt} f^+\big)^0 = T \cap \operatorname{spt} f^+ \setminus \partial \operatorname{spt} f^+ .$$

In many cases (*e.g.*, whenever $\operatorname{spt} f^+$ is strictly convex), $(T \cap \operatorname{spt} f^+)^0 = T \cap \operatorname{spt} f^+$, so that taking the interior part is ininfluent. Otherwise, it may happen that $t$ is continuous on $(T \cap \operatorname{spt} f^+)^0$ but not on the whole $T \cap \operatorname{spt} f^+$, though all the assumptions of Theorem 1.2 are satisfied: we show this phenomenon in Example 4.12 below. For the construction of such example we need as a fundamental "brick" the preliminary Example 4.10.

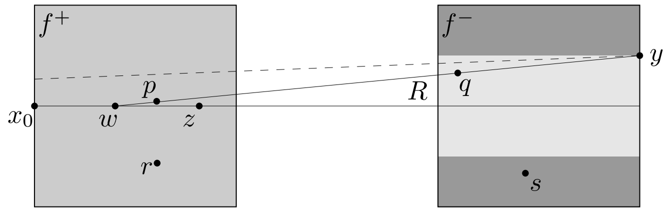


Figure 6. Example 4.10

**Example 4.10.** Let $f^+ \equiv 1$ on $Q^+$ and, for a fixed $\varepsilon > 0$, let $f^-$ be a continuous function such that $f^-(x_1, x_2) = f^-(x_1, -x_2)$ and

$$f^-(x) = \begin{cases} \varepsilon & \text{if } |x_2| \leq 1/2, \\ 2-\varepsilon & \text{if } 1/2 + \varepsilon^2 \leq |x_2| \leq 1, \end{cases}$$

see Figure 6. Fix $x_0 \equiv (-3, 0)$. By a careful application of Lemma 4.1, one can prove the following claim (shown in [20]).

*There exist $z \equiv (z_1(\varepsilon), 0)$, with $z_1(\varepsilon) \to 1$ as $\varepsilon \to 0$, such that the segment $[x_0 z]$ consists of doubling points, and the set of left extremes is $\big(\{-3\} \times [-1, 1]\big) \cup [x_0 z]$.*

Here we limit ourselves to prove that, for $\varepsilon \ll 1$:

(1) there exist left extremes in the interior of $Q^+$;

(2) there exist points $p$, $r \in T$ such that $p_1 = r_1$ and $q_1 < s_1$, being $q := t(p)$ and $s := t(r)$.

To show (1), fix $y \equiv (3, 1/2)$, and let $w$ be the left extreme of a maximal transport ray $R$ having $y$ as the right extreme (*a priori*, $R$ could be not unique). Assume by contradiction that $w \in \partial Q^+$; then, by symmetry and non-intersection of transport rays (Remark 2.4), necessarily $w_2 \geq 0$ (for instance, $R$ could be the dashed line in figure). Hence, again by non-intersection, $f^+(E) = f^-(E)$, being $E$ the set of points lying below $wy$ and above $\{x_2 = 0\}$: for $\varepsilon \ll 1$, this is impossible by mass balance.

To show (2), we begin by noticing that the line $\{x : \ x_1 = w_1\}$ cannot be made by left extremes (indeed, by (1), $w_1 > -3$). Therefore, we may find a point $p \in R$ arbitrarily close to $w$ and a point $r \in T$ with $r_1 = p_1$ such that $\overline{\mathrm{lx}(r)r} \gg \overline{wp}$. Then, letting $q := t(p)$ and $s := t(r)$, for $\overline{wp} \ll 1$ one has $q_1 < s_1$ because, being $t$ non-decreasing on $R$ and on $\mathrm{lx}(r)\mathrm{rx}(r)$, $|q_1 - 1| \ll |s_1 - 1|$.

**Remark 4.11.** The previous example enlightens the possible presence of doubling points in the *interior part* of $\mathrm{spt}\, f^+$, which might be not so intuitive at first glance. This phenomenon makes the proofs of the regularity of the maps lx, rx considerably more delicate than it would be in absence of doubling points (see Section 6).

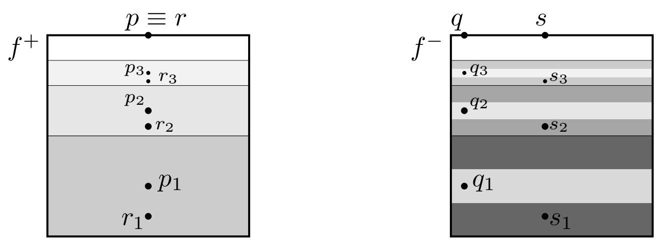


FIGURE 7. Example 4.12

**Example 4.12.** We define $f^+$ and $f^-$ as the pasting of countably many "bricks" built as in the Example 4.10, see Figure 7 where only the first three bricks are drawn: the height of the $n-$th brick is $2^{-n}$, and the densities $f^+$ and $f^-$ are multiplied by the factor $2^{-n}$ w.r.t. those of the fundamental brick (of course, one needs to join the different bricks continuously, in a region of height much smaller than $2^{-n}$). Consistently with the fundamental brick, we have two sequences of points $\{p_n\}$ and $\{r_n\}$ with $t(p_n) = q_n$ and $t(r_n) = s_n$; but $p := \lim_n p_n = \lim_n r_n =: r \in \partial Q^+$, while $q := \lim_n q_n \neq \lim_n s_n =: s$. Therefore, $t$ cannot be extended continuously to $\partial \mathrm{spt}\, f^+$, even though it is a continuous map on $\big(T \cap \mathrm{spt}\, f^+\big)^0$ according to Theorem 1.2.

**Remark 4.13.** As one can guess from Example 4.12, the element which may prevent the continuity of $t$ on $\partial \mathrm{spt}\, f^+$ is the possible presence of zeros of $f^\pm$ on $\partial \mathrm{spt}\, f^\pm$. Indeed, the same arguments of our proof of Theorem 1.2 show that, if (iii) is replaced by the stronger assumption
*(iii)′ $f^\pm$ are strictly positive on* $\mathrm{spt}\, f^\pm$,
then $t$ is continuous on the whole $T \cap \mathrm{spt}\, f^+$.

## 5. GEOMETRICAL LEMMAS

• *From now on, we tacitly assume that the hypotheses of Theorem 1.2 hold true. In particular, all the results in the remaining of the paper are stated under these hypotheses, even though they are not explicitly written.*

In this section we establish some preliminary lemmas, which will enable us to show that the sufficient conditions of Proposition 3.4 are satisfied. We begin by defining what we mean by "triangular" and "trapezoidal" sets.

**Definition 5.1.** Let $Q$ and $R$ be two distinct half-lines originating from the same point $x \in \mathrm{spt}\, f^+$ (resp. $x \in \mathrm{spt}\, f^-$). We call *triangular set determined by $Q$ and $R$* the intersection of $\mathrm{spt}\, f^+$ (resp. $\mathrm{spt}\, f^-$) with the half-cone delimited by $Q$ and $R$ (see Figure 8).

**Definition 5.2.** Let $Q$ and $R$ be two distinct maximal transport rays; we call *left (resp. right) trapezoidal set determined by* $Q$ *and* $R$ the intersection of $\mathrm{spt}\, f^+$ (resp. $\mathrm{spt}\, f^-$) with the set

$$\Big\{ z \in T : \varphi(z) \text{ belongs to the segment joining } Q \cap H \text{ and } R \cap H \Big\}$$

(see Figure 8).

We point out that a right trapezoidal set is exactly the image through the transport $t$ of the left trapezoidal set determined by the same rays. Further, in view of the continuity of rx that we will prove in Theorem 6.1, the set $\mathrm{rx}([Q\cap H, R\cap H])$ is a continuous curve (as drawn in Figure 8), and it is part of the boundary of the right trapezoidal set (the same holds replacing rx by lx). We will often use trapezoidal sets determined by maximal transport rays crossing at a doubling point (such kind of trapezoidal sets are in fact special triangular sets).

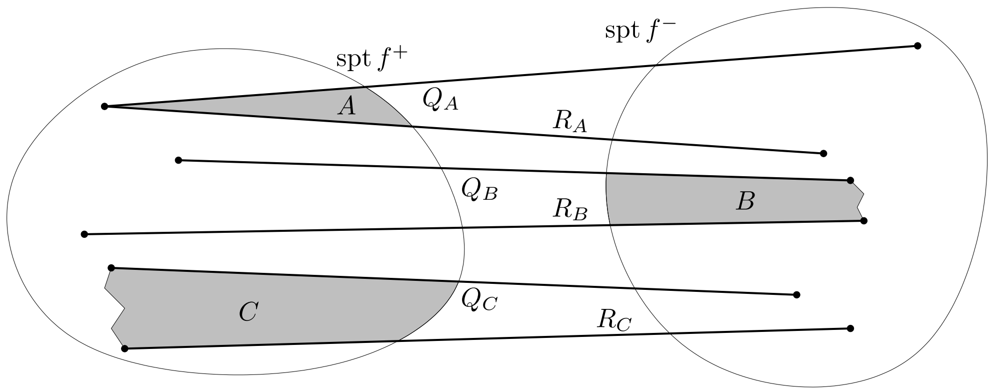


FIGURE 8. A triangular set $A$, a right trapezoidal set $B$, and a left trapezoidal set $C$

**Definition 5.3.** We say that a sequence of triangular sets determined by $Q_n$ and $R_n$ is *thinning* if the half-lines $Q_n$ and $R_n$ converge to the same half-line. Similarly, we say that a sequence of left trapezoidal sets (or of right trapezoidal sets) determined by $Q_n$ and $R_n$ is *thinning* if the maximal transport rays $Q_n$ and $R_n$ converge to segments lying on the same straight line.

Now, given a thinning sequence of triangular or trapezoidal sets, we want to provide an asymptotic estimate of their masses. To this aim, we introduce the following additional definitions.

**Definition 5.4.** Let $x \in \mathrm{spt}\, f^+$ and $r \in \partial\mathrm{spt}\, f^+$; for any $\sigma \in [0, \overline{xr}]$, we denote by $x_\sigma$ the point of $[xr]$ such that $\overline{xx_\sigma} = \sigma$, and we set

$$I^+(x,r) := \frac{1}{\overline{xr}} \int_0^{\overline{xr}} f^+(x_\sigma)\sigma\, d\sigma \quad \text{if} \quad x \neq r \text{ (and 0 otherwise).}$$

In the analogous way, we define $I^-(x,r)$ for $x \in \mathrm{spt}\, f^-$ and $r \in \partial\mathrm{spt}\, f^-$.

**Definition 5.5.** Let $R$ be a maximal transport ray and $\lambda > 0$; set $z := R \cap H$, $r := R \cap \partial\mathrm{spt}\, f^+$ and $v := R \cap \partial\mathrm{spt}\, f^-$. For any $\sigma \in [0, \overline{z\mathrm{rx}(z)}]$, we denote by $z_\sigma$ the point of $[z\mathrm{rx}(z)]$ such that $\overline{zz_\sigma} = \sigma$. Then we set

$$J^-(R,\lambda) := \int_{\overline{zv}}^{\overline{z\mathrm{rx}(z)}} f^-(z_\sigma)\rho^-(\sigma)\, d\sigma\,,$$

where $\rho^- = \rho^-(R,\lambda)$ is the affine function whose values in $-\overline{zr}$ and $\overline{zv}$ are, respectively, $1/\lambda$ and 1, *i.e.*,

$$\rho^-(\sigma) := 1 + \frac{\sigma - \overline{zv}}{\overline{rv}}\left(1 - \frac{1}{\lambda}\right).$$

In the analogous way, we define $\rho^+(\sigma)$ and $J^+(R,\lambda)$.

