SECOND-ORDER APPROXIMATION OF FREE-DISCONTINUITY PROBLEMS WITH LINEAR GROWTH

TERESA ESPOSITO

Abstract. Motivated by applications to image denoising, we propose an approximation of functionals of the form

$$F(u)=\int_\Omega |\nabla u|\,dx+\int_{S_u} g(|u^+-u^-|)\,d\mathcal{H}^{n-1}+|D^c u|(\Omega),\qquad u\in BV(\Omega),$$

with $g\colon [0,+\infty)\to[0,+\infty)$ increasing and bounded. The approximating functionals are of Ambrosio-Tortorelli type and depend on the Hessian or on the Laplacian of the edge variable v which thus belongs to $W^{2,2}(\Omega)$. When the space dimension is equal to two and three v is then continuous and this improved regularity leads to a sequence of approximating functionals which are ready to be used for numerical simulations.

Keywords: Ambrosio-Tortorelli approximation, free-discontinuity problems, Γ-convergence.

2000 Mathematics Subject Classification: 49J45, 74G65, 68U10.

1. Introduction

The variational approach to image processing requires the minimization, over a suitable space of discontinuous functions, of a functional characterized by a regularizing term and a fidelity term. In the framework of image denoising, one of the most successful models is the total-variation based model of Rudin, Osher and Fatemi [20]. According to it, if $h\in L^\infty(\Omega)$ is a given input image then its reconstruction u is obtained as a solution of the following problem:

$$\min_{u\in BV(\Omega)}\left\{|Du|(\Omega)+\|u-h\|^2_{L^2(\Omega)}\right\},\tag{1.1}$$

where Du denotes the distributional derivative of $u\in BV(\Omega)$ and is given by $Du=\nabla u\,\mathcal{L}^n+(u^+-u^-)\,\nu_u\,\mathcal{H}^{n-1}\llcorner S_u+D^c u$. The model (1.1) performs well for removing noise and preserving edges. However it always causes a loss of contrast in the reconstructed image, and this can be attributed to the fact that the jump-penalization increases linearly with the amplitude of the jump $|u^+-u^-|$, resulting in a strong penalization of large jump-amplitude. Therefore it would be desirable to consider an energy functional which penalizes the jump-amplitude and whose dependence on $|u^+-u^-|$ is increasing for small amplitudes, and bounded for large ones. With this idea in mind, we are then interested in replacing in (1.1) the total variation $|Du|(\Omega)$ with the functional

$$F(u)=\int_\Omega |\nabla u|\,dx+\int_{S_u} g(|u^+-u^-|)\,d\mathcal{H}^{n-1}+|D^c u|(\Omega),\tag{1.2}$$

for some C^1, increasing and bounded function $g\colon[0,+\infty)\to[0,+\infty)$ such that $g(0)=0$ (see Figure 1.).

It is well known that functionals as (1.2) are difficult to be treated numerically. Then a very important task is to approximate them, in the sense of Γ-convergence, with volume-functionals defined on spaces of more regular functions. In the spirit of Ambrosio and Tortorelli's approximation of the Mumford-Shah functional [5], Alicandro, Braides and Shah proposed in [2] an approximation of (1.2) by means of the sequence

$$ABS_\varepsilon(u,v)=\int_\Omega\left(v^2|\nabla u|+\frac{(1-v)^2}{\varepsilon}+\varepsilon|\nabla v|^2\right)dx,\tag{1.3}$$

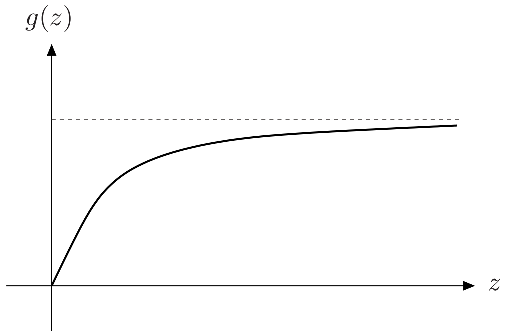

FIGURE 1. The function g.

with $u \in W^{1,1}(\Omega)$ and $v \in W^{1,2}(\Omega)$.

Motivated by the good computational results obtained in [8], where a second-order approximation of the Mumford-Shah functional has been proposed and analyzed, in (1.3) we replace the first-order term $\varepsilon|\nabla v|^2$ with a second-order term depending either on the Hessian or on the Laplacian of v. Precisely, for $u \in W^{1,1}(\Omega)$ and $v \in W^{2,2}(\Omega)$, we consider the functionals

$$F_\varepsilon(u,v) = \int_\Omega \left(v^2|\nabla u| + \frac{(1-v)^2}{\varepsilon} + \varepsilon^3|\nabla^2 v|^2 \right) dx \tag{1.4}$$

and, under the additional condition $(v-1) \in W_0^{1,2}(\Omega)$, the functionals

$$E_\varepsilon(u,v) = \int_\Omega \left(v^2|\nabla u| + \frac{(1-v)^2}{\varepsilon} + \varepsilon^3|\Delta v|^2 \right) dx. \tag{1.5}$$

In Theorems 3.1 and 3.2 we prove that both F_ε and E_ε Γ-converge, with respect to the strong topology of $L^1(\Omega) \times L^1(\Omega)$, as $\varepsilon \to 0$, to the functional F defined in (1.2), with $g\colon [0,+\infty) \to [0,+\infty)$ given by

$$g(z) = \min_{r\in[0,1]} \{r^2 z + 2\sqrt{2}(1-r)^2\} = \frac{4z}{4+\sqrt{2}z}, \tag{1.6}$$

where $\sqrt{2}(1-r)^2$ represents the minimal cost in terms of the unscaled, one-dimensional Modica-Mortola contribution in (1.4) and (1.5) for a transition between r and 1.

Let us briefly comment the heuristic idea behind these Γ-convergence results. Let $(u_\varepsilon, v_\varepsilon)$ be a minimizing sequence either for F_ε or for E_ε. Then, far from S_u, v_ε approaches 1 driven by the factor $1/\varepsilon$ which multiplies the potential term $(1-v_\varepsilon)^2$. Around S_u, instead, both the first and second terms in (1.4) are diverging; to keep them bounded v_ε makes transitions from r to 1, the value r chosen so that the sum of the energy contributions given by $\|v_\varepsilon^2 \nabla u\|_{L^1(\Omega)}$ and the Modica-Mortola term in (1.4) is minimal; *i.e.,* minimizing (1.6).

On account of the Γ-convergence results Theorems 3.1 and 3.2, in Section 5 we prove that, when perturbed by a term $\|u-h\|^2_{L^2(\Omega)}$ for some $h \in L^\infty(\Omega)$, the functionals F_ε and E_ε are equicoercive (see Theorems 5.1 and 5.2). As a result, we derive the convergence of the associated minimization problems to

$$\min_{u\in BV(\Omega)} \left\{ F(u) + \|u-h\|^2_{L^2(\Omega)} \right\}. \tag{1.7}$$

However the functionals F_ε, as well as the functionals E_ε, are not suited to numerical applications. In fact, due to the lack of compactness properties of the space $W^{1,1}(\Omega)$, the direct methods of the calculus of variations cannot be applied to obtain the existence of minimizers for F_ε and E_ε at fixed $\varepsilon > 0$.

A relaxation argument allows nevertheless to obtain existence of minimizers in the larger space $BV(\Omega)$. For $n \leq 3$, which is the interesting case for applications, if we denote by $\overline{F}_\varepsilon$ and $\overline{E}_\varepsilon$ the relaxations of F_ε and E_ε with respect to the strong topology of $L^1(\Omega) \times L^1(\Omega)$, we find that, thanks to the presence of the second-order perturbation in v, the expressions of $\overline{F}_\varepsilon$ and $\overline{E}_\varepsilon$ are particularly easy. Specifically, in Section 5, we prove that if $u \in BV(\Omega)$ and $v \in W^{2,2}(\Omega)$ then

$$\overline{F}_\varepsilon(u,v) = \int_\Omega v^2\, d|Du| + \int_\Omega \left(\frac{(1-v)^2}{\varepsilon} + \varepsilon^3 |\nabla^2 v|^2 \right) dx,$$

$$\overline{E}_\varepsilon(u,v) = \int_\Omega v^2\, d|Du| + \int_\Omega \left(\frac{(1-v)^2}{\varepsilon} + \varepsilon^3 |\Delta v|^2 \right) dx,$$

the proof strongly relying on the fact that $W^{2,2}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$.

Then, the functionals $\overline{F}_\varepsilon$ and $\overline{E}_\varepsilon$ still Γ-converge to F, and, if perturbed by a term $\|u-h\|^2_{L^2(\Omega)}$ for some $h \in L^\infty(\Omega)$, they are equicoercive. If for $\eta_\varepsilon > 0$ we consider the minimization problems

$$\min_{BV(\Omega)\times W^{2,2}(\Omega)} \left\{ \overline{F}_\varepsilon(u,v) + \eta_\varepsilon |Du|(\Omega) + \|u-h\|^2_{L^2(\Omega)} \right\} \tag{1.8}$$

and

$$\min_{BV(\Omega)\times W^{2,2}(\Omega)} \left\{ \overline{E}_\varepsilon(u,v) + \eta_\varepsilon |Du|(\Omega) + \|u-h\|^2_{L^2(\Omega)} \right\}, \tag{1.9}$$

the term $\eta_\varepsilon |Du|(\Omega)$ makes the functionals in (1.8) and (1.9) coercive for fixed $\varepsilon > 0$. Then, the existence of a minimizing pair easily follows appealing to the direct methods of the calculus of variations. Moreover if η_ε is chosen so that $\eta_\varepsilon = o(\varepsilon)$ then it can be easily shown that

$$\overline{F}_\varepsilon(u,v) + \eta_\varepsilon |Du|(\Omega) \xrightarrow{\Gamma} F(u,v), \quad \overline{E}_\varepsilon(u,v) + \eta_\varepsilon |Du|(\Omega) \xrightarrow{\Gamma} F(u,v).$$

Hence if $(u_\varepsilon, v_\varepsilon)$ is a minimizing sequence for (1.8) or (1.9), then $v_\varepsilon \to 1$ in $L^1(\Omega)$, u_ε converges to a solution $\bar{u}$ of (1.7) in $L^p(\Omega)$ for all $p \in [1, +\infty)$, and

$$\lim_{\varepsilon \to 0} \left(\overline{F}_\varepsilon(u_\varepsilon, v_\varepsilon) + \eta_\varepsilon |Du_\varepsilon|(\Omega) + \|u_\varepsilon - h\|^2_{L^2(\Omega)} \right) = F(\bar{u}) + \|\bar{u} - h\|^2_{L^2(\Omega)}.$$

We notice that the strong convergence of u_ε to $\bar{u}$ in $L^p(\Omega)$ for all $p \in [1, +\infty)$ is consequence of the strong convergence of u_ε to $\bar{u}$ in $L^1(\Omega)$ together with the fact that one can always assume $\|u_\varepsilon\|_{L^\infty(\Omega)} \leq \|h\|_{L^\infty(\Omega)}$.

The relaxed functionals $\overline{F}_\varepsilon$ and $\overline{E}_\varepsilon$ are now ready to be used for numerical simulations. In order to solve (1.8) or (1.9) one can follow the common strategy of iterative alternating minimization (see e.g. [18]). Hence, given an iterate (u^k, v^k), one computes

$$\begin{aligned} v^{k+1} &\in \arg\min_v \overline{F}_\varepsilon(u^k, v) \\ u^{k+1} &\in \arg\min_u \overline{F}_\varepsilon(u, v^{k+1}) + \eta_\varepsilon |Du|(\Omega) + \|u-h\|^2_{L^2(\Omega)}. \end{aligned} \tag{1.10}$$

We notice that now, thanks to the continuity of v^{k+1}, (1.10) is a standard weighted total-variation minimization problem and can be solved in a straightforward way with primal dual methods (see e.g. [9, 10, 17]).

The paper is organized as follows: after recalling some useful notation and preliminaries in Section 2, we state and prove the main results, Theorems 3.1 and 3.2, in Section 3 and Section 4. In Section 5 we first establish an equicoercivity result for the functionals under examination and then we provide an integral representation for their relaxations, in the case when $n \leq 3$.

2. Notation and preliminaries

In this section we set a few notation and recall some preliminary results we employ in the sequel.

Throughout the paper the parameter ε varies in a strictly decreasing sequence of positive real numbers converging to zero.

Let $n \geq 1$; if not otherwise specified, $\Omega \subset \mathbb{R}^n$ denotes an open bounded set with Lipschitz boundary. We denote by $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ the families of all open and Borel subsets of Ω, respectively. The Lebesgue measure and the k-dimensional Hausdorff measure on $\mathbb{R}^n$ are denoted by $\mathcal{L}^n$ and $\mathcal{H}^k$, respectively. If $x \in \mathbb{R}^n$, we denote by $(x_1, \dots, x_n)$ its components in the canonical basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$. The scalar product of x, $y \in \mathbb{R}^n$ is denoted by $\langle x, y \rangle$ and the euclidean norm by $|x|$, whereas $A \cdot B$ denotes the product between two suitable matrices A, B. If $x_0 \in \mathbb{R}^n$ and $\varrho > 0$, then $B_\varrho(x_0)$ denotes the open ball centered at x_0 with radius ϱ; if x_0 coincides with the origin we omit the dependence on x_0 and we simply write B_ϱ. Moreover we denote by S^{n-1} the boundary of B_1 in $\mathbb{R}^n$.

Let $\mathcal{M}_b(\Omega)$ be the set of all bounded Radon measures on Ω; if μ_k, $\mu \in \mathcal{M}_b(\Omega)$, we say that $\mu_k \overset{*}{\rightharpoonup} \mu$ weakly* in $\mathcal{M}_b(\Omega)$ as $k \to +\infty$ if

$$\lim_{k \to +\infty} \int_\Omega \varphi \, d\mu_k = \int_\Omega \varphi \, d\mu \qquad \forall\, \varphi \in C^0_0(\Omega).$$

Let $1 \leq p \leq +\infty$ and $k \in \mathbb{N}$, we use standard notation for the Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{k,p}(\Omega)$.

