ON THE MINIMALITY OF THE POTENTIAL FUNCTION OF A GRADIENT SHRINKING RICCI SOLITON

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1. Gradient Shrinking Ricci Solitons and their $W$–Entropy

A gradient shrinking Ricci soliton is a complete, connected Riemannian manifold $(M, g)$ satisfying the relation

$$\text{Ric} + \nabla^2 f = \frac{g}{2},$$

for some smooth function $f : M \to \mathbb{R}$ which is called potential function of the soliton $(M, g, f)$.

It is well known that the quantity $a(g, f) := R + |\nabla f|^2 - f$ must be constant on $M$ and it is often called the auxiliary constant.

We recall the following growth estimates, originally proved by Cao–Zhou and Munteanu [1, 7] and improved by Haslhofer–Müller [5] to the present form.

**Proposition 1.1** (Potential and volume growth, Lemma 2.1 and 2.2 in [5]). Let $(M, g, f)$ be an $n$–dimensional gradient shrinking Ricci soliton with auxiliary constant $a(g, f)$. Then there exists a point $p \in M$ where $f$ attains its infimum and we have the following estimates for the growth of the potential

$$\frac{1}{4}(d_g(x, p) - 5n)^2 \leq f(x) - a(g, f) \leq \frac{1}{4}(d_g(x, p) + \sqrt{2n})^2.$$

Moreover, we have the volume growth estimate $\text{Vol}(B^\infty(p)) \leq V(n)r^n$ for geodesic balls in $(M, g)$ around $p \in M$, where $V(n)$ is a constant depending only on the dimension $n$ of the soliton.

As a consequence of these estimates, $\int_M e^{-f}d\text{Vol}$ is well–defined and the potential function $f$ can always be “normalized” by adding a constant in order that

$$\int_M \frac{e^{-f}}{(4\pi)^{n/2}} d\text{Vol} = 1.$$  \hfill (1.1)

We then call such a potential function $f$ and the resulting soliton $(M, g, f)$ normalized.

In all the paper we will always consider complete, connected, normalized, gradient, shrinking Ricci solitons, unless explicitly stated.

Proposition 1.1 implies that every function $\phi$ satisfying $|\phi(x)| \leq Ce^{\alpha d_g^2(x, p)}$ for some $\alpha < \frac{1}{4}$ and constant $C$, is integrable with respect to $e^{-f}d\text{Vol}$. In particular, since $0 \leq
\[ R + |\nabla f|^2 \leq f + a(g, f) \] and \( \Delta f = \frac{n}{2} - R \), this holds for every polynomial in \( f, |\nabla f|^2, R \) and \( \Delta f \). Hence, every gradient shrinking Ricci soliton has a well-defined \( W \)-entropy

\[ W(g, f) := \int_M (R + |\nabla f|^2 + f - n) \frac{e^{-f}}{(4\pi)^{n/2}} dVol. \]

Let us now collect some properties of shrinking solitons and their \( W \)-entropy that we will use in the next sections.

**Lemma 1.2.** For every normalized, gradient, shrinking Ricci soliton \((M, g)\) with potential function \( f : M \to \mathbb{R} \), the following properties holds:

1. Either the scalar curvature \( R \) is positive everywhere or \((M, g)\) is the standard flat \( \mathbb{R}^n \) and \( f(x) = |x - x_0|^2/4 \) for some \( x_0 \in \mathbb{R}^n \), called “Gaussian soliton”.
2. There holds
   \[ W(g, f) = \int_M (R + 2\Delta f - |\nabla f|^2 + f - n) \frac{e^{-f}}{(4\pi)^{n/2}} dVol. \]
3. The \( W \)-entropy \( W(g, f) \) is equal to \(-a(g, f)\).
4. Two potential functions \( f^1 \) and \( f^2 \) of the same soliton \((M, g)\) either coincide, or \((M, g)\) is the Riemannian product of the flat \( \mathbb{R}^k \), for some \( k > 1 \) with a Riemannian manifold \((\tilde{M}, \tilde{g})\) which is still a gradient shrinking Ricci soliton with a potential function \( \tilde{f} : M \to \mathbb{R} \) and
   \[ f^\ell(x, y) = \tilde{f}(x) + \frac{1}{4} |y - y_\ell|_{\mathbb{R}^k}^2, \]
   for some points \( y_1 \) and \( y_2 \in \mathbb{R}^k \).
   In particular, if \( M \) is compact, there can be only one potential function for the soliton \((M, g)\).
5. Any two potential functions \( f^1 \) and \( f^2 \) of the same soliton \((M, g)\) share the same auxiliary constant, that is \( a(g, f^1) = a(g, f^2) \), which implies \( W(g, f^1) = W(g, f^2) \). Hence, we can speak respectively of the auxiliary constant \( a(g) \) and the \( W \)-entropy \( W(g) \) of the soliton \((M, g)\).
6. We have \( W(g) \leq 0 \) and \( W(g) = 0 \) if and only if the manifold \((M, g)\) is the flat \( \mathbb{R}^n \) (Gaussian soliton).
7. If a soliton \((M, g, f)\) is also an Einstein manifold, either it is compact and \( f \) is constant, or \((M, g, f)\) is the Gaussian soliton.