**Remark 5.6.** One can check that, keeping $r$ and $\overleftrightarrow{xr}$ fixed, if $x$ is moved to the left along $\overleftrightarrow{xr}$, $I^+(\cdot, r)$ strictly increases. On the other hand it is clear that, keeping $R$ fixed, due to the monotonicity of the function $\rho^-$ w.r.t. $\lambda$ for any $\sigma$ in the interval $[\overline{zv}, \overline{z\mathrm{rx}(z)}]$, $J^-(R,\cdot)$ is a strictly increasing function of $\lambda$. The analogous remark holds for $I^-$ and $J^+$.

**Lemma 5.7** (Mass of thinning triangular and trapezoidal sets)**.** *Let $\{A_n\}$ be a thinning sequence of triangular sets determined by $\overrightarrow{x_n r_n}$ and $\overrightarrow{x_n q_n}$ with $x_n \in \operatorname{spt} f^+$ and $r_n \in \partial \operatorname{spt} f^+$; define $r'_n$ as the point of $\overrightarrow{x_n q_n}$ whose orthogonal projection onto $\overrightarrow{x_n r_n}$ is $r_n$, and set $\varepsilon_n := \overline{r_n r'_n}$. Assume that $x_n \to x \in \big(\operatorname{spt} f^+\big)^0$ and $r_n \to r$. Then*

$$f^+(A_n) = \varepsilon_n I^+(x,r) + o\big(f^+(A_n)\big)\,. \tag{5.1}$$

*Let $\{B_n\}$ be a thinning sequence of right trapezoidal sets determined by $R(y_n)$ and $R(z_n)$, with $y_n$, $z_n \in H \cap T$. We set $s_n := R(z_n) \cap \partial \operatorname{spt} f^+$, $u_n := R(z_n) \cap \partial \operatorname{spt} f^-$, $s'_n$ and $u'_n$ the points of $R(y_n)$ whose orthogonal projections onto $R(z_n)$ are $s_n$ and $u_n$, $\delta_n := \overline{s_n s'_n}$ and $\lambda_n := \overline{u_n u'_n}/\overline{s_n s'_n}$. Assume that the common limit $z$ of $\{z_n\}$ and $\{y_n\}$ belongs to $H \cap T^0$, that $\lim_n u_n \neq \mathrm{rx}(z)$, and that $\mathrm{rx}$ is continuous at $z$. Then*

$$f^-(B_n) = \lambda_n \delta_n J^-(R(z), \lambda_n) + o\big(f^-(B_n)\big)\,. \tag{5.2}$$

*The analogous results hold for thinning triangular sets in $\operatorname{spt} f^-$ and for thinning left trapezoidal sets in $\operatorname{spt} f^+$.*

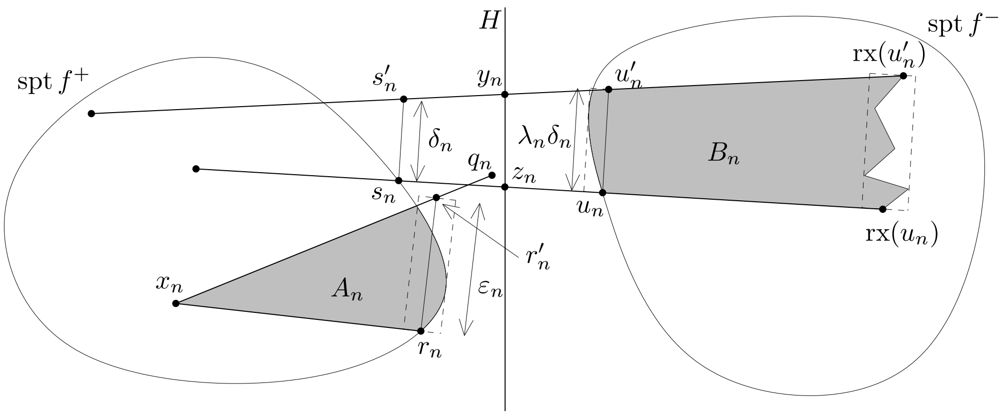


Figure 9. Situation in Lemma 5.7

*Proof.* By assumption, $f^\pm$ are strictly positive in the interior of their supports, $x \neq r$, $\mathrm{rx}(z) \neq \lim_n u_n$, $x$ belongs to the interior of $\operatorname{spt} f^+$ and $R(z)$ intersects $\big(\operatorname{spt} f^-\big)^0$. Therefore, there exist positive constants $C^\pm$ such that $f^\pm > C^\pm$ respectively in a subset of $A_n$ whose area is of order $\varepsilon_n$ and in a subset of $B_n$ whose area is of order $\lambda_n\delta_n$. Therefore, it is enough to prove (5.1) with $o\big(f^+(A_n)\big)$ replaced by $o(\varepsilon_n)$, and (5.2) with $o\big(f^-(B_n)\big)$ replaced by $o(\lambda_n\delta_n)$.

Define $E_n$ as the triangle $x_n r_n r'_n$ (notice that, depending on the geometry, it may happen that $A_n \subseteq E_n$, as well as $E_n \subseteq A_n$, as well as none of these: Figure 9 shows this third possibility). The symmetric difference $\big(A_n \setminus E_n\big) \cup \big(E_n \setminus A_n\big)$ can be encapsulated into a rectangle (dashed in Figure 9) having one side of length $\approx \varepsilon_n$ and the other one of infinitesimal length, due to the regularity of $\partial \mathrm{spt}\, f^+$. Thus, $\mathscr{L}^2\big(A_n \setminus E_n\big) + \mathscr{L}^2\big(E_n \setminus A_n\big) = o(\varepsilon_n)$, which by the boundedness of $f^+$ implies $f^+(A_n) = f^+(E_n) + o\big(\varepsilon_n\big)$. Therefore, (5.1) is proved if we show that

$$f^+(E_n) = \varepsilon_n I^+(x, r) + o(\varepsilon_n)\,. \tag{5.3}$$

In a similar way, we consider the trapezoid $D_n$ with vertexes $u_n, u_n', \mathrm{rx}(u_n')$, and $\mathrm{rx}(u_n)$. By the moment, we have not yet proved that the curve $\{\mathrm{rx}(c) : c \in [y_n z_n]\}$ is continuous. Nevertheless, thanks to the continuity assumption for rx at $z$, the orthogonal projection of such curve onto $\overleftrightarrow{z\mathrm{rx}(z)}$ is a set converging to the single point $\mathrm{rx}(z)$. Using this fact and the regularity of $\partial(\mathrm{spt}\, f^-)$, we deduce that the symmetric difference $\big(B_n \setminus D_n\big) \cup \big(D_n \setminus B_n\big)$ can be encapsulated into two thin rectangles (dashed in Figure 9) whose areas are both $o(\lambda_n \delta_n)$. Since $f^-$ is bounded, this implies $f^-(B_n) = f^-(D_n) + o\big(\lambda_n \delta_n\big)$. Therefore, (5.2) is proved if we show that

$$f^-(D_n) = \lambda_n \delta_n J^-(R(z), \lambda_n) + o(\lambda_n \delta_n)\,. \tag{5.4}$$

In order to prove (5.3) and (5.4), we need to evaluate the masses of $E_n$ and $D_n$. To this aim, we indicate by $v_\sigma$ the point $(x_n)_\sigma$ according to Definition 5.4, and by $w_\sigma$ the point $(z_n)_\sigma$ according to Definition 5.5. Let us also denote by $v_\sigma'$ and $w_\sigma'$ the points of $\overrightarrow{x_n q_n}$ and $R(y_n)$ whose orthogonal projections onto $\overrightarrow{x_n r_n}$ and $R(z_n)$ are $v_\sigma$ and $w_\sigma$. Set $\beta_n^+(\sigma)$ the average of $f^+$ on the segment $v_\sigma v_\sigma'$ and $\beta_n^-(\sigma)$ the average of $f^-$ on the segment $w_\sigma w_\sigma'$. By Fubini Theorem,

$$f^+(E_n) = \int_0^{\overline{x_n r_n}} \overline{v_\sigma v_\sigma'}\, \beta_n^+(\sigma)\, d\sigma \qquad \text{and} \qquad f^-(D_n) = \int_{\overline{z_n u_n}}^{\overline{z_n \mathrm{rx}(z_n)}} \overline{w_\sigma w_\sigma'}\, \beta_n^-(\sigma)\, d\sigma\,. \tag{5.5}$$

Since $f^+$ and $f^-$ are uniformly continuous and for $n \gg 1$ the whole segments $v_\sigma v_\sigma'$ and $w_\sigma w_\sigma'$ are arbitrarily close to $x_\sigma$ and $z_\sigma$ uniformly w.r.t. $\sigma$, then

$$\beta_n^+(\sigma) = f^+(x_\sigma) + o(1) \qquad \text{and} \qquad \beta_n^-(\sigma) = f^-(z_\sigma) + o(1)\,. \tag{5.6}$$

Finally observe that, by similitude,

$$\overline{v_\sigma v_\sigma'} = \frac{\varepsilon_n}{\overline{x_n r_n}}\,\sigma \qquad \text{and} \qquad \overline{w_\sigma w_\sigma'} = \lambda_n \delta_n \rho_n^-(\sigma)\,, \tag{5.7}$$

where $\rho_n^-$ is the function determined by $R(z_n)$ and $\lambda_n$ according to Definition 5.5. The required expansions (5.3) and (5.4) follow from (5.5), (5.6) and (5.7) by taking into account that $x_n \to x$, $r_n \to r$, $z_n \to z$ and $\mathrm{rx}(z_n) \to \mathrm{rx}(z)$. $\square$

Next lemma is the key argument we will use to prove by contradiction the continuity of the maps lx and rx. In fact, one immediately sees that lx and rx cannot be continuous if the assumptions of the lemma hold. Under these assumptions, we are able to locate a sequence of doubling points whose limit lies in the interior of a transport ray; related to such doubling points, we also find a thinning sequence of triangular sets, to which Lemma 5.7 will be applied.

**Lemma 5.8** (Extremes converging to interior points)**.** *Let $\{z_n\}$ be a sequence of left extremes converging to a point $x^*$ lying in a maximal transport ray $xy$. Then, for any $\hat{x} \in xx^*$, there exist a sequence $\{x_n\}$ of doubling points converging to $\hat{x}$ and two sequences of maximal transport rays $\{x_n y_n\}$ and $\{x_n w_n\}$, such that the triangular sets determined by $\overrightarrow{x_n y_n}$ and $\overrightarrow{x_n w_n}$ contain $z_n$ and are thinning. The analogous result holds for a sequence of right extremes converging to an interior point of a maximal transport ray.*

**Remark 5.9.** By a standard diagonalization argument, the thesis of the above lemma still holds in the limit case when $\hat{x} = x^*$.