2.1. **Functions of bounded variation.** For the general theory of functions of bounded variation we refer the reader to [4]; here we only collect some useful notation and facts.

For every $u \in BV(\Omega)$, ∇u denotes the approximate gradient of u, $D^c u$ the Cantor part of the distributional derivative of u, S_u the approximate discontinuity set of u, ν_u the generalized normal to S_u, which is defined up to the sign, and u^+ and u^- are the traces of u on S_u.

We state a compactness result in $BV(\Omega)$ (see [4, Theorem 3.23 and Proposition 3.21]).

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary and let (u_k) be a bounded sequence in $BV(\Omega)$. Then there exist a subsequence of u_k (not relabeled) and a function $u \in BV(\Omega)$ such that $u_k \overset{*}{\rightharpoonup} u$ in $BV(\Omega)$; i.e., $u_k \to u$ in $L^1(\Omega)$ and $Du_k \overset{*}{\rightharpoonup} Du$ in Ω in $\mathcal{M}_b(\Omega)$.*

We say that a function $u \in BV(\Omega)$ is a special function of bounded variation, and we write $u \in SBV(\Omega)$, if $D^c u = 0$.

We also consider the larger space of the generalized functions of bounded variation on Ω, $GBV(\Omega)$, which is made of all the functions $u \in L^1(\Omega)$ whose truncations $u^m := (-m) \vee (u \wedge m)$ belong to $BV(\Omega)$ for every $m > 0$.

By the very definitions we have $BV(\Omega) \subset GBV(\Omega)$ and $BV(\Omega) \cap L^\infty(\Omega) = GBV(\Omega) \cap L^\infty(\Omega)$.

The space GBV inherits some of the main properties of the space BV (see [4, Theorem 4.34]).

Theorem 2.2. *Let $u \in GBV(\Omega)$. Then*

(i) u is approximately differentiable $\mathcal{L}^n$-a.e. in Ω and $\nabla u = \nabla u^m$ $\mathcal{L}^n$-a.e. in $\{|u| \leq m\}$;

(ii) $S_u = \cup_{m>0} S_{u^m}$ is countably $\mathcal{H}^{n-1}$-rectifiable and $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$, where J_u denotes the set of the approximate jump points of u.

The Cantor part of the distributional derivative of $u \in GBV(\Omega)$ is defined as

$$|D^c u| := \bigvee_{m>0} |D^c u^m|,$$

where the supremum is understood in the sense of measure (see [4, Definition 1.68]).

Notice that if $u \in GBV(\Omega)$ then

$$|\nabla u^m(x)| \leq |\nabla u(x)| \ \ \mathcal{L}^n\text{-a.e. in } \Omega$$

$$\nabla u^m(x) \to \nabla u(x)\ \mathcal{L}^n\text{-a.e. in } \Omega \text{ as } m \to +\infty$$

$$S_{u^m} \subseteq S_u,\ (u^m)^{\pm} = (u^{\pm})^m\ \ \mathcal{H}^{n-1}\text{-a.e. in } \Omega \tag{2.1}$$

$$\chi_{S_{u^m}} \to \chi_{S_u},\ (u^m)^{\pm}(x) \to u^{\pm}(x)\ \ \mathcal{H}^{n-1}\text{-a.e. in } \Omega \text{ as } m \to +\infty \tag{2.2}$$

$$\mathcal{H}^{n-1}(S_{u^m}) \to \mathcal{H}^{n-1}(S_u) \text{ as } m \to +\infty$$

$$|D^c u^m|(\Omega) \to |D^c u|(\Omega) \text{ as } m \to +\infty.$$

2.2. **Slicing.** We recall here some properties of one-dimensional restrictions of BV functions. We first fix some notation. For each $\xi \in S^{n-1}$ we consider the hyperplane through the origin and orthogonal to ξ; *i.e.,*

$$\Pi^{\xi} := \{x \in \mathbb{R}^n \colon \langle x, \xi \rangle = 0\},$$

and, for every $y \in \Pi^{\xi}$ and $A \subset \mathbb{R}^n$, we consider the one-dimensional set

$$A_{\xi,y} := \{t \in \mathbb{R} \colon y + t\,\xi \in A\}.$$

Moreover, for any given function $u \colon \Omega \to \mathbb{R}$ we define $u_{\xi,y} \colon \Omega_{\xi,y} \to \mathbb{R}$ by $u_{\xi,y}(t) := u(y + t\,\xi)$.

The following result holds true (see [3]).

Theorem 2.3 (Slicing Theorem in BV)**.** *(i) Let $u \in BV(\Omega)$. Then for all $\xi \in S^{n-1}$ the function $u_{\xi,y}$ belongs to $BV(\Omega_{\xi,y})$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^{\xi}$. For those y such that $u_{\xi,y} \in BV(\Omega_{\xi,y})$ we have*

$$u'_{\xi,y}(t) = \langle \nabla u(y + t\xi), \xi \rangle \text{ for } \mathcal{L}^1\text{-a.e.}\, t \in \Omega_{\xi,y}$$

$$S_{u_{\xi,y}} = \{t \in \mathbb{R} \colon y + t\xi \in S_u\}$$

$$u^{\pm}_{\xi,y} = u^{\pm}(y + t\xi) \text{ or } u^{\pm}_{\xi,y} = u^{\mp}(y + t\xi)$$

according to the cases $\langle \nu_u, \xi \rangle > 0$ or $\langle \nu_u, \xi \rangle < 0$ (the case $\langle \nu_u, \xi \rangle = 0$ being negligible). Moreover, we have

$$\int_{\Pi^{\xi}} |D^c u_{\xi,y}|(A_{\xi,y})\, d\mathcal{H}^{n-1}(y) = |\langle D^c u, \xi \rangle|(A)$$

for all $A \in \mathcal{A}(\Omega)$, and for all functions $g \in L^1(S_u; \mathcal{H}^{n-1})$

$$\int_{\Pi^{\xi}} \sum_{t \in S_{u_{\xi,y}}} g(t)\, d\mathcal{H}^{n-1}(y) = \int_{S_u} g(x) |\langle \nu_u, \xi \rangle|\, d\mathcal{H}^{n-1}(x).$$

(ii) Let $u \in L^1(\Omega)$. If $u_{\xi,y} \in BV(\Omega_{\xi,y})$ and

$$\int_{\Pi^{\xi}} |Du_{\xi,y}|(\Omega_{\xi,y})\, d\mathcal{H}^{n-1}(y) < +\infty$$

for all $\xi \in \{e_1, \dots, e_n\}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^{\xi}$, then $u \in BV(\Omega)$.

The previous theorem will be a key tool to get the lower bound inequality in the proof of Theorem 3.1.

2.3. **A density result.** We denote by $\mathcal{W}(\Omega)$ the space of all functions $u \in SBV(\Omega)$ such that:

(i) $\mathcal{H}^{n-1}(\overline{S}_u \setminus S_u) = 0$;
(ii) $\overline{S}_u$ is the intersection of Ω with the union of a finite number of pairwise disjoint $(n-1)$-dimensional simplexes;
(iii) $u \in W^{k,\infty}(\Omega \setminus \overline{S}_u)$ for every $k \in \mathbb{N}$.

The following theorem due to Cortesani and Toader (see [11, Theorem 3.1]) provides a density result of $\mathcal{W}(\Omega)$ in $SBV^2(\Omega) \cap L^\infty(\Omega)$, where by $SBV^2(\Omega)$ we denote

$$SBV^2(\Omega) = \{u \in SBV(\Omega) \colon \nabla u \in L^2(\Omega),\, \mathcal{H}^{n-1}(S_u) < +\infty\},$$

and it will be used to get the upper bound inequality in both Theorems 3.1 and 3.2.

Theorem 2.4. *Let $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$. Then there exists a sequence $(u_j) \subset \mathcal{W}(\Omega)$ such that $u_j \to u$ in $L^1(\Omega)$, $\nabla u_j \to \nabla u$ in $L^2(\Omega)$, $\limsup_{j\to+\infty} \|u_j\|_\infty \le \|u\|_\infty$ and*

$$\limsup_{j\to+\infty} \int_{S_{u_j}} \phi(u_j^+, u_j^-, \nu_{u_j})\, d\mathcal{H}^{n-1} \le \int_{S_u} \phi(u^+, u^-, \nu_u)\, d\mathcal{H}^{n-1}$$

for every upper semicontinuous function $\phi \colon \mathbb{R} \times \mathbb{R} \times S^{n-1} \to [0, +\infty)$ such that $\phi(a, b, \nu) = \phi(b, a, -\nu)$ for every a, $b \in \mathbb{R}$ and $\nu \in S^{n-1}$.

2.4. **A relaxation result.** We state here a relaxation result, due to Fonseca and Leoni (see [15, Theorem 1.8]), which will be crucial to obtain the lower bound inequality in Theorem 3.2.

Let $N \ge 1$ and let $f \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to [0, +\infty)$ be a Borel function. For any $(x, w, z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$ we define the recession function of f as

$$f^\infty(x, w, z) := \limsup_{t\to+\infty} \frac{f(x, w, tz)}{t}.$$

Theorem 2.5. *Let $N \ge 1$ and let $f \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to [0, +\infty)$ be a Borel function. Assume that*

(i) for all $(x_0, w_0) \in \Omega \times R^N$ $f(x_0, w_0, \cdot)$ is convex in $\mathbb{R}^{nN}$;
(ii) for all $(x_0, w_0) \in \Omega \times \mathbb{R}^N$ either $f(x_0, w_0, z) \equiv 0$ for all $z \in \mathbb{R}^{nN}$, or for every $\eta > 0$ there exist c_1, $c_2, \delta > 0$ such that

$$f(x_0, w_0, z) - f(x, w, z) \le \eta(1 + f(x, w, z))$$

$$f(x, w, z) \ge c_2|z| + c_1$$

for all $(x, w) \in \Omega \times \mathbb{R}^N$ with $|x - x_0| + |w - w_0| \le \delta$ and for all $z \in \mathbb{R}^{nN}$.

Consider the functional $F \colon L^1(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ defined by

$$F(w, A) := \begin{cases} \displaystyle\int_A f(x, w, \nabla w)\, dx & w \in W^{1,1}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then for $w \in BV(\Omega)$ we get

$$\overline{F}(w, A) \ge \int_A f(x, w, \nabla w)\, dx + \int_A f^\infty\left(x, w, \frac{dD^c w}{d|D^c w|}\right) d|D^c w|.$$

2.5. **Interpolation inequalities and elliptic regularity estimate.** As we will heavily use them in what follows, we recall here two interpolation inequalities (see e.g. [1, Theorems 4.14 and 4.15] and [19, Theorem 3.1.2.1 and Remark 3.1.2.2]).

Proposition 2.6. *Let U be an open bounded subset of $\mathbb{R}^n$ and let $\varepsilon_0 > 0$.*

(i) If U has Lipschitz boundary, then there exists a constant $c_0(\varepsilon_0, U) > 0$ such that

$$c_0\,\varepsilon \int_U |\nabla v|^2\,dx \le \frac{1}{\varepsilon}\int_U (1-v)^2\,dx + \varepsilon^3 \int_U |\nabla^2 v|^2\,dx,$$

for every $\varepsilon \in (0, \varepsilon_0]$ and for every $v \in W^{2,2}(U)$.

(ii) If U has C^2-boundary, then there exists a constant $c_0(\varepsilon_0, U) > 0$ such that

$$c_0\,\varepsilon \int_U |\nabla v|^2\,dx \le \frac{1}{\varepsilon}\int_U (1-v)^2\,dx + \varepsilon^3 \int_U |\Delta v|^2\,dx,$$

for every $\varepsilon \in (0, \varepsilon_0]$ and for every $v \in W^{2,2}(U)$ with $(1-v) \in W_0^{1,2}(U)$.

Moreover, we also recall a local *a priori* estimate for the Laplace operator (see [13, Theorem 1, Section 6.3.1]) that we will use in Section 4.

Proposition 2.7. *Let U be an open bounded subset of $\mathbb{R}^n$. Then for each open subset $V \subset\subset U$ there exists a constant $c(U, V) > 0$ such that*

$$\|v\|_{W^{2,2}(V)} \le c(U,V)\left(\|\Delta v\|_{L^2(U)} + \|v\|_{L^2(U)}\right),$$

for all $v \in W^{2,2}(U)$.

3. Statement of the main result

We consider the functionals F_ε and E_ε defined as

$$F_\varepsilon(u,v) := \begin{cases} \displaystyle\int_\Omega \left(v^2|\nabla u| + \frac{(1-v)^2}{\varepsilon} + \varepsilon^3|\nabla^2 v|^2\right) dx & u \in W^{1,1}(\Omega),\ v \in W^{2,2}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega), \end{cases} \tag{3.1}$$

and

$$E_\varepsilon(u,v) := \begin{cases} \displaystyle\int_\Omega \left(v^2|\nabla u| + \frac{(1-v)^2}{\varepsilon} + \varepsilon^3|\Delta v|^2\right) dx & u \in W^{1,1}(\Omega),\ v \in W^{2,2}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega). \end{cases} \tag{3.2}$$

Hereinafter the Γ-convergence of F_ε and E_ε is understood with respect to the strong topology of $L^1(\Omega) \times L^1(\Omega)$.

The first main result of this paper is a Γ-convergence result for the functionals F_ε.

Theorem 3.1. *The sequence (F_ε) defined as in (3.1) Γ-converges to the functional F defined as*

$$F(u,v) := \begin{cases} \displaystyle\int_\Omega |\nabla u|\,dx + \int_{S_u} g(|u^+ - u^-|)\,d\mathcal{H}^{n-1} + |D^c u|(\Omega) & u \in GBV(\Omega),\ v = 1 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega), \end{cases} \tag{3.3}$$

where g is given by

$$g(z) := \min_{r\in[0,1]} \{r^2\, z + 2\sqrt{2}(1-r)^2\} = \frac{4z}{4+\sqrt{2}z}. \tag{3.4}$$

An analogous result can be recovered on $GBV(\Omega)$ for the functionals E_ε, as stated in the following theorem.

