Chirality transitions in frustrated ferromagnetic spin chains: a link with the gradient theory of phase transitions

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Abstract

We study chirality transitions in frustrated ferromagnetic spin chains, in view of a possible connection with the theory of Liquid Crystals. A variational approach to the study of these systems has been recently proposed by Cicalese and Solombrino, focusing close to the helimagnet/ferromagnet transition point corresponding to the critical value of the frustration parameter $\alpha = 4$. We reformulate this problem for any $\alpha \geq 0$ in the framework of surface energies in nonconvex discrete systems with nearest neighbours ferromagnetic and next-to-nearest neighbours antiferromagnetic interactions and we link it to the gradient theory of phase transitions, by showing a uniform equivalence by Γ-convergence on $[0, 4]$ with Modica-Mortola type functionals.

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1 Introduction

Frustration is the competition between different interactions in a continuous or discrete physical system that favors incompatible ground states. It occurs, for instance, in the liquid-crystalline phases of *chiral* molecules: a molecule is chiral if it cannot be superimposed on its mirror image via any proper rotation or translation. The main effect of chirality is that chiral molecules do not align themselves parallel to their neghbors but tend to form an angle with them (see e.g. [1, 14, 12]).

Edge-sharing chains of cuprates, instead, provide a natural example of frustrated lattice systems, the frustration resulting from the competition between ferromagnetic (F) nearest-neighbour (NN) and antiferromagnetic (AF) next-nearest-neighbour (NNN) interactions (see e.g. [11]). In this paper we study the asymptotic properties of a one-dimensional frustrated spin system at zero temperature, focusing also on the variational equivalence with problems in gradient theory of phase transitions (see e.g. Braides [3, 2] for a simple introduction to the topic). This can be seen as a first step in the analysis of chirality transitions in more complicated physical systems like as chiral liquid crystals. A discrete to continuous analysis via Γ-convergence of some problems in liquid crystals has been recently treated e.g. by Braides, Cicalese and Solombrino [6], but this promising research field is still largely unexplored.

We consider a minimal energy model describing the magnetic properties of one-dimensional frustrated magnetic systems: the so-called F-AF spin chain model. On the one-dimensional torus $[0, 1]$, the state of the system is described by the values of a vectorial spin variable parameterized over the points of the set $\frac{1}{n}\mathbb{Z} \cap [0, 1]$, $n \in \mathbb{N}$. The energy of a given state of the system $u : \frac{1}{n}\mathbb{Z} \cap [0, 1] \to S^1$, $u^i = u(\frac{i}{n})$ is

$$E_n^\alpha(u) = -\alpha \sum_{i=0}^{n-1} (u^i, u^{i+1}) + \sum_{i=0}^{n-1} (u^i, u^{i+2}) - nm_\alpha, \tag{1.1}$$

where $\alpha \geq 0$ is the *frustration parameter*, $(\cdot, \cdot)$ denotes the scalar product between vectors in $\mathbb{R}^2$ and

$$m_\alpha = \begin{cases} -\left(\frac{\alpha^2}{8} + 1\right) & \text{if } \alpha \in [0, 4], \\ -\alpha + 1 & \text{if } \alpha \in [4, +\infty). \end{cases} \tag{1.2}$$

We assume *periodic* boundary conditions $(u^0, u^1) = (u^n, u^{n+1})$. While the first term of the energy (1.1) is ferromagnetic and favors the alignment of neighbouring spins, the second, being antiferromagnetic, frustrates it as it favors antipodal next-to-nearest neighbouring spins. As a result, the frustration of the system depends on the parameter α. In order to characterize the ground states of this system and their dependence on the value of α, we first associate associate to each pair of nearest neighbours u^i, u^{i+1} the corresponding oriented central angle $\theta^i \in [-\pi, \pi)$ given by

$$\theta^i = \chi[u^i, u^{i+1}] \arccos((u^i, u^{i+1})),$$

where

$$\chi[u^i, u^{i+1}] = \mathrm{sign}(u_1^i u_2^{i+1} - u_2^i u_1^{i+1}), \tag{1.3}$$

with the convention that $\mathrm{sign}(0) = -1$, $u = (u_1|u_2) \in \mathbb{R}^2$. Correspondingly, by the periodicity assumption, we may rewrite the energies in terms of this scalar variable as

$$E_n^\alpha(\theta) = -\frac{\alpha}{2}\sum_{i=0}^{n-1}\Big(\cos\theta^i + \cos\theta^{i+1}\Big) + \sum_{i=0}^{n-1}\cos(\theta^i + \theta^{i+1}) - nm_\alpha. \tag{1.4}$$

Following the approach by Braides and Cicalese [5] for lattice systems of the form (1.4), for each $i = 0, 1, \dots, n-1$ we fix the next-to-nearest interactions $\theta^i + \theta^{i+1} = 2\theta$ and solve the minimum problem

$$\min_{\varphi\in[-\pi,\pi)}\Big\{-\frac{\alpha}{2}\Big(\cos\varphi + \cos(2\theta-\varphi)\Big) + \cos 2\theta - m_\alpha\Big\}; \tag{1.5}$$

that is, we "minimize out" the nearest neighbours interactions. We find that, for each i, the unique (up to a reparametrization) minimizers in (1.5) are $\varphi = \theta^i = \theta^{i+1} = \theta \in (-\pi/2, \pi/2)$, and correspondingly we define the *effective potential* as

$$W_\alpha(\theta) = \cos 2\theta - \alpha\cos\theta - m_\alpha. \tag{1.6}$$

We note that, by the definition of W_α, we have that

$$E_n^\alpha(\theta) \geq \sum_{i=0}^{n-1} W_\alpha(\theta^i) \tag{1.7}$$

and

$$\arg\min W_\alpha(\theta) = \begin{cases} \{\pm\theta_\alpha\} = \{\pm\arccos(\frac{\alpha}{4})\}, & \text{if } \alpha\in[0,4], \\ \{0\}, & \text{if } \alpha \geq 4. \end{cases} \tag{1.8}$$

Thus, if $\alpha \geq 4$ the nearest neighbours prefer to stay aligned (ferromagnetic order); if $0 \leq \alpha \leq 4$, instead, the minimal configurations of E_n^α are $\theta^i = \theta^{i+1} \in \{\pm\theta_\alpha\}$; that is, the angle between pairs of nearest neighbours u^i, u^{i+1} and u^{i+1}, u^{i+2} is constant and depending on the particular value of α (helimagnetic order). In this last case, the two possible choices for θ_α correspond to either clockwise or counterclockwise spin rotations, or, in other words, to a positive or a negative chirality (see Fig. 1). Such a degeneracy is known in literature as *chirality symmetry.*

The asymptotic behaviour of energies E_n^α as $n \to \infty$ and for fixed α (Theorem 3.5) reflects such different regimes for the ground states. If $\alpha \geq 4$ the limit is trivially finite (and equal to zero) only on the constant function $\theta \equiv 0$, while if $0 \leq \alpha < 4$ it is finite on functions with bounded variation taking only the two values $\{\pm\theta_\alpha\}$ and it counts the number of chirality transitions; more precisely,

$$\Gamma\text{-}\lim_{n\to+\infty} E_n^\alpha(\theta) = C_\alpha\#(S(\theta)),$$

where $S(\theta)$ is the jump set of function θ and $C_\alpha = C(\alpha)$ is the cost of each chirality transition defined as

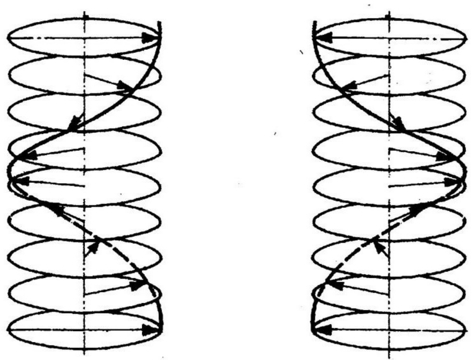


Figure 1: A schematic representation of the ground states of the spin system for $0 \leq \alpha < 4$ for clockwise (on the left) and counterclockwise (on the right) chirality (picture taken from [12]).

$$\begin{aligned} C_\alpha &:= C(-\theta_\alpha, \theta_\alpha) \\ &= \inf_{N\in\mathbb{N}} \min\Big\{ \sum_{i=-\infty}^{+\infty} \Big(\cos(\theta^i + \theta^{i+1}) - \frac{\alpha}{2}\Big(\cos\theta^i + \cos\theta^{i+1}\Big) + \frac{\alpha^2}{8} + 1\Big) : \\ &\qquad\qquad \theta : \mathbb{Z} \to [-\pi/2, \pi/2],\ \theta^i = \operatorname{sign}(i)\theta_\alpha, \text{ if } |i| \geq N \Big\}. \end{aligned} \tag{1.9}$$

The value C_α (see Section 3.1) represents the energy of an interface which is obtained by means of a 'discrete optimal-profile problem' connecting the two constant (minimal) states $\pm\theta_\alpha$. It is continuous as a function of α in the interval $[0, 4)$ (as shown by Proposition 3.3) and can be defined to be equal 0 for $\alpha \geq 4$, consistently with definition (1.9). Moreover, $C_\alpha \to 0$ as $\alpha \to 4$ and (compare with [13] and Remark 4.3)

$$C_\alpha \sim \frac{\sqrt{2}}{3}(4-\alpha)^{3/2}, \text{ as } \alpha \to 4^-. \tag{1.10}$$

In a recent paper [10], Cicalese and Solombrino investigated the asymptotic behaviour of this system close to the ferromagnet/helimagnet transition point; that is, they found the correct scaling (heuristically suggested by (1.10)) to detect the symmetry breaking and to compute the asymptotic behavior of the scaled energy describing this phenomenon as α is close to 4. They let the parameter α depend on n and be close to 4 from below; i.e. they rewrite energies (1.4) in terms of $4 - \alpha_n$, with $4 - \alpha_n \to 0$ as $n \to \infty$. We state their result in a slight different form, useful for the sequel. More precisely, we prove in Theorem 4.2 that an analogous result can be obtained if we choose as order parameter the "flat" angular variable

$$v = \frac{\theta}{\theta_\alpha}, \tag{1.11}$$

which is equivalent to the variable considered in [10] in the regime of small angles.