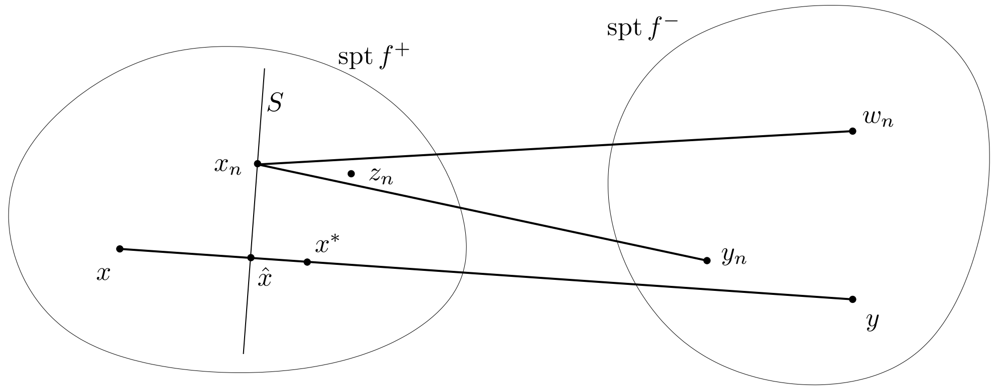


FIGURE 10. Situation in Lemma 5.8

*Proof. of Lemma 5.8.* Let $\hat{x} \in xx^*$ be fixed, and let $S$ be the perpendicular to $xy$ at $\hat{x}$. For $n \gg 1$, $z_n$ is in the right half-plane determined by $S$; moreover we may assume, just to fix the ideas, that every $z_n$ is above $xy$ (see Figure 10). Since $z_n$ is a left extreme, every transport ray crossing $S$ cannot contain $z_n$. Consider the points $s$ of $S$ such that every transport ray containing $s$ is below $z_n$; taking into account Remark 2.2, we can define $x_n$ as the highest among these points $s$. Similarly, by Remark 2.2 and Remark 2.4, $x_n$ is also the lowest among the points $s'$ of $S$ such that every transport ray containing $s'$ is above $z_n$. Therefore, by construction, $x_n$ is a doubling point; it is then possible to choose two maximal transport rays $x_n y_n$ and $x_n w_n$ such that $z_n$ is above (resp. below) $x_n y_n$ (resp. $x_n w_n$).
We claim that $x_n \to \hat{x}$ and that the sequence of triangular sets determined by $\overrightarrow{x_n y_n}$ and $\overrightarrow{x_n z_n}$ is thinning. Indeed, let $a$ and $b$ be limit points of $\{x_n\}$ and $\{y_n\}$, respectively; both $a$ and $b$ must be above $xy$, but the segment $ab$ must be below $x^*$ (which is the limit of $z_n$). Therefore, $ab \subseteq xy$ and, since $a \in S$, it must be $a = \hat{x}$. Let now $c$ be a limit point of $\{w_n\}$. By Remark 2.2, $\hat{x}c$ is a transport segment; if $c \notin xy$, $\hat{x}$ would be a doubling point, against the assumption $\hat{x} \in xy \subseteq T$. Hence $c \in xy$, so the sequence of the triangular sets determined by $\overrightarrow{x_n y_n}$ and $\overrightarrow{x_n w_n}$ is thinning. □

Finally, we present some geometrical lemmas, each one involving a pair of transport segments; we stress that in the sequel such results will be often applied in the special case when the two transport segments have a common endpoint (this is directly assumed in the last lemma). Next statement closely reminds the results proved in [11, Section 4.2].

**Lemma 5.10.** *Let $x$, $z \in \operatorname{spt} f^+$ and $y$, $w \in \operatorname{spt} f^-$ be given points such that $xy$ and $zw$ are transport segments, and the orthogonal projection $z_\perp$ of $z$ onto $\overleftrightarrow{xy}$ lies in $xy$. Take a point $z' \in zz_\perp$ such that $\overline{z'z_\perp} \le \overline{zz_\perp}/3$ and $\overrightarrow{xz'} \cap \overrightarrow{zw} \notin (\operatorname{spt} f^+)^0$. Then, provided $\overline{zz_\perp} \ll \overline{xz_\perp}$, if $A$ is the triangular set determined by $\overrightarrow{xy}$ and $\overrightarrow{xz'}$, $t(A)$ is contained into the right trapezoidal set determined by (the maximal transport rays containing) $xy$ and $zw$. The analogous result holds for triangular sets $A \subseteq \operatorname{spt} f^-$ built in the symmetric way.*

*Proof.* We have to show that, for any $x' \in A \cap T$, $\varphi(x')$ belongs to the segment $\varphi(x)\varphi(z)$, where we write for convenience $\varphi(x) = xy \cap H$ and $\varphi(z) = zw \cap H$ even though it might happen that $x$ or $z$ do not belong

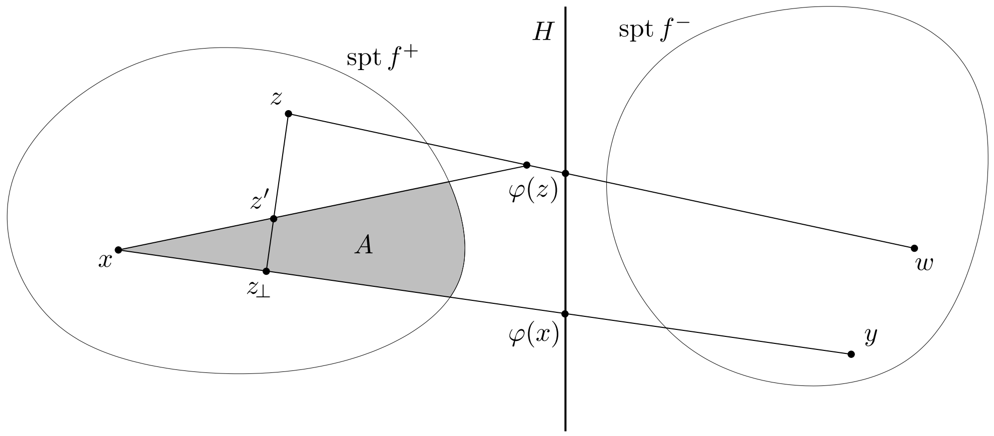


FIGURE 11. Situation in Lemma 5.10

to $T$. Suppose without loss of generality that $\varphi(x)$ is below $zw$ as in Figure 11 (in the case $\varphi(x)$ above $zw$ the proof is fully analogous, while in the case $\varphi(x) = \varphi(z)$ the statement is trivially true). Then, thanks to the assumption $\overrightarrow{xz'} \cap zw \notin (\operatorname{spt} f^+)^0$, every point $x'$ of $A$ is below $zw$, so that $z$ is above $xx'$. Assume by contradiction that there exists some point $x' \in A \cap T$ such that $\varphi(x') \notin \varphi(x)\varphi(z)$; then, calling $y' := t(x')$, the transport ray $x'y'$ is either below $\varphi(x)$ or above $\varphi(z)$.

The first possibility is immediately excluded. Indeed, since $y_1 > x_1$, also $(z_\perp)_1 > x_1$, hence $z'_1 > x_1$ for $\overline{zz_\perp} \ll \overline{xz_\perp}$. This implies $x'_1 > x_1$, which is incompatible with $x'y'$ below $\varphi(x)$, because the transport segments $x\varphi(x)$ and $x'\varphi(x')$ would intersect.

It remains to exclude the second possibility. So assume by contradiction that $x'y'$ is above $\varphi(z)$. By Remark 2.3, $\overline{xy} + \overline{x'y'} \leq \overline{xy'} + \overline{x'y}$; this inequality holds, *a fortiori*, if we replace $y'$ by $z$. Indeed, $\overline{x'y'} - \overline{xy'} > \overline{x'z} - \overline{xz}$ by geometrical arguments. More precisely, the region $\{s : \overline{x'y'} - \overline{xy'} > \overline{x's} - \overline{xs}\}$ must contain $z$: it is a convex region delimited by an hyperbole with focuses $x$ and $x'$, and passing by $y'$; moreover, $z_1 < y'_1$ and $z$ is above $\overleftrightarrow{xx'}$ and below $\overleftrightarrow{x'y'}$ (since $\varphi(z)$ is below $\overleftrightarrow{x'y'}$ and the transport segments $x'y'$ and $z\varphi(z)$ cannot intersect). Thus

$$\overline{xy} + \overline{x'z} \leq \overline{xz} + \overline{x'y}\,. \tag{5.8}$$

To obtain a contradiction from (5.8), just for the remaining of this proof, we set $l := \overline{zz_\perp}/3$, and we consider the system of coordinates in which $x \equiv (0,0)$, $z_\perp \equiv (1,0)$, $z \equiv (1,3l)$; we define $\sigma, \eta$ so that $x' \equiv (\sigma, \eta)$, and we notice that, by construction of $A$,

$$0 \leq \eta \leq \sigma l\,. \tag{5.9}$$

Let us consider separately the two exhaustive cases $\sigma \geq 7/6$ and $\sigma < 7/6$ (in fact, any other value in $(1, 6/5)$ would work as well as $7/6$).

<u>*Case I.*</u> $\sigma \geq 7/6$.

Let us compute the lengths of the segments appearing in (5.8). Setting $\lambda := \overline{xy}$ and taking into account (5.9), we obtain

$$\overline{xy} = \lambda, \qquad \overline{x'z} = \sigma - 1 + o(l), \qquad \overline{xz} = 1 + o(l), \qquad \overline{x'y} = \lambda - \sigma + o(l)\,,$$

where in the last equality we have used the fact that $\lambda > \sigma$ for $l \ll 1$. Therefore, $\overline{xy} + \overline{x'z} \approx \lambda + \sigma - 1$ and $\overline{xz} + \overline{x'y} \approx 1 + \lambda - \sigma$. This contradicts (5.8) if $\sigma \geq 7/6$.

<u>*Case II.*</u> $\sigma < 7/6$.

Let us notice that (5.8) holds, *a fortiori*, with $z_\perp$ in place of $y$, since $\overline{xy} - \overline{xz_\perp} \geq \overline{x'y} - \overline{x'z_\perp}$ by the triangle inequality. So we have $\overline{xz_\perp} + \overline{x'z} \leq \overline{xz} + \overline{x'z_\perp}$. In turn, this inequality is still satisfied if we replace $x'$ by $x'' \equiv (\sigma, \sigma l)$; indeed, by (5.9), it holds $\overline{x''z} \leq \overline{x'z}$, while $\overline{x''z_\perp} \geq \overline{x'z_\perp}$. Thus,

$$\overline{xz_\perp} + \overline{x''z} \leq \overline{xz} + \overline{x''z_\perp}. \tag{5.10}$$

The lengths of the segments in (5.10) can be computed as

$$\overline{xz_\perp} = 1, \qquad \overline{x''z} = \sqrt{(1-\sigma)^2 + (3-\sigma)^2 l^2}, \qquad \overline{xz} = \sqrt{1+9l^2}, \qquad \overline{x''z_\perp} = \sqrt{(1-\sigma)^2 + \sigma^2 l^2}.$$

Taking the square of both sides in (5.10), we get

$$-3\sigma l^2 + \sqrt{(1-\sigma)^2 + (3-\sigma)^2 l^2} \leq \sqrt{1+9l^2}\sqrt{(1-\sigma)^2 + \sigma^2 l^2}\,.$$

Since the left-hand side of the above inequality is non-negative for $l \ll 1$, taking again the square we deduce that

$$4 - 3\sigma \leq 2\sqrt{(1-\sigma)^2 + (3-\sigma)^2 l^2}$$

and, by the same argument (recall that $\sigma < 4/3$),

$$5\sigma^2 - 16\sigma + 12 \leq 4(3-\sigma)^2 l^2\,.$$

This inequality cannot be true for $l \ll 1$, as $5\zeta^2 - 16\zeta + 12 = 0$ for $\zeta = 6/5$ or $\zeta = 2$, and we are assuming $\sigma < 7/6$. □

**Lemma 5.11.** *Let $x$, $z \in \operatorname{spt} f^+$ and $y$, $w \in \operatorname{spt} f^-$ be given points such that $xy$ and $zw$ are transport segments. Define $u := zw \cap \partial\operatorname{spt} f^-$, $v := xy \cap \partial\operatorname{spt} f^-$, $w_\perp$ and $u_\perp$ as the orthogonal projections of $w$ and $u$ onto $\overleftrightarrow{xy}$, and $B \subseteq \operatorname{spt} f^-$ as the triangular set determined by $\overrightarrow{yu}$ and $\overrightarrow{yv}$. If one of the following conditions is satisfied: either*

$$\overline{vw_\perp} \geq \frac{3}{4}\,\overline{vy} \qquad \text{and} \qquad \overline{uu_\perp} \leq \overline{ww_\perp} \tag{5.11}$$