**Proof.**

1. This is a result of Zhang [16, Theorem 1.3] and Yokota [13, Appendix A.2] (see also Pigola, Rimoldi and Setti [9]).
2. The necessary partial integration formula
   \[ \int_M \Delta f e^{-f} dVol = \int_M |\nabla f|^2 e^{-f} dVol \]
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(3) By the auxiliary equation \( a(g,f) = R + |\nabla f|^2 - f \) and the traced soliton equation \( R + \Delta f = \frac{n}{2} \), we have
\[
R + 2\Delta f - |\nabla f|^2 + f - n = -a(g,f),
\]
hence, the equality \( \mathcal{W}(g,f) = -a(g,f) \) follows from Point 2, by integration.

(4) Since the Hessian of any potential of the soliton is uniquely determined by the soliton equation, the difference function \( h := f_1 - f_2 \) is either a constant or the vector field \( \nabla h \) is parallel. In the first case, the constant has to be zero by the normalization condition (1.1). In the second case, by De Rham’s splitting theorem, \((M,g)\) isometrically splits off a line (see for instance [3, Theorem 1.16]). Hence, we let \((M,g) = (\tilde{M},\tilde{g}) \times (\mathbb{R}^k,\text{can})\), with \( 1 \leq k \leq n \), such that \( \tilde{M} \) cannot split off a line. Denoting by \( x \) the coordinates on \( \tilde{M} \) and by \( y \) the coordinates on \( \mathbb{R}^k \), the soliton equation implies that both potentials also split as \( f_\ell(x,y) = \tilde{f}_\ell(x) + \frac{1}{4}|y - y_\ell|_{\mathbb{R}^k}^2 \) for \( \ell = 1, 2 \), where \( y_\ell \in \mathbb{R}^k \). Moreover, \((\tilde{M},\tilde{g})\) is still a gradient shrinking Ricci soliton with both functions \( \tilde{f} : \tilde{M} \to \mathbb{R} \) as possible potentials, and since \( \tilde{M} \) cannot split off a line, they must coincide. Thus, we have
\[
f_\ell(x,y) = \tilde{f}(x) + \frac{1}{4}|y - y_\ell|_{\mathbb{R}^k}^2,
\]
for some function \( \tilde{f} : \tilde{M} \to \mathbb{R} \).

(5) Integrating the two functions \( e^{-f_\ell} \) of the previous point, by means of Fubini’s theorem and the normalization condition (1.1), we get that
\[
a(g, f_\ell) = R + |\nabla f_\ell|^2 - f_\ell = R + |\nabla \tilde{f}|^2 - \tilde{f}
\]
which is independent of \( \ell = 1, 2 \).

(6) This point is a result of Yokota (Carrillo–Ni [2] got similar results under more restrictive curvature hypotheses). Our version is equivalent to his statement [13, Corollary 1.1] and [14, Theorem 2].

