EQUIDISTRIBUTION OF JELLIUM ENERGY FOR COULOMB AND RIESZ INTERACTIONS

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ABSTRACT. For general dimension d we prove the equidistribution of energy at the microscale in $\mathbb{R}^d$, for the optimal point configurations appearing in Coulomb gases at zero temperature. At the microscopic scale, i.e. after blow-up at the scale corresponding to the interparticle distance, in the case of Coulomb gases we show that the energy concentration is precisely determined by the macroscopic density of points, independently of the scale. This uses the "jellium energy" which was previously shown to control the next-order term in the large particle number asymptotics of the minimum energy. As a corollary we obtain the hyperuniformity of optimal point configurations for Coulomb gases, extending previous results valid only for 2-dimensional log-gases. For Riesz gases with interaction potentials $g(x) = |x|^{-s}, s \in]\min\{0, d-2\}, d[$ and one-dimensional log-gases, we prove the same equidistribution result under an extra hypothesis on the decay of the localized energy, which we conjecture to hold for minimizing configurations. In this case we use the Caffarelli-Silvestre description of the non-local fractional Laplacians in $\mathbb{R}^d$ to localize the problem.

1. INTRODUCTION

A long-standing question and direction of research at the intersection of mathematics and physics is to ask how solving the minimization problem of sums of two-body interactions between a large number of particles, or more simply between a large number of points, can lead to "collective behavior" of the minimizers, in which some better order structure is seen to emerge. A type of emergent phenomenon, in which a more rigid structure for minimizers tends to diminish the overall complexity of the configurations and is observed empirically in a large number of situations, is usually termed "crystallization". This name refers in the most restrictive meaning to the appearance of periodic structures for minimizers (see the recent review [13]). From a statistical physics viewpoint, the case in which we have a more ordered structure with higher correlations than a random one fits within the framework of crystallization. This class of phenomena has also been intensively investigated and systematized under the name of "hyperuniformity" [71] (on which we delve in more detail in section 1.3).

The particular model which we consider here comes from the theory of Riesz gases already studied in [61, 63, 62, 55, 52]. At zero temperature, we individuate and rigorously prove a *rigidity phenomenon* which is a weak version of crystallization. A corollary of our results is that minimizers of the renormalized Coulomb are very uniform configurations close to the hyperuniformity definition of [72] (see Theorem 3 below). The present work extends the result [54] valid for the 2-dimensional Coulomb gases to the case of general dimension d and of Riesz gases with power-law interactions with power $s \in [\min\{0, d-2\}, d[$, using the strategies for localizing the energy available in [52], and inspired by [25]. In particular, the present result completes the parallel between the work of Rota Nodari and Serfaty [54] and the one of

The authors are grateful to S. Serfaty for helpful discussions and useful comments.

Alberti, Choksi and Otto [2] to general dimensions, for the case $s = d - 2$. A consequence of this parallel and of the result of the present work, is the conjecture that [2] might have extensions to nonlocal interactions corresponding to Green functions for the fractional laplacian. In two dimensions the "Abrikosov conjecture" [61, 52] valid in this range of exponents is that the renormalized energy is in fact minimized by a suitably rescaled copy of the triangular lattice $\mathbb{Z} + e^{i\pi/3}\mathbb{Z}$. An analogous conjecture holds for the minimizers of the Ohta-Kawasaki type model of [2]. It is believed that in high enough dimension the lattice structure is not characteristic of minimizing configurations.

Crystallization problems have up to now been solved only for specific short range interaction potentials (see [70, 18, 40, 69, 53] and references therein) that do not cover Coulomb forces, or in 1D systems [22, 44, 62]. As a positive result, in [61, 52] it was shown however that in dimension 2 and for the above range of exponents s, if the minimizer is a lattice, then it has to be the triangular one.

Recently, the study of one-component plasmas at positive temperature has received a lot of attention. Results to some extent parallel to ours in the case of 2-dimensional Coulomb gases was provided by Bauerschmidt-Bourgade-Nikula-Yau [8], whose main result is interpretable as a quantification of discrepancy at the microscopic scale at positive temperature. This result uses tools related to the so-called loop equation, as in [3, 4]. Again in the 2-dimensional case at positive temperature, bootstrapping techniques similar to ours have allowed Leblè [46] to prove microscopic energy distribution results. Other results in the same spirit concern universality of the law of eigenvalues of random matrices [10, 15, 14, 16, 17, 74]. In the random setting the analogue of our discrepancy bounds (see Theorem 3 below) are so-called charge fluctuation results, see also [76] and the references therein.

1.1. **General setting of the problem.** We now pass to the precise description of our problem. We study the equilibrium properties of a system of n points in the full space of dimension $d \geq 1$, interacting via Riesz kernel interactions and confined by an "external field" or potential V. More precisely, we consider energies of the form

$$H_n(x_1, \cdots, x_n) = \sum_{i \neq j} g(x_i - x_j) + n \sum_{i=1}^{n} V(x_i) \tag{1}$$

where $x_1, \cdots, x_n$ are n points in $\mathbb{R}^d$ and the interaction kernel is given by either

$$g(x) = \frac{1}{|x|^s} \qquad \max(0, d-2) \leq s < d, \tag{2}$$

or

$$g(x) = -\log|x| \quad \text{in dimension } d = 1, \tag{3}$$

or

$$g(x) = -\log|x| \quad \text{in dimension } d = 2. \tag{4}$$

In the mean-field setting, the factor n multiplying the one-body potential term has the role of giving equal influence to this term as compared to the two-body interaction term. If V has some particular homogeneity, then often we can reduce to an energy of this form by an appropriate scaling. The case of $s \in [d-2, d[$, $s < 0$, which is not treated here, can happen only for $d = 1$ and this seems to be a more tractable situation. In particular the case $s = -1$,

i.e. $g(x) = |x|$, was shown to be completely solvable [1, 47, 48, 22, 44]. Note that, in what follows, we will take the convention that $s = 0$ when $g(x) = -\log|x|$, i.e. in the cases (3) and (4).

The reason why systems of particles with Coulomb and Riesz interactions are interesting in *statistical physics* is that they represent the most basic model containing the long-range interaction potentials typical of the electrostatic potential. For studies in the Coulomb case see [64, 49, 41, 51], and see [65] for a review. The possibility of changing the exponent s allows to "turn on" or "off" the locality of the interactions. The case $s \geq d$ (also called *hypersingular* case [58, 24]) corresponds to interactions of more local nature. In particular, in [61] the precise energy of our form is linked to the study of vortex systems, that appear in classical and quantum fluids [28, 31, 30] and in fractional quantum Hall physics [39, 56, 57].

Our interaction energy is also appearing in the theory of *random matrix ensembles* [6, 50, 32]. Worth mentioning, especially the Ginibre ensembles, as was described in [38, 75] and exploited by Dyson starting in [33] for $d = 2,\ s = 0$ and GOE or GUE for $d = 1,\ s = 0$, as described in [34] and the references therein.

Another direction of study in which this type of energy appears is related to Smale's 7th problem [68], which asks to find fast algorithms for minimizing our energy up to a very small error. Studies of this question are related to the optimal conditioning for interpolation points and to the theory of quadrature (see [66, 58, 59] and the references therein).

The leading order behavior of minimizers of H_n is known: there holds

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i} \rightharpoonup \mu_V,$$

where the convergence is the weak convergence of probability measures, and μ_V is the equilibrium measure, i.e. the minimizer of the energy

$$I(\mu) := \int\int_{\mathbb{R}^d\times\mathbb{R}^d} g(x-y)d\mu(x)d\mu(y) + \int_{\mathbb{R}^d} V(x)d\mu(x)\ . \tag{5}$$

The next-order behavior of H_n and of its minimizers is observed at the scale $n^{-1/d}$ at which (after blow-up) the points become well-separated. As first observed in [63], [62] via methods later extended in [55] and [52] to our general setting, if μ_V is the minimizer of I, then H_n can be split into two contributions corresponding to a constant leading order term and a typically next order term as follows:

$$H_n(x_1,\dots,x_n) = n^2\mathcal{I}(\mu_V) + 2n\sum_{i=1}^{n}\zeta(x_i) + n^{1+s/d}w_n(x_1,\dots,x_n) \tag{6}$$

in the case (2) and respectively

$$H_n(x_1,\dots,x_n) = n^2\mathcal{I}(\mu_V) - \frac{n}{d}\log n + 2n\sum_{i=1}^{n}\zeta(x_i) + nw_n(x_1,\dots,x_n) \tag{7}$$

in the cases (3) or (4), where w_n will be made explicit in Proposition 2.2 and ζ is an "effective potential" (defined below in (15)) depending only on V, which is nonnegative and vanishes on $\mathrm{supp}(\mu_V)$. As shown in [63, 62, 55, 52], w_n has a limit $\mathcal{W}$ as $n \to \infty$, which is our renormalized energy. The precise definition of $\mathcal{W}$ is given in Section 2.3 below in terms of the potential generated by the limits of configurations blown-up at the scale $n^{1/d}$. The renormalization

procedure consists in considering first a version of the energy where the charges are "spread" at scale η, and defining an energy $\mathcal{W}_\eta$ as the version of $\mathcal{W}$ where the self-interaction term of these spread charges, which becomes infinite as $\eta \to 0$, is removed. Then $\mathcal{W}$ is defined as $\lim_{\eta\to 0} \mathcal{W}_\eta$. Due to the splitting formulas (6), (7), minimizers of H_n converge, after blow-up, to minimizers of $\mathcal{W}$. As mentioned above, it is a hard mathematical conjecture corroborated by simulations and experimental evidence (the so-called "Abrikosov conjecture" in 2-dimensions being the most celebrated case), that in low dimensions the minimum of $\mathcal{W}$ is achieved at simple crystalline configurations, i.e. minimizers of $\mathcal{W}$ are expected to ressemble perfect lattices. In [63, 62, 55, 52] the analysis of the microscopic behavior of minimizers of H_n was thus connected to the behavior of minimizers of $\mathcal{W}$ by allowing to rigorously formulate the crystallization conjecture in terms of $\mathcal{W}$.

1.2. **Statement of the main results.** We now state the main results of our paper. As said before, these results are the generalization of the result of [54] for the 2-dimensional Coulomb gases to the case of general dimension d and of Riesz gases with power-law interactions. More precisely, we prove that the renormalized energy $\mathcal{W}$ is equidistributed at the microscopic scale in an arbitrary square provided that the square is chosen sufficiently far away from $\partial\Sigma$. Moreover, we improve the result of [52, Thm. 4], where it was established that almost minimizers of H_n tend to minimize $\mathcal{W}$ after blow-up at scale $n^{1/d}$ around *almost every* point in Σ. Here we show that if we deal with a minimizer of H_n this holds after blow-up around *any* point sufficiently far from the boundary of Σ (see Section 2 below for precise definitions). Note that for the case $k = 1$ we require the extra condition (8), which will be discussed in Section 2.4 below. We conjecture that this hypothesis is automatically verified for sequences of minimizing configurations, but it seems to be out of reach of the present methods. We expect that fundamentally new methods and ideas will be needed for proving this conjecture.

Theorem 1. *Let $(x_1, \cdots, x_n)$ be a minimizer of H_n. Let $\mu_V = m_V(x)dx$, $\mu'_V = m'_V(x')dx'$ be respectively the equilibrium measure and its blow-up at scale $n^{1/d}$. Let Σ' be the support of μ'_V and suppose that $m_V \in C^{0,\alpha}(\Sigma)$ for some $\alpha \in]0, 1]$.*

Let $E'_n = \nabla h'_n$ be the vector fields expressed as the gradient of the potentials of blow-up configurations corresponding to these minimizers, as in (26) below. If $k = 1$, assume that

$$\lim_{t\to\infty} \lim_{R\to\infty} \lim_{n\to\infty} \frac{1}{|K_R|} \int_{K_R\times(\mathbb{R}\setminus[-t,t])} |y|^\gamma |E'_n|^2 = 0 \tag{8}$$

uniformly with respect to the choice of the centers of the hypercubes K_R.

If $k = 0$, there exists $q \in]0, 1[$ such that for $a_n \in \Sigma'$ in the regime where $\operatorname{dist}(K_\ell(a_n), \partial\Sigma') \geq n^{q/d}$, we have

$$\lim_{\eta\to 0} \limsup_{\ell\to\infty} \limsup_{n\to\infty} \left| \frac{\mathcal{W}_\eta(E'_n, K_\ell(a_n))}{|K_\ell|} - \frac{1}{|K_\ell|} \int_{K_\ell(a_n)} \min_{\mathcal{A}_{m'_V(x')}} \mathcal{W} dx' \right| = 0\,. \tag{9}$$

If $k = 1$, for every $\varepsilon_0 > 0$ there exists a convergence regime depending on (8) and compatible with the condition $\operatorname{dist}(K_\ell(a_n), \partial\Sigma') \geq \varepsilon_0 n^{1/d}$ for $a_n \in \Sigma' \times \{0\}$ such that (9) holds.

In the above result it is natural to ask under which conditions we can interchange the $\eta \to 0$ limit with the other ones, obtaining a result valid for $\mathcal{W}$ rather than for the family $\mathcal{W}_\eta$. Our proof strategy for the above result is to select "good boundaries", and then use a screening

procedure like in [52], in order to compare different boundary conditions for the minimizers. In this case the requirement for a "good boundary" is that the field E_η should not have a large concentration of energy on such boundaries.

Unfortunately the purely energetic considerations which we apply in our proof make it impossible to control whether or not the locations of the supports $\partial B(p,\eta),\ p\in\Lambda$ of the smeared charges $\delta_p^{(\eta)}$ appearing in the second term in (39) "follow" the energy concentration of E_η locally near such good boundaries, and governed by the first term in (39). In this sense the definition (39) of our energy is really just a global one, and it may happen that large discrepancies between the behaviors $\mathcal{W}_\eta(K_\ell(a))$ and $\int_{K_\ell(a)\times\mathbb{R}^k}|y|^\gamma|E_\eta|^2$ occur for exceptional choices of $K_\ell(a)$. This lack of control prevents the exchange of the $\eta\to 0$ limit with the $n,\ell\to\infty$ limit without further assumptions on $K_\ell(a)$.

However, if we allow ourselves to slightly perturb the cubes and if we use the charge separation result of Proposition 2.3, stated below, we can perform the desired interchange of limits for the perturbed hyperrectangles, and we obtain the following:

Theorem 2. *Let $(x_1,\cdots,x_n)$ be a minimizer of H_n. Let $\mu_V=m_V(x)dx,\ \mu_V'=m_V'(x')dx'$ be respectively the equilibrium measure and its blow-up at scale $n^{1/d}$ as above. Let Σ' be the support of μ_V' and suppose that $m_V\in C^{0,\alpha}(\Sigma)$ for some $\alpha\in]0,1]$.*

Let $E_n'=\nabla h_n'$ be a sequence of blown-up vector fields corresponding to these minimizers as in (26) below. If $k=1$, assume that (8) holds uniformly with respect to the choice of the centers of the hypercubes K_R.

In either of the regimes valid for cases $k=0,\ k=1$ and linking a_n,ℓ,n like in Theorem 1 there exists sets Γ_n which can be expressed as bi-Lipschitz deformations $f_n: K_\ell(a_n)\to\Gamma_n$ such that $\|f_n-id\|_{L^\infty}\leq 1$ and such that we have

$$\limsup_{\ell\to\infty}\limsup_{n\to\infty}\left|\frac{\mathcal{W}(E_n',\Gamma_n)}{|\Gamma_n|}-\frac{1}{|\Gamma_n|}\int_{\Gamma_n}\min_{\mathcal{A}_{m_V'(x')}}\mathcal{W}dx'\right|=0\,. \tag{10}$$

Moreover, we may assume that Γ_n is a hyperrectangle.

Remark that in both theorems the result is slightly weaker in the case $k=1$. As we will explain below, this is due to the fact that we do not know the decay in the extra dimension y of the energy vector fields E_n' corresponding to minimizing configurations of points. This is why we need hypothesis (8). If we were able to prove that the r.h.s. of (8) decays to zero as a negative power of t, then (9) and (10) would hold in the regime where $\mathrm{dist}(K_\ell(a_n),\partial\Sigma')\geq n^{q/d}$ for some $q\in]0,1[$.

1.3. **Bound on discrepancy as an analogue to hyperuniformity for optimal configurations.** In series of works by Torquato and Stillinger [72, 73], for which we also refer to the recent survey [71], an intermediate regularity property of point configurations, laying between the average regularity of random point patterns and that of a crystalline configuration has been distilled, and this is the notion of *hyperuniformity*. A point distribution $\Lambda\subset\mathbb{R}^d$ is said to be hyperuniform if the number of points within a region of size $\sim R$ grows like a surface term, $\#(\Lambda\cap K_R(a))\sim R^{d-1}$. In a random or statistical setting, one requires that the variance of the number of points in a window of fixed shape and size $\sim R$ grows like R^{d-1}. There is compelling evidence suggesting that hyperuniform configurations appear in a wide range of situations, ranging from the global structure of the universe to biological systems. A

related notion of *rigidity* was also introduced in a random setting [37]; see the recent survey [36] and the references therein. However, as noted in [71], the cases where it is possible to rigorously prove hyperuniformity of point configurations are not as numerous. Our results can be seen as a progress in this direction. More precisely, as a consequence of the $k = 0$ case of Theorem 1, we deduce a decay of discrepancies, valid for $s = d - 2$, and which precisely shows that minimizers of the Coulomb jellium energy have a controlled discrepancy in all dimensions:

Theorem 3 (Discrepancy bound of jellium minimizers)**.** *Assume that $s = d - 2$, that there exist constants $\underline{m}, \overline{m} > 0$ such that $\underline{m} \leq m_V(x) \leq \overline{m}$ for all $x \in \mathrm{spt} m_V$. Further assume that we are in the regime in which* (9) *holds and that E_n' satisfy the charge separation condition of Proposition 2.3. Then letting*

$$\nu_n' := \sum_{i=1}^{n} \delta_{x_i'}$$

we have a finite asymptotic bound of the discrepancy of the ν_n' with respect to μ_V' as follows:

$$\lim_{\eta \to 0} \limsup_{\ell \to \infty} \limsup_{n \to \infty} \frac{1}{\ell^{d-1}} \left| \nu_n'(K_\ell(a)) - \int_{K_\ell(a)} m_V'(x) dx \right| < \infty. \tag{11}$$

We note that for the $d = 2$ case the above result is already present in [54]. A weaker version in which, still for $d = 2$, the decay of the absolute value term in (11) is shown to be $o(\ell^d)$ rather than $O(\ell^{d-1})$ like here, was also proved via Beurling-Landau densities, in [5].