We compute the Γ-limit F^0 as $n \to \infty$, $\alpha = \alpha_n \to 4$ with respect to the strong L^1-topology of the scaled energies

$$F_n^\alpha(v) := \mu_\alpha E_n^\alpha(v) = \frac{8E_n^\alpha(v)}{\sqrt{2}(4-\alpha)^{3/2}}, \tag{1.12}$$

and show that, within this scaling, several regimes are possible depending on the value

$$l := \lim_n \frac{\sqrt{2}}{4n(4-\alpha)^{1/2}}. \tag{1.13}$$

If $l = 0$ then $F^0(v) = \frac{8}{3}\#(S(v))$, $v \in BV(I, \{\pm 1\})$, if $l = +\infty$ then F^0 is finite (and equal to zero) only on constant functions, while in the intermediate case $l \in (0, +\infty)$ we get

$$F^0(v) = \frac{1}{l}\int_I \Big(v^2(t) - 1\Big)^2 dt + l\int_I (\dot v(t))^2\, dt,\ v \in W^{1,2}_{|per|}(I),$$

where $I = (0,1)$, $BV(I, \{\pm 1\})$ is the space of functions of bounded variation defined on I and taking the values $\{\pm 1\}$, and $W^{1,2}_{|per|}(I) = \{v \in W^{1,2}(I) : |v(0)| = |v(1)|\}$.

Motivated by the particular form of the result by Cicalese and Solombrino and in the spirit of Braides-Truskinovsky [8], with Theorem 5.6 we find a link between such energies (seen as a ‘parametrized’ family of functionals) and the gradient theory of phase transitions in the framework of the *equivalence* by Γ-convergence. Roughly speaking, two families of functionals are equivalent by Γ-convergence if they share the same Γ-limit (see Definition 5.1 and the subsequent ones for the precise definitions useful in this framework). More precisely, we show the *uniform* equivalence by Γ-convergence on $[0,4]$ of the energies $F_n^\alpha(v)$ with the “Modica-Mortola type” functionals given by

$$G_n^\alpha(v) = \mu_\alpha\Big(\lambda_{n,\alpha}\int_I \Big(v^2 - 1\Big)^2 dt + \frac{M_\alpha^2}{\lambda_{n,\alpha}}\int_I (\dot v)^2\, dt\Big),\ v \in W^{1,2}_{|per|}(I),$$

where $\lambda_{n,\alpha} = 2n\theta_\alpha^4$ and $M_\alpha = 3C_\alpha/8$.

The value $\alpha_0 = 4$ is a *singular point*, since the Γ-limit of G_n^α will depend on choice of the particular sequence $\alpha_n \to \alpha_0^- = 4^-$. Each $\alpha_0 \in [0,4)$, instead, is a *regular point*; i.e., it is not singular. As a consequence of Theorem 5.6, we deduce (see Corollary 5.7) the uniform equivalence of the energies $E_n^\alpha(\theta)$ for $\alpha \in [0,4)$ with the family

$$H_n^\alpha(\theta) = \frac{\lambda_{n,\alpha}}{\theta_\alpha^4}\int_I \Big(\theta^2(t) - \theta_\alpha^2\Big)^2 dt + \frac{M_\alpha^2}{\lambda_{n,\alpha}\theta_\alpha^2}\int_I (\dot\theta(t))^2\, dt,\ \theta \in W^{1,2}_{|per|}(I),$$

whose potentials $\mathcal{W}_\alpha(\theta) := (\theta^2 - \theta_\alpha^2)^2$ have the wells at the minimal angles $\theta = \pm\theta_\alpha$.

As a final remark, we would like to observe that our result can be useful also to analyze more general problems of interest for the applied community. For instance, a natural extension would be the case of S^2-valued spins, that has been recently investigated in Cicalese, Ruf and Solombrino [9] in the vicinity of the transition point. Moreover, even in the case of values in S^1, the Villain Helical XY-model studied there could be attacked with our approach, at least in the regime of "strong" ferromagnetic interaction considered therein by the authors.

2 Setting of the problem

We denote by $I = (0,1)$ and by $\lambda_n = \frac{1}{n}, n \in \mathbb{N}$ a positive parameter. Given $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the integer part of x. The symbol S^1 stands for the standard unit sphere of $\mathbb{R}^2$. Given a vector $v \in \mathbb{R}^2$ with components v_1 and v_2 with respect to the canonical basis of $\mathbb{R}^2$, we will use the notation $v = (v_1|v_2)$. Given two vectors $v, w \in \mathbb{R}^2$ we will denote by (v, w) their scalar product. Here and in the following $\mathcal{A}_n(I, B)$ is the space of the functions $w : \lambda_n\mathbb{Z} \cap [0, 1] \to B$ and we use the notation $w^i = w(i\lambda_n)$; $\bar{\mathcal{A}}_n(I, B)$ will denote the subspace of those u satisfying the following *periodic boundary condition*

$$(w^1, w^0) = (w^{n+1}, w^n). \tag{2.1}$$

We will identify each function $w \in \bar{\mathcal{A}}_n(I, B)$ with its piecewise-constant interpolation belonging to the class

$$\mathcal{C}_n(I, B) = \{w \in \bar{\mathcal{A}}_n(I, B) :\ w(t) = w(\lambda_n i) \text{ if } t \in (i, i+1)\lambda_n,\ i \in \{0, 1, \dots, n-1\}\}.$$

Given $v = (v_1|v_2), w = (w_1|w_2) \in S^1$, we define the function $\chi[v, w] : S^1 \times S^1 \to \{\pm 1\}$ as

$$\chi[v, w] = \operatorname{sign}(v_1 w_2 - v_2 w_1), \tag{2.2}$$

with the convention that $\operatorname{sign}(0) = -1$.

We consider the energies

$$E_n^\alpha(u) = P_n^\alpha(u) - n m_\alpha = -\alpha \sum_{i=0}^{n-1} (u^i, u^{i+1}) + \sum_{i=0}^{n-1} (u^i, u^{i+2}) - n m_\alpha, \tag{2.3}$$

where $u \in \mathcal{C}_n(I, S^1)$, $\alpha \geq 0$ and (see Proposition 3.2 in [10])

$$m_\alpha = \frac{1}{n} \min_{u \in L^\infty(I, S^1)} P_n^\alpha(u) = \begin{cases} -\left(\frac{\alpha^2}{8} + 1\right) & \text{if } \alpha \in [0, 4], \\ -\alpha + 1 & \text{if } \alpha \in [4, +\infty). \end{cases} \tag{2.4}$$

Thanks to the periodicity assumption (2.1), we can write the energy (2.3) as

$$E_n^\alpha(u) = -\frac{\alpha}{2} \sum_{i=0}^{n-1} \Big((u^i, u^{i+1}) + (u^{i+1}, u^{i+2}) \Big) + \sum_{i=0}^{n-1} (u^i, u^{i+2}) - n m_\alpha. \tag{2.5}$$

Now we associate to each pair of neighbouring spins u^i, u^{i+1} the corresponding oriented central angle $\theta^i \in [-\pi, \pi)$ given by

$$\theta^i = \chi[u^i, u^{i+1}] \arccos((u^i, u^{i+1})), \tag{2.6}$$

and taking θ^i as (scalar) order parameter, the energies (2.5) can be rewritten as

$$E_n^\alpha(\theta) = -\frac{\alpha}{2} \sum_{i=0}^{n-1} \Big(\cos\theta^i + \cos\theta^{i+1} \Big) + \sum_{i=0}^{n-1} \cos(\theta^i + \theta^{i+1}) - nm_\alpha. \tag{2.7}$$

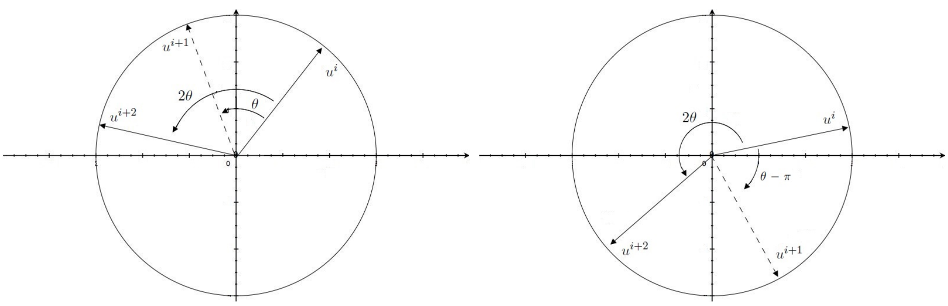
2.1 Ground states of E_n^α

First, we focus on the ground states of the energies E_n^α. We will show the emergence of *chiral* ground states for $\alpha \in [0, 4]$ by means of a double-minimization technique introduced by Braides and Cicalese in [5] for lattice systems of the form (2.7).

For each $i = 0, 1, \dots, n-1$, we fix the next-to-nearest neighbour interactions $\theta^i + \theta^{i+1} = 2\theta$ and solve the minimum problem

$$\min_{\theta^i \in [-\pi, \pi)} \Big\{ -\frac{\alpha}{2} \Big(\cos\theta^i + \cos(2\theta - \theta^i) \Big) + \cos 2\theta - m_\alpha \Big\}. \tag{2.8}$$

We find that the unique minimizers in (2.8) for $\alpha \neq 0$ are $\theta^i = \theta^{i+1} = \theta$ if $\theta \in (-\pi/2, \pi/2)$, and $\theta^i = \theta^{i+1} = \theta - \pi$ if $|\theta| \in (\pi/2, \pi)$, while for $\alpha = 0$ we have $\theta^i = \theta^{i+1} = \pm\pi/2$. The following pictures shows that, up to a reparametrization, they actually represent the same minimizer.