(7) By point (1) either the scalar curvature is positive, or \((M,g,f)\) is the Gaussian soliton. In the first case \((M,g)\) must be Einstein with a positive constant, hence, it is compact (by Myers’s diameter estimate) with constant scalar curvature if \( n \geq 3 \). If \( n = 2 \) it is known that the only compact solitons are \( S^2 \) and its quotient \( \mathbb{R}P^2 \) with a constant potential function. If \( n \geq 3 \) it follows that \( \Delta f \) is constant on \( M \), which is compact, hence, the potential function \( f \) (unique by compactness, see point (4)) is constant.

\[\square\]

2. THE CRITICAL POINTS OF THE FUNCTIONAL \( \mathcal{W} \)

We consider the Perelman’s functional \( \mathcal{W} \) on a normalized, gradient, shrinking Ricci solitons \((M,g,\tilde{f})\), freezing the metric \( g \) and varying only the function \( f \). We want to discuss the existence of a minimizer or, more generally, of a critical point \( f \in C^\infty(M) \), under the constraint \( \int_M \frac{e^{-f}}{(4\pi)^{n/2}} \ dVol = 1 \).
Substituting $u = \frac{e^{-f/2}}{(4\pi)^{n/4}}$ the functional becomes

$$\widetilde{W}(u) = \int_M (Ru^2 + 4|\nabla u|^2 - u^2 \log u^2 - u^2 \frac{n}{2} \log 4\pi - nu^2) \, d\text{Vol},$$

under the constraint $\int_M u^2 \, d\text{Vol} = 1$.

Moreover, in studying its properties, we can actually consider the functional $\widetilde{F}$ defined as

$$\widetilde{F}(u) = \int_M (Ru^2 + 4|\nabla u|^2 - u^2 \log u^2) \, d\text{Vol},$$

which differs by $\widetilde{W}(u)$ only for a constant term, by the constraint $\int_M \frac{e^{-f}}{(4\pi)^{n/2}} \, d\text{Vol} = 1$ and we define the infimum

$$\sigma = \inf_{u \in C^\infty(M), \int_M u^2 \, d\text{Vol} = 1} \widetilde{F}(u).$$

Notice that, even in the flat $\mathbb{R}^n$, the functional $\widetilde{F}(u)$ could be unbounded above on $H^1(M, g)$, indeed, consider the functions, for $t > 1$, which all satisfy $\int_M u_t^2 \, d\text{Vol} = 1$,

$$u_t = \frac{e^{-|x|^2/8t}}{(4\pi t)^{n/4}}.$$

We see that

$$|\nabla u_t|^2 = \frac{|x|^2}{16t^2} u_t^2\quad \text{and}\quad -u_t^2 \log u_t^2 = \left(\frac{|x|^2}{4t} + \frac{n}{2} \log 4\pi t\right) u_t^2,$$

hence,

$$\widetilde{F}(u_t) = \int_M \left(\frac{|x|^2}{2t^2} + \frac{n}{2} \log 4\pi t\right) u_t^2 \, d\text{Vol} = \int_M \frac{|x|^2}{2t^2} u_t^2 \, d\text{Vol} + \frac{n}{2} \log 4\pi t$$

By direct computation, we can see that

$$\int_M |\nabla u_t|^2 \, d\text{Vol} = \int_M \frac{|x|^2}{16t^2} u_t^2 \, d\text{Vol} = C/t,$$

hence, the family of functions $u_t$, for $t \geq 1$, is uniformly bounded in $H^1(M, g)$ but $\lim_{t \to +\infty} \widetilde{F}(u_t) \to +\infty$.

We first discuss when the functional $\widetilde{F}$ is bounded below on $H^1(M, g)$. Notice that if $(M, g)$ is different by the flat $\mathbb{R}^n$, the function $R$ is everywhere positive and actually bounded above by

$$R \leq \overline{F} + a(g) \leq 2a(g) + \frac{1}{4} \left(d_g(x, p) + \sqrt{2n}\right)^2$$

for some point $p \in M$, by the results of the previous section. Hence, we only need to uniformly bound the integrand $u^2 \log u^2$ (notice that the function $h(t) = t^2 \log t^2$ is $C^1$, defining $h(0) = 0$) and bounded below by $1/e$, hence, we will need that Sobolev inequalities hold. This is assured by the following result of Varopoulos in [12] (see
Also [4], assuming that the Ricci tensor is uniformly bounded below and the soliton is non-collapsed.

**Proposition 2.1** (Theorem 3.2 in [6]). Let \((M, g)\) be a smooth, complete, \(n\)-dimensional Riemannian manifold with Ricci tensor bounded below and

\[
\inf_{x \in M} \text{Vol}(B_1(x)) > 0,
\]

where \(B_1(p)\) is the unit geodesic ball in \((M, g)\) of center \(x \in M\).

