METRIC METHODS FOR HETEROCLINIC CONNECTIONS

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Abstract. We consider the problem $\min \int_{\mathbb{R}} \frac{1}{2}|\dot{\gamma}|^2 + W(\gamma)\,dt$ among curves connecting two given wells of $W \geq 0$ and we reduce it, following a standard method, to a geodesic problem of the form $\min \int_0^1 K(\gamma)|\dot{\gamma}|\,dt$ with $K = \sqrt{2W}$. We then prove existence of curves minimizing this new action just by proving that the distance induced by K is proper (i.e. its closed balls are compact). The assumptions on W are minimal, and the method seems robust enough to be applied in the future to some PDE problems.

1. Introduction

The minimization of an energy such as

$$\left(\gamma : I \to \mathbb{R}^d\right) \mapsto \int_I \left(\frac{1}{2}|\dot{\gamma}|^2(t) + W(\gamma(t))\right) dt \tag{1.1}$$

is a very common problem in many mathematical issues, first of all because of its meaning in classical mechanics (where it corresponds to kinetic minus potential energy): the corresponding Euler-Lagrange equation $\gamma'' = \nabla W(\gamma)$ is Newton's equation. The same minimization problem and the same ODE also appear in other issues, for instance in phase transition models, where a suitable rescaling of the curve γ gives an optimal transition between two states (we refer for instance to [3] for a general introduction to this field). For many applications, the case where $I = \mathbb{R}$, $W \geq 0$ and γ connects two wells of W (i.e. $\gamma(\pm\infty) = x^{\pm}$ with $W(x^{\pm}) = 0$) is the most interesting one. The optimal curve γ is called a heteroclinic connection in contrast with the homoclinic connections, which are solutions of $\gamma'' = \nabla W(\gamma)$ but with same limits at $\pm\infty$ (see [4]). The existence of a heteroclinic connection is a delicate problem, because of the lack of compactness of the set $H^1(\mathbb{R})$ and of the invariance by translations of the action to be minimized. Many ways to overcome this problem have been proposed, under suitable assumptions on W (on its degeneracy or radial monotonicity near the wells, for instance). We cite [6] as a first analysis of this problem, and many more recent papers, in particular [1, 4, 5]. This last paper, [5], is the one with the most general result, as it removes the monotonicty assumptions of [1] around the wells. In [5] there is the assumption $\liminf_{|x|\to\infty} W(x) > 0$, but it is easy to see that it can weakened into something like $\sqrt{W(x)} \geq k(|x|)$ with $\int_0^\infty k(t)dt = +\infty$, as we do in this paper. Note that [1] already used a similar assumption, in the form $\liminf_{|x|\to\infty} |x|^2 W(x) = +\infty$, but ours is weaker.

The idea behind the method that we propose here, very much different from [1, 5], is classical: reduce the problem to a geodesic problem for a weighted metric with a cost given by $K(x) := \sqrt{2W(x)}$, i.e., instead of minimizing (1.1), solving

$$\min \int_I \left(\sqrt{2W(\gamma(t))}\, |\gamma'(t)| \right) dt$$

with given initial and final data. The difficulty in this problem is the fact that K is not bounded from below, which makes it difficult to obtain bounds on a minimizing sequence (note that the strategy used in [6] shares to some extent this idea, but requires very strong assumptions on W in order to obtain bounds on the Euclidean length of γ). Instead, we propose an abstract metric approach: we show that the distance d_K induced by the weight K makes $\mathbb{R}^d$ a proper space, which automatically means that it admits the existence of geodesics.

We present our approach in the framework of a general metric space X instead of $\mathbb{R}^d$ in order to prepare possible later extensions to higher dimensional problems, i.e. attacking

$$\min \int_{\mathbb{R}\times I} \left(\frac{1}{2}|\nabla u|^2(x) + W(u(x))\right) \mathrm{d}x$$

where $x = (x_1, x_2)$, and boundary data are fixed as $x_1 \to \pm\infty$. This can be interpreted in our framework using x_1 as t and X to be $L^2(I)$, with an effective potential of the form $v \mapsto \int_I \frac{1}{2}|\partial_{x_2} v|(x_2)^2 + W(v(x_2))\,\mathrm{d}x_2$. But this obviously raises extra difficulties due to the lack of compactness in infinite dimensions.