(a) The angle between NN is θ (b) The angle between NN is $\theta - \pi$

Without loosing generality we will assume up to the end that $\theta^i = \theta^{i+1} = \theta \in J$, $J := [-\pi/2, \pi/2]$ and correspondingly we define the *effective potential* as

$$W_\alpha(\theta) = \cos 2\theta - \alpha\cos\theta - m_\alpha. \tag{2.9}$$

The potential W_α is thus obtained by integrating out the effect of nearest-neighbour interactions optimizing over atomic-scale oscillations, and its properties strongly depend on the value α. In fact, if $0 \le \alpha < 4$ W_α is a "double-well" potential, while if $\alpha \ge 4$ the potential is convex (see Fig. 2). We note that by the definition of W_α we get

$$E_n^\alpha(\theta) \ge \sum_{i=0}^{n-1} W_\alpha(\theta^i) \tag{2.10}$$

and

$$\arg\min W_\alpha(\theta) = \begin{cases} \{\pm\theta_\alpha\} = \{\pm \arccos(\frac{\alpha}{4})\}, & \text{if } \alpha \in [0,4], \\ \{0\}, & \text{if } \alpha \in [4,+\infty), \end{cases} \tag{2.11}$$

the inequality in (2.10) being strict if $\theta^i \neq \theta^{i+1}$ or $\theta^i \neq \pm\theta_\alpha$. In particular, $E_n^\alpha(\theta) \ge 0$.

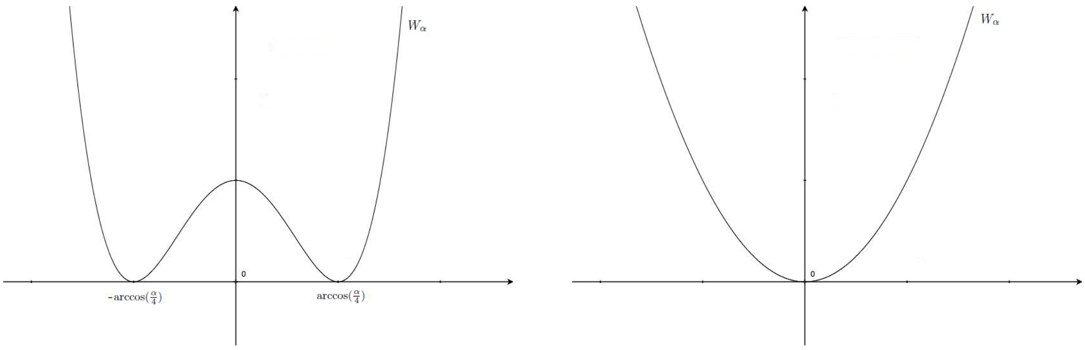


Figure 2: The potential W_α for $0 \le \alpha < 4$ (on the left) and for $\alpha \ge 4$ (on the right).

Thus, the particular minimization procedure leading to the definition of W_α (and then to inequality (2.10)) allows us to deduce some information about the energies E_n^α from the properties of this potential. More precisely, if $\alpha \le 4$ the minimal configurations of E_n^α are $\theta^i = \theta^{i+1} \in \{\pm\theta_\alpha\}$; that is, the angle between pairs of nearest neighbours u^i, u^{i+1} and u^{i+1}, u^{i+2} is constant and depending on the particular value of α; if $\alpha \ge 4$, instead, the nearest neighbours prefer to stay aligned.

Let be $\theta \in \mathcal{C}_n(I,J)$. We may regard the energies E_n^α as defined on a subset of $L^\infty(I,J)$ and consider their extension on $L^\infty(I,J)$. With a slight abuse of notation, we set $E_n^\alpha : L^\infty(I,J) \to [0,+\infty]$ as

$$E_n^\alpha(\theta) = \begin{cases} -\alpha \sum_{i=0}^{n-1} \cos\theta^i + \sum_{i=0}^{n-1} \cos(\theta^i + \theta^{i+1}) - nm_\alpha, & \text{if } \theta \in \mathcal{C}_n(I,J), \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.12}$$

3 Limit behaviour of E_n^α with fixed α

In this section we compute explicitly the Γ-limit as $n \to \infty$ of the energies E_n^α with fixed $\alpha \in [0, +\infty)$. The limit is non-trivial in the helimagnetic regime ($0 \le \alpha < 4$) representing the energy the system spends on the scale 1 for a finite number of chirality transitions from $-\theta_\alpha$ to θ_α.

3.1 Crease transition energies

Let $\alpha \in [0, 4)$. The *crease transition energy* between $-\theta_\alpha$ and θ_α is defined as (see [5], Section 2.2)

$$\begin{aligned} C_\alpha &:= C(-\theta_\alpha, \theta_\alpha) \\ &= \inf_{N \in \mathbb{N}} \min\Big\{ \sum_{i=-\infty}^{+\infty} \Big(\cos(\theta^i + \theta^{i+1}) - \frac{\alpha}{2}\Big(\cos\theta^i + \cos\theta^{i+1}\Big) + \frac{\alpha^2}{8} + 1\Big) : \\ &\qquad\qquad \theta : \mathbb{Z} \to [-\pi/2, \pi/2],\ \theta^i = \operatorname{sign}(i)\theta_\alpha, \text{ if } |i| \ge N \Big\}. \end{aligned} \tag{3.1}$$

The value C_α represents the energy of an interface which is obtained by means of a 'discrete optimal-profile problem' connecting the two constant (minimal) states $\pm\theta_\alpha$. We note that the infinite sums in (3.1) are well defined since they involve only non negative terms and, actually, for fixed N they are finite sums, since the terms in the sum are 0 for $i \ge N$ and $i \le -N - 1$.

Now we prove that the optimal test function in (3.1) is constantly equal to $\pm\theta_\alpha$ only for $N \to +\infty$, thus relaxing the boundary condition as a condition at infinity in the definition of C_α. Such a property of crease energies has been showed by Braides and Solci in [7] for a one-dimensional system of Lennard-Jones nearest and next-to-nearest neighbour interactions.

Proposition 3.1. *The infimum in (3.1) is obtained for $N \to \infty$; that is,*

$$\begin{aligned} C_\alpha = \inf\Big\{ \sum_{i=-\infty}^{+\infty} \Big(\cos(\theta^i + \theta^{i+1}) - \frac{\alpha}{2}\Big(\cos\theta^i + \cos\theta^{i+1}\Big) + \frac{\alpha^2}{8} + 1\Big) : \\ \theta : \mathbb{Z} \to [-\pi/2, \pi/2],\ \lim_{i \to \pm\infty} \operatorname{sign}(i)\theta^i = \theta_\alpha \Big\}. \end{aligned} \tag{3.2}$$

Moreover, $C_\alpha > 0$.

Proof. Let θ^i be a test function for the problem (3.2) and denote by $\widetilde{C}_\alpha$ the infimum in (3.2). With fixed $\eta > 0$, let N_η be such that $|\theta^i - \operatorname{sign}(i)\theta_\alpha| < \eta$ for $|i| \ge N_\eta$, and define

$$\theta_\eta^i = \begin{cases} \theta^i, & \text{if } |i| \le N_\eta \\ \operatorname{sign}(i)\theta_\alpha, & \text{if } |i| > N_\eta. \end{cases}$$

We then have

$$
\begin{aligned}
&\sum_{i=-\infty}^{+\infty}\Big(-\frac{\alpha}{2}(\cos\theta_\eta^i+\cos\theta_\eta^{i+1})+\cos(\theta_\eta^i+\theta_\eta^{i+1})+\frac{\alpha^2}{8}+1\Big)\\
&=\sum_{i=-N_\eta-1}^{N_\eta}\Big(-\frac{\alpha}{2}(\cos\theta_\eta^i+\cos\theta_\eta^{i+1})+\cos(\theta_\eta^i+\theta_\eta^{i+1})+\frac{\alpha^2}{8}+1\Big)\\
&=\sum_{i=-N_\eta}^{N_\eta-1}\Big(-\frac{\alpha}{2}(\cos\theta^i+\cos\theta^{i+1})+\cos(\theta^i+\theta^{i+1})+\frac{\alpha^2}{8}+1\Big)\\
&\quad-\frac{\alpha}{2}(\cos\theta^{N_\eta}+\cos\theta_\alpha)+\cos(\theta^{N_\eta}+\theta_\alpha)+\frac{\alpha^2}{8}+1\\
&\quad-\frac{\alpha}{2}(\cos\theta_\alpha+\cos\theta^{-N_\eta})+\cos(\theta^{-N_\eta}-\theta_\alpha)+\frac{\alpha^2}{8}+1\\
&\le\sum_{i=-\infty}^{+\infty}\Big(-\frac{\alpha}{2}(\cos\theta^i+\cos\theta^{i+1})+\cos(\theta^i+\theta^{i+1})+\frac{\alpha^2}{8}+1\Big)+2\omega(\eta)
\end{aligned}
$$

where

$$
\omega(\eta):=\max\Big\{-\frac{\alpha}{2}(\cos\theta+\cos\theta_\alpha)+\cos(\theta+\theta_\alpha)+\frac{\alpha^2}{8}+1:\ |\theta-\theta_\alpha|\le\eta\Big\} \tag{3.3}
$$

is infinitesimal as $\eta\to 0$. This shows that the value C_α defined in (3.1) is less or equal than $\widetilde{C}_\alpha$. Then we are done, the converse inequality being trivial since any test function for problem (3.1) is a test function for problem (3.2). The estimate $C_\alpha>0$ follows from the fact that $\pm\theta_\alpha$ are the unique minimizers of W_α. □

Remark 3.2. We note that, in the case $\alpha\ge 4$, $C_\alpha:=0$ and this definition is consistent with (3.1). In fact, being $\theta_\alpha=-\theta_\alpha=0$, we can choose as a test function in (3.1) $\theta\equiv 0$ thus obtaining the estimate $C_\alpha\le 0$.

It will be useful in the sequel the following continuity property of C_α with respect to the frustration parameter α.

Proposition 3.3 (Continuity). *The crease energy C_α defined as before is continuous in $[0,4)$; i.e., for any $\bar\alpha\in[0,4)$ and any sequence α_j such that $0\le\alpha_j<4$, $\alpha_j\to\bar\alpha$ it results $C_{\alpha_j}\to C_{\bar\alpha}$.*

Proof. Let us fix $\eta>0$ and let $\alpha,\alpha'\in[0,4)$ be such that if $|\alpha-\alpha'|<\delta(\eta)$ for a suitable $\delta(\eta)>0$, then $|\theta_\alpha-\theta_{\alpha'}|<\eta/2$.