The above control of discrepancies could also prove useful in making more rigorous the study of *scars*, i.e. the study of topological defects appearing in numerical simulations of point configurations on manifolds. In this case it is apparent by numerical simulations that defects arise, with a large literature focussing on the case of points distributed on the 2-dimensional sphere [21, 9, 19, 20] and also on more general surfaces and manifolds as in [7, 43]. However in this case rigorous mathematical studies of the asymptotics of defects in the large-n limit seem to be difficult, also due to the lack of a well-accepted and easy to control notion of "defect". In the case of the 2-dimensional sphere, a first step in the study of the next-order term which may allow to obtain a version of our functional $\mathcal{W}$ on the sphere has been recently provided by Bétermin-Sandier in [11]. As the presence of scars may be detected by the presence of localized higher discrepancy regions, it seems that the above result, if transferred to the case of points living on compact manifolds, may provide a tool towards a rigorous proof of such asymptotics.

2. Assumptions and main definitions

2.1. **Hypotheses on V and μ_V.** The minimization of I, defined by (5), on $\mathcal{P}(\mathbb{R}^d)$, the space of probability measures on $\mathbb{R}^d$, is a standard problem in potential theory (see [35] or [59] for $d = 2$). In particular, if V satisfies the following assumptions:

$$V \text{ is l.s.c. and bounded below,} \tag{12}$$

$$\{x : V(x) < \infty\} \text{ has positive } g\text{-capacity,} \tag{13}$$

$$\lim_{|x| \to \infty} V(x) = +\infty, \quad \text{resp. } \lim_{|x| \to \infty} \frac{V(x)}{2} - \log|x| = +\infty \text{ in cases (3) – (4),} \tag{14}$$

then the minimum of I over $\mathcal{P}(\mathbb{R}^d)$ exists, is finite and is achieved by a unique equilibrium measure μ_V, which has a compact support Σ of positive g-capacity. In addition μ_V is uniquely characterized by the fact that[1]

$$\begin{cases} h^{\mu_V} + \frac{V}{2} \geq c & \text{q.e.} \quad , \\ h^{\mu_V} + \frac{V}{2} = c & \text{q.e. on } \Sigma. \end{cases}$$

where $h^{\mu_V}(x) := \int g(x-y)d\mu_V(y)$ and $c := I(\mu_V) - \int \frac{V}{2} d\mu_V$.

Note that h^{μ_V} can be characterized as the unique solution of a fractional obstacle problem, and the corresponding regularity theory [67, 27, 26] allows to obtain regularity results on h^{μ_V} and on the free-boundary $\partial\Sigma$ in terms of the regularity of V. We will write

$$\zeta := h^{\mu_V} + \frac{V}{2} - c \geq 0. \tag{15}$$

Like in [63, 55, 52], it is assumed that μ_V is absolutely continuous with respect to the Lebesgue measure, with density also denoted by μ_V by abuse of notation, and in order to make the explicit constructions easier, we need to assume that this density is bounded and sufficiently regular on its support. More precisely, we make the following technical, and certainly not optimal, assumptions:

$$\partial\Sigma \text{ is } C^1 \tag{16}$$

$$\mu_V \text{ has a density which is } C^{0,\beta} \text{ in } \Sigma, \tag{17}$$

$$\exists c_1, c_2, \overline{m} > 0 \text{ s.t. } c_1 \mathrm{dist}(x,\partial\Sigma)^\alpha \leq \mu_V(x) \leq \min(c_2 \mathrm{dist}(x,\partial\Sigma)^\alpha, \overline{m}) < \infty \text{ in } \Sigma,$$

with the conditions

$$0 < \beta \leq 1, \qquad 0 \leq \alpha \leq \frac{2\beta d}{2d-s}. \tag{18}$$

Of course if $\alpha < 1$ one should take $\beta = \alpha$, and if $\alpha \geq 1$, one should take $\beta = 1$ and $\alpha \leq \frac{2d}{d-s}$. These assumptions include the case of the semi-circle law $\frac{1}{2\pi}\sqrt{4-x^2}1_{|x|<2}$ arising for the quadratic potential in (3). In the Coulomb cases, a quadratic potential gives rise to the equilibrium measure $c1_B$ where $B \subset \mathbb{R}^d$ is a ball, a case also covered by our assumptions with $\alpha = 0$. In the Riesz case, any compactly supported radial profile can be obtained as the equilibrium measure associated to some potential (see [29, Corollary 1.4]). Our assumptions are thus never empty.

2.2. **Blowup, regularization and splitting formula.** The renormalized energy appears in [52] as a next order limit of H_n after a blow-up is performed, at the inverse of the typical nearest neighbor distance between the points, i.e. $n^{1/d}$. It is expressed in terms of the potential generated by the configuration $x_1, \cdots, x_n$ and defined by

$$h_n(X) = g * \left(\sum_{i=1}^n \delta_{(x_i,0)} - n\mu_V \delta_{\mathbb{R}^d} \right). \tag{19}$$

Recall that the Riesz kernel g is not the convolution kernel of a local operator, as in the Coulomb case $s = d-2$ or (4), where g is the kernel of the Laplacian. It is the kernel of a fractional Laplacian, which is a nonlocal operator. It turns out however that, as originally noticed in [25], if $d-2 < s < d$ then this fractional Laplacian operator can be transformed

[1]Recall that using the usual notation of potential theory [45], here "quasi everywhere", abbreviated "q.e.", means "up to sets of zero g-capacity".

into a local but inhomogeneous operator of the form $\mathrm{div}(|y|^\gamma \nabla \cdot)$ by adding one space variable $y \in \mathbb{R}$ to the space $\mathbb{R}^d$. The number γ is chosen such that

$$\gamma = s - d + 2 - k \tag{20}$$

where k will denote the dimension extension. We will take $k = 0$ in all the Coulomb cases, i.e. $s = d - 2$ and $d \geq 3$ or (4). In all other cases, we will need to take $k = 1$. In the particular case of $s = d - 1$ then $\gamma = 0$, and this correspond to using a harmonic extension (see [25, 62, 52] for more details). Points in the space $\mathbb{R}^d$ will be denoted by x, and points in the extended space $\mathbb{R}^{d+k}$ by X, with $X = (x, y)$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^k$.

For the blown-up quantities we will use the following notation (with the convention $s = 0$ in the cases (3) or (4)):

$$x' = n^{1/d} x \quad X' = n^{1/d} X \quad x'_i = n^{1/d} x_i \tag{21}$$

$$\mu'_V(x') = \mu_V(x) \tag{22}$$

$$h'_n(X') = n^{-\frac{s}{d}} h_n(X). \tag{23}$$

In particular if $\Sigma = \mathrm{spt}(\mu_V), \Sigma' = \mathrm{spt}(\mu'_V)$ then there holds

$$\Sigma' = n^{\frac{1}{d}} \Sigma. \tag{24}$$

We note that h_n and h'_n satisfy

$$-\mathrm{div}(|y|^\gamma \nabla h_n) = c_{s,d} \Big(\sum_{i=1}^n \delta_{x_i} - n \mu_V \delta_{\mathbb{R}^d} \Big) \quad \text{in } \mathbb{R}^{d+k}, \tag{25}$$

$$-\mathrm{div}(|y|^\gamma \nabla h'_n) = c_{s,d} \Big(\sum_{i=1}^n \delta_{x'_i} - \mu'_V \delta_{\mathbb{R}^d} \Big) \quad \text{in } \mathbb{R}^{d+k}, \tag{26}$$

In order to define our renormalized energy we need to truncate and regularize the Riesz (or logarithmic) kernel. We define the truncated kernel as in [52], in the following way: for $1 > \eta > 0$ and $X \in \mathbb{R}^{d+k}$, let

$$f_\eta(X) = (g(X) - g(\eta))_+ . \tag{27}$$

We note that the function f_η vanishes outside of $B(0, \eta)$ and satisfies that

$$\delta_0^{(\eta)} := \frac{1}{c_{s,d}} \mathrm{div}(|y|^\gamma \nabla f_\eta) + \delta_0 \tag{28}$$

is a positive measure supported on $\partial B(0, \eta)$, and which is such that for any test-function φ,

$$\int \varphi \delta_0^{(\eta)} = \frac{1}{c_{s,d}} \int_{\partial B(0,\eta)} \varphi(X) |y|^\gamma g'(\eta).$$

One can check that $\delta_0^{(\eta)}$ is the uniform measure of mass 1 on $\partial B(0, \eta)$, and we may write

$$-\mathrm{div}(|y|^\gamma \nabla f_\eta) = c_{s,d}(\delta_0 - \delta_0^{(\eta)}) \quad \text{in } \mathbb{R}^{d+k}. \tag{29}$$

We will also denote by $\delta_p^{(\eta)}$ the measure $\delta_0^{(\eta)}(X - p)$, for $p \in \mathbb{R}^d \times \{0\}$. In the Coulomb cases, i.e. when $k = 0$, then $\delta_0^{(\eta)}$ is the same as in [55]. If h can be written in the form (19), then

we will also denote

$$h_\eta := h - \sum_{i=1}^n f_\eta(x - x_i). \tag{30}$$

Remark 2.1. *For $h = h_n$ as in* (19) *the transformation from h to h_η amounts to truncating the kernel g, but only for the Dirac part of the r.h.s. Indeed, letting $g_\eta(x) = \min(g(x), g(\eta))$ be the truncated kernel, we have*

$$h_\eta = g_\eta * (\sum_{i=1}^n \delta_{x_i}) - g * (m\delta_{\mathbb{R}^d}).$$

In view of (29), $h_{n,\eta}$ and $h'_{n,\eta}$, defined from h_n and h'_n via (30), satisfy

$$-\operatorname{div}(|y|^\gamma \nabla h_{n,\eta}) = c_{s,d}\Big(\sum_{i=1}^n \delta_{x_i}^{(\eta)} - n\mu_V \delta_{\mathbb{R}^d}\Big) \quad \text{in } \mathbb{R}^{d+k}, \tag{31}$$

$$-\operatorname{div}(|y|^\gamma \nabla h'_{n,\eta}) = c_{s,d}\Big(\sum_{i=1}^n \delta_{x'_i}^{(\eta)} - \mu'_V \delta_{\mathbb{R}^d}\Big) \quad \text{in } \mathbb{R}^{d+k}, \tag{32}$$

with the usual embedding of $\mathbb{R}^d$ into $\mathbb{R}^{d+k}$. We now recall the splitting formula from [52].

Proposition 2.2 (Splitting formula)**.** *For any n, any $x_1, \cdots, x_n$ distinct points in $\mathbb{R}^d \times \{0\}$, letting h_n be as in* (19) *and $h_{n,\eta}$ deduced from it via* (30)*, we have in the case* (2)

$$H_n(x_1, \cdots, x_n) = n^2 I(\mu_V) + 2n \sum_{i=1}^n \zeta(x_i) + n^{1+\frac{s}{d}} \lim_{\eta \to 0} \frac{1}{c_{s,d}} \left(\frac{1}{n} \int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla h'_{n,\eta}|^2 - c_{s,d} g(\eta) \right), \tag{33}$$

respectively in the cases (3)–(4)

$$H_n(x_1, \cdots, x_n) = n^2 I(\mu_V) + 2n \sum_{i=1}^n \zeta(x_i) - \frac{n}{d} \log n + n \lim_{\eta \to 0} \frac{1}{c_{s,d}} \left(\frac{1}{n} \int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla h'_{n,\eta}|^2 - c_{s,d} g(\eta) \right). \tag{34}$$

One expects the repelling points x_i to organise in a very uniform way, and thus that the interpoint distance asymptotically decreases like $n^{-1/d}$. The following is proven in [52], by potential-theoretic methods [45, 23] and using the maximum principle.

Proposition 2.3 (Point separation, [52, Thm. 5])**.** *Let $(x_1, \ldots, x_n)$ minimize H_n. Then for each $i \in [1, n]$, $x_i \in \Sigma$, and for each $i \neq j$, it holds*

$$|x_i - x_j| \geq \frac{r}{(n \max_x |m(x)|)^{1/d}},$$

where r is some positive constant depending only on s and d.

The scale $\sim n^{-1/d}$ is then termed the *microscopic scale* of our gases, and the two-scale reformulation of the energy H_n as done in [63, 55, 52] involves separating the energy contributions from the macroscale and from this microscopic scale. In particular the distribution of points at the microscopic scale is governed by the renormalized energy $\mathcal{W}$ to be introduced below.

2.3. **The renormalized energy.** Consider the formulas appearing in Proposition 2.2. As $\zeta \geq 0$ and $\zeta = 0$ on $\mathrm{spt}(\mu_V)$, it acts as an effective potential, favouring the configurations where the points x_i are in the support of μ_V. The last term produces the next-order term of the energy, and justifies the definition of the renormalized energy $\mathcal{W}$ of an infinite configuration of points. This functional $\mathcal{W}$ is defined via the gradient of the potential generated by the point configuration, embedded into the extended space $\mathbb{R}^{d+k}$. That gradient is a vector field that we denote E (like electric field, by analogy with the Coulomb case). Then E will solve a relation of the form

$$-\operatorname{div}(|y|^\gamma E) = c_{d,s}\Big(\sum_{p\in\Lambda}\delta_p - m(x)\delta_{\mathbb{R}^d}\Big) \quad \text{in } \mathbb{R}^{d+k}. \tag{35}$$

where Λ is some discrete set in $\mathbb{R}^d\times\{0\}$ (identified with $\mathbb{R}^d$). Due to the fact that (as recalled in Proposition 2.3) the minimizers of our energy have separated charges, we restrict in the present work to fields E corresponding to multiplicity-one charges, as opposed to general positive integer multiplicity case considered in [55, 52]. For any such E (defined over $\mathbb{R}^{d+k}$ or over subsets of it), we define, by a formula generalizing (30),

$$E_\eta := E - \sum_{p\in\Lambda}\nabla f_\eta(X-p). \tag{36}$$

We will write Φ_η for the map that sends E to E_η, and note that it is a bijection from the set of vector fields satisfying a relation of the form (35) to those satisfying a relation of the form

$$-\operatorname{div}(|y|^\gamma E_\eta) = c_{d,s}\Big(\sum_{p\in\Lambda}\delta_p^{(\eta)} - m(x)\delta_{\mathbb{R}^d}\Big) \quad \text{in } \mathbb{R}^{d+k}. \tag{37}$$

The class of vector fields on which we are going to concentrate is thus the following:

Definition 2.4 (Admissible vector fields)**.** *Given a non-negative density function $m:\mathbb{R}^d\to\mathbb{R}^+$, we define the class $\mathcal{A}_m$ to be the class of gradient vector fields $E=\nabla h$ that satisfy*

$$-\operatorname{div}(|y|^\gamma E) = c_{s,d}\Big(\sum_{p\in\Lambda}\delta_p - m(x)\delta_{\mathbb{R}^d}\Big) \quad \textit{in } \mathbb{R}^{d+k} \tag{38}$$

where Λ is a discrete set of points in $\mathbb{R}^d\times\{0\}$.

In case $m\in L^\infty_{loc}$, vector fields as above blow up exactly in $1/|X|^{s+1}$ near each $p\in\Lambda$ (with the convention $s=0$ for the cases (3)–(4)); such vector fields naturally belong to the space $L^p_{loc}(\mathbb{R}^{d+k},\mathbb{R}^{d+k})$ for $p<\frac{d+k}{s+1}$.

We are now in a position to define the renormalized energy for blow-up configurations like in [55, 52]. In the definition, we let K_R denote the hypercubes $[-R/2,R/2]^d$.

Definition 2.5 (Renormalized energy)**.** *Let $E\in\mathcal{A}_m$, satisfy* (35) *and $f:\mathbb{R}^{d+k}\to\mathbb{R}^+$ be a measurable function. Then for $0<\eta<1$ we define*

$$\mathcal{W}_\eta(E,f) = \int_{\mathbb{R}^{d+k}} |y|^\gamma f|E_\eta|^2 - c_{d,s}g(\eta)\int_{\mathbb{R}^{d+k}} f\sum_{p\in\Lambda}\delta_p^{(\eta)}. \tag{39}$$

For $A \subset \mathbb{R}^d$ a Borel set we define $\mathcal{W}_\eta(E, A) := \mathcal{W}_\eta(E, 1_{A\times\mathbb{R}^k})$ where 1_S is the characteristic function which equals 1 on a set S and 0 outside S. We then define

$$\mathcal{W}(E, A) = \lim_{\eta\to 0} \mathcal{W}_\eta(E, A), \qquad \mathcal{W}(E) = \lim_{\eta\to 0} \limsup_{R\to\infty} \frac{\mathcal{W}_\eta(E, K_R)}{R^d}. \tag{40}$$

Remark 2.6. *Note that if $\chi_{A,\epsilon}(x, y) = 1_A * \rho_\epsilon^{(d)}(x)1_{[-R_\epsilon,R_\epsilon]^k} * \rho_\epsilon^{(k)}(y)$ are C_c^∞ functions approximating $1_{A\times\mathbb{R}^k}$ where $R_\epsilon \to \infty$ as $\epsilon \to 0$ and $\rho_\epsilon^{(n)}(z) = \epsilon^{-n}\rho^{(n)}(z/\epsilon)$ denotes mollifiers based on a smooth radial probability density $\rho^{(n)}$ supported on the unit ball of $\mathbb{R}^n$ then we have $\mathcal{W}_\eta(E, A) = \lim_{\epsilon\to 0} \mathcal{W}_\eta(E, \chi_{A,\epsilon})$, by monotone convergence in* (39).

