Some Results on Geometric Evolution Problems

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Ph.D. Thesis

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Contents

Introduction

This thesis is devoted to the study of some aspects of curvature flows both for abstract and immersed smooth differentiable manifolds. We will be concerned with three kinds of flows: the Ricci flow, the mean curvature flow (MCF) and a curvature flow which can be used to make evolve *singular* immersed initial data in some Euclidean space.
Geometric flows have been introduced many years ago (see e.g. [**10**]) to make the metric of a Riemannian manifold evolve towards a *special* one. In particular, given a Riemannian manifold (M, g) and a smooth functional $\mathcal{F}$ of the metric and its derivatives, it is possible to try to make the metric move along the gradient lines of $\mathcal{F}$ to investigate the existence of special metrics corresponding to critical points.
Within this frame, one can date the introduction of the mean curvature flow to the paper of Mullins (see [**30**]), where the evolution of some interfaces is studied by means of the gradient flow of a functional which is proportional to the area of the interface. Another motivation for the study of the mean curvature flow comes from its geometric applications. Indeed, it is possible to use this kind of flow to obtain classification theorems for hypersurfaces satisfying certain curvature conditions, to obtain isoperimetric inequalities and to construct minimal surfaces (see e.g. [**13**], [**12**], [**20**], [**23**], [**21**], [**22**], [**24**], [**25**] and [**38**]).
The Ricci flow has been introduced more recently by Hamilton (see [**16**]) as a possible way to give a proof of the Poincaré conjecture. Despite this kind of flow is not the gradient flow of any smooth functional of the metric and its derivatives, the discovery of its *gradient-like* structure by Perelman (see [**32**]) both brought to the proof of the Poincaré and Thurston conjectures (see [**34**], [**33**]) and gave a strong impulse to the study of the Ricci flow. Amongst the most important results obtained along these lines, we mention the proof of the long standing *differentiable sphere* conjecture by Brendle and Schoen [**2**], [**4**].

This thesis is essentially divided into three parts. In Chapters 1 and 2 we deal with the Ricci flow and we present a new framework for the description of its *gradient-like* structure. Moreover, we give some result on the possibility to obtain monotonicity formulas from the coupling of the Ricci flow with the MCF.
In Chapters 3 and 4 we study some aspects of the MCF. We show some results which can be obtained by maximizing the Huisken's and we give some applications to the study of the singularities (especially for the case of evolving plane curves).
In Chapter 5, we begin the study of a curvature motion for hypersurfaces with boundary. Our motivation is to analyze the mean curvature evolution of partitions of Euclidean spaces in dimension greater than two.
The thesis is structured as follows:

Chapter 1 is mainly for a notational purpose: we will briefly recall some theorems and formulae in Riemannian geometry and the basic tools which are used in the study of the Ricci flow and of the mean curvature flow.

In Chapter 2, we focus our attention on the *gradient-like* structure of the Ricci flow and, taking inspiration from Perelman's work (see [**32**]), we give a frame which makes it possible to write this flow - and other flows - in a *gradient-like* way.

In Chapter 3 we consider the coupling of the Ricci flow with the mean curvature flow. This kind of coupling can be useful to find new monotonicity formulae for submanifold evolving by mean curvature. Even if at the moment we miss a general theory for such a kind of coupling, we will show some new monotonic quantities arising in some special settings.

In Chapter 4 we take as our starting point the Hamilton's generalization of the Huisken's monotonicity formula[**18**], [**19**] and we will investigate the structure of the largest set of functions on which this formula makes sense. In the second part of the chapter we give some applications, especially to the case of plane curve. Namely, using some techniques and ideas from Ilmanen [**25**], Stone [**36**] and White [**39**], we will present a unified analysis for the singularities of the flow which will lead to a short proof of Grayson's theorem [**15**].

In Chapter 5 we try to give a setting for a possible generalization of the work done in [**28**](which was concerned with the curvature evolution of partitions in the plane) to higher dimension. As a preliminary step, we will compute the evolution equation for the motion of codimension one hypersurfaces with boundary in an Euclidean space. The speed of the motion will have a normal component equal to the mean curvature and a non vanishing tangential component preserving the parabolicity of the equation. In particular, we will work out in full detail the dependence of the evolution of the second fundamental form - and its covariant gradients of any order - on the tangential speed. We plan to continue this analysis in the near future.

We give a list of the papers and notes where part of the material collected in this thesis can be found

- "On Perelman's Dilaton"
(M. Caldarelli, G. Catino, Z. Djadli, A. Magni, C. Mantegazza)
Geom. Dedicata, to appear (available at arXiv:0805.3268)

- "Some Remarks on Huisken's Monotonicity Formula for the Mean Curvature Flow"
(A. Magni, C. Mantegazza)
Singularities in nonlinear Evolution Phenomena and Applications; Proceedings; M. Novaga, G.Orlandi Eds, Edizioni della Normale, Birkäuser, 2009
(available at http://cvgmt.sns.it/papers/magman08/monoton.pdf)

- "A note on Grayson's Theorem"
(A. Magni, C. Mantegazza)
Available at: http://cvgmt.sns.it/people/mantegaz

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CHAPTER 1

Preliminaries

1.1. Riemannian Metrics and the Riemann Tensor for abstract Manifolds

In this chapter we fix notation, sign conventions and we give some results in Riemannian geometry which will be used extensively in the sequel.
Given an abstract differentiable n-dimensional manifold M endowed with a Riemannian metric g, we denote with ∇ the covariant differentiation associated to the Levi-Civita connection related to g. If $(e_1, ..., e_n)$ are a local basis for the tangent space of M at some point, we define the components of the Riemannian metric with respect to this basis as $g_{ij} = g(e_i, e_j)$ and we denote with g^{ij} its inverse. Using the standard abstract index notation and understanding summation over repeated indices, we can define the norm of a p-covariant and q-controvariant tensor $\mathrm{T}^{i_1\cdots i_q}_{j_1\cdots j_p}$ as

$$\mathrm{T}^{i_1\cdots i_q}_{j_1\cdots j_p}\mathrm{T}^{a_1\cdots a_q}_{b_1\cdots b_p} g_{i_1a_1}\cdots g_{i_qa_q} g^{j_1b_1}\cdots g^{j_pb_p}\,. \tag{1.1.1}$$

With the same convention we can define the Christoffel symbols of the connection

$$\Gamma^k_{ij} := g(\nabla_i e_j, e_k) = \frac{1}{2} g^{kl}(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij})\,, \tag{1.1.2}$$

where ∂_i is the standard derivative along the direction e_i.
If V is a smooth vector field on M with components $(v^1, ..., v^n)$, we have

$$\nabla_i v^j = \partial_i v^j + \Gamma^j_{ik} v^k\,. \tag{1.1.3}$$

Consequently, for a smooth one form ω on M, with components $(\omega_1, ..., \omega_n)$, we get:

$$\nabla_i \omega_j = \partial_i \omega_j - \Gamma^k_{ij}\omega_k\,. \tag{1.1.4}$$

The covariant derivative extends to a (p, q) tensor as follows

$$\nabla_k \mathrm{T}^{i_1\cdots i_q}_{j_1\cdots j_p} = \partial_k \mathrm{T}^{i_1\cdots i_q}_{j_1\cdots j_p} + \sum_{l=1}^{q} \Gamma^{i_l}_{kr}\mathrm{T}^{i_1\cdots i_{l+1}r\cdots i_q}_{j_1\cdots j_p} - \sum_{l=1}^{p}\Gamma^r_{kj_l}\mathrm{T}^{i_1\cdots i_p}_{j_1\cdots j_{l+1}r\cdots j_p}\,. \tag{1.1.5}$$

We define the Riemann tensor as follows

$$\nabla_i\nabla_j\omega_k - \nabla_j\nabla_i\omega_k = \mathrm{R}_{ijk}{}^l\omega_l = [\partial_j\Gamma^l_{ik} - \partial_i\Gamma^l_{jk} + \Gamma^r_{ik}\Gamma^l_{jr} - \Gamma^r_{jk}\Gamma^l_{ir}]\,\omega_l\,, \tag{1.1.6}$$

this way we can express it in terms of the Christoffel symbols as:

$$\mathrm{R}_{ijk}{}^{l} = \partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^r_{ik}\Gamma^l_{jr} - \Gamma^r_{jk}\Gamma^l_{ir}. \tag{1.1.7}$$

As an immediate consequence, we have that

$$\nabla_i \nabla_j v^k - \nabla_j \nabla_i v^k = -\mathrm{R}_{ijl}{}^{k} v^l\,. \tag{1.1.8}$$

The Ricci tensor and the scalar curvature are defined respectively by

$$\mathrm{R}_{ij} = \mathrm{R}_{ikj}{}^{k} \quad \text{and} \quad \mathrm{R} = \mathrm{R}^i_i\,. \tag{1.1.9}$$

From (1.1.7), it is immediate to obtain the expressions of R_{ij} and R in terms of the Christoffel symbols and their spatial derivatives.
All the symmetry properties of the Riemann tensor are determined by

$$\mathrm{R}_{ijkl} = -\mathrm{R}_{jikl} = \mathrm{R}_{klij}\,, \tag{1.1.10}$$

which in turn imply the following Bianchi identity

$$\nabla_i \mathrm{R}_{jklr} + \nabla_l \mathrm{R}_{jkri} + \nabla_r \mathrm{R}_{jkil} = 0\,, \tag{1.1.11}$$

and contracting

$$\nabla_i \mathrm{R}_{ijkl} = \nabla_k \mathrm{R}_{jl} - \nabla_l \mathrm{R}_{jk}\,. \tag{1.1.12}$$

One more contraction gives

$$\nabla_i \mathrm{R}_{ij} = \frac{1}{2}\nabla_j \mathrm{R}\,. \tag{1.1.13}$$

1.2. Riemannian Geometry of immersed Manifolds

Along this section, M will be an n-dimensional smooth differentiable manifold without boundary and $\phi : M \to \mathbb{R}^{n+1}$ a smooth immersion. Denoting with $(x^1, ..., x^n)$ a local coordinate system on M at some point and $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}) := (e_1, ..., e_n)$ the associated base for the tangent space at the same point, the Riemannian metric g naturally induced by ϕ on M via the pullback reads as follows:

$$g_{ij} := \langle \partial_i \phi, \partial_j \phi \rangle\,, \tag{1.2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^{n+1}$. Using the metric g we can endow M with a Riemannian volume element given by $d\mu = \sqrt{\det(g)}dx$, where dx denotes the Lebesgue measure on $\mathbb{R}^n$. If X is a vector field on M, the existence of a Riemannian volume element allows to define the divergence of X (denoted by $\operatorname{div} X$) by the relation

$$\operatorname{div} X d\mu = L_X d\mu = \nabla_i X^i d\mu \,, \tag{1.2.2}$$

where L_X is the Lie derivative with respect to the vector field X and ∇ is the covariant derivative associated to the Levi-Civita connection of g.
Since $\phi(M)$ has codimension one in $\mathbb{R}^{n+1}$, it follows that M is orientable and at each point of $\phi(M)$ there is a well defined (up to sign) normal vector field that we call ν. Within this setting we define (giving components) the second fundamental form of $\phi(M)$ according to

$$\mathrm{A} = \mathrm{h}_{ij} := \langle \nu, \partial^2_{ij}\phi \rangle \,, \tag{1.2.3}$$

which immediately implies that A is a well defined symmetric 2-tensor on $\phi(M)$.
The mean curvature of the couple (M, ϕ) is defined as the trace of the second fundamental form and will be denoted by H.
Within this setting, the Gauss-Weingarten relations read

$$\partial^2_{ij}\phi = \Gamma^k_{ij}\partial_k\phi + \mathrm{h}_{ij}\nu \quad \text{and} \quad \partial_i \nu = -\mathrm{h}_{ik} g^{kl} \partial_l \phi \,, \tag{1.2.4}$$

where the Γ^k_{ij} are the Christoffel symbols of the Levi-Civita connection associated to g.
Direct computations show that all the properties of the curvature tensor associated with the metric g are encoded by the second fundamental form. Actually, we have:

$$\begin{aligned} \mathrm{R}_{ijkl} &= \mathrm{h}_{ik}\mathrm{h}_{lj} - \mathrm{h}_{il}\mathrm{h}_{jk} \,, \\ \mathrm{R}_{ij} &= \mathrm{H}\mathrm{h}_{ij} - \mathrm{h}_{ik}\mathrm{h}_{lj} g^{kl} \,, \\ \mathrm{R} &= \mathrm{H}^2 - |\mathrm{h}|^2 \end{aligned} \tag{1.2.5}$$

and the Bianchi identities for the Riemann tensor of the immersed manifold are given by

$$\nabla_i \mathrm{h}_{jk} = \nabla_j \mathrm{h}_{ik} \,. \tag{1.2.6}$$

In the next chapters, we will make use of the following identity (Simons' identity, see [**35**]) while computing the evolution equations of certain geometric quantities:

$$\Delta \mathrm{h}_{ij} = \nabla_i \nabla_j \mathrm{H} \mathrm{h}_{ik}\mathrm{h}_{lj} g^{kl} - |\mathrm{A}|^2 \mathrm{h}_{ij} \,. \tag{1.2.7}$$

If M is compact and we denote its boundary with ∂M, we have that the divergence theorem holds when both the manifold and its boundary are endowed with the natural Riemannian volume elements $d\mu$ and $d\eta$ induced by the immersion. If X is a vector field on M, we have that

$$\int_{\phi(M)} \operatorname{div} X d\mu = \int_{\phi(\partial M)} \langle X, n \rangle d\eta \,, \tag{1.2.8}$$

were n is the outward unit normal vector field on $\phi(\partial M)$ in the tangent space to $\phi(M)$.

1.3. The Ricci Flow

Given a smooth Riemannian manifold with a time dependent metric $(M, g(\cdot, t))$, we say that it evolves by the Ricci flow if

$$\partial_t g_{ij} = -2\mathrm{R}_{ij}\,. \tag{1.3.1}$$

It is possible to prove (see [**7**] for a reference) that for any M and any admissible Riemannian metric g_0 on M, there exists a unique smooth solution for small times to the problem

$$\begin{aligned} \partial_t g_{ij} &= -2\mathrm{R}_{ij}\,, \\ g_{ij}(\cdot, 0) &= g_0\,. \end{aligned} \tag{1.3.2}$$

1.3.1. Evolutions of the Curvature Tensors. We now list the evolution equations induced by the Ricci flow on the curvature tensors. The proofs consist of computations worked out in normal coordinates (see [**29**] for a detailed proof).

$$\partial_t \Gamma^k_{ij} = -g^{kl}(\nabla_i \mathrm{R}_{lj} + \nabla_j \mathrm{R}_{il} - \nabla_l \mathrm{R}_{ij})\,, \tag{1.3.3}$$

using (1.1.7) we have

$$\begin{aligned} \partial_t \mathrm{R}_{ijkl} = \Delta R_{ijkl} &+ 2(\mathrm{B}_{ijkl} - \mathrm{B}_{ijlk} + \mathrm{B}_{ikjl} - \mathrm{B}_{iljk}) \\ &- \mathrm{R}_{is}\mathrm{R}_{sjkl} - \mathrm{R}_{js}\mathrm{R}_{iskl} - \mathrm{R}_{ks}\mathrm{R}_{ijsl} - \mathrm{R}_{ls}\mathrm{R}_{ijks}\,, \end{aligned} \tag{1.3.4}$$

where $\mathrm{B}_{ijkl} := \mathrm{R}_{ipjq}\mathrm{R}_{krls} g^{pr} g^{qs}$
and consequently

$$\partial_t \mathrm{R}_{ij} = \Delta \mathrm{R}_{ij} + 2\mathrm{R}_{rs}\mathrm{R}_{irjs} - 2\mathrm{R}_{is}\mathrm{R}_{sj}\,, \tag{1.3.5}$$

$$\partial_t \mathrm{R} = \Delta \mathrm{R} + 2\mathrm{R}_{ij}\mathrm{R}^{ij}\,. \tag{1.3.6}$$

For the evolution of the volume element associated to the metric g we have

$$\partial_t dV = -\mathrm{R} dV\,. \tag{1.3.7}$$

The backward analogue of the Ricci flow is defined by assigning the following evolution equation for the metric:

$$\partial_t g_{ij} = 2\mathrm{R}_{ij}\,, \tag{1.3.8}$$

which is usually called anti-Ricci flow.
It is possible to compute the evolution for all the curvature tensors following the same procedure used for the Ricci flow.

1.3.2. Solitons. In this section we recall some aspects of the theory of Ricci solitons.

DEFINITION 1.3.1. The pair $(M, g(t))$ is called a Ricci soliton if $g(t)$ is evolving by Ricci flow and there exist a smooth positive function $\mu : M \to (0, +\infty)$ as well as a family of diffeomorphisms $\phi : m \times [0, T) \to M$ such that

$$g(t) = \mu(t)\phi^*(t)g(0)\,. \tag{1.3.9}$$

Differentiating (1.3.9) with respect to time, we get

$$-2\mathrm{Ric}_{g(t)} = \dot{\mu}(t)\phi^*(t)g_0 + \mu(t)\phi^*(t)(L_X g_0)\,, \tag{1.3.10}$$

where L is the Lie derivative and X is the (possibly time dependent) vector field associated with the one-parameter family of diffeomorphisms $\phi(t)$. We say that a given soliton is shrinking (resp. steady, expanding) at a given time t_0 if $\dot{\mu}(t) < 0(\mathrm{resp} = 0, > 0)$.
The following theorem allows to write each soliton in a canonical form (see [**5**] for a proof).