From the definition of $C_{\alpha'}$ as in (3.2), there exists a function $\theta : \mathbb{Z} \to [-\pi/2, \pi/2]$ such that $\sum_{i\in\mathbb{Z}} \mathcal{E}^{i,\alpha'}(\theta) < C_{\alpha'} + \eta$, where we have set

$$\mathcal{E}^{i,\alpha'}(\theta) := -\frac{\alpha'}{2}\left(\cos\theta^i + \cos\theta^{i+1}\right) + \cos(\theta^i + \theta^{i+1}) + \frac{(\alpha')^2}{8} + 1,$$

and $\lim_{i\to\pm\infty} \operatorname{sign}(i)\theta^i = \theta_{\alpha'}$. This means that there exist two indices $h_1(\eta), h_2(\eta) \in \mathbb{N}$ such that

$$|\theta^i - \theta_{\alpha'}| < \eta/2 \quad \text{for every} \quad i > h_2(\eta) \quad \text{and} \quad |\theta^i - (-\theta_{\alpha'})| < \eta/2 \quad \text{for every} \quad i < -h_1(\eta).$$

Setting $\bar{h} = \bar{h}(\eta) := \max\{h_1, h_2\}$ and $K_{\bar{h}} := \{i \in \mathbb{Z} : |i| \leq \bar{h}\}$, we observe that for every $i \notin K_{\bar{h}}$ it also holds that $|\operatorname{sign}(i)\theta^i - \theta_\alpha| < \eta$.

Now we modify θ in order to obtain a test function for the problem defining C_α by setting

$$\tilde{\theta}^i = \begin{cases} \theta^i, & \text{if } i \in K_{\bar{h}} \\ \operatorname{sign}(i)\theta_\alpha, & \text{otherwise.} \end{cases} \tag{3.4}$$

We then have

$$\begin{aligned} C_\alpha &\leq \sum_{i\in\mathbb{Z}} \mathcal{E}^{i,\alpha}(\tilde{\theta}) = \sum_{|i|<\bar{h}} \mathcal{E}^{i,\alpha}(\theta) + \mathcal{E}^{-\bar{h},\alpha}(\tilde{\theta}) + \mathcal{E}^{\bar{h},\alpha}(\tilde{\theta}) \\ &\leq \sum_{i\in\mathbb{Z}} \mathcal{E}^{i,\alpha'}(\theta) + 2|\alpha - \alpha'|\#K_{\bar{h}} + \left(\mathcal{E}^{-\bar{h},\alpha}(\tilde{\theta}) - \mathcal{E}^{-\bar{h},\alpha'}(\theta)\right) \\ &+ \left(\mathcal{E}^{\bar{h},\alpha}(\tilde{\theta}) - \mathcal{E}^{\bar{h},\alpha'}(\theta)\right), \end{aligned} \tag{3.5}$$

where in the second inequality we used the estimate

$$\sum_{|i|<\bar{h}} \left|\mathcal{E}^{i,\alpha}(\theta) - \mathcal{E}^{i,\alpha'}(\theta)\right| \leq 2|\alpha - \alpha'|\#K_{\bar{h}}.$$

Each of the last two terms in (3.5) can be estimated in the same way, so we make an explicit computation only for the latter. We have

$$\begin{aligned} &\left|\mathcal{E}^{\bar{h},\alpha}(\tilde{\theta}) - \mathcal{E}^{\bar{h},\alpha'}(\theta)\right| \leq \left|(\cos(\theta^{\bar{h}} + \theta_\alpha) - \cos(\theta^{\bar{h}} + \theta^{\bar{h}+1})\right| + \left|\frac{\alpha^2 - (\alpha')^2}{8}\right| + \\ &\left|\frac{\alpha}{2}(\cos\theta^{\bar{h}} + \cos\theta_\alpha) - \frac{\alpha'}{2}(\cos\theta^{\bar{h}} + \cos\theta^{\bar{h}+1})\right| \leq \eta + 2|\alpha - \alpha'|. \end{aligned}$$

Collecting all the previous estimates and inserting them into (3.5) we obtain

$$\begin{aligned} C_\alpha &\leq C_{\alpha'} + \eta + 2|\alpha - \alpha'|\#K_{\bar{h}} + \eta + 2|\alpha - \alpha'| \\ &\leq C_{\alpha'} + 2\eta + 2|\alpha - \alpha'|(1 + \#K_{\bar{h}}). \end{aligned}$$

Choosing now $\gamma \geq 4\eta$ and α and α' such that

$$|\alpha - \alpha'| \leq \min \left\{ \delta(\eta), \frac{\gamma}{4(1 + \# K_{\bar{h}})} \right\} =: \sigma(\gamma, \eta),$$

we finally obtain $C_\alpha \leq C_{\alpha'} + \gamma$.

If we change the role of α and α', we get an analogous estimate for $C_{\alpha'}$. Hence, we conclude that for every $\gamma \geq 4\eta$, there exists $\sigma(\gamma, \eta) > 0$ such that if $|\alpha - \alpha'| < \sigma(\gamma, \eta)$ then $|C_\alpha - C_{\alpha'}| \leq \gamma$. Since the choice of η was arbitrary, the assertion immediately follows. □

The following compactness result states that, if $0 \leq \alpha < 4$, then functions θ_n with equibounded energy E_n^α converge to a limit function θ which has a finite number of jumps and is such that $\theta \in \{\pm\theta_\alpha\}$ a.e., while if $\alpha \geq 4$, then θ_n converge necessarily to the function $\theta(t) = 0$.

Proposition 3.4 (Compactness). *Let $E_n^\alpha : L^\infty(I, J) \to [0, +\infty]$ be the energies defined by (2.12). If $\{\theta_n\}$ is a sequence of functions such that*

$$\sup_n E_n^\alpha(\theta_n) < +\infty, \tag{3.6}$$

then we have two cases:
(i) *if $0 \leq \alpha < 4$ there exists a set $S \subset (0,1)$ with $\#S < +\infty$ such that, up to subsequences, θ_n converges to θ in $L^1_{loc}((0,1)\backslash S)$, where θ is a piecewise constant function and $\theta(0+) = \theta(1+)$. Moreover, $\theta(t) \in \{\pm\theta_\alpha\}$ for a.e. $t \in (0,1)$ and $S(\theta) \subseteq S$.*
(ii) *If $\alpha \geq 4$ then the limit function θ is identically 0.*

Proof. (i) We first note that $-\alpha \cos\theta \geq |p_\alpha \theta| - \alpha - 1$ for $\theta \in [-\pi/2, \pi/2]$ and a constant p_α depending on α, so that

$$\begin{aligned} C > C\lambda_n \geq \lambda_n E_n^\alpha(\theta_n) \geq& |p_\alpha| \sum_{i=0}^{n-1} \lambda_n |\theta_n^i| - (\alpha+1)n\lambda_n + \lambda_n - n\lambda_n + \frac{\alpha^2}{8} + 1 \\ \geq& |p_\alpha| \sum_{i=0}^{n-1} \lambda_n |\theta_n^i| - \alpha - 1, \end{aligned}$$

from which we deduce that

$$\int_0^1 |\theta_n(t)|\, dt < +\infty. \tag{3.7}$$

By (3.6) we get

$$
\begin{aligned}
C \geq E_n^\alpha(\theta_n) &= \sum_{i=0}^{n-1}\Big(\cos(\theta_n^i+\theta_n^{i+1}) - \frac{\alpha}{2}(\cos\theta_n^i+\cos\theta_n^{i+1}) + \frac{\alpha^2}{8}+1\Big)\\
&=: \sum_{i=0}^{n-1}\mathcal{E}_n^i(\theta_n),
\end{aligned}
\tag{3.8}
$$

where we have set

$$
\mathcal{E}_n^i(\theta_n) = \cos(\theta_n^i+\theta_n^{i+1}) - \frac{\alpha}{2}(\cos\theta_n^i+\cos\theta_n^{i+1}) + \frac{\alpha^2}{8}+1. \tag{3.9}
$$

Hence, for every fixed $\eta>0$, if we define $I_n(\eta) := \{i\in\{0,1,\dots,n-1\} : \mathcal{E}_n^i(\theta_n)>\eta\}$, then there exists a constant $C(\eta)$ such that

$$
\sup_n \#I_n(\eta) \leq C(\eta) < +\infty. \tag{3.10}
$$

Let $i\notin I_n(\eta)$; that is,

$$
\mathcal{E}_n^i(\theta_n) = \cos(\theta_n^i+\theta_n^{i+1}) - \frac{\alpha}{2}(\cos\theta_n^i+\cos\theta_n^{i+1}) + \frac{\alpha^2}{8}+1 \leq \eta. \tag{3.11}
$$

Let $\sigma=\sigma(\eta)$ be defined so that if

$$
0 \leq \cos(\theta_1+\theta_2) - \frac{\alpha}{2}(\cos\theta_1+\cos\theta_2) + \frac{\alpha^2}{8}+1 \leq \eta,
$$

then

$$
\min_{\theta\in\{\pm\theta_\alpha\}} \sqrt{(\theta_1-\theta)^2+(\theta_2-\theta)^2} \leq \sigma(\eta).
$$

Choosing $\eta>0$ such that $\sigma(\eta)<\sqrt{2}\theta_\alpha$, if $i\notin I_n(\eta)$ we deduce the existence of $\theta\in\{\pm\theta_\alpha\}$ such that

$$
|\theta_n^i-\theta|\leq\sigma \text{ and } |\theta_n^{i+1}-\theta|\leq\sigma.
$$

Hence, there exists a finite number of indices $0=i_1<i_2<\cdots<i_{N_n}=n$ such that for all $k=1,2,\dots,N_n$ there exists $\theta_k^n\in\{\pm\theta_\alpha\}$ such that for all $i\in\{i_{k-1}+1, i_{k-1}+2,\dots,i_k-1\}$ we have

$$
|\theta_n^i-\theta_k^n|\leq\sigma \text{ and } |\theta_n^{i+1}-\theta_k^n|\leq\sigma. \tag{3.12}
$$