Then, the Sobolev embeddings \(W^{1,q}(M, g) \hookrightarrow L^p(M, g)\) holds for every \(q \in [1, n)\), where \(1/p = 1/q - 1/n\).

As we do not know whether every normalized, gradient, shrinking Ricci soliton has a bound from below on the Ricci tensor and/or it must be non-collapsed. What we know is that, by Perelman’s work [8], all the gradient, shrinking Ricci solitons coming from a blow-up of a compact Ricci flow satisfy these conditions.

From now on we will assume, unless differently specified, that all the solitons we are going to consider are non-collapsed and with Ricci tensor bounded below. As a consequence, Sobolev embeddings hold, in particular, there exists a constant \(C_M\) such that

\[
\int_M u^{2^*} \, d\text{Vol} \leq \left( C_M \int_M (|\nabla u|^2 + u^2) \, d\text{Vol} \right)^{\frac{n}{n - 2}},
\]

where \(2^* = \frac{2n}{n - 2}\), for every \(u \in H^1(M, g)\) (when \(n = 2\) we can take \(2^*\) to be whatever value in \((2, +\infty)\), by Theorem 3.7 in [6]).

**Proposition 2.2.** On \(H^1(M, g)\) the functional \(\tilde{F}\) (hence also \(W\) and \(\tilde{W}\)) is uniformly bounded below, that is, \(\sigma > -\infty\).

**Proof.** Clearly, since we know that \(R \geq 0\) it is sufficient to show that the integral \(\int_M u^2 \log u^2 \, d\text{Vol}\) is uniformly bounded above.

For any \(u \in H^1(M, g)\), by applying Jensen inequality with respect to the probability measure \(u^2 \, d\text{Vol}\), one has

\[
\int_M u^2 \log u^2 \, d\text{Vol} = \frac{n - 2}{2} \int_M u^2 \log u^{\frac{4}{n - 2}} \, d\text{Vol}
\]

\[
\leq \frac{n - 2}{2} \log \left( \int_M u^2 u^{\frac{4}{n - 2}} \, d\text{Vol} \right)
\]

\[
= \frac{n - 2}{2} \log \left( \int_M u^{\frac{2n}{n - 2}} \, d\text{Vol} \right)
\]

\[
= \frac{n - 2}{2} \log \left( \int_M u^{2^*} \, d\text{Vol} \right).
\]
On the other hand,
\[
\log\left(\int_M u^2 \, d\text{Vol}\right) \leq \log \left[ \left( C_M \int_M (|\nabla u|^2 + u^2) \, d\text{Vol} \right)^{\frac{n}{2}} \right] = \frac{n}{n-2} \log \left( C_M \int_M (|\nabla u|^2 + u^2) \, d\text{Vol} \right),
\]
where $C_M$ is the Sobolev constant of $(M, g)$.
Putting together these two inequalities we get
\[
-\int_M u^2 \log u^2 \, d\text{Vol} \geq -\log \left( C_M \int_M (|\nabla u|^2 + u^2) \, d\text{Vol} \right) \geq -4 \int_M (|\nabla u|^2 + u^2) \, d\text{Vol} - C'_M,
\]
for some positive constant $C'_M$ depending only on $(M, g)$. Hence,
\[
\tilde{F}(u) = \int_M \left( Ru^2 + 4 |\nabla u|^2 - u^2 \log u^2 \right) \, d\text{Vol} \geq 4 \int_M |\nabla u|^2 \, d\text{Vol} - 4 \int_M (|\nabla u|^2 + u^2) \, d\text{Vol} - C'_M = -4 \int_M u^2 \, d\text{Vol} - C'_M,
\]
where in the last passage we used that $\int_M u^2 \, d\text{Vol} = 1$. This shows that $\sigma > -\infty$. \hfill \Box