The paper is organised as follows: first we recall the main notions concerning curves and geodesics in metric spaces, then we consider the problem of minimizing a weighted length in a metric space, with a weight K which can possibly vanish, then we apply this result to the problem of heteroclinic connections.

2. Minimal length problem in metric spaces

Let (X, d) be a metric space, a standard situation being $X = \mathbb{R}^d$ endowed with the Euclidean distance.

Curves in (X, d). A *curve* is a continuous map $\gamma : I \to X$, where $I \subset \mathbb{R}$ is a non-empty interval. We denote the set of Lipschitz maps (resp. locally Lipschitz maps) from I to X by $\mathcal{L}(I, X)$ (resp. $\mathcal{L}_{loc}(I, X)$):

$$\mathcal{L}(I,X) = \left\{\gamma : I \mapsto X \ : \ \sup_{t\neq s} \frac{d(\gamma(t),\gamma(s))}{|t-s|} < \infty\right\} \quad \text{and} \quad \mathcal{L}_{loc}(I,X) = \Big\{\gamma : I \to X \ : \ \forall J \subset\subset I,\ \gamma \in \mathcal{L}(J,X)\Big\}.$$

We also need to introduce the set of *piecewise locally Lipschitz maps*:

$$\mathcal{L}_{ploc}(I,X) := \Big\{\gamma \in C(I,X) \ : \ \exists t_0 = \inf I < t_1 < \cdots < t_n = \sup I,\ \forall i,\ \gamma \in \mathcal{L}_{loc}(I \cap (t_i, t_{i+1}))\Big\}.$$

Length of a curve. Given any curve $\gamma : I \to X$, we define the length of γ by the usual formula

$$L_d(\gamma) := \sup \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \in \mathbb{R} \cup \{+\infty\},$$

where the supremum is taken over all $n \geq 1$ and all sequences $t_0 \leq \cdots \leq t_n$ in I. A curve γ is said to be *rectifiable* if $L(\gamma) < \infty$.

Length of locally Lipschitz curves. For piecewise locally Lipschitz maps we have the following representation formula for the length:

Proposition 1. *Given $\gamma \in \mathcal{L}_{ploc}(I, X)$, the following quantity,*

$$|\dot\gamma|(t) = \lim_{s\to t} \frac{d(\gamma(t),\gamma(s))}{|t-s|},$$

is well defined for a.e. $t \in I$ and measurable. $|\dot\gamma|$ is called metric derivative *of γ. Moreover, one has*

$$L_d(\gamma) = \int_I |\dot\gamma|(t)\,\mathrm{d}t.$$

We refer for instance to [2] for the notion of metric derivative and for many other notions on the analysis of metric spaces.

Parametrization. If $\gamma : I \to X$ is a curve and $\varphi : I' \to I$ is a non-decreasing surjective continuous mapping, called *parametrization*, then the curve $\sigma = \gamma \circ \varphi : I' \to X$ satisfies $L_d(\sigma) = L_d(\gamma)$. The curve γ is said to have *constant speed* if for all $t, t' \in I$ s.t. $t < t'$, $L_d(\gamma_{|(t,t')}) = \lambda|t - t'|$, for a constant λ, called *speed* of the curve γ. Note that γ has constant speed λ if and only if γ is Lipschitz and $|\dot\gamma(t)| = \lambda$ a.e.

The curve γ is parametrized by arc length if $\lambda = 1$. If γ satisfies $L_d(\gamma_{|J}) < \infty$ for all compact subset $J \subset I$, then γ admits a parametrization by arc length. Indeed, fix $t_0 \in I$ and define $\varphi(t) := \pm L_d(\gamma_{|(t_0,t)})$ for $t \in I$ s.t. $\pm(t - t_0) \geq 0$. The map φ is continuous, non-decreasing and the curve $\sigma : \varphi(I) \to X$, $\sigma(\varphi(t)) = \gamma(t)$ is well defined, continuous and parametrized by arc length.