The name renormalized energy (originating in Bethuel-Brezis-Hélein [12] in the context of two-dimensional Ginzburg-Landau vortices) reflects the fact that $\int |y|^\gamma |\nabla h|^2$ which is infinite, is computed in a renormalized way by first changing h into h_η and then removing the appropriate divergent terms $c_{s,d}g(\eta)$ corresponding to all points.

The above is a generalized version of the renormalized energy defined in [52], and fits in the framework of the study of "jellium energies", for which we refer to [13] and to the references therein. As in [55, 52] the next-order functional $\mathcal{W}$ differs from the one defined in previous works by Sandier-Serfaty [61, 62] for the one and two-dimensional logarithmic interaction, essentially in the fact that the order of the limits $R \to \infty$ and $\eta \to 0$ is reversed. We refer to [55] for a further discussion of the comparison between the two.

In the case of constant m, by scaling we may always reduce to studying the class $\mathcal{A}_1$, indeed, if $E \in \mathcal{A}_m$ and A is Borel, then $\hat{E} = m^{-\frac{s+1}{d}} E(c_{s,d} \cdot m^{-1/d}) \in \mathcal{A}_1$ [2] and

$$\mathcal{W}_\eta(E, A) = m^{1+s/d}\mathcal{W}_{\eta m^{1/d}}(\hat{E}, m^{1/d}A) \qquad \mathcal{W}(E) = m^{1+s/d}\mathcal{W}(\hat{E}). \tag{41}$$

in the case (2), and respectively

$$\mathcal{W}_\eta(E, A) = m\left(\mathcal{W}_{m\eta}(\hat{E}, m^{1/d}A) - \frac{2\pi}{d}\log m\right) \qquad \mathcal{W}(E) = m\left(\mathcal{W}(\hat{E}) - \frac{2\pi}{d}\log m\right) \tag{42}$$

in the cases (3)–(4).

2.4. **Discussion on the hypothesis** (8)**.**

2.4.1. *Power-law bound.* For studying the case $k = 1$ we need a further assumption regarding decay in the extra dimension (i.e. as $|y| \to \infty$) of the energy vector fields E_n' corresponding to minimizing configurations of points. It is tempting to conjecture that for fields E_n' corresponding to minimizers there exist constants $a, C > 0$ such that for each bilipschitz deformation of a cube $K \subset \mathbb{R}^d$ there holds

$$\int_{K\times(\mathbb{R}\setminus[-t,t])} |y|^\gamma |E_n'|^2 \le C_2|K|t^{-a}. \tag{43}$$

Such decay is not true for general configuratons for which $\mathcal{W}(E) < \infty$, and it seems to be equivalent to a uniformity condition on the field-generating configurations. We note here that this condition holds in the case of lattice-like configurations:

Lemma 2.7 (Power-like decay for lattices)**.** *Assume that $\Lambda \subset \mathbb{R}^d$ is a Bravais lattice of density one and consider the admissible vector field E' corresponding to Definition 2.4 with this choice of Λ and measure $m \equiv 1$. Then* (43) *holds with $a = s - d + 2 > 0$.*

[2] with the convention $s = 0$ in the case (3)

Proof. We prove the result in the case $\Lambda = \mathbb{Z}^d$ but the same proof works for a general Bravais lattice of density one. We can calculate the norm of $E_{loc} := \nabla h = \nabla(g * (\sum_{p\in\Lambda} \delta_p - \delta_{\mathbb{R}^d}))$ at a point (x_0, y) with $|y| = t$. As Λ is periodic, so is E_{loc} and we may suppose that $x_0 \in [0,1]^d$. Then we find,

$$\begin{aligned} h(x_0,y) &= -|\mathbb{S}^{d-1}| \int_0^\infty \frac{r^{d-1}}{(t^2+r^2)^{s/2}} dr + \sum_{p\in\mathbb{Z}^d+x_0} \frac{1}{(t^2+|p|^2)^{s/2}} \\ &= -t^{d-s}|\mathbb{S}^{d-1}| \int_0^\infty \frac{\rho^{d-1}d\rho}{(1+\rho^2)^{s/2}} + t^{-s} \sum_{q\in t^{-1}(\mathbb{Z}^d+x_0)} \frac{1}{(1+|q|^2)^{s/2}}, \end{aligned}$$

and we see that this expression is t^{-s} times the error in the approximation for the Riemann integral of $(1+|x|^2)^{-s/2}$ given by the partition of $\mathbb{R}^d$ in cubic cells centered at $t^{-1}(\mathbb{Z}^d + x_0)$. To compute $\nabla h = \nabla(g * (\sum_{p\in\Lambda} \delta_p - \delta_{\mathbb{R}^d}))$ at the same point we must take into account the fact that cancellations occur: for the continuum counterpart the horizontal contributions to ∇h from points symmetric with respect to x cancel, and the length of the vertical component is $t/(t^2+r^2)^{1/2}$. We thus find that $|E_{loc}|(x,t)$ is t^{-s-1} times the Riemann sum approximation error for the integral of $|x|(1+|x|^2)^{-(s+3)/2}$ corresponding to the decomposition of $\mathbb{R}^d$ by cubic cells centered at $t^{-1}(\mathbb{Z}^d + x_0)$. It follows that uniformly in x_0 we have the sharp power decay bound $|\nabla h| \le t^{d-s-2}$ and thus (using also the relation $\gamma = s - d + 2 - k = s - d + 1$ valid here)

$$\int_{\mathbb{R}\setminus[-t,t]} |y|^\gamma |E_{loc}|^2(x_0,y)dy \le Ct^{2d-2s-3+\gamma} = Ct^{d-2-s},$$

which implies (43). □

2.4.2. *Weaker decay bound.* In general we were not able to find a way to prove a bound of the form (43) for our minimizers E_n', and we leave the question of whether the lattice-configurations have the same decay power a as the minimizers for future work. This can be seen as a uniformity conjecture on the minimizers, which therefore is a weaker version of the conjecture that lattices are minimizers.

Note also that so far we didn't exclude that there exist distinct minimizing configurations with different decay of the energy.

We will denote, compatibly with the notation (43), that for a vector field E, a cube $K \subset \mathbb{R}^d$ and a "height" t,

$$\frac{1}{|K|} \int_{K\times(\mathbb{R}\setminus[-t,t])} |y|^\gamma |E|^2 := C_2(E,t,K). \tag{44}$$

In the above notation (43) states that $C_2(E,t,K) = C_2 t^{-a}$, independently of K. We denote as follows some more global bounds, provided that they are finite.

$$\begin{aligned} C_2(E,t,L) &:= \sup\left\{C_2(E,t,K_l(a)) :\ l\in[L/2,2L],\ a\in\mathbb{R}^d\right\}, \quad L>0, \\ C_2(E,t) &:= \sup\left\{C_2(E,t,L) : L>1\right\}. \end{aligned}$$

We then see that, by expressing the average over $K_{2l}(a)$ as the average of the 2^d averages over distinct subcubes $K_l(a')$ partitioning $K_{2l}(a)$ up to measure zero, we find

$$C_2(E,t,2L) \le C_2(E,t,L) \quad \text{for all} \quad t>0. \tag{45}$$

Recall that in [52, Prop. 7.1] it was proved that there exists a minimizer E of $\mathcal{W}_\eta$ over $\mathcal{A}_1$ which satisfies

$$\lim_{t\to\infty}\lim_{R\to\infty} C_2(E,R,t)=0,$$

and moreover (see [52, Sect. 5]) a weak limit of a subsequence of rescaled minimizers E'_n provides an E with the above property for generic choices of the blow-up centers.

We need however to use the stronger fact that the choice of E'_n corresponding to a blow-up sequence of minimizers in our problem, satisfies this type of uniform bound too as stated in hypothesis (8).

3. Screening lemmas

The following is the main tool for selecting the good boundaries in our constructions. This is used later in combination with our precise splitting formula of Proposition 5.1 in performing the screening construction.

Proposition 3.1 (Good boundary slices)**.** *Let K_T be a rectangle of sidelenghts $\sim T$, let $\rho\in L^\infty(K_T)$ and fix $\eta\in]0,1[$. If $\Lambda\subset\mathbb{R}^d$ is a discrete set, we define*

$$\nu=\sum_{p\in\Lambda}\delta_p,\quad \nu^{(\eta)}=\sum_{p\in\Lambda}\delta_p^{(\eta)}.$$

Assume that $E=\nabla h$ as in (26) on the rectangle K_T, in particular

$$-\mathrm{div}(|y|^\gamma E_\eta)=c_{d,s}\left(\nu^{(\eta)}-\rho(x)\delta_{\mathbb{R}^d}\right)\qquad \text{in } K_T\times\mathbb{R}^k.$$

Further assume that E is controlled in the following sense:

$$\frac{1}{T^d}\int_{K_T\times[-t,t]^k}|y|^\gamma|E_\eta|^2\le C_1 \tag{46}$$

for some positive constant C_1 and $t\in[0,T]$.

Let $\varepsilon_1\in]0,1[$, $L_i\in[\varepsilon_1^{1/d}T_i,T_i]$, $i=1,\dots,d$ and $a\in\mathbb{R}^d\times\{0\}$ such that the rectangle $K_L(a)$ of sidelenghts L_i is included in K_T and, if $k=1$, assume that

$$\frac{1}{L^d}\int_{K_L\times\left(\mathbb{R}\setminus[-\frac12 t,\frac12 t]\right)}|y|^\gamma|E_\eta|^2\le C_2. \tag{47}$$

for positive constant C_2 and $t\in[0,T]$.

Let $l\in]0,L/3]$. There exists a rectangle $K'_L(a)\subset K_L(a)$ such that $\mathrm{dist}(\partial K'_L(a),\partial K_L(a))\in[l,2l[$ and a universal constant C such that the following hold:

$$\int_{\partial K'_L\times[-t,t]^k}|y|^\gamma|E_\eta|^2\ \ \le\ \ CC_1\varepsilon_1^{-1}l^{-1}L^d\ , \tag{48}$$

and when $k=1$ there exists $t'\in[t/2,t]$ such that

$$\int_{K'_L\times\{-t',t'\}}|y|^\gamma|E_\eta|^2<CC_2L^dt^{-1}. \tag{49}$$

Remark 3.2. *We recall that if the charges corresponding to a vector field E are well-separated at distance r_0 then the value of $\mathcal{W}_\eta(E,A)$ over a domain A is bounded in terms of the volume $|A|$. More precisely, if $C_{r_0}=|B_{r_0}|^{-1}$ then we have*

$$\int_{A\times\mathbb{R}^k}|y|^\gamma|E_\eta|^2-C_{r_0}c_{d,s}g(\eta)|A|\le\mathcal{W}_\eta(E,A)\le\int_{A\times\mathbb{R}^k}|y|^\gamma|E_\eta|^2\ , \tag{50}$$

therefore in our error bounds on thin boundary layers of cubes the main difficulty is in estimating the $L^2_{|y|^\gamma}$-norm of E_η rather than the remainder of $\mathcal{W}_\eta$. In the setting of Proposition 3.1 we obtain the bound

$$\frac{\mathcal{W}_\eta(E,K_L)}{|K_L|} \geq \frac{\mathcal{W}_\eta(E,K'_L)}{|K'_L|} + C_{s,d,\overline{m}}(g(\eta)+1)\frac{l}{L}, \tag{51}$$

where we include in the notation the fact, mentioned in Proposition 2.3 (see [52, Thm. 5]*) that r_0 depends on $\max_x m(x), s, d$ only.*

Proof. We consider the boundaries $\partial K_\tau(a)$ for $\tau \in [L-2l, L-l]$. Then the existence of $K'_L(a)$ such that (48) holds, follows by applying the mean value theorem and using (46) and the fact that $L_i \geq \varepsilon_1^{1/d} T_i$. To obtain (49) for $k=1$, we use a mean value principle and (47) to find $t' \in [t/2, t]$ such that

$$\int_{K_L \times \{-t', t'\}} |y|^\gamma |E|^2 < 2C_2 L^d t^{-1},$$

after which (49) follows. □

3.1. **Interchanging small energy boundary data by screening.** The following is a version of proposition 6.1 of [52] with a varying background measure instead of a constant one. We include the case where we pass with a small energy change from a good boundary datum on the inner hyperrectangle to zero on the outer one, and the case where such roles are inverted and we rather pass from a good boundary on the outer hyperrectangle to zero boundary datum on the inner one.

Proposition 3.3 (Screening)**.** *Let K_T be a rectangle of sidelenghts $\sim T$, let ρ be a $C^{0,\alpha}(K_{T+1})$ function with $\alpha > 0$ for which there exist $\underline{\rho}, \overline{\rho} > 0$ such that $\underline{\rho} \leq \rho(x) \leq \overline{\rho}$. Fix $\eta \in]0,1[$, let $\Lambda \subset \mathbb{R}^d$ and assume there exists $r_0 \in]0, 1/2]$ such that $\min\{|p-q| : \ p \neq q \in \Lambda\} \geq \sqrt{d} r_0$. Assume that $E = \nabla h$ such that*

$$-\mathrm{div}(|y|^\gamma E) = c_{d,s}\left(\sum_{p\in\Lambda} \delta_p - \rho(x)\delta_{\mathbb{R}^d}\right) \qquad \text{in } K_T \times \mathbb{R}^k,$$

and that E is controlled as in (46) with $C_1 > 0$. Moreover, let $\varepsilon_1 \in]0,1[$, $L_i \in [\varepsilon_1^{1/d} T_i, T_i]$ for $i = 1, \dots, d$ and $a \in \mathbb{R}^d \times \{0\}$ be such that the rectangle $K_L(a)$ of sidelenghts L_i is included in K_T. If $k=1$, assume also that C_2 is such that (47) holds, i.e. that $C_2 \geq C_2(E,t,L)$. Let $K'_L(a), l$ be as in the claim of Proposition 3.1.

There exists a constant c_0 such that if L is sufficiently large and

$$\begin{cases} C_1^{\frac{1}{2}} t^{-\frac{d+1}{2}} L^{\frac{d}{2}} \varepsilon_1^{-\frac{1}{2}} l^{-\frac{1}{2}} \leq c_0 & \text{for } k=0 \\ t^{\frac{\gamma-3}{2}} L C_2^{\frac{1}{2}} \leq c_0 \text{ and } C_1^{\frac{1}{2}} t^{\frac{-d+\gamma}{2}} L^{\frac{d}{2}} \varepsilon_1^{-\frac{1}{2}} l^{-\frac{1}{2}} \leq c_0 & \text{for } k=1 \end{cases}, \tag{52}$$

and if we denote

$$err_{sc}(\eta, t, L, l, \varepsilon_1, C_1, C_2) := C_1 \varepsilon_1^{-1} l^{-1} g(\eta) + C_2 t^{-1} L + C_1 \varepsilon_1^{-1} l^{-1} t + g(\eta) L^{-1} t, \tag{53}$$

then the following hold.

(1) There exists $\bar{K}_L(a)$ *such that* $K'_L(a) \subset \bar{K}_L(a)$, $\mathrm{dist}(\partial K'_L(a), \partial \bar{K}_L(a)) \in [t, 2t]$ *and* $\int_{\bar{K}_L(a)} \rho(x)\,dx \in \mathbb{N}$. *There exist a vector field* $\tilde{E} \in L^p_{loc}(\bar{K}_L(a) \times \mathbb{R}^k, \mathbb{R}^{d+k})$ *with* $p < \min(2, \frac{2}{1+\gamma}, \frac{d+k}{s+1})$ *and a subset* $\tilde{\Lambda} \subset \bar{K}_L(a)$ *such that*

$$\begin{cases} -\operatorname{div}(|y|^\gamma \tilde{E}) = c_{d,s}\left(\sum_{p\in\tilde{\Lambda}} \delta_p - \rho(x)\delta_{\mathbb{R}^d}\right) & \text{in } \bar{K}_L(a)\times\mathbb{R}^k \\ \tilde{E}\cdot\vec{\nu} = 0 & \text{on } \partial\bar{K}_L(a)\times\mathbb{R}^k\cdot \\ \tilde{E}\cdot\vec{\nu} = E\cdot\vec{\nu} & \text{on } \partial K'_L(a)\times[-t,t]^k \\ \tilde{E} = E,\ \tilde{\Lambda} = \Lambda & \text{in } K'_L(a)\times[-t,t]^k \end{cases}$$

Moreover,

$$\frac{1}{L^d}\int_{(\bar{K}_L\setminus K'_L)\times\mathbb{R}^k} |y|^\gamma |\tilde{E}_\eta|^2 \le Cerr_{sc}(\eta, t, L, l, \varepsilon_1, C_1, C_2). \tag{54}$$

Finally, the minimal distance between points in $\tilde{\Lambda}\setminus\Lambda$, *and between points in* $\tilde{\Lambda}\setminus\Lambda$ *and* $\partial(\bar{K}_L(a)\setminus K_L(a)')$ *is bounded below by* r_1 *with* r_1 *a positive constant depending only on* $d, \underline{\rho}, \overline{\rho}$.