Theorem 3.2. *For every* $u \in GBV(\Omega)$,

$$\Gamma\text{-}\lim_{\varepsilon\to 0} E_\varepsilon(u,1) = F(u,1),$$

with E_ε *and* F *defined as in* (3.2) *and* (3.3), *respectively.*

We may also consider the functionals $\mathscr{E}_\varepsilon$ defined as

$$\mathscr{E}_\varepsilon(u,v) := \begin{cases} E_\varepsilon(u,v) & (v-1) \in W_0^{1,2}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega). \end{cases} \tag{3.5}$$

Then, if Ω has C^2-boundary, Theorem 3.2 immediately yields the following theorem.

Theorem 3.3. *For every* u, $v \in L^1(\Omega)$,

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \mathscr{E}_\varepsilon(u,v) = F(u,v).$$

In fact it is sufficient to notice that, thanks to the boundary conditions on v and the increased regularity of Ω, we can now invoke Proposition 2.6(ii) to get

$$2\,\mathscr{E}_\varepsilon(u,v) \geq \int_\Omega \left(v^2|\nabla u| + \frac{(1-v)^2}{\varepsilon} + c_0\varepsilon|\nabla v|^2 \right) dx,$$

which, thanks to [2, Theorem 4.1], guarantees that the domain of the Γ-limit is contained in $GBV(\Omega) \times \{v = 1 \text{ a.e. in } \Omega\}$.

Remark 3.4. Let $r \in \mathbb{R}$. Consider the minimization problem

$$\mathbf{m}_r := \inf\left\{ \int_0^{+\infty} \left((f-1)^2 + (f'')^2\right) dt \colon f \in W^{2,2}_{\mathrm{loc}}(0,+\infty), \right.$$
$$\left. f(0) = r,\ f'(0) = 0,\ f(t) = 1 \text{ if } t > T, \text{ for some } T > 0 \right\}.$$

The constant $\mathbf{m}_r$ represents the minimal cost, in terms of the unscaled, one-dimensional Modica-Mortola contribution in (3.1) and (3.2), for a transition from the value r to the value 1 on the positive real half-line. A direct computation gives (see [8, Section 3])

$$\mathbf{m}_r = \min\left\{ \int_0^{+\infty} \left((f-1)^2 + (f'')^2\right) dt \colon f \in W^{2,2}_{\mathrm{loc}}(0,+\infty), \right.$$
$$\left. f(0) = r,\ f'(0) = 0,\ \lim_{t\to+\infty} f(t) = 1 \right\} = \sqrt{2}(1-r)^2. \tag{3.6}$$

Remark 3.5. It can be easily checked that the function g defined as in (3.4) satisfies the following properties:

(i) g is increasing, $g(0) = 0$ and $\lim_{z\to+\infty} g(z) = 2\sqrt{2}$;
(ii) g is subadditive;
(iii) $g(z) \leq z$ for all $z \in \mathbb{R}^+$ and $\lim_{z\to 0^+} \frac{g(z)}{z} = 1$;
(iv) g is Lipschitz continuous on $\mathbb{R}^+$ with Lipschitz constant 1;
(v) for any $T > 0$ there exists a constant $c_T > 0$ such that $z \leq c_T\, g(z)$ for all $z \in [0,T]$.

Remark 3.6. The functional $F(\cdot,1)$ with F defined as in (3.3) is continuous with respect to truncation in $GBV(\Omega)$. In fact let $u \in GBV(\Omega)$ and for $m > 0$ let u^m be the truncation of u at level m. Then by the properties of GBV functions (see Subsection 2.1) we immediately have

$$\lim_{m\to+\infty} \left(\int_\Omega |\nabla u^m|\, dx + |D^c u^m|(\Omega) \right) = \int_\Omega |\nabla u|\, dx + |D^c u|(\Omega).$$

Moreover, by virtue of (i) and (iv) in Remark 3.5, the Monotone Convergence Theorem together with (2.1)-(2.2) yields

$$\lim_{m\to+\infty}\int_{S_{u^m}} g(|(u^m)^+-(u^m)^-|)\,d\mathcal{H}^{n-1} = \lim_{m\to+\infty}\int_{S_u} g(|(u^m)^+-(u^m)^-|)\chi_{S_{u^m}}\,d\mathcal{H}^{n-1} = \int_{S_u} g(|u^+-u^-|)\,d\mathcal{H}^{n-1}.$$

4. Γ-convergence

In this section we study the asymptotic behavior of the functionals F_ε and E_ε. In particular Theorems 3.1 and 3.2 will follow from Proposition 4.1-4.5.

We start proving the lower bound inequality in the one-dimensional case, where F_ε and E_ε clearly coincide. The proof follows the line of that of [2, Proposition 4.3]; the main difference is that, due to the presence of the second derivative of v in the approximating functionals, we are not allowed to assume that $0 \le v \le 1$.

Proposition 4.1. *Let $n = 1$ and let F_ε and F be defined as in* (3.1) *and* (3.3)*, respectively. Then $F(u,v) \le \Gamma\text{-}\liminf_{\varepsilon\to0} F_\varepsilon(u,v)$ for all u, $v \in L^1(\Omega)$.*

Proof. For simplicity we suppose that there exist a, $b \in \mathbb{R}$, $a < b$ such that $\Omega = (a,b)$, the general case follows by repeating the same argument in each connected component of Ω.

Let u, $v \in L^1(a,b)$ and (u_ε), $(v_\varepsilon) \subset L^1(a,b)$ be such that $u_\varepsilon \to u$, $v_\varepsilon \to v$ in $L^1(a,b)$. We want to show that

$$\liminf_{\varepsilon\to0} F_\varepsilon(u_\varepsilon,v_\varepsilon) \ge F(u,v).$$

Clearly it is enough to consider the case $\lim_{\varepsilon\to0} F_\varepsilon(u_\varepsilon,v_\varepsilon) = \liminf_{\varepsilon\to0} F_\varepsilon(u_\varepsilon,v_\varepsilon) < +\infty$; we suppose moreover

$$u_\varepsilon \to u,\ v_\varepsilon \to v \quad \text{a.e. in } (a,b). \tag{4.1}$$

We begin noticing that $\|v_\varepsilon - 1\|_{L^2(a,b)} \le c\sqrt{\varepsilon}$ immediately gives $v = 1$ a.e. in (a,b). Moreover the interpolation inequality Proposition 2.6(i) yields $\varepsilon\|v'_\varepsilon\|_{L^1(a,b)} \to 0$ as $\varepsilon \to 0$; hence, up to subsequences (not relabeled),

$$\varepsilon\, v'_\varepsilon \to 0 \quad \text{a.e. in } (a,b). \tag{4.2}$$

Again appealing to Proposition 2.6(i) we deduce the existence of a positive constant c_0 such that for $\varepsilon > 0$ sufficiently small there holds

$$c_0\int_a^b \varepsilon(v'_\varepsilon)^2\,dx \le \int_a^b \frac{(v_\varepsilon-1)^2}{\varepsilon}\,dx + \int_a^b \varepsilon^3(v''_\varepsilon)^2\,dx.$$

Therefore for $\varepsilon > 0$ sufficiently small

$$\int_a^b \left(\frac{(v_\varepsilon-1)^2}{\varepsilon} + \varepsilon\,(v'_\varepsilon)^2\right) dx \le c;$$

hence, fixed $z \in (0,1)$, we can apply [6, Lemma 6.2 and Remark 6.3], with $Z = \{1\}$ and $W(s) = (1-s)^2$, to conclude that there exists a finite set S_z such that, for every fixed $\eta > 0$ and for $\varepsilon > 0$ sufficiently small, $z \le v_\varepsilon \le 2$ on $(a,b) \setminus (S_z + [-\eta,\eta])$. Let $\eta > 0$; then

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon)\\
&\quad \ge \liminf_{\varepsilon\to 0} \int_{(a,b)\setminus(S_z+[-\eta,\eta])} v_\varepsilon^2|u_\varepsilon'|\,dx + \liminf_{\varepsilon\to 0}\int_{S_z+[-\eta,\eta]} \left(v_\varepsilon^2|u_\varepsilon'| + \frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3(v_\varepsilon'')^2\right) dx\\
&\quad \ge z^2 \liminf_{\varepsilon\to 0} \int_{(a,b)\setminus(S_z+[-\eta,\eta])} |u_\varepsilon'|\,dx + \liminf_{\varepsilon\to 0}\int_{S_z+[-\eta,\eta]} \left(v_\varepsilon^2|u_\varepsilon'| + \frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3(v_\varepsilon'')^2\right) dx. \qquad (4.3)
\end{aligned}$$

In particular $\int_{(a,b)\setminus(S_z+[-\eta,\eta])} |u_\varepsilon'|\,dx$ is equibounded. Hence $u \in BV((a,b)\setminus(S_z+[-\eta,\eta]))$ and

$$\liminf_{\varepsilon\to 0} \int_{(a,b)\setminus(S_z+[-\eta,\eta])} |u_\varepsilon'|\,dx \ge |Du|((a,b)\setminus(S_z+[-\eta,\eta])). \qquad (4.4)$$

Moreover, by the arbitrariness of η, we have $u \in BV((a,b)\setminus S_z)$ and, since S_z is finite, $u \in BV(a,b)$.

Let $N := \mathcal{H}^0(S_z)$, $S_z = \{y_1,\dots,y_N\}$. We claim that

$$\liminf_{\varepsilon\to 0} \int_{y_i-\eta}^{y_i+\eta} \left(v_\varepsilon^2|u_\varepsilon'| + \frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3(v_\varepsilon'')^2\right) dx \ge g\left(\left| \operatorname*{ess-sup}_{(y_i-\eta,y_i+\eta)} u - \operatorname*{ess-inf}_{(y_i-\eta,y_i+\eta)} u\right|\right) \qquad (4.5)$$

for every $i = 1,\dots,N$.

Suppose for a moment that (4.5) holds true. Then (4.3)-(4.5) immediately give

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon) \ge z^2|Du|((a,b)\setminus(S_z+[-\eta,\eta])) + \sum_{i=1}^N g\left(\left| \operatorname*{ess-sup}_{(y_i-\eta,y_i+\eta)} u - \operatorname*{ess-inf}_{(y_i-\eta,y_i+\eta)} u\right|\right). \qquad (4.6)$$

Finally we first let $\eta \to 0$ in (4.6) to get

$$\begin{aligned}
\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon) &\ge z^2|Du|((a,b)\setminus S_z) + \sum_{i=1}^N g(|u^+-u^-|(y_i))\\
&\ge z^2|Du|((a,b)\setminus S_u) + z^2\sum_{y\in S_u\setminus S_z} |u^+-u^-|(y) + \sum_{S_u\cap S_z} g(|u^+-u^-|(y))\\
&\ge z^2|Du|((a,b)\setminus S_u) + \sum_{y\in S_u} \big((z^2|u^+-u^-|(y)) \wedge g(|u^+-u^-|(y))\big).
\end{aligned}$$

and then $z \to 1$, obtaining the required inequality, since $g(|t|) \le |t|$ (see Remark 3.5).

We prove now (4.5) for $i = 1,\dots,N$. Upon passing to subsequences (not relabeled), we may assume

$$\liminf_{\varepsilon\to 0} \int_{y_i-\eta}^{y_i+\eta} \left(v_\varepsilon^2|u_\varepsilon'| + \frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3(v_\varepsilon'')^2\right) dx = \lim_{\varepsilon\to 0} \int_{y_i-\eta}^{y_i+\eta} \left(v_\varepsilon^2|u_\varepsilon'| + \frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3(v_\varepsilon'')^2\right) dx.$$

By the very definition of essential infimum and essential supremum and by (4.1)-(4.2), we have that for any $\delta > 0$ there exist x_1^i, $x_2^i \in (y_i-\eta, y_i+\eta)$ such that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} u_\varepsilon(x_1^i) = u(x_1^i) < \operatorname*{ess-inf}_{(y_i-\eta,y_i+\eta)} u + \delta\\
&\lim_{\varepsilon\to 0} u_\varepsilon(x_2^i) = u(x_2^i) > \operatorname*{ess-sup}_{(y_i-\eta,y_i+\eta)} u - \delta\\
&\lim_{\varepsilon\to 0} v_\varepsilon(x_1^i) = \lim_{\varepsilon\to 0} v_\varepsilon(x_2^i) = 1\\
&\lim_{\varepsilon\to 0} \varepsilon v_\varepsilon'(x_1^i) = \lim_{\varepsilon\to 0} \varepsilon v_\varepsilon'(x_2^i) = 0.
\end{aligned} \qquad (4.7)$$

Let $x_\varepsilon^i \in [x_1^i, x_2^i]$ be such that $v_\varepsilon(x_\varepsilon^i) = \min_{[x_1^i,x_2^i]} v_\varepsilon$. We have now to distinguish three cases.

Case 1: $x^i_\varepsilon \in (x^i_1, x^i_2)$. In this case the regularity of v_ε yields $v'_\varepsilon(x^i_\varepsilon) = 0$. Moreover we have

$$\begin{aligned}&\int_{y_i-\eta}^{y_i+\eta}\left(v_\varepsilon^2|u'_\varepsilon| + \frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3(v''_\varepsilon)^2\right)dx \geq \int_{x^i_1}^{x^i_2}\left(v_\varepsilon^2|u'_\varepsilon| + \frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3(v''_\varepsilon)^2\right)dx\\ &\geq (\max\{0, v_\varepsilon(x^i_\varepsilon)\})^2\left|\int_{x^i_1}^{x^i_2} u'_\varepsilon\,dx\right| + \int_{x^i_1}^{x^i_\varepsilon}\left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3(v''_\varepsilon)^2\right)dx + \int_{x^i_\varepsilon}^{x^i_2}\left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3(v''_\varepsilon)^2\right)dx\\ &\geq (\max\{0, v_\varepsilon(x^i_\varepsilon)\})^2|u_\varepsilon(x^i_2) - u_\varepsilon(x^i_1)| + \int_{x^i_1}^{x^i_\varepsilon}\left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3(v''_\varepsilon)^2\right)dx + \int_{x^i_\varepsilon}^{x^i_2}\left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3(v''_\varepsilon)^2\right)dx.\end{aligned} \tag{4.8}$$

We estimate from below the term

$$\int_{x^i_\varepsilon}^{x^i_2}\left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3|v''_\varepsilon|^2\right)dx = \int_0^{\frac{x^i_2 - x^i_\varepsilon}{\varepsilon}}\left((w_\varepsilon - 1)^2 + (w''_\varepsilon)^2\right)dz,$$

where $z = (t - x^i_\varepsilon)/\varepsilon$ and $w_\varepsilon(z) := v_\varepsilon(\varepsilon z + x^i_\varepsilon)$.