Minimal length problem. We define the intrinsic pseudo-metric geo$d$ (called *geodesic distance*) by minimizing the length of all curves γ connecting two points $x^\pm \in X$:

$$\text{geo}d(x^-, x^+) := \inf\{L_d(\gamma) \ : \ \gamma : x^- \mapsto x^+\} \in [0, +\infty], \tag{2.1}$$

where the notation $\gamma : x^- \mapsto x^+$ means that γ is a *path* from x^- to x^+: there exists $a^- \leq a^+$ s.t. $\gamma \in C^0([a^-, a^+], X)$ with $\gamma(a^\pm) = x^\pm$. Here, if a^+ or a^- is infinite, we use the convention $\gamma(\pm\infty) := \lim_{t\to\pm\infty} \gamma(t) = x^\pm$, if the limit exists.

When (X, d) is a normed space (for instance the Euclidean space), geo$d = d$ and the infimum value in (2.1) is achieved by the segment $[a^-, a^+]$. In general, a metric space such that geo$d = d$ is called *length space*.

The minimal length problem consists in finding a curve $\gamma : x^- \mapsto x^+$ such that $L_d(\gamma) = \text{geo}d(x^-, x^+)$. The existence of such a curve, called *minimizing geodesic*, is given by the classical theorem (see [2], for instance):

Theorem 1. *Assume that (X, d) is proper, i.e. every bounded closed subset of (X, d) is compact. Then, for any two points $x^\pm$ such that $\text{geo}d(x^+, x^-) < +\infty$, there exists a minimizing geodesic joining x^- and x^+.*

3. Minimal length problem in weighted metric spaces

Let (X, d) be a metric space and $K : X \to \mathbb{R}^+$ be a nonnegative function called *weight function*. From now on, we make the following assumptions on (X, d, K):

- **(H1):** (X, d) is a proper length metric space.
- **(H2):** K is continuous and $\Sigma := \{K = 0\}$ is finite.
- **(H3):** For all $x \in X$, $K(x) \geq k(d(x, \Sigma))$ for some function $k \in C^0(\mathbb{R}^+, \mathbb{R}^+)$ with $\int_0^\infty k(t)\,\mathrm{d}t = +\infty$.

Assumption **(H1)** is satisfied in particular by any Euclidean space. The confining property **(H3)** is fulfilled, for instance, whenever $\liminf_{d(x_0,x)\to\infty} K(x) > 0$.

Our aim is to investigate the existence of a curve $\gamma \in \mathcal{L}_{ploc}(I, X)$ minimizing the K-length, defined by

$$L_K(\gamma) := \int_I K(\gamma(t))\,|\dot\gamma(t)|\,\mathrm{d}t.$$

Namely, we want to find a curve $\gamma \in \mathcal{L}_{ploc}(I, X)$ which minimizes the K-length between given points $x^\pm \in X$:

$$d_K(x^-, x^+) := \inf\{L_K(\gamma) \ : \ \gamma \in \mathcal{L}_{ploc}(I, X) \text{ s.t. } \gamma : x^- \mapsto x^+\}.$$

We are going to prove that d_K is a metric on X s.t. (X, d_K) is proper and $L_K = L_{d_K}$, thus implying the existence of a geodesic between two joinable points, in view of Theorem 1 (see Theorem 2 below).

