SOME ENTROPIES FOR MEAN CURVATURE FLOW

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CONTENTS

1. BASIC COMPUTATIONS

For an immersed hypersurface $M \subset \mathbb{R}^{n+1}$, we call A and H respectively its *second fundamental form* and its *mean curvature*.

Let $M_t = \varphi(M, t)$ be the *mean curvature flow* (MCF) of a n–dimensional compact hypersurface in $\mathbb{R}^{n+1}$ (or in a flat Riemannian manifold T), defined by the smooth family of immersions $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ which satisfies $\partial_t \varphi = \mathrm{H}$.

It is well known that if a bounded convex set $\Omega \subset \mathbb{R}^{n+1}$ contains $\varphi(M, 0)$ then all the flow M_t stays inside Ω for every $t \in [0, T)$. Hence, if we embed isometrically such an Ω in a flat compact Riemannian manifold, we can consider the flow M_t as if it "lives" in T.

Suppose that we have a positive smooth solution of $u_t = -\Delta u$ in $\Omega \times [0, C]$ and $M_t \subset \Omega \subset \mathbb{R}^{n+1}$. The generalization of Huisken's monotonicity formula by Hamilton read (see [4, 5, 6])

$$\begin{aligned}(1.1)\quad \frac{d}{dt}\Big[\sqrt{2(C-t)}\int_M u\,d\mu_t\Big] = &-\sqrt{2(C-t)}\int_M u\,|\mathrm{H}+\nabla^\perp \log u|^2\,d\mu_t \\ &-\sqrt{2(C-t)}\int_M \Big(\nabla^\perp\nabla^\perp u - \frac{|\nabla^\perp u|^2}{u} + \frac{u}{2(C-t)}\Big)\,d\mu_t\end{aligned}$$

in the time interval $[0, \min\{C, T\})$, where $\nabla^\perp$ denotes the covariant derivative along the normal direction and $d\mu_t$ is the canonical measure on M associated to the metric induced by the immersion at time t.

Suppose that we have a positive smooth solution of $u_t = -\Delta u$ in $\mathrm{T} \times [0, C]$ with T a flat compact Riemannian manifold without boundary. Then, we have (see [9])

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$$\frac{d}{dt}\int_{\mathrm{T}} u\log u\,dx = \int_{\mathrm{T}}\frac{|\nabla u|^2}{u}\,dx$$
$$\frac{d}{dt}\int_{\mathrm{T}}\frac{|\nabla u|^2}{u}\,dx = 2\int_{\mathrm{T}} u\,|\mathrm{Hess}\log u|^2\,dx$$

$$\begin{aligned}(1.2)\quad \frac{d}{dt}\int_{\mathrm{T}}\Big((C-t)\,\frac{|\nabla u|^2}{u} - u\log u - \frac{(n+1)u}{2}\log\left[4\pi(C-t)\right] - u(n+1)\Big)\,dx\\
&= \int_{\mathrm{T}}\Big(2(C-t)u\,|\mathrm{Hess}\log u|^2 - 2\frac{|\nabla u|^2}{u} + \frac{(n+1)u}{2(C-t)}\Big)\,dx\\
&= \int_{\mathrm{T}}\Big(2(C-t)u\,\Big|\mathrm{Hess}_{ij}\log u + \frac{\delta_{ij}}{2(C-t)}\Big|^2 - 2u\Delta\log u - 2\frac{|\nabla u|^2}{u}\Big)\,dx\\
&= 2(C-t)\int_{\mathrm{T}} u\,\Big|\mathrm{Hess}_{ij}\log u + \frac{\delta_{ij}}{2(C-t)}\Big|^2\,dx\,,\end{aligned}$$

in $[0,\min\{C,T\})$.