*or*

$$\overline{vw_\perp} \geq \overline{vy}, \tag{5.12}$$

*then, provided $\overline{uu_\perp} \ll \overline{vy}$, $t^{-1}(B)$ is contained into the left trapezoidal set determined by $R(u)$ and $R(v)$. The analogous result holds for triangular sets $B \subseteq \operatorname{spt} f^+$ built in the symmetric way.*

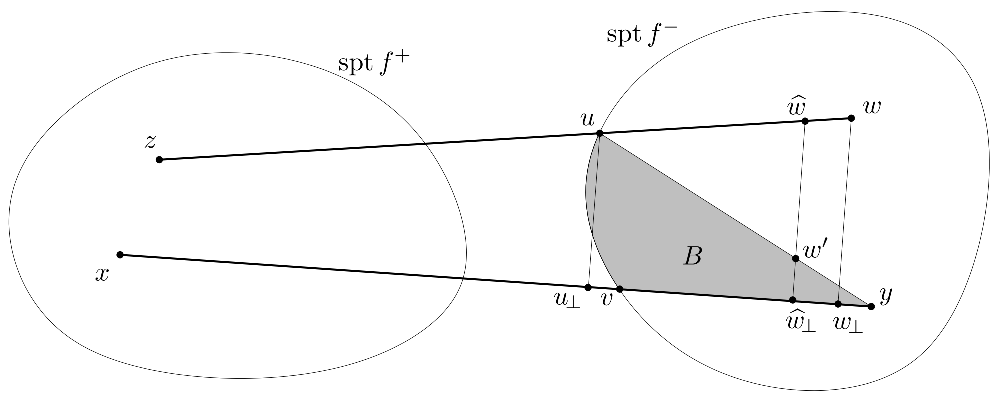


FIGURE 12. Situation in Lemma 5.11

*Proof.* If the assumption (5.11) holds, we consider the point $\widehat{w} \in [uw]$ such that $\overline{\widehat{w}_\perp y} = \overline{vy}/4$, being $\widehat{w}_\perp$ the orthogonal projection of $\widehat{w}$ onto $xy$. Then the thesis follows by applying Lemma 5.10 to the transport segments $z\widehat{w}$ and $xy$, the point $w' := yu \cap \widehat{w}\widehat{w}_\perp$, and the triangular set $B$. Indeed, $\overline{\widehat{w}\widehat{w}_\perp} \leq \overline{uu_\perp}\big(\mathrm{diam}\big(\mathrm{spt}\, f^+ \cup \mathrm{spt}\, f^-\big)/\mathrm{dist}\big(\mathrm{spt}\, f^+, \mathrm{spt}\, f^-\big)\big) \ll \overline{vy}$; on the other hand,

$$\overline{w'\widehat{w}_\perp} \leq \frac{1}{3}\overline{uu_\perp} \leq \frac{1}{3}\overline{\widehat{w}\widehat{w}_\perp},$$

where the first inequality is immediate by similitude since $\overline{uu_\perp} \ll \overline{vy}$, so that $\overline{u_\perp v} \ll \overline{vy}$, and the second one follows from the hypothesis $\overline{uu_\perp} \leq \overline{ww_\perp}$.

If the assumption (5.12) holds, notice that one can assume $w \not\equiv y$, since otherwise the thesis is immediately satisfied by non-intersection. Then, we consider the point $\widehat{w} \in [uw]$ such that $\overline{\widehat{w}\widehat{w}_\perp} = 3\overline{w'\widehat{w}_\perp}$, where $w'$ is defined as above. Noticing that $\overline{\widehat{w}\widehat{w}_\perp} \ll \overline{\widehat{w}_\perp y}$ by similitude, one can conclude applying again Lemma 5.10. $\square$

We conclude with an useful property about condition (5.11).

**Lemma 5.12.** *Let $x, z, y, w$ be as in Lemma 5.11, and assume in addition that $x \equiv z$. Then, provided $\theta := \widehat{yxw} \ll \overline{vy}/\mathrm{diam}\big(\mathrm{spt}\, f^+ \cup \mathrm{spt}\, f^-\big)$, condition (5.11) holds up to exchanging $y$ and $w$.*

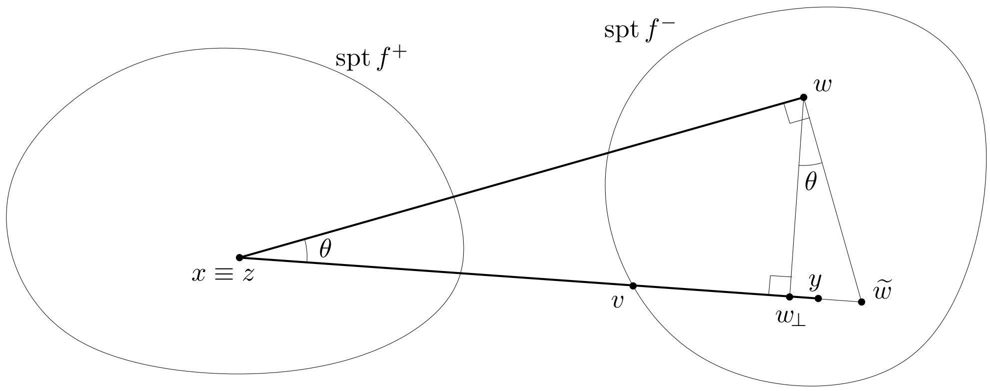


FIGURE 13. Situation in Lemma 5.12

*Proof.* If (5.11) is satisfied, there is nothing to prove. So assume that

$$w_\perp \in vy\,, \qquad \text{with} \qquad \overline{w_\perp y} > \frac{\overline{vy}}{4}\,. \tag{5.13}$$

We claim that, if (5.13) holds, the orthogonal projection $y_\perp$ of $y$ onto $\overleftrightarrow{xw}$ must lie outside $xw$ (so that (5.11) is satisfied exchanging $y$ and $w$). Defining $\widetilde{w}$ as the point of $\overrightarrow{xy}$ whose orthogonal projection onto $xw$ equals $w$, our claim is equivalent to $\widetilde{w} \in w_\perp y$. To show this fact, notice that $\widehat{\widetilde{w}ww_\perp} = \theta$, hence $\overline{w_\perp\widetilde{w}} = \overline{ww_\perp}\ \mathrm{tg}\,\theta = \overline{xw_\perp}\ (\mathrm{tg}\,\theta)^2$. Since $\overline{xw_\perp} \leq \mathrm{diam}(\mathrm{spt}\, f^+ \cup \mathrm{spt}\, f^-)$, we deduce that $\overline{w_\perp\widetilde{w}} \leq \overline{vy}/4$ for $\theta \ll \overline{vy}/\mathrm{diam}\big(\mathrm{spt}\, f^+ \cup \mathrm{spt}\, f^-\big)$. So, by (5.13), $\widetilde{w}$ must lie in $w_\perp y$ when $\theta$ is small enough, and the proof is achieved. $\square$

## 6. Continuity of lx and rx

This section is devoted to prove the following result, which ensures that the first sufficient condition of Proposition 3.4 is satisfied.

**Theorem 6.1.** *The maps $\mathrm{lx} : T^0 \to \mathrm{spt}\, f^+$ and $\mathrm{rx} : T^0 \to \mathrm{spt}\, f^-$ are continuous.*

We begin by proving the continuity of the maps lx and rx under an additional "far apart condition" on the supports of $f^{\pm}$ (Proposition 6.2). Next, removing such condition, we prove the continuity of lx and rx respectively for those points $p \in H \cap T^0$ such that only a "small" portion of the transport ray $R(p)$ is contained in $\operatorname{spt} f^+$ and in $\operatorname{spt} f^-$ (Proposition 6.3). Finally, we attack the proof of Theorem 6.1.
We point out that, for the proof of Theorem 6.1, just Proposition 6.3 and not Proposition 6.2 is needed. Nevertheless, we prefer to present first Proposition 6.2, since under its assumption the continuity of lx and rx admits a simpler proof. With this proof in mind, the general case will be easier to understand because it follows the same scheme.

**Proposition 6.2** (Continuity of lx and rx in case of "far" supports)**.** *Assume that*

$$\operatorname{dist}\big(\operatorname{spt} f^+, \operatorname{spt} f^-\big) > 6 \max\big\{\operatorname{diam}\big(\operatorname{spt} f^+\big), \operatorname{diam}\big(\operatorname{spt} f^-\big)\big\}\,. \tag{6.1}$$

*Then the maps* $\mathrm{lx} : T^0 \to \operatorname{spt} f^+$ *and* $\mathrm{rx} : T^0 \to \operatorname{spt} f^-$ *are continuous.*

*Proof.* Since $\mathrm{lx} = \mathrm{lx}_{|H\cap T^0} \circ \varphi$ and $\varphi$ is continuous by Lemma 3.2, we may confine ourselves to prove the continuity of lx and rx on the set $H \cap T^0$.
By contradiction, assume that the assert is false. Then, by the symmetry of the problem, we may suppose that there exists a sequence $\{p_n\} \subseteq H \cap T^0$ such that $p_n \to p \in H \cap T^0$ and $\liminf_n |\mathrm{lx}(p_n) - \mathrm{lx}(p)| > 0$. Then, by Remark 2.2, any limit point of $\{\mathrm{lx}(p_n)\}$ must lie in $R(p) =: xy$. So, up to subsequences, we may assume that $\{\mathrm{lx}(p_n)\}$ converges to some point $x^* \in xy \cap \operatorname{spt} f^+$. By Lemma 5.8, choosing $\hat{x} \in xx^*$, there exist a sequence $\{x_n\}$ of doubling points converging to $\hat{x}$ and a thinning sequence of triangular sets $A_n$ determined by $\overrightarrow{x_n y_n}$ and $\overrightarrow{x_n w_n}$, being $x_n y_n$ and $x_n w_n$ maximal transport rays. We may assume, again up to subsequences, that $y_n \to \hat{y}$; since $\hat{x}$ is not a doubling point, by Remark 2.2 $\hat{y}$ must lie in $[xy] \cap \operatorname{spt} f^-$.