(2) There exists $\underline{K}_L(a)$ *such that* $\underline{K}_L(a) \subset K'_L(a)$, $\mathrm{dist}(\partial K'_L(a), \partial \underline{K}_L(a)) \in [t, 2t]$ *and* $\int_{\underline{K}_L(a)} \rho(x)\,dx \in \mathbb{N}$. *There exist a vector field* $\tilde{E} \in L^p_{loc}(K_L(a) \times \mathbb{R}^k, \mathbb{R}^{d+k})$ *with* $p < \min(2, \frac{2}{1+\gamma}, \frac{d+k}{s+1})$ *and a subset* $\tilde{\Lambda} \subset K_L(a)$ *such that*

$$\begin{cases} -\operatorname{div}(|y|^\gamma \tilde{E}) = c_{d,s}\left(\sum_{p\in\tilde{\Lambda}} \delta_p - \rho(x)\delta_{\mathbb{R}^d}\right) & \text{in } K_L(a)\times\mathbb{R}^k \\ \tilde{E}\cdot\vec{\nu} = 0 & \text{on } \partial\underline{K}_L(a)\times\mathbb{R}^k\cdot \\ \tilde{E}\cdot\vec{\nu} = E\cdot\vec{\nu} & \text{on } \partial K'_L(a)\times[-t,t]^k \\ \tilde{E} = E,\ \tilde{\Lambda} = \Lambda & \text{in } (K_L(a)\setminus K'_L(a))\times[-t,t]^k \end{cases}$$

Moreover,

$$\frac{1}{L^d}\int_{(K'_L\setminus\underline{K}_L)\times\mathbb{R}^k} |y|^\gamma |\tilde{E}_\eta|^2 \le Cerr_{sc}(\eta, t, L, l, \varepsilon_1, C_1, C_2). \tag{55}$$

Finally, the minimal distance between points in $\tilde{\Lambda}\setminus\Lambda$, *and between points in* $\tilde{\Lambda}\setminus\Lambda$ *and* $\partial(K'_L(a)\setminus\underline{K}_L(a))$ *is bounded below by* r_1.

The above result is obtained precisely along the lines of Proposition 6.1 in [52], except for a series of modifications needed in order to accommodate the nonconstant ρ. Before describing the modifications of the proof, we prove a hyperrectangle subdivision lemma which generalizes [52, Lem. 6.3] to the case of a controlled background measure. To re-obtain Lemma 6.3 of [52] from the statement below, one must take $\rho \equiv 1$ and then scale the units of length by $m^{-1/d}$.

Lemma 3.4 (Subdivision of a hyperrectangle)**.** *Let* $H = [0,\ell_1]\times\cdots\times[0,\ell_d]$ *be a* d*-dimensional hyperrectangle of sidelengths* ℓ_i *and let* $\rho : H \to \mathbb{R}^+$ *be a function such that for two constants* $\underline{\rho}, \overline{\rho}$ *there holds* $0 < \underline{\rho} \le \rho(x) \le \overline{\rho}$ *for all* $x \in H$ *and assume* $\int_H \rho = I \in \mathbb{N}^*$. *Assume that for*

all i, $\ell_i \geq 2/\underline{\rho}$. Fix a face F of H. Then there is a partition of H into I subrectangles $\mathcal{R}_j$, such that the following hold

- *all rectangles have volume* 1,
- *the sidelengths of each $\mathcal{R}_j$ lie in the interval $[2^{-d}\overline{\rho}^{-d}\underline{\rho}^{d-1}, 2^d\underline{\rho}^{-d}\overline{\rho}^{d-1}]$,*
- *all the $\mathcal{R}_j$'s which have a face in common with F have the same sidelength in the direction perpendicular to F.*

Proof. We proceed by induction on the dimension. In case $d = 1$ the mean value theorem allows to successively find values $0 = p_0 < p_1 < \cdots < p_I = \ell_1$ such that $\int_{p_i}^{p_{i+1}} \rho = 1$. Then the lengths $p_{i+1} - p_i$ lie in $[\overline{\rho}^{-1}, \underline{\rho}^{-1}]$ due to the bounds on ρ. In case $d \geq 2$ we may assume without loss of generality that the special face F is the one on $\{x_d = 0\}$. Define

$$\tilde{\rho} := \frac{\int_H \rho}{|H|} \in [\underline{\rho}, \overline{\rho}], \quad b := \left\lfloor \ell_d^{-1} \int_H \rho \right\rfloor = \lfloor \tilde{\rho}|F| \rfloor \,,$$

where $\lfloor a \rfloor := \max(\mathbb{Z} \cap]-\infty, a])$, and partition $[0, \ell_d]$ into a certain number a of segments S_α of ρ-mass b and one extra segment S_0 of ρ-mass belonging to $[b, 2b]$. Let $\bar{\ell}_\alpha$ be the lengths of these segments . We have $\bar{\ell}_\alpha |F| \underline{\rho} \leq b \leq \bar{\ell}_\alpha |F| \overline{\rho}$ so

$$\bar{\ell}_\alpha \in \frac{b}{|F|}[\overline{\rho}^{-1}, \underline{\rho}^{-1}] = \frac{\lfloor \tilde{\rho}|F| \rfloor}{|F|}[\overline{\rho}^{-1}, \underline{\rho}^{-1}] \subset \left[\frac{\underline{\rho} - \frac{1}{|F|}}{\overline{\rho}} \,,\, \frac{\overline{\rho}}{\underline{\rho}} \right].$$

Since $\ell_i \geq 2/\underline{\rho}$, we have $\underline{\rho} - |F|^{-1} \geq \frac{1}{2}\underline{\rho}$ and thus

$$\bar{\ell}_\alpha \in \left[\frac{1}{2}\frac{\underline{\rho}}{\overline{\rho}} \,,\, \frac{\overline{\rho}}{\underline{\rho}} \right]. \tag{56}$$

Let H'_α be the hyperrectangles $[0, \ell_1] \times \cdots \times [0, \ell_{d-1}] \times S_\alpha$. Then pushing forward by the projection onto F the restrictions $\mu \llcorner H'_\alpha$ gives measures $\mu_\alpha = \rho_\alpha dx$ of integer total mass equal to b. Note that we have bounds, $\rho_\alpha(x) \in \bar{\ell}_\alpha^{-1}[\underline{\rho}, \overline{\rho}]$. Now we apply the inductive hypothesis on F with the measure ρ_α and we obtain a subdivision of F into $(d-1)$-dimensional unit ρ_α-mass hyperrectangles $\mathcal{R}'_j$. Then by definition of ρ_α the hyperrectangles $\mathcal{R}'_j \times S_\alpha$ are of unit ρ-mass. The sidelenghts of the $\mathcal{R}'_j$ then lie in

$$\bar{\ell}_\alpha [2^{1-d}\overline{\rho}^{1-d}\underline{\rho}^{d-2}, 2^{d-1}\underline{\rho}^{1-d}\overline{\rho}^{d-2}], \tag{57}$$

which via (56) gives the correct bound as in the thesis. We may perform the same procedure also for the last segment S_0, the only difference being that we proceed with a different value $\tilde{b} \in [b, 2b[$ instead of b. The only effect of this change is that the bound (56) on $\bar{\ell}_0$ is perturbed by a factor belonging to $[1, 2[$, however combined with the bound (57) in the case $\alpha = 0$, this still gives estimates within the range allowed in the thesis. □

How to modify the argument in [52] *Section 6 to obtain the proof of Proposition 3.3:* We consider only the first case where we desire to change our charges and vector fields only over $K'_L \subset \bar{K}_L$, which corresponds to the same setting as in [52, Prop. 6.1]. The modifications needed for the case where we desire to modify the field and charges on $K'_L \setminus \underline{K}_L$ are straightforward and we leave them to the interested reader.

Section 6 of [52] is devoted to the proof of Proposition 6.1 there, which is the precise analogue of Proposition 3.3 here. The main difference to [52] is that here we work with non-constant ρ. The changes to be made are as follows. Here $\bar{K}_L, K'_L$ correspond to K_R, K'_R of

[52, Prop. 6.1], respectively.

In our case we work in the simplified situation where points in Λ have a minimal separation of $r_0 > 0$ and multiplicity 1 for all $p \in \Lambda$. Although most estimates work for more general Λ as considered in [52] too, our present restrictions allow to formulate the conclusion in terms of $\mathcal{W}_\eta$ rather than in terms of the quantity $\int |y|^\gamma |E_\eta|^2$ used in [52].

Part 1. *Changes in the lemmas of* [52, Sec. 6]*:*

- In all of [52, Sec. 6], we replace at all instances the volume measures $|A|$ of subsets (usually hyperrectangles) $A \subset \mathbb{R}^d$, by the integrals $\int_A \rho$. For example the hypothesis $|K_R| \in \mathbb{N}$ of Proposition 6.1 is replaced by our hypothesis $\int_{\bar{K}_L} \rho \in \mathbb{N}$ here, which plays the same role.
- The vertical part of our domains is now endowed with a separate scale t. Thus t plays the role that in [52] was that of $\epsilon^2 R$, throughout the whole proof.
- [52, Lem. 6.3] has to be replaced by our Lemma 3.4 here.
- In [52, Lem. 6.5] we replace the constant m by a function ρ such that $0 < \underline{\rho} \leq \rho(x) \leq \overline{\rho}$ throughout $\mathcal{R}$. The constant C in [52, Lem. 6.5] now depends on $\underline{\rho}, \overline{\rho}$ as well.
- [52, Lem. 6.6] remains unchanged.

Part 2. *Changes in the proof of* [52, Prop. 6.1]:
We apply the following adaptations, besides the replacement of d-dimensional volumes by ρ-masses everywhere.

- The hypotheses analogous to (48), (49) of our Proposition 3.3 are the same as the estimates of step 1 of [52, Prop. 6.1] except for the different choices of constants.
- The constant C_0 is defined only in case $k = 1$. It is now defined using the width $t' \sim t$ given by the thesis of Proposition 3.1 rather than ℓ like in [52]. The role of the constant C_0 is to compensate the boundary datum $E_\eta \cdot \nu$ along $\partial K'_L \times [-t', t']$ by a constant boundary datum over $K'_L \times \{-t', t'\}$, thus we define:
$$C_0 := (2(t')^\gamma |\bar{K}_L \setminus K'_L|)^{-1} \int_{(\bar{K}_L \setminus K'_L) \times [-t', t']} |y|^\gamma E_\eta \cdot \nu.$$
- We tile a neighborhood of $\partial K'_L$ in $\bar{K}_L \setminus K'_L$ by hyperrectangles H_i of size $\sim t$ and we define again n_i like in [52] as the total mass of the smeared charges that intersect $\partial K'_L$ restricted to H_i.
- For defining the hyperrectangles H_i we first use a subdivision into size $\sim t$ strips of the t-neighborhood above, done along the lines of Proposition 3.4. Observe that measure $\rho + \sum_p \delta_p^{(\eta)}$ is larger than $\underline{\rho}$ and $x \mapsto \int_{\{x\} \times [-t,t]^k} |y|^\gamma E_\eta \cdot \nu$ is by Cauchy-Schwartz inequality an L^2_{loc} function on $\partial K'_L$, therefore up to perturbing the H_i slightly (following which also the n_i will change with continuity) we obtain
$$\int_{H_i} \rho - c_{s,d}^{-1} \left(\int_{\partial D_0 \cap \partial \tilde{H}_i} |y|^\gamma E_\eta \cdot \nu + 2C_0 t^\gamma |H_i| + n_i \right) \in \mathbb{N}.$$
- Then we define the negative constants $-\bar{m}_i$ which play the same role as $(m_i - 1)$ in [52], as follows
$$c_{d,s} \bar{m}_i |H_i| = \int_{\partial D_0 \cap \partial \tilde{H}_i} |y|^\gamma E_\eta \cdot \nu - 2C_0 t^\gamma |H_i| - n_i. \tag{58}$$

This implies that

$$\int_{H_i} (\rho - \bar{m}_i) \in \mathbb{N}.$$

- The requirement replacing $|m_i - 1| < 1/2$ in [52] is that $|\bar{m}_i| \leq \underline{\rho}/2$. Assuming this to hold, we then have $\rho - \bar{m}_i \in [\underline{\rho}/2, \overline{\rho} + \underline{\rho}/2]$.
- The $\mathcal{R}_\alpha$ are produced via our new Lemma 3.4, applied to the density $\rho - \bar{m}_i$, which by the two previous points satisfies the hypotheses in that lemma.
- The bounds on sidelengths of the $\mathcal{R}_\alpha$ described in [52, par. following (6.43)] now contain another bounded factor $\overline{\rho}^{d-1}/\underline{\rho}^d$.
- The scale of the hyperrectangles in [52] was $\ell = \epsilon^2 R$, whereas here we take the H_i of size $\sim t$, as given in the good boundary Proposition 3.1.
- We find an analogue of [52, (6.45)] in our setting, by multiplying all the terms by a constant factor dependent on $\underline{\rho}, \overline{\rho}$, because $\int_{H_i} \rho \geq \underline{\rho}|H_i|$ and the factors t^{-d} in the right hand side are obtained from the analogue of [52, (6.34)](we recall again that the parameter t here corresponds to ℓ of [52] in this case) via the comparison $ct^d \leq |H_i| \leq Ct^d$, following from [52, Lem. 6.3] in [52], whereas in our setting, Lemma 3.4 gives the extra factors as above.
- Step 4 of the proof of [52, Prop. 6.1] provides the conditions ensuring $|m_i - 1| \leq 1/2$. Our $-\bar{m}_i$ are defined precisely like $m_i - 1$ in that proof, thus our corresponding desired bound $|\bar{m}_i| \leq \underline{\rho}/2$ can be proved in the same way. The bounds we obtain change only by a constant depending on $\underline{\rho}, \overline{\rho}, d$. One simplification is that the sum of $\bar{n}_\alpha$, $\alpha \in I_i$ is in our case bounded by $Cr_0^d t^{d-1}$ using the charge separation result of Proposition 2.3, therefore the corresponding term, which appears in our analogue of [52, (6.45)] multiplied by a t^{-d} factor, is bounded for t large. The remaining estimates give precisely the condition (52).
- The construction of the four vector fields $E_{i,1}, \dots, E_{i,4}$ is conducted with the following modifications:
  - The fields $E_{i,1}, E_{i,2}$ are defined exactly as in [52].
  - The field $E_{i,3}$ has as the only change the replacement of the constant charge $1 - m_i$ by the constant $\bar{m}_i$.
  - The field $E_{i,4}$ is still defined as a superposition of fields obtained like in [52, Lem. 6.5] on the $\mathcal{R}_\alpha$ that cover H_i, but now we use the background negative charge $-(\rho - \bar{m}_i)$, to which we add a charge at the center of $\mathcal{R}_\alpha$. Note that the estimate of [52, Lem. 6.5] depends only on elliptic estimates and holds for the new charges that we use here. The constant obtained in such estimate now depends on $\underline{\rho}, \overline{\rho}$.

□

3.2. **Choice of parameters.** In this subsection we are going to establish a set of parameters, which ensure that (52) holds and that (54), (55) are $\ll 1$. In particular, this will imply a condition on ε_1, which says that a good bound on err_{sc} can be expected only for ε_1 not too small.

3.2.1. *Case $k = 0$.* In the case $k = 0$, one can write ε_1 as a power of T and choose l and t to be powers of L. More precisely, let ε_1, l and t be such that

$$\varepsilon_1 = T^{\left(\frac{1-\delta}{\delta}\right)d},\ l = L^b \text{ and } t = L^\theta$$

with $\delta > 1$, $b < 1$ and $\theta < 1$. Recall that L is such that $\varepsilon_1^d T = T^{1/\delta} \leq L \leq T$ which gives $T \leq L^\delta$. A straightforward calculation shows that (52) holds if

$$\theta \geq \frac{\delta d - b}{d+1}$$

and

$$err_{sc}(\eta, t, L, l, \varepsilon_1, C_1, 0) = C_1 g(\eta) L^{(\delta-1)d-b} + C_1 L^{\theta+(\delta-1)-b} + g(\eta) L^{\theta-1}.$$

As a consequence to ensure that $err_{sc}(\eta, t, L, l, \varepsilon_1, C_1, 0) \ll 1$ we have to choose

$$\theta < b + (1-\delta)d.$$

Note that, since $\delta > 1$, $b + (1-\delta)d < b < 1$. To sum up, θ as to be chosen such that

$$\frac{\delta d - b}{d+1} \leq \theta < b + (1-\delta)d. \tag{59}$$

This is possible if and only if $\frac{\delta d - b}{d+1} < b + (1-\delta)d$. As a consequence δ has to be such that

$$1 < \delta < 1 + \frac{b(d+2) - d}{d(d+2)} \tag{60}$$

which in particular gives a condition on the scale $\varepsilon_1 T$. Such a δ exists if and only if $\frac{b(d+2)-d}{d(d+2)} > 0$. Hence b has to be chosen such that

$$b > \frac{d}{d+2}. \tag{61}$$

Finally remark that, with this choice of parameters $(\delta - 1)d - b < 0$ and

$$err_{sc}(\eta, t, L, l, \varepsilon_1, C_1, 0) = (g(\eta) + 1)(1 + C_1) o_{L\to\infty}(1). \tag{62}$$

3.2.2. *Case $k = 1$.* In the case $k = 1$, the situation is more delicate since we do not know explicitly the decay rate of $C_2(E, t, L)$. Nevertheless, in what follows, we need to apply Proposition 3.3 to a sequence of fields E'_n such that

$$\lim_{t\to+\infty} \lim_{L\to\infty} \lim_{n\to\infty} C_2(E'_n, t, L) = 0.$$

Hence, for all $\varepsilon_2 > 0$, for $L \gg \varepsilon_2^{1/2}$ and L, n large enough there holds

$$\frac{1}{L^d} \int_{K_L \times \left(\mathbb{R} \setminus [-\frac{1}{2}\varepsilon_2^{1/2} L, \frac{1}{2}\varepsilon_2^{1/2} L]\right)} |y|^\gamma |E'_{n,\eta}|^2 \leq \varepsilon_2.$$

Hence, by choosing $t = \varepsilon_2^{1/2} L, l = \varepsilon_2^{1/4} L$ and $C_2 = \varepsilon_2$ in Proposition 3.3, we have that the first condition in (52) holds. The second one is satisfied if

$$C_1 \varepsilon_2^{-\frac{2(d-\gamma)+1}{4}} L^{(\gamma-1)} \varepsilon_1^{-1} \leq c_0^2$$

that is if

$$L \geq c_0^{-\frac{2}{1-\gamma}} \varepsilon_1^{-\frac{1}{1-\gamma}} \varepsilon_2^{-\frac{2(d-\gamma)+1}{4(1-\gamma)}}.$$