Proposition 2. *The quantity d_K is a metric on X. Moreover (X, d_K) enjoys the following properties*

1. *d_K and d are equivalent (i.e. they induce the same topology) on all d-compact subsets of X.*
2. *(X, d_K) is a proper metric space.*
3. *Any locally Lipschitz curve $\gamma : I \to X$ is also d_K-locally Lipschitz and the metric derivative of γ in (X, d_K), denoted by $|\dot\gamma|_K$, is given by $|\dot\gamma|_K(t) = K(\gamma(t))\,|\dot\gamma|(t)$ a.e.*
4. *We have $L_K(\gamma) = L_{d_K}(\gamma)$ for all $\gamma \in \mathcal{L}_{ploc}(I, X)$.*

Theorem 2. *For any $x, y \in X$, there exists $\gamma \in \mathcal{L}_{ploc}(I, X)$ s.t. $L_K(\gamma) = d_K(x, y)$ and $\gamma : x \mapsto y$.*

Proof. Let us see how Proposition 2 implies Theorem 2. As (X, d_K) is a proper metric space, Theorem 1 insures the existence of a L_{d_K}-minimizing curve $\gamma : x \mapsto y$. Up to renormalization, one can assume that γ is parametrized by L_{d_K}-arc length. By minimality and the choice of the parametrization, such a curve γ must be injective. In particular, γ meets the finite set $\{K = 0\}$ at finite many instants $t_1 < \cdots < t_N$. As K is bounded from below by some positive constant on each compact subinterval of (t_i, t_{i+1}) for $i \in \{1, \ldots, N\}$, Lemma 1 below implies that γ is piecewise locally d-Lipschitz. Finally, thanks to Statement 4 of Proposition 2, the fact that γ minimizes L_{d_K} means that it also minimizes L_K among $\mathcal{L}_{ploc}$ curves connecting x to y. □

In order to prove Proposition 2, we will need the following estimates on d_K.

Lemma 1. *For all $x, y \in X$, one has*

$$K_{d(x,y)}(x)\, d(x,y) \leq d_K(x,y) \leq K^{d(x,y)}(x)\, d(x,y),$$

where $K_r(x)$ and $K^r(x)$ are defined for any $r \geq 0$ and $x \in X$ by

$$K_r(x) := \inf\{K(y) \ : \ d(x,y) \leq r\}, \quad K^r(x) := \sup\{K(y) \ : \ d(x,y) \leq r\}.$$

Proof. Set $r := d(x,y)$. Since any curve $\gamma : x \mapsto y$ has to get out of the open ball $B := B_d(x,r)$, it is clear that

$$L_K(\gamma) = \int_I K(\gamma(t))\, |\dot{\gamma}|(t)\, \mathrm{d}t \geq r \inf_B K = r K_r(x).$$

Taking the infimum over the set of curves $\gamma \in \mathcal{L}_{ploc}$ joining x and y, one gets the first inequality.

For the second inequality, let us fix $\varepsilon > 0$. By construction, there exists a Lipschitz curve $\gamma : x \mapsto y$, that one can assume to be parametrized by arc-length, s.t. $L_d(\gamma) \leq r + \varepsilon$. In particular, $\mathrm{Im}(\gamma)$ is included in the ball $B_d(x, r+\varepsilon)$. Thus, one has

$$d_K(x,y) \leq L_K(\gamma) \leq (r+\varepsilon)\, K^{r+\varepsilon}(x)$$

and the second inequality follows by sending $\varepsilon \to 0$. Indeed, the mapping $r \to K^r(x)$ is continuous on $[0, +\infty)$ since K uniformly continuous on compact sets and since bounded closed subsets of X are compact (assumption **(H1)**). □

Proof of Proposition 2. The proof is divided into six steps.

Step 1: d_K is a metric. First note that d_K is finite on $X \times X$. Indeed, given two points $x, y \in X$, just take a Lipschitz curve connecting them, and use $L_K(\gamma) \leq L_d(\gamma) \sup_{\mathrm{Im}(\gamma)} K < +\infty$. The triangle inequality for d_K is a consequence of the stability of the set $\mathcal{L}_{ploc}$ by concatenation. The fact that $d_K(x,y) = 0$ implies $x = y$ is an easy consequence of the finiteness of the set $\{K = 0\}$. Indeed, if $x \neq y$, then any curve $\gamma : x \mapsto y$ has to connect $B_d(x,\varepsilon)$ to $B_d(x, 2\varepsilon)^c$ for all $\varepsilon > 0$ small enough. This implies that $L_K(\gamma) \geq \varepsilon \inf_C K$, where $C = \{y \ : \ \varepsilon \leq d(x,y) \leq 2\varepsilon\}$. But for ε small enough, C does not intersect the set $\{K = 0\}$ so that $\inf_C K > 0$. In particular, $d_K(x,y) \geq \varepsilon \inf_C K > 0$.