PROPOSITION 1.3.2. Let $(M, g(t))$ to be a Ricci soliton and assume that the Ricci flow with initial datum g_0 admits a unique solution among the solitonic ones. Then there exist a family of diffeomorphisms $\psi : M \times [0, T) \to M$ and a constant $\omega \in \mathbb{R}$ such that

$$g(t) = (1 + \omega t)\psi^*(t)g_0 \;\; \blacksquare \tag{1.3.11}$$

Notice that, by rescaling the initial metric, we can always restrict to the case $\omega = -1(resp.0, +1)$ for the shrinking (resp. steady, expanding) solitons.
If we write the equation (1.3.10) for a given time $t = t_0$ and we set $\omega = \dot{\mu}(t_0)$, we obtain

$$-2\mathrm{Ric}_{g(t_0)} = \omega g_0 + L_{\hat{X}(t_0)}g_0\,, \tag{1.3.12}$$

where $\hat{X}(t_0) = \mu(t_0)X(t_0)$. Using coordinates, removing subscripts and setting $X_i = g_{ik}X^k$ we have

$$-2\mathrm{R}_{ij} = \omega g_{ij} + \nabla_i X_j + \nabla_j X_i\,. \tag{1.3.13}$$

DEFINITION 1.3.3. Given a smooth manifold M, we say that the triple (g, X, ω), with obvious notation, is a Ricci soliton structure on M if the equation (1.3.13) holds true.

Amongst the Ricci soliton structures, the following ones are of special interest.

DEFINITION 1.3.4. A Ricci soliton structure (g, X, ω) is a gradient Ricci soliton structure if there exists a smooth function $f : M \to \mathbb{R}$ such that $X_i = g_{ik}X^k = (df)_i$. In this case the function f is called the potential of the soliton.

It is possible to prove a canonical form theorem for the case of the gradient Ricci solitons (see again [**5**] for a proof):

PROPOSITION 1.3.5. Suppose that $(g_0, \nabla_{g_0} f_0, \omega)$ is a complete gradient Ricci soliton structure on M. Then there exist a solution $g(t)$ for the Ricci flow with $g(0) = g_0$, a one-parameter family of diffeomorphisms $\psi(t)$ with $\psi(0) = Id_M$ and a one parameter family of smooth functions $f(t)$ with $f(0) = f_0$ such that, if we set

$$\tau = 1 + \omega t > 0 \,, \tag{1.3.14}$$

we have:

(1) $\psi(t)$ is the one-parameter family of diffeomorphisms associated with the vector field $X(t) = \frac{1}{\tau(t)} \nabla_{g_0} f_0$,
(2) $g(t) = \tau(t)\psi^*(t) g_0$ and $f(t) = f_0(\psi(t))$;
(3) the following two equations hold for all times

$$\mathrm{R}_{ij}(g(t)) + \nabla_i^{g(t)} \nabla_j^{g(t)} f(t) + \frac{\omega}{2\tau(t)} g_{ij}(t) = 0 \,, \tag{1.3.15}$$

$$\partial_t f(x,t) = |\nabla^{g(t)} f(t)|^2_{g(t)} \quad \blacksquare \tag{1.3.16}$$

1.4. The Mean Curvature Flow

Let (M, ϕ_0) be a smooth n-dimensional compact embedded hypersurface in $\mathbb{R}^{n+1}$. We say that the hypersurface moves by Mean Curvature Flow (MCF) with initial datum $M_0 = \phi_0(M)$ if there exists a smooth one parameter family of immersions $\phi(\cdot, t) : M \times [0, T) \to \mathbb{R}^{n+1}$ which satisfies

$$\begin{aligned} &\partial_t \phi = \mathrm{H}\nu = \Delta \phi \,, \\ &\phi(\cdot, 0) = \phi_0 \,, \end{aligned} \tag{1.4.1}$$

where ν is the inner normal vector to the hypersurface.
From now on we will use the notation $M_t := \phi(M, t)$ to denote the image of the hypersurface along the flow.
Using (1.4.1) and the definitions in the previous sections, one can obtain the following results:

$$\partial_t g_{ij} := \partial_t \langle \partial_i \phi, \partial_j \phi \rangle = -2\mathrm{H}\mathrm{h}_{ij} \,, \tag{1.4.2}$$

$$\partial_t \nu = -\nabla \mathrm{H} \,, \tag{1.4.3}$$

$$\partial_t \sqrt{\det g} = -\mathrm{H}^2 \sqrt{\det g} \,, \tag{1.4.4}$$

$$\partial_t \mathrm{h}_{ij} = \Delta \mathrm{h}_{ij} - 2\mathrm{H}\mathrm{h}_{jl} g^{lk} \mathrm{h}_{ki} + |\mathrm{A}|^2 \mathrm{h}_{ij} \,, \tag{1.4.5}$$

$$\partial_t \mathrm{H} = \Delta \mathrm{H} + |\mathrm{A}|^2 \mathrm{H}\,, \tag{1.4.6}$$

$$\partial_t |\mathrm{A}|^2 = \Delta |\mathrm{A}|^2 - 2|\nabla \mathrm{A}|^2 + 2|\mathrm{A}|^4\,, \tag{1.4.7}$$

while for the generic k-th covariant derivative of the second fundamental form we have

$$\partial_t |\nabla^k \mathrm{A}|^2 = \Delta |\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1} \mathrm{A}|^2 + \sum_{p+q+r=k} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A}\,, \tag{1.4.8}$$

where the symbol $*$ denotes a suitable contraction with the metric tensor g_{ij}.
One of the most important technical tools for the study of the MCF is Huisken's monotonicity formula.

Theorem 1.4.1 (Huisken's monotonicity - Hamilton's formulation). *Given a positive smooth solution of the backward heat equation $\partial_t u = -\Delta u$ on $\mathbb{R}^{n+1} \times [0, C)$ and a MCF which exists at least on the time interval $[0, C)$, we have that*

$$\begin{aligned} \partial_t [\sqrt{4\pi(C-t)} \int_M u\, d\mu] = & -\sqrt{4\pi(C-t)} \int_M u |\mathrm{H} - \langle \nabla \log u, \nu \rangle|^2 d\mu \\ & - \sqrt{4\pi(C-t)} \int_M \Big(\nabla_\nu \nabla_\nu u - \frac{|\nabla_\nu u|^2}{u} + \frac{u}{2(C-t)} \Big) d\mu\,, \end{aligned} \tag{1.4.9}$$

where ∇_ν denotes the covariant derivative in the normal direction to $\phi(M)$ ■

Notice that in (1.4.9), the first term on the rhs is non positive and the second term on the same side is non positive thanks to the Li-Yau Harnack estimate (see [**26**]) and it vanishes on any backward heat kernel, for which the formula becomes

$$\partial_t \int_M \frac{e^{-\frac{|x-y|^2}{4(C-t)}}}{[4\pi(C-t)]^{n/2}} d\mu = -\int_M \frac{e^{-\frac{|x-y|^2}{4(C-t)}}}{[4\pi(C-t)]^{n/2}} \left| \mathrm{H} + \frac{\langle x-y, \nu \rangle}{2(C-t)} \right|^2 d\mu\,, \tag{1.4.10}$$

where y is the center of the backward heat kernel.

CHAPTER 2

Perelman's Dilaton

It is very well known since the '90 that it is not possible to give a formulation of the Ricci flow as the gradient flow of any smooth functional of the Riemannian metric and its derivatives. Anyway, one of the most important contributions given by Perelman to the study of the Ricci flow (see [**32**]), it has been the discover of its gradient-like structure. In this chapter we investigate the gradient flow-like structure of the Ricci flow using a Kaluza-Klein reduction approach. In particular, given a smooth Riemannian manifold, we will show how the Ricci flow can be presented as a component of the constrained gradient flow of the Hilbert-Einstein functional on an extended manifold.

2.1. Hilbert-Einstein Action and Perelman's $\mathcal{F}$-Functional

Let (M^m, g) and (N^n, h) be two closed Riemannian manifolds of dimension m and n respectively and let $f : M \to \mathbb{R}$ be a smooth function on M. On the product manifold $\widetilde{M} = M \times N$ we consider a metric $\widetilde{g}$ of the form

$$\widetilde{g} = e^{-Af} g \oplus e^{-Bf} h,$$

where A and B are real constants and we call the function f *dilaton field*. Notice that $\widetilde{g}$ is a conformal deformation of a warped product on M.
As a notation, we will use Latin indices, $i, j, \dots$ for the coordinates on M (we will call them the "real" variables) and Greek indices, $\alpha, \beta, \dots$, for the coordinates on N (the "phantom" variables). Within this setting, we clearly have $\forall\, i, j \in \{1, \dots, m\}$ and $\forall\, \alpha, \beta \in \{1, \dots, n\}$,

$$\widetilde{g}_{i\alpha} = \widetilde{g}^{i\alpha} = 0\,,$$
$$\widetilde{g}^{ij} = e^{Af} g^{ij}\,, \quad \widetilde{g}^{\alpha\beta} = e^{Bf} h^{\alpha\beta}\,.$$

Let μ, σ and $\widetilde{\mu}$ be respectively the canonical volume measure on M, N and $\widetilde{M}$. By definition of $\widetilde{g}$, it follows that $\widetilde{\mu} = e^{-\frac{Am+Bn}{2} f} \mu \times \sigma$.
Remembering (1.1.2), we can compute the following expressions for the Christoffel symbols of the product metric

$$\begin{aligned}
\widetilde{\Gamma}^k_{ij} &= \frac{1}{2}\widetilde{g}^{kl}\left(\partial_i \widetilde{g}_{jl} + \partial_j \widetilde{g}_{il} - \partial_l \widetilde{g}_{ij}\right) \\
&= \frac{1}{2} e^{Af} g^{kl} \left[e^{-Af}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) - A e^{-Af}\left(\partial_i f g_{jl} + \partial_j f g_{il} - \partial_l f g_{ij}\right)\right] \\
&= \Gamma^k_{ij} - \frac{A}{2}\left(\partial_i f \delta^k_j + \partial_j f \delta^k_i - g^{kl}\partial_l f g_{ij}\right)\,.
\end{aligned}$$

Using the fact that the metric $\widetilde{g}$ is zero for a pair of "mixed" indices and that the function f depends only on the real variables, we get

$$\begin{aligned}
\widetilde{\Gamma}^{\gamma}_{ij} &= \frac{1}{2}\widetilde{g}^{\gamma\beta}\left(\partial_i\widetilde{g}_{j\beta} + \partial_j\widetilde{g}_{i\beta} - \partial_\beta\widetilde{g}_{ij}\right) = 0\,,\\
\widetilde{\Gamma}^{k}_{\alpha i} &= \frac{1}{2}\widetilde{g}^{kl}\left(\partial_i\widetilde{g}_{\alpha l} + \partial_\alpha\widetilde{g}_{il} - \partial_l\widetilde{g}_{i\alpha}\right) = 0\,,\\
\widetilde{\Gamma}^{\gamma}_{i\beta} &= \frac{1}{2}\widetilde{g}^{\gamma\alpha}\left(\partial_i\widetilde{g}_{\alpha\beta} + \partial_\beta\widetilde{g}_{i\alpha} - \partial_\alpha\widetilde{g}_{i\beta}\right) = -\frac{B}{2}\partial_i f\delta^{\gamma}_{\beta}\,,\\
\widetilde{\Gamma}^{k}_{\alpha\beta} &= \frac{1}{2}\widetilde{g}^{kl}\left(\partial_\alpha\widetilde{g}_{l\beta} + \partial_\beta\widetilde{g}_{\alpha l} - \partial_l\widetilde{g}_{\alpha\beta}\right) = \frac{B}{2}e^{(A-B)f}g^{kl}\partial_l f h_{\alpha\beta}\,.
\end{aligned}$$

Finally, a computation analogous to the one above gives $\widetilde{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta}$.
Summarizing we have:

$$\widetilde{\Gamma}^{k}_{ij} = \Gamma^{k}_{ij} - \frac{A}{2}\left(\partial_i f\delta^k_j + \partial_j f\delta^k_i - g^{kl}\partial_l f g_{ij}\right) \tag{2.1.1}$$

$$\widetilde{\Gamma}^{\alpha}_{ij} = \widetilde{\Gamma}^{k}_{i\alpha} = 0 \tag{2.1.2}$$

$$\widetilde{\Gamma}^{k}_{\alpha\beta} = \frac{B}{2}e^{(A-B)f}g^{kl}\partial_l f h_{\alpha\beta} \tag{2.1.3}$$

$$\widetilde{\Gamma}^{\gamma}_{i\beta} = -\frac{B}{2}\partial_i f\delta^{\gamma}_{\beta} \tag{2.1.4}$$

$$\widetilde{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta}\,. \tag{2.1.5}$$

We now want to compute the Ricci curvature of the metric $\widetilde{g}$. Keeping in mind (1.1.7) and using equations (2.1.1)– (2.1.5), computing in normal coordinates on both M and N, we get the following

$$\begin{aligned}
\widetilde{\mathrm{R}}_{jl} &= \partial_i\widetilde{\Gamma}^{i}_{jl} - \partial_j\widetilde{\Gamma}^{k}_{kl} - \partial_j\widetilde{\Gamma}^{\alpha}_{\alpha l} + \widetilde{\Gamma}^{k}_{jl}\widetilde{\Gamma}^{i}_{ki} + \widetilde{\Gamma}^{k}_{jl}\widetilde{\Gamma}^{\alpha}_{\alpha k} - \widetilde{\Gamma}^{k}_{ij}\widetilde{\Gamma}^{i}_{kl} - \widetilde{\Gamma}^{\beta}_{\alpha j}\widetilde{\Gamma}^{\alpha}_{\beta l}\\
&= \mathrm{R}_{jl} - \frac{A}{2}\left(2\nabla^2_{jl}f - \Delta f g_{jl}\right)\\
&\quad + \frac{Am}{2}\nabla^2_{jl}f + \frac{Bn}{2}\nabla^2_{jl}f\\
&\quad + \frac{A^2m}{4}\left(2df_jdf_l - |\nabla f|^2 g_{jl}\right) + \frac{ABn}{4}\left(2df_jdf_l - |\nabla f|^2 g_{jl}\right)\\
&\quad - \frac{A^2}{4}\left[(m+2)df_jdf_l - 2|\nabla f|^2 g_{jl}\right] - \frac{B^2n}{4}df_jdf_l\,.
\end{aligned}$$

Collecting similar terms, it becomes

$$\begin{aligned}
\widetilde{\mathrm{R}}_{jl} &= \mathrm{R}_{jl} + \nabla^2_{jl}f\left(\frac{Am+Bn}{2} - A\right) + \frac{A}{2}g_{jl}\left[\Delta f - |\nabla f|^2\left(\frac{Am+Bn}{2} - A\right)\right]\\
&\quad + \frac{1}{4}df_jdf_l\left(2ABn + (m-2)A^2 - B^2n\right)\,.
\end{aligned} \tag{2.1.6}$$

Proceeding in an analogous way for the phantom indices, we get

$$\begin{aligned}
\widetilde{\mathrm{R}}_{\beta\gamma} &= \partial_\alpha \widetilde{\Gamma}^\alpha_{\beta\gamma} - \partial_\gamma \widetilde{\Gamma}^\alpha_{\alpha\beta} + \partial_k \widetilde{\Gamma}^k_{\beta\gamma} + \widetilde{\Gamma}^k_{\beta\gamma}\widetilde{\Gamma}^\alpha_{\alpha\gamma} + \widetilde{\Gamma}^k_{\beta\gamma}\widetilde{\Gamma}^i_{ik} - \widetilde{\Gamma}^k_{\alpha\gamma}\widetilde{\Gamma}^\alpha_{\beta k} - \widetilde{\Gamma}^\alpha_{k\gamma}\widetilde{\Gamma}^k_{\alpha\beta} \\
&= \mathrm{R}_{\beta\gamma} + \frac{B}{2} e^{(A-B)f} h_{\beta\gamma}\left(\Delta f + (A-B)|\nabla f|^2\right) \\
&\quad - \frac{B^2 n}{4} e^{(A-B)f} h_{\beta\gamma}|\nabla f|^2 - \frac{ABm}{4} e^{(A-B)f} h_{\beta\gamma}|\nabla f|^2 \\
&\quad + \frac{B^2}{4} e^{(A-B)f} h_{\beta\gamma}|\nabla f|^2 + \frac{B^2}{4} e^{(A-B)f} h_{\beta\gamma}|\nabla f|^2\,,
\end{aligned}$$

that is,

$$\widetilde{\mathrm{R}}_{\beta\gamma} = \mathrm{R}_{\beta\gamma} + \frac{B}{2} e^{(A-B)f} h_{\beta\gamma}\left[\Delta f - |\nabla f|^2\left(\frac{Am+Bn}{2} - A\right)\right]. \tag{2.1.7}$$

Finally, it is easy to see that the mixed terms of the Ricci tensor of $\widetilde{g}$ vanish, that is: $\widetilde{\mathrm{R}}_{i\alpha} = 0$.
Contracting with the metric, we obtain the formula for the scalar curvature of $\widetilde{g}$:

$$\begin{aligned}
\widetilde{\mathrm{R}} &= e^{Af}\mathrm{R}^M + e^{Bf}\mathrm{R}^N + e^{Af}\Delta f(Am + Bn - A) \\
&\quad + \frac{e^{Af}}{4}|\nabla f|^2\left(4ABn - 2ABmn + 3mA^2 - 2A^2 - m^2A^2 - B^2 n - B^2 n^2\right).
\end{aligned}$$

where R^M and R^N are respectively the scalar curvatures of (M, g) and (N, h).
We now make the following ansatz:

$$2ABn + (m-2)A^2 - B^2 n = 0 \tag{C1}$$

and

$$\frac{Am+Bn}{2} - A = 1 \qquad \Longleftrightarrow \qquad A(m-2) + Bn = 2\,. \tag{C2}$$

It is useful at this stage to give a motivation for the choice of the constants A and B, which we guess it is not very clear at this point. Condition (C1) is assumed in order to make vanish from the expression of $\widetilde{\mathrm{R}}_{ij}$ the term in $df \otimes df$ that otherwise appears in doing the flow by the gradient of the functional $\int_{\widetilde{M}} \widetilde{\mathrm{R}}\, d\widetilde{\mu}$ (see Section 2.3). The second condition, that clearly also simplifies both $\widetilde{\mathrm{R}}_{ij}$ and $\widetilde{\mathrm{R}}_{\alpha\beta}$, is instead more related to Perelman's $\mathcal{F}$–functional. In writing the functional $\int_{\widetilde{M}} \widetilde{\mathrm{R}}\, d\widetilde{\mu}$ as an integral on M with respect to the measure μ we will see that the only way to get the factor e^{-f} is to assume condition (C2).