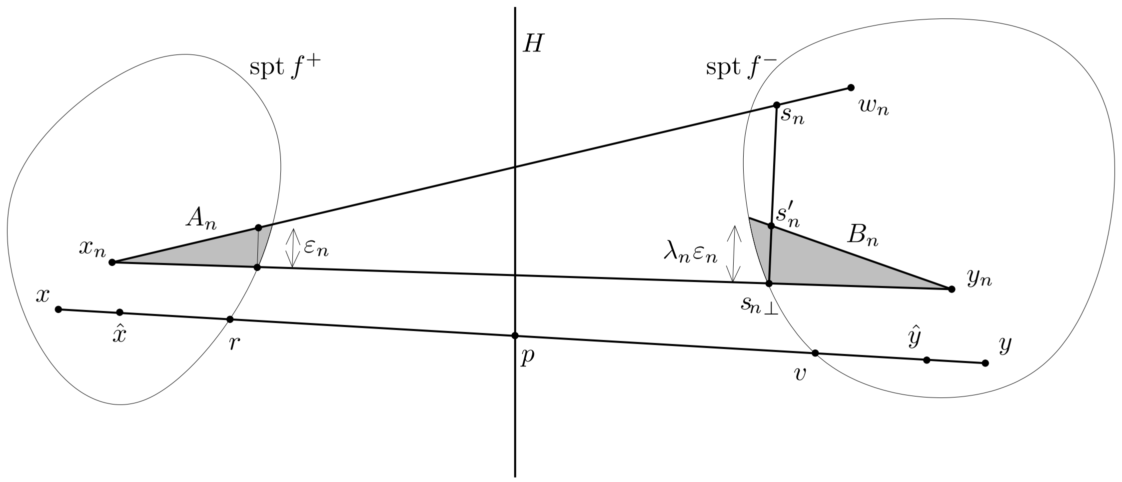


FIGURE 14. Situation in Proposition 6.2

Denote by $r$ and $v$ the intersections of $xy$ with $\partial \operatorname{spt} f^+$ and $\partial \operatorname{spt} f^-$ respectively (see Figure 14). We claim that

$$I^+(\hat{x}, r) \geq 2\, I^-(\hat{y}, v)\,. \tag{6.2}$$

Notice that this inequality is trivially satisfied if $\hat{y} = v$; so we may prove it under the assumption $\hat{y} \neq v$. We set $s_n$ the point of $x_n w_n$ whose orthogonal projection onto $x_n y_n$ is the point $s_{n\perp} := x_n y_n \cap \partial \operatorname{spt} f^-$;

let also $s'_n \in s_{n\perp}s_n$ be the point such that $\overline{s_n s_{n\perp}} = 3\overline{s'_n s_{n\perp}}$. Then we call $B_n \subseteq \operatorname{spt} f^-$ the triangular set determined by $\overrightarrow{y_n s'_n}$ and $\overrightarrow{y_n s_{n\perp}}$ (see Figure 14).

The sequences $\{A_n\}$ and $\{B_n\}$ are thinning; thus, we can estimate asymptotically their masses by using Lemma 5.7: defining $\varepsilon_n$ and $\lambda_n\varepsilon_n$ as indicated in Figure 14, the expansion (5.1) gives

$$f^+(A_n) = \varepsilon_n I^+(\hat{x}, r) + o\big(f^+(A_n)\big) \qquad \text{and} \qquad f^-(B_n) = \lambda_n\varepsilon_n I^-(\hat{y}, v) + o\big(f^-(B_n)\big)\,. \tag{6.3}$$

Now notice that, for $n \gg 1$, the transport segments $x_n s_n$ and $x_n y_n$, and the point $s'_n$ satisfy the assumptions of Lemma 5.10. Then, by such lemma and the non-intersection of transport rays, we have $t^{-1}(B_n) \subseteq A_n$, which implies $f^+(A_n) \geq f^-(B_n)$. Then, (6.2) follows from (6.3): indeed, by similitude $\lambda_n \to \lambda := \overline{\hat{x}v}/3\overline{\hat{x}r}$, and by (6.1) we have

$$3\lambda = \frac{\overline{\hat{x}v}}{\overline{\hat{x}r}} \geq \frac{\operatorname{dist}(\operatorname{spt} f^+, \operatorname{spt} f^-)}{\overline{\hat{x}r}} \geq \frac{\operatorname{dist}(\operatorname{spt} f^+, \operatorname{spt} f^-)}{\operatorname{diam}(\operatorname{spt} f^+)} \geq 6\,. \tag{6.4}$$

Having proved (6.2), we now obtain a contradiction separately in the two cases $\hat{y} \neq y$ and $\hat{y} = y$.

<u>*Case I.*</u> $\hat{y} \neq y$.

By Lemma 5.8 and Remark 5.9, we may choose a sequence $\{\hat{y}_n\} \subseteq \operatorname{spt} f^-$ of doubling points converging to $\hat{y}$ and a thinning sequence of triangular sets $\widehat{B}_n$ containing $y_n$ and delimited by maximal transport rays crossing at $\hat{y}_n$. Let $s_n$ and $q_n$ be the points of the two half-lines determining $\widehat{B}_n$ whose orthogonal

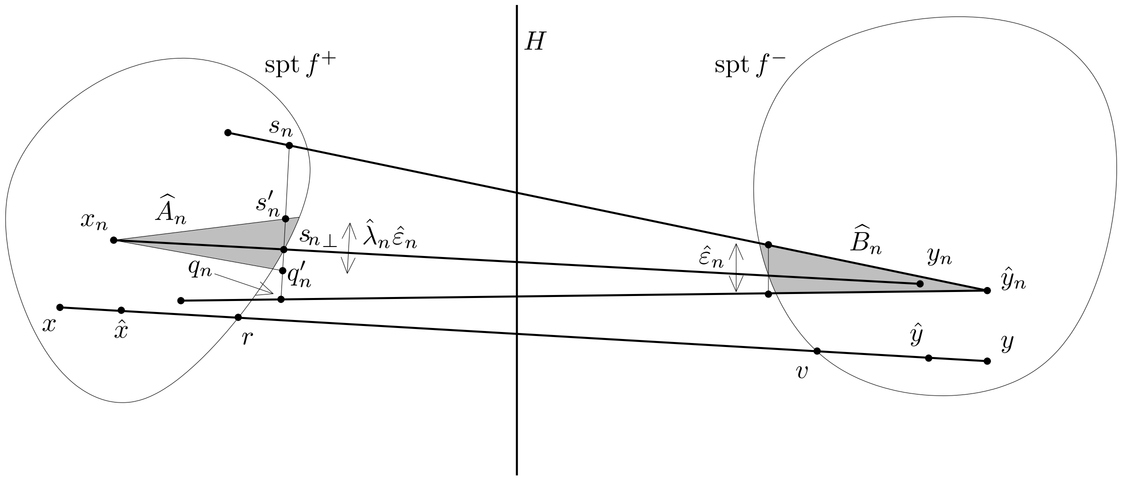


FIGURE 15. Situation in Case I of Proposition 6.2

projections onto $x_n y_n$ are the point $s_{n\perp} \equiv q_{n\perp} := x_n y_n \cap \partial \operatorname{spt} f^+$; let also $s'_n \in s_{n\perp}s_n$ and $q'_n \in q_{n\perp}q_n$ be the points such that $3\overline{s'_n s_{n\perp}} = \overline{s_n s_{n\perp}}$ and $3\overline{q'_n q_{n\perp}} = \overline{q_n q_{n\perp}}$. We call $\widehat{A}_n$ the triangular set determined by $\overrightarrow{x_n q'_n}$ and $\overrightarrow{x_n s'_n}$. Using Lemma 5.10 as above (actually one has to apply it twice, once to the part of $\widehat{A}_n$ above $x_n y_n$ and once to the part below), we deduce that, for $n \gg 1$, $t(\widehat{A}_n) \subseteq \widehat{B}_n$, so that the following inequality holds:

$$f^-(\widehat{B}_n) \geq f^+(\widehat{A}_n)\,. \tag{6.5}$$

In order to evaluate the masses of $\widehat{A}_n$ and $\widehat{B}_n$, we define $\hat{\varepsilon}_n$ and $\hat{\lambda}_n\hat{\varepsilon}_n$ as indicated in Figure 15. Now notice that, by using a similitude and (6.1) exactly as done in (6.4), one has $\hat{\lambda}_n \geq 2$ for any $n \gg 1$. Consequently, inequality (6.5) implies $\hat{y} \neq v$. Indeed, if $\hat{y} = v$, since $\hat{y}_n \to \hat{y}$ and $f^-$ is bounded, one

would have $f^-(\widehat{B}_n) = o(\hat{\varepsilon}_n)$. This contradicts (6.5) because, since $f^+ \geq C$ in a subset of $\widehat{A}_n$ whose area is of order $\hat{\lambda}_n\hat{\varepsilon}_n$ (remind that $\hat{x} \neq r$), there holds $f^+(\widehat{A}_n) \geq C\hat{\lambda}_n\hat{\varepsilon}_n$.
In particular, $\hat{\lambda}_n \to \hat{\lambda} := \overline{\hat{y}r}/3\overline{\hat{y}v}$, with

$$3\hat{\lambda} = \frac{\overline{\hat{y}r}}{\overline{\hat{y}v}} \geq \frac{\operatorname{dist}(\operatorname{spt} f^+, \operatorname{spt} f^-)}{\overline{\hat{y}v}} \geq \frac{\operatorname{dist}(\operatorname{spt} f^+, \operatorname{spt} f^-)}{\operatorname{diam}(\operatorname{spt} f^-)} \geq 6\,. \tag{6.6}$$

By (5.1) (for $\widehat{A}_n$, apply it twice and then sum up), and since $\hat{y} \neq v$ and $\hat{x} \neq r$, we may write down the following expansions:

$$f^-(\widehat{B}_n) = \hat{\varepsilon}_n I^-(\hat{y}, v) + o\big(f^-(\widehat{B}_n)\big) \qquad \text{and} \qquad f^+(\widehat{A}_n) = \hat{\lambda}_n\hat{\varepsilon}_n I^+(\hat{x}, r) + o\big(f^+(\widehat{A}_n)\big)\,.$$

Since $\hat{\lambda}_n \geq 2$, the above expansions combined with (6.5) give

$$I^-(\hat{y}, v) \geq 2\, I^+(\hat{x}, r)\,. \tag{6.7}$$

This inequality is incompatible with (6.2) because $I^+(\hat{x}, r)$ and $I^-(\hat{y}, v)$ are both strictly positive.

<u>*Case II.*</u> $\hat{y} = y$.

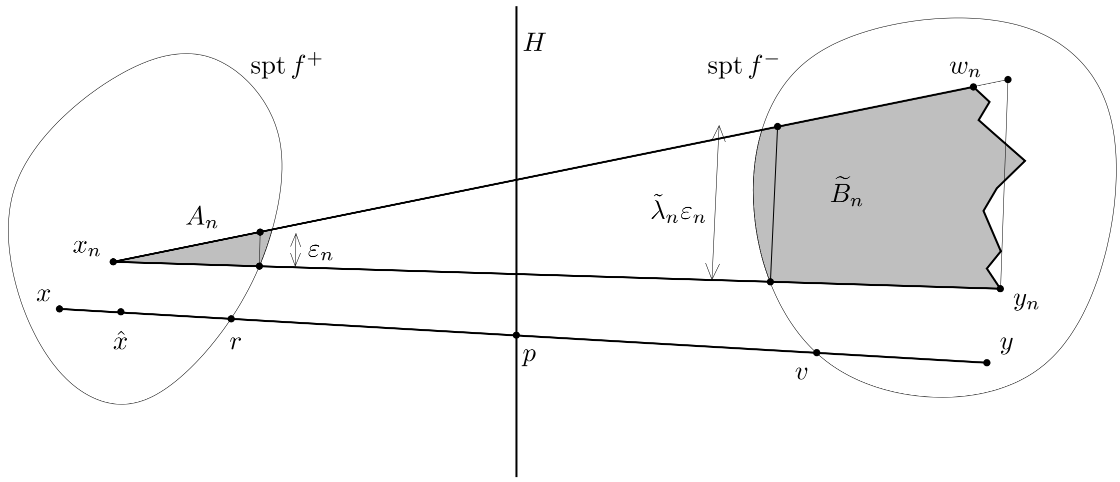


FIGURE 16. Situation in Case II of Proposition 6.2

Recalling Remark 5.6, (6.2) implies

$$I^+(x, r) \geq I^+(\hat{x}, r) \geq 2\, I^-(\hat{y}, v) = 2\, I^-(y, v)\,. \tag{6.8}$$