Moreover,

$$err_{sc}(\eta, \varepsilon_2^{1/2} L, L, \varepsilon_2^{1/4} L, \varepsilon_1, C_1, \varepsilon_2) = C_1 \frac{1}{\varepsilon_1 \varepsilon_2^{1/4} L} g(\eta) + \varepsilon_2^{1/2} + C_1 \frac{\varepsilon_2^{1/4}}{\varepsilon_1} + g(\eta) \varepsilon_2^{1/2}.$$

To ensure that $err_{sc}(\eta, \varepsilon_2^{1/2}L, L, \varepsilon_2^{1/4}L, \varepsilon_1, C_1, \varepsilon_2) \ll 1$, ε_1 has to be such that

$$1 \geq \varepsilon_1 \gg \varepsilon_2^{\frac{1}{4}}. \tag{63}$$

In this case the above bounds on L are implied by the condition

$$L \geq c_0^{-\frac{2}{1-\gamma}} \varepsilon_2^{-\frac{(d-\gamma)+1}{2(1-\gamma)}}. \tag{64}$$

To summarize, if (63), (64) hold then

$$err_{sc}(\eta, \varepsilon_2^{1/2}L, L, \varepsilon_2^{1/4}L, \varepsilon_1, C_1, \varepsilon_2) = (1+g(\eta))(1+C_1)o_{L,n\to\infty}(1). \tag{65}$$

4. Proof of (9) of Theorem 1

By using the fact that for a minimizer of H_n all the point are in Σ, and in view of Proposition 2.2 and the result of [52, Thm. 4], we have the a priori bound

$$\mathcal{W}_\eta(E'_n, \mathbb{R}^d) \leq n \int_\Sigma \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W}\, dx + o_{n\to+\infty}(n) + o_{\eta\to 0}(1). \tag{66}$$

Moreover, in case $k = 1$ we work under the assumption (8). Finally, we will use that since m_V is $C^{0,\alpha}$ in Σ, we have

$$\|m'_V\|_{C^{0,\alpha}(\Sigma')} \leq \frac{\|m_V\|_{C^{0,\alpha}(\Sigma)}}{n^{\alpha/d}}, \tag{67}$$

and, whenever $K_T(a) \subset \Sigma'$, $T \lesssim n^{1/d}$, $0 \leq \beta < \alpha \leq 1$ there holds

$$\|m'_V\|_{\mathcal{C}^{0,\alpha}(K_T(a))} T^\beta \leq C \frac{1}{(n^{1/d})^{\alpha-\beta}} \leq o_{n\to+\infty}(1) \tag{68}$$

The proof is based on a bootstrap argument as in [54] : by a mean value argument, using the a priori bound on the energy (66) and $T = n^{1/d}$ as initial scale, we can find a square close to $K_\ell(a)$ which has a good boundary, i.e. such that (48), (49) are satisfied (relative to ℓ). This is only possible if ℓ is not too small compared to $n^{1/d}$, more precisely if $\ell_i \in [\varepsilon_1^{1/d} n^{1/d}, n^{1/d}]$ for $i = 1, \dots, d$. If indeed ℓ is such that $\ell_i \in [\varepsilon_1^{1/d} n^{1/d}, n^{1/d}]$ then we are essentially done: a comparison argument in the hypercube with the good boundary allows to conclude. More precisely, we use the following result. We note here that in this section we omit the index n on a in order to lighten up the notation.

Proposition 4.1. *Let C_1 a positive constant. Let $t, L, l, \varepsilon_1, \varepsilon_2$ and E'_n be as in Section 3.2. Assume that bounds* (48) *and* (49) *hold for E'_n in $K'_L(a) \subset K_L(a)$ with $\operatorname{dist}(\partial K'_L(a), \partial K_L(a)) \in [l, 2l[$. Then*

$$\left| \frac{\mathcal{W}_\eta(E'_n, K'_L(a))}{|K'_L(a)|} - \frac{1}{|K_L(a)|} \int_{K_L(a)} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W} dx \right| \leq (1+g(\eta))(1+C_1)o_{L,n\to\infty}(1) + o_{\eta\to 0}(1). \tag{69}$$

Next, if ℓ_i is smaller than $\varepsilon_1^{1/d} n^{1/d}$ for some i, we bootstrap the argument: we first obtain by the above argument a control of the energy and the number of points on a hypercube of size $\varepsilon_1^{1/d} n^{1/d}$ containing $K_\ell(a)$, and then we re-apply the reasoning starting from that hypercube. This allows to go down to a smaller scale, and we iterate the procedure until we reach the desired value of ℓ. This iteration will not cumulate error, its only main restriction is that

the final square will have to be at a certain distance away from $\partial\Sigma'$, because of the repeated mean value arguments.

More precisely, we proceed as follows for the proof of (9) of Theorem 1. Let $K_\ell(a)$ as in the statement of Theorem 1. We set

$$C_1 = \max_{m\in[\underline{m},\overline{m}]} \min_{\mathcal{A}_m} \mathcal{W} + C_{s,d,\overline{m}} g(\eta) + 1 \tag{70}$$

and

$$l_\ell := \begin{cases} \ell^b & \text{if } k=0 \\ \varepsilon_2^{1/4}\ell & \text{if } k=1 \end{cases} \tag{71}$$

with $b<1$ and ε_2 chosen as in Section 3.2. Without loss of generality we may assume $\mathrm{dist}(K_\ell(a),\partial\Sigma') \geq 3l_\ell$. In this case $\mathrm{dist}(K_\ell(a),\partial\Sigma') \geq \max(d_n, 3l_\ell)$ where

$$d_n := \begin{cases} n^{q/d} & \text{if } k=0 \\ \varepsilon_0 n^{1/d} & \text{if } k=1 \end{cases}.$$

The proof for the case where $d_n \leq \mathrm{dist}(K_\ell(a),\partial\Sigma') < 3l_\ell$ would start by subdividing the cube $K_\ell(a)$ into cubes of the smallest size L_{min} permitted by the choices of Section 3.2, namely $L_{min}\sim 1$ for $k=0$ and $L_{min}\sim \varepsilon_2^{-\frac{2(d-\gamma)+1}{2(1-\gamma)}}$ for $k=1$. The smaller cubes have $3l_{L_{min}} \leq d_n$ respectively if n is large enough in case $k=0$ and if ε_2 is small enough (which by Section 3.2 means that ℓ, n must be large enough depending on (8)) in case $k=1$. Therefore the result for $\mathrm{dist}(K_{L_{min}}(a),\partial\Sigma') \geq 3l_{L_{min}}$ holds for all cubes in the subdivision, and summing the bounds (9) for all these cubes we obtain (9) for $K_\ell(a)$ too.

We split the proof of Theorem 1 into two cases.

Case 1: $\ell + 3l_\ell \geq \varepsilon_1^{1/d} n^{1/d}$. Let us then define the scale $L^{(1)} = \ell + 3l_\ell$. Since we assumed that $\mathrm{dist}(K_\ell(a),\partial\Sigma') \geq 3l_\ell$, we have $K_{\ell+3l_\ell}(a) \subset \Sigma'$, and so there exists a center $a^{(1)}$ such that

$$K_{\ell+3l_\ell}(a) \subset K_{L^{(1)}}(a^{(1)}) \subset \Sigma'.$$

Then we apply Proposition 3.1 with K_T replaced by Σ' itself, C_1 given by (70) and $L = L^{(1)}$. Moreover t, C_2, ε_1 and l are chosen as in Section 3.2. This gives the existence of a square $K'_{L^{(1)}}(a^{(1)}) \subset K_{L^{(1)}}(a^{(1)})$ such that $\mathrm{dist}(\partial K'_{L^{(1)}}(a^{(1)}), \partial K_{L^{(1)}}(a^{(1)})) \in [l, 2l[$ and (48) and (49) are satisfied. Note that for ℓ large enough

$$K_\ell(a) \subset K'_{L^{(1)}}(a^{(1)}) \subset K_{L^{(1)}}(a^{(1)}) \subset \Sigma'.$$

Then by applying Proposition 4.1, we deduce

$$\left| \frac{\mathcal{W}_\eta(E'_n, K'_{L^{(1)}}(a^{(1)}))}{|K'_{L^{(1)}}|} - \frac{1}{|K_{L^{(1)}}|} \int_{K_{L^{(1)}}(a^{(1)})} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W} dx \right| \leq (1+g(\eta))(1+C_1) o_{\ell,n\to\infty}(1) + o_{\eta\to 0}(1). \tag{72}$$

By using charges separation and the fact that $K_\ell(a) \subset K'_{L^{(1)}}(a^{(1)})$, we obtain the upper bound

$$\begin{aligned} \frac{\mathcal{W}_\eta(E'_n, K_\ell(a))}{|K_\ell|} &\leq \frac{1}{|K_{L^{(1)}}|} \int_{K_{L^{(1)}}(a^{(1)})} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W} dx + g(\eta)^2 o_{\ell,n\to\infty}(1) + o_{\eta\to 0}(1) \\ &= \frac{1}{|K_\ell|} \int_{K_\ell(a)} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W} dx + g(\eta)^2 o_{\ell,n\to\infty}(1) + o_{\eta\to 0}(1). \end{aligned}$$

To obtain the lower bound, we may apply once again Proposition 3.1 at the scale ℓ to obtain a square $K'_\ell(a) \subset K_\ell(a)$ with a good boundary and such that

$$\frac{\mathcal{W}_\eta(E'_n, K_\ell(a))}{|K_\ell|} \geq \frac{\mathcal{W}_\eta(E'_n, K'_\ell(a))}{|K'_\ell|} - g(\eta)o_{\ell\to\infty}(1).$$

Hence the application of Proposition 4.1 to the square $K'_\ell(a)$ leads to the desired result.

Case 2: $\ell + 3l_\ell < \varepsilon_1^{1/d} n^{1/d}$. Let $L^{(1)} = \varepsilon_1^{\frac{1}{d}} n^{\frac{1}{d}}$. Since we have $\operatorname{dist}(K_\ell(a), \partial\Sigma') \geq d_n$, we have $K_{\ell+d_n}(a) \subset \Sigma'$, and so there exists a center $a^{(1)}$ such that

$$K_{\ell+d_n}(a) \subset K_{L^{(1)}}(a^{(1)}) \subset \Sigma'.$$

Then we apply Proposition 3.1 with K_T replaced by Σ' itself, C_1 given by (70) and $L = L^{(1)}$. Moreover $t, C_2, \varepsilon_1^{(1)}$ and $l^{(1)}$ are chosen as in Section 3.2. This gives the existence of a square $K'_{L^{(1)}}(a^{(1)}) \subset K_{L^{(1)}}(a^{(1)})$ such that $\operatorname{dist}(\partial K'_{L^{(1)}}(a^{(1)}), \partial K_{L^{(1)}}(a^{(1)})) \in [l^{(1)}, 2l^{(1)}[$ and (48) and (49) are satisfied. Note that for ℓ large enough there holds

$$K_{\ell+d_n-2l^{(1)}}(a) \subset K'_{L^{(1)}}(a^{(1)}) \subset K_{L^{(1)}}(a^{(1)}) \subset \Sigma'.$$

Then by applying Proposition 4.1, we have (72) as before. Next, if ℓ is large enough, the hypotheses of Proposition 3.1 are satisfied in $K'_{L^{(1)}}$ (which plays the role of K_T in Proposition 3.1), and with the same constant C_1. This a consequence of (72) for ℓ, n and $1/\eta$ large enough. We can thus re-apply Proposition 3.1 in $K'_{L^{(1)}}$ and with new subscale $L^{(2)} = \max(\ell + 3l_\ell, (\varepsilon_1^{(1)})^{1/d} L^{(1)})$. If $\ell + 3l_\ell \geq (\varepsilon_1^{(1)})^{1/d} L^{(1)}$, we conclude as in the Case 1. If $\ell + 3l_\ell < (\varepsilon_1^{(1)})^{1/d} L^{(1)}$, we iterate the above procedure. In the end, we obtain a sequence of $L^{(j)}$ with $L^{(j)} = \max(\ell + 3l_\ell, (\varepsilon_1^{(j-1)})^{1/d} L^{(j-1)})$ and such that

$$K_{\ell+d_n-2\sum_{j=1}^{\bar{j}} l^{(j)}}(a) \subset K'_{L^{(\bar{j})}}(a^{(\bar{j})}) \subset K_{L^{(\bar{j})}}(a^{(\bar{j})}) \subset \ldots \subset K'_{L^{(1)}}(a^{(1)}) \subset K_{L^{(1)}}(a^{(1)}) \subset \Sigma'.$$

where $\bar{j}$ is the smallest integer such that $\ell > \prod_{j=2}^{\bar{j}} (\varepsilon_1^{(j-1)})^{1/d} (\varepsilon_1 n)^{1/d} = L^{(\bar{j})}$. Now, we have to ensure that $K_\ell(a)$ is a subset $K'_{L^{(\bar{j})}}(a^{(\bar{j})})$. This is true if $d_n - 2\sum_{j=1}^{\bar{j}} l^{(j)} > 0$. We proceed as follows.

- **Case $k = 0$.** In this case, by Section 3.2, it follows that $L^{(j)} = (L^{(j-1)})^{1/\delta} = (n^{1/d})^{\delta^{-j}}$ and $l^{(j)} = (L^{(j)})^b$. Bounding each time $L^{(j)}$ by $n^{1/(\delta d)}$ we have that $\sum_{j=1}^{\bar{j}} l^{(j)} \leq \bar{j} n^{b/(\delta d)}$. Moreover, $\bar{j}$ is finite. Indeed, $\bar{j}$ is defined as the smallest integer such that $\ell > \prod_{j=2}^{\bar{j}} (\varepsilon_1^{(j-1)})^{1/d} (\varepsilon_1 n)^{1/d} = (n^{1/d})^{\delta^{-\bar{j}}}$ which gives

$$\bar{j} = \left\lceil \frac{\log \frac{\log n^{1/d}}{\log \ell}}{\log \delta} \right\rceil.$$

  Hence, to have

$$d_n - 2\sum_{j=1}^{\bar{j}} l^{(j)} \geq n^{q/d} - 2\bar{j} n^{b/(\delta d)} > 0$$

  it is enough to choose $1 > q > \frac{b}{\delta}$.

- **Case $k=1$.** By the $k=1$ cases in Section 3.2, it now follows that $l^{(j)}=\varepsilon_2^{1/4}L^{(j)}$ and $L^{(j)}=\prod_{r=2}^{j}(\varepsilon_1^{(r-1)})^{1/d}(\varepsilon_1 n)^{1/d}$. So to deal with this case, we fix $\varepsilon_1^{(r)}=\varepsilon_2^{1/8}$ for all $r\in[\![1,\bar{j}]\!]$. This is compatible with condition on ε_1 in Section 3.2. As a consequence $L^{(j)}=\varepsilon_2^{j/(8d)}n^{1/d}$ for all $j\in[\![1,\bar{j}]\!]$ and

$$\sum_{j=1}^{\bar{j}} l^{(j)}=\varepsilon_2^{1/4}n^{1/d}\sum_{j=1}^{\bar{j}}(\varepsilon_2^{1/(8d)})^j\le \varepsilon_2^{1/4}\frac{\varepsilon_2^{1/(8d)}}{1-\varepsilon_2^{1/(8d)}}n^{1/d}.$$

  for all $\varepsilon_2>0$ provided that ℓ and n are large enough. Therefore if $d_n=\varepsilon_0 n^{1/d}$ with $\varepsilon_0>0$, then $d_n-2\sum_{j=1}^{\bar{j}} l^{(j)}>0$.

4.1. **Comparison argument on hypercubes with good boundary.** This subsection is devoted to the proof of Proposition 4.1. We start with the following lemma.

Lemma 4.2. *Let $(x_1,\cdots,x_n)$ be a minimizer of H_n. Let $\mu_V=m_V(x)dx$, $\mu_V'=m_V'(x')dx'$ be respectively the equilibrium measure and its blow-up at scale $n^{1/d}$ as above and let $E_n'=\nabla h_n'$ be a sequence of blown-up vector fields corresponding to these minimizers as in* (26)*. Let Σ' be the support of μ_V'. Then for all $\Omega\subset\Sigma'$, we have*

$$\mathcal{W}_\eta(E_n',\Omega)\le \mathcal{W}_\eta(E,\Omega)+o_{\eta\to 0}(1) \tag{73}$$

for any $E=\nabla h$ which satisfies

$$\begin{cases} -\operatorname{div}(|y|^\gamma\nabla h)=c_{s,d}\Big(\sum_{i=1}^{n}\delta_{p_i'}-m_V'\delta_{\mathbb{R}^d}\Big) & \text{in } \Omega\times\mathbb{R}^k\\ E\cdot\vec{\nu}=E_n'\cdot\vec{\nu} & \text{on } \partial\Omega\times\mathbb{R}^k\end{cases}$$

with all the points p_i' in Σ'.

The proof of the lemma has some similarity to [54, Sec. 5.2].