Step 2: d_K and d are equivalent on d-compact sets. Take $Y \subset X$ a compact set, and suppose $Y \subset B_d(x_0, R)$ just to fix the ideas. Consider the identity map from (Y, d) to (Y, d_K). It is an injective map between metric spaces. Moreover, it is continuous, since, as a consequence of Lemma 1, we have $d_K \leq Cd$ on $Y \times Y$, where $C = \sup_{B_d(x_0, 3R)} K < +\infty$ (note that the closed ball $B_d(x_0, 3R)$ is d-compact, and that we supposed $d = \mathrm{geo}d$ since (X,d) is a length space). Hence, as every injective continuous map defined on a compact space is a homeomorphism, d and d_K are equivalent (on Y).

Step 3: every ball in (X, d_K) is d-bounded. This is a consequence of assumptions (**H1**) and (**H3**). Let us take $x_0, x \in X$ with $d_K(x, x_0) \leq R$. By definition, there exists $\gamma \in \mathcal{L}_{ploc}(I, X)$ s.t. $\gamma : x_0 \mapsto x$ and $L_K(\gamma) \leq d_K(x_0, x) + 1$. Now, set $\phi(t) := d(\gamma(t), \Sigma)$: since the function $x \mapsto d(x, \Sigma)$ has Lipschitz constant equal to 1, we have $\phi \in \mathcal{L}_{ploc}(I, \mathbb{R})$ and $|\phi'(t)| \leq |\gamma'(t)|$ a.e. Take $h : \mathbb{R}^+ \to \mathbb{R}^+$ the antiderivative of k, i.e. $h' = k$ with $h(0) = 0$, and compute $[h(\phi(t))]' = k(\phi(t))\phi'(t)$. Hence,

$$|[h(\phi(t))]'| = k(\phi(t))|\phi'(t)| \leq K(\gamma(t))\, |\gamma'(t)|$$

and $h(d(\gamma(t), \Sigma)) \leq h(d(x_0, \Sigma)) + L_K(\gamma) \leq h(d(x_0, \Sigma)) + R + 1$. Since $\lim_{s \to \infty} h(s) = +\infty$, this provides a bound on $d(x, \Sigma)$ which means that the ball $B_{d_K}(x_0, R)$ is d-bounded.

Step 4: every closed ball in (X, d_K) is d_K-compact. Now that we know that closed ball in (X, d_K) are d-bounded, since (X, d) is proper, we know that they are contained in d-compact sets. But on this sets d and d_K are equivalent, hence these balls are also d-closed, hence d-compact, and thus d_K-compact.

Step 5: proof of statement 3. Let $\gamma : I \mapsto X$ be a d-locally Lipschitz curve valued in X. Thanks to the second inequality in Lemma 1, γ is also d_K-locally Lipschitz. Now, Lemma 1 provides

$$K_r(\gamma(t))\,\frac{d(\gamma(t),\gamma(s))}{|t-s|} \leq \frac{d_K(\gamma(t),\gamma(s))}{|t-s|} \leq K^r(\gamma(t))\,\frac{d(\gamma(t),\gamma(s))}{|t-s|}$$

with $r := d(\gamma(t),\gamma(s))$. In the limit $s \to t$ we get

$$K(\gamma(t))\,|\dot\gamma|(t) \leq |\dot\gamma|_K(t) \leq K(\gamma(t))\,|\dot\gamma|(t) \text{ a.e.,}$$

where the continuity of $r \to K^r(x)$ and $r \to K_r(x)$ on $[0,+\infty)$ has been used.