2. Perelman's Type Entropies for MCF

Definition 2.1. Given a flat $(n+1)$–dimensional Riemannian manifold T and a smooth immersed hypersurface $\varphi : M \to \mathrm{T}$, we consider the *Huisken's integral*

$$\mathcal{H}(\varphi,\mathrm{T},\tau,u) = \sqrt{4\pi\tau}\int_M u\,d\mu$$

and the *Perelman's entropy*

$$\mathcal{W}(\mathrm{T},\tau,u) = \int_{\mathrm{T}}\Big(\tau\,\frac{|\nabla u|^2}{u} - u\log u - \frac{(n+1)u}{2}\log\left[4\pi\tau\right] - u(n+1)\Big)\,dx$$

for $\tau > 0$ and $u : \mathrm{T} \to \mathbb{R}$ smooth and positive.
We define the combined $\mathcal{HW}$–entropy as follows,

$$\begin{aligned}\mathcal{HW}(\varphi,\mathrm{T},\tau,u) &= \mathcal{H}(\varphi,\mathrm{T},\tau,u) - 2\mathcal{W}(\mathrm{T},\tau,u)\\
&= \sqrt{4\pi\tau}\int_M u\,d\mu - \int_{\mathrm{T}}\Big(2\tau\,\frac{|\nabla u|^2}{u} - 2u\log u - u(n+1)\log\left[4\pi\tau\right] - 2u(n+1)\Big)\,dx\,.\end{aligned}$$

Remark 2.2. The Perelman's functional $\mathcal{W}$ shares the following important properties, see [8],

- Suppose that $K_{\mathrm{T}}(\cdot,p,\tau)$ is the positive heat kernel at some point $p \in \mathrm{T}$ and time $\tau > 0$, for the manifold T, then $\mathcal{W}(\mathrm{T},\tau,K_{\mathrm{T}}(\cdot,p,\tau)) \leq 0$. Moreover, $\mathcal{W}(\mathbb{R}^{n+1},\tau,K_{\mathbb{R}^{n+1}}(\cdot,p,\tau)) = 0$.
- Defining
$$\theta(\mathrm{T},\tau) = \inf_{\int_{\mathrm{T}} u\,dx\,=\,1,\,u\,\in\,C^\infty(\mathrm{T})}\mathcal{W}(\mathrm{T},\tau,u)\,,$$
we have $\theta(\mathrm{T},\tau) \leq 0$ for every T and $\tau > 0$, and $\lim_{\tau\to 0^+}\theta(\mathrm{T},\tau) = 0$.

- We have $\lim_{\tau\to 0}\mathcal{W}(\mathrm{T},\tau,K_{\mathrm{T}}(\cdot,p,\tau))=0$.

Definition 2.3. Let $\varphi : M \to \mathbb{R}^{n+1}$ be a smooth, compact, immersed hypersurface. Given $\tau > 0$, we consider the family $\mathcal{F}_\tau$ of smooth positive functions $u:\mathbb{R}^{n+1}\to\mathbb{R}$ such that $\int_{\mathbb{R}^{n+1}} u\,dx = 1$ and there exists a smooth positive solution of the problem

$$\begin{cases} v_t = -\Delta v \ \text{ in } \mathbb{R}^{n+1}\times[0,\tau)\,,\\ v(x,0)=u(x) \ \text{ for every } x\in\mathbb{R}^{n+1}\,. \end{cases}$$

Then, we define the following quantity

$$\sigma(\varphi,\tau)=\sup_{u\in\mathcal{F}_\tau}\mathcal{HW}(\varphi,\mathbb{R}^{n+1},\tau,u)\,.$$

Remark 2.4. The heat kernel $K(\cdot,p,\tau)$ of $\mathbb{R}^{n+1}$ at time $\tau>0$ and point $p\in\mathbb{R}^{n+1}$ clearly belongs to the family $\mathcal{F}_\tau$, we recall that $K_{\mathbb{R}^{n+1}}(x,p,\tau)=\frac{e^{-\frac{|x-p|^2}{4\tau}}}{(4\pi\tau)^{\frac{n+1}{2}}}$.

Proposition 2.5. *The quantity $\sigma(\varphi,\tau)$ is positive and precisely, for every $p\in\mathbb{R}^{n+1}$ and $\tau>0$,*

$$\sigma(\varphi,\tau)\geq\sqrt{4\pi\tau}\int_M\frac{e^{-\frac{|x-p|^2}{4\tau}}}{(4\pi\tau)^{\frac{n+1}{2}}}\,d\mu(x)=\int_M\frac{e^{-\frac{|x-p|^2}{4\tau}}}{(4\pi\tau)^{\frac{n}{2}}}\,d\mu(x)>0\,,$$

which is the quantity of the "classical" Huisken's monotonicity formula. Hence,

$$\sigma(\varphi,\tau)\geq\sup_{p\in\mathbb{R}^{n+1}}\int_M\frac{e^{-\frac{|x-p|^2}{4\tau}}}{(4\pi\tau)^{\frac{n}{2}}}\,d\mu(x)>0\,.$$