We now prove that our choice for A and B is not significantly restrictive.

LEMMA 2.1.1. *If $m+n>2$, we can always find two non zero constants A and B satisfying (C1) and (C2)* ■

PROOF. Notice that $A=0$ implies $B=0$. If $B\neq 0$, dividing both sides of condition (C1) by B^2, it can be expressed in the following form for $\theta=A/B$,

$$(m-2)\theta^2+2n\theta-n=0\,. \tag{C1*}$$

If $m\neq 2$, this second degree equation for θ has always two solutions for every choice of the dimensions $m,n\in\mathbb{N}$, which would coincide only in the case $m=n=1$, that we excluded. Notice also that the two solutions have opposite signs. Precisely, they are

$$\theta=\frac{-n\pm\sqrt{n(n+m-2)}}{m-2}$$

and in the special case $n=1$, we have $\theta=\frac{-1\pm\sqrt{m-1}}{m-2}$.
If $m=2$ we have only one solution of equation (C1*) which is $\theta=1/2$.
Then, condition (C2) is equivalent to $\theta(m-2)+n=2/B$ which can be fulfilled, by homogeneity, if $\theta(m-2)+n\neq 0$. In the case of equality, we would have

$$0=\theta^2(m-2)+2n\theta-n=n\theta-n$$

which would imply $\theta=1$. Hence, $m-2+n=0$ and $m=n=1$. □

Under assumptions (C1) and (C2), the last term of $\widetilde{\mathrm{R}}_{jl}$ in formula (2.1.6) cancels out and many coefficients become one. Indeed we get the following simplified formulas for the components of the Ricci tensor of $\widetilde{g}$,

$$\widetilde{\mathrm{R}}_{jl}=\mathrm{R}_{jl}+\nabla^2_{jl}f+\frac{A}{2}g_{jl}\left(\Delta f-|\nabla f|^2\right), \tag{2.1.8}$$

$$\widetilde{\mathrm{R}}_{\beta\gamma}=\mathrm{R}_{\beta\gamma}+\frac{B}{2}e^{(A-B)f}h_{\beta\gamma}\left(\Delta f-|\nabla f|^2\right). \tag{2.1.9}$$

Then, the scalar curvature of $\widetilde{g}$ becomes

$$\widetilde{\mathrm{R}}=e^{Af}\mathrm{R}^M+e^{Bf}\mathrm{R}^N+e^{Af}\big(\Delta f(A+2)-|\nabla f|^2(A+1)\big)\,. \tag{2.1.10}$$

From this last formula, it follows immediately the relation between the Einstein–Hilbert action functional $\mathcal{S}$ on $\widetilde{M}$ and the Perelman's $\mathcal{F}$–functional, see [**32**],

$$\mathcal{F}(g,f)=\int_M(\mathrm{R}^M+|\nabla f|^2)e^{-f}\,d\mu\,.$$

THEOREM 2.1.2. *Let (M^m,g) and (N^n,h) be two closed Riemannian manifolds of dimension m and n respectively, with $m+n>2$ and let $f:M\to\mathbb{R}$ be a smooth function on M. On the product manifold $\widetilde{M}=M\times N$ consider the metric $\widetilde{g}$ of the form*

$$\widetilde{g}=e^{-Af}g\oplus e^{-Bf}h\,,$$

where A and B are constants satisfying conditions (C1) and (C2).
Then the following formula holds

$$\mathcal{S}(\widetilde{g}) = \int_{\widetilde{M}} \widetilde{\mathrm{R}}\, d\widetilde{\mu} = \mathrm{Vol}(N,h)\mathcal{F}(g,f) + \left(\int_M e^{(B-A-1)f}\, d\mu\right)\int_N \mathrm{R}^N\, d\sigma \tag{2.1.11}$$

In particular, if (N,h) has zero total scalar curvature and unit volume, we get $\mathcal{S}(\widetilde{g}) = \mathcal{F}(g,f)$ ■

Proof. We simply compute

$$\begin{aligned}
\int_{\widetilde{M}} \widetilde{\mathrm{R}}\, d\widetilde{\mu} &= \int_M\int_N e^{-\frac{Am+Bn}{2}f}\widetilde{\mathrm{R}}\, d\mu\, d\sigma\\
&= \int_M\int_N e^{-(1+A)f}\left[e^{Af}\mathrm{R}^M + e^{Bf}\mathrm{R}^N + e^{Af}\left(\Delta f(A+2) - |\nabla f|^2(A+1)\right)\right]\, d\mu\, d\sigma\\
&= \int_M\int_N e^{-(1+A)f}e^{Bf}\mathrm{R}^N\, d\mu\, d\sigma\\
&\quad + \int_M\int_N \left[\mathrm{R}^M + \Delta f(A+2) - |\nabla f|^2(A+1)\right]e^{-f}\, d\mu\, d\sigma\\
&= \left(\int_M e^{(B-A-1)f}\, d\mu\right)\int_N \mathrm{R}^N\, d\sigma\\
&\quad + \mathrm{Vol}(N,h)\int_M \left[\mathrm{R}^M + \Delta f(A+2) - |\nabla f|^2(A+1)\right]e^{-f}\, d\mu\\
&= \left(\int_M e^{(B-A-1)f}\, d\mu\right)\int_N \mathrm{R}^N\, d\sigma\\
&\quad + \mathrm{Vol}(N,h)\int_M \left(\mathrm{R}^M + |\nabla f|^2\right)e^{-f}\, d\mu
\end{aligned}$$

where in the last passage we integrated by parts the Laplacian term. □

2.2. The Associated Flow

Under assumptions (C1) and (C2), we have

$$\widetilde{\mathrm{R}}_{jl} = \mathrm{R}_{jl} + \nabla^2_{jl}f + \frac{A}{2}g_{jl}\left(\Delta f - |\nabla f|^2\right)\,, \qquad \widetilde{\mathrm{R}}_{i\alpha} = 0\,, \tag{2.2.1}$$

$$\widetilde{\mathrm{R}}_{\beta\gamma} = \mathrm{R}_{\beta\gamma} + \frac{B}{2}e^{(A-B)f}h_{\beta\gamma}\left(\Delta f - |\nabla f|^2\right) \tag{2.2.2}$$

and

$$\widetilde{\mathrm{R}} = e^{Af}\mathrm{R}^M + e^{Bf}\mathrm{R}^N + e^{Af}\left(\Delta f(A+2) - |\nabla f|^2(A+1)\right)\,. \tag{2.2.3}$$

Suppose we have a manifold $\widetilde{M} = M\times N$ with a time dependent metric $\widetilde{g}(t)$ for $t\in[0,T]$. Given the initial metric as a warped product $\widetilde{g} = \widehat{g}\oplus\varphi h$, with $\varphi : M\to\mathbb{R}$ a smooth function, (N,h) a Ricci–flat having unit volume, we consider the evolution of the metric by the gradient of the Einstein–Hilbert action with the constraining the measure $\varphi^{-\theta}\widetilde{\mu}$ to stay fixed, where θ is the one given by condition (C1*) and A, B are the correspondent constants satisfying conditions (C1) and (C2) above.

Suppose there exists a unique solution of this flow, preserving the warped product. We can assume that for every $t \in [0,T]$ we have $\widetilde{g}(t) = \widehat{g}(t) \oplus \varphi(t)h(t)$ with $(N, h(t))$ always of volume 1.

Writing down the evolution of h we see that it moves only by multiplication by a positive factor. As we assumed that $(N, h(t))$ is of unit volume, we can conclude that the metric $h(t)$ is constant and equal to the initial h. Setting $f = -\frac{1}{B}\log\varphi$, which implies $\varphi = e^{-Bf}$ and $\varphi^\theta = e^{-Af}$, we can write $\widetilde{g} = e^{-Af}g \oplus e^{-Bf}h$ where $g(t) = e^{Af}\widehat{g}(t)$. Clearly, we also have that $\widetilde{g} = \varphi^\theta g \oplus \varphi h$.

Denote with $\delta\widetilde{g}$, δg and δf the variations of $\widetilde{g}$, g and f respectively. Then we have,

$$\delta\widetilde{g} = e^{-Af}\left(\delta g - Ag\delta f\right) \oplus e^{-Bf}\left(-Bh\delta f\right) .$$

In terms of these variations, the constraint on the measure becomes $\delta f = \mathrm{tr}_g \delta g/2$. Keeping in mind (2.2.1), (2.2.2), (2.2.3) and that (N, h) is Ricci–flat, we get

$$\begin{aligned}
\delta \int_{\widetilde{M}} 2\widetilde{\mathrm{R}}\, d\widetilde{\mu} &= \int_{\widetilde{M}} \langle -2\widetilde{\mathrm{Ric}} + \widetilde{\mathrm{R}}\widetilde{g} \,|\, \delta\widetilde{g}\rangle\, d\widetilde{\mu} \\
&= \int_{\widetilde{M}} \langle -2\widetilde{\mathrm{Ric}} + \widetilde{\mathrm{R}}\widetilde{g} \,|\, e^{-Af}\left(\delta g - Ag\delta f\right) \oplus e^{-Bf}\left(-Bh\delta f\right)\rangle\, d\widetilde{\mu} \\
&= \int_{\widetilde{M}} \langle -2(\mathrm{Ric}^M + \nabla^2 f) \,|\, \delta g\rangle e^{-Af}\, d\widetilde{\mu} \\
&\quad - \frac{1}{2}\int_{\widetilde{M}} (\Delta f - |\nabla f|^2)(ABn + 2A - B^2 n)\mathrm{tr}_g(\delta g) e^{-Af}\, d\widetilde{\mu} \\
&= \int_{\widetilde{M}} \langle -2(\mathrm{Ric}^M + \nabla^2 f) \,|\, \delta g\rangle e^{-Af}\, d\widetilde{\mu} \\
&= -2\int_M \langle \mathrm{Ric}^M + \nabla^2 f \,|\, \delta g\rangle e^{-f}\, d\mu \,,
\end{aligned}$$

since, by conditions (C1) and (C2), it follows $ABn + 2A - B^2 n = 0$.

Hence, the system

$$\begin{cases} \delta g = -2(\mathrm{Ric}^M + \nabla^2 f) \\ \delta f = -\Delta f - \mathrm{R}^M \end{cases}$$

represents the constrained gradient of the Einstein–Hilbert action functional. The associated flow of the metric $\widetilde{g} = e^{-Af}g \oplus e^{-Bf}h$ is described by

$$\begin{cases} \partial_t g = -2(\mathrm{Ric}^M + \nabla^2 f) \\ \partial_t h = 0 \\ \partial_t f = -\Delta f - \mathrm{R}^M \,, \end{cases}$$

that is, g evolves by the "modified" Ricci flow.

Following Perelman [**32**] and transforming the pair (g, f) by a suitable diffeomorphism, we get a solution of

$$\begin{cases} \partial_t g = -2\mathrm{Ric}^M \\ \partial_t f = -\Delta f + |\nabla f|^2 - \mathrm{R}^M . \end{cases}$$

This way, up to a factor and a diffeomorphism, the spatial part of the metric $\widetilde{g}$ moves according to the Ricci flow (g is equal to the spatial part of $\widetilde{g}$ times the factor e^{Af}).

2.3. Other Flows

It is interesting to see what kind of functionals and flows one can get by varying the constants A and B.
Supposing that (N, h) has unit volume and zero total scalar curvature, we already computed,

$$\begin{aligned} \widetilde{\mathrm{R}}_{jl} &= \mathrm{R}_{jl} + \nabla^2_{jl} f\left(\frac{Am+Bn}{2} - A\right) + \frac{A}{2} g_{jl}\left[\Delta f - |\nabla f|^2 \left(\frac{Am+Bn}{2} - A\right)\right] \\ &\quad + \frac{1}{4} df_j df_l \left(2ABn + (m-2)A^2 - B^2 n\right) \\ \widetilde{\mathrm{R}}_{\beta\gamma} &= \mathrm{R}_{\beta\gamma} + \frac{B}{2} e^{(A-B)f} h_{\beta\gamma}\left[\Delta f - |\nabla f|^2 \left(\frac{Am+Bn}{2} - A\right)\right] \end{aligned}$$

Assuming the condition $\frac{Am+Bn}{2} - A = 1$ we have

$$\begin{aligned} \widetilde{\mathrm{R}}_{jl} &= \mathrm{R}_{jl} + \nabla^2_{jl} f + \frac{A}{2} g_{jl}\left[\Delta f - |\nabla f|^2\right] \\ &\quad + \frac{1}{4} df_j df_l \left(2ABn + (m-2)A^2 - B^2 n\right) \\ \widetilde{\mathrm{R}}_{\beta\gamma} &= \mathrm{R}_{\beta\gamma} + \frac{B}{2} e^{(A-B)f} h_{\beta\gamma}\left[\Delta f - |\nabla f|^2\right] \\ \widetilde{\mathrm{R}} &= e^{Af}\mathrm{R}^M + e^{Bf}\mathrm{R}^N + e^{Af}\Delta f \\ &\quad + e^{Af}\left(\frac{Am+Bn}{2}\right)(\Delta f - |\nabla f|^2) + \frac{e^{Af}}{4}\left(2ABn + (m-2)A^2 - B^2 n\right)|\nabla f|^2 \\ &= e^{Af}\mathrm{R}^M + e^{Bf}\mathrm{R}^N + e^{Af}\Delta f \\ &\quad + e^{Af}(A+1)(\Delta f - |\nabla f|^2) + \frac{e^{Af}}{4}\left(2ABn + (m-2)A^2 - B^2 n\right)|\nabla f|^2 . \end{aligned}$$

Hence,

$$
\begin{aligned}
\int_{\widetilde{M}} \widetilde{\mathrm{R}}\, d\widetilde{\mu} &= \int_M \int_N e^{-\frac{Am+Bn}{2}f} \widetilde{\mathrm{R}}\, d\mu\, d\sigma \\
&= \int_M \int_N e^{-(1+A)f} \left[e^{Af}\mathrm{R}^M + e^{Bf}\mathrm{R}^N + e^{Af}\Delta f \right] d\mu\, d\sigma \\
&\quad + \int_M \int_N e^{-(1+A)f} e^{Af}(A+1)(\Delta f - |\nabla f|^2)\, d\mu\, d\sigma \\
&\quad + \int_M \int_N e^{-(1+A)f} \frac{e^{Af}}{4} \left(2ABn + (m-2)A^2 - B^2 n\right) |\nabla f|^2\, d\mu\, d\sigma \\
&= \int_M \int_N e^{-(1+A)f} e^{Bf} \mathrm{R}^N\, d\mu\, d\sigma \\
&\quad + \int_M \int_N \left[\mathrm{R}^M + \Delta f + \frac{1}{4}|\nabla f|^2 (2ABn + (m-2)A^2 - B^2 n) \right] e^{-f}\, d\mu\, d\sigma \\
&= \int_M \left[\mathrm{R}^M + |\nabla f|^2 + \frac{1}{4}|\nabla f|^2 (2ABn + (m-2)A^2 - B^2 n) \right] e^{-f}\, d\mu \\
&= \mathcal{F}(g,f) + Z_{m,n}(A,B) \int_M |\nabla f|^2 e^{-f}\, d\mu\,,
\end{aligned}
$$

with $Z_{m,n}(A,B) = (2ABn + (m-2)A^2 - B^2 n)/4$.