To this end we introduce the auxiliary function $G : \mathbb{R}^2 \longrightarrow [0, +\infty)$ given by

$$G(w, z) := \inf\left\{\int_0^1\left((g-1)^2 + (g'')^2\right)dt \colon g \in C^2([0,1]),\ g(0) = w,\ g(1) = 1,\ g'(0) = z,\ g'(1) = 0\right\};$$

testing G with a third-degree polynomial satisfying the boundary conditions, one can easily show that

$$\lim_{(w,z)\to(1,0)} G(w, z) = 0. \tag{4.9}$$

Let $g_{\varepsilon,i} \in C^2([0,1])$ be an admissible function for $G(v_\varepsilon(x^i_2), \varepsilon\, v'_\varepsilon(x^i_2))$; *i.e.*, $g_{\varepsilon,i}(0) = v_\varepsilon(x^i_2)$, $g_{\varepsilon,i}(1) = 1$, $g'_{\varepsilon,i}(0) = \varepsilon\, v'_\varepsilon(x^i_2)$, $g'_{\varepsilon,i}(1) = 0$. By construction

$$\lim_{\varepsilon\to0} g_{\varepsilon,i}(0) = 1 \quad \text{and} \quad \lim_{\varepsilon\to0} g'_{\varepsilon,i}(0) = 0,$$

hence by (4.9) we infer

$$\lim_{\varepsilon\to0} G(v_\varepsilon(x^i_2), \varepsilon\, v'_\varepsilon(x^i_2)) = 0.$$

Let $(\tilde v_{\varepsilon,i})$ be the sequence defined as

$$\tilde v_{\varepsilon,i}(z) := \begin{cases} w_\varepsilon(z) & \text{if } 0 \leq z \leq \dfrac{x^i_2 - x^i_\varepsilon}{\varepsilon},\\ g_{\varepsilon,i}\left(z - \dfrac{x^i_2 - x^i_\varepsilon}{\varepsilon}\right) & \text{if } \dfrac{x^i_2 - x^i_\varepsilon}{\varepsilon} \leq z \leq \dfrac{x^i_2 - x^i_\varepsilon}{\varepsilon} + 1,\\ 1 & \text{if } z \geq \dfrac{x^i_2 - x^i_\varepsilon}{\varepsilon} + 1.\end{cases}$$

By definition of $g_{\varepsilon,i}$ it follows that $(\tilde v_{\varepsilon,i}) \subset W^{2,2}_{\rm loc}(0, +\infty)$. Since $\tilde v_{\varepsilon,i}$ is a test function for $\boldsymbol{m}_{v_\varepsilon(x^i_\varepsilon)}$ (where $\boldsymbol{m}_{v_\varepsilon(x^i_\varepsilon)}$ is as in (3.6) with $r = v_\varepsilon(x^i_\varepsilon)$), we have

$$\begin{aligned}\int_0^{\frac{x^i_2 - x^i_\varepsilon}{\varepsilon}}\left((w_\varepsilon - 1)^2 + (w''_\varepsilon)^2\right)dz &= \int_0^{+\infty}\left((\tilde v_{\varepsilon,i} - 1)^2 + (\tilde v''_{\varepsilon,i})^2\right)dz - G(v_\varepsilon(x^i_2), \varepsilon\, v'_\varepsilon(x^i_2))\\ &\geq \boldsymbol{m}_{v_\varepsilon(x^i_\varepsilon)} - G(v_\varepsilon(x^i_2), \varepsilon\, v'_\varepsilon(x^i_2)).\end{aligned}$$

A similar argument applies to the term

$$\int_{x^i_1}^{x^i_\varepsilon}\left(\frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3(v''_\varepsilon)^2\right)dt$$

in (4.8).

We have

$$\begin{aligned}
&\int_{y_i-\eta}^{y_i+\eta}\left(v_\varepsilon^2|u_\varepsilon'|+\frac{(1-v_\varepsilon)^2}{\varepsilon}+\varepsilon^3(v_\varepsilon'')^2\right)dx\\
\geq\;& (\max\{0,v_\varepsilon(x_\varepsilon^i)\})^2|u_\varepsilon(x_2^i)-u_\varepsilon(x_1^i)|+2\boldsymbol{m}_{v_\varepsilon(x_\varepsilon^i)}-G(v_\varepsilon(x_1^i),\varepsilon\, v_\varepsilon'(x_1^i))-G(v_\varepsilon(x_2^i),\varepsilon\, v_\varepsilon'(x_2^i))\\
\geq\;& \inf_{r\in\mathbb{R}}\{(\max\{0,r\})^2|u_\varepsilon(x_2^i)-u_\varepsilon(x_1^i)|+2\boldsymbol{m}_r\}-G(v_\varepsilon(x_1^i),\varepsilon\, v_\varepsilon'(x_1^i))-G(v_\varepsilon(x_2^i),\varepsilon\, v_\varepsilon'(x_2^i))\\
=\;& \min\left\{2\inf_{r\leq 0}\boldsymbol{m}_r,\inf_{r\geq 0}\{r^2|u_\varepsilon(x_2^i)-u_\varepsilon(x_1^i)|+2\boldsymbol{m}_r\}\right\}-G(v_\varepsilon(x_1^i),\varepsilon\, v_\varepsilon'(x_1^i))-G(v_\varepsilon(x_2^i),\varepsilon\, v_\varepsilon'(x_2^i))\\
=\;& \min\left\{2\sqrt{2},\frac{4|u_\varepsilon(x_2^i)-u_\varepsilon(x_1^i)|}{4+\sqrt{2}|u_\varepsilon(x_2^i)-u_\varepsilon(x_1^i)|}\right\}-G(v_\varepsilon(x_1^i),\varepsilon\, v_\varepsilon'(x_1^i))-G(v_\varepsilon(x_2^i),\varepsilon\, v_\varepsilon'(x_2^i))\\
=\;& \min_{r\in[0,1]}\{r^2|u_\varepsilon(x_2^i)-u_\varepsilon(x_1^i)|+2\boldsymbol{m}_r\}-G(v_\varepsilon(x_1^i),\varepsilon\, v_\varepsilon'(x_1^i))-G(v_\varepsilon(x_2^i),\varepsilon\, v_\varepsilon'(x_2^i)),
\end{aligned}$$

where the last equality follows from the definition (3.4) of g. By letting $\varepsilon\to 0$, we get

$$\liminf_{\varepsilon\to 0}\int_{y_i-\eta}^{y_i+\eta}\left(v_\varepsilon^2|u_\varepsilon'|+\frac{(1-v_\varepsilon)^2}{\varepsilon}+\varepsilon^3(v_\varepsilon'')^2\right)dx\geq\inf_{r\in[0,1]}\left\{r^2\left|\operatorname*{ess-sup}_{(y_i-\eta,y_i+\eta)}u-\operatorname*{ess-inf}_{(y_i-\eta,y_i+\eta)}u+2\delta\right|+2\boldsymbol{m}_r\right\}.$$

Thus, by the arbitrariness of δ, we obtain (4.5).

Case 2: there exists a subsequence of ε (not relabeled) such that $x_\varepsilon^i\equiv x_1^i$. In this case, we have

$$\begin{aligned}
&\int_{y_i-\eta}^{y_i+\eta}\left(v_\varepsilon^2|u_\varepsilon'|+\frac{(1-v_\varepsilon)^2}{\varepsilon}+\varepsilon^3(v_\varepsilon'')^2\right)dx\\
&\qquad\geq\int_{x_1^i}^{x_2^i}v_\varepsilon^2|u_\varepsilon'|\,dx\geq(\max\{0,v_\varepsilon(x_1^i)\})^2|u_\varepsilon(x_2^i)-u_\varepsilon(x_1^i)|=v_\varepsilon^2(x_1^i)|u_\varepsilon(x_2^i)-u_\varepsilon(x_1^i)|
\end{aligned}$$

where the last equality holds for ε sufficiently small by virtue of (4.7). Letting $\varepsilon\to 0$, by Remark 3.5(iii) and again (4.7), we get

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\int_{y_i-\eta}^{y_i+\eta}\left(v_\varepsilon^2|u_\varepsilon'|+\frac{(1-v_\varepsilon)^2}{\varepsilon}+\varepsilon^3(v_\varepsilon'')^2\right)dx&\geq\left|\operatorname*{ess-sup}_{(y_i-\eta,y_i+\eta)}u-\operatorname*{ess-inf}_{(y_i-\eta,y_i+\eta)}u+2\delta\right|\\
&\geq g\left(\left|\operatorname*{ess-sup}_{(y_i-\eta,y_i+\eta)}u-\operatorname*{ess-inf}_{(y_i-\eta,y_i+\eta)}u+2\delta\right|\right),
\end{aligned}$$

thus, by the arbitrariness of δ, (4.5).

Case 3: there exists a subsequence of ε (not relabeled) such that $x_\varepsilon^i\equiv x_2^i$. In this case we apply the same argument as for *Case 2*. □

We use now Proposition 4.1 to recover the lower bound for F_ε in dimension $n>1$, by means of the slicing method (see Subsection 2.2). As a preliminary step we localize the functionals F_ε by introducing an explicit dependence on the set of integration: for any $A\in\mathcal{A}(\Omega)$, we set

$$F_\varepsilon(u,v,A):=\begin{cases}\displaystyle\int_A\left(v^2|\nabla u|+\frac{(1-v)^2}{\varepsilon}+\varepsilon^3|\nabla^2 v|^2\right)dx & u\in W^{1,1}(A),\ v\in W^{2,2}(A),\\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega).\end{cases}$$

Proposition 4.2. *Let $n>1$ and let F_ε and F be defined as in* (3.1) *and* (3.3)*, respectively. Then $F(u,v)\leq\Gamma\text{-}\liminf_{\varepsilon\to 0}F_\varepsilon(u,v)$ for all u, $v\in L^1(\Omega)$.*

Proof. In what follows we use the notation introduced in Subsection 2.2.

Let $\xi \in S^{n-1}$ and $A \in \mathcal{A}(\Omega)$. We begin noticing that for any $u \in W^{1,1}(\Omega)$ and $v \in W^{2,2}(\Omega)$, Fubini's Theorem yields

$$F_\varepsilon(u,v,A) = \int_{\Pi^\xi}\left(\int_{A_{\xi,y}}\left(v^2(y+t\xi)|\nabla u(y+t\xi)| + \frac{(1-v(y+t\xi))^2}{\varepsilon} + \varepsilon^3|\nabla^2 v(y+t\xi)|^2\right)dt\right)d\mathcal{H}^{n-1}(y)$$

$$\geq \int_{\Pi^\xi}\left(\int_{A_{\xi,y}}\left(v^2(y+t\xi)|\langle\nabla u(y+t\xi),\xi\rangle| + \frac{(1-v(y+t\xi))^2}{\varepsilon} + \varepsilon^3|\langle\nabla^2 v(y+t\xi)\cdot\xi,\xi\rangle|^2\right)dt\right)d\mathcal{H}^{n-1}(y)$$

$$= \int_{\Pi^\xi}\mathscr{F}_\varepsilon(u_{\xi,y},v_{\xi,y},A_{\xi,y})\,d\mathcal{H}^{n-1}(y), \quad (4.10)$$

where $\mathscr{F}_\varepsilon$ is the one-dimensional functional defined by

$$\mathscr{F}_\varepsilon(w,z,I) := \begin{cases}\displaystyle\int_I\left(z^2|w'| + \frac{(1-z)^2}{\varepsilon} + \varepsilon^3(z'')^2\right)dx & w \in W^{1,1}(I),\ z \in W^{2,2}(I),\\ +\infty & \text{otherwise,}\end{cases}$$

for any w, $z \in L^1(I)$ and $I \subset \mathbb{R}$ open and bounded.

Let u, $v \in L^1(\Omega)$ and (u_ε), $(v_\varepsilon) \subset L^1(\Omega)$ be such that $u_\varepsilon \to u$, $v_\varepsilon \to v$ in $L^1(\Omega)$ and

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,v_\varepsilon) < +\infty. \quad (4.11)$$

Then $v = 1$ a.e. in Ω. Moreover, by Fubini's Theorem and Fatou's Lemma, $(u_\varepsilon)_{\xi,y} \to u_{\xi,y}$ and $(v_\varepsilon)_{\xi,y} \to 1$ in $L^1(\Omega_{\xi,y})$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. Therefore, appealing to Proposition 4.1 and taking into account (4.10), we have that $u_{\xi,y} \in BV(A_{\xi,y})$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ and

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,v_\varepsilon,A) \geq \liminf_{\varepsilon\to 0}\int_{\Pi^\xi}\mathscr{F}_\varepsilon((u_\varepsilon)_{\xi,y},(v_\varepsilon)_{\xi,y},A_{\xi,y})\,d\mathcal{H}^{n-1}(y)$$

$$\geq \int_{\Pi^\xi}\liminf_{\varepsilon\to 0}\mathscr{F}_\varepsilon((u_\varepsilon)_{\xi,y},(v_\varepsilon)_{\xi,y},A_{\xi,y})\,d\mathcal{H}^{n-1}(y)$$

$$\geq \int_{\Pi^\xi}\left(\int_{A_{\xi,y}}|u'_{\xi,y}|\,dt + \int_{S_{u_{\xi,y}}\cap A_{\xi,y}} g(|u^+_{\xi,y}-u^-_{\xi,y}|)\,d\mathcal{H}^0 + |D^c u_{\xi,y}|(A_{\xi,y})\right)d\mathcal{H}^{n-1}(y), \quad (4.12)$$

where in the second inequality we have used Fatou's Lemma.