Let $\{j_1,j_2,\dots,j_{M_n}\}$ be the maximal subset of $0=i_1<i_2<\cdots<i_{N_n}=n$ defined by the requirement that if $\theta_{j_k}^n=\pm\theta_\alpha$ then $\theta_{j_k+1}^n=\mp\theta_\alpha$; this means that

$\{j_1, \dots, j_{M_n}\} \subseteq I_n(\eta)$. Hence, there exists $C(\eta) > 0$ such that $\sum_{i=0}^{n-1} \mathcal{E}_n^i(\theta_n) \geq C(\eta)M_n$ and then $E_n^\alpha(\theta_n) \geq C(\eta)M_n$, so that from (3.6) M_n are equibounded. Thus, up to further subsequences, we can assume that $M_n = M$ and that for every $k = 1, \dots, M$, $t_{j_k}^n = \lambda_n i_{j_k} \to t_k$ for some $t_k \in [0,1]$ and $\theta_{j_k}^n = \theta_k$. Set $S = \{t_1, \dots t_M\}$ and, for fixed $\delta > 0$, $S_\delta = \bigcup_k (t_k - \delta, t_k + \delta)$. Then, by identifying θ_n with its piecewise constant interpolation, from (3.12) and for n large enough we get

$$\sup_{t \in (0,1) \backslash S_\delta} |\theta_n(t) - \theta_k| \leq \sigma.$$

The previous estimates, together with (3.7) ensure that $\{\theta_n\}$ is an equicontinuous and equibounded sequence in $(0,1) \backslash S_\delta$. Thus, thanks to the arbitrariness of δ, up to passing to a further subsequence (not relabelled), θ_n converges in $L^\infty_{loc}((0,1) \backslash S)$ (and in $L^1_{loc}((0,1) \backslash S)$) to a function θ such that $\theta(t) \in \{\pm\theta_\alpha\}$ for a.e. $t \in (0,1)$. Moreover, $S(\theta) \subseteq S$. Finally, by the periodicity assumption (2.1), we have $\theta_n^0 = \theta_n^n$ from which passing to the limit as $n \to \infty$ we conclude that $\theta(0+) = \theta(1+)$.
(ii) The proof of (ii) requires minor changes in the argument for (i), so we will omit it. □

Now we state the Γ-convergence result.

Theorem 3.5. *(i) Let $\alpha \in [0,4)$. Then E_n^α Γ-converges with respect to the L^1_{loc}-topology to*

$$E^\alpha(\theta) = \begin{cases} C_\alpha \#(S(\theta) \cap [0,1)), & \text{if } \theta \in PC_{loc}(\mathbb{R}),\ \theta \in \{\pm\theta_\alpha\} \\ & \theta \text{ is 1-periodic} \\ +\infty, & \text{otherwise} \end{cases} \tag{3.13}$$

on $L^1_{loc}(\mathbb{R})$ and $\theta(0+) = \theta(1+)$, where $C_\alpha = C(-\theta_\alpha, \theta_\alpha)$ is given by (3.1).
(ii) Let $\alpha \in [4, +\infty)$. Then E_n^α Γ-converges with respect to the L^1_{loc}-topology to

$$E^\alpha(\theta) = \begin{cases} 0, & \text{if } \theta = 0 \\ +\infty, & \text{otherwise} \end{cases} \tag{3.14}$$

on $L^1_{loc}(\mathbb{R})$.

Proof. (i) **Liminf inequality.** We may assume, without loss of generality, that θ is left-continuous at each jump. Let $\theta_n \to \theta$ in $L^1(0,1)$ be such that $E_n^\alpha(\theta_n) < +\infty$. Then, from Proposition 3.4 there exist $N \in \mathbb{N}$, $\bar\theta_1, \dots, \bar\theta_N \in \{\pm\theta_\alpha\}$ and $0 = s_0 < s_1 < \dots < s_N = 1$, $\{s_j\} = \{t_k\}$ (the set of indices may be different if $s_k = s_{k+1}$ for some k) such that

$$\theta_{j_k}^n \to \bar\theta_j, \quad \text{on the interval } (s_{j-1}, s_j),\ j \in \{1, \dots, N\}. \tag{3.15}$$

For $l \in \{0, 1, \dots, N\}$, let $\{k_n^l\}_n$ be a sequence of indices such that $k_n^0 = 0$,

$$\lim_n \lambda_n k_n^l = s_l,$$

and let $\{h_n^l\}_n$ be another sequence of indices such that $h_n^0 = 0$,

$$\lim_n \lambda_n h_n^l = \frac{s_l + s_{l-1}}{2}.$$

To get the Γ-liminf inequality, we rewrite the energy as follows:

$$E_n^\alpha(\theta_n) = \sum_{j=1}^{N-1} E_n^\alpha(\theta_n, h_n^j, h_n^{j+1}) + r_n, \tag{3.16}$$

where we have set

$$E_n^\alpha(\theta_n, h_n^j, h_n^{j+1}) = \sum_{i=h_n^j}^{h_n^{j+1}-1} \Big(\cos(\theta_n^i + \theta_n^{i+1}) - \frac{\alpha}{2}\left(\cos\theta_n^i + \cos\theta_n^{i+1}\right) + \frac{\alpha^2}{8} + 1\Big) \tag{3.17}$$

and

$$r_n = \sum_{i=0}^{h_n^1 - 1} \mathcal{E}_n^i(\theta_n) + \sum_{i=h_n^N+1}^{n-1} \mathcal{E}_n^i(\theta_n),$$

with $r_n > 0$ and $\mathcal{E}_n^i(\theta_n)$ as in (3.9). Defining for $j \in \{1, 2, \dots, N-1\}$

$$\tilde{\theta}_n^i = \begin{cases} \bar{\theta}_j, & \text{if } i \le h_n^j - k_n^j - 1, \\ \theta_n^{i+k_n^j}, & \text{if } h_n^j - k_n^j \le i \le h_n^{j+1} - k_n^j - 1, \\ \bar{\theta}_{j+1}, & \text{if } i \ge h_n^{j+1} - k_n^j, \end{cases} \tag{3.18}$$

we have that $\tilde{\theta}_n^i$ is a test function for the minimum problem defining $C(\theta(s_j-), \theta(s_j+))$ as in (3.1), where $\theta(s_j-) = \bar{\theta}_j$ and $\theta(s_j+) = \bar{\theta}_{j+1}$.

For n large enough and any $\sigma > 0$, we have that

$$
\begin{aligned}
E_n^\alpha(\theta_n, k_n^j, k_n^{j+1}) &= \sum_{i=h_n^j}^{k_n^j} \mathcal{E}_n^i(\theta_n) + \sum_{i=k_n^j}^{h_n^{j+1}-1} \mathcal{E}_n^i(\theta_n) \\
&= \sum_{l=h_n^j-k_n^j}^{0} \mathcal{E}_n^{l+k_n^j}(\theta_n) + \sum_{l=1}^{h_n^{j+1}-k_n^j-1} \mathcal{E}_n^{l+k_n^j}(\theta_n) \\
&= \sum_{l=h_n^j-k_n^j}^{h_n^{j+1}-k_n^j-1} \mathcal{E}_n^l(\tilde{\theta}_n) \\
&= \sum_{l\in\mathbb{Z}} \mathcal{E}_n^l(\tilde{\theta}_n) - \sum_{l\le h_n^j-k_n^j-1} \mathcal{E}_n^l(\tilde{\theta}_n) - \sum_{l\ge h_n^{j+1}-k_n^j} \mathcal{E}_n^l(\tilde{\theta}_n) \\
&= \sum_{l\in\mathbb{Z}} \mathcal{E}_n^l(\tilde{\theta}_n) - \mathcal{E}_n^{h_n^j-k_n^j-2}(\tilde{\theta}_n) \\
&\ge \sum_{i\in\mathbb{Z}} \Big(\cos(\tilde{\theta}_n^i + \tilde{\theta}_n^{i+1}) - \frac{\alpha}{2}\Big(\cos\tilde{\theta}_n^i + \cos\tilde{\theta}_n^{i+1}\Big) + \frac{\alpha^2}{8} + 1\Big) - \omega(\sigma) \\
&\ge C(\bar{\theta}_j, \bar{\theta}_{j+1}) - \omega(\sigma),
\end{aligned} \tag{3.19}
$$

where $\omega : [0, +\infty) \to [0, +\infty)$ is a suitable continuous function, $\omega(0) = 0$. Finally, by the arbitrariness of $\sigma > 0$,

$$
\begin{aligned}
\liminf_{n\to\infty} E_n^\alpha(\theta_n) &\ge \liminf_{n\to\infty} \Big[(N-1)C_\alpha - (N-1)\omega(\sigma)\Big] \\
&= C_\alpha \#(S(\theta) \cap [0,1)).
\end{aligned} \tag{3.20}
$$

Limsup inequality. Let θ be such that $E^\alpha(\theta) < +\infty$. Then there exist $M \in \mathbb{N}$, $\bar{\theta}_1, \dots, \bar{\theta}_M \in \{\pm\theta_\alpha\}$ and $0 = t_0 < t_1 < \dots < t_M = 1$ such that $\#S(\theta) = M - 1$ and

$$
\theta(t) = \bar{\theta}_j, \quad t \in (t_{j-1}, t_j),\ j \in \{1, 2, \dots, M\}. \tag{3.21}
$$

Fixed $\eta > 0$, from the definition of $C(\theta(t_j-), \theta(t_j+))$ for $j \in \{1, 2, \dots, M-1\}$ we can find functions $\psi_{j,j+1} : \mathbb{Z} \to [-\pi/2, \pi/2]$, such that

$$
\psi_{j,j+1}^i = \begin{cases} \bar{\theta}_j & \text{for } i \le -N_j, \\ \bar{\theta}_{j+1} & \text{for } i \ge N_j, \end{cases} \tag{3.22}
$$

and

$$\sum_{i\in\mathbb{Z}}\Big(\cos(\psi^i_{j,j+1}+\psi^{i+1}_{j,j+1})-\frac{\alpha}{2}\Big(\cos\psi^i_{j,j+1}+\cos\psi^{i+1}_{j,j+1}\Big)+\frac{\alpha^2}{8}+1\Big)$$
$$\leq C(\theta(t_j-),\theta(t_j+))+\eta=C_\alpha+\eta.$$

Note that in (3.22) we may assume $N = N_j$ independent of j, up to choose $N = \max\limits_{1\leq j\leq M-1}\{N_j\}$.