As we have already get a contradiction in Case I, the lack of continuity of lx at $p$ necessarily leads to the validity (6.8); in the very same way, if rx were not continuous at $p$ we would deduce also $I^-(y, v) \geq 2\, I^+(x, r)$. Being this last inequality incompatible with (6.8), we infer that rx is continuous at $p$.
The continuity of rx at $p$ allows us to estimate the mass of $\widetilde{B}_n := t(A_n)$ through Lemma 5.7; in fact, $A_n$ and $\widetilde{B}_n$ are the left and right trapezoidal sets determined by the maximal transport rays $x_n y_n$ and $x_n w_n$. By (5.1) and (5.2), the equality $f^+(A_n) = f^-(\widetilde{B}_n)$ implies

$$I^+(\hat{x}, r) = \tilde{\lambda} J^-(R(p), \tilde{\lambda})\,, \tag{6.9}$$

where, if $\tilde{\lambda}_n$ are defined according to Figure 16, $\tilde{\lambda} := \lim_n \tilde{\lambda}_n = \overline{\hat{x}v}/\overline{\hat{x}r}$. Recall now that $\hat{x}$ was chosen arbitrarily in $xx^*$, then move $\hat{x}$ towards $x$ along $xx^*$: by this way, $I^+(\hat{x}, r)$ strictly increases by Remark 5.6,

$\tilde\lambda$ strictly decreases by construction, and $J^-(R(p),\tilde\lambda)$ strictly decreases again by Remark 5.6. Therefore, equality (6.9) cannot remain true, and we have obtained a contradiction. $\square$

**Proposition 6.3** (Continuity of lx and rx with an endpoint close to the boundary)**.** *Let*

$$S^+ := \Big\{p \in H\cap T:\ \big|\mathrm{lx}(p) - [\mathrm{lx}(p)\,p]\cap \partial \mathrm{spt}\, f^+\big| < \mathrm{dist}\big(\mathrm{spt}\, f^+, \mathrm{spt}\, f^-\big)/12\Big\}\,,$$
$$S^- := \Big\{p \in H\cap T:\ \big|\mathrm{rx}(p) - [p\,\mathrm{rx}(p)]\cap \partial \mathrm{spt}\, f^-\big| < \mathrm{dist}\big(\mathrm{spt}\, f^+, \mathrm{spt}\, f^-\big)/12\Big\}\,.$$

*Then the restriction of the map* lx *(resp.* rx*) to* $\overline{S^+}$ *(resp.* $\overline{S^-}$*) is continuous.*

*Proof.* The proof is quite similar to the one of Proposition 6.2. So we adopt the same notation for all the points involved and we enlighten just the required modifications. In particular observe that, in the proof of Proposition 6.2, assumption (6.1) is used *only* to obtain the lower bounds (6.4) and (6.6). Thanks to the symmetry of the statement, we may assume by contradiction that lx is not continuous at some point $p\in\overline{S^+}$. Moreover, by Remark 5.9 and since $p\in\overline{S^+}$, we may also assume that

$$0 < \overline{\hat x r} \leq \mathrm{dist}\big(\mathrm{spt}\, f^+, \mathrm{spt}\, f^-\big)/12\,. \tag{6.10}$$

Therefore, the lower bound in (6.4) can be replaced by

$$3\lambda = \frac{\overline{\hat x v}}{\overline{\hat x r}} \geq \frac{\mathrm{dist}(\mathrm{spt}\, f^+, \mathrm{spt}\, f^-)}{\overline{\hat x r}} \geq 12\,.$$

This allows to obtain, analogously to (6.2),

$$I^+(\hat x, r) \geq 4 I^-(\hat y, v)\,. \tag{6.11}$$

Now, let us analyze separately the two cases $\hat y\neq y$ and $\hat y = y$.

<u>*Case I.*</u> $\hat y \neq y$.
The lower bound (6.6) can be replaced by

$$3\hat\lambda = \frac{\overline{\hat y r}}{\overline{\hat y v}} \geq 1$$

(being $\hat y$ necessarily different from $v$ as in Proposition 6.2). Then, analogously to (6.7), we find $I^-(\hat y, v)\geq (1/3) I^+(\hat x, r)$, which is incompatible with (6.11).

<u>*Case II.*</u> $\hat y = y$.
If $\hat y = y$, (6.11) reads

$$I^+(\hat x, r)\geq 4 I^-(y, v)\,. \tag{6.12}$$

We claim that (6.12) guarantees the continuity of rx at $p$. Were this claim proved, the proof would be achieved by using the continuity of rx at $p$ exactly in the same way as in Proposition 6.2.
In spite, the continuity of rx at $p$ is obtained in a slightly different way from Proposition 6.2. Assume by contradiction that rx is not continuous at $p$. Then, by the same arguments used to prove (6.2) and (6.11), we may find a sequence of doubling points converging to a point $y' \in xy\cap \mathrm{spt}\, f^-$, with $y'\neq v$, and a sequence of left extremes converging to a point $x'\in [xy]\cap \mathrm{spt}\, f^+$, in such a way that

$$I^-(y', v) \geq \frac{\overline{ry'}}{3\,\overline{vy'}} I^+(x', r) \geq \frac{1}{3} I^+(x', r)\,. \tag{6.13}$$

Combining (6.12) and (6.13), and recalling Remark 5.6, we deduce that $I^+(\hat x, r)\geq (4/3) I^+(x', r)$. This inequality and Remark 5.6 ensure that $x'\in \hat x r$; then, by (6.10), $\overline{x'r}\leq \mathrm{dist}\big(\mathrm{spt}\, f^+, \mathrm{spt}\, f^-\big)/12$. Then, arguing in the same way as in Case I of Proposition 6.2, we obtain first that $x' \neq r$ and then that

$$I^+(x', r) \geq \frac{\overline{x'v}}{3\,\overline{x'r}} I^-(y', v) \;\geq \frac{\operatorname{dist}\bigl(\operatorname{spt} f^+, \operatorname{spt} f^-\bigr)}{3\,\overline{x'r}} I^-(y', v) \geq 4I^-(y', v)\,. \tag{6.14}$$

Since (6.13) and (6.14) are incompatible, rx must be continuous at $p$ and the proof is achieved. □

*Proof. of Theorem 6.1.* The proof is again quite similar to the one of Proposition 6.2, so we just enlighten the required modifications. Unless otherwise specified, we adopt the same notation for all the points involved. By contradiction, let $p \in H \cap T^0$ be a point where lx is not continuous; then, by Proposition 6.3, it must be $p \notin \overline{S^+}$. Let $\hat{x} \in xp$ be the point such that

$$\overline{\hat{x}p} = \liminf_{q \to p} \overline{\mathrm{lx}(q)q} = \liminf_{q \to p} \overline{\mathrm{lx}(q)p}\,; \tag{6.15}$$

since $p \notin \overline{S^+}$, we have $\hat{x} \neq r$; we stress that this is the only point where Proposition 6.3 is used.

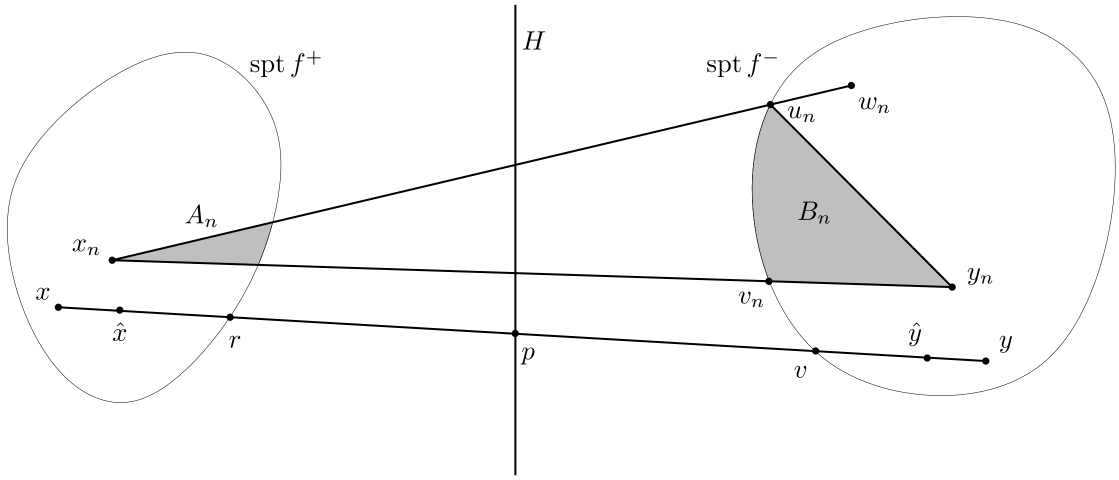


FIGURE 17. Situation in Theorem 6.1

By Lemma 5.8 and Remark 5.9, there exist two sequences $\{x_ny_n\}$ and $\{x_nw_n\}$ of maximal transport rays crossing at doubling points $x_n$, such that $x_n \to \hat{x}$, $y_n \to \hat{y}$ and the sequence $\{A_n\}$ of the triangular sets determined by $\overrightarrow{x_ny_n}$ and $\overrightarrow{x_nw_n}$ is thinning. We claim that an inequality analogous to (6.2) in Proposition 6.2 holds, that is

$$I^+(\hat{x}, r) \geq \lambda^* I^-(\hat{y}, v)\,, \tag{6.16}$$

where

$$\lambda^* := 1 + \frac{\operatorname{dist}\bigl(\operatorname{spt} f^+,\, \operatorname{spt} f^-\bigr)}{\max\bigl\{\operatorname{diam}\operatorname{spt} f^+, \operatorname{diam}\operatorname{spt} f^-\bigr\}}\,. \tag{6.17}$$

Inequality (6.16) can be proved by applying Lemma 5.11. More precisely, we call $B_n \subseteq \operatorname{spt} f^-$ the triangular set determined by $\overrightarrow{y_nu_n}$ and $\overrightarrow{y_nv_n}$, being $u_n := x_nw_n \cap \partial\operatorname{spt} f^-$ and $v_n := x_ny_n \cap \partial\operatorname{spt} f^-$. We emphasize that these sets $B_n$ are *different* from the sets $B_n$ considered in Proposition 6.2, compare Figure 14 with Figure 17; this different definition is the key new feature of the present proof w.r.t. the ones of Propositions 6.2 and 6.3.

If $\hat{y} \neq v$ (otherwise (6.16) is trivial), and since $\{B_n\}$ is thinning, by Lemma 5.12 condition (5.11) is definitively satisfied up to exchanging $y_n$ and $w_n$. Then, by Lemma 5.11, we have $f^+(A_n) \geq f^-(B_n)$ for $n \gg 1$. From this inequality, arguing in a similar way as in Proposition 6.2 and taking into account that $\hat{x} \neq r$, we infer that $I^+(\hat{x}, r) \geq (\overline{\hat{x}v}/\overline{\hat{x}r}) I^-(\hat{y}, v)$. This immediately yields (6.16) by using the definition (6.17) of $\lambda^*$.

Now, as usual we obtain a contradiction from (6.16) separately in the two cases $\hat{y} \neq y$ and $\hat{y} = y$.

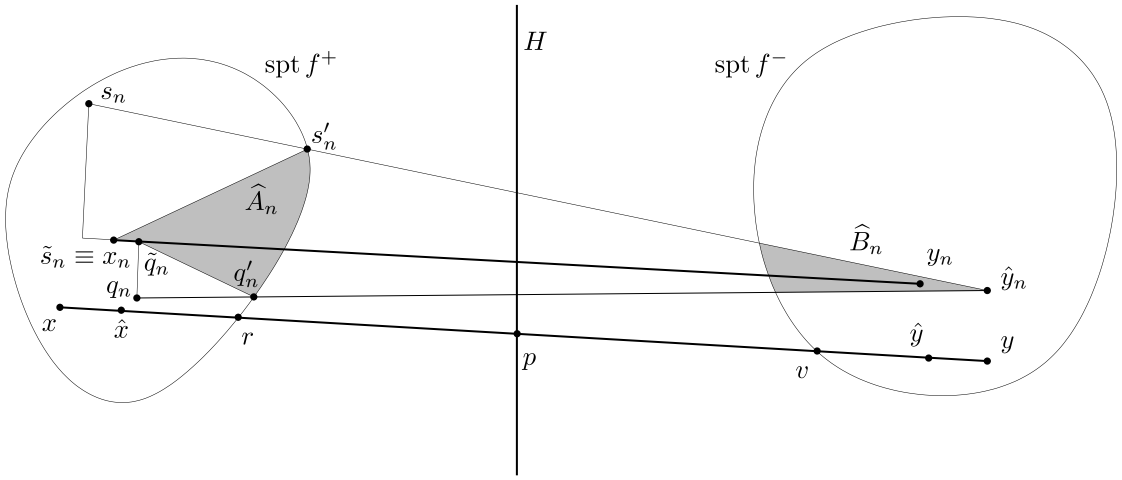


FIGURE 18. Situation in Case I of Theorem 6.1

*Case I.* $\hat{y} \neq y$.