Last step: proof of statement 4. This is an easy consequence of Statement 3. Indeed, by additivity of L_K and L_{d_K} and since $L_K(\gamma) = \sup L_K(\gamma_J)$, $L_{d_K}(\gamma) = \sup L_{d_K}(\gamma_J)$, both supremum being taken on compact subsets $J \subset I$, it is enough to prove that $L_K(\gamma) = L_{d_K}(\gamma)$ when $\gamma \in \mathcal{L}(I,X)$. But any curve $\gamma \in \mathcal{L}(I,X)$ is locally d_K-Lipschitz and satisfies

$$L_{d_K}(\gamma) = \int_I |\dot\gamma|_K(t)\,\mathrm{d}t = \int_I K(\gamma(t))\,|\dot\gamma|(t)\,\mathrm{d}t = L_K(\gamma). \qquad \square$$

4. Existence of heteroclinic connections

Our aim is to investigate the existence of a global minimizer of the energy

$$E_W(\gamma) = \int_{\mathbb{R}} \left(\frac{1}{2}|\dot\gamma|^2(t) + W(\gamma(t))\right)\mathrm{d}t,$$

defined over locally Lipschitz curves $\gamma : x^- \mapsto x^+$, defined on $\mathbb{R}$ and valued in a metric space (X,d). Here $W : X \mapsto \mathbb{R}^+$ is a continuous function, called *potential* in all the sequel, and $x^\pm \in X$ are two wells, i.e. $W(x^\pm) = 0$. Note that $W(x^\pm) = 0$ is a necessary condition for the energy of γ to be finite. The main result of this section is the following:

Theorem 3. *Let us take (X,d) a metric space, $W : X \mapsto \mathbb{R}^+$ a continuous function and $x^\pm \in X$ such that:*

- **(H):** *(X,d,K) satisfies hypotheses* (**H1 – 3**) *of the previous section, where $K := \sqrt{2W}$.*
- **(STI):** *$W(x^-) = W(x^+) = 0$ and d_K (defined above) satisfies the following strict triangle inequality on the set $\{W = 0\}$: for all $x \in X \setminus \{x^-, x^+\}$ s.t. $W(x) = 0$, $d_K(x^-,x^+) < d_K(x^-,x) + d_K(x,x^+)$.*

Then, there exists a heteroclinic connection between x^- and x^+, i.e. $\gamma \in \mathcal{L}(\mathbb{R},X)$ such that

$$E_W(\gamma) = \inf\{E_W(\sigma) \ : \ \sigma \in \mathcal{L}_{ploc}(\mathbb{R},X),\ \sigma : x^- \mapsto x^+\}.$$

Moreover, $E_W(\gamma) = d_K(x^-,x^+)$.

Proof. This theorem is a consequence of Theorem 2 and the following consequence of Young's inequality:

$$\text{for all } \gamma \in \mathcal{L}_{ploc}(\mathbb{R},X),\ E_W(\gamma) \geq L_K(\gamma), \tag{4.1}$$

where $K := \sqrt{2W}$. Indeed, thanks to assumption **(H)**, Theorem 2 provides a L_K-minimizing curve $\gamma_0 : I \to X$, that one can assume to be parametrized by L_K-arc length (and injective by minimality), connecting x^- to x^+. Thanks to assumption (**STI**), it is clear that the curve γ_0 cannot meet the set $\{W = 0\}$ at a third point $x \neq x^\pm$: in other words $K(\gamma(t)) > 0$ on the interior of I. Thus, γ_0 is also d-locally Lipschitz on I (and not only piecewise locally Lipschitz). In particular, one can reparametrize the curve γ_0 by L_d-arc length, so that $|\dot\gamma_0| = 1$ a.e. in I. Note that, since d and d_K are equivalent on $\mathrm{Im}(\gamma_0)$, γ_0 remains d_K-continuous, and γ_0 is even d_K-Lipschitz as $d_K(\gamma(t),\gamma(s)) \leq L_K(\gamma_{|(t,s)}) \leq \sup K \circ \gamma \times L_d(\gamma_{|(t,s)}) \leq C|t-s|$. In particular, γ_0 is still L_K-minimizing.