We define a recovery sequence $\tilde{\theta}_n$ by means of functions $\psi_{j,j+1}$ as

$$\tilde{\theta}^i_n=\begin{cases}\bar{\theta}_1 & \text{if } 0\leq i\leq \lfloor t_1 n\rfloor - N\\ \psi^{i-\lfloor t_j n\rfloor}_{j,j+1} & \text{if } \lfloor t_j n\rfloor - N\leq i\leq \lfloor t_j n\rfloor + N\\ \bar{\theta}_{j+1} & \text{if } \lfloor t_j n\rfloor + N\leq i\leq \lfloor t_{j+1} n\rfloor - N, \quad j\in\{1,\dots,M-1\}\\ \bar{\theta}_M & \text{if } n-N\leq i\leq n-1.\end{cases} \tag{3.23}$$

We note that $\tilde{\theta}_n\to\theta$ in L^∞ and (here we use the simplified notation for the energies as in (3.9))

$$\begin{aligned}E^\alpha_n(\tilde{\theta}_n)=\sum_{i=0}^{n-1}\mathcal{E}^i_n(\tilde{\theta}_n)&=\sum_{i=0}^{\lfloor t_1 n\rfloor-N}\mathcal{E}^i_n(\tilde{\theta}_n)+\sum_{j=1}^{M-1}\sum_{i=\lfloor t_j n\rfloor-N}^{\lfloor t_j n\rfloor+N}\mathcal{E}^i_n(\tilde{\theta}_n)\\&\quad+\sum_{j=1}^{M-1}\sum_{i=\lfloor t_j n\rfloor+N}^{\lfloor t_{j+1} n\rfloor-N}\mathcal{E}^i_n(\tilde{\theta}_n)+\sum_{i=n-N}^{n-1}\mathcal{E}^i_n(\tilde{\theta}_n)\\&=\sum_{j=1}^{M-1}\sum_{i=\lfloor t_j n\rfloor-N}^{\lfloor t_j n\rfloor+N}\mathcal{E}^i_n(\psi^i_{j,j+1})=\sum_{j=1}^{M-1}\sum_{i\in\mathbb{Z}}\mathcal{E}^i_n(\psi^i_{j,j+1})\\&\leq (M-1)(C_\alpha+\eta),\end{aligned}$$

from which, by the arbitrariness of η, we deduce that

$$\limsup_{n\to+\infty}E^\alpha_n(\tilde{\theta}_n)\leq (M-1)(C_\alpha+\eta)=C_\alpha\#(S(\theta)\cap[0,1)).$$

(ii) The proof in this case is immediate. Let $\theta_n\to 0$. From $E^\alpha_n(\theta_n)\geq 0$ we have in particular that

$$\liminf_{n\to+\infty}E^\alpha_n(\theta_n)\geq 0.$$

As a recovery sequence, we can choose $\theta_n\equiv 0$, for which we obtain $\lim\limits_{n\to+\infty}E^\alpha_n(\theta_n)=0$.

□

4 Limit behaviour near the transition point $\alpha = 4$

The description of the limit as $n \to +\infty$ of the energies E_n^α with fixed α, as done in the previous section, has a gap for $\alpha = 4$. In fact, the crease energy C_α jumps from a strictly positive value (corresponding to $0 \le \alpha < 4$) to 0 (when $\alpha \ge 4$). Note also that the explicit value of C_α defined implicitly by (3.1) is not known in literature. This suggests to focus near the transition point $\alpha = 4$, let the parameter α depend on n and be close to 4 from below; that is, replace α by $4 - \alpha_n$, $\alpha_n \to 4^-$.

Such analysis is the main content of a recent paper by Cicalese and Solombrino [10], where they find suitable scaling and order parameter to compute the energy the system spends in a transition between two states with different chirality when $\alpha \simeq 4$. Moreover, they show the dependence of the limit on the particular sequence $\alpha_n \to 4^-$ and the existence of different regimes. Our aim is to show that their result can be retrieved also correspondingly to a different choice of the order parameter in the energies. First of all, we write the energies (2.12) in terms of $4 - \alpha$ as

$$E_n^\alpha(\theta) = (4 - (4 - \alpha)) \sum_{i=0}^{n-1} (1 - \cos\theta^i) - \sum_{i=0}^{n-1} (1 - \cos(\theta^i + \theta^{i+1})) + n\frac{(4 - \alpha)^2}{8}. \tag{4.1}$$

When necessary, we may think also the quantities W_α, C_α, etc. to be functions of $4 - \alpha$. Note that if we choose as a test function in (3.1) $\theta^{i,\alpha} = \mathrm{sign}(i) \arccos\Big(\frac{\alpha}{4}\Big)$ then we obtain a first rough estimate

$$0 < C_\alpha \le (4 - \alpha) - \frac{(4 - \alpha)^2}{8},$$

showing in particular that $C_\alpha \to 0$ as $\alpha \to 4^-$.

The following proposition (compare with [10], Proposition 4.3) characterizes the angles between neighbours for an equibounded (in energy) sequence of spins.

Proposition 4.1. *If $\{\theta_n\}$ is a sequence such that*

$$\sup_n E_n^{\alpha_n}(\theta_n) \le C(4 - \alpha_n)^{3/2}, \tag{4.2}$$

then $\theta_n^i \to 0$ as $n \to \infty$ uniformly with respect to $i \in \{0, 1, \dots, n-1\}$.

Proof. The claim follows immediately from the estimate

$$0 \le 2\Big(\cos\theta_n^i - \frac{\alpha_n}{4}\Big)^2 \le \sum_{i=0}^{n-1} W_{\alpha_n}(\theta_n^i) \le E_n^{\alpha_n}(\theta_n) \le C(4 - \alpha_n)^{3/2} \tag{4.3}$$

valid for all $i \in \{0, 1, \dots, n-1\}$. □

We introduce a new order parameter

$$v_n^i = \frac{\theta_n^i}{\theta_{\alpha_n}} \tag{4.4}$$

and reformulate the Γ-convergence result by Cicalese and Solombrino ([10], Theorem 4.2) in terms of this new variable. However, it is worth noting that the "flat" angular parameter v_n^i is equivalent with their variable z_n^i in the regime of "small angles", i.e. as $\alpha_n \to 4^-$, $\theta_n^i \to 0$, since in this case

$$\theta_{\alpha_n} = \arccos\left(1 - \frac{(4-\alpha_n)}{4}\right) \simeq \frac{\sqrt{4-\alpha_n}}{\sqrt{2}}$$

and

$$z_n^i = \frac{2\sqrt{2}}{\sqrt{4-\alpha_n}} \sin\left(\frac{\theta_n^i}{2}\right) \simeq \frac{\sqrt{2}\theta_n^i}{\sqrt{4-\alpha_n}}.$$

The change of variables (4.4) associates to any given $\theta_n \in \mathcal{C}_n(I,J)$ a piecewise-constant function $v_n \in \tilde{\mathcal{C}}_n(I,\mathbb{R})$ where

$$\tilde{\mathcal{C}}_n(I,\mathbb{R}) := \Big\{ v : [0,1) \to \mathbb{R} : v(t) = v_n^i \text{ if } t \in \lambda_n(i + [0,1)), \\ i \in \{0,1,\dots,n-1\} \Big\}$$

with v_n as in (4.4). With a slight abuse of notation, we regard $E_n^{\alpha_n}$ as a functional defined on $v \in L^1(I,\mathbb{R})$ by

$$E_n^{\alpha_n}(v) = \begin{cases} E_n^{\alpha_n}(\theta), & \text{if } v \in \tilde{\mathcal{C}}_n(I,\mathbb{R}) \\ +\infty, & \text{otherwise,} \end{cases} \tag{4.5}$$

and correspondingly we define the scaled energies

$$F_n^{\alpha_n}(v) := \frac{8E_n^{\alpha_n}(v)}{\sqrt{2}(4-\alpha_n)^{3/2}}. \tag{4.6}$$

Theorem 4.2 (Cicalese and Solombrino [10]). *Let $F_n^{\alpha_n} : L^1(I,\mathbb{R}) \to [0,+\infty]$ be the functional in (4.6). Assume that there exists $l := \lim_n \sqrt{2}\lambda_n/4(4-\alpha_n)^{1/2}$. Then $F^0(v) := \Gamma\text{-}\lim_n F_n^{\alpha_n}(v)$ with respect to the $L^1(I)$ convergence is given by:*

(i) if $l = 0$

$$F^0(v) := \begin{cases} \frac{8}{3}\#(S(v)) & \text{if } v \in BV(I,\{\pm 1\}), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.7}$$

(ii) if $l \in (0,+\infty)$

$$F^0(v) := \begin{cases} \frac{1}{l}\int_I \Big(v^2(t)-1\Big)^2 dt + l\int_I (\dot v(t))^2\, dt & \textit{if } v \in W^{1,2}_{|per|}(I), \\ +\infty & \textit{otherwise.} \end{cases} \tag{4.8}$$

where we have set $W^{1,2}_{|per|}(I) := \{v \in W^{1,2}(I) : |v(0)| = |v(1)|\}$.

(iii) if $l = +\infty$

$$F^0(v) := \begin{cases} 0 & \textit{if } v = const., \\ +\infty & \textit{otherwise.} \end{cases} \tag{4.9}$$

Proof. In order to simplify the notation, we put $\varepsilon_n := 4 - \alpha_n \to 0$ as $n \to \infty$. Let $\{v_n\}$ be a sequence in $\tilde{\mathcal{C}}_n(I,\mathbb{R})$ such that $\sup_n \frac{E_n^{\varepsilon_n}(v_n)}{\varepsilon_n^{3/2}} \le C < \infty$. As remarked before, correspondingly, there exists a sequence $\{\theta_n\}$ in $\mathcal{C}_n(I,J)$ such that $\sup_n \frac{E_n^{\varepsilon_n}(\theta_n)}{\varepsilon_n^{3/2}} \le C < \infty$, satisfying $\theta_n^i \to 0$ uniformly with respect to i by Proposition 4.1.