We want to see what are the possible values of $Z_{m,n}$, we recall that we have the constraint $A(m-2) + Bn = 2$.
We change variables according to $x = A$ and $y = (B - A)$ so that the constraint becomes $(m+n-2)x + ny = 2$ and $4Z_{m,n}(A,B) = (m+n-2)x^2 - ny^2$. As $y = [2 - x(m+n-2)]/n$ we get (like before we assume $m + n > 2$),

$$
\begin{aligned}
4Z_{m,n}(A,B) &= (m+n-2)x^2 - n\left(\frac{2 - x(m+n-2)}{n}\right)^2 \\
&= (m+n-2)x^2 - (4 + x^2(m+n-2)^2 - 4x(m+n-2))/n \\
&= x^2[(m+n-2) - (m+n-2)^2/n] + 4x(m+n-2)/n - 4/n \\
&= -\,x^2\frac{(m+n-2)(m-2)}{n} + x\frac{4(m+n-2)}{n} - \frac{4}{n}\,.
\end{aligned}
$$

When $m = 2$, we have $B = 2/n$ and A "free", then

$$
Z_{m,n}(A,B) = \frac{x(m+n-2) - 1}{n} = \frac{An - 1}{n} = A - 1/n
$$

which can take every real value as x can vary from $-\infty$ to $+\infty$.
If instead, $m > 2$ the expression

$$
Z_{m,n}(A,B) = -A^2\frac{(m+n-2)(m-2)}{4n} + A\frac{m+n-2}{n} - \frac{1}{n}\,.
$$

is a second degree polynomial in $A \in \mathbb{R}$ with negative leading coefficient, so it can vary only between $-\infty$ and some maximum value. By a straightforward computation one finds that this maximum value is given by $1/(m-2)$ and is evidently independent of the dimension n.

This means that by a suitable choice of the constants A and B one has

$$\mathcal{S}(\widetilde{g}) = \int_{\widetilde{M}} \widetilde{\mathrm{R}}\, d\widetilde{\mu} = \int_M (\mathrm{R}^M + (\lambda+1)|\nabla f|^2)e^{-f}\, d\mu\,,$$

for every $\lambda \in \big(-\infty, \frac{1}{m-2}\big]$. Notice that (if $m > 2$), with the exception of $\lambda = 1/(m-2)$ one has always two possible choices of pairs of constants (A, B) for every value λ.

When $\lambda \neq 0$ as

$$\widetilde{\mathrm{R}}_{jl} = \mathrm{R}_{jl} + \nabla^2_{jl} f + \frac{A}{2}(\Delta f - |\nabla f|^2)g_{jl} + \lambda(df \otimes df)_{jl}\,,$$

the associated flow is substantially different from the (modified) Ricci flow, indeed if as before $\delta f = \frac{1}{2}\mathrm{tr}_g(\delta g)$ and (N, h) is Ricci–flat, we get

$$\begin{aligned}
\delta \int_{\widetilde{M}} 2\widetilde{\mathrm{R}}\, d\widetilde{\mu} &= \int_{\widetilde{M}} \langle -2\widetilde{\mathrm{Ric}} + \widetilde{\mathrm{R}}\widetilde{g} \,|\, \delta\widetilde{g}\rangle\, d\widetilde{\mu}\\
&= \int_{\widetilde{M}} \langle -2\widetilde{\mathrm{Ric}} + \widetilde{\mathrm{R}}\widetilde{g} \,|\, e^{-Af}\,(\delta g - Ag\delta f) \oplus e^{-Bf}\,(-Bh\delta f)\rangle\, d\widetilde{\mu}\\
&= \int_{\widetilde{M}} \langle -2(\mathrm{Ric}^M + \nabla^2 f + \lambda df \otimes df) \,|\, \delta g\rangle e^{-Af}\, d\widetilde{\mu}\\
&\quad + \int_{\widetilde{M}} \big[-A(\Delta f - |\nabla f|^2) + (\mathrm{R}^M + \Delta f(A+2) - |\nabla f|^2(A+1))\big]\, \mathrm{tr}_g(\delta g)e^{-Af}\, d\widetilde{\mu}\\
&\quad + \int_{\widetilde{M}} \Big\langle -2\Big[\mathrm{Ric}^M + \nabla^2 f + \frac{A}{2}g(\Delta f - |\nabla f|^2)\Big] \,\Big|\, -Ag\delta f\Big\rangle e^{-Af}\, d\widetilde{\mu}\\
&\quad + \int_{\widetilde{M}} \big[\mathrm{R}^M + \Delta f(A+2) - |\nabla f|^2(A+1)\big]\,(-Am\delta f)e^{-Af}\, d\widetilde{\mu}\\
&\quad + \int_{\widetilde{M}} \big[-B(\Delta f - |\nabla f|^2)\big]\,(-Bn\delta f)e^{-Af}\, d\widetilde{\mu}\\
&\quad + \int_{\widetilde{M}} \big[\mathrm{R}^M + \Delta f(A+2) - |\nabla f|^2(A+1)\big]\,(-Bn\delta f)e^{-Af}\, d\widetilde{\mu}\\
&= -2\int_M \langle \mathrm{Ric}^M + \nabla^2 f + \lambda df \otimes df \,|\, \delta g\rangle e^{-f}\, d\mu\,.
\end{aligned}$$

Hence, as before, the system

$$\begin{cases} \delta g = -2(\mathrm{Ric}^M + \nabla^2 f + \lambda df \otimes df)\\ \delta f = -\Delta f - \mathrm{R}^M - \lambda|\nabla f|^2 \end{cases}$$

represents the constrained gradient of the Einstein–Hilbert action functional and the associated flow is

$$\begin{cases} \partial_t g = -2(\mathrm{Ric}^M + \nabla^2 f + \lambda df \otimes df) \\ \partial_t f = -\Delta f - \mathrm{R}^M - \lambda |\nabla f|^2 \,. \end{cases}$$

Notice that, like in the case of the Ricci flow, the metric flow $\partial_t g = -2(\mathrm{Ric} + \nabla^2 f + \lambda df \otimes df)$ can be modified by a diffeomorphism to the flow $\partial_t g = -2(\mathrm{Ric} + \lambda df \otimes df)$. Anyway, the extra term $df \otimes df$, can not be "cancelled" in this way as it was possible for $\nabla^2 f$. Moreover, as in Perelman's work, one gets immediately the monotonicity of the associated $\mathcal{F}$–functional along the flow (see also [**27**]):

$$\frac{d}{dt} \int_M (\mathrm{R}^M + (\lambda + 1)|\nabla f|^2) e^{-f} \, d\mu = -2 \int_M |\mathrm{Ric}^M + \nabla^2 f + \lambda df \otimes df|^2 e^{-f} \, d\mu \,.$$

CHAPTER 3

Ricci Flow Coupling with the MCF

In this chapter we study the coupling of the MCF with the other geometric flows. In particular, we will focus our interest on an ambient Riemannian manifold (M, g) and an embedded submanifold S endowed with the canonical Riemannian metric induced by the embedding. We will consider the motion by MCF of S meanwhile the ambient metric g evolves along an other flow and we will deduce some monotonicity formulas for this kind of coupled flow in some special cases.

3.1. Rigid Ambient Space

In this section we show the extension of Huisken's monotonicity formula worked out by Hamilton. Let u be a positive solution of the backward heat equation on a Riemannian manifold (M, g) on a time interval $[0, T)$,

$$u_t = -\Delta^M u\,.$$

Let us assume that we have a smooth immersed submanifold N with $\dim N = n$ evolving by mean curvature flow on the time interval $[0, T)$, in the ambient space M with $\dim M = m$, the metric on N is the induced metric and let μ the associated measure.
We denote the normal indices with $\alpha, \beta, \gamma, \dots$ and the tangent ones with $i, j, k, \dots$.
By mean of a straightforward computation, it is possible to prove the following "decomposition" formula for the Riemannian Laplace operator of M:

$$\Delta^M u = \Delta^N u + g^{\alpha\beta}\nabla_\alpha\nabla_\beta u - \mathrm{H}^\alpha\nabla_\alpha u\,. \tag{3.1.1}$$

On the other hand we have that

$$\frac{d}{dt}\int_N u\,d\mu = \int_N u_t + \mathrm{H}^\alpha\nabla_\alpha u - \mathrm{H}^2 u\,d\mu = \int_N -\Delta^M u + \mathrm{H}^\alpha\nabla_\alpha u - \mathrm{H}^2 u\,d\mu\,.$$

Using (3.1.1) and integrating by parts we obtain

$$\frac{d}{dt}\int_N u\,d\mu = \int_N -g^{\alpha\beta}\nabla_\alpha\nabla_\beta u + 2\mathrm{H}^\alpha\nabla_\alpha u - \mathrm{H}^2 u\,d\mu\,.$$

Adding and subtracting the quantity $\frac{\nabla_\alpha u\nabla^\alpha u}{u}$ we get

$$\frac{d}{dt}\int_N u\,d\mu = \int_N -\left(\mathrm{H}^2 u - 2\mathrm{H}^\alpha\nabla_\alpha u + \frac{\nabla_\alpha u\nabla^\alpha u}{u}\right) - \int_N \nabla_\alpha\nabla^\alpha u - \frac{\nabla_\alpha u\nabla^\alpha u}{u}\,d\mu\,.$$

This becomes

$$\frac{d}{dt}\int_N u\,d\mu = -\int_N \left|\mathrm{H} - \frac{\nabla^\perp u}{u}\right|^2 u\,d\mu - \int_N \nabla_\alpha\nabla^\alpha u - \frac{\nabla_\alpha u\nabla^\alpha u}{u}\,d\mu\,.$$

Finally, setting $\tau = T - t$ for some constant $T \in \mathbb{R}$ one obtains, for every $t < T$,

$$\begin{aligned}\frac{d}{dt}\left(\tau^{\frac{m-n}{2}}\int_N u\,d\mu\right) &= -\,\tau^{\frac{m-n}{2}}\int_N \left|\mathrm{H} - \frac{\nabla^\perp u}{u}\right|^2 u\,d\mu\\ &\quad -\,\tau^{\frac{m-n}{2}}\int_N \nabla_\alpha\nabla^\alpha u - \frac{\nabla_\alpha u\nabla^\alpha u}{u} + \frac{u}{2\tau}(m-n)\,d\mu\\ &= -\,\tau^{\frac{m-n}{2}}\int_N \left|\mathrm{H} - \frac{\nabla^\perp u}{u}\right|^2 u\,d\mu\\ &\quad -\,\tau^{\frac{m-n}{2}}\int_N \left(\frac{\nabla^2_{\alpha\beta}u}{u} - \frac{\nabla_\alpha u\nabla_\beta u}{u^2} + \frac{g_{\alpha\beta}}{2\tau}\right) g^{\alpha\beta}u\,d\mu\\ &= -\,\tau^{\frac{m-n}{2}}\int_N \left|\mathrm{H} + \nabla^\perp f\right|^2 e^{-f}\,d\mu\\ &\quad +\,\tau^{\frac{m-n}{2}}\int_N \left(\nabla^2_{\alpha\beta}f - \frac{g_{\alpha\beta}}{2\tau}\right) g^{\alpha\beta}e^{-f}\,d\mu\,,\end{aligned}\tag{3.1.2}$$

where in the last passage we substituted $u = e^{-f}$, as $u > 0$. Notice that $f_t = -\Delta^M f + |\nabla f|^2$. This is Hamilton's result in [**19**].

3.2. Moving Ambient Space

Let us now assume that the metric of the ambient space evolves according to $g_t = -2\mathrm{Q}$ (if $\mathrm{Q} = \mathrm{Ric}$ we have the Ricci flow) and the backward heat kernel equation is modified to

$$u_t = -\Delta^M u + \mathrm{K}u$$

for some K where with R we denote the scalar curvature of the ambient manifold.
If now we repeat the previous computation we have two extra terms, the first arising from the modification to the equation for u and the second from the derivative of the measure on N. Indeed, the associated metric on N is affected not only by the motion of the submanifold but also by the evolution of the ambient metric on M. After some computations, we get

$$\frac{d}{dt}\mu = -(\mathrm{H}^2 + \mathrm{Q}_{ij}g^{ij})\mu = (-\mathrm{H}^2 - \operatorname{tr}\mathrm{Q} + \mathrm{Q}_{\alpha\beta}g^{\alpha\beta})\mu\,.$$

Therefore we obtain

$$
\begin{aligned}
\frac{d}{dt}\left(\tau^{\frac{m-n}{2}}\int_N u\,d\mu\right) =&-\tau^{\frac{m-n}{2}}\int_N\left|\mathrm{H}-\frac{\nabla^\perp u}{u}\right|^2 u\,d\mu\\
&-\tau^{\frac{m-n}{2}}\int_N\left(\frac{\nabla^2_{\alpha\beta}u}{u}-\frac{\nabla_\alpha u\nabla_\beta u}{u^2}+\frac{g_{\alpha\beta}}{2\tau}\right)g^{\alpha\beta}u\,d\mu\\
&+\tau^{\frac{m-n}{2}}\int_N(\mathrm{K}-\operatorname{tr}\mathrm{Q}+\mathrm{Q}_{\alpha\beta}g^{\alpha\beta})u\,d\mu\\
=&-\tau^{\frac{m-n}{2}}\int_N\left|\mathrm{H}+\nabla^\perp f\right|^2 e^{-f}\,d\mu\\
&+\tau^{\frac{m-n}{2}}\int_N\left(\nabla^2_{\alpha\beta}f+\mathrm{Q}_{\alpha\beta}-\frac{g_{\alpha\beta}}{2\tau}\right)g^{\alpha\beta}e^{-f}\,d\mu\\
&+\tau^{\frac{m-n}{2}}\int_N(\mathrm{K}-\operatorname{tr}\mathrm{Q})e^{-f}\,d\mu\,.
\end{aligned}
\tag{3.2.1}
$$

This result suggests that a good choice is $\mathrm{K}=\operatorname{tr}\mathrm{Q}$, as the last term vanishes and we get

$$
\begin{aligned}
\frac{d}{dt}\left(\tau^{\frac{m-n}{2}}\int_N u\,d\mu\right) =&-\tau^{\frac{m-n}{2}}\int_N\left|\mathrm{H}+\nabla^\perp f\right|^2 e^{-f}\,d\mu\\
&+\tau^{\frac{m-n}{2}}\int_N\left(\nabla^2_{\alpha\beta}f+\mathrm{Q}_{\alpha\beta}-\frac{g_{\alpha\beta}}{2\tau}\right)g^{\alpha\beta}e^{-f}\,d\mu\,.
\end{aligned}
\tag{3.2.2}
$$

Moreover, notice that with the choice $\mathrm{K}=\operatorname{tr}\mathrm{Q}$, we have

$$
\frac{d}{dt}\int_M u=\int_M u_t-\operatorname{tr}\mathrm{Q}u=\int_M-\Delta^M u=0\,,
$$

hence the integral $\int_M u=\int_M e^{-f}$ is constant during the flow.

DEFINITION 3.2.1. If $(M,g(t))$ is the flow $g_t=-2\mathrm{Q}$ in a time interval (a,b) and u is a smooth solution of $u_t=-\Delta^M u+\operatorname{tr}\mathrm{Q}u$ in $M\times(a,b)$, we say that (g,u) is a monotonic pair if the quantity

$$
(T-t)^{\frac{m-n}{2}}\int_N u\,d\mu
$$

is monotone nonincreasing in the interval $(a,b)\cap(-\infty,T)$.
In the case $\mathrm{Q}=\mathrm{Ric}$, we say that (g,u) is a Ricci monotonic pair, while in the case $\mathrm{Q}=-\mathrm{Ric}$, we say that (g,u) is a anti–Ricci monotonic pair.

3.3. Ricci and Back–Ricci Flow

3.3.1. Ricci Flow Case. We choose now $\mathrm{Q}=\mathrm{Ric}$, that is, the metric g on M evolves by the Ricci flow in some time interval $(a,b)\subset\mathbb{R}$, and we set $\mathrm{K}=\mathrm{R}$ to be the scalar curvature.

By the previous computation we get

$$\begin{aligned}\frac{d}{dt}\left(\tau^{\frac{m-n}{2}}\int_N u\,d\mu\right) =&-\tau^{\frac{m-n}{2}}\int_N \left|\mathrm{H}+\nabla^\perp f\right|^2 e^{-f}\,d\mu\\ &+\tau^{\frac{m-n}{2}}\int_N\left(\nabla^2_{\alpha\beta}f+\mathrm{R}_{\alpha\beta}-\frac{g_{\alpha\beta}}{2\tau}\right)g^{\alpha\beta}e^{-f}\,d\mu\,,\end{aligned}\tag{3.3.1}$$

for a positive solution of the conjugate backward heat equation

$$u_t=-\Delta u+\mathrm{R}u \tag{3.3.2}$$

and $f=-\log u$. Hence,

$$f_t=-\Delta f+|\nabla f|^2-\mathrm{R}\,. \tag{3.3.3}$$

Monotonicity of $\tau^{\frac{m-n}{2}}\int_N u\,d\mu$ is so related to the nonpositivity of the Li–Yau–Hamilton type expression $\left(\nabla^2_{\alpha\beta}f+\mathrm{R}_{\alpha\beta}-\frac{g_{\alpha\beta}}{2\tau}\right)g^{\alpha\beta}$. Notice that the same conclusion holds also if $u_t\leq-\Delta u+\mathrm{R}u$.
If $(M,g(t))$ is a gradient soliton of the Ricci flow and f its "potential" function, it is well known that $u=e^{-f}$ satisfies the conjugate heat equation (3.3.2) and we have

- Expanding Solitons: flow defined on $(T_{\min},+\infty)$ and $\nabla^2 f+\mathrm{Ric}=g/2(T_{\min}-t)$
- Steady Solitons: eternal flow and $\nabla^2 f+\mathrm{Ric}=0$
- Shrinking Solitons: flow defined on $(-\infty,T_{\max})$ and $\nabla^2 f+\mathrm{Ric}=g/2(T_{\max}-t)$

Substituting, in the three cases, the above expression becomes

- Expanding Soliton: $\frac{m-n}{2}\left(\frac{1}{T_{\min}-t}-\frac{1}{T-t}\right)$ which is always negative as $t\in(T_{\min},T)$.
- Steady Soliton: $\frac{m-n}{2}\left(-\frac{1}{T-t}\right)$ which is always negative as $t\in(-\infty,T)$.
- Shrinking Soliton: $\frac{m-n}{2}\left(\frac{1}{T_{\max}-t}-\frac{1}{T-t}\right)$ which is nonpositive if $T\leq T_{\max}$ as $t\in(-\infty,\min\{T,T_{\max}\})$.