Proof. By the previous remark, for every $p\in\mathbb{R}^{n+1}$ and $\tau>0$, the heat kernel $K_{\mathbb{R}^{n+1}}(\cdot,p,\tau)$ belongs to the family $\mathcal{F}_\tau$, moreover, my Remark 2.2 it satisfies $\mathcal{W}(\mathbb{R}^{n+1},\tau,K_{\mathbb{R}^{n+1}}(\cdot,p,\tau))=0$, hence, $\sigma(\varphi,\tau)\geq\mathcal{HW}(\varphi,\mathbb{R}^{n+1},\tau,K_{\mathbb{R}^{n+1}}(\cdot,p,\tau))=\sqrt{4\pi\tau}\int_M K_{\mathbb{R}^{n+1}}(\cdot,p,\tau)\,d\mu(x)$. ☐

Before going on we work out some properties of the functions $u\in\mathcal{F}_\tau$.
We recall the integrated version of Li–Yau Harnack inequality (see [7]).

Proposition 2.6 (Li–Yau integral Harnack inequality in $\mathbb{R}^{n+1}$). *Let $u:\mathbb{R}^{n+1}\times(0,T)\to\mathbb{R}$ be a smooth positive solution of heat equation, then for every $0<t\leq s<T$ we have*

$$u(x,t)\leq u(y,s)\left(\frac{s}{t}\right)^{(n+1)/2}e^{\frac{|x-y|^2}{4(s-t)}}\,.$$

Since the functions $v:\mathbb{R}^{n+1}\times[0,\tau)\to\mathbb{R}$ associated to any $u\in\mathcal{F}_\tau$ are positive solutions of the backward heat equation, such inequality read, for $0\leq s\leq t<\tau$,

$$v(x,t)\leq v(y,s)\left(\frac{\tau-s}{\tau-t}\right)^{(n+1)/2}e^{\frac{|x-y|^2}{4(t-s)}}\,. \tag{2.1}$$

This estimates, together with the uniqueness theorem for positive solution of the heat equation (see again [7]), implies that the function $u=v(\cdot,0)$ is obtained by convolution of the function $v(\cdot,t)$ with the *forward* heat kernel at time $t>0$. This fact implies that the condition $\int_{\mathbb{R}^{n+1}} v(x,t)\,dx = 1$ holds for every $t \in [0,\tau)$, and that every derivative of every function v is bounded in the strip $[0, \tau - \varepsilon]$, for every $\varepsilon > 0$.

Another important property is the generalization by Hamilton of differential Li–Yau Harnack inequality in [4].

Proposition 2.7 (Hamilton's matrix Harnack inequality in $\mathbb{R}^{n+1}$)**.** *Let $u : \mathbb{R}^{n+1} \times (0,T) \to \mathbb{R}$ be a smooth positive solution of heat equation such that u, $|\nabla u|$, $|\nabla^2 u|$ are bounded in space by some constant $C(t) > 0$, depending only on t ($C(t)$ is possibly unbounded as $t \to 0$).*
Then, the quadratic matrix

$$H_{ij} = \nabla^2_{ij} u - \frac{\nabla_i u \nabla_j u}{u} + \frac{u}{2t}\delta_{ij}$$

is non negative definite for every $x \in \mathbb{R}^n$ and $t > 0$.

Remark 2.8. In the compact setting the growth hypothesis on u and its derivatives is obviously not needed.

As the functions $w(x,s) = v(x, \tau - s) : \mathbb{R}^{n+1} \times (0,\tau) \to \mathbb{R}$, are positive solutions of heat equation on $(0,\tau)$ satisfying the hypothesis of this proposition, by the previous discussion, we get

$$\nabla^2_{ij} w - \frac{\nabla_i w \nabla_j w}{w} + \frac{w}{2s}\delta_{ij} \geq 0\,,$$

which in terms of v becomes, as $t = \tau - s$,

$$\nabla^2_{ij} v - \frac{\nabla_i v \nabla_j v}{v} + \frac{v}{2(\tau - t)}\delta_{ij} \geq 0\,. \tag{2.2}$$

Finally, the last result we need is the fact that formula (1.2) holds for the positive functions v, even if $\mathbb{R}^{n+1}$ is not compact. This is another consequence of Li–Yau Harnack inequality, in particular of the exponential estimate (2.1), see the paper [8] and references therein for details ([2] in particular).