Let $m > 0$ and consider the truncated functions u^m. Since g is increasing, it is clear that we decrease the last term in (4.12) if we replace u with u^m. Moreover, since $u^m \in L^\infty(\Omega)$, with $\|u^m\|_\infty \leq m$, by Remark 3.5(v), we have

$$|(u^m)^+ - (u^m)^-| \leq c_m\, g(|(u^m)^+ - (u^m)^-|),$$

for a suitable positive constant c_m depending only on m. Then, by (4.11) and (4.12) we have

$$\int_{\Pi^\xi}|D(u^m)_{\xi,y}|(A_{\xi,y})\,d\mathcal{H}^{n-1}(y) < +\infty.$$

Thus, applying Theorem 2.3, we get that $u^m \in BV(A)$ and

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u,1,A) \geq \int_A|\langle\nabla u^m,\xi\rangle|^2\,dx + \int_{S_{u^m}\cap A} g(|(u^m)^+ - (u^m)^-|)|\langle\xi,\nu_u\rangle|\,d\mathcal{H}^{n-1} + |\langle D^c u^m,\xi\rangle|(A) \quad (4.13)$$

where we have taken into account the arbitrariness of u_ε and v_ε.

Consider the superadditive increasing function μ on $\mathcal{A}(\Omega)$ defined by

$$\mu(A) := \Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u,1,A)$$

and the Radon measure

$$\lambda := \mathcal{L}^n + g(|(u^m)^+ - (u^m)^-|)\mathcal{H}^{n-1} \llcorner S_{u^m} + |D^c u^m|.$$

Fixed a sequence (ξ_i), dense in S^{n-1}, we have, by (4.13)

$$\mu(A) \geq \int_A \psi_i \, d\lambda$$

for all $i \in \mathbb{N}$, where

$$\psi_i(x) := \begin{cases} |\langle \nabla u^m(x), \xi_i \rangle| & \mathcal{L}^n\text{-a.e. in } \Omega, \\ |\langle \xi_i, \nu_u \rangle| & |D^c u^m|\text{-a.e. in } \Omega, \\ |\langle \xi_i, \nu_u \rangle| & \mathcal{H}^{n-1}\text{-a.e. in } S_{u^m}. \end{cases}$$

Hence applying [6, Lemma 15.2] we get

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u,v,A) \geq \int_A |\nabla u^m| \, dx + \int_{S_{u^m}\cap A} g(|(u^m)^+ - (u^m)^-|) \, d\mathcal{H}^{n-1} + |D^c u^m|(A),$$

thus, taking $A = \Omega$,

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u,v) \geq \int_\Omega |\nabla u^m| \, dx + \int_{S_{u^m}} g(|(u^m)^+ - (u^m)^-|) \, d\mathcal{H}^{n-1} + |D^c u^m|(\Omega). \tag{4.14}$$

Finally by the arbitrariness of $m > 0$, we conclude $u \in GBV(\Omega)$ and the thesis follows letting $m \to +\infty$ in (4.14). □

A different approach is needed instead to prove the lower bound inequality for E_ε in dimension $n > 1$. The slicing technique is in fact not anymore applicable, because of the symmetry breaking due to the presence of the Laplacian. We use then the blow-up method of Fonseca-Müller [16]. To this end it is convenient to localize the functionals E_ε by defining for any $A \in \mathcal{A}(\Omega)$

$$E_\varepsilon(u,v,A) := \begin{cases} \displaystyle\int_A \left(v^2|\nabla u| + \frac{(1-v)^2}{\varepsilon} + \varepsilon^3|\Delta v|^2 \right) dx & u \in W^{1,1}(A),\ v \in W^{2,2}(A), \\ +\infty & \text{otherwise in } L^1(\Omega) \times L^1(\Omega). \end{cases}$$

Proposition 4.3. *Let $n > 1$ and let E_ε and F be defined as in* (3.2) *and* (3.3)*, respectively. Then $F(u,1) \leq \Gamma\text{-}\liminf_{\varepsilon\to 0} E_\varepsilon(u,1)$ for all $u \in GBV(\Omega)$.*

Proof. First let $u \in BV(\Omega)$. Let moreover (u_ε), $(v_\varepsilon) \subset L^1(\Omega)$ be such that $u_\varepsilon \to u$ in $L^1(\Omega)$, $v_\varepsilon \to 1$ in $L^2(\Omega)$ and

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty.$$

For each $\varepsilon > 0$ consider the measures

$$\mu_\varepsilon := \left(v_\varepsilon^2|\nabla u_\varepsilon| + \frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3|\Delta v_\varepsilon|^2 \right) \mathcal{L}^n \llcorner \Omega.$$

By hypothesis $\mu_\varepsilon(\Omega) = E_\varepsilon(u_\varepsilon, v_\varepsilon)$ is equibounded therefore, up to subsequences (not relabeled), $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ where μ is a non-negative finite Radon measure on Ω. Using the Radon-Nikodým Theorem we decompose μ into the sum of four mutually orthogonal measures

$$\mu = \mu_a \, \mathcal{L}^n + \mu_c \, |D^c u| + \mu_J \, \mathcal{H}^{n-1} \llcorner S_u + \mu_s$$

and we claim that

$$\mu_a(x_0) \geq |\nabla u(x_0)| \ \text{ for } \mathcal{L}^n\text{-a.e. } x_0 \in \Omega \tag{4.15}$$

$$\mu_c(x_0) \geq 1 \ \text{ for } |D^c u|\text{-a.e. } x_0 \in \Omega \tag{4.16}$$

$$\mu_J(x_0) \geq g(|u^+ - u^-|(x_0)) \ \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x_0 \in S_u. \tag{4.17}$$

Suppose for a moment that (4.15)-(4.17) hold true, then to conclude it is enough to consider an increasing sequence of smooth cut-off functions (φ_k), such that $0 \le \varphi_k \le 1$ and $\sup_k \varphi_k(x) = 1$ on Ω, and to note that for every $k \in \mathbb{N}$

$$\begin{aligned}\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon) &\ge \liminf_{\varepsilon\to 0} \int_\Omega \left(v_\varepsilon^2 |\nabla u_\varepsilon| + \frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3 |\Delta v_\varepsilon|^2 \right) \varphi_k \, dx \\ &= \int_\Omega \varphi_k \, d\mu \ge \int_\Omega \mu_a \, \varphi_k \, dx + \int_\Omega \mu_c \, \varphi_k \, d|D^c u| + \int_{S_u} \mu_J \, \varphi_k \, d\mathcal{H}^{n-1} \\ &\ge \int_\Omega |\nabla u| \, \varphi_k \, dx + \int_\Omega \varphi_k \, d|D^c u| + \int_{S_u} g(|u^+ - u^-|) \, \varphi_k \, d\mathcal{H}^{n-1}.\end{aligned}$$

Hence, letting $k \to +\infty$ the thesis follows from the Monotone Convergence Theorem.

We have now to check (4.15)-(4.17). The proof of (4.17) follows exactly the proof of (5.3) in [8, Proposition 5.1], the only difference residing in the fact that here we appeal to the one-dimensional result Proposition 4.1.

We prove now (4.15) and (4.16). To this end we define the function $\Phi\colon [0,1] \to [0,+\infty)$ by

$$\Phi(t) := 2\int_0^t (1-s)\, ds = 1 - (1-t)^2; \tag{4.18}$$

we notice that Φ is strictly increasing, $\Phi(0) = 0$ and $\Phi(1) = 1$.

Let $A \in \mathcal{A}(\Omega)$. Fix $A' \in \mathcal{A}(\Omega)$ such that $A' \subset\subset A$; appealing to Propositions 2.7 and 2.6 we deduce that there exists a positive constant c_0 such that for $\varepsilon > 0$ sufficiently small there holds

$$\begin{aligned}E_\varepsilon(u_\varepsilon, v_\varepsilon, A) &= \int_A \left(v_\varepsilon^2 |\nabla u| + \frac{(1-v)^2}{\varepsilon} + \varepsilon^3 |\Delta v_\varepsilon|^2 \right) dx \\ &\ge \int_{A'} v_\varepsilon^2 |\nabla u| \, dx + c_0 \int_{A'} \left(\frac{(1-v)^2}{\varepsilon} + \varepsilon |\nabla v_\varepsilon|^2 \right) dx \\ &\ge \int_{A'} \tilde v_\varepsilon^2 |\nabla u| \, dx + c_0 \int_{A'} \left(\frac{(1-\tilde v)^2}{\varepsilon} + \varepsilon |\nabla \tilde v_\varepsilon|^2 \right) dx,\end{aligned}$$

where $\tilde v_\varepsilon := 0 \vee (v_\varepsilon \wedge 1)$. Moreover for every $\varepsilon > 0$ Young's inequality yields

$$\int_{A'} \left(\frac{(1-\tilde v_\varepsilon)^2}{\varepsilon} + \varepsilon |\nabla \tilde v_\varepsilon|^2 \right) dx \ge \int_{A'} |\nabla \Phi(\tilde v_\varepsilon)| \, dx,$$

with Φ as in (4.18). Hence we have

$$E_\varepsilon(u_\varepsilon, v_\varepsilon, A) \ge \int_{A'} f(x, (u_\varepsilon, \Phi(\tilde v_\varepsilon)), \nabla(u_\varepsilon, \Phi(\tilde v_\varepsilon))) \, dx,$$

with $f\colon \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2n} \to [0,+\infty)$ defined by

$$f(x,(u,v),(z,\zeta)) := \left(\Phi^{-1}(0 \vee (v \wedge 1))\right)^2 (|z| + c_0|\zeta|).$$

It is easy to check that f satisfies all the hypotheses of Theorem 2.5 with $N = 2$, then we deduce that

$$\begin{aligned}\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon, A) &\ge \liminf_{\varepsilon\to 0} \int_{A'} f(x, (u_\varepsilon, \Phi(\tilde v_\varepsilon)), \nabla(u_\varepsilon, \Phi(\tilde v_\varepsilon))) \, dx \\ &\ge \int_{A'} f(x,(u,1),\nabla(u,1)) \, dx + \int_{A'} f^\infty \left(x, (u,1), \frac{dD^c(u,1)}{d|D^c(u,1)|} \right) d|D^c(u,1)| \\ &= \int_{A'} |\nabla u| \, dx + |D^c u|(A'),\end{aligned}$$

where we have used the fact that $\Phi(\tilde{v}_\varepsilon) \to 1$ in $L^1(\Omega)$, as $\tilde{v}_\varepsilon \to 1$ in $L^2(\Omega)$. Since u belongs to $BV(\Omega)$, letting $A' \nearrow A$ gives

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon, A) \geq \int_A |\nabla u|\, dx + |D^c u|(A).$$

From this, by using the Besicovitch derivation Theorem, we immediately get (4.15) and (4.16), and this completes the proof for $u \in BV(\Omega)$.

If now $u \in GBV(\Omega)$ the thesis follows by a standard truncation argument. In fact the truncations u^m of u belong to $BV(\Omega)$ for all $m > 0$. Appealing to the continuity under truncation of $F(\cdot, 1)$ in $GBV(\Omega)$ (see Remark 3.6) and noticing that $E_\varepsilon(\cdot, v)$ (and hence $\Gamma\text{-}\liminf_{\varepsilon\to 0} E_\varepsilon(\cdot, v)$) decreases by truncation, we immediately get

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} E_\varepsilon(u, 1) \geq \liminf_{m\to+\infty} F(u^m, 1) \geq F(u, 1).$$

□

We prove now the upper bound inequality for F_ε and E_ε. To do this, we will use the density and relaxation results introduced in Subsection 2.3 and Subsection 2.4.

Proposition 4.4. *Let $n \geq 1$ and let F_ε and F be defined as in* (3.1) *and* (3.3)*, respectively. Then $F(u, v) \geq \Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u, v)$ for all u, $v \in L^1(\Omega)$.*

Proof. To check the upper bound inequality, it suffices to deal with $u \in GBV(\Omega)$ and $v = 1$ a.e. in Ω.

We divide the proof into three main steps.

Step 1: $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$. We prove that

$$F(u, 1) \geq \Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u, 1) \tag{4.19}$$

for all $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$.

By Theorem 2.4 it is enough to prove (4.19) when u belongs to $\mathcal{W}(\Omega)$. Indeed assume (4.19) holds true in $\mathcal{W}(\Omega)$. If $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$ then there exists a sequence $(u_j) \subset \mathcal{W}(\Omega)$ such that $u_j \to u$ in $L^1(\Omega)$ and

$$\limsup_{j\to+\infty} F(u_j, 1) \leq F(u, 1);$$

hence the lower semicontinuity of $\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(\cdot, 1)$ yields

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u, 1) \leq \liminf_{j\to+\infty} \Big(\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u_j, 1)\Big) \leq \liminf_{j\to+\infty} F(u_j, 1) \leq F(u, 1). \tag{4.20}$$

We now prove (4.19) for a function $u \in \mathcal{W}(\Omega)$; we suppose $\overline{S}_u = \Omega \cap K$, with K a $(n-1)$-dimensional simplex (the case $\overline{S}_u = \Omega \cap \cup_{i=1}^r K_i$, with K_i pairwise disjoint $(n-1)$-dimensional simplexes, following as a natural generalization). Upon making a rototranslation, we may assume K to be contained in the plane $\{x_n = 0\}$.