Let $\{\hat{y}_n\}$ and $\{\widehat{B}_n\}$ be sequences of doubling points and triangular sets in spt $f^-$, defined as in Case I of Proposition 6.2, with $\widehat{B}_n$ determined by maximal transport rays $s_n\hat{y}_n$ and $q_n\hat{y}_n$. Let $\tilde{s}_n$ (resp. $\tilde{q}_n$) be the point among $x_n$ and $s_{n\perp}$ (resp. $x_n$ and $q_{n\perp}$) which is closest to $y_n$, being $s_{n\perp}$ and $q_{n\perp}$ the orthogonal projections of $s_n$ and $q_n$ onto $x_ny_n$; for instance, in Figure 18, $\tilde{s}_n \equiv x_n$ and $\tilde{q}_n \equiv q_{n\perp}$. We call $\widehat{A}_n \subseteq \mathrm{spt}\, f^+$ the union of the two triangular sets determined respectively by $\overrightarrow{\tilde{s}_n s'_n}$ and $\overrightarrow{\tilde{s}_n y_n}$, and by $\overrightarrow{\tilde{q}_n q'_n}$ and $\overrightarrow{\tilde{q}_n y_n}$, where $s'_n := s_n\hat{y}_n \cap \partial \mathrm{spt}\, f^+$ and $q'_n := q_n\hat{y}_n \cap \partial \mathrm{spt}\, f^+$. As above, these sets $\widehat{A}_n$ are different from the sets $\widehat{A}_n$ considered in Proposition 6.2, compare Figure 15 with Figure 18. Now, we can apply Lemma 5.11 once to the transport segments $s_n\hat{y}_n$ and $\tilde{s}_n y_n$, and once to the transport segments $q_n\hat{y}_n$ and $\tilde{q}_n y_n$: indeed, in both cases condition (5.12) is satisfied by construction (while condition (5.11) may not). Thus, we deduce that $t(\widehat{A}_n) \subseteq \widehat{B}_n$, so $f^+(\widehat{A}_n) \leq f^-(\widehat{B}_n)$ for $n \gg 1$. From this inequality, by arguing in the analogous way as in Proposition 6.2, taking into account that by (6.15) the lengths $\overline{\tilde{s}_n x_n}$ and $\overline{\tilde{q}_n x_n}$ converge to 0, and using definition (6.17), we get $I^-(\hat{y}, v) \geq (\overline{\hat{y}r}/\overline{\hat{y}v}) I^+(\hat{x}, r) \geq \lambda^* I^+(\hat{x}, r)$. This contradicts (6.16) because $\lambda^* > 1$.

*Case II.* $\hat{y} = y$.

In this case the proof is the very same as in Proposition 6.2. It suffices to replace the constant 2 in (6.8) by $\lambda^*$. □

## 7. C$^1$-SMOOTHNESS OF $\varphi$

In this section, by using Theorem 6.1, we show that also the second sufficient condition of Proposition 3.4 is satisfied, so that Theorem 1.2 is established.

**Theorem 7.1.** *The function $\varphi$ is of class* $\mathrm{C}^1$ *on $T^0$, and* $\inf_{x \in T^0} |\nabla\varphi(x)| > 0$.

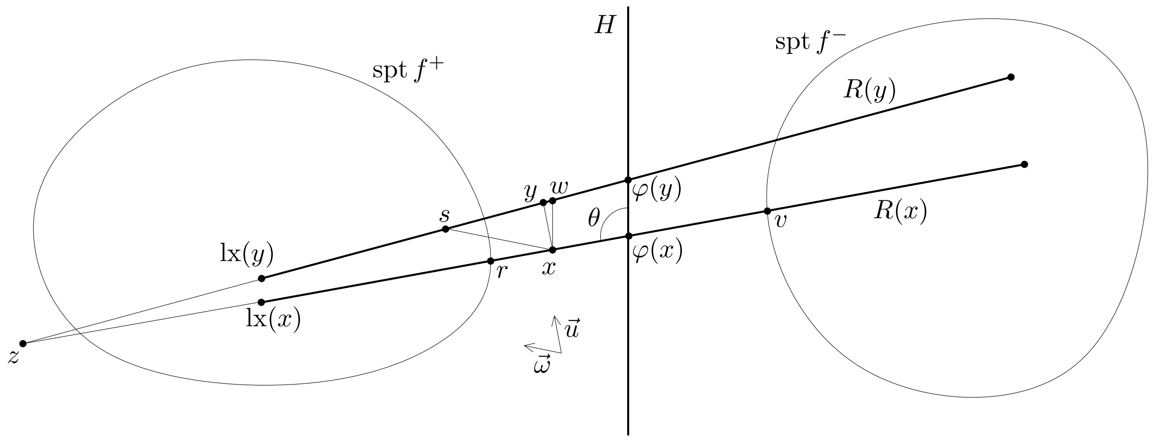


FIGURE 19. Situation in Theorem 7.1

*Proof.* Let $x \in T^0$ be fixed, with $x_1 \neq 0$. Set $r := R(x) \cap \partial \mathrm{spt}\, f^+$ and $v := R(x) \cap \partial \mathrm{spt}\, f^-$. We denote by $\theta = \theta(x) \in (0, \pi)$ the angle $\widehat{x\varphi(x)}(\varphi(x) + e_2)$, and by $\vec{u} = \vec{u}(x)$ the unit vector $\vec{u} \equiv (\cos\theta, \sin\theta)$, which is orthogonal to $R(x)$ (see Figure 19). We adopt the same notation $\varphi(x)_\sigma$ as in Definition 5.5 for the points of the ray $R(x)$; moreover, for any $\tilde{x} \in R(x)$, we set $\ell(\tilde{x})$ the "coordinate" of $\tilde{x}$ along $R(x)$, that is

$$\ell(\tilde{x}) := \frac{\tilde{x}_1}{\sin\theta}\,.$$

For $h \neq 0$, $h \ll 1$, let $y = x + h\vec{u}$ be a point of $T^0$; then, for any $\tilde{x} \in R(x)$, we denote by $\tilde{x}'$ the point of $R(y)$ whose orthogonal projection onto $R(x)$ is $\tilde{x}$. We also set

$$\Delta_\varphi(h) := \frac{\varphi(y) - \varphi(x)}{|y - x|}\,;$$

identifying $H$ with $\mathbb{R}$, we will consider $\varphi$ and $\Delta_\varphi$ as scalar functions.
For convenience, we divide the remaining of the proof in several steps.

<u>*Step I.*</u> *$\Delta_\varphi(h)$ remains bounded as $h \to 0$.*
Assume $h > 0$, so that $\Delta_\varphi(h) > 0$ and we may define the point $w := R(y) \cap \{x + t\, e_2 \,:\, t > 0\}$. By immediate geometrical arguments, there exists a positive constant $C$ such that, if $\Delta_\varphi(h) > C$, then the straight lines containing $R(x)$ and $R(y)$ cross at a point $z$ which satisfies $z_1 < 0$ if $x_1 < 0$ (as in Figure 19), and $z_1 > 0$ if $x_1 > 0$. By the non-intersection of the maximal transport rays and since $R(y)$ and $R(x)$ become parallel as $h \to 0$ (as $\varphi$ is continuous), we have

$$\liminf_{h\to 0^+} \overline{zx} \geq \overline{\mathrm{lx}(x)x} \qquad \text{and} \qquad |y - x| = |w - x|\big(\sin\theta + o(1)\big)\,.$$

Moreover, by similitude there holds

$$\frac{\varphi(y) - \varphi(x)}{|x - w|} = \frac{\overline{z\varphi(x)}}{\overline{zx}} = 1 + \frac{\overline{x\varphi(x)}}{\overline{zx}}\,,$$

so that, unless $\Delta_\varphi(h) \leq C$,

$$\limsup_{h\to 0^+} \Delta_\varphi(h) = \frac{1}{\sin\theta}\,\limsup_{h\to 0^+} \frac{\varphi(y) - \varphi(x)}{|x - w|} = \frac{1}{\sin\theta}\left(1 + \limsup_{h\to 0^+} \frac{\overline{x\varphi(x)}}{\overline{zx}}\right) \leq \frac{1}{\sin\theta}\left(1 + \frac{\overline{x\varphi(x)}}{\overline{\mathrm{lx}(x)x}}\right).$$

If $h < 0$, then $\Delta_\varphi(h) < 0$ and the same argument as above gives that, if $\Delta_\varphi(h) < -C$,

$$\liminf_{h\to 0^-} \Delta_\varphi(h) > -\frac{1}{\sin\theta}\left(1 + \frac{\overline{x\varphi(x)}}{\overline{\mathrm{lx}(x)x}}\right).$$

*Step II. If the partial derivative $\partial\varphi/\partial\vec{u}\,(x)$ exists, then $\partial\varphi/\partial\vec{\omega}\,(x)$ exists for every $\vec{\omega}\in S^1$ and*

$$\frac{\partial\varphi}{\partial\vec{\omega}}\,(x)=\vec{u}\cdot\vec{\omega}\,\frac{\partial\varphi}{\partial\vec{u}}\,(x)\,. \tag{7.1}$$

If $\vec{u}\cdot\vec{\omega}=0$ then (7.1) is immediately satisfied because $\varphi$ is constant along the direction $\vec{\omega}$ (*i.e.*, along $R(x)$). Assume then $\vec{u}\cdot\vec{\omega}\neq 0$, and let $s$ be the intersection between $R(y)$ and the straight line passing from $x$ with direction $\vec{\omega}$. Since $R(x)$ and $R(y)$ tend to become parallel, $|y-x|=|s-x|\big(\vec{u}\cdot\vec{\omega}+o(1)\big)$. Hence,

$$\frac{\partial\varphi}{\partial\vec{\omega}}(x)=\lim_{h\to 0}\frac{\varphi(s)-\varphi(x)}{|s-x|}=\vec{u}\cdot\vec{\omega}\,\lim_{h\to 0}\frac{\varphi(s)-\varphi(x)}{|y-x|}=\vec{u}\cdot\vec{\omega}\,\lim_{h\to 0}\frac{\varphi(y)-\varphi(x)}{|y-x|}=\vec{u}\cdot\vec{\omega}\,\frac{\partial\varphi}{\partial\vec{u}}\,(x)\,.$$

*Step III. Setting for brevity $J^+:=J^+\big(R(x),\overline{rr'}/\overline{vv'}\big)$ and $J^-:=J^-\big(R(x),\overline{vv'}/\overline{rr'}\big)$ (cf. Definition 5.5), we have*

$$\frac{\partial\varphi}{\partial\vec{u}}\,(x)=\frac{\ell(v)J^- -\ell(r)J^+}{\sin\theta(x)\big[(\ell(x)-\ell(r))J^+-(\ell(x)-\ell(v))J^-\big]}\,. \tag{7.2}$$