From the estimates contained in the proof of Theorem 4.2 in [10] we get

$$E_n^{\varepsilon_n}(\theta_n) \ge 8\sum_{i=0}^{n-1}\Big(\sin^2\Big(\frac{\theta_n^i}{2}\Big) - \frac{\varepsilon_n}{8}\Big)^2 + 2(1-\gamma_n)\sum_{i=0}^{n-1}\Big(\sin\Big(\frac{\theta_n^{i+1}}{2}\Big) - \sin\Big(\frac{\theta_n^i}{2}\Big)\Big)^2,$$

for some $\gamma_n \to 0$. Since $\sin\theta \simeq \theta$ as $\theta \to 0$, we may improve the estimate obtaining

$$E_n^{\varepsilon_n}(\theta_n) \ge 8(1-\gamma_n')\sum_{i=0}^{n-1}\Big(\Big(\frac{\theta_n^i}{2}\Big)^2 - \frac{\varepsilon_n}{8}\Big)^2 + \frac{(1-\gamma_n'')}{2}\sum_{i=0}^{n-1}\Big(\Big(\frac{\theta_n^{i+1}}{2}\Big) - \Big(\frac{\theta_n^i}{2}\Big)\Big)^2,$$

for suitable $\gamma_n', \gamma_n'' \to 0$. In terms of the new order parameter v_n^i defined by (4.4) the previous inequality now reads

$$E_n^{\varepsilon_n}(\theta_n) \ge \frac{2\theta_{\varepsilon_n}^4}{\lambda_n}(1-\gamma_n')\sum_{i=0}^{n-1}\lambda_n\Big((v_n^i)^2 - \frac{\varepsilon_n}{2\theta_{\varepsilon_n}^2}\Big)^2 + \frac{\theta_{\varepsilon_n}^2\lambda_n}{8}(1-\gamma_n'')\sum_{i=0}^{n-1}\lambda_n\Big(\frac{v_n^{i+1}-v_n^i}{\lambda_n}\Big)^2,$$

where $\lambda_n = \frac{1}{n}$. If we multiply both the sides by $8/\sqrt{2}\varepsilon_n^{3/2}$, since $\frac{\varepsilon_n}{2\theta_{\varepsilon_n}^2} \to 1$ we get

$$\frac{8E_n^{\varepsilon_n}(\theta_n)}{\sqrt{2}\varepsilon_n^{3/2}} \ge \frac{8\sqrt{2}\theta_{\varepsilon_n}^4}{\lambda_n\varepsilon_n^{3/2}}(1-\gamma_n')\sum_{i=0}^{n-1}\lambda_n\Big((v_n^i)^2-1\Big)^2 + \frac{\sqrt{2}\theta_{\varepsilon_n}^2\lambda_n}{2\varepsilon_n^{3/2}}(1-\gamma_n'')\sum_{i=0}^{n-1}\lambda_n\Big(\frac{v_n^{i+1}-v_n^i}{\lambda_n}\Big)^2.$$

Since $\theta_{\varepsilon_n} = \arccos(1 - \frac{\varepsilon_n}{4})$ and $\theta_{\varepsilon_n} \simeq \frac{\sqrt{\varepsilon_n}}{\sqrt{2}}$ as $\varepsilon_n \to 0$, we note that $\dfrac{8\sqrt{2}\theta^4_{\varepsilon_n}}{\lambda_n \varepsilon_n^{3/2}} \simeq \dfrac{4\sqrt{\varepsilon_n}}{\sqrt{2}\lambda_n}$ and $\dfrac{\sqrt{2}\theta^2_{\varepsilon_n}\lambda_n}{2\varepsilon_n^{3/2}} \simeq \dfrac{\sqrt{2}\lambda_n}{4\sqrt{\varepsilon_n}}$ as $n \to \infty$, we finally get

$$\frac{8E_n^{\varepsilon_n}(\theta_n)}{\sqrt{2}\varepsilon_n^{3/2}} \geq \frac{4\sqrt{\varepsilon_n}}{\sqrt{2}\lambda_n}(1-\widetilde{\gamma}'_n)\sum_{i=0}^{n-1}\lambda_n\Big((v_n^i)^2-1\Big)^2+\frac{\sqrt{2}\lambda_n}{4\sqrt{\varepsilon_n}}(1-\widetilde{\gamma}''_n)\sum_{i=0}^{n-1}\lambda_n\Big(\frac{v_n^{i+1}-v_n^i}{\lambda_n}\Big)^2 \quad (4.10)$$

for suitable $\widetilde{\gamma}'_n, \widetilde{\gamma}''_n \to 0$. The estimate (4.10) implies the liminf inequality both in case (i) and (ii) as remarked in [10], and the limsup inequality can be obtained in both cases by the constructive argument contained therein, so we will omit the proof. □

Remark 4.3 (asymptotic behaviour of C_α). As remarked before, $C_\alpha \to 0$ as $\alpha \to 4^-$. However, we may use Theorem 4.2 (i) to refine this estimate and determine the right order of C_{α_n} with respect to $4 - \alpha_n$ as $\alpha_n \to 4$.

In the regime $\lambda_n << (4 - \alpha_n)^{1/2}$ we can compute the limit of energies $F_n^{\alpha_n}(v)$ first as $n \to \infty$ while keeping $\alpha_n \equiv \alpha_0 \neq 4$ fixed, and then the limit as $\alpha_0 \to 4^-$. Thanks to Theorem 3.5 and the continuity result ensured by Proposition 3.3, we obtain

$$F^{\alpha_0}(v) := \Gamma\text{-}\lim_{n\to+\infty} \frac{8E_n^{\alpha_n}(v)}{\sqrt{2}(4-\alpha_n)^{3/2}} = \frac{8C_{\alpha_0}}{\sqrt{2}(4-\alpha_0)^{3/2}}\#(S(v)), \quad (4.11)$$

then, by means of (i) of Theorem 4.2, we get

$$F^0(v) := \Gamma\text{-}\lim_{\alpha_0\to 4} F^{\alpha_0}(v) = \frac{8}{3}\#(S(v)). \quad (4.12)$$

The convergence of minimum problems as $\alpha \to 4^-$ finally gives

$$\lim_{\alpha\to 4^-} \frac{3C_\alpha}{\sqrt{2}(4-\alpha)^{3/2}} = 1. \quad (4.13)$$

Thus, near the ferromagnet-helimagnet transition point, the energy C_α coincides with the energy $E_{dw} \propto (4-\alpha)^{3/2}$ for the excitation of a chiral domain wall separating two domains of opposite chirality, which is a well-known universal low-temperature property of frustrated classical spin chains (see e.g. Dmitriev and Krivnov [13]).

5 A link with the gradient theory of phase transitions

In this section we show that the variational asymptotic behaviour of the energies F_n^α for any $\alpha \in [0, 4]$, both in the case of fixed α (Theorem 3.5) and in the case $\alpha \simeq 4$ (Theorem 4.2), is the same as that of a parametrized family of Modica-Mortola type functionals, thus providing an interesting connection between frustrated lattice spin systems and the gradient theory of phase transitions (see also [5], Section 6).

In order to do that in the framework of the *equivalence by* Γ*-convergence*, we recall some definitions about Γ-equivalence for families of parametrized functionals, uniform equivalence, regular and singular points, as introduced by Braides and Truskinovsky [8].

Definition 5.1 (Γ-equivalence). Let $\mathcal{A}$ be a set of parameters. Two families of parametrized functionals F_n^α and G_n^α are *equivalent at scale 1 at* $\alpha_0 \in \mathcal{A}$ if $F_n^{\alpha_0}$ and $G_n^{\alpha_0}$ are equivalent at scale 1, i.e.

$$\Gamma\text{-}\lim_{n\to+\infty} F_n^{\alpha_0} = \Gamma\text{-}\lim_{n\to+\infty} G_n^{\alpha_0} \tag{5.1}$$

and these Γ-limits are non-trivial.

Definition 5.2 (uniform Γ-equivalence). Let $\mathcal{A}$ be a set of parameters. Two families of parametrized functionals F_n^α and G_n^α are *uniformly equivalent at scale 1 at* $\alpha_0 \in \mathcal{A}$ if for all $n \to +\infty$, $\alpha_n \to \alpha_0$ we have, up to subsequences,

$$\Gamma\text{-}\lim_{n\to+\infty} F_n^{\alpha_n} = \Gamma\text{-}\lim_{n\to+\infty} G_n^{\alpha_n} \tag{5.2}$$

and these Γ-limits are non-trivial. They are uniformly equivalent on $\mathcal{A}$ if they are uniformly equivalent at all $\alpha_0 \in \mathcal{A}$.

Definition 5.3 (regular point). $\alpha_0 \in \mathcal{A}$ is a *regular point* if for all $n \to +\infty$, $\alpha_n \to \alpha_0$ we have, up to a subsequence,

$$\Gamma\text{-}\lim_{n\to+\infty} F_n^{\alpha_n} = \Gamma\text{-}\lim_{n\to+\infty} F_n^{\alpha_0}. \tag{5.3}$$

Definition 5.4 (singular point). $\alpha_0 \in \mathcal{A}$ is a *singular point* if it is not regular; that is, if for all $n \to +\infty$, there exist $\alpha'_n \to \alpha_0$, $\alpha''_n \to \alpha_0$ such that (up to subsequences)

$$\Gamma\text{-}\lim_{n\to+\infty} F_n^{\alpha'_n} \neq \Gamma\text{-}\lim_{n\to+\infty} F_n^{\alpha''_n}. \tag{5.4}$$

According to the previous definitions, each $0 \leq \alpha_0 < 4$ is a regular point for F_n^α, since, as already observed in Remark 4.3, for any sequence $\alpha_n \to \alpha_0$, we have

$$F^{\alpha_0}(v) := \Gamma\text{-}\lim_{n\to+\infty} F_n^{\alpha_n}(v) = \frac{8C_{\alpha_0}}{\sqrt{2}(4-\alpha_0)^{3/2}}\#(S(v)).$$

As a consequence of Theorem 4.2, instead, $\alpha_0 = 4$ is a singular point for F_n^α.