By formulas (1.1) and (1.2), it follows then if $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ is the MCF of a compact hypersurface M and $u \in \mathcal{F}_\tau$ with associate positive solution v of the backward heat equation, we have

(2.3)

$$\begin{aligned} &\frac{d}{dt}\mathcal{HW}\left(\varphi(\cdot,t), \mathbb{R}^{n+1}, \tau - t, v(\cdot,t)\right) \\ &\qquad \leq -\sqrt{4\pi(\tau - t)}\int_M v\,|\mathrm{H} + \nabla^\perp \log v|^2\,d\mu_t - 4(\tau - t)\int_{\mathbb{R}^{n+1}} v\left|\mathrm{Hess}_{ij}\log v + \frac{\delta_{ij}}{2(\tau - t)}\right|^2 dx \end{aligned}$$

which is clearly negative in the time interval $[0, \min\{\tau, T\})$.

Proposition 2.9. *For every compact M, immersion $\varphi : M \to \mathbb{R}^{n+1}$ and $\tau > 0$, the quantity $\sigma(\varphi,\tau)$ is finite.*
Moreover, the $\sup$ *in Definition 2.3 is actually a maximum which is obtained by some positive function $u_\tau \in \mathcal{F}_\tau$.*

Proof. Let $u_i : \mathbb{R}^{n+1} \to \mathbb{R}$ be a sequence of smooth positive functions in $\mathcal{F}_\tau$ such that $\mathcal{HW}(\varphi, \mathbb{R}^{n+1}, \tau, u_i) \nearrow \sigma(\varphi, \tau)$.
We know that $u_i = v_i(\cdot, t) * K_{\mathbb{R}^{n+1}}(\cdot, 0, t)$ where $K_{\mathbb{R}^{n+1}}$ is the heat kernel at time $t < \tau$. By the uniform integrability, one gets compactness of $v_i(\cdot, t)\, dx$ in the weak*–topology of measures and using a diagonal procedure, we can assume this holds for a sequence of times $t_j \to 0$. Now, by Harnack estimates and the uniqueness theorem for positive solutions, we have a smooth non negative limit function $v_\tau : \mathbb{R}^{n+1} \times [0, \tau)$ satisfying the backward heat equation, which, by maximum principle, it has to be positive or identically zero, hence, the sequence u_i converges to $u_\tau = v_\tau(\cdot, 0)$ in $L^1_{loc}(\mathbb{R}^{n+1})$ and $C^\infty_{loc}(\mathbb{R}^{n+1})$.
It is now standard to see that the $\mathcal{W}$ functional is lower semicontinuous with respect to the L^1_{loc}–convergence and, since the functions u_i converge uniformly on compact subset of $\mathbb{R}^{n+1}$, also the $\mathcal{H}$ term is continuous, thus $\mathcal{HW}(\varphi, \mathbb{R}^{n+1}, \tau, u_\tau) = \sigma(\varphi, \tau)$.
Clearly, the function u_τ cannot be zero, otherwise this quantity would be zero also, hence, the function v_τ is positive everywhere, moreover it satisfies $0 \leq \int_{\mathbb{R}^{n+1}} u_\tau\, dx \leq 1$.
The last point to check is that $\int_{\mathbb{R}^{n+1}} u_\tau\, dx = 1$, hence $u_\tau \in \mathcal{F}_\tau$ and we are done. This can be seen noticing that for any $\lambda > 1$ we have

$$\mathcal{HW}(\varphi, \mathbb{R}^{n+1}, \tau, \lambda u_\tau) = \lambda \mathcal{HW}(\varphi, \mathbb{R}^{n+1}, \tau, u_\tau) + 2\lambda \log \lambda \int_{\mathbb{R}^{n+1}} u_\tau\, dx \geq \lambda \sigma(\varphi, \tau) > \sigma(\varphi, \tau)\,.$$