For $y \in \overline{S}_u$, we set

$$h(y) := |u^+(y) - u^-(y)|$$

and appealing to the regularity hypotheses on u we have that for any $\delta > 0$ there exists a triangulation $\{T_i\}_{i=1}^N$ of $\overline{S}_u$ such that

$$|h(y_1) - h(y_2)| < \delta$$

for every y_1, $y_2 \in T_i$. We consider moreover the piecewise constant function $h_\delta \colon \overline{S}_u \to \mathbb{R}$ defined by

$$h_\delta(y) := \min\{h(s) \colon s \in \overline{T_i}\} =: z_i, \quad y \in T_i.$$

Then Remark 3.5 together with the fact that $\|h - h_\delta\|_\infty < \delta$ gives

$$\int_{S_u} g(h_\delta(y))\, d\mathcal{H}^{n-1}(y) \leq \int_{S_u} g(h(y))\, d\mathcal{H}^{n-1}(y) + \delta\mathcal{H}^{n-1}(\overline{S}_u). \tag{4.21}$$

Let x_{z_i} realize the minimum in (3.4) for $z = z_i$; *i.e.*, $x_{z_i} \in [0,1]$ and

$$g(z_i) = x_{z_i}^2 z_i + 2\sqrt{2}(1 - x_{z_i})^2. \tag{4.22}$$

Fix $\eta > 0$. Then for $i = 1,\dots,N$ by virtue of (3.6) we deduce that there exist $T(i,\eta) > 0$ and $v_\eta(z_i;\cdot) \in W^{2,2}(0, T(i,\eta))$ such that $v_\eta(z_i;0) = x_{z_i}$, $v'_\eta(z_i;0) = 0$, $v_\eta(z_i;t) = 1$ for all $t \geq T(i,\eta)$ and

$$\int_0^{T(i,\eta)} \left((1 - v_\eta(z_i;t))^2 + (v''_\eta(z_i;t))^2\right) dt \leq 2\sqrt{2}(1 - x_{z_i})^2 + \eta.$$

If now $T(\eta) := \max_{1\leq i \leq N} T(i,\eta)$, we have

$$\int_0^{T} \left((1 - v_\eta(z_i;t))^2 + (v''_\eta(z_i;t))^2\right) dt \leq 2\sqrt{2}(1 - x_{z_i})^2 + \eta, \tag{4.23}$$

for every $T \geq T(\eta)$ and $i \in \{1,\dots,N\}$.

We now construct the sequences (u_ε) and (v_ε) that we expect to be the recovery sequences for $F(u,1)$.

Let $\xi_\varepsilon > 0$ be such that $\xi_\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0$; set $T_\varepsilon := (T(\eta)+1)\varepsilon + \xi_\varepsilon$ and

$$K_\varepsilon := \{y \in \{x_n = 0\} \colon \mathrm{dist}(y,K) < \varepsilon\}.$$

Let $\gamma_\varepsilon \colon \mathbb{R}^{n-1} \to \mathbb{R}$ be a cut-off function between K and K_ε; *i.e.*, $\gamma_\varepsilon \in C_c^\infty(K_\varepsilon)$, $0 \leq \gamma_\varepsilon \leq 1$, $\gamma_\varepsilon \equiv 1$ on K with $\|\nabla \gamma_\varepsilon\|_\infty \leq \frac{c}{\varepsilon}$ for some $c > 0$, and let $(\tilde{u}_\varepsilon)$ be the sequence defined by

$$\tilde{u}_\varepsilon(y,t) := \begin{cases} u(y,t) & |t| \geq T_\varepsilon, \\ w_\varepsilon(u(y,-T_\varepsilon), u(y,T_\varepsilon), t) & |t| < T_\varepsilon, \end{cases}$$

where

$$w_\varepsilon(z_1,z_2,t) := \begin{cases} z_1 & -T_\varepsilon < t < -\xi_\varepsilon, \\ \dfrac{z_2 - z_1}{2\xi_\varepsilon}(t + \xi_\varepsilon) + z_1 & |t| \leq \xi_\varepsilon, \\ z_2 & \xi_\varepsilon < t < T_\varepsilon. \end{cases}$$

For $(y,t) \in \Omega$ we set

$$u_\varepsilon(y,t) := \tilde{u}_\varepsilon(y,t)\gamma_\varepsilon(y) + (1 - \gamma_\varepsilon(y))u(y,t).$$

For $r > 0$, $\varepsilon > 0$ and $i \in \{1,\dots,N\}$, set

$$B_r := \{(y,t) \in \Omega \colon y \in \overline{S}_u,\ |t| < r\}$$

$$T_i^\varepsilon := \{y \in T_i \colon \mathrm{dist}(y, \partial T_i) > \varepsilon\}$$

and let $\varphi_\varepsilon^i \colon \mathbb{R}^{n-1} \to \mathbb{R}$ be a cut-off function between T_i^ε and T_i; *i.e.*, $\varphi_\varepsilon^i \in C_c^\infty(T_i)$, $0 \leq \varphi_\varepsilon^i \leq 1$, $\varphi_\varepsilon^i \equiv 1$ on T_i^ε with

$$\|\nabla \varphi_\varepsilon^i\|_\infty \leq \frac{c}{\varepsilon}, \tag{4.24}$$

$$\|\nabla^2 \varphi_\varepsilon^i\|_\infty \leq \frac{c}{\varepsilon^2}, \tag{4.25}$$

for some $c > 0$.

We define the sequence

$$v_\varepsilon(y,t) := \begin{cases} 1 & (y,t) \in \Omega \setminus B_{T_\varepsilon}, \\ \phi_\varepsilon^i(y) v_\varepsilon^i(t) + (1 - \phi_\varepsilon^i(y)) & y \in T_i,\ |t| < T_\varepsilon, \end{cases}$$

where

$$v_\varepsilon^i(t) := \begin{cases} x_{z_i} & |t| \leq \xi_\varepsilon, \\ v_\eta\left(z_i; \dfrac{|t| - \xi_\varepsilon}{\varepsilon}\right) & \xi_\varepsilon < |t| < T_\varepsilon. \end{cases}$$

We have that $(u_\varepsilon) \subset W^{1,1}(\Omega)$, $(v_\varepsilon) \subset W^{2,2}(\Omega)$; moreover $u_\varepsilon \to u$ and $v_\varepsilon \to 1$ in $L^1(\Omega)$.

We now prove that u_ε and v_ε satisfy the upper bound inequality.
As $u \in W^{1,\infty}(\Omega \setminus \overline{S}_u)$ and $0 \le \varphi^i_\varepsilon \le 1$, we have

$$\begin{aligned}
&\int_\Omega v_\varepsilon^2 |\nabla u_\varepsilon|\, dx \\
= \quad & \int_{\Omega\setminus(K_\varepsilon\times(-T_\varepsilon,T_\varepsilon))} |\nabla u|\, dx + \int_{\Omega\cap(K_\varepsilon\setminus K)} \left(\int_{-T_\varepsilon}^{T_\varepsilon} \Big(|\nabla\gamma_\varepsilon(y)||\tilde u_\varepsilon(y,t) - u(y,t)| \right.\\
&\qquad \left. + \gamma_\varepsilon(y)|\nabla \tilde u_\varepsilon(y,t)| + (1-\gamma_\varepsilon(y))|\nabla u(y,t)| \Big)\, dt \right) d\mathcal{H}^{n-1}(y) \\
&+ \sum_{i=1}^N \int_{T_i^\varepsilon} \left(\int_{-\xi_\varepsilon}^{\xi_\varepsilon} x_{z_i}^2 \left| \frac{t+\xi_\varepsilon}{2\xi_\varepsilon}(\nabla u(y,t_\varepsilon) - \nabla u(y,-T_\varepsilon)) + \nabla u(y,-T_\varepsilon) \right.\right.\\
&\qquad \left.\left. + \frac{1}{2\xi_\varepsilon}(u(y,T_\varepsilon) - u(y,-T_\varepsilon)) e_n \right| dt \right) d\mathcal{H}^{n-1}(y) \\
&+ \sum_{i=1}^N \int_{T_i^\varepsilon} \left(\int_{\xi_\varepsilon}^{T_\varepsilon} v_\eta^2 \left(z_i; \frac{t-\xi_\varepsilon}{\varepsilon} \right) |\nabla u(y,T_\varepsilon)|\, dt \right) d\mathcal{H}^{n-1}(y) \\
&+ \sum_{i=1}^N \int_{T_i^\varepsilon} \left(\int_{-T_\varepsilon}^{-\xi_\varepsilon} v_\eta^2 \left(z_i; \frac{-t-\xi_\varepsilon}{\varepsilon} \right) |\nabla u(y,-T_\varepsilon)|\, dt \right) d\mathcal{H}^{n-1}(y) \\
&+ \sum_{i=1}^N \int_{T_i\setminus T_i^\varepsilon} \left((\varphi^i_\varepsilon(y) x_{z_i} + 1 - \varphi^i_\varepsilon(y))^2 \left| \frac{t+\xi_\varepsilon}{2\xi_\varepsilon}(\nabla u(y,t_\varepsilon) - \nabla u(y,-T_\varepsilon)) + \nabla u(y,-T_\varepsilon) \right.\right.\\
&\qquad \left.\left. + \frac{1}{2\xi_\varepsilon}(u(y,T_\varepsilon) - u(y,-T_\varepsilon)) e_n \right| dt \right) d\mathcal{H}^{n-1}(y) \\
&+ \sum_{i=1}^N \int_{T_i\setminus T_i^\eta} \left(\int_{\xi_\varepsilon}^{T_\varepsilon} \left(\varphi^i_\varepsilon(y) v_\eta \left(z_i; \frac{t-\xi_\varepsilon}{\varepsilon} \right) + 1 - \varphi^i_\varepsilon(y) \right)^2 |\nabla u(y,T_\varepsilon)|\, dt \right) d\mathcal{H}^{n-1}(y) \\
&+ \sum_{i=1}^N \int_{T_i\setminus T_i^\eta} \left(\int_{-T_\varepsilon}^{-\xi_\varepsilon} \left(\varphi^i_\varepsilon(y) v_\eta \left(z_i; \frac{-t-\xi_\varepsilon}{\varepsilon} \right) + 1 - \varphi^i_\varepsilon(y) \right)^2 |\nabla u(y,-T_\varepsilon)|\, dt \right) d\mathcal{H}^{n-1}(y) \\
\le \quad & \int_\Omega |\nabla u|\, dx + \sum_{i=1}^N \int_{T_i^\varepsilon} \left(\int_{-\xi_\varepsilon}^{\xi_\varepsilon} \frac{1}{2\xi_\varepsilon} x_{z_i}^2 |u(y,T_\varepsilon) - u(y,-T_\varepsilon)|\, dt \right) d\mathcal{H}^{n-1}(y) + O(\varepsilon) \\
\le \quad & \int_\Omega |\nabla u|\, dx + \sum_{i=1}^N \int_{T_i} x_{z_i}^2 |u^+(y) - u^-(y)|\, d\mathcal{H}^{n-1}(y) + O(\varepsilon), \qquad (4.26)
\end{aligned}$$

where we have used that ξ_ε, T_ε, $\mathcal{H}^{n-1}(T_i \setminus T_i^\varepsilon) \to 0$ as ε decreases to 0 for all $i = 1, \dots, N$.

Moreover, by virtue of (4.23)-(4.25) and since $\xi_\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0$, we have also

$$\begin{aligned}
&\int_\Omega \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3 |\nabla^2 v_\varepsilon|^2 \right) dx \\
= &\sum_{i=1}^N \int_{T_i^\varepsilon} \left(\int_{-\xi_\varepsilon}^{\xi_\varepsilon} \frac{(1-x_{z_i})^2}{\varepsilon}\, dt \right) d\mathcal{H}^{n-1}(y) \\
&+\frac{2}{\varepsilon} \sum_{i=1}^N \int_{T_i^\varepsilon} \left(\int_{\xi_\varepsilon}^{T_\varepsilon} \left(\left(1 - v_\eta\left(z_i; \frac{t-\xi_\varepsilon}{\varepsilon}\right)\right)^2 + v_\eta^2\left(z_i; \frac{t-\xi_\varepsilon}{\varepsilon}\right)\right) dt \right) d\mathcal{H}^{n-1}(y) \\
&+2 \sum_{i=1}^N \int_{T_i \setminus T_i^\varepsilon} \left(\int_{\xi_\varepsilon}^{T_\varepsilon} \left(\frac{1}{\varepsilon}(\varphi_\varepsilon^i(y))^2 \left(1 - v_\eta\left(z_i; \frac{t-\xi_\varepsilon}{\varepsilon}\right)\right)^2 + \varepsilon^3 \left| \nabla^2 \varphi_\varepsilon^i(y) \left(1 - v_\eta\left(z_i; \frac{t-\xi_\varepsilon}{\varepsilon}\right)\right) \right.\right.\right. \\
&\qquad \left.\left.\left. + \frac{2}{\varepsilon} \nabla \varphi_\varepsilon^i(y) v_\eta'\left(z_i; \frac{t-\xi_\varepsilon}{\varepsilon}\right) + \frac{1}{\varepsilon^2} \varphi_\varepsilon^i(y) v_\eta''\left(z_i; \frac{t-\xi_\varepsilon}{\varepsilon}\right) \right|^2 \right) dt \right) d\mathcal{H}^{n-1}(y) \\
&+\sum_{i=1}^N \int_{T_i \setminus T_i^\varepsilon} \left(\int_{-\xi_\varepsilon}^{\xi_\varepsilon} \left(\frac{1}{\varepsilon}(\varphi_\varepsilon^i(y))^2 (1-x_{z_i})^2 + \varepsilon^3 \left| \nabla^2 \varphi_\varepsilon^i(y)(1-x_{z_i}) \right|^2 \right) dt \right) d\mathcal{H}^{n-1}(y) \\
\le &\sum_{i=1}^N \int_{T_i} 2\sqrt{2}(1-x_{z_i})^2\, d\mathcal{H}^{n-1}(y) + c\eta + O(\varepsilon). \qquad (4.27)
\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$, by (4.26) and (4.27), we get

$$\begin{aligned}
\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u,1) &\le \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon) \\
&\le \int_\Omega |\nabla u|\, dx + \sum_{i=1}^N \int_{T_i} \left(x_{z_i}^2 |u^+(y) - u^-(y)| + 2\sqrt{2}(1-x_{z_i})^2 \right) d\mathcal{H}^{n-1}(y) + c\eta \\
&\le \int_\Omega |\nabla u|\, dx + \sum_{i=1}^N \int_{T_i} \left(x_{z_i}^2 z_i + 2\sqrt{2}(1-x_{z_i})^2 \right) d\mathcal{H}^{n-1}(y) + c(\eta+\delta) \\
&= \int_\Omega |\nabla u|\, dx + \sum_{i=1}^N \int_{T_i} g(h_\delta(y))\, d\mathcal{H}^{n-1}(y) + c(\eta+\delta) \\
&\le \int_\Omega |\nabla u|\, dx + \sum_{i=1}^N \int_{T_i} g(h(y))\, d\mathcal{H}^{n-1}(y) + c(\eta+\delta) \\
&= \int_\Omega |\nabla u|\, dx + \sum_{i=1}^N \int_{T_i} g(|u^+ - u^-|(y))\, d\mathcal{H}^{n-1}(y) + c(\eta+\delta),
\end{aligned}$$

where we have also used that $\|h - h_\delta\|_\infty < \delta$ together with (4.21)-(4.22). We finally let η and δ go to 0 to obtain the required inequality. This concludes the case $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$.