Proof. Let $(x_1,\cdots,x_n)$ minimize H_n and $E_n'=\nabla h_n'$ be as in (26). Since all points are in Σ by Proposition 2.3, in view of Proposition 2.2, by minimality of $(x_1,\cdots,x_n)$, we have

$$\lim_{\eta\to 0}\mathcal{W}_\eta(E_n',\mathbb{R}^d)\le\lim_{\eta\to 0}\mathcal{W}_\eta(\bar{E},\mathbb{R}^d) \tag{74}$$

for any $\bar{E}=\nabla h$ which satisfies $-\operatorname{div}(|y|^\gamma\nabla h)=c_{s,d}\Big(\sum_{i=1}^n\delta_{p_i'}-m_V'\delta_{\mathbb{R}^d}\Big)$ in $\mathbb{R}^{d+k}$ with all the points p_i' in Σ'. Note that we may extend E as in the statement (provided $\int_\Omega\mu_V'$ is such that such an E exists) by E_n' outside Ω in order to obtain a competitor $\bar{E}$ as above. Moreover as a consequence the energy contributions to (74) of $\bar{E},E_n'$ outside Ω coincide. This proves our result. □

As a consequence of Lemma 4.2, we have that, for all $\Omega\in\Sigma'$,

$$\frac{\mathcal{W}_\eta(E_n',\Omega)}{|\Omega|}=\min_{E\in\mathcal{A}_{m_V'},\ E\cdot\vec{\nu}=E_n'\cdot\vec{\nu}}\frac{\mathcal{W}_\eta(E,\Omega)}{|\Omega|}+o_{\eta\to 0}(1). \tag{75}$$

We now use parameters as in Section 3.2

Next, we remark that the vector field E'_n satisfies the hypotheses of Proposition 3.3. Hence, there exists $\underline{K}_L(a) \subset K'_L(a) \subset K_L(a)$ such that by (62) or (65)

$$\frac{\mathcal{W}_\eta(E'_n, K'_L(a))}{|K'_L(a)|} \leq \sigma_0(\underline{K}_L(a); m'_V) + (1+g(\eta))\, o_{L\to\infty}(1) + o_{\eta\to 0}(1), \tag{76}$$

with $\sigma_0(K; m'_V) = \min_{E\in\mathcal{A}_{m'_V},\ E\cdot\vec{\nu}=0} \frac{\mathcal{W}_\eta(E,K)}{|K|}$. To prove the claim (76) we apply Proposition 3.3, which allows to construct a vector field $\tilde{E}$ and a subset $\tilde{\Lambda} \subset K_L(a)$ such that

$$\begin{cases} -\operatorname{div}(|y|^\gamma \tilde{E}) = c_{d,s}\left(\sum_{p\in\tilde{\Lambda}} \delta_p - m'_V(x)\delta_{\mathbb{R}^d}\right) & \text{in } K_L(a)\times\mathbb{R}^k \\ \tilde{E}\cdot\vec{\nu} = 0 & \text{on } \partial\underline{K}_L(a)\times\mathbb{R}^k\,, \\ \tilde{E}\cdot\vec{\nu} = E'_n\cdot\vec{\nu} & \text{on } \partial K'_L(a)\times[-t,t]^k \\ \tilde{E} = E'_n,\ \tilde{\Lambda}=\Lambda & \text{in } (K_L(a)\setminus K'_L(a))\times[-t,t]^k \end{cases}$$

and

$$\frac{1}{L^d}\int_{(K'_L\setminus\underline{K}_L)\times\mathbb{R}^k} |y|^\gamma |\tilde{E}_\eta|^2 \leq Cerr_{sc}(\eta,t,L,l,\varepsilon_1,C_1,C_2).$$

Due to the choices of parameters like in Section 3.2, the quantity err_{sc} is bounded by $(1+g(\eta))(1+C_1)o_{L,n\to\infty}(1)$. The bound (76) then follows using the bound on the separation of charges and a packing argument to bound the number of charges in $K'_L\setminus\underline{K}_L$ by $C|K'_L\setminus\underline{K}_L| \leq CtL^{d-1}$.

Hence, by defining $\tilde{E} = \arg\min_{E\in\mathcal{A}_{m'_V},\ E\cdot\vec{\nu}=0} \frac{\mathcal{W}_\eta(E,\underline{K}_L)}{|\underline{K}_L|}$ on $\underline{K}_L(a)$ and by using Lemma 4.2, we obtain

$$\frac{\mathcal{W}_\eta(E'_n, K'_L(a))}{|K'_L(a)|} \leq \frac{\mathcal{W}_\eta(\tilde{E}, K'_L(a))}{|K'_L(a)|}.$$

By the same reasoning, there exists $K'_L(a) \subset \bar{K}_L(a)$, a subset $\tilde{\Lambda} \subset \bar{K}_L(a)$ and a vector field $\tilde{E}'_n$ such that

$$\begin{cases} -\operatorname{div}(|y|^\gamma \tilde{E}'_n) = c_{d,s}\left(\sum_{p\in\tilde{\Lambda}} \delta_p - m'_V(x)\delta_{\mathbb{R}^d}\right) & \text{in } \bar{K}_L(a)\times\mathbb{R}^k \\ \tilde{E}'_n\cdot\vec{\nu} = 0 & \text{on } \partial\bar{K}_L(a)\times\mathbb{R}^k \\ \tilde{E}'_n\cdot\vec{\nu} = E'_n\cdot\vec{\nu} & \text{on } \partial K'_L(a)\times[-t,t]^k \\ \tilde{E}'_n = E'_n,\ \tilde{\Lambda}=\Lambda & \text{in } K'_L(a)\times[-t,t]^k \end{cases}$$

and

$$\frac{\mathcal{W}_\eta(E'_n, K'_L(a))}{|K'_L(a)|} \geq \frac{\mathcal{W}_\eta(\tilde{E}'_n, \bar{K}_L(a))}{|\bar{K}_L(a)|} - err_{sc}(\eta,t,L,l,\varepsilon_1,C_1,C_2(E'_n,t,L)) \tag{77}$$

$$= \frac{\mathcal{W}_\eta(\tilde{E}'_n, \bar{K}_L(a))}{|\bar{K}_L(a)|} - (1+g(\eta))\,(1+C_1)o_{L,n\to\infty}(1). \tag{78}$$

The last step required in order to conclude the proof, is to find an upper bound for $\sigma_0(\underline{K}_L(a); m'_V)$ and a lower bound for $\frac{\mathcal{W}_\eta(\tilde{E}'_n,\bar{K}_L(a))}{|\bar{K}_L(a)|}$. This is done in the two next subsections.

4.1.1. *Upper bound.* To obtain an upper bound for $\sigma_0(K;m'_V) = \min_{E\in\mathcal{A}_{m'_V},\ E\cdot\vec{\nu}=0} \frac{\mathcal{W}_\eta(E,K)}{|K|}$, we proceed as in [54]. In particular we have the following proposition.

Proposition 4.3. *Let $a\in\mathbb{R}^d$ and $R>0$ such that $K_R(a)\subset\Sigma'$. Let $\alpha\in]0,1]$ and ρ be a non-negative $\mathcal{C}^{0,\alpha}(K_R(a))$ function for which there exists $\underline{\rho},\overline{\rho}>0$ such that $\underline{\rho}\le\rho(x)\le\overline{\rho}$. Let $K_R(a)$ be such that $\int_{K_R(a)}\rho(x)\,dx\in\mathbb{N}$. Then there exists a constant $C>0$ depending only on d,s such that for each fixed $\beta\in]0,1+\alpha[$ we have*

$$\sigma_0(K_R(a);\rho)\le\frac{1}{|K_R|}\int_{K_R(a)}\min_{\mathcal{A}_{\rho(x)}}\mathcal{W}\,dx+C(g(\eta)+1)o_{R\to\infty}(1)+o_{\eta\to0}(1)+CR^{2\beta}\|\rho\|^2_{C^{0,\alpha}(K_R(a))}$$
$$+CR^{\beta}\|\rho\|_{C^{0,\alpha}(K_R(a))}\left[c_{d,s}\overline{\rho}g(\eta)+1+(g(\eta)+1)o_{R\to\infty}(1)+o_{\eta\to0}(1)\right]^{1/2}. \tag{79}$$

Remark 4.4. *Note that the rate of convergence of the above bounds $o_{R\to\infty}(1)$ as $R\to\infty$ depends on m_V,β. As $\beta\to0$ this rate degenerates, as can be seen in the proof. In the present formulation, choosing $\beta>\alpha$ provides no advantage, however it may be possible by refining the proof to obtain more precise estimates of $o_{R\to\infty}(1)$ valid only for such large values of β.*

Proof. The proof of this proposition is similar to [54, Prop. 4.1] except that we have to be more careful with error terms which blow up as $\eta\to0$.
Step 1. We define a smaller scale $r=R^\lambda$ with $\lambda\in]0,1[$ for now, and with λ to be fixed at the end of the proof, and we construct a collection $\mathcal{K}$ of rectangles which partition $K_R(a)$, whose sidelengths are between $r-O\left(\frac{1}{r}\right)$ and $r+O\left(\frac{1}{r}\right)$, and such that for all $K\in\mathcal{K}$ we have $\int_K\rho(x)\,dx\in\mathbb{N}$. This is possible for example via the partitioning lemma [52, Lem. 6.3].

Step 2. We denote by x_K the center of each K and $\rho_K=⨍_K\rho(x)\,dx$ and we consider E a minimizer of $\mathcal{W}$. Since as in [52, Sec. 7] and [55, 63] we may obtain E by screening minimizing configurations on larger and larger cubes, we may assume due to Proposition 2.3 that charges contributing to E are well-separated and have multiplicity one. Moreover, if $k=1$, we note that, by the periodicity of E, we must have

$$\lim_{t\to+\infty}\lim_{R\to+\infty}\frac{1}{R^d}\int_{K_R\times(\mathbb{R}\setminus(-t,t))}|y|^\gamma|E|^2=0.$$

Hence, using [52, Prop. 6.1], we obtain in each K a vector field E_K satisfying

$$\begin{cases}-\operatorname{div}(|y|^\gamma E_K)=c_{s,d}\Big(\sum_{p\in\Lambda_K}\delta_p-\rho_K\delta_{\mathbb{R}^d}\Big) & \text{in } K\times\mathbb{R}^k\\ E_K\cdot\vec{\nu}=0 & \text{on } \partial K\times\mathbb{R}^k\\ E_K=0 & \text{outside } K\times\mathbb{R}^k\end{cases}$$

for some discrete subset $\Lambda_K\subset K$, and

$$\frac{\mathcal{W}_\eta(E_K,K)}{|K|}\le\min_{\mathcal{A}_{\rho_K}}\mathcal{W}+C(g(\eta)+1)o(1)_{r\to+\infty}+o_{\eta\to0}(1), \tag{80}$$

where the $o(1)$ terms depend on the approximation of $\min\mathcal{W}$ in [52] as described above.

Then, we have to rectify the weight ρ_K. For $K \in \mathcal{K}$, we let h_K solve

$$\begin{cases} -\operatorname{div}(|y|^\gamma \nabla h_K) = c_{s,d}(\rho_K - \rho(x))\delta_{\mathbb{R}^d} & \text{in } K \times [-r,r]^k \\ \partial_\nu h_K = 0 & \text{on the rest of } \partial(K \times [-r,r]^k) \end{cases}$$

and we put $h_K = 0$ in $K \times (\mathbb{R}^k \setminus [-1,1]^k)$. As a consequence of elliptic estimates as in the proof of [52, Lem. 6.4], we have

$$\int_K |\nabla h_K|^2 \le C r^2 \|\rho - \rho_K\|^2_{L^\infty(K)} |K| \le C r^{2\alpha+2} \|\rho\|^2_{C^{0,\alpha}(K)} |K| \quad \text{for } k = 0; \tag{81}$$

$$\int_{K\times\mathbb{R}^k} |y|^\gamma |\nabla h_K|^2 \le C r^{1-\gamma} \|\rho - \rho_K\|^2_{L^\infty(K)} |K| \le C r^{2\alpha+1-\gamma} \|\rho\|^2_{C^{0,\alpha}(K)} |K| \quad \text{for } k = 1. \tag{82}$$

We then define $\tilde{E}$ to be $E_K + \nabla h_K$ in each $K \in \mathcal{K}$. Pasting these together defines a $\tilde{E}$ over the whole $K_R(a)$, satisfying

$$\begin{cases} -\operatorname{div}(|y|^\gamma \tilde{E}) = c_{s,d}\Big(\sum_{p\in\Lambda} \delta_p - \rho(x)\delta_{\mathbb{R}^d}\Big) & \text{in } K_R(a) \times \mathbb{R}^k \\ \tilde{E} \cdot \vec{\nu} = 0 & \text{on } \partial K_R(a) \times \mathbb{R}^k \end{cases}$$

for some discrete set Λ. Next we have to evaluate $\mathcal{W}_\eta(\tilde{E}, K_R(a))$. In each $K \in \mathcal{K}$, we apply the Cauchy-Schwartz inequality to the squared L^2_γ-norm of $\tilde{E}$, followed by (80) and estimating the ∇h_K-term via the weaker bound (81) in both cases $k = 0, 1$, rather than using the more precise (82) for $k = 1$. This gives:

$$\begin{aligned} \mathcal{W}_\eta(\tilde{E}, K) \le{}& \mathcal{W}_\eta(E_K, K) + \int_{K\times\mathbb{R}^k} |y|^\gamma |\nabla h_K|^2 \\ &+ 2\left(\int_{K\times\mathbb{R}^k} |y|^\gamma |E_K|^2\right)^{1/2} \left(\int_{K\times\mathbb{R}^k} |y|^\gamma |\nabla h_K|^2\right)^{1/2} \\ &\le |K| \min_{\mathcal{A}_{\rho_K}} \mathcal{W} + C|K|(g(\eta)+1)o_{r\to\infty}(1) + |K| o_{\eta\to 0}(1) + C|K| r^{2\alpha+2} \|\rho\|^2_{C^{0,\alpha}(K)} \\ &+ C|K| r^{\alpha+1} \|\rho\|_{C^{0,\alpha}(K)} \left(\frac{\mathcal{W}_\eta(E_K, K)}{|K|} + c_{d,s}\rho_K g(\eta)\right)^{1/2} \\ &\le |K| \min_{\mathcal{A}_{\rho_K}} \mathcal{W} + C|K|(g(\eta)+1)o_{r\to\infty}(1) + |K| o_{\eta\to 0}(1) + C|K| r^{2\alpha+2} \|\rho\|^2_{C^{0,\alpha}(K)} \\ &+ C|K| r^{\alpha+1} \|\rho\|_{C^{0,\alpha}(K)} \left[1 + (g(\eta)+1)o_{r\to\infty}(1) + o_{\eta\to 0}(1) + c_{d,s}\rho_K g(\eta)\right]^{1/2} \end{aligned}$$

We now sum over all hypercubes K as above and we use the subadditivity of $A \mapsto \mathcal{W}_\eta(E, A)$ in order to add the contributions of the above left-hand sides to bound $\mathcal{W}_\eta(\tilde{E}, K_R(a))$. On the right hand side all terms except the first give contribution of $|K_R|$ multiplied by the error terms. For the first term, we proceed using the scaling $\min_{\mathcal{A}_\rho} \mathcal{W} = \rho^{1+s/d} \min_{\mathcal{A}_1} \mathcal{W}$ for power kernels and $\min_{\mathcal{A}_\rho} \mathcal{W} = \rho \min_{\mathcal{A}_1} \mathcal{W} - \frac{c_{0,d}}{d}\rho\log\rho$ for logarithmic ones. Using this together with the fact that $\rho \in C^{0,\alpha}(K_R(a))$, we conclude that

$$\left| \int_K \min_{\mathcal{A}_{\rho(x)}} \mathcal{W} dx - |K| \min_{\mathcal{A}_{\rho_K}} \mathcal{W} \right| \le C|K| r^\alpha \|\rho\|_{C^{0,\alpha}(K)} M_{s,d}(\rho), \tag{83}$$

where

$$M_{s,d}(\rho) = \begin{cases} \bar{\rho}^{s/d} \min_{\mathcal{A}_1} \mathcal{W}, & \text{for power-law kernels,} \\ \min_{\mathcal{A}_1} \mathcal{W} - \log \underline{\rho} - 1, & \text{for logarithmic kernels.} \end{cases} \tag{84}$$

We may then absorb this term into the $r^{1+\alpha}$-terms above and after summing all contributions, using the above bounds, and dividing both sides by $|K_R|$, we obtain:

$$\begin{aligned} \frac{\mathcal{W}_\eta(\tilde{E}, K_R(a))}{|K_R|} \leq & \frac{1}{|K_R|} \int_{K_R(a)} \min_{\mathcal{A}_{\rho(x)}} \mathcal{W}\, dx \\ & + C(g(\eta)+1) o_{R\to\infty}(1) + o_{\eta\to 0}(1) + C r^{2\alpha+2} \|\rho\|^2_{C^{0,\alpha}(K_R(a))} \\ & + C r^{\alpha+1} \|\rho\|_{C^{0,\alpha}(K_R(a))} \left(1 + (g(\eta)+1) o_{R\to\infty}(1) + o_{\eta\to 0}(1) + c_{d,s} \overline{\rho} g(\eta)\right)^{1/2} \end{aligned}$$

This gives precisely (79) if we choose

$$0 < \lambda \leq \frac{\beta}{1+\alpha}.$$

To conclude, we remark that projecting E onto gradients decreases the energy. The result follows. □

4.1.2. *Lower Bound.* The goal of this section is to find a lower bound for $\frac{\mathcal{W}_\eta(\tilde{E}'_n, \bar{K}_L(a))}{|\bar{K}_L(a)|}$.

Proposition 4.5. *Let $\tilde{E}'_n$ be as above and let $\alpha \in]0,1]$ such that $m'_V \in C^{0,\alpha}(K_R(a))$ with $m'_V(x) \leq \overline{m}$ and let $\beta \in]0, 1+\alpha[$. Then there exists a constant $C > 0$ depending only on d, s such that there holds*

$$\begin{aligned} \frac{\mathcal{W}_\eta(\tilde{E}'_n, \bar{K}_L(a))}{|\bar{K}_L(a)|} \geq & \frac{1}{|\bar{K}_L|} \int_{\bar{K}_L(a)} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W}\, dx - o_{L,n\to\infty}(1)\,(1+g(\eta)) - (1+o_{L,n\to\infty}(1))\, o_{\eta\to 0}(1) \\ & - C L^{2\beta} \|m'_V\|^2_{C^{0,\alpha}(\bar{K}_L(a))} - C L^{\beta} \|m'_V\|_{C^{0,\alpha}(\bar{K}_L(a))} \left(C + c_{d,s} \bar{m} g(\eta)\right)^{1/2}. \end{aligned} \tag{85}$$

Proof. Recall that thanks to the minimality of $(x_1, \dots, x_n)$, the charges are well-separated at a distance r_0 which depends only on d, s and $\overline{m}$.