Hence, if $\int_{\mathbb{R}^{n+1}} u_\tau\, dx < 1$ choosing $\widetilde{u} = u(\int_{\mathbb{R}^{n+1}} u_\tau\, dx)^{-1}$, which belongs to the family $\mathcal{F}_\tau$, we would get a contradiction. ☐

Proposition 2.10 (Rescaling Invariance)**.** *For every $\lambda > 0$ we have*

$$\sigma(\lambda\varphi, \lambda^2\tau) = \sigma(\varphi, \tau)\,.$$

Proof. Let $u \in \mathcal{F}_\tau$ with associate solution of backward heat equation $v : \mathbb{R}^{n+1} \times [0, \tau) \to \mathbb{R}$ and consider the rescaled function $\widetilde{u}(y) = u(y/\lambda)\lambda^{-(n+1)}$.
It is easy to see that

$$\int_{\mathbb{R}^{n+1}} \widetilde{u}(y)\, dy = \lambda^{-(n+1)} \int_{\mathbb{R}^{n+1}} u(y/\lambda)\, dy = \int_{\mathbb{R}^{n+1}} u(x)\, dx = 1$$

with the change of variable $x = \lambda^{-(n+1)}y$, moreover the function $\widetilde{v}(y, s) = v(y/\lambda, s/\lambda^2)\lambda^{-(n+1)}$ is a positive solution of the backward heat equation on the time interval $\lambda^2\tau$, hence $\widetilde{u} \in \mathcal{F}_{\lambda^2\tau}$.
It is now a straightforward computation to see that $\mathcal{HW}(\lambda\varphi, \mathbb{R}^{n+1}, \lambda^2\tau, \widetilde{u}) = \mathcal{HW}(\varphi, \mathbb{R}^{n+1}, \tau, u)$ for every smooth immersion of a compact hypersurface $\varphi : M \to \mathbb{R}^{n+1}$, the statement clearly follows. ☐

Proposition 2.11 (Monotonicity and Differentiability)**.** *Along a MCF, $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$, if $\tau(t) = C - t$ for some constant $C > 0$, the quantity $\sigma(\varphi_t, \tau)$ is monotone non increasing in the time interval $[0, \min\{C, T\})$, hence it is differentiable almost everywhere.*

Moreover, letting $f_\tau = -\log u_\tau$, where $\tau = C - t$ and u_τ is one of maximizer of Proposition 2.9, we have for almost every $t \in [0, \min\{C, T\})$,

$$\frac{d}{dt}\sigma(\varphi, \tau) \leq -\sqrt{4\pi\tau}\int_M e^{-f_\tau}\left|\mathrm{H} - \nabla^\perp f_\tau\right|^2 d\mu_t - 4\tau\int_{\mathbb{R}^{n+1}} e^{-f_\tau}\left|\mathrm{Hess}_{ij} f_\tau - \frac{\delta_{ij}}{2\tau}\right|^2 dx \tag{2.4}$$

or, since this inequality has to be intended in distributional sense, for every $0 \leq r < t < T$,

$$\begin{aligned}\sigma(\varphi_r, \tau(r)) - \sigma(\varphi_t, \tau(t)) \geq & \int_r^t \sqrt{4\pi\tau}\int_M e^{-f_\tau}\left|\mathrm{H} - \nabla^\perp f_\tau\right|^2 d\mu_s\, ds \\ & + 4\int_r^t \tau\int_{\mathbb{R}^{n+1}} e^{-f_\tau}\left|\mathrm{Hess}_{ij} f_\tau - \frac{\delta_{ij}}{2\tau}\right|^2 dx\, ds\,.\end{aligned} \tag{2.5}$$

Proof. Fix some time $t \in [0, \min\{C, T\})$ and let u_i be a sequence of functions in $\mathcal{F}_\tau$ such that $\mathcal{HW}(\varphi(\cdot, t), \mathbb{R}^{n+1}, \tau, u_i) \nearrow \sigma(\varphi(\cdot, t), \tau)$. By definition, there exist a family of positive solutions $v_i : \mathbb{R}^{n+1} \times [0, C - t)$ of the backward heat equation such that $v_i(x, 0) = u_i(x)$. Being the functions u_i smooth, we can extend such solutions of the backward heat equations on $\mathbb{R}^{n+1}$ to the time interval $[-t, C - t]$, simply solving backward the *heat equation* with initial data u_i, notice that they remain positive by the strong maximum principle. It is easy to see then that defining $u_i^\varepsilon(x) = v_i(x, -\varepsilon)$ for $\varepsilon \in (0, t]$, every function u_i^ε belongs to the family $\mathcal{F}_{\tau+\varepsilon}$. Hence, by inequality (2.3)