2.1. Critical Points and Minima. By means of compact perturbations, every critical point of the functionals $\tilde{F}$ or $\tilde{W}$ satisfies the Euler equation
\[
-4\Delta u + Ru - 2u \log u = \lambda u
\]
for some constant $\lambda$ coming from the constraint. Indeed, as usual, multiplying by $u$ and integrating, we get
\[
\tilde{W}(u) = \int_M \left( Ru^2 + 4 |\nabla u|^2 - u^2 \log u^2 - u^2 \frac{n}{2} \log 4\pi - nu^2 \right) \, d\text{Vol} = \lambda - \frac{n}{2} \log 4\pi - n,
\]
that is,
\[
\lambda = \tilde{W}(u) + \frac{n}{2} \log 4\pi + n.
\]
Rereading all this in terms of the function $f = -\log [(4\pi)^{n/2} u^2]$ we get, equivalently,
\[
2\Delta f - |\nabla f|^2 + R + f = \lambda - \frac{n}{2} \log 4\pi
\]
for every constrained critical point $f : M \to \mathbb{R}$ of the functional $\mathcal{W}$ such that $u = \frac{e^{-f/2}}{(4\pi)^{n/2}} \in H^1(M, g)$ and $\int_M \frac{e^{-f}}{(4\pi)^{n/2}} \, d\text{Vol} = 1$. Moreover,
\[
\lambda = \mathcal{W}(f) + \frac{n}{2} \log 4\pi + n,
\]
hence, setting $\mu = \lambda - \frac{n}{2} \log 4\pi$, we conclude as follows.
Proposition 2.3. A constrained critical point of the functional $W$ on the set of functions $f : M \to \mathbb{R}$ such that $u = \frac{e^{-f}}{(4\pi)^{n/2}} \in H^1(M, g)$ and $\int_M e^{-\frac{f}{2}}(4\pi)^{n/2} d\text{Vol} = 1$ satisfies

$$2\Delta f - |\nabla f|^2 + R + f = \mu$$

where the constant $\mu$ is given by

$$\mu = W(f) + n.$$

The potential function $\bar{f}$, satisfies $\Delta \bar{f} + R = n/2$ and $\int_M e^{-\bar{f}} d\text{Vol} = 1$, moreover, $R + |\nabla \bar{f}|^2 - \bar{f} = a(g, \bar{f}) = a(g)$, hence,

$$2\Delta \bar{f} - |\nabla \bar{f}|^2 + R + \bar{f} = n - a(g) = n + W(g) = n + W(\bar{f})$$

which implies that $\bar{f}$ is a critical point of the functional $W$.

In the following, we want to discuss whether other critical point or minimizers of $W$ actually exist and their relation with the potential function of the soliton $(M, g)$.

2.2. The Compact Case.

Proposition 2.4. If the soliton $(M, g)$ is compact, the infimum $\sigma$ is achieved by a minimizer $u \in C^\infty(M)$, moreover, $u > 0$ everywhere on $M$.

Proof. The same argument showing that $\sigma > -\infty$, gives that any minimizing sequence $u_i \in C^\infty(M)$, with $\|u_i\|_{L^2} = 1$, is uniformly bounded in the space $H^1(M, g)$. Hence, we can extract a subsequence (not relabeled) weakly converging in $H^1(M, g)$ strongly converging in $L^{2+\varepsilon}(M)$, for some $\varepsilon > 0$, to some function $u$ (by compactness of $(M, g)$) (the Sobolev compact embeddings hold on a compact Riemannian manifold, see [6]). Clearly, by the the $L^{2+\varepsilon}$–convergence and the compactness of $(M, g)$, we have $\int_M u^2 d\text{Vol} = 1$ and we can also assume $u \geq 0$, by the definition of $\bar{F}$.

It is easy to see that the functional $\bar{F}$ is lower semicontinuous with respect to the weak convergence in $H^1(M, g)$, as the term $u^2 \log u^2$ is subcritical (and the function $h(t) = t^2 \log t^2$ is continuous) hence its integral is continuous.

Then, a limit function $u : M \to \mathbb{R}$ is a nonnegative, constrained minimizer of $\bar{F}$ in $H^1(M, g)$.

The Euler–Lagrange equation for $u$ read

$$-4\Delta u + Ru - (u \log u^2 + u) = Cu,$$

for some constant $C$. It can be rewritten as

$$\Delta u = Ru/4 + Cu - u \log u,$$  \(2.1\)

to be intended in $H^1(M, g)$.

As $u$ is in $H^1(M, g)$ and the term $u^2 \log u$ is subcritical, a bootstrap argument together with standard elliptic estimates gives that $u \in C^{1, \alpha}$.