Let $B_h$ and $C_h$ be respectively the right and the left trapezoidal sets determined by $R(x)$ and $R(y)$. In order to estimate their masses, we want to apply Lemma 5.7. We point out that, as in (5.7),

$$\overline{\varphi(x)_\sigma(\varphi(x)_\sigma)'}=\overline{vv'}\rho^-(\sigma)\,,$$

being the function $\rho^-=\rho^-(R(x),\overline{vv'}/\overline{rr'})$ as in Definition 5.5. In particular, one has

$$\rho^-(\ell(x))=\frac{|y-x|}{\overline{vv'}}\qquad\text{and}\qquad\rho^-(0)=\frac{\big(\varphi(y)-\varphi(x)\big)\big(\sin\theta+o(1)\big)}{\overline{vv'}}\,,$$

so, recalling the definition of $\Delta_\varphi(h)$, it holds

$$\rho^-(\sigma)=\frac{|y-x|}{\ell(x)\overline{vv'}}\Big(\sigma+\Delta_\varphi(h)(\ell(x)-\sigma)(\sin\theta+o(1))\Big)\,. \tag{7.3}$$

Lemma 5.7 gives

$$f^-(B_h)=\overline{vv'}J^-+o\big(f^-(B_h)\big)\qquad\text{and}\qquad f^+(C_h)=\overline{rr'}J^++o\big(f^+(C_h)\big)\,. \tag{7.4}$$

Since $f^+(C_h)=f^-(B_h)$ and $\overline{rr'}/\overline{vv'}=\rho^-(\ell(r))/\rho^-(\ell(v))$, (7.3) and (7.4) give

$$\lim_{h\to 0}\frac{M^+\Delta_\varphi(h)+K^+}{M^-\Delta_\varphi(h)+K^-}=1\,, \tag{7.5}$$

where

$$M^+:=\sin\theta(x)(\ell(x)-\ell(r))J^+,\quad K^+:=\ell(r)J^+,\quad M^-:=\sin\theta(x)(\ell(x)-\ell(v))J^-,\quad K^-:=\ell(v)J^-.$$

We can now compute the limit of $\Delta_\varphi(h)$ as $h\to 0$ by exploiting equation (7.5). First, we observe that

$$K^+<0<K^- \tag{7.6}$$

so that, by Step I and (7.5), $M^+\neq M^-$. Then, we notice that

$$\liminf_{h\to 0}\big|M^-\Delta_\varphi(h)+K^-\big|>0\,. \tag{7.7}$$

Otherwise, by (7.5) one would have $M^+/M^-=K^+/K^-$ and then the quotient in the left-hand side of (7.5) would be identically equal to $K^+/K^-$; hence, by (7.6), (7.5) could not hold. By (7.7), Step I and (7.5) we infer that

$$\lim_{h\to 0}\Delta_\varphi(h)=\frac{K^- -K^+}{M^+-M^-}\,,$$

which gives (7.2), because clearly $\partial\varphi/\partial\vec{u}\,(x)=\lim_h\Delta_\varphi(h)$. Notice also that the denominator in (7.2) does not vanish as we have shown that $M^+\neq M^-$.

*Step IV.* $\varphi \in \mathrm{C}^1(T^0 \setminus H)$.
This claim immediately follows from Step II and Step III, by taking into account that $\theta$, $M^\pm$ and $K^\pm$ depend continuously on $x$ (we are using here Theorem 6.1). In particular, by (7.2) and Step II we can write

$$\nabla\varphi(x) = \frac{\ell(v)J^- - \ell(r)J^+}{\big(\ell(x)-\ell(r)\big)J^+ - \big(\ell(x)-\ell(v)\big)J^-}\Big(\frac{\cos\theta(x)}{\sin\theta(x)}, 1\Big). \tag{7.8}$$

*Step V.* $\varphi \in \mathrm{C}^1(T^0)$.

We already know that $\varphi \in \mathrm{C}^0(T^0)$ and $\nabla\varphi \in \mathrm{C}^0(T^0 \setminus H)$. Then it is enough to observe that $\nabla\varphi$ can be continuously extended on $H$ because the denominator in (7.8) remains strictly positive as $\ell(x) \to 0$ (indeed, $\ell(r) < 0 < \ell(v)$).

*Step VI.* $\inf_{x\in T^0} |\nabla\varphi(x)| > 0$.
Take a point $x \in T^0$ with $x_1 > 0$ (so $\ell(x) > 0$). Since $\rho^-(\ell(\mathrm{lx}(x)))$ must be non-negative, by (7.3) we get

$$\ell(\mathrm{lx}(x)) + \Delta_\varphi(h)\big(\ell(x) - \ell(\mathrm{lx}(x))\big)(\sin\theta + o(1)) \geq 0\,.$$

This estimate, together with the inequalities $\sin\theta \leq 1$, $\ell(\mathrm{lx}(x)) \leq -\mathrm{dist}(H, \mathrm{spt}\, f^+)$, and $\ell(x) - \ell(\mathrm{lx}(x)) \leq \mathrm{diam}\big(\mathrm{spt}\, f^+ \cup \mathrm{spt}\, f^-\big)$, implies

$$|\nabla\varphi(x)| = \lim_{h\to 0} |\Delta_\varphi(h)| \geq \frac{\mathrm{dist}(H, \mathrm{spt}\, f^+)}{\mathrm{diam}\big(\mathrm{spt}\, f^+ \cup \mathrm{spt}\, f^-\big)}\,.$$

Similarly, for any $x \in T^0$ with $x_1 < 0$, the same argument starting from the inequality $\rho(\ell(\mathrm{rx}(x))) \geq 0$ gives

$$|\nabla\varphi(x)| \geq \frac{\mathrm{dist}(H, \mathrm{spt}\, f^-)}{\mathrm{diam}\big(\mathrm{spt}\, f^+ \cup \mathrm{spt}\, f^-\big)}\,.$$

The continuity of $\nabla\varphi$ allows to conclude that $\inf_{x\in T^0} |\nabla\varphi(x)| > 0$. $\square$

## 8. Continuity of the transport density

In this section, we discuss the regularity of the *transport density* $\mu$. Such measure is an important tool in the study of Monge problem from many different points of view. For instance, it plays a crucial role in the construction of optimal transports by Evans and Gangbo [13]; furthermore (see [1, 18]), it allows to establish the connection between (1.1) and different problems, such as the mass optimization problem considered by Bouchitté, Buttazzo and Seppecher in [4, 5], and the evolution problem studied by Brenier and Benamou in [7, 6, 8].
First of all, we give the definition of transport density in our setting.

**Definition 8.1.** Given an optimal transport $\tau$, the associated *transport density* is the measure $\mu$ on $\mathbb{R}^2$ defined by

$$\langle \mu, g\rangle := \int_{\mathbb{R}^2}\int_{[x\tau(x)]} g(z)\, d\mathscr{H}^1(z)\, df^+(x) \tag{8.1}$$

for any continuous bounded function $g$ on $\mathbb{R}^2$.

The transport density $\mu$ is clearly concentrated on the transport set $T$. Roughly speaking, it measures the work done while dragging the mass $f^+$ on $f^-$ through the map $\tau$. Even though $\mu$ is defined starting from $\tau$, it has been proved [1, 14] that in fact it depends only on the sources $f^\pm$ whenever one of them is absolutely continuous w.r.t $\mathscr{L}^2$: in other words, any optimal transport defines via (8.1) the same transport density. In particular, in the one-dimensional case, it is immediate to check that, for any pair of probability measures $f^\pm$ on an oriented segment $C$, $\mu \ll \mathscr{H}^1 \llcorner C$ and its density $\underline{\mu}$ is

$$\underline{\mu}(x) = \int_{\inf C}^{x} \big(f^+(z) - f^-(z)\big)\, d\mathscr{H}^1 \llcorner C\,(x)\,. \tag{8.2}$$

Since 2000, there have been a deep interest in studying the regularity of the transport density; the main known regularity properties (see [1, 14, 11, 10, 12]) are collected in the next statement (which holds as well with $\mathbb{R}^N$ in place of $\mathbb{R}^2$).

**Theorem 8.2.** *Let $f^+$ and $f^-$ be two probability measures concentrated on an open, bounded and convex subset $X$ of $\mathbb{R}^2$. Then:*

- *if at least one between $f^+$ and $f^-$ is absolutely continuous, then so is $\mu$;*
- *if $f^+ \in L^p$ (or if $f^- \in L^p$) for some $p \in (1, +\infty]$, then also $\mu \in L^q$, with $q = q(p, X) \in (1, p)$;*
- *if both $f^\pm \in L^p$ for some $p \in [1, +\infty]$, then also $\mu \in L^p$.*

Theorem 8.2 shows that usually $\mu$ earns the same regularity as $f^\pm$; thus, it seems quite reasonable to expect that also the continuity of the sources is inherited by $\mu$. However, this result is currently unknown. Regarding to this matter, our geometrical analysis yields the following theorem.

**Theorem 8.3.** *Under the assumptions of Theorem 1.2, the transport density $\mu$ is continuous.*

*Proof.* We begin by writing $\mu$ as in (8.1), with $\tau = t$ (remind that $\mu$ is actually independent of the choice of $\tau$). Taking into account the disintegration $f^\pm = f^\pm_C \otimes \lambda$ obtained in Section 2.2, we infer by definition (8.1) that $\mu$ can be decomposed as $\mu = \mu_C \otimes \lambda$, being $\mu_C$ the transport density relative to the one-dimensional sources $f^\pm_C$. As already noticed in the proof of Lemma 3.3, $\eta = i_\# \lambda = h\mathscr{H}^1 \llcorner H$, so that the density $\underline{\mu}$ of $\mu$ is given by $\underline{\mu}(x) = h(\varphi(x))\underline{\mu}_{C(x)}(x)$. By (8.2), $\underline{\mu}_{C(x)}(x)$ can be written as

$$\underline{\mu}_{C(x)}(x) = \int_{\mathrm{lx}(x)}^{x} \Big(f^+_{C(x)}(z) - f^-_{C(x)}(z)\Big)\, d\mathscr{H}^1 \llcorner \big(C(x)\big)(z)\,.$$

Finally, the thesis follows by recalling the expressions of $h$ and $f^\pm_C$ in Lemma 3.3, and arguing as in Proposition 3.4. □

We are aware that Theorem 8.3 is far from being optimal, as the continuity of $\mu$ can be reasonably expected, provided $f^\pm$ are continuous, under much weaker assumptions. In fact, in all the examples of non-continuity of $t$ given in Section 4, $\mu$ is continuous (except for Examples 4.3 and 4.4, in which however $f^\pm$ are not continuous on the whole $\mathbb{R}^2$). In our opinion, the approach of the present paper is needlessly too accurate, and so it requires too strong assumptions, in order to study the regularity of $\mu$, which depends on the behavior of transport rays in a less subtle way than the one of $t$. This last assertion can be motivated at least by two arguments. First, $\mu$ is defined by an integral, so that its continuity should be easier to achieve. Second, while the continuity of $t$ may be prevented by zeros of $f^\pm$ (see Section 4), the same cannot be said for $\mu$. Indeed, if $f^\pm$ are multiplied by a measurable function $g$ of $\mathbb{R}^2$, constant on each maximal transport ray, $t$ remains unchanged, while $\mu$ is multiplied by $g$: in particular, $\mu$ turns out to vanish on the rays where $g = 0$, so that a suitable choice of $g$ may have a regularizing effect on $\mu$.

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Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo Da Vinci 32, 20133 Milano, Italy
*E-mail address*: fragala@mate.polimi.it

Dipartimento di Matematica, Università di Pisa, via F. Buonarroti 2, 56127 Pisa, Italy
*E-mail address*: gelli@dm.unipi.it

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy
*E-mail address*: a.pratelli@sns.it