Then, in view of (4.1), it is enough to prove that γ_0 can be reparametrized in a locally Lipschitz curve γ satisfying the equipartition condition $|\dot\gamma| = K \circ \gamma$ a.e., for which (4.1) becomes an equality. Namely, we look for an admissible curve $\gamma : \mathbb{R} \to X$ of the form $\gamma(t) = \gamma_0(\varphi(t))$, where $\varphi : \mathbb{R} \to I$ is C^1, increasing and surjective. For γ to satisfy $|\dot\gamma|(t) = K(\gamma(t))$ a.e., we need φ to solve the ODE

$$\varphi'(t) = F(\varphi(t)), \tag{4.2}$$

where $F : \bar{I} \to \mathbb{R}$ is the continuous function defined by $F = K \circ \gamma_0$ on I and $F \equiv 0$ outside I. Thanks to the Peano-Arzelà theorem, (4.2) admits at least one maximal solution $\varphi_0 : J = (t^-, t^+) \mapsto \mathbb{R}$ such that $0 \in J$ and $\varphi_0(0)$ is any point inside I. Since F vanishes out of I, we know that $\mathrm{Im}(\varphi_0) \subset \bar{I}$. Moreover, since φ_0 is non decreasing on I, it converges to two distinct stationary points of the preceding ODE. As $F > 0$ inside I, one has $\lim_{t\to t^+} \varphi_0(t) = \sup I$ and $\lim_{t\to t^-} \varphi_0(t) = \inf I$. We deduce that φ_0 is an entire solution of the preceding ODE, i.e. $I = \mathbb{R}$. Indeed, if $I \neq \mathbb{R}$, say $t^+ < +\infty$, then one could extend φ_0 by setting $\varphi_0(t) = \sup I$ for $t > t^+$. Finally, the curve $\gamma := \gamma_0 \circ \varphi_0$ satisfies $\gamma(\pm\infty) = x^\pm$, it is locally Lipschitz (by composition), and $|\dot\gamma|(t) = K(\gamma(t))$ a.e. Thus,

$$E_W(\gamma) = L_K(\gamma) = L_{d_K}(\gamma) = L_{d_K}(\gamma_0) = d_K(x^-, x^+) \leq \inf\{E_W(\sigma) \ : \ \sigma \in \mathcal{L}_{ploc}(\mathbb{R}, X),\ \gamma : x \mapsto y\}.$$

Namely, γ minimizes E_W over all admissible connections between x^- and x^+. □

Remark.
- It is easy to see that the equipartition of the energy, that is the identity $|\dot\gamma|^2(t) = 2W(\gamma(t))$, is a necessary condition for critical points of E_W.
- The assumption **(STI)** is not optimal but cannot be removed, and is quite standard in the literature. Without this assumption, it could happen that a geodesic γ would meet the set $\{W = 0\}$ at a third point $x \neq x^\pm$. In this case, it is not always possible to parametrize γ in such a way that $|\dot\gamma|(t) = K(\gamma(t))$.
- However, if $K = \sqrt{2W}$ is not Lipschitz, it is possible that there exists a heteroclitic connection $\gamma : x^- \mapsto x^+$ meeting $\{W = 0\}$ at a third point $x \neq x^\pm$. Indeed, if $\liminf_{y\to x} K(y)/|y| > 0$, then, there exists a heteroclinic connection $\gamma^- : x^- \mapsto x$ which reaches x in finite time (say, $\gamma^-(t) = x$ for $t \geq 0$). Similarly, there exists a heteroclinic connection $\gamma^+ : x \mapsto x^+$ such that $\gamma^+(t) = z$ for $t \leq 0$. Thus, there exists a heteroclinic connection between x^- and x^+ obtained by matching γ^- and γ^+.

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