The proof of the lower bound uses a partitioning argument like the one employed to prove [54, Thm. 1.11], however we remove the bootstrap part of the argument. Let $\bar{L}$ the sidelength of $\bar{K}_L(a)$.

First let us apply Lemma 3.4 to partition $\bar{K}_L(a)$ into smaller hypercubes $K_i(a_i)$ of sidelengths $\sim r = L^{\frac{\beta}{1+\alpha}}$ and such that $\int_{K_i(a_i)} m'_V \in \mathbb{N}$. It follows that

$$\mathcal{W}_\eta(\tilde{E}'_n, \bar{K}_L(a)) = \sum_i \mathcal{W}_\eta(\tilde{E}'_n, K_i(a_i)).$$

The goal is to bound from below this sum. We may assume that

$$\mathcal{W}_\eta(\tilde{E}'_n, K_i(a_i)) \leq \min_{\mathcal{A}_{\overline{m}}} \mathcal{W} |K_i| := C_m |K_i|, \tag{86}$$

for otherwise, we have a lower bound $\mathcal{W}_\eta(\tilde{E}'_n, K_i(a_i)) \geq C_m |K_i|$ which will suffice.

Let $K_r(b)$ be one of the above squares $K_i(a_i)$. As a consequence of the separation of charges of $\tilde{E}'_n$ and of (86), by Remark 3.2 it follows that

$$\int_{K_r(b)\times\mathbb{R}^k} |y|^\gamma |E'_{n,\eta}|^2 \leq (C_m + g(\eta) C_{d,s,\overline{m}}) |K_r|\ , \tag{87}$$

at least for L, r large enough. Therefore the hypotheses of Proposition 3.1 are satisfied in $K_r(b)$ and we may apply it at scale r and find a good boundary cube $K'_r(b)$ which is at distance l from $\partial K_r(b)$. By applying Proposition 3.3 to the obtained good boundary we show that there exist $\bar{K}_r(b) \supset K'_r(b)$, a subset $\tilde{\Lambda} \subset \bar{K}_L(b)$ and a vector field $\bar{E}'_n$ such that

$$\begin{cases} -\operatorname{div}(|y|^\gamma \bar{E}'_n) = c_{d,s}\left(\sum_{p\in\tilde{\Lambda}} \delta_p - m'_V(x)\delta_{\mathbb{R}^d}\right) & \text{in } \bar{K}_r(b)\times\mathbb{R}^k \\ \bar{E}'_n \cdot \vec{\nu} = 0 & \text{on } \partial\bar{K}_r(b)\times\mathbb{R}^k \\ \bar{E}'_n \cdot \vec{\nu} = \tilde{E}'_n \cdot \vec{\nu} & \text{on } \partial K'_r(b)\times[-t,t]^k \\ \bar{E}'_n = \tilde{E}'_n,\ \tilde{\Lambda} = \Lambda & \text{in } K'_r(b)\times[-t,t]^k \end{cases}$$

and

$$\mathcal{W}_\eta(\tilde{E}'_n, K'_r(b)) \geq \mathcal{W}_\eta(\bar{E}'_n, \bar{K}_r(b)) - Cr^d err_{sc}(\eta, r, t, l, 1, C_1, C_2). \tag{88}$$

With parameter choices like in Section 3.2 with $r, t, l, 1$ in the place of L, t, l, ε_1 and with C_1 equal to the constant $C_m + g(\eta)C_{s,d,\overline{m}}$ from (87), we obtain that the err_{sc} error above is controlled as $(1+g(\eta))o_{r,n\to\infty}(1)$. By using the separation of charges as in Remark 3.2 we find that

$$\mathcal{W}_\eta(\tilde{E}'_n, K'_r(b)) \leq \mathcal{W}_\eta(\tilde{E}'_n, K_r(b)) + Cg(\eta)r^{d-1}l = \mathcal{W}_\eta(\tilde{E}'_n, K_r(b)) + |K_r|g(\eta)o_{r,n\to\infty}(1).$$

After absorbing the above error and summing over all $K_i(a_i)$, we obtain

$$\mathcal{W}_\eta(\tilde{E}'_n, \bar{K}_L(a)) \geq \sum_i \min(C_m, \mathcal{W}_\eta(\bar{E}'_n, \bar{K}_i(a_i))) - (1+g(\eta))(1+C_1)o_{r,n\to\infty}(1)|\bar{K}_L(a)|. \tag{89}$$

Now we correct the background measure of $\bar{E}'_n$ on each cube $K_i(a_i)$. To do this we proceed as in the proof of Proposition 4.3. We put $\tilde{m} = \fint_{K_r(b)} m'_V(x)\,dx$ and we define h to be the solution of

$$\begin{cases} -\operatorname{div}(|y|^\gamma \nabla h) = c_{s,d}(m'_V(x) - \tilde{m})\delta_{\mathbb{R}^d} & \text{in } \bar{K}_r(b)\times[-r,r]^k \\ \partial_\nu h = 0 & \text{on } \partial\bar{K}_r(b)\times[-r,r]^k \\ \partial_\nu h = 0 & \text{on the rest of } \partial(\bar{K}_r(b)\times[-r,r]^k) \end{cases}$$

and we put $h = 0$ in $\bar{K}_r(b)\times(\mathbb{R}^k \setminus [-r,r]^k)$. Then define $\bar{E}$ to be $\bar{E}'_n + \nabla h$ in $\bar{K}_r(b)$. Hence

$$\begin{aligned} \frac{\mathcal{W}_\eta(\bar{E}, \bar{K}_r(b))}{|\bar{K}_r|} \leq \frac{\mathcal{W}_\eta(\bar{E}'_n, \bar{K}_r(b))}{|\bar{K}_r|} + Cr^{2+2\alpha}\|m'_V\|^2_{C^{0,\alpha}(\bar{K}_r(b))} \\ + Cr^{1+\alpha}\|m'_V\|_{C^{0,\alpha}(\bar{K}_r(b))}\left(C + c_{d,s}\tilde{m}g(\eta)\right)^{1/2}. \end{aligned} \tag{90}$$

Next, we use the fact that

$$\frac{\mathcal{W}_\eta(\bar{E}, \bar{K}_r(b))}{|\bar{K}_r|} \geq \min_{\mathcal{A}_{\tilde{m}}} \mathcal{W} + o_{\eta\to 0}(1).$$

Indeed, define $\tilde{K}_r(b)$ to be the hypercube obtained by taking all the iterative reflections in the d directions across faces of $\bar{K}_r(b)$ (the sidelengths of $\tilde{K}_r(b)$ are thus 2 twice the sidelengths of $\bar{K}_r(b)$). Since we have zero Neumann boundary condition, we can extend $\bar{E}$ on $\tilde{K}_r(b)\times\mathbb{R}^k$ by

reflection across the boundary of $\tilde{K}_t(b)$. Then we periodize $\bar{E}$ to have a vector-field E defined on $\mathbb{R}^d \times \mathbb{R}^k$. Since all the vector fields and measures are periodic,

$$\min_{\mathcal{A}_{\tilde{m}}} \mathcal{W}_\eta \leq \mathcal{W}_\eta(E) = \frac{2^d \mathcal{W}_\eta(\bar{E}, \bar{K}_r(b))}{2^d |\bar{K}_r|} = \frac{\mathcal{W}_\eta(\bar{E}, \bar{K}_r(b))}{|\bar{K}_r|}.$$

Similarly to the process of obtaining (83), with the notations $M_{s,d}(m'_V)$ like in (84), we have

$$\left| \int_{\bar{K}_r(b)} m'_V(x)^{1+s/d} dx - |\bar{K}_r| \tilde{m}^{1+s/d} \right| \leq C |\bar{K}_r| r^\alpha \|m'_V\|_{C^{0,\alpha}(K_r(b))} M_{s,d}(m'_V)$$

and thus we absorb this into the above $r^{1+\alpha}$-term and we obtain

$$\begin{aligned} \mathcal{W}_\eta(\bar{E}, \bar{K}_r(b)) \geq \int_{\bar{K}_r(b)} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W}\, dx - |\bar{K}_r| \left(1 + g(\eta)\right) (1 + C_1) o_{r,n\to\infty}(1) - |\bar{K}_r| o_{\eta\to 0}(1) \\ - C|\bar{K}_r| r^{2+2\alpha} \|m'_V\|^2_{C^{0,\alpha}(K_r)} - C|\bar{K}_r| r^{1+\alpha} \|m'_V\|_{C^{0,\alpha}(\bar{K}_r)} \left(C + c_{d,s} \bar{m} g(\eta)\right)^{1/2}. \end{aligned}$$

Hence, summing over all cubes and using our choice of C_m in (86), using the fact that $\bar{K}_L(a)$ has larger measure than the union of the $K_i(a_i)$ as well as the direct bounds on $|\bar{K}_r|/|K_r|$ and on the integral of m'_V over the difference of these cubes, we are led to

$$\begin{aligned} \mathcal{W}_\eta(\tilde{E}'_n, \bar{K}_L(a)) \geq \int_{\bar{K}_L(a)} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W}\, dx - O(l/r)|\bar{K}_L| - |\bar{K}_L| \left(1 + g(\eta)\right) o_{r,n\to\infty}(1) \\ - (1 + O(l/r))|\bar{K}_L| o_{\eta\to 0}(1) - C|\bar{K}_L| r^{2+2\alpha} \|m'_V\|^2_{C^{0,\alpha}(\bar{K}_L(a))} \\ - C|\bar{K}_L| r^{1+\alpha} \|m'_V\|_{C^{0,\alpha}(\bar{K}_L(a))} \left(C + c_{d,s} \bar{m} g(\eta)\right)^{1/2}. \end{aligned} \tag{91}$$

Note that under the choice of parameters according to Section 3.2 we have $O(l/r) = o_{r,n\to\infty}(1)$. This proves (85) and thus concludes the proof. □

4.1.3. *End of proof of Proposition 4.1.* We recall that due to (76) and (77), under the choice of parameters as in Section 3.2, we have

$$\frac{\mathcal{W}_\eta(\tilde{E}'_n, \bar{K}_L(a))}{|\bar{K}_L(a)|} - err \leq \frac{\mathcal{W}_\eta(E'_n, K'_L(a))}{|K'_L(a)|} \leq \sigma_0(\underline{K}_L(a); m'_V) + err + o_{\eta\to 0}(1),$$

for $err := (1 + g(\eta))(1 + C_1) o_{L,n\to\infty}(1)$. Next we use Proposition 4.3 in which we choose $\rho = m'_V, K_R(a) = \underline{K}_L(a)$ and $\beta \in]0, 1[$, and Proposition 4.5.

The final result after including the extra errors from (79), (85) and using (68) is that on the cube $K'_L(a)$ which has sidelenghts $\sim L_i \in [\varepsilon_1^{1/d} T_i, T_i]$ we have

$$\frac{1}{|\bar{K}_L(a)|} \int_{\bar{K}_L(a)} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W} dx - err_1 \leq \frac{\mathcal{W}_\eta(E'_n, K'_L(a))}{|K'_L(a)|} \leq \frac{1}{|\underline{K}_L(a)|} \int_{\underline{K}_L(a)} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W} dx + err_2,$$

where

$$\begin{aligned} err_1 &:= (1 + g(\eta))(1 + C_1) o_{L,n\to\infty}(1) + o_{\eta\to 0}(1) + o_{n\to\infty}(1) \left(1 + g(\eta) + o_{L\to\infty}(1)\right)^{1/2} \\ err_2 &:= (1 + g(\eta))(1 + C_1) o_{L,n\to\infty}(1) + (1 + o_{L,n\to\infty}(1)) o_{\eta\to 0}(1) + o_{n\to\infty}(1) \left(1 + g(\eta)\right)^{1/2}. \end{aligned}$$

Using the facts that m'_V is Hölder continuous and that in Proposition 3.3 we obtained $\bar{K}_L, \underline{K}_L$ are t-close to K_L, we further find up to requiring L to be larger,

(92)
$$\frac{1}{|K_L(a)|}\int_{K_L(a)} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W} dx - err_1 \leq \frac{\mathcal{W}_\eta(E'_n, K'_L(a))}{|K'_L(a)|} \leq \frac{1}{|K_L(a)|}\int_{K_L(a)} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W} dx + err_2.$$

5. Proof of Theorem 2

5.1. **Tools for estimates on crenel boundaries.** By refining the proofs of [52, Lem. 2.3, Prop. 2.4], we obtain an estimate generalizing the discrepancy bound of [54, Prop. 3.1], which relied on the ball construction [60], [42]. It seems to be a difficult open question whether a precise analogue of the ball construction can be done in dimensions $d > 2$ for the not conformal energy considered here, and in particular in cases where a suited notion of degree like in [42] is missing. We recall that here, compared to [52], we are in the situation of multiplicities equal to one for $p \in \Lambda$, as considered also in [54] and [60]. This condition is a consequence of minimality as proved in [52, Thm. 5] (Proposition 2.3 here).

Proposition 5.1. *Let Λ, E be as in (35). Consider a compactly supported cutoff function $\chi_A \in C^1(\mathbb{R}^{d+k}, [0,1])$ with $A := \mathrm{spt}\chi_A$. Let $0 < \alpha < \eta < 1$ and assume that $\mathcal{W}_\eta(\chi_A, E) < \infty$. For a set $S \subset \mathbb{R}^d$ write $S_r := \{x : \mathrm{dist}(x, S) \leq r\}$. Then we may write*

$$\mathcal{W}_\alpha(\chi_A, E) - \mathcal{W}_\eta(\chi_A, E) = I + II + III, \tag{93}$$

where for constants c, C depending only on s, d there holds

$$c\left(\min_x m(x)\right) \#\left(\Lambda \cap A_\eta\right) \leq \frac{I}{\min(\eta^{d-s}, \eta^d |\log \eta|)} \leq C\left(\max_x m(x)\right) \#\left(\Lambda \cap A_\eta\right). \tag{94}$$

If for a set $S \subset \mathbb{R}^d$ we denote $X_r(S) := \{(p,q) : \ p \neq q, \ p, q \in \Lambda \cap S, \ |p-q| < r\}$ then

$$0 \leq II \leq c_{s,d}(g(\alpha) - g(\eta)) \# X_{2\eta}(A_\eta). \tag{95}$$

Finally, for a constant C depending only on d, s,

$$|III| \leq C \# \left(\Lambda \cap (\mathrm{spt}(\nabla \chi_A))_\eta\right) \eta^d g^2(\alpha) \left(\|\nabla_x \chi_A(0,x)\|_{L^1(\mathbb{R}^d)} + \int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla \chi_A| |E_\alpha| \right). \tag{96}$$

Proof. Define

$$f_{\alpha,\eta}(X) := f_\alpha(X) - f_\eta(X), \quad X \in \mathbb{R}^{d+k}.$$

From the definition (27) of f_η we find that

$$f_{\alpha,\eta}(X) = \begin{cases} g(\eta) - g(\alpha) & \text{for } |X| \leq \alpha, \\ g(\eta) - g(X) & \text{for } \alpha \leq |X| \leq \eta, \\ 0 & \text{for } |X| \geq \eta, \end{cases} \tag{97}$$

and as a consequence of the equation (29) that f_η satisfies

$$-\mathrm{div}(|y|^\gamma \nabla f_{\alpha,\eta}) = c_{d,s}\left(\delta_0^{(\eta)} - \delta_0^{(\alpha)}\right). \tag{98}$$

By the definition (36) of E_η we find

$$E_\eta = E_\alpha + \sum_{p \in \Lambda} \nabla f_{\alpha,\eta}(X - p).$$

Thus we may write, using the definition (39) of $\mathcal{W}_\eta$,

$$\begin{aligned}\mathcal{W}_\alpha(\chi_A,E)-\mathcal{W}_\eta(\chi_A,E) &= \int_{\mathbb{R}^{d+k}}|y|^\gamma\chi_A\left(|E_\alpha|^2-\left|E_\alpha+\sum_{p\in\Lambda}\nabla f_{\alpha,\eta}(X-p)\right|^2\right)\\ &\quad -c_{d,s}\int_{\mathbb{R}^{d+k}}\chi_A\sum_{p\in\Lambda}\left(g(\alpha)\delta_p^{(\alpha)}-g(\eta)\delta_p^{(\eta)}\right).\end{aligned}\tag{99}$$

The first term on the right in (99) can be expanded to give, after using an integration by parts and (98), (38), the following sum:

$$\begin{aligned}&-c_{d,s}\sum_{p,q\in\Lambda}\int_{\mathbb{R}^{d+k}}\chi_A f_{\alpha,\eta}(X-p)\left(\delta_q^{(\eta)}-\delta_q^{(\alpha)}\right)+\sum_{p,q\in\Lambda}\int_{\mathbb{R}^{d+k}}|y|^\gamma f_{\alpha,\eta}(X-p)\nabla\chi_A\cdot\nabla f_{\alpha,\eta}(X-q)\\ &-2c_{d,s}\sum_{p\in\Lambda}\int_{\mathbb{R}^{d+k}}f_{\alpha,\eta}(X-p)\left(\sum_{q\in\Lambda}\delta_q^{(\alpha)}-m\delta_{\mathbb{R}^d}\right)\chi_A+2\sum_{p\in\Lambda}\int_{\mathbb{R}^{d+k}}f_{\alpha,\eta}(X-p)|y|^\gamma\nabla\chi_A\cdot E_\alpha.\end{aligned}\tag{100}$$

We define the following terms which summed together give (100):

$$I := 2c_{d,s}\sum_{p\in\Lambda}\int_{\mathbb{R}^d}f_{\alpha,\eta}(x-p)m(x)\chi_A,\tag{101}$$

$$II := -c_{d,s}\sum_{p\neq q\in\Lambda}\int_{\mathbb{R}^{d+k}}\chi_A f_{\alpha,\eta}(X-p)\left(\delta_q^{(\eta)}+\delta_q^{(\alpha)}\right),\tag{102}$$

$$III' := -c_{d,s}\sum_{p\in\Lambda}\int_{\mathbb{R}^{d+k}}\chi_A(g(\eta)-g(\alpha))\delta_p^{(\alpha)},\tag{103}$$

$$\begin{aligned}III'' &:= \sum_{p,q\in\Lambda}\int_{\mathbb{R}^{d+k}}|y|^\gamma f_{\alpha,\eta}(X-p)\nabla\chi_A\cdot\nabla f_{\alpha,\eta}(X-q)\\ &\quad +2\sum_{p\in\Lambda}\int_{\mathbb{R}^{d+k}}f_{\alpha,\eta}(X-p)|y|^\gamma\nabla\chi_A\cdot E_\alpha,\end{aligned}\tag{104}$$

where in the expression III' we used the properties of $f_{\alpha,\eta}(X-p)$ which equals $g(\eta)-g(\alpha)$ on the support of $\delta_p^{(\alpha)}$ and is zero on the support of $\delta_p^{(\eta)}$.