$$\begin{aligned}\sigma(\varphi_{t-\varepsilon}, \tau + \varepsilon) - \mathcal{HW}\,(\varphi_t, \mathbb{R}^{n+1}, \tau, u_i) \geq\ & \mathcal{HW}(\varphi_{t-\varepsilon}, \mathbb{R}^{n+1}, \tau + \varepsilon, u_i^\varepsilon) - \mathcal{HW}(\varphi_t, \mathbb{R}^{n+1}, \tau, u_i) \\ = & \int_{-\varepsilon}^0 \sqrt{4\pi\tau}\int_M v_i\,|\mathrm{H} + \nabla^\perp \log v_i|^2\, d\mu_s\, ds \\ & + 4\int_{-\varepsilon}^0 \tau\int_{\mathbb{R}^{n+1}} v_i\left|\mathrm{Hess}_{ij}\log v_i + \frac{\delta_{ij}}{2\tau}\right|^2 dx\, ds\end{aligned}$$

and passing to the limit

$$\begin{aligned}\sigma(\varphi_{t-\varepsilon}, \tau + \varepsilon) - \sigma(\varphi_t, \tau) \geq & \int_{-\varepsilon}^0 \sqrt{4\pi\tau}\int_M v_i\,|\mathrm{H} + \nabla^\perp \log v_i|^2\, d\mu_s\, ds \\ & + 4\int_{-\varepsilon}^0 \tau\int_{\mathbb{R}^{n+1}} v_i\left|\mathrm{Hess}_{ij}\log v_i + \frac{\delta_{ij}}{2\tau}\right|^2 dx\, ds \geq 0\,,\end{aligned}$$

which gives the monotonicity of $\sigma(\varphi, \tau)$.

The last assertion is standard, using *Hamilton's trick* (see [3]) to exchange the $\sup$ and derivative operations. □

Remark 2.12. Notice that the quantity σ can be defined also for any n–dimensional countably rectifiable subset S of $\mathbb{R}^{n+1}$, by substituting in the definition of $\mathcal{HW}$ the term $\int_M u\, d\mu$ with $\int_S u\, d\mathcal{H}^n$, where $\mathcal{H}^n$ is the n–dimensional Hausdorff measure (possibly *counting multiplicities*). If then S is the support of a rectifiable *varifold*, with finite *Area*, moving by mean curvature according to Brakke's definition (see [1]), Huisken–Hamilton's monotonicity formula (1.1) holds, hence, also this proposition.

Definition 2.13. We define, in the same hypothesis, for $\tau = C - t$ with $C \leq T$,

$$\Sigma(C) = \lim_{t \to C^-} \sigma(\varphi_t, \tau)\,,$$

and $\Sigma = \Sigma(T)$.

By Proposition 2.5, $\Sigma \geq \sup_{p \in \mathbb{R}^{n+1}} \Theta(p)$, where this latter quantity is defined as

$$\Theta(p) = \lim_{t \to T^-} \frac{1}{[4\pi(T-t)]^{\frac{n}{2}}} \int_M e^{-\frac{|x-p|^2}{4(T-t)}}\, d\mu(x)\,,$$

the existence of this limit for every $p \in \mathbb{R}^{n+1}$ is a consequence of Huisken's monotonicity formula.

3. Other Entropies

Definition 3.1. Let $\varphi : M \to \mathbb{R}^{n+1}$ be a smooth, compact, immersed hypersurface.