PROPOSITION 3.3.1. *If $(M,g(t))$ is a steady or expanding gradient soliton and f is its potential function, then (g,e^{-f}) is a Ricci monotonic pair for every $T\geq T_{\min}$.*
If $(M,g(t))$ is a shrinking gradient soliton on $(-\infty,T_{\max})$ and f is its potential function, then (g,e^{-f}) is a Ricci monotonic pair for every $T\leq T_{\max}$ ∎

3.3.2. Back–Ricci Flow Case. If we choose $\mathrm{Q}=-\mathrm{Ric}$, that is, the metric g evolves by back–Ricci flow in some time interval $(a,b)\subset\mathbb{R}$, and we set $\mathrm{K}=\mathrm{R}$ to be the scalar curvature.
By the previous computation we get

$$\begin{aligned}\frac{d}{dt}\left(\tau^{\frac{m-n}{2}}\int_N u\,d\mu\right) =&-\tau^{\frac{m-n}{2}}\int_N \left|\mathrm{H}+\nabla^\perp f\right|^2 e^{-f}\,d\mu\\ &+\tau^{\frac{m-n}{2}}\int_N\left(\nabla^2_{\alpha\beta}f-\mathrm{R}_{\alpha\beta}-\frac{g_{\alpha\beta}}{2\tau}\right)g^{\alpha\beta}e^{-f}\,d\mu\,,\end{aligned}\tag{3.3.4}$$

for a positive solution of the conjugate backward heat equation

$$u_t = -\Delta u - \mathrm{R}u \tag{3.3.5}$$

and $f = -\log u$. Hence,

$$f_t = -\Delta f + |\nabla f|^2 + \mathrm{R}$$

Monotonicity of $\tau^{\frac{m-n}{2}} \int_N u\, d\mu$ is so related to the nonpositivity of the Li–Yau–Hamilton type expression $\left(\nabla^2_{\alpha\beta} f - \mathrm{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau}\right) g^{\alpha\beta}$. Notice that the same conclusion holds also if $u_t \leq -\Delta u - \mathrm{R}u$.

3.4. Li–Yau–Hamilton Harnack Inequalities and Ricci Flow

- We denote with $f_{ij} = \nabla^2_{ij} f$ the second covariant derivative of f, then
$$\nabla^2_{ij} f = \frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial f}{\partial x_k}\,.$$
- Let ω_i a 1–form, then we have the following formula for interchanging of covariant derivatives
$$\nabla_{pq}\omega_i - \nabla_{qp}\omega_i = \mathrm{R}_{pqi}{}^s \omega_s\,.$$
Let ω_{ij} a 2–form, then
$$\nabla_{pq}\omega_{ij} - \nabla_{qp}\omega_{ij} = \mathrm{R}_{pqi}{}^s \omega_{sj} + \mathrm{R}_{pqj}{}^s \omega_{is}\,.$$
- II Bianchi Identity:
$$\nabla_s \mathrm{R}_{ijkl} + \nabla_l \mathrm{R}_{ijsk} + \nabla_k \mathrm{R}_{ijls} = 0$$
contracted,
$$g^{js}\nabla_s \mathrm{R}_{ijkl} - \nabla_l \mathrm{Ric}_{ik} + \nabla_k \mathrm{Ric}_{il} = 0$$
that is,
$$\operatorname{div} \mathrm{Riem}_{ikl} = \nabla_k \mathrm{Ric}_{il} - \nabla_l \mathrm{Ric}_{ik}$$
contracted again (Schur Lemma),
$$\operatorname{div} \mathrm{Ric}_k = \nabla_k \mathrm{R} - \operatorname{div} \mathrm{Ric}_k$$
that is,
$$\operatorname{div} \mathrm{Ric} = \nabla \mathrm{R}/2\,.$$
- Evolution equations for Ricci tensor and scalar curvature under Ricci flow:
$$\partial_t \mathrm{Ric}_{ij} = \Delta \mathrm{Ric}_{ij} + 2\mathrm{Ric}^{pq}\mathrm{R}_{ipjq} - 2g^{pq}\mathrm{Ric}_{ip}\mathrm{Ric}_{qj}$$
$$\partial_t \mathrm{R} = \Delta \mathrm{R} + 2|\mathrm{Ric}|^2\,.$$
- Evolution equations for Christoffel symbols under Ricci flow:
$$\partial_t \Gamma^k_{ij} = -g^{kl}(\nabla_i \mathrm{Ric}_{jl} + \nabla_j \mathrm{Ric}_{il} - \nabla_l \mathrm{Ric}_{ij})\,.$$

- Interchange of Laplacian and second derivatives:

$$\begin{aligned}
\nabla^2_{ij}\Delta f =& \nabla_i\nabla_j\nabla_k\nabla_k f\\
=& \nabla_i(\mathrm{R}_{jkkp}\nabla_p f) + \nabla_i\nabla_k\nabla_j\nabla_k f\\
=& -\nabla_i(\mathrm{Ric}_{jp}\nabla_p f) + \nabla_i\nabla_k\nabla_k\nabla_j f\\
=& -\nabla_i\mathrm{Ric}_{jp}\nabla_p f - \mathrm{Ric}_{jp}f_{ip} + \nabla_i\nabla_k\nabla_k\nabla_j f\\
=& -\nabla_i\mathrm{Ric}_{jp}\nabla_p f - \mathrm{Ric}_{jp}f_{ip} + \mathrm{R}_{ikkp}f_{pj} + \mathrm{R}_{ikjp}f_{kp} + \nabla_k\nabla_i\nabla_k\nabla_j f\\
=& -\nabla_i\mathrm{Ric}_{jp}\nabla_p f - \mathrm{Ric}_{jp}f_{ip} - \mathrm{Ric}_{ip}f_{pj} - \mathrm{R}_{ikpj}f_{kp} + \nabla_k(\mathrm{R}_{ikjp}\nabla_p f) + \nabla_k\nabla_k\nabla_i\nabla_j f\\
=& -\nabla_i\mathrm{Ric}_{jp}\nabla_p f - \nabla_k\mathrm{R}_{ikpj}\nabla_p f - \mathrm{Ric}_{jp}f_{ip} - \mathrm{Ric}_{ip}f_{pj} - \mathrm{R}_{ikpj}f_{kp} - \mathrm{R}_{ikpj}f_{kp} + \Delta\nabla_i\nabla_j f\\
=& -(\nabla_i\mathrm{Ric}_{jk} + \nabla_j\mathrm{Ric}_{ik} - \nabla_k\mathrm{Ric}_{ij})\nabla_k f - \mathrm{Ric}_{jp}f_{ip} - \mathrm{Ric}_{ip}f_{pj} - 2\mathrm{R}_{ikpj}f_{kp} + \Delta\nabla_i\nabla_j f
\end{aligned}$$

where in the last passage we used the II Bianchi identity. Hence,

$$\nabla^2_{ij}\Delta f - \Delta\nabla_i\nabla_j f = -(\nabla_i\mathrm{Ric}_{jk} + \nabla_j\mathrm{Ric}_{ik} - \nabla_k\mathrm{Ric}_{ij})\nabla_k f - \mathrm{Ric}_{jp}f_{ip} - \mathrm{Ric}_{ip}f_{pj} - 2\mathrm{R}_{ikpj}f_{kp}\,.$$

3.4.1. Computation I: Ricci Flow. Suppose that $u_t = -\Delta u + \mathrm{R}u$ and $u > 0$, we want to show the nonpositivity of the term

$$\nabla^2_{ij}f + \mathrm{R}_{ij} - \frac{g_{ij}}{2\tau}$$

for $f = -\log u$ which satisfies

$$f_t = -\Delta f + |\nabla f|^2 - \mathrm{R}\,.$$

Equivalently, if we had chosen $f = \log u$, we can show the positivity of

$$\nabla^2_{ij}f - \mathrm{R}_{ij} + \frac{g_{ij}}{2\tau}$$

for $f = \log u$ which satisfies

$$f_t = -\Delta f - |\nabla f|^2 + \mathrm{R}\,.$$

We set $\tau = T - t$, $L_{ij} = f_{ij} - \mathrm{Ric}_{ij}$, $H_{ij} = \tau L_{ij} + g_{ij}/2 = \tau[f_{ij} - \mathrm{Ric}_{ij}] + g_{ij}/2$.

$$
\begin{aligned}
(\partial_t + \Delta)H_{ij} = &- L_{ij} - \mathrm{Ric}_{ij} \\
&+ \tau[\Delta f_{ij} + \nabla^2_{ij} f_t + (\nabla_i \mathrm{Ric}_{jk} + \nabla_j \mathrm{Ric}_{ik} - \nabla_k \mathrm{Ric}_{ij})\nabla^k f] \\
&- \tau[\partial_t \mathrm{Ric}_{ij} + \Delta \mathrm{Ric}_{ij}] \\
= &- L_{ij} - \mathrm{Ric}_{ij} \\
&+ \tau[\Delta f_{ij} - \nabla^2_{ij}\Delta f - \nabla^2_{ij}|\nabla f|^2 \\
&+ (\nabla_i \mathrm{Ric}_{jk} + \nabla_j \mathrm{Ric}_{ik} - \nabla_k \mathrm{Ric}_{ij})\nabla^k f] \\
&- \tau[2\Delta \mathrm{Ric}_{ij} + 2\mathrm{Ric}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{Ric}_{ip}\mathrm{Ric}_{pj} - \nabla^2 \mathrm{R}] \\
= &- L_{ij} - \mathrm{Ric}_{ij} \\
&+ \tau[(\nabla_i \mathrm{Ric}_{jk} + \nabla_j \mathrm{Ric}_{ik} - \nabla_k \mathrm{Ric}_{ij})\nabla_k f \\
&+ \mathrm{Ric}_{jp} f_{ip} + \mathrm{Ric}_{ip} f_{pj} + 2\mathrm{R}_{ikpj} f_{kp} \\
&- \nabla^2_{ij}|\nabla f|^2 + (\nabla_i \mathrm{Ric}_{jk} + \nabla_j \mathrm{Ric}_{ik} - \nabla_k \mathrm{Ric}_{ij})\nabla^k f] \\
&- \tau[2\Delta \mathrm{Ric}_{ij} + 2\mathrm{Ric}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{Ric}_{ip}\mathrm{Ric}_{pj} - \nabla^2 \mathrm{R}] \\
= &- L_{ij} - \mathrm{Ric}_{ij} \\
&+ \tau[\mathrm{Ric}_{jp} f_{ip} + \mathrm{Ric}_{ip} f_{pj} + 2\mathrm{R}_{ikpj} f_{kp} \\
&- 2 f_{ip} f_{jp} - 2\nabla^3_{ijk} f \nabla^k f + 2(\nabla_i \mathrm{Ric}_{jk} + \nabla_j \mathrm{Ric}_{ik} - \nabla_k \mathrm{Ric}_{ij})\nabla^k f] \\
&- \tau[2\Delta \mathrm{Ric}_{ij} + 2\mathrm{Ric}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{Ric}_{ip}\mathrm{Ric}_{pj} - \nabla^2 \mathrm{R}] \\
= &- L_{ij} - \mathrm{Ric}_{ij} \\
&+ \tau[\mathrm{Ric}_{jp} f_{ip} + \mathrm{Ric}_{ip} f_{pj} + 2\mathrm{R}_{ikpj} f_{kp} \\
&- 2 f_{ip} f_{jp} - \nabla^3_{ijk} f \nabla^k f - \nabla^3_{jik} f \nabla^k f + 2(\nabla_i \mathrm{Ric}_{jk} + \nabla_j \mathrm{Ric}_{ik} - \nabla_k \mathrm{Ric}_{ij})\nabla^k f] \\
&- \tau[2\Delta \mathrm{Ric}_{ij} + 2\mathrm{Ric}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{Ric}_{ip}\mathrm{Ric}_{pj} - \nabla^2 \mathrm{R}] \\
= &- L_{ij} - \mathrm{Ric}_{ij} \\
&+ \tau[\mathrm{Ric}_{jp} f_{ip} + \mathrm{Ric}_{ip} f_{pj} + 2\mathrm{R}_{ikpj} f_{kp} \\
&- 2 f_{ip} f_{jp} - 2\nabla^3_{kij} f \nabla^k f - 2\mathrm{R}_{ikjp}\nabla_p f \nabla_k f + 2(\nabla_i \mathrm{Ric}_{jk} + \nabla_j \mathrm{Ric}_{ik} - \nabla_k \mathrm{Ric}_{ij})\nabla^k f] \\
&- \tau[2\Delta \mathrm{Ric}_{ij} + 2\mathrm{Ric}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{Ric}_{ip}\mathrm{Ric}_{pj} - \nabla^2 \mathrm{R}] \\
= &- L_{ij} - \mathrm{Ric}_{ij} \\
&+ \tau[\mathrm{Ric}_{jp} f_{ip} + \mathrm{Ric}_{ip} f_{pj} - 2 f_{ip} f_{jp} - 2\nabla^3_{kij} f \nabla^k f] \\
&- \tau[2\Delta \mathrm{Ric}_{ij} + 2\mathrm{Ric}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{Ric}_{ip}\mathrm{Ric}_{pj} - \nabla^2 \mathrm{R}] \\
&+ \tau[2(\nabla_i \mathrm{Ric}_{jk} + \nabla_j \mathrm{Ric}_{ik} - \nabla_k \mathrm{Ric}_{ij})\nabla^k f \\
&- 2\mathrm{R}_{ikjp} f_{pk} - 2\mathrm{R}_{ikjp}\nabla_p f \nabla_k f]
\end{aligned}
$$

substituting, $L_{ij} = [H_{ij} - g_{ij}/2]/\tau$ and $f_{ij} = [H_{ij} - g_{ij}/2]/\tau + \mathrm{Ric}_{ij}$, we get

$$
\begin{aligned}
(\partial_t + \Delta)H_{ij} =& - H_{ij}/\tau + g_{ij}/2\tau - \mathrm{Ric}_{ij} \\
&+ \tau[\mathrm{Ric}_{jp}\mathrm{Ric}_{ip} + \mathrm{Ric}_{ip}\mathrm{Ric}_{pj} - 2\nabla_k\mathrm{Ric}_{ij}\nabla^k f] \\
&- 2\tau[H^2_{ij}/\tau^2 - H_{ij}/\tau^2 + g_{ij}/4\tau^2 + \mathrm{Ric}_{ik}\mathrm{Ric}_{kj} + \mathrm{Ric}_{ik}H_{jk}/\tau + \mathrm{Ric}_{jk}H_{ik}/\tau - \mathrm{Ric}_{ij}/\tau] \\
&+ [\mathrm{Ric}_{jp}H_{ip} + \mathrm{Ric}_{ip}H_{pj} - 2\nabla_k H_{ij}\nabla^k f] \\
&- \mathrm{Ric}_{ij} \\
&- \tau[2\Delta\mathrm{Ric}_{ij} + 2\mathrm{Ric}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{Ric}_{ip}\mathrm{Ric}_{pj} - \nabla^2_{ij}R] \\
&+ \tau[2(\nabla_i\mathrm{Ric}_{jk} + \nabla_j\mathrm{Ric}_{ik} - \nabla_k\mathrm{Ric}_{ij})\nabla^k f] \\
&- 2\tau\mathrm{R}_{ikjp}\mathrm{Ric}_{pk} - 2\mathrm{R}_{ikjp}H_{pk} + \mathrm{Ric}_{ij} \\
&- 2\tau\mathrm{R}_{ikjp}\nabla_p f\nabla_k f \\
=& [H_{ij} - 2H^2_{ij}]/\tau + \mathrm{Ric}_{ij} + \tau[\mathrm{Ric}_{jp}\mathrm{Ric}_{ip} + \mathrm{Ric}_{ip}\mathrm{Ric}_{pj} - 2\nabla_k\mathrm{Ric}_{ij}\nabla^k f] \\
&- 2\tau[\mathrm{Ric}_{ik}\mathrm{Ric}_{kj} + \mathrm{Ric}_{ik}H_{jk}/\tau + \mathrm{Ric}_{jk}H_{ik}/\tau] \\
&+ [\mathrm{Ric}_{jp}H_{ip} + \mathrm{Ric}_{ip}H_{pj} - 2\nabla_k H_{ij}\nabla^k f] \\
&- \tau[2\Delta\mathrm{Ric}_{ij} + 4\mathrm{Ric}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{Ric}_{ip}\mathrm{Ric}_{pj} - \nabla^2_{ij}R] \\
&+ \tau[2(\nabla_i\mathrm{Ric}_{jk} + \nabla_j\mathrm{Ric}_{ik} - \nabla_k\mathrm{Ric}_{ij})\nabla^k f] \\
&- 2\mathrm{R}_{ikjp}H_{pk} - 2\tau\mathrm{R}_{ikjp}\nabla_p f\nabla_k f \\
=& [H_{ij} - 2H^2_{ij}]/\tau - 2\nabla_k H_{ij}\nabla^k f \\
&- [\mathrm{Ric}_{ik}H_{jk} + \mathrm{Ric}_{jk}H_{ik} + 2\mathrm{R}_{ikjp}H_{pk}] \\
&- \tau[2\Delta\mathrm{Ric}_{ij} - 2\mathrm{Ric}_{jp}\mathrm{Ric}_{ip} + 4\mathrm{Ric}_{pq}\mathrm{R}_{ipjq} - \nabla^2_{ij}R - \mathrm{Ric}_{ij}/\tau] \\
&+ \tau[2(\nabla_i\mathrm{Ric}_{jk} + \nabla_j\mathrm{Ric}_{ik} - 2\nabla_k\mathrm{Ric}_{ij})\nabla^k f] \\
&- 2\tau\mathrm{R}_{ikjp}\nabla_p f\nabla_k f
\end{aligned}
$$