The term I is bounded as in (94) by noticing that only the points $p\in\Lambda\cap A_\eta$ are such that $\operatorname{spt}f_{\alpha,\eta}\cap A\neq\emptyset$ and that $\|f_{\alpha,\eta}\|_{L^1}\leq C_{d,s}\min(\eta^{d-s},\eta^d|\log\eta|)$.

Regarding the term II we note that again, each of the terms

$$\int_{\mathbb{R}^{d+k}}f_{\alpha,\eta}(X-p)\left(\delta_q^{(\eta)}+\delta_q^{(\alpha)}\right)\chi_A\tag{105}$$

corresponding to a choice $p,q\in\Lambda$ vanishes in case $(p,q)\notin X_{2\eta}(A_\eta)$ due to the support properties of $f_{\alpha,\eta}(X-p)$ and $\delta_q^{(\eta)}$, and it has values in the interval $[g(\eta)-g(\alpha),0]$ due to (97) and to the fact that $\delta_q^{(\eta)},\delta_q^{(\alpha)}$ are probability measures. This gives the bounds in (95).

The term III' summed to the second line in (99) give the contribution

$$-c_{d,s}\sum_{p\in\Lambda}\int_{\mathbb{R}^{d+k}}\chi_A g(\eta)\left(\delta_p^{(\alpha)}-\delta_p^{(\eta)}\right)=c_{d,s}g(\eta)\sum_{p\in\Lambda}\left(\frac{1}{|\partial B_\eta(p)|}\int_{\partial B_\eta(p)}\chi_A-\frac{1}{|\partial B_\alpha(p)|}\int_{\partial B_\alpha(p)}\chi_A\right)$$

which can be bounded via the first term in (96). The term III'' is also similarly bounded by (96). We now define III as the sum of III', III'' and of the second line of (99), and we then find the bound (96), concluding the proof. □

From the Proposition 5.1, by approximating the characteristic function $1_{A\times\mathbb{R}^k}$ of a measurable set by C^1-functions χ_A which appropriately avoid the set Λ, we find the following result:

Corollary 4.
- *If $A\subset\mathbb{R}^d$ is a bounded Borel set such that $\mathrm{dist}(\partial A,\Lambda)\geq\epsilon>0$ and we chose $\eta,\alpha<\epsilon$ then we may find a decomposition $\mathcal{W}_\alpha(A,E)-\mathcal{W}_\eta(A,E)=I+II$ satisfying (94), (95).*
- *If in the setting of Proposition 5.1 we further assume that $|p-q|\geq 2\eta$ for all pairs of points $p\neq q\in\Lambda\cap A$ then we find $\mathcal{W}_\alpha(A,E)-\mathcal{W}_\eta(A,E)=I$ satisfying (94).*

5.2. **Construction of crenel boundaries.** The main idea of the proof of Theorem 2 is that if we perturb the boundary of $K_\ell(a)$ such that it avoids the charges then the sharp bounds of Proposition 5.1 allow to take the $\eta\to 0$ limit in Theorem 1 without uncontrolled error terms. For the perturbation of $\partial K_\ell(a)$ we will need the following tool:

Proposition 5.2 (crenel boundaries)**.** *Let $m_V, m_V', \Sigma, \Sigma', E_n'$ be as in Theorem 2 and $K_\ell(a)\subset\Sigma'$. Let moreover r_0 be the minimum point separation of the charges corresponding to E_n', bounded in Proposition 2.3. Then the following hold.*

(1) There exists a set Γ which can be expressed as the image of a bi-Lipschitz deformation $f: K_\ell(a)\to\Gamma$ such that $\|f-id\|_{L^\infty}\leq 1$, and for which

$$\min_{p\in\Lambda\cap\Gamma}\mathrm{dist}(p,\partial\Gamma)\geq\frac{r_0}{8}. \tag{106}$$

(2) There exists a constant C depending only on the dimension such that if

$$r_1<Cr_0^dL^{-d+1} \tag{107}$$

then there exists a universal constant C and a cube $K_\ell'(a)$ with $\mathrm{dist}(\partial K_\ell'(a),\partial K_\ell(a))<1$ and such that moreover

$$\min_{p\in\Lambda\cap K_\ell'}\mathrm{dist}(p,\partial K_\ell')\geq r_1. \tag{108}$$

Proof. **Part 1.** This follows Step 1 of the proof of [55, Prop. 5.6]. Consider the set P points p such that $K_{r_0/2}(p)\cap\partial K_\ell\neq\emptyset$. Then all cubes $K_{r_0/2}(p)$, $p\in P$ are disjoint and included in the set $K_{\ell+1}(a)\setminus K_{\ell-1}(a)$. Then define $\Gamma:=K_\ell\cup\bigcup_{p\in P}K_{r_0/2}(p)$. The fact that Γ is a bi-lipschitz and close to the identity deformation of K_ℓ is a straightforward but tedious argument that we leave to the reader.

Part 2. We prove that if (107) holds for a small enough C depending on d only, then Γ can be chosen to be a hyperrectangle. Note first that due to the separation condition, if $r_0<1$

then the $r_0/2$-balls with centers in P are disjoint and contained in $K_{\ell+1} \setminus K_{\ell-1}$, therefore by a straightforward volume comparison the number of charges in P satisfies

$$|P| \leq \frac{|K_{\ell+1} \setminus K_{\ell-1}|}{|B_{r_0/2}|} \leq C r_0^{-d} \ell^{d-1}, \tag{109}$$

where $C_{(109)}$ is a packing constant which depends only on the dimension. Let now

$$T_{r_1} := \{\tau \in [-1,1] : \ \exists p \in \Lambda, \partial K_{\ell+\tau} \cap B_{r_1}(p) \neq \emptyset\},$$

Then T_{r_1} can be covered by at most $|P|$ intervals of size $2r_1$ and thus due to the bound (109), we find that $|T_{r_1}| \leq C_{(109)} r_1 r_0^{-d} \ell^{d-1}$. Now fix $C_{(107)}$ depending on the packing constant $C_{(109)}$ only, such that for r_1 satisfying (107) we have $C_{(109)} r_1 r_0^{-d} L^{d-1} < 1$. Therefore $[-1,1] \setminus T_{r_1} \neq \emptyset$, furnishing the desired cube $K'_\ell(a)$. □

5.3. **Proof of Theorem 2 given Theorem 1.** For each $K_\ell(a_n)$ as in Theorem 2 we consider a modification Γ_n as obtained by applying Proposition 5.2 (the case of cubes $K'_\ell(a_n)$ is completely analogous). Then firstly, we obtain via Proposition 5.1 and Corollary 4 that for all $0 < \alpha < \eta < r_0/2$, where r_0 is the separation constant of Proposition 2.3, there holds

$$\left|\mathcal{W}_\eta(E'_n, \Gamma_n) - \mathcal{W}_\alpha(E'_n, \Gamma_n)\right| \leq C\bar{m} \min(\eta^{d-s}, \eta^d |\log \eta|)|\Gamma_n|, \tag{110}$$

Second, recall that by the result (72) together with the generalization of such bounds, which is done as in Section 4, we have that for the choices $\sigma = \pm 1$ there holds

$$\left|\frac{\mathcal{W}_\eta(E'_n, K_{\ell+\sigma}(a_n))}{|K_{\ell+\sigma}(a_n)|} - \frac{1}{|K_{\ell+\sigma}|}\int_{K_{\ell+\sigma}(a_n)} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W} dx\right| \leq (1+g(\eta))(1+C_1)o_{\ell,n\to\infty}(1) + o_{\eta\to 0}(1). \tag{111}$$

Moreover, if $M_{s,d}$ is the shorthand used in the bound (84), we obtain

$$\left|\int_{K_{\ell+1}(a_n)\setminus\Gamma_n} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W} dx\right| \leq M_{s,d}(m_V)|K_{\ell+1}(a_n) \setminus \Gamma_n| \leq C M_{s,d} \ell^{d-1}. \tag{112}$$

Using the bounds (111), the fact that $K_{\ell-1}(A_n) \subset \Gamma_n \subset K_{\ell+1}(a_n)$, the definition of $\mathcal{W}_\eta$ (and in particular the fact that the two terms defining $\mathcal{W}_\eta(E'_n, A)$ as in (39) are additive and monotone under inclusion with respect to A) and Remark 3.2, we obtain

$$\left|\frac{\mathcal{W}_\eta(E'_n, K_{\ell+1} \setminus \Gamma_n)}{|\Gamma_n|}\right| \leq (1 + g(\eta))(1 + C_1)o_{\ell,n\to\infty}(1) + o_{\eta\to 0}(1). \tag{113}$$

By summing up (110), (111), (112) and (113), we find that for all $0 < \alpha < \eta < r_0/2$ there holds

$$\left|\frac{\mathcal{W}_\alpha(E'_n, \Gamma_n)}{|\Gamma_n|} - \frac{1}{|\Gamma_n|}\int_{\Gamma_n} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W} dx\right| \leq (1 + g(\eta))(1 + C_1)o_{\ell,n\to\infty}(1) + o_{\eta\to 0}(1). \tag{114}$$

Therefore for all $\epsilon > 0$ we may fix $\eta < r_0/2$ such that the rightmost term above is $\leq \epsilon$ and then let $\alpha \to 0$ obtaining that the limit in (10) is $\leq \epsilon$. As $\epsilon > 0$ is arbitrary this concludes the proof of (10).

6. Discrepancy bounds

In this section we show how to deduce the result of Theorem 3 from the one of Theorem 1. We consider a scale $\underline{\ell} \leq \frac{1}{4}\ell$ to be more precisely fixed later, and we find that due to Theorem 1 there holds

$$\mathcal{W}_\eta(E'_n, K_{\underline{\ell}}(b)) \leq \int_{K_{\underline{\ell}}(b)} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W} dx + |K_{\underline{\ell}}| o_{\eta\to 0, n, \underline{\ell}\to\infty}(1).$$

Due to the charge separation condition and to Remark 3.2 and to the bounds on m'_V, we find, using the scaling of $c \mapsto \min_{\mathcal{A}_c} \mathcal{W}$ (see (84)) that

$$\int_{K_{\underline{\ell}}(b)} |E'_n|^2 \leq \int_{K_{\underline{\ell}}(b)} \min_{\mathcal{A}_{m'_V(x)}} \mathcal{W} dx + C_{d,\overline{m}}(1+g(\eta))|K_{\underline{\ell}}| \leq \bar{C}_1 |K_{\underline{\ell}}(b)|, \tag{115}$$

where $\bar{C}_1$ depends only on $\underline{m}, \overline{m}, s, d$. By Besicovitch's covering theorem, we find a cover of $\partial K_{\ell-2\underline{\ell}}(a)$ by the union of J_d families $\mathcal{F}_j$ of cubes $K_{\underline{\ell}}(b), b \in \partial K_{\ell-2\underline{\ell}}$, such that the cubes $K_{2\underline{\ell}}(b)$ belonging to a given $\mathcal{F}_j$ are disjoint. Then for each $j \in J_d$ and each $K \in \mathcal{F}_j$, from (115) by a mean value theorem there exist $\underline{\ell}_1 \in [\underline{\ell}, 2\underline{\ell}]$ (note that $\underline{\ell}_1$, unlike $\underline{\ell}$, depends on the center b, but we omit this dependence to make notations lighter) such that

$$\int_{\partial K_{\underline{\ell}_1}(b)\times\mathbb{R}^k} |E'_n|^2 \leq C\bar{C}_1 \underline{\ell}^{d-1}. \tag{116}$$

Then from (116) we obtain by Cauchy-Schwartz inequality

$$\left| \int_{\partial K_{\underline{\ell}_1}(b)} E'_n \cdot \nu \right| \leq |\partial K_{\underline{\ell}_1}(b)|^{1/2} \left(\int_{\partial K_{\underline{\ell}_1}(b)} |E'_n|^2 \right)^{1/2} \leq C\bar{C}_1 \underline{\ell}^{d-1}. \tag{117}$$

As a consequence of the properties of our cover, we also obtain that the cubes $K_{\underline{\ell}_1}(b)$ cover $\partial K_{\ell-2\underline{\ell}}(a)$. In particular,

$$\partial R_{\ell,\underline{\ell}} := \partial \left(K_{\ell-2\underline{\ell}}(a) \cup \bigcup_{j\in J_d} \bigcup_{K_{\underline{\ell}}\in\mathcal{F}_j} K_{\underline{\ell}_1} \right) \subset \partial \left(\bigcup_{j\in J_d} \bigcup_{K_{\underline{\ell}}\in\mathcal{F}_j} K_{\underline{\ell}_1} \right).$$

It then follows that, using (117) and the fact that the cubes $K_{\underline{\ell}_1}$ corresponding to a single $\mathcal{F}_j$ are disjoint,

$$\left| \int_{\partial R_{\ell,\underline{\ell}}} E'_n \cdot \nu \right| \leq \sum_{j\in J_d} \sum_{K_{\underline{\ell}}\in\mathcal{F}_j} \left| \int_{\partial K_{\underline{\ell}_1}} E'_n \cdot \nu \right| \leq |J_d| \max_{j\in J_d} |\mathcal{F}_j| C\bar{C}_1 \underline{\ell}^{d-1}. \tag{118}$$

As the $K_{\underline{\ell}}$ cover $\partial K_{\ell-2\underline{\ell}}(a)$, also for the larger cubes $K_{\underline{\ell}_1}$ the sets $\partial K_{\ell-2\underline{\ell}}(a) \cap K_{\underline{\ell}_1}$ corresponding to the union of all the $\mathcal{F}_j$'s cover $\partial K_{\ell-2\underline{\ell}}(a)$. We also have that, as the centers of such $K_{2\underline{\ell}}$ belong to $\partial K_{\ell-2\underline{\ell}}(a)$, there holds

$$\mathcal{H}^{d-1}(\partial K_{\ell-2\underline{\ell}}(a) \cap K_{2\underline{\ell}}) \leq C_d \underline{\ell}^{d-1}.$$

Therefore, as the $K_{2\underline{\ell}}(b)$ corresponding to each single family $\mathcal{F}_j$ are disjoint, by summing the above inequalities over each fixed $\mathcal{F}_j$ separately, we find the lower bound below, while the upper bound is straightforward:

$$\max_{j\in J_d} |\mathcal{F}_j| C_d \underline{\ell}^{d-1} \leq \mathcal{H}^{d-1}(\partial K_{\ell-2\underline{\ell}}(a)) \leq C(\ell - 2\underline{\ell})^{d-1}.$$

Using this and (118), by integrating the equation (26) satisfied by E'_n over $R_{\ell,\underline{\ell}}$ and using Stokes' theorem, we find that (for a new constant C depending on the above C_d and using the previous choice $\underline{\ell} < \frac{1}{4}\ell$ as above)

(119)
$$\left|\nu'_n(R_{\ell,\underline{\ell}}) - \int_{R_{\ell,\underline{\ell}}} \mu'_V\right| = \left|\int_{\partial R_{\ell,\underline{\ell}}} E'_n \cdot \nu\right| \leq C\bar{C}_1|J_d|(\ell - 2\underline{\ell})^{d-1} \leq C\bar{C}_1(\ell - 2\underline{\ell})^{d-1} \leq 2^{1-d}\bar{C}_1\ell^{d-1}.$$

To reach the desired bound, as (119), (11) involve additive quantities, by triangle inequality we just need to estimate $|\nu'_n - \int \mu'_V|$ over $K_\ell(a) \setminus R_{\ell,\underline{\ell}}$. To this aim we separately bound from above $\nu'_n(K_\ell(a) \setminus R_{\ell,\underline{\ell}})$ and $\int_{K_\ell(a)\setminus R_{\ell,\underline{\ell}}} \mu'_V$ by using Remark 3.2 and the bounds on m_V, together with the fact that $|K_\ell(a) \setminus R_{\ell,\underline{\ell}}| \leq C\underline{\ell}\ell^{d-1}$. Therefore we obtain

$$\left|\nu'_n(K_\ell(a)) - \int_{K_\ell(a)} \mu'_V\right| \leq C(\bar{C}_1 + (1 + g(\eta))\underline{\ell})\ell^{d-1}. \tag{120}$$

Here $\bar{C}_1$ depends only on the bounds in (9) at scale $\underline{\ell}$, thus $\underline{\ell}$ can be chosen so that the energy error in (9) at scale $\underline{\ell}$ is uniformly bounded on all $\underline{\ell}$-cubes. As the right hand side of (120) does not have any further dependece on $\underline{\ell}$, we find (11).

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