Step 2: $u \in BV(\Omega)$. We now claim that the relaxation $\overline{E}$ with respect to the strong $L^1(\Omega)$-topology of the functional

$$E(u) := \begin{cases} F(u,1) & u \in SBV^2(\Omega) \cap L^\infty(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

satisfies

$$\overline{E}(u) \le F(u,1) \qquad (4.28)$$

for all $u \in BV(\Omega)$.

Suppose for a moment that (4.28) holds true. Then by virtue of (4.20) and (4.28) we have

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u,1) \le \overline{E}(u) \le F(u,1)$$

for all $u \in BV(\Omega)$, hence the limsup inequality in $BV(\Omega)$.

We now prove (4.28). To this end for $u \in L^1(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we consider the functionals

$$E_1(u,A) := \begin{cases} \displaystyle\int_A |\nabla u|\, dx & u \in C^1(\Omega) \\ +\infty & \text{otherwise} \end{cases}, \qquad E_2(u,A) := \begin{cases} \displaystyle\int_A |\nabla u|\, dx & u \in C^1(\overline{\Omega}) \\ +\infty & \text{otherwise} \end{cases}$$

and their relaxations

$$\overline{E}_1(u,A) = \inf\left\{ \liminf_{k\to+\infty} E_1(u_k,A) \colon u_k \to u \text{ in } L^1(A) \right\}$$

$$\overline{E}_2(u,A) = \inf\left\{ \liminf_{k\to+\infty} E_2(u_k,A) \colon u_k \to u \text{ in } L^1(A) \right\}.$$

A well-known relaxation result (see e.g. [4, Theorem 5.47]) yields

$$\overline{E}_1(u,A) = \begin{cases} |Du|(A) & u \in BV(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{4.29}$$

We now want to prove that $\overline{E}_2 = \overline{E}_1$. To this end, we start noticing that (4.29) in particular implies that for $A \in \mathcal{A}(\Omega)$ and $u \in BV(\Omega)$ there exists a sequence $(u_k) \subset C^1(\Omega)$ such that $u_k \to u$ in $L^1(A)$ and

$$\lim_{k\to+\infty} \int_A |\nabla u_k|\, dx = |Du|(A). \tag{4.30}$$

We also notice that $(u_k) \subset W^{1,1}(A)$. Now we suitably modify (u_k) to obtain a sequence in $C^1(\overline{\Omega})$ converging to u in $L^1(A)$ and satisfying (4.30); this would immediately imply

$$\overline{E}_2(u,A) = \begin{cases} |Du|(A) & u \in BV(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{4.31}$$

Let $(u_{k,h}) \subset C^\infty(\overline{A})$ be such that $u_{k,h} \to u_k$ in $W^{1,1}(A)$ as $h \to +\infty$ (see e.g. [14, Theorem 3, Section 4.2]). Moreover, let A' be an open subset of $\mathbb{R}^n$ such that $A' \supset\supset A$ and let $\gamma \colon \mathbb{R}^n \to \mathbb{R}$ be a cut-off function between A and A'; *i.e.,* $\gamma \in C_c^\infty(A')$, $0 \le \gamma \le 1$, $\gamma \equiv 1$ on A. We define $\tilde{u}_{k,h} := \gamma\, u_{k,h}$; then, $\tilde{u}_{k,h} \subset C^1(\overline{\Omega})$ and, letting first $h \to +\infty$ and then $k \to +\infty$, we also have $\tilde{u}_{k,h} \to u$ in $L^1(A)$ and

$$\liminf_{k\to+\infty} \liminf_{h\to+\infty} \int_A |\nabla \tilde{u}_{k,h}|\, dx \le |Du|(A).$$

Then a diagonalization argument provides us with a diverging sequence (k_h) such that $\tilde{u}_{k_h,h} \to u$ in $L^1(A)$, and

$$\lim_{h\to+\infty} \int_A |\nabla u_{k_h,h}|\, dx \le |Du|(A),$$

and this concludes the proof of (4.31).

If now we define the localized functionals

$$E(u,A) := \begin{cases} \displaystyle\int_A |\nabla u|\, dx + \int_{S_u\cap A} g(|u^+ - u^-|)\, d\mathcal{H}^{n-1} & u \in SBV^2(\Omega) \cap L^\infty(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{4.32}$$

since $C^1(\overline{\Omega}) \subset SBV^2(\Omega) \cap L^\infty(\Omega)$, we have $E(u,A) \le E_2(u,A)$; hence for all $u \in BV(\Omega)$ and $A \in \mathcal{A}(\Omega)$

$$\overline{E}(u,A) \le \overline{E}_2(u,A) = |Du|(A). \tag{4.33}$$

Arguing as in [7, Proposition 3.3], one can prove that for every $u \in BV(\Omega)$ the set function $\overline{E}(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a regular Borel measure μ. Therefore

$$\overline{E}(u) = \mu(\Omega) = \mu(\Omega \setminus S_u) + \mu(\Omega \cap S_u).$$

Then (4.33) yields

$$\mu(\Omega \setminus S_u) \leq \int_\Omega |\nabla u|\, dx + |D^c u|(\Omega) \tag{4.34}$$

while (4.32) gives

$$\mu(\Omega \cap S_u) \leq \int_{S_u} g(|u^+ - u^-|)\, d\mathcal{H}^{n-1}. \tag{4.35}$$

Thus finally gathering (4.34) and (4.35) gives (4.28) and thus the limsup inequality for $u \in BV(\Omega)$.

Step 3: $u \in GBV(\Omega)$. Finally to recover the general case $u \in GBV(\Omega)$, we use a truncation argument. Let $u \in GBV(\Omega)$ and consider the truncated functions u^m. Then

$$\lim_{j\to+\infty} F(u^m, 1) = F(u, 1)$$

(see Remark 3.6). Since $u^m \to u$ in $L^1(\Omega)$ we get the thesis by virtue of the lower semicontinuity of $\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(\cdot, 1)$. □

Proposition 4.5. *Let $n \geq 1$ and let E_ε and F be defined as in* (3.2) *and* (3.3)*, respectively. Then $F(u,1) \geq \Gamma\text{-}\limsup_{\varepsilon\to0} E_\varepsilon(u,1)$ for all $u \in GBV(\Omega)$.*

Proof. The proof is obtained by taking the same recovery sequence as in Proposition 4.4. □

5. Convergence of minimization problems and relaxation

In this section we prove an equicoercivity result for suitable modifications of the functionals F_ε and $\mathscr{E}_\varepsilon$. On account of this result, we also study the convergence of the associated minimization problems.

Let $h \in L^\infty(\Omega)$ and set

$$M := \min\left\{F(u,1) + \int_\Omega |u-h|^2\, dx\colon u \in L^1(\Omega)\right\}, \tag{5.1}$$

with F defined as in (3.3); it is easy to check that the minimization problem in (5.1) admits a solution $\tilde{u} \in BV(\Omega)$, and $\|\tilde{u}\|_{L^\infty(\Omega)} \leq \|h\|_{L^\infty(\Omega)}$.

Theorem 5.1. *Consider the minimization problem*

$$M_\varepsilon := \inf\left\{F_\varepsilon(u,v) + \int_\Omega |u-h|^2\, dx\colon u,\, v \in L^1(\Omega)\right\},$$

with F_ε defined as in (3.1)*. Let $(u_\varepsilon, v_\varepsilon)$ be a minimizing sequence for $F_\varepsilon(u,v) + \|u-h\|^2_{L^2(\Omega)}$; i.e.,*

$$\lim_{\varepsilon\to0}\left(F_\varepsilon(u_\varepsilon, v_\varepsilon) + \int_\Omega |u_\varepsilon - h|^2\, dx - M_\varepsilon\right) = 0. \tag{5.2}$$

Then there exist a subsequence of $(u_\varepsilon, v_\varepsilon)$ (not relabeled) and a function $\bar{u} \in BV(\Omega)$ such that $u_\varepsilon \to \bar{u}$ and $v_\varepsilon \to 1$ in $L^1(\Omega)$. Moreover $\bar{u}$ is a solution of the minimization problem in (5.1)*, and $M_\varepsilon \to M$.*

Proof. Let $(u_\varepsilon, v_\varepsilon)$ be as in the statement. As a consequence of (5.2), we immediately have that $(u_\varepsilon, v_\varepsilon) \subset W^{1,1}(\Omega) \times W^{2,2}(\Omega)$, and $v_\varepsilon \to 1$ in $L^2(\Omega)$. We prove now that, up to passing to a subsequence (not relabeled), $u_\varepsilon \to \bar{u}$ in $L^1(\Omega)$ for some $\bar{u} \in BV(\Omega)$.

We begin noticing that, by means of a truncation argument, we may always assume

$$\|u_\varepsilon\|_\infty \leq \|h\|_\infty. \tag{5.3}$$

Let $\tilde{v}_\varepsilon := 0 \vee (v_\varepsilon \wedge 1)$, then $\tilde{v}_\varepsilon \in W^{1,2}(\Omega)$; we define the sequence $(w_\varepsilon) \subset W^{1,1}(\Omega)$ by

$$w_\varepsilon := \Phi^2(\tilde{v}_\varepsilon)\, u_\varepsilon$$

where Φ is defined as in (4.18).

Then, w_ε is bounded in $W^{1,1}(\Omega)$. In fact, since Φ is increasing and $\Phi(1) = 1$, we infer

$$\int_\Omega |w_\varepsilon|\, dx \leq \|h\|_\infty \mathcal{L}^n(\Omega);$$

moreover, appealing to the interpolation inequality Proposition 2.6(i), we deduce the existence of a positive constant c_0 such that for $\varepsilon > 0$ sufficiently small we have

$$\begin{aligned} \int_\Omega |\nabla w_\varepsilon|\, dx = \int_\Omega |u_\varepsilon \nabla(\Phi^2(\tilde{v}_\varepsilon)) + \Phi^2(\tilde{v}_\varepsilon)\nabla u_\varepsilon|\, dx &\leq 2\,\|h\|_\infty \int_\Omega |\nabla \Phi(\tilde{v}_\varepsilon)|\, dx + 4\int_\Omega \tilde{v}_\varepsilon^2 |\nabla u_\varepsilon|\, dx \\ &\leq 2\,\|h\|_\infty \int_\Omega \left(\frac{(1-\tilde{v}_\varepsilon)^2}{\varepsilon} + \varepsilon|\nabla \tilde{v}_\varepsilon|^2 \right) dx + 4\int_\Omega \tilde{v}_\varepsilon^2 |\nabla u_\varepsilon|\, dx \\ &\leq 2\,\|h\|_\infty \int_\Omega \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon|\nabla v_\varepsilon|^2 \right) dx + 4\int_\Omega v_\varepsilon^2 |\nabla u_\varepsilon|\, dx \\ \leq 2\, c_0 \|h\|_\infty \int_\Omega \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon^3 |\nabla^2 v_\varepsilon|^2 \right) dx + 4 \int_\Omega v_\varepsilon^2 |\nabla u_\varepsilon|\, dx &\leq 2(c_0\|h\|_\infty + 2) F_\varepsilon(u_\varepsilon, v_\varepsilon), \end{aligned} \tag{5.4}$$

where we have also used Young's inequality together with the fact that $0 \leq \Phi(t) \leq 2\,t$ for all $t \in [0,1]$.

Hence Theorem 2.1 yields the existence of a subsequence of w_ε (not relabeled) and a function $\bar{u} \in BV(\Omega)$ such that $w_\varepsilon \overset{*}{\rightharpoonup} \bar{u}$ in $BV(\Omega)$. As $\Phi(\tilde{v}_\varepsilon) \to 1$ in $L^1(\Omega)$, we have then $u_\varepsilon \to \bar{u}$ in $L^1(\Omega)$.

We notice now that, by the uniqueness of the limit, (5.3) yields also $u_\varepsilon \to \bar{u}$ in $L^2(\Omega)$ so that

$$\begin{aligned} \liminf_{\varepsilon \to 0} M_\varepsilon &= \liminf_{\varepsilon\to 0} \left(F_\varepsilon(u_\varepsilon, v_\varepsilon) + \|u_\varepsilon - h\|_{L^2(\Omega)} \right) \\ &\geq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon) + \|u - h\|^2_{L^2(\Omega)} \geq F(\bar{u}, 1) + \|u-h\|^2_{L^2(\Omega)} \geq M, \end{aligned} \tag{5.5}$$

where we have used Proposition 4.2.