Rothaus proved in [11] that a solution of equation (2.1) is positive or identically zero. this second possibility is obviously excluded by the constraint $\int_M u^2 d\text{Vol} = 1$.

Finally, as the function $h(t) = t^2 \log t^2$ is smooth in $\mathbb{R} \setminus \{0\}$, again by a bootstrap argument, we can conclude that the function $u$ is actually smooth. □
Assume that \( f : M \to \mathbb{R} \) is a critical point, that is, \( f \) satisfies
\[
2\Delta f - |\nabla f|^2 + R + f = \text{constant},
\]
then we have
\[
g^{kj} \nabla_k \left[ 2(R_{ij} + \nabla^2_{ij} f - g_{ij}/2)e^{-f} \right] = \nabla_i (R + 2\Delta f - |\nabla f|^2 + f)e^{-f} = 0.
\]

Hence,
\[
\text{div}[(\text{Ric} + \nabla^2 f - g/2)e^{-f}] = 0
\]
and
\[
\text{div}[(\nabla_k f - \nabla_k \bar{f})g^{kj}(R_{ij} + \nabla^2_{ij} f - g_{ij}/2)e^{-f}]
\]
\[
= (\nabla^2_{ik} f - \nabla^2_{ik} \bar{f})g^{kj}g^{li}(R_{ij} + \nabla^2_{ij} f - g_{ij}/2)e^{-f}
\]
\[
= |R_{ij} + \nabla^2_{ij} f - g_{ij}/2|^2 e^{-f},
\]
where, passing from the second to the third line, we substituted \( \nabla^2_{ik} \bar{f} \) with \( g_{ik}/2 - R_{ik} \), by the soliton relation.

Hence, we conclude that, setting \( T = (\nabla_k f - \nabla_k \bar{f})g^{kj}(R_{ij} + \nabla^2_{ij} f - g_{ij}/2)e^{-f} \), we have
\[
0 \leq Q = |\text{Ric} + \nabla^2 f - g/2|^2 e^{-f} = \text{div} T.
\]

Being \( M \) compact, integrating \( Q \) on \( M \), we immediately get that \( Q = 0 \), since it is nonnegative.

This says that the function \( f \) is a potential for the gradient Ricci soliton \((M, g)\), then, by point (4) of Lemma 1.2 it must coincide with \( \bar{f} \).

**Proposition 2.5.** If the soliton \((M, g)\) is compact its potential function is the unique constrained critical point of the functional \( W \), it is smooth and minimizes \( W \).

### 2.3. The Noncompact Case.

The noncompact case is more delicate, in particular it is quite more difficult to obtain the existence of a minimizer of the functional \( \bar{F} \). In [15] Zhang showed that there exists complete, noncollapsed manifolds with bounded Riemann tensor such that the functional \( \bar{F} \) does not have an extremal.

Anyway, Carrillo and Ni [2] were able to show that on every gradient shrinking soliton, the potential function is a constrained minimizer of the functional \( W \). Zhang [15] and, recently, with weaker hypotheses on the geometry of the manifold, Rimoldi and Veronelli [10] showed the existence of extremals for \( W \) on a generic manifold, under a condition at infinity. In particular, in this latter paper, the authors use such conclusion to show that a general shrinking soliton (not a priori gradient) with bounded Ricci tensor...
and injectivity radius uniformly bounded below, actually admits a gradient soliton structure (under the above mentioned condition at infinity).

As a consequence of the work of Carrillo and Ni [2], arguing as Rimoldi and Veronelli [10], if we consider, as in the compact case, the tensor $T = (\nabla_k f - \nabla_k \bar{f}) g^{kj} (R_{ij} + \nabla^2_{ij} f - g_{ij}/2) e^{-f}$,

$$0 \leq Q = |\text{Ric} + \nabla^2 f - g/2|^2 e^{-f} = \text{div } T,$$

under the hypotheses of bound on the Ricci tensor and on the injectivity radius of the manifold, the function $Q$ can be integrated on $M$ and we can conclude that $Q = 0$.

**Proposition 2.6.** If a gradient shrinking Ricci soliton $(M, g)$ has uniformly bounded Ricci tensor and injectivity radius (this latter from below), the unique constrained critical point of the functional $W$ is the potential function of the soliton and minimizes $W$.

**References**

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