The behavior of the system close to the transition point $\alpha = 4$ can be pictured in the $\frac{1}{n}$–α plane (see Fig. 3), where the crossover line $\frac{1}{n} = (4-\alpha)^{1/2}$ separates a zone where there is *helimagnetic order* ($\frac{1}{n} << (4-\alpha)^{1/2}$) from one where we have *ferromagnetic order* ($\frac{1}{n} >> (4-\alpha)^{1/2}$).

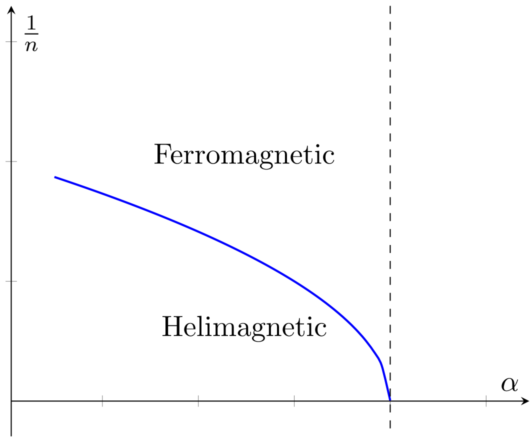


Figure 3: The $\frac{1}{n} - \alpha$ space. In blue the failure curve $\frac{1}{n} = (4-\alpha)^{1/2}$.

It is useful to recall the well known Γ-convergence result in gradient theory of phase transitions due to Modica and Mortola [15]. Let $\Omega \subset \mathbb{R}$ be an open set, $u : \Omega \to \mathbb{R}$ and $W = W(u)$ a non-convex energy such that $W \geq 0$, $W(u) \geq c(u^2 - 1)$ and $W = 0$ if and only if $u = a, b$. W is called a *double-well* potential. Let $C > 0$ and consider the energies

$$F_n(u) = n \int_\Omega W(u)\, dx + \frac{C^2}{n} \int_\Omega (\dot{u})^2\, dx, \quad u \in W^{1,2}(\Omega). \tag{5.5}$$

Theorem 5.5 (Modica-Mortola's theorem). *The functionals F_n above Γ-converge as $n \to \infty$ and with respect to the $L^1(\Omega)$ convergence to the functional*

$$F_\infty(u) = \begin{cases} C \cdot c_W \#(S(u) \cap \Omega), & \text{if } u \in \{a, b\} \text{ a.e.} \\ +\infty, & \text{otherwise,} \end{cases} \tag{5.6}$$

where $c_W := 2 \int_a^b \sqrt{W(s)}\, ds$.

The following theorem states the uniform equivalence by Γ-convergence of energies F_n^α with parametrized Modica-Mortola type functionals.

Theorem 5.6. *Setting* $\lambda_{n,\alpha} := 2n\theta_\alpha^4$, $M_\alpha := 3C_\alpha/8$, $\mu_\alpha := \frac{8}{\sqrt{2}(4-\alpha)^{3/2}}$, *the energies*

$$G_n^\alpha(v) = \begin{cases} \mu_\alpha \Big(\lambda_{n,\alpha} \int_I \Big(v^2 - 1 \Big)^2 dt + \frac{M_\alpha^2}{\lambda_{n,\alpha}} \int_I (\dot{v})^2\, dt \Big), & \text{if } v \in W^{1,2}_{|per|}(I), \\ +\infty, & \text{otherwise in } L^1_{loc}(\mathbb{R}), \end{cases}$$

and $F_n^\alpha(v) := \mu_\alpha E_n^\alpha(v)$ *are uniformly equivalent by Γ-convergence on* $[0, 4]$. *Moreover,*
(i) each $\alpha_0 \in [0, 4)$ *is a regular point;*
(ii) $\alpha_0 = 4$ *is a singular point.*

Proof. As before, we put $\varepsilon_n := 4 - \alpha_n$, so that $\varepsilon_n \geq 0$.
(i) Let $\varepsilon_n \to \varepsilon_0 \neq 0$ and $\{v_n\}$ be a sequence with equibounded energy. Correspondingly, by (4.4) we may find a sequence $\{\theta_n\}$ such that $v_n = \theta_n/\theta_{\varepsilon_n}$. The continuity of energies G_n^ε with respect to the parameter $\varepsilon = \varepsilon_n$, ensured also by Proposition 3.3 allow us to consider, without loss of generality, the energies

$$G_n^{\varepsilon_0}(v) = \frac{8}{\sqrt{2}\varepsilon_0^{3/2}}\Big(2n\theta_{\varepsilon_0}^4 \int_I \Big(v^2 - 1\Big)^2\, dt + \frac{1}{2n\theta_{\varepsilon_0}^4}\Big(\frac{3C_{\varepsilon_0}}{8}\Big)^2 \int_I (\dot{v})^2\, dt\Big),$$

where $v = \theta/\theta_{\varepsilon_0}$. After simplifying constants, we may rewrite the energies in terms of θ as

$$G_n^{\varepsilon_0}(\theta) = \frac{8}{\sqrt{2}\varepsilon_0^{3/2}}\Big(2n \int_I \Big(\theta^2 - \theta_{\varepsilon_0}^2\Big)^2\, dt + \frac{1}{2n}\Big(\frac{3C_{\varepsilon_0}}{8\theta_{\varepsilon_0}^3}\Big)^2 \int_I (\dot{\theta})^2\, dt\Big).$$

In order to compute the Γ-limit as $n \to \infty$ of energies $G_n^{\varepsilon_0}$, we may apply the Γ-convergence result by Modica and Mortola (Theorem 5.5), thus obtaining

$$G^{\varepsilon_0}(\theta) := \Gamma\text{-}\lim_{n\to+\infty} G_n^{\varepsilon_0}(\theta) = \frac{8C_{\varepsilon_0}}{\sqrt{2}\varepsilon_0^{3/2}}\#(S(\theta)),$$

since

$$c_W = 2\int_{-\theta_{\varepsilon_0}}^{\theta_{\varepsilon_0}} |\theta^2 - \theta_{\varepsilon_0}^2|\, d\theta = \frac{8}{3}\theta_{\varepsilon_0}^3.$$

This result coincides with (4.11), once we remark that $\#(S(v)) = \#(S(\theta))$.
(ii) Let $\varepsilon_n \to 0$ and $v \in W^{1,2}_{loc}(\mathbb{R})$. The estimates contained in the proof of Theorem 4.2 and equation (4.13) allow us to rewrite the functional $G_n^{\varepsilon_n}$ as

$$G_n^{\varepsilon_n}(v) = \frac{4n\sqrt{\varepsilon_n}}{\sqrt{2}}(1+\eta_n)\int_I \Big(v^2 - 1\Big)^2\, dt + \frac{\sqrt{2}}{4n\sqrt{\varepsilon_n}}(1+\eta_n')\int_I (\dot{v})^2\, dt,$$

for suitable sequences $\eta_n, \eta_n' \to 0$. In order to simplify the notation, we put

$$K_n := \frac{\sqrt{2}}{4n\sqrt{\varepsilon_n}},$$

and we write

$$G_n^{\varepsilon_n}(v) = \frac{1}{K_n}(1+\eta_n)\int_I \Big(v^2 - 1\Big)^2\, dt + K_n(1+\eta_n')\int_I (\dot{v})^2\, dt.$$

We distinguish between three cases:
(a) $K_n \to 0$. In this case, we apply anymore Theorem 5.5 (with $C = 1$), thus obtaining

$$G^0(v) := \Gamma\text{-}\lim_{n\to+\infty} G_n^{\varepsilon_n}(v) = \frac{8}{3}\#(S(v)), \tag{5.7}$$

since $c_W = 2\int_{-1}^{1} |v^2-1|\,dv = \frac{8}{3}$.
(b) $K_n \to l \in (0,+\infty)$. A sequence v_n with equibounded energy is weakly compact in $W^{1,2}_{|per|}(I)$, then by lower semicontinuity in $W^{1,2}_{|per|}(I)$ we get

$$\liminf_n G_n^{\varepsilon_n}(v_n) \geq \frac{1}{l}\int_I \Big(v^2-1\Big)^2\,dt + l\int_I (\dot v)^2\,dt.$$

In order to obtain the limsup inequality, we can argue by density considering $v_n \in W^{1,2}_{|per|}(I)\cap C^\infty(\overline{I})$, $v_n \to v$ such that

$$\lim_n G_n^{\varepsilon_n}(v_n) = \frac{1}{l}\int_I \Big(v^2-1\Big)^2\,dt + l\int_I (\dot v)^2\,dt.$$

(c) $K_n \to +\infty$. Let v be a constant function, and consider the constant sequence $v_n \equiv v$. Trivially,

$$\liminf_n G_n^{\varepsilon_n}(v_n) = \liminf_n \Big(\frac{1}{K_n}(1+\eta_n)\int_I \Big(v_n^2-1\Big)^2\,dt\Big) \geq 0,$$

and

$$\lim_n G_n^{\varepsilon_n}(v_n) = 0.$$

□

The proof of point (i) of Theorem 5.6 allows us to deduce an equivalence result also for energies $E_n^\alpha(\theta)$ defined in (2.12) with Modica Mortola type functionals whose potentials $\mathcal{W}_\alpha(\theta) := (\theta^2-\theta_\alpha^2)^2$ have the wells at the minimal angles $\theta = \pm\theta_\alpha$. It can be stated as follows.

Corollary 5.7. *Let α be a positive number, $\alpha \in [0,4)$. The energies $E_n^\alpha(\theta)$ and the family of functionals $H_n^\alpha(\theta)$ defined as*

$$H_n^\alpha(\theta) = \begin{cases} \dfrac{\lambda_{n,\alpha}}{\theta_\alpha^4}\displaystyle\int_I \Big(\theta^2(t)-\theta_\alpha^2\Big)^2\,dt + \frac{M_\alpha^2}{\lambda_{n,\alpha}\theta_\alpha^2}\int_I (\dot\theta(t))^2\,dt, & \text{if } \theta \in W^{1,2}_{|per|}(I),\\ +\infty, & \text{otherwise in } L^1_{loc}(\mathbb{R}), \end{cases}$$

are uniformly equivalent by Γ-convergence.

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