so finally, we get

$$
\begin{aligned}
(\partial_t + \Delta)H_{ij} =& [H_{ij} - 2H^2_{ij}]/\tau - 2\nabla_k H_{ij}\nabla^k f - \mathrm{Ric}^k_i H_{kj} - \mathrm{Ric}^k_j H_{ki} - 2\mathrm{R}_{ipjq}H^{pq} \\
&- \tau[2\Delta\mathrm{Ric}_{ij} - 2g^{pq}\mathrm{Ric}_{ip}\mathrm{Ric}_{jq} + 4\mathrm{Ric}^{pq}\mathrm{R}_{ipjq} - \nabla^2_{ij}R - \mathrm{Ric}_{ij}/\tau] \\
&+ 2\tau(\nabla_i\mathrm{Ric}_{jk} + \nabla_j\mathrm{Ric}_{ik} - 2\nabla_k\mathrm{Ric}_{ij})\nabla^k f - 2\tau\mathrm{R}_{ipjq}\nabla^p f\nabla^q f\,.
\end{aligned}
$$

Notice that the second and third lines gives the Hamilton's Harnack quadratic with a wrong term $-\mathrm{Ric}_{ij}/\tau$.

3.4.2. Computation II: Back–Ricci Flow. Suppose that $u_t = -\Delta u - \mathrm{R}u$ and $u > 0$, we want to show the nonpositivity of the term

$$\nabla^2_{ij}f - \mathrm{R}_{ij} - \frac{g_{ij}}{2\tau}$$

for $f = -\log u$ which satisfies

$$f_t = -\Delta f + |\nabla f|^2 + \mathrm{R}\,.$$

Equivalently, if we had chosen $f = \log u$ we can show the positivity of

$$\nabla^2_{ij} f + \mathrm{R}_{ij} + \frac{g_{ij}}{2\tau}$$

for $f = \log u$ which satisfies

$$f_t = -\Delta f - |\nabla f|^2 - \mathrm{R}\,.$$

- Evolution equations for Ricci tensor and scalar curvature under back–Ricci flow:

$$\partial_t \mathrm{Ric}_{ij} = -(\Delta \mathrm{Ric}_{ij} + 2\mathrm{Ric}^{pq}\mathrm{R}_{ipjq} - 2g^{pq}\mathrm{Ric}_{ip}\mathrm{Ric}_{qj})$$

$$\partial_t \mathrm{R} = -(\Delta \mathrm{R} + 2|\mathrm{Ric}|^2)\,.$$

- Evolution equations for Christoffel symbols under back–Ricci flow:

$$\partial_t \Gamma^k_{ij} = g^{kl}(\nabla_i \mathrm{Ric}_{jl} + \nabla_j \mathrm{Ric}_{il} - \nabla_l \mathrm{Ric}_{ij})\,.$$

We set $f_i = \nabla_i f$, $f_{ij} = \nabla^2_{ij} f$ and $L_{ij} = f_{ij} + \mathrm{Ric}_{ij}$, $H_{ij} = \tau L_{ij} + g_{ij}/2 = \tau[f_{ij} + \mathrm{Ric}_{ij}] + g_{ij}/2$,

$$\begin{aligned}
(\partial_t + \Delta)H_{ij} =& - L_{ij} + \mathrm{Ric}_{ij}\\
&+ \tau[\Delta f_{ij} + \nabla^2_{ij} f_t - (\nabla_i \mathrm{Ric}_{jk} + \nabla_j \mathrm{Ric}_{ik} - \nabla_k \mathrm{Ric}_{ij})\nabla^k f]\\
&+ \tau[\partial_t \mathrm{Ric}_{ij} + \Delta \mathrm{Ric}_{ij}]\\
=& - f_{ij} + \tau[\Delta f_{ij} - \nabla^2_{ij}\Delta f - \nabla^2_{ij}|\nabla f|^2 - (\nabla_i \mathrm{Ric}_{jk} + \nabla_j \mathrm{Ric}_{ik} - \nabla_k \mathrm{Ric}_{ij})\nabla^k f]\\
&- \tau[2\mathrm{Ric}^{pq}\mathrm{R}_{ipjq} - 2g^{pq}\mathrm{Ric}_{ip}\mathrm{Ric}_{qj} + \nabla^2_{ij}\mathrm{R}]\\
=& - f_{ij} + \tau[(\nabla_i \mathrm{Ric}_{jk} + \nabla_j \mathrm{Ric}_{ik} - \nabla_k \mathrm{Ric}_{ij})\nabla^k f + g^{pq}\mathrm{Ric}_{jp} f_{iq} + g^{pq}\mathrm{Ric}_{ip} f_{qj} - 2\mathrm{R}_{ipjq} f^{pq}]\\
&+ \tau[-\nabla^2_{ij}|\nabla f|^2 - (\nabla_i \mathrm{Ric}_{jk} + \nabla_j \mathrm{Ric}_{ik} - \nabla_k \mathrm{Ric}_{ij})\nabla^k f]\\
&- \tau[2\mathrm{Ric}^{pq}\mathrm{R}_{ipjq} - 2g^{pq}\mathrm{Ric}_{ip}\mathrm{Ric}_{qj} + \nabla^2_{ij}\mathrm{R}]\\
=& - f_{ij} + \tau[g^{pq}\mathrm{Ric}_{jp} f_{iq} + g^{pq}\mathrm{Ric}_{ip} f_{qj} - 2\mathrm{R}_{ipjq} f^{pq} - \nabla^2_{ij}|\nabla f|^2]\\
&- \tau[2\mathrm{Ric}^{pq}\mathrm{R}_{ipjq} - 2g^{pq}\mathrm{Ric}_{ip}\mathrm{Ric}_{qj} + \nabla^2_{ij}\mathrm{R}]\\
=& - f_{ij} + \tau[g^{pq}\mathrm{Ric}_{jp} f_{iq} + g^{pq}\mathrm{Ric}_{ip} f_{qj} - 2\mathrm{R}_{ipjq} f^{pq}]\\
&- \tau[2\mathrm{Ric}^{pq}\mathrm{R}_{ipjq} - 2g^{pq}\mathrm{Ric}_{ip}\mathrm{Ric}_{qj} + \nabla^2_{ij}\mathrm{R}]\\
&- \tau[2 f_{ip} f_{jp} + 2\nabla^3_{ijk} f \nabla^k f]\\
=& - f_{ij} + \tau[g^{pq}\mathrm{Ric}_{jp} f_{iq} + g^{pq}\mathrm{Ric}_{ip} f_{qj} - 2\mathrm{R}_{ipjq} f^{pq}]\\
&- \tau[2\mathrm{Ric}^{pq}\mathrm{R}_{ipjq} - 2g^{pq}\mathrm{Ric}_{ip}\mathrm{Ric}_{qj} + \nabla^2_{ij}\mathrm{R}]\\
&- \tau[2 f_{ip} f_{jp} + 2\nabla^3_{kij} f \nabla^k f + 2\mathrm{R}_{ipjq}\nabla^p f \nabla^q f]\,.
\end{aligned}$$

Suppose now that at time $t > 0$, the tensor H_{ij} (which goes $+\infty$ as $t \to T^-$) get its "last" zero eigenvalue at some point (p, t) in space and time, with V^i unit zero eigenvector. We extend V^i in space such that $\nabla V(p) = \nabla^2 V(p) = 0$ and constant in time. Then if $Z = H_{ij} V^i V^j$ we have that Z has a global minimum on $M \times [t, T]$ at (p, t). At such point we have $Z = 0$, $\nabla Z = 0$ and $\Delta Z \geq 0$, hence, $f_{ij}V^iV^j = -\mathrm{Ric}_{ij}V^iV^j - 1/2\tau$, and as $\nabla Z = 0$, $\nabla_k f_{ij}V^iV^j = -\nabla_k\mathrm{Ric}_{ij}V^iV^j$. Then

$$\begin{aligned}
0 \leq \partial_t Z + \Delta Z =& (\partial_t H_{ij} + \Delta H_{ij})V^iV^j \\
=& \{-f_{ij} + \tau[g^{pq}\mathrm{Ric}_{jp}f_{iq} + g^{pq}\mathrm{Ric}_{ip}f_{qj} - 2\mathrm{R}_{ipjq}f^{pq}] \\
&- \tau[2\mathrm{Ric}^{pq}\mathrm{R}_{ipjq} - 2g^{pq}\mathrm{Ric}_{ip}\mathrm{Ric}_{qj} + \nabla^2_{ij}\mathrm{R}] \\
&- \tau[2f_{ip}f_{jp} + 2\nabla^3_{kij}f\nabla^k f + 2\mathrm{R}_{ipjq}\nabla^p f\nabla^q f]\}V^iV^j \\
=& \{\mathrm{Ric}_{ij} + g_{ij}/2\tau + \tau[-2\mathrm{Ric}^2_{ij} - \mathrm{Ric}_{ij}/\tau + 2\mathrm{Ric}^{pq}\mathrm{R}_{ipjq} + \mathrm{Ric}_{ij}/\tau] \\
&- \tau[2\mathrm{Ric}^{pq}\mathrm{R}_{ipjq} - 2g^{pq}\mathrm{Ric}_{ip}\mathrm{Ric}_{qj} + \nabla^2_{ij}\mathrm{R}] \\
&- \tau[2\mathrm{Ric}^2_{ij} + 2\mathrm{Ric}_{ij}/\tau + g_{ij}/2\tau^2 - 2\nabla_k\mathrm{Ric}_{ij}\nabla^k f + 2\mathrm{R}_{ipjq}\nabla^p f\nabla^q f]\}V^iV^j \\
=& \{-\mathrm{Ric}_{ij} - \tau[2\mathrm{Ric}^2_{ij} + \nabla^2_{ij}\mathrm{R} - 2\nabla_k\mathrm{Ric}_{ij}\nabla^k f + 2\mathrm{R}_{ipjq}\nabla^p f\nabla^q f]\}V^iV^j \\
=& -\tau\{\nabla^2_{ij}\mathrm{R} + 2\mathrm{Ric}^2_{ij} + \mathrm{Ric}_{ij}/\tau - 2\nabla_k\mathrm{Ric}_{ij}\nabla^k f + 2\mathrm{R}_{ipjq}\nabla^p f\nabla^q f]\}V^iV^j\,.
\end{aligned}$$

By this computation, it follows that we would get a contradiction by maximum principle, if the following Hamilton–Harnack type inequality is true.

$$\nabla^2_{ij}\mathrm{R} + 2\mathrm{Ric}^2_{ij} + \mathrm{Ric}_{ij}/\tau - 2\nabla_k\mathrm{Ric}_{ij}U^k + 2\mathrm{R}_{ipjq}U^pU^q \geq 0\,.$$

See [**31**] and also [**11**].

3.4.3. Dimension 2. In the special two–dimensional case of a surface with bounded and positive scalar curvature this inequality holds, see [**6**, Chapter 15, Section 3].
If a positive function u satisfies

$$u_t = -\Delta u - \mathrm{R}u$$

for a closed curve moving by its curvature k inside a surface evolving by $g_t = 2\mathrm{Ric} = \mathrm{R}g$, we have

$$\frac{d}{dt}\left(\sqrt{\tau}\int_\gamma u\,ds\right) \leq -\sqrt{\tau}\int_\gamma \left|\mathrm{k} - \nabla^\perp \log u\right|^2 u\,ds\,,$$

where ν is the unit normal to the curve γ.

CHAPTER 4

Maximizing Huisken's Functional

4.1. Maximizing Huisken's Monotonicity Formula

Let $(M, \phi(\cdot, t))$ to be a one parameter family of immersions of an n-dimensional smooth hypersurface in $\mathbb{R}^{n+1}$, with second fundamental form and mean curvature respectively denoted by A and H.
According to (1.4.9), we call the quantity

$$\sqrt{4\pi(T-t)} \int_M u \, d\mu_t(x) \tag{4.1.1}$$

the Huisken's functional (evaluated on a suitable function u). Within the spirit of Hamilton's extension of the Huisken's monotonicity formula, we want to obtain informations on the MCF of M by maximizing the Huisken's functional on the largest class of admissible functions.

Definition 4.1.1. Let $\phi : M \to \mathbb{R}^{n+1}$ be a smooth, compact, immersed hypersurface. Given $\tau > 0$, we consider the family $\mathcal{F}_\tau$ of smooth positive functions $u : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\int_{\mathbb{R}^{n+1}} u \, dx = 1$ and there exists a smooth positive solution of the problem

$$\begin{cases} v_t = -\Delta v \ \text{ in } \mathbb{R}^{n+1} \times [0, \tau) \,, \\ v(x, 0) = u(x) \ \text{ for every } p \in \mathbb{R}^{n+1} \,. \end{cases}$$

Then, we define the following quantity

$$\sigma(\phi, \tau) = \sup_{u \in \mathcal{F}_\tau} \sqrt{4\pi\tau} \int_M u \, d\mu \,.$$

It is important to notice that heat kernel $K_{\mathbb{R}^{n+1}}(x, \tau) = \frac{e^{-\frac{|x-y|^2}{4\tau}}}{(4\pi\tau)^{(n+1)/2}}$ of $\mathbb{R}^{n+1}$ evaluated at time $\tau > 0$ and centered at the point $y \in \mathbb{R}^{n+1}$ clearly belongs to the family $\mathcal{F}_\tau$. As an immediate consequence, we have that the quantity $\sigma(\phi, \tau)$ is positive and precisely, for every $p \in \mathbb{R}^{n+1}$ and $\tau > 0$,

$$\sigma(\phi, \tau) \geq \sqrt{4\pi\tau} \int_M \frac{e^{-\frac{|x-y|^2}{4\tau}}}{(4\pi\tau)^{(n+1)/2}} \, d\mu(x) = \int_M \frac{e^{-\frac{|x-y|^2}{4\tau}}}{(4\pi\tau)^{n/2}} \, d\mu(x) > 0 \,,$$

which is the quantity of the "classical" Huisken's monotonicity formula. Hence,

$$\sigma(\phi,\tau) \geq \sup_{y\in\mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-y|^2}{4\tau}}}{(4\pi\tau)^{n/2}}\,d\mu(x) > 0\,. \tag{4.1.2}$$

We now want to show that this inequality is actually an equality, which would mean that in order to maximize the Huisken's functional we can take the sup only on heat kernels. Moreover, by the assumed compactness of M this would also imply that the supremum would be a maximum.
We work out some properties of the functions $u \in \mathcal{F}_\tau$.

Let us to start by recalling the integrated version of Li–Yau Harnack inequality (see [**26**]).

Proposition 4.1.2 (Li–Yau integral Harnack inequality). *Let $u : \mathbb{R}^{n+1}\times(0,T)\to\mathbb{R}$ be a smooth positive solution of heat equation, then for every $0 < t \leq s < T$ we have*

$$u(x,t) \leq u(y,s)\left(\frac{s}{t}\right)^{(n+1)/2} e^{\frac{|x-y|^2}{4(s-t)}} \quad \blacksquare$$

Since the functions $v : \mathbb{R}^{n+1}\times[0,\tau)\to\mathbb{R}$ associated to any $u\in\mathcal{F}_\tau$ are positive solutions of the backward heat equation, such inequality reads, for $0\leq s\leq t<\tau$,

$$v(x,t) \leq v(y,s)\left(\frac{\tau-s}{\tau-t}\right)^{(n+1)/2} e^{\frac{|x-y|^2}{4(t-s)}}\,.$$

This estimate, together with the uniqueness theorem for positive solution of the heat equation (see again [**26**]), implies that the function $u = v(\cdot,0)$ is obtained by convolution of the function $v(\cdot,t)$ with the *forward* heat kernel at time $t>0$. This fact implies that the condition $\int_{\mathbb{R}^{n+1}} v(x,t)\,dx = 1$ holds for every $t\in[0,\tau)$, and that every derivative of every function v is bounded in the strip $[0,\tau-\varepsilon]$, for every $\varepsilon>0$.