On the other hand, if $\tilde{u}$ is a minimizer for $F(u,1) + \|u-h\|^2_{L^2(\Omega)}$, then by virtue of Proposition 4.4 there exists a sequence $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) \to (\tilde{u}, 1)$ in $L^1(\Omega) \times L^1(\Omega)$ such that

$$M = F(\tilde{u}, 1) + \|\tilde{u} - h\|_{L^2(\Omega)} = \lim_{\varepsilon \to 0} \left(F_\varepsilon(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) + \|\tilde{u}_\varepsilon - h\|_{L^2(\Omega)} \right) \geq \limsup_{\varepsilon\to 0} M_\varepsilon. \tag{5.6}$$

Gathering (5.5)-(5.6), we deduce that $\bar{u}$ is a solution of the minimization problem (5.1) and $M_\varepsilon \to M$. □

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with C^2 boundary and consider the minimization problem*

$$M_\varepsilon := \inf \left\{ \mathscr{E}_\varepsilon(u,v) + \int_\Omega |u-h|^2\, dx \colon u,\, v \in L^1(\Omega) \right\},$$

with $\mathscr{E}_\varepsilon$ defined as in (3.5). *Let $(u_\varepsilon, v_\varepsilon)$ be a minimizing sequence for $\mathscr{E}_\varepsilon(u,v) + \|u-h\|^2_{L^2(\Omega)}$; i.e.,*

$$\lim_{\varepsilon\to 0} \left(\mathscr{E}_\varepsilon(u_\varepsilon, v_\varepsilon) + \int_\Omega |u_\varepsilon - h|^2\, dx - M_\varepsilon \right) = 0.$$

Then there exist a subsequence of $(u_\varepsilon, v_\varepsilon)$ (not relabeled) and a function $\bar{u} \in BV(\Omega)$ such that $u_\varepsilon \to \bar{u}$ and $v_\varepsilon \to 1$ in $L^1(\Omega)$. Moreover $\bar{u}$ is a solution of the minimization problem in (5.1), *and $M_\varepsilon \to M$.*

Proof. The proof follows the line of that of Theorem 5.2, but here we appeal to Theorem 3.3. We only point out that, to get an analogous bound as in (5.4), we need to use in addition the interpolation inequality Proposition 2.6(ii). □

Let now $\eta_\varepsilon > 0$ be such that $\eta_\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0$ and for $u,\, v \in L^1(\Omega)$ consider the functionals

$$F'_\varepsilon(u,v) := \begin{cases} F_\varepsilon(u,v) + \eta_\varepsilon \displaystyle\int_\Omega |\nabla u|\, dx & u \in W^{1,1}(\Omega),\ v \in W^{2,2}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Thanks to the requirement that $\eta_\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0$, arguing as in Propositions 4.1, 4.2 and 4.4, one can easily show that for all $u,\, v \in L^1(\Omega)$

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F'_\varepsilon(u,v) = F(u,v)$$

with F defined as in (3.3).

For fixed $\varepsilon > 0$, let $\overline{F}'_\varepsilon$ denote the relaxed functional of F'_ε with respect to the strong topology of $L^1(\Omega) \times L^1(\Omega)$; then, we also have

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \overline{F}'_\varepsilon(u,v) = F(u,v)$$

for all $u,\, v \in L^1(\Omega)$ (see e.g. [12, Proposition 6.11]).

The last part of this section is devoted to provide an integral representation formula for $\overline{F}'_\varepsilon$ in the case $n \le 3$, which is the interesting case in numerical applications. We show in particular that the presence of the second derivative of v makes the expression of $\overline{F}'_\varepsilon$ particularly easy.

We introduce the following notation: for $u,\, v \in L^1(\Omega)$, we set

$$R_\varepsilon(u,v) := \begin{cases} \displaystyle\int_\Omega (v^2 + \eta_\varepsilon)\, d|Du| + \int_\Omega \left(\frac{(1-v)^2}{\varepsilon} + \varepsilon^3 |\nabla^2 v|^2 \right) dx & u \in BV(\Omega),\ v \in W^{2,2}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 5.3. *Let $n \le 3$. Then $\overline{F}'_\varepsilon(u,v) = R_\varepsilon(u,v)$, for all $u,\, v \in L^1(\Omega)$.*

Proof. We begin noticing that $F'_\varepsilon(u,v) = R_\varepsilon(u,v)$ for all $u \in W^{1,1}(\Omega)$, $v \in W^{2,2}(\Omega)$; moreover it is clear that $R_\varepsilon \le F'_\varepsilon$.

We now show that R_ε is lower semicontinuous. To this end let $u,\, v \in L^1(\Omega)$, (u_k), $(v_k) \subset L^1(\Omega)$ be such that $u_\varepsilon \to u$ and $v_\varepsilon \to v$ in $L^1(\Omega)$; we prove that

$$\liminf_{k\to+\infty} R_\varepsilon(u_k, v_k) \ge R_\varepsilon(u,v).$$

Clearly it is enough to consider the case $\liminf\limits_{k\to+\infty} R_\varepsilon(u_k,v_k) < +\infty$, moreover up to subsequences we can always assume that the liminf is a limit. As a result, we have $|Du_k|(\Omega) \le c$, for some $c > 0$ independent of k and for every $k \in \mathbb{N}$; moreover, by the interpolation inequality Proposition 2.6(i) we also have $\|v_k\|_{W^{2,2}(\Omega)} \le c$. Then up to subsequences (not relabeled)

$$u_k \overset{*}{\rightharpoonup} u \text{ in } BV(\Omega), \quad v_\varepsilon \rightharpoonup v \text{ in } W^{2,2}(\Omega) \tag{5.7}$$

and, by virtue of the compact embedding of $W^{2,2}(\Omega)$ in $C(\overline{\Omega})$ when $n \le 3$ (see. e.g. [1, Theorem 6.2]), we also deduce

$$v_k \to v \text{ in } L^\infty(\Omega). \tag{5.8}$$

Then, appealing to the weak lower semicontinuity of the L^2-norm and to (5.7), we get

$$
\begin{aligned}
\liminf_{k\to+\infty} R_\varepsilon(u_k,v_k) &\geq \liminf_{k\to+\infty}\int_\Omega v_k^2\,d|Du_k| + \eta_\varepsilon|Du|(\Omega) + \int_\Omega \left(\frac{(1-v)^2}{\varepsilon} + \varepsilon^3|\nabla^2 v|^2\right)dx \\
&\geq \lim_{k\to+\infty}\int_\Omega (v_k-v)^2\,d|Du_k| + \liminf_{k\to+\infty}\int_\Omega v^2\,d|Du_k| + 2\lim_{k\to+\infty}\int_\Omega (v_k-v)\,v\,d|Du_k| \\
&\quad +\eta_\varepsilon|Du|(\Omega) + \int_\Omega \left(\frac{(1-v)^2}{\varepsilon} + \varepsilon^3|\nabla^2 v|^2\right)dx \\
&\geq R_\varepsilon(u,v),
\end{aligned}
$$

where we have also used that, by (5.8),

$$
\left|\int_\Omega (v_k-v)\,v\,d|Du_k|\right| \leq \|v_k-v\|_{L^\infty(\Omega)}\|v\|_{L^\infty(\Omega)}|Du_k|(\Omega) \longrightarrow 0, \qquad \text{as } k\to+\infty.
$$

Then it remains to prove that for all $u \in BV(\Omega)$, $v \in W^{2,2}(\Omega)$ there exists a sequence $(u_k,v_k) \subset W^{1,1}(\Omega)\times W^{2,2}(\Omega)$ such that $u_k \to u$ and $v_k \to v$ in $L^1(\Omega)$, and

$$
\lim_{k\to+\infty} F'_\varepsilon(u_k,v_k) \leq R_\varepsilon(u,v).
$$

Fix $u \in BV(\Omega)$ and $v \in W^{2,2}(\Omega)$; in particular, as $n \leq 3$, $v \in C(\overline{\Omega})$. By a standard approximation argument (see e.g. [14, Theorem 2 and Theorem 3, Section 5.2]) there exists $(u_k) \subset BV(\Omega)\cap C^\infty(\Omega)$ such that $u_k \to u$ in $L^1(\Omega)$, $|\nabla u_k|\mathcal{L}^n \overset{*}{\rightharpoonup} |Du|$ in $\mathcal{M}_b(\Omega)$ and

$$
\lim_{k\to+\infty}\int_\Omega |\nabla u|\,dx = |Du|(\Omega).
$$

Then appealing to [4, Proposition 1.80], we infer

$$
\lim_{k\to+\infty}\int_\Omega v^2 d|Du_k| = \int_\Omega v^2 d|Du|.
$$

Hence the pair (u_k, v) is the desired sequence. □

We now consider the minimization problem

$$
\overline{M}_\varepsilon := \inf\left\{\overline{F}'_\varepsilon(u,v) + \int_\Omega |u-h|^2 \colon u,\, v\in L^1(\Omega)\right\}. \tag{5.9}
$$

By using the direct methods of calculus of variations, one can show that the problem in (5.9) admits a solution in $BV(\Omega)\times W^{2,2}(\Omega)$; moreover we have

$$
\overline{M}_\varepsilon = \inf\left\{F'_\varepsilon(u,v) + \int_\Omega |u-h|^2 \colon u,\, v\in L^1(\Omega)\right\}.
$$

Finally the following theorem holds true.

Theorem 5.4. *Let $(u_\varepsilon, v_\varepsilon)$ be a minimizing pair for $\overline{F}'_\varepsilon(u,v) + \|u-h\|^2_{L^2(\Omega)}$. Then, there exist a subsequence of $(u_\varepsilon, v_\varepsilon)$ (not relabeled) and a function $\bar{u} \in BV(\Omega)$ such that $u_\varepsilon \to \bar{u}$ and $v_\varepsilon \to 1$ in $L^1(\Omega)$. Moreover $\bar{u}$ is a solution of the minimization problem in* (5.1)*, and $\overline{M}_\varepsilon \to M$.*

Proof. The proof follows the line of that of Theorem 5.1. □

Remark 5.5. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with C^2 boundary and let $n \leq 3$. For u, $v \in L^1(\Omega)$, set

$$
\mathscr{E}'_\varepsilon(u,v) := \begin{cases} \mathscr{E}_\varepsilon(u,v) + \eta_\varepsilon\displaystyle\int_\Omega |\nabla u|\,dx & u \in W^{1,1}(\Omega),\ v\in W^{2,2}(\Omega), \\ +\infty & \text{otherwise.}\end{cases}
$$

Arguing as before one can prove that the relaxed functional of $\mathscr{E}'_\varepsilon$ with respect to the strong topology of $L^1(\Omega) \times L^1(\Omega)$ is given by

$$\overline{\mathscr{E}}'_\varepsilon = \begin{cases} \displaystyle\int_\Omega (v^2 + \eta_\varepsilon)\, d|Du| + \int_\Omega \left(\frac{(1-v)^2}{\varepsilon} + \varepsilon^3 |\Delta v|^2 \right) dx & u \in BV(\Omega),\ v \in W^{2,2}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega) \times L^1(\Omega). \end{cases}$$

Moreover, an analogous result as in Theorem 5.4 can be recovered for $\overline{\mathscr{E}}'_\varepsilon$ as well.

Acknowledgements

The author wishes to thank Caterina Ida Zeppieri and Martin Burger for many helpful suggestions and discussions.

References

[1] R. A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

[2] R. Alicandro, A. Braides, and J. Shah, Free-discontinuity problems via functionals involving the L^1-norm of the gradient and their approximation, *Interfaces and Free Boundaries* **1** (1999), 1737.

[3] L. Ambrosio, A compactness theorem for a new class of functions of bounded variation. *Boll. Un. Mat. Ital.* **3-B** (1989), 857-881,

[4] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[5] L. Ambrosio and V. M. Tortorelli, Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence, *Comm. Pure Appl. Math.* **43** (1990) no. 8, 999-1036.

[6] A. Braides, *Γ-convergence for Beginners*, Oxford University Press, Oxford, 2002.

[7] G. Bouchitté, A. Braides, and G. Buttazzo, Relaxation result for some free-discontinuity problems, *J. Reine Angew. Math.* **458** (1995), 1-18.

[8] M. Burger, T. Esposito, and C. I. Zeppieri, Second-order edge-penalization in the Ambrosio-Tortorelli functional, *SIAM J. Multiscale Model. Simul.* **13** (2015) no. 4, 1354-1389.

[9] M. Burger and S. Osher, A guide to the TV zoo, *Level set and PDE based reconstruction methods in imaging*, 1-70, *Lecture Notes in Math.*, 2090, Springer, Cham, 2013.

[10] A. Chambolle and T. Pock, A first-order primal-dual algorithm for convex problems with applications to imaging, *J. Math. Imaging Vision* **40** (2011) no. 1, 120-145.

[11] G. Cortesani and R. Toader, A density result in *SBV* with respect to non-isotropic energies, *Nonlinear Analysis* **38** (1999), 585-604.

[12] G. Dal Maso, *An Introduction to Γ-convergence*, Birkhäuser, Boston, 1993.

[13] L. C. Evans, *Partial differential equations*, American Mathematical Society, Graduate Studies in Mathematics **19**, 1998.

[14] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, 1992.

[15] I. Fonseca and G. Leoni, On lower semicontinuity and relaxation, *Proc. Roy. Soc. Edin.* **131A** (2001), 519-2001.

[16] I. Fonseca and S. Müller, Quasiconvex integrands and lower semicontinuity in L^1, *SIAM J. Math. Anal.* **23** (1992), 1081-1098.

[17] T. Goldstein and S. Osher, The split Bregman method for $L1$-regularized problems, *SIAM J. Imaging Sci.* **2** (2009) no. 2, 323-343.

[18] L. Grady and C. Alvino, The piecewise smooth Mumford-Shah functional on an arbitrary graph, *IEEE Transactions on Image Processing* **18** (2009) no. 11, 2547-2561.

[19] P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics **24**, Pitman, Massachusetts, 1985.

[20] L. I. Rudin, S. J. Osher, and E. Fatemi, Nonlinear total variation based noise removal algorithms, *Phys. D*, **60** (1992), 259-268.

(Teresa Esposito) Institut für Numerische und Angewandte Mathematik, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62, 48149 Münster, Germany

E-mail address, Teresa Esposito: teres.esposito@gmail.com