- Given $\tau > 0$, we consider the family $\mathcal{F}_\tau$ of smooth positive functions $u : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\int_{\mathbb{R}^{n+1}} u\, dx = 1$ and there exists a smooth positive solution of the problem

$$\begin{cases} v_t = -\Delta v \ \text{ in } \mathbb{R}^{n+1} \times [0, \tau)\,, \\ v(x, 0) = u(x) \ \text{ for every } x \in \mathbb{R}^{n+1}\,. \end{cases}$$

  Then, we define the following quantity

$$\sigma_{\mathcal{H}}(\varphi, \tau) = \sup_{u \in \mathcal{F}_\tau} \mathcal{H}(\varphi, \mathbb{R}^{n+1}, \tau, u)\,.$$

- Given $\tau > 0$, we consider the subfamily $\mathcal{K}_\tau$ of $\mathcal{F}_\tau$ of the heat kernels of $\mathbb{R}^{n+1}$ at time $\tau > 0$, that is, $\mathcal{K}_\tau = \{K_{\mathbb{R}^{n+1}}(\cdot, p, \tau) \,|\, p \in \mathbb{R}^{n+1}\}$.
  Then, we define the following quantity

$$\begin{aligned} \sigma_K(\varphi, \tau) &= \sup_{p \in \mathbb{R}^{n+1}} \mathcal{HW}(\varphi, \mathbb{R}^{n+1}, \tau, K_{\mathbb{R}^{n+1}}(\cdot, p, \tau)) \\ &= \sup_{p \in \mathbb{R}^{n+1}} \mathcal{H}(\varphi, \mathbb{R}^{n+1}, \tau, K_{\mathbb{R}^{n+1}}(\cdot, p, \tau))\,. \end{aligned}$$

- Given a flat Riemannian manifold T, a Riemannian covering map $I : \mathbb{R}^{n+1} \to \mathrm{T}$ and $\tau > 0$, we consider the immersion $\widetilde{\varphi} = I \circ \varphi : M \to \mathrm{T}$ and we define the family $\mathcal{F}_{\mathrm{T},\tau}$ of smooth positive functions $u : \mathrm{T} \to \mathbb{R}$ such that $\int_{\mathrm{T}} u\, dx = 1$ and there exists a smooth positive solution of the problem

$$\begin{cases} v_t = -\Delta v \ \text{ in } \mathrm{T} \times [0, \tau) \\ v(x, 0) = u(x) \ \text{ for every } x \in \mathrm{T}\,. \end{cases}$$

  Then, we define the following quantity

$$\sigma_{\mathrm{T}}(\varphi, \tau) = \sup_{u \in \mathcal{F}_{\mathrm{T},\tau}} \mathcal{HW}(\widetilde{\varphi}, \mathrm{T}, \tau, u)\,.$$

- Given an open bounded convex set with smooth boundary $\Omega \subset \mathbb{R}^{n+1}$ such that $\varphi(M) \subset \Omega$, we define the family $\mathcal{F}_{\Omega,\tau}$ of smooth positive functions $u : \overline{\Omega} \to \mathbb{R}$ such that $\int_\Omega u\, dx = 1$, $\partial u/\partial\nu = 0$ on $\partial\Omega$ and there exists a smooth positive solution of the problem
$$\begin{cases} v_t = -\Delta v \text{ in } \Omega \times [0,\tau) \\ v(x,0) = u(x) \text{ for every } x \in \Omega \\ \frac{\partial v}{\partial \nu} = 0 \text{ in } \partial\Omega \times [0,\tau)\,. \end{cases}$$
Then, we define the following quantity
$$\sigma_\Omega(\varphi,\tau) = \sup_{u \in \mathcal{F}_{\Omega,\tau}} \mathcal{HW}(\varphi, \mathbb{R}^{n+1}, \tau, u)\,.$$

Remark 3.2. All these quantities are well defined, finite, positive and monotonically decreasing if φ_t moves by mean curvature.
For σ_{T} and σ_Ω, we can use again as a test function in order to show their positivity, the relative heat kernel at time $\tau > 0$ but the analysis is more delicate, see the paper by Lei Ni [8]. It can also be shown that

$$\begin{aligned} \liminf_{\tau\to 0} \sup_{\mathcal{F}_{\mathrm{T},\tau}} (-2\mathcal{W}) &= -2 \limsup_{\tau\to 0} \inf_{\mathcal{F}_{\mathrm{T},\tau}} \mathcal{W} \geq 0\,, \\ \liminf_{\tau\to 0} \sup_{\mathcal{F}_{\Omega,\tau}} (-2\mathcal{W}) &= -2 \limsup_{\tau\to 0} \inf_{\mathcal{F}_{\Omega,\tau}} \mathcal{W} \geq 0\,. \end{aligned} \tag{3.1}$$

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