Lemma 4.1.3. *The functions $v(\cdot,t)$ weakly* converge as probability measures, as $t\to\tau$, to some positive unit measure λ on $\mathbb{R}^{n+1}$ such that*

$$v(x,t) = \int_{\mathbb{R}^{n+1}} \frac{e^{-\frac{|x-y|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{(n+1)/2}}\,d\lambda(y)\,. \tag{4.1.3}$$

Conversely, every probability measure λ, by convolution with the heat kernel, gives rise to a function v such that $v(\cdot,\tau)\in\mathcal{F}_\tau$, the most interesting case being $\lambda=\delta_p$ for $p\in\mathbb{R}^{n+1}$ $\blacksquare$

Proof. Indeed, we know that for every $t\in[0,\tau)$ and $s\in(t,\tau)$

$$v(x,t) = \int_{\mathbb{R}^{n+1}} v(y,s)\frac{e^{\frac{|x-y|^2}{4(t-s)}}}{[4\pi(s-t)]^{(n+1)/2}}\,dx\,,$$

hence, choosing a sequence of times $s_i \nearrow \tau$ such that the measures $v(\cdot,s_i)\mathcal{L}^{n+1}$ weakly* converge to some measure λ. Since $\frac{e^{\frac{|x-y|^2}{4(t-s)}}}{[4\pi(s-t)]^{(n+1)/2}}$ converges uniformly to $\frac{e^{-\frac{|x-y|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{(n+1)/2}}$ on

$\mathbb{R}^{n+1}$ as $s \to \tau$, we get equality (4.1.3) .

This representation formula also implies that the limit measure λ is unique and that actually $\lim_{s\to\tau} v(\cdot, s)\mathcal{L}^{n+1} = \lambda$ in the weak* convergence of measures on $\mathbb{R}^{n+1}$.

Finally, we show that $|\lambda| = 1$. This follows by Fubini–Tonelli's theorem for positive product measures, as $\int_{\mathbb{R}^{n+1}} u(x)\, dx = 1$,

$$\begin{aligned} 1 = \int_{\mathbb{R}^{n+1}} u(x)\, dx &= \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \frac{e^{-\frac{|x-y|^2}{4\tau}}}{[4\pi\tau]^{(n+1)/2}}\, d\lambda(y)\, dx \\ &= \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \frac{e^{-\frac{|x-y|^2}{4\tau}}}{[4\pi\tau]^{(n+1)/2}}\, dx\, d\lambda(y) \\ &= \int_{\mathbb{R}^{n+1}} d\lambda(y) = |\lambda|\,. \end{aligned}$$

□

By this discussion it follows that the family $\mathcal{F}_\tau$ consists of the functions

$$u(x,t) = \int_{\mathbb{R}^{n+1}} \frac{e^{-\frac{|x-y|^2}{4\tau}}}{[4\pi\tau]^{(n+1)/2}}\, d\lambda(y)\,,$$

where λ varies among the convex set of Borel probability measures on $\mathbb{R}^{n+1}$ (which is weak*–compact).
As a consequence of this fact, since the integral $\sqrt{4\pi\tau}\int_M u\, d\mu$ is a linear functional in the function u, the sup in defining $\sigma(\phi,\tau)$ can be taken considering only the extremal points of the above convex, which are the *delta* measures in $\mathbb{R}^{n+1}$. Consequently, the functions u to be considered for the maximization process can be restricted to be heat kernels at time $\tau > 0$. Thus, it is then easy to conclude that, being the hypersurface M is compact in $\mathbb{R}^{n+1}$, the sup is actually a maximum.

PROPOSITION 4.1.4. *The quantity $\sigma(\phi,\tau)$ is given by*

$$\sigma(\phi,\tau) = \max_{y\in\mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-p|^2}{4\tau}}}{(4\pi\tau)^{n/2}}\, d\mu(x)\,.$$

It is also easy to check that

$$\sigma(\phi,\tau) = \sup_{y\in\mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-y|^2}{4\tau}}}{(4\pi\tau)^{n/2}}\, d\mu(x) \le \int_M \frac{1}{(4\pi\tau)^{n/2}}\, d\mu(x) \le \frac{\mathrm{Area}(M)}{(4\pi\tau)^{n/2}}\,.$$

We want now to study the scaling properties of σ.

PROPOSITION 4.1.5 (Rescaling Invariance). *For every $\lambda > 0$ we have*

$$\sigma(\lambda\phi, \lambda^2\tau) = \sigma(\phi,\tau) \ \blacksquare$$

PROOF. Let $u \in \mathcal{F}_\tau$ and $v : \mathbb{R}^{n+1} \times [0, \tau) \to \mathbb{R}$ the associated solution of the backward heat equation. Consider the rescaled function $\widetilde{u}(y) = u(y/\lambda)\lambda^{-(n+1)}$. Using the change of variable $x = \lambda^{-(n+1)}y$, it is easy to see that

$$\int_{\mathbb{R}^{n+1}} \widetilde{u}(y)\, dy = \lambda^{-(n+1)} \int_{\mathbb{R}^{n+1}} u(y/\lambda)\, dy = \int_{\mathbb{R}^{n+1}} u(x)\, dx = 1\,.$$

Moreover, the function $\widetilde{v}(y, s) = v(y/\lambda, s/\lambda^2)\lambda^{-(n+1)}$ is a positive solution of the backward heat equation on the time interval $\lambda^2\tau$, hence $\widetilde{u} \in \mathcal{F}_{\lambda^2\tau}$.
Now, with a straightforward computation, we see that

$$\sqrt{4\pi\lambda^2\tau} \int_M \widetilde{u}\, d\mu_{\lambda\phi} = \sqrt{4\pi\tau} \int_M u\, d\mu_\phi\,,$$

for every smooth immersion of a compact hypersurface $\phi : M \to \mathbb{R}^{n+1}$. The statement clearly follows. $\square$

By formula (1.4.9), as the second term vanishes when v is a backward heat kernel, it follows that if $\phi : M \times [0, T) \to \mathbb{R}^{n+1}$ is the MCF of a compact hypersurface M, we have

$$\begin{aligned} &\frac{d}{dt}\Big[\sqrt{2(C-t)} \int_M K_{\mathbb{R}^{n+1}}(x, p, C-t)\, d\mu_t(x)\Big] \\ &\qquad = -\sqrt{2(C-t)} \int_M K_{\mathbb{R}^{n+1}}(x, p, C-t) |\mathrm{H} - (x-p)^\perp/2(C-t)|^2\, d\mu_t(x) \end{aligned}$$

which is clearly negative in the time interval $[0, \min\{C, T\})$.

PROPOSITION 4.1.6 (Monotonicity and Differentiability). *Along a MCF, $\phi : M \times [0, T) \to \mathbb{R}^{n+1}$, if $\tau(t) = C - t$ for some constant $C > 0$, the quantity $\sigma(\phi_t, \tau)$ is monotone nonincreasing in the time interval $[0, \min\{C, T\})$, hence it is differentiable almost everywhere.*
Moreover, letting y_τ a point in $\mathbb{R}^{n+1}$ such that $K_{\mathbb{R}^{n+1}}(x, y_\tau, \tau)$ is one of maximizer for $\sigma(\phi_t, \tau(t))$ of Proposition 4.1.4, we have for almost every $t \in [0, \min\{C, T\})$,

$$\frac{d}{dt}\sigma(\phi_t, \tau) \leq -\int_M \frac{e^{-\frac{|x-y_\tau|^2}{4\tau}}}{(4\pi\tau)^{n/2}} \left|\mathrm{H} - \frac{\langle (x - y_\tau), \nu\rangle}{2\tau}\right|^2 d\mu_t \tag{4.1.4}$$

or, since this inequality has to be intended in distributional sense, for every $0 \leq r < t \leq \min\{C, T\}$,

$$\sigma(\phi_r, \tau(r)) - \sigma(\phi_t, \tau(t)) \geq \int_r^t \int_M \frac{e^{-\frac{|x-y_{\tau(s)}|^2}{4\tau(s)}}}{(4\pi\tau(s))^{n/2}} \left|\mathrm{H} - \frac{\langle (x - y_{\tau(s)}), \nu\rangle}{2\tau(s)}\right|^2 d\mu_s\, ds \;\blacksquare \tag{4.1.5}$$

PROOF. As the function $\sigma(\phi_t, \tau)$ is the maximum of monotone nonincreasing smooth functions, it also must be monotone nonincreasing. Thus, it is differentiable at almost every time $t \in [0, \min\{C, T\})$.

The last assertion is standard, using *Hamilton's trick* (see [**17**]) to exchange the sup and derivative operations. □

REMARK 4.1.7. It is interesting to notice that the quantity σ can be defined also for any n–dimensional countably rectifiable subset S of $\mathbb{R}^{n+1}$, by substituting in the definition the term $\int_M u\,d\mu$ with $\int_S u\,d\mathcal{H}^n$, where $\mathcal{H}^n$ is the n–dimensional Hausdorff measure (possibly *counting multiplicities*). If then S is the support of a compact rectifiable *varifold*, with finite Area, moving by mean curvature according to Brakke's definition (see [**3**]), Huisken's monotonicity formula holds. Hence, the previous proposition holds too.

DEFINITION 4.1.8. Under the same hypothesis we define, for $\tau = C - t$ with $C \leq T$,

$$\Sigma(C) = \lim_{t\to C^-} \sigma(\phi_t, \tau)\,,$$

and $\Sigma = \Sigma(T)$.

By the previous discussion, $\Sigma \geq \sup_{y\in\mathbb{R}^{n+1}} \Theta(y)$, where this latter quantity, that we will call density function, is defined as

$$\Theta(y) = \lim_{t\to T^-} \theta(y,t) = \lim_{t\to T^-} \int_M \frac{e^{-\frac{|x-p|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t(x)\,, \tag{4.1.6}$$

the existence of this limit for every $p \in \mathbb{R}^{n+1}$ is an obvious consequence of Huisken's monotonicity formula.
Moreover, it is easy to prove the existence of $\max_{y\in\mathbb{R}^{n+1}} \Theta(y)$.

DEFINITION 4.1.9. Let $\phi : M \to \mathbb{R}^{n+1}$ be a smooth, compact, immersed hypersurface. Then we define

$$\nu(\phi) = \sup_{\tau>0} \sigma(\phi,\tau)\,.$$

PROPOSITION 4.1.10. *The quantity $\nu(\phi)$ is finite and actually reached by some τ_ϕ* ■

PROOF. Indeed, we have

$$\lim_{\tau\to 0^+} \sigma(\phi,\tau) = \Theta(\phi) > 0\,,$$

where $\Theta(\phi)$ is the maximum (which clearly exists as M is compact) of the n–dimensional density of $\phi(M)$ in $\mathbb{R}^{n+1}$. Then, if ϕ is an embedding, $\Theta(\phi) = 1$, otherwise it will be the highest multiplicity of the points of $\phi(M)$.
We show then that

$$\lim_{\tau\to+\infty} \sigma(\phi,\tau) = 0\,.$$

By the rescaling property of σ, we have $\sigma(\phi,\tau) = \sigma(\phi/\sqrt{4\pi\tau}, 1/4\pi)$, hence we need to show that

$$\limsup_{\tau\to+\infty} \sup_{u\in\mathcal{F}_1} \int_{\frac{M}{\sqrt{4\pi\tau}}} u\,d\mu = 0\,.$$

Since we already know that any function $u \in \mathcal{F}_1$ satisfies $0 \leq u(x) \leq \frac{1}{(4\pi)^{(n+1)/2}}$, we have

$$\limsup_{\tau \to +\infty} \sup_{u \in \mathcal{F}_1} \int_{\frac{M}{\sqrt{4\pi\tau}}} u \, d\mu \leq \limsup_{\tau \to +\infty} \frac{\mathrm{Vol}(M/\sqrt{4\pi\tau})}{(4\pi)^{(n+1)/2}} = \limsup_{\tau \to +\infty} \frac{\mathrm{Vol}(M)}{(4\pi)^{(2n+1)/2}} \, \tau^{-n/2} = 0 \, .$$

□

The following statement can be proved by the same argument of the proof of Proposition (4.1.6).

Proposition 4.1.11 (Monotonicity and Differentiability – II). *Along a MCF, $\phi : M \times [0,T) \to \mathbb{R}^{n+1}$, the quantity above $\nu(\phi_t)$ is monotone non increasing in the time interval $[0,T)$, hence it is differentiable almost everywhere.*
Moreover, letting $p_\phi \in \mathbb{R}^{n+1}$ and τ_ϕ to be some of the maximizers whose existence is granted by Propositions 4.1.4 and 4.1.10, we have for almost every $t \in [0,T)$,

$$\frac{d}{dt}\nu(\phi_t) \leq - \int_M \frac{e^{-\frac{|x-y_{\phi_t}|^2}{4\tau_{\phi_t}}}}{(4\pi\tau_{\phi_t})^{n/2}} \left| \mathrm{H} - \frac{\langle (x - y_{\phi_t}), \nu \rangle}{2\tau_{\phi_t}} \right|^2 d\mu_t(x) \tag{4.1.7}$$

or, since this inequality has to be intended in distributional sense, for every $0 \leq r < t < T$,

$$\nu(\phi_r) - \nu(\phi_t) \geq \int_r^t \int_M \frac{e^{-\frac{|x-y_{\phi_s}|^2}{4\tau_{\phi_s}}}}{(4\pi\tau_{\phi_s})^{n/2}} \left| \mathrm{H} - \frac{\langle (x - y_{\phi_s}), \nu \rangle}{2\tau_{\phi_s}} \right|^2 d\mu_s(x) \, ds \tag{4.1.8}$$ ■

It is important to observe that it is possible to go through all this analysis for a compact, immersed hypersurface in a flat Riemannian manifold T. Moreover, if the original hypersurface $\phi : M \to \mathbb{R}^{n+1}$ is immersed in $\mathbb{R}^{n+1}$, we can choose a Riemannian covering map $I : \mathbb{R}^{n+1} \to \mathrm{T}$ and consider the immersion $\widetilde{\phi} = I \circ \phi : M \to \mathrm{T}$. Then, we define as above, for every $\tau > 0$, the family $\mathcal{F}_{\mathrm{T},\tau}$ of smooth positive functions $u : \mathrm{T} \to \mathbb{R}$ such that $\int_{\mathrm{T}} u \, dx = 1$ and there exists a smooth positive solution of the problem

$$\begin{cases} v_t = -\Delta v \ \text{ in } \mathrm{T} \times [0,\tau) \\ v(y,0) = u(x) \ \text{ for every } y \in \mathrm{T} \, . \end{cases}$$

Then, we define the following quantity

$$\sigma_{\mathrm{T}}(\phi,\tau) = \sup_{u \in \mathcal{F}_{\mathrm{T},\tau}} \sqrt{4\pi\tau} \int_{\widetilde{M}} u \, d\widetilde{\mu}$$

where $\widetilde{M}$ refers to the fact that we are considering the immersion $\widetilde{\phi} : M \to \mathrm{T}$.
Notice that another possibility is simply to embed isometrically a convex set $\Omega \subset \mathbb{R}^{n+1}$ containing $\phi(M)$ in a flat Riemannian manifold T (during the mean curvature flow a hypersurface ϕ initially contained in Ω stays "inside" for all the evolution).
As before, these quantities are well defined, finite, positive and monotonically decreasing as long as ϕ_t moves by mean curvature.

4.2. Applications

4.2.1. A No–Breathers Result.

DEFINITION 4.2.1. A *breather* (following Perelman [**32**]) for the MCF in $\mathbb{R}^{n+1}$ is a smooth n–dimensional hypersurface evolving by MCF $\phi : M \times [0,T) \to \mathbb{R}^{n+1}$, such that there exists a time $\overline{t} > 0$, an isometry L of $\mathbb{R}^{n+1}$ and a positive constant $\lambda \in \mathbb{R}$ for which $\phi(M,\overline{t}) = \lambda L(\phi(M,0))$.

REMARK 4.2.2. Notice that in the case of MCF is useless to consider nonshrinking (steady or expanding) compact breathers because, by the comparison with evolving spheres, they simply do not exist.

THEOREM 4.2.3. *Every compact breather is a homothetic solution to MCF* ■

PROOF. By the rescaling property of σ in Proposition 4.1.5, fixing $C > 0$ we have

$$\sigma(\phi_0, C) \geq \sigma(\phi_{\overline{t}}, C - \overline{t}) = \sigma(\lambda\phi_0, C - \overline{t}) = \sigma(\phi_0, (C - \overline{t})/\lambda^2)$$

hence, if we choose $C = \frac{\overline{t}}{1-\lambda^2}$ we have $C > \overline{t}$, as $\lambda < 1$ and $(C - \overline{t})/\lambda^2 = C$. It follows that

$$\sigma(\phi_0, C) = \sigma(\phi_{\overline{t}}, C - \overline{t})$$

and (for such special C), by Proposition 4.1.6, if $\tau(t) = C - t$

$$\int_0^{\overline{t}} \int_M \frac{e^{-\frac{|x-y_{\tau(t)}|^2}{4\tau(t)}}}{(4\pi\tau(t))^{n/2}} \left|\mathrm{H} - \frac{\langle (x - y_{\tau(t)}), \nu\rangle}{2\tau(t)}\right|^2 d\mu_t\, dt = 0\,.$$

This implies that there exists at least one value of $t \in (0, \overline{t})$ such that $\mathrm{H}(x,t) = \frac{\langle (x-y),\nu\rangle}{2(C-t)}$ for some $p \in \mathbb{R}^{n+1}$, which is the well known equation characterizing a homothetic solution of MCF. □

This is the same argument to show that compact shrinking breathers of Ricci flow are actually Ricci gradient solitons.
Recalling the monotone nondecreasing quantity μ of Perelman in [**32**], along a Ricci flow $g(t)$ of a compact, n–dimensional Riemannian manifold M,

$$\mu(g,\tau) = \inf_{\int_M u=1,\, u>0} \int_M \Big(\tau\Big[\mathrm{R} + \frac{|\nabla u|^2}{u}\Big] - u\log u - \frac{un}{2}\log\left[4\pi\tau\right] - un\Big)\, dV\,.$$

By the rescaling property $\mu(\lambda g, \lambda\tau) = \mu(g,\tau)$, if we have that $g(\overline{t}) = \lambda dL^* g(0)$ for some diffeomorphism $L : M \to M$ and $0 < \lambda < 1$, fixing $C > 0$ we have

$$\mu(g(0), C) \leq \mu(g(\overline{t}), C - \overline{t}) = \mu(\lambda dL^* g(0), C - \overline{t}) = \mu(\lambda g(0), C - \overline{t}) = \mu(g(0), (C - \overline{t})/\lambda)$$

hence, if we choose $C = \frac{\overline{t}}{1-\lambda}$ we have $C > \overline{t}$, as $\lambda < 1$ and $(C - \overline{t})/\lambda = C$. It follows that

$$\mu(g(0), C) = \mu(g(\overline{t}), C - \overline{t})$$

and by the results of Perelman, $g(t)$ is a shrinking soliton.

4.2.2. Singularities. If $\phi : M \times [0,T) \to \mathbb{R}^{n+1}$ is a MCF of a smooth, compact, embedded hypersurface, it is well known that during the flow it remains embedded and that there exists a finite maximal time $T > 0$ for the smooth existence, for which the curvature A is unbounded as $t \nearrow T$.
Moreover for every $t \in [0,T)$

$$\sup_{p\in M} |\mathrm{A}(p,t)| \geq \frac{1}{\sqrt{2(T-t)}}\,.$$

If there exists a constant $C > 0$ such that also

$$\sup_{p\in M} |\mathrm{A}(p,t)| \leq \frac{C}{\sqrt{2(T-t)}}\,.$$

we say that at T we have a *type I* singularity, otherwise we say the singularity is of *type II*.
We want to show that if at time T we have a singularity, the associated quantity $\Sigma = \lim_{t\to T^-} \sigma(\phi_t, \tau)$ is larger than one.
Indeed, for every $p \in \mathbb{R}^{n+1}$ such that there exists a sequence of points $q_i \in M$ and times $t_i \nearrow T$ with $p = \lim_{i\to\infty} \phi(q_i, t_i)$, we consider the function $\Theta(p)$ defined in equation (4.1.6). By a simple semicontinuity argument, we can see that $\Theta(p) \geq 1$ for every $p \in \mathbb{R}^{n+1}$ like above, see [**8**, Corollary 4.20], hence, as $\Sigma \geq \sup_{p\in\mathbb{R}^{n+1}} \Theta(p)$ we get $\Sigma \geq 1$.
If then $\Sigma = 1$, it forces $\Theta(p) = 1$ for all such points p which implies, by the local regularity result of White [**39**], that the flow cannot develop a singularity at time T (see also Ecker [**8**]).
Suppose now to have a type I singularity at time T.
By Proposition 4.1.6 we know that along this flow, for $C = T$, hence, $\tau = T - t$,

$$\sigma(\phi_r, T-r) - \sigma(\phi_t, T-t) \geq \int_r^t \int_M \frac{e^{-\frac{|x-y_{T-s}|^2}{4(T-s)}}}{[4\pi(T-s)]^{n/2}} \left|\mathrm{H} - \frac{\langle x - y_{T-s}, \nu\rangle}{2(T-s)}\right|^2 d\mu_s(x)\, ds$$

for every $0 \leq r \leq t \leq T$, hence,

$$C(\phi_0) \geq \sigma(\phi_0, T) - \Sigma \geq \int_0^T \int_M \frac{e^{-\frac{|x-y_{T-s}|^2}{4(T-s)}}}{[4\pi(T-s)]^{n/2}} \left|\mathrm{H} - \frac{\langle (x - y_{T-s}), \nu\rangle}{2(T-s)}\right|^2 d\mu_s(x)\, ds\,. \tag{4.2.1}$$

Rescaling every hypersurface ϕ_t as in [**21**], around the point y_{T-t} as follows,

$$\widetilde{\phi}_s(q) = \frac{\phi(q, t(s)) - y_{T-t(s)}}{\sqrt{2(T-t(s))}} \qquad s = s(t) = -\frac{1}{2}\log(T-t)$$

and changing variables in formula (4.2.1), we get

$$C \geq \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_{-\frac{1}{2}\log T} \geq \int_{-\frac{1}{2}\log T}^{+\infty} \int_M e^{-\frac{|y|^2}{2}} |\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle|^2\, d\widetilde{\mu}_s(y)\, ds\,. \tag{4.2.2}$$

Reasoning like in [**21**] and [**36**] (or [**37**]), we obtain that if the singularity is of type I, the curvature of the rescaled hypersurfaces $\widetilde{\phi}_s : M \to \mathbb{R}^{n+1}$ is uniformly bounded and any sequence converges (up to a subsequence) to a limit embedded hypersurface $\widetilde{M}_\infty$ satisfying

$\widetilde{\mathrm{H}} = -\langle x \,|\, \widetilde{\nu}\rangle$ which is the defining equation for a homothetic solution of the MCF. Moreover, By the estimates of Stone [**36**, Lemma 2.9], this limit hypersurface satisfies

$$\frac{1}{(2\pi)^{n/2}} \int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\, d\mathcal{H}^n(y) = \lim_{t \to T^-} \sigma(\phi_t, T-t) = \Sigma > 1\,.$$

Clearly, by this equation, this embedded limit hypersurface cannot be empty. Moreover, it cannot be flat also, as it would be an hyperplane for the origin of $\mathbb{R}^{n+1}$ (the only hyperplanes satisfying $\mathrm{H} = -\langle x \,|\, \nu\rangle$ must pass through the origin) as the above integral would be one.

Proposition 4.2.4. *At a singular time T of the MCF of an embedded compact hypersurface the quantity Σ is larger than one.*
If the singularity of the flow is of type I, any sequence of rescaled hypersurfaces (with the maximal curvature) around the maximizer points for the Huisken's functional at times $t_i \nearrow T$ converges, up to a subsequence, to a nonempty and nonflat, smooth embedded limit hypersurface, satisfying $\mathrm{H} = -\langle x \,|\, \nu\rangle$ ■

4.3. Shrinking Curves in the Plane

In this section we apply the previous analysis the case to the motion by curvature of embedded compact curves in the plane and we will give a short proof of the following Grayson's result:

Theorem 4.3.1. *Before the singular time T each initially embedded compact curve becomes convex* ■

Using this theorem and the work by Gage and Hamilton [**12, 13, 14**], we can conclude that after the curve has become convex, it stays convex along the flow and it shrinks into a point becoming asymptotically circular.
Just to fix the notation, let $\gamma : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be the curvature flow of a simple, smooth, closed curve in the Euclidean plane, on a maximal time interval $[0,T)$. This implies that

$$\partial_t \gamma = k\nu\,, \tag{4.3.1}$$

$$k_t = k_{ss} + k^3\,, \tag{4.3.2}$$

$$\partial_t k_s = k_{sss} + 4k^2 k_s\,. \tag{4.3.3}$$

We also have that the following interior estimates hold:

Theorem 4.3.2 (Interior estimates of Ecker and Huisken [**9**], see also [**22**]). *Suppose that in a ball $B_{2R}(x_0)$ the curve γ_t, for $t \in [0,\tau)$ is a graph of a function over $\langle e_1\rangle$ and let $v = \langle \nu \,|\, e_2\rangle^{-1} > 0$ at time $t = 0$.*

- *Letting $\phi(x,t) = R^2 - |x-x_0|^2 - 2t$, if ϕ_+ denotes the positive part of ϕ, we have*

$$v(x,t)\phi_+(x,t) \leq \sup_{x\in\gamma_0} v(x,0)\phi_+(x,0) \tag{4.3.4}$$

  for every $t \in [0,\tau)$ and $x \in \gamma_t$, as long as $v(x,t)$ is defined everywhere on the support of ϕ_+.
- *For arbitrary $\theta \in [0,1)$ we have the estimate*

$$\sup_{\gamma_t \cap B_{\theta R}(x_0)} k^2 \leq C(1-\theta)^{-2}\left(\frac{1}{R^2}+\frac{1}{t}\right) \sup_{B_R(x_0)\times[0,\tau)} v^4 \tag{4.3.5}$$

  for all $t \in [0,\tau)$. The constant C is independent of t and γ_t ∎

For the case of curves evolving in the plane, we also have the following properties:

THEOREM 4.3.3 (Huisken's embeddedness measure [**23**]). *Let L_t the length of $\gamma_t \subset \mathbb{R}^2$ and consider the function $\Phi_t : \gamma_t \times \gamma_t \to \mathbb{R}$ given by*

$$\Phi_t(x,y) = \begin{cases} \frac{\pi|x-y|}{L_t} / \sin\frac{\pi d_t(x,y)}{L_t} & \text{if } x \neq y, \\ 1 & \text{if } x = y\,, \end{cases}$$

where $d_t(x,y)$ is the geodesic *distance* inside *γ_t.*
For every $t \in [0,T)$, we define the following infimum, which is actually a minimum by compactness for closed curves,

$$E(t) = \inf_{x,y\in\gamma_t} \Phi_t(x,y)\,.$$

Then, if the initial closed curve γ_0 is embedded, the function $E(t)$ is uniformly bounded below by a positive constant depending only on γ_0, for every $t \in [0,T)$.
As the function $E(t)$ is positive if and only if γ_t is embedded, a simple closed curve stays embedded during all the flow ∎

LEMMA 4.3.4 (Stone [**36**]). *Let B_R a ball of radius $R > 0$ in $\mathbb{R}^2$, then the following estimates on the family of curves $\widetilde{\gamma}_r$ hold uniformly for $r \in [-\frac{1}{2}\log T, +\infty)$,*

1. *There exist a constant C independent of B_R such that $\mathcal{H}^1(\widetilde{\gamma}_r \cap B_R) \leq Ce^{R^2/2}$ where $\mathcal{H}^1$ is the one–dimensional Hausdorff measure in $\mathbb{R}^2$.*
2. *For any $\varepsilon > 0$ there exists a uniform radius $R = R(\varepsilon, \mathrm{Length}(\gamma_0), T)$ such that $\int_{\widetilde{\gamma}_r \setminus B_R(0)} e^{-\frac{|y|^2}{2}}\,ds \leq \varepsilon$* ∎

LEMMA 4.3.5. *For every $x_0 \in \mathcal{S} \subset \mathbb{R}^2$ where*

$$\mathcal{S} = \{x \in \mathbb{R}^2 \,|\, \exists\ t_i \nearrow T \text{ and } \alpha_i \in \mathbb{S}^1 \text{ such that } \gamma_{t_i}(\alpha_i) \to x\}\,,$$

we have $\Theta(x_0) \geq 1$.
The set $\mathcal{S}$ is non empty and compact, hence, $\mu(t) \geq 1$ for every $t \in [0,T)$ and $\Sigma \geq 1$ ∎

From the results in the previous section, it follows that for every family of disjoint intervals $(a_i, b_i) \subset [-\frac{1}{2}\log T, +\infty)$ such that $\sum_{i\in\mathbb{N}}(b_i - a_i) = +\infty$ we can find a sequence $r_i \in (a_i, b_i)$ such that

$$\lim_{i\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\gamma}_{r_i}} e^{-\frac{|y|^2}{2}} \left|\widetilde{k} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 ds = 0 \tag{4.3.6}$$

and

$$\lim_{i\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\gamma}_{r_i}} e^{-\frac{|y|^2}{2}} ds = \lim_{i\to\infty} \mu(t(r_i)) = \Sigma\,. \tag{4.3.7}$$

Clearly, the sequence r_i converges monotonically increasing to $+\infty$.
From the estimate (4.3.6) on the local length, it follows that the sequence of curves $\widetilde{\gamma}_{r_i}$ has curvatures *locally* equibounded in L^2. Hence, we can extract a subsequence (not relabeled) that, after a possible reparametrization, converges in C^1_{loc} to a limit curve $\widetilde{\gamma}_\infty$. Such curve satisfies $\widetilde{k} + \langle x \,|\, \widetilde{\nu}\rangle = 0$, as the integral $\int_{\widetilde{\gamma}} e^{-\frac{|y|^2}{2}} \left|\widetilde{k} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 ds$ is lower semicontinuous under C^1_{loc}–convergence and it is embedded, indeed, the Huisken's quantity E is scaling invariant and upper semicontinuous under the C^1_{loc}–convergence of curves, hence, it is bounded below also for the limit curve by a positive constant, implying that it has no self–intersections. Moreover, by a bootstrap argument, $\widetilde{\gamma}_\infty$ is smooth, then by the classification result, it is either a line through the origin or the unit circle.
Since the second point of the lemma implies that

$$\lim_{i\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\gamma}_{q_i}} e^{-\frac{|y|^2}{2}} ds = \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\gamma}_\infty} e^{-\frac{|y|^2}{2}} ds\,,$$

and the first limit is equal to Σ, by equation (4.3.7), we conclude that if $\Sigma > 1$ then $\widetilde{\gamma}_\infty$ is the unit circle, if $\Sigma = 1$ then $\widetilde{\gamma}_\infty$ is a line through the origin.

Grayson's Theorem is then a consequence of the analysis of the following two cases.

The Case $\Sigma = 1$.
For every $x_0 \in \mathcal{S}$ we have $\Theta(x_0) = 1$. Then, this case can be excluded by the following general local regularity theorem of White [**39**] (holding in any dimension).

THEOREM 4.3.6. *There exists a constant $\varepsilon > 0$ such that if a point $x_0 \in \mathcal{S}$ satisfies $\Theta(x_0) < 1 + \varepsilon$, then there exists a radius $R > 0$ such that in $B_R(x_0) \times [0, T) \subset \mathbb{R}^2 \times \mathbb{R}$ the curvature is uniformly bounded* ■

Clearly, this theorem gives a contradiction, as (by a compactness argument) it implies that the curvature is uniformly bounded as $t \to T^-$, which is impossible as T is the maximal time of existence of the flow.
In our special case of simple curves, the fact that $\Sigma = 1$ implies the boundedness of the curvature around every $x_0 \in \mathcal{S}$ also follows by the interior estimates of Ecker and Huisken. We give a sketch of the proof.
As $\Theta(x_0) = 1$, by the C^1_{loc}–convergence of the rescaled curves, for every $R > 2$ there is a sequence of times $t_i \nearrow T$ and a line L passing for x_0 such that every curve γ_{t_i} is a

graph over L in the ball $B_{2R\sqrt{2(T-t_i)}}(x_0)$, indeed, the distance of $\gamma_{t_i} \cap B_{2R\sqrt{2(T-t_i)}}(x_0)$ from $\mathrm{L} \cap B_{2R\sqrt{2(T-t_i)}}(x_0)$ in the C^1–norm goes to zero.
Then, supposing that $x_0 = 0$ and that L is $\langle e_1 \rangle$ in $\mathbb{R}^2$, the pieces of curves $\gamma_t \cap B_{2R\sqrt{2(T-t_i)}}$ can be represented as a graph of a function at least for a small time. Moreover, the quantity $v(x,t) = \langle \nu(x,t) \,|\, e_2 \rangle^{-1}$ is small at time $t = t_i$ and $x \in \gamma_{t_i} \cap B_{2R\sqrt{2(T-t_i)}}$. As the sphere $\partial B_{\sqrt{2(T+\varepsilon-t)}}$ is moving by curvature and, choosing $\varepsilon > 0$ small enough, at time $t = t_i$ it is contained in the ball $B_{2R\sqrt{2(T-t_i)}}$, by a geometric comparison argument it is not possible that other parts of the moving curve "get into" the ball $B_{\sqrt{2(T+\varepsilon-t)}}$ at time $t > t_i$. Hence, the only way that $\gamma_t \cap B_{\sqrt{2(T+\varepsilon-t)}}$ can possibly stop to be a graph is that the tangent vector to such graph becomes vertical at some time, equivalently, the function v is not bounded. The interior estimates of Ecker and Huisken (4.3.4) and (4.3.5) exclude this fact if we start with v small enough. Hence, with a suitable choice of one of the times t_i, the curvature of γ_t for $t \in [t_i, T)$ is bounded in the ball $B_{\sqrt{2(T+\varepsilon-t)}}$, in particular it is bounded in $B_{\sqrt{2\varepsilon}}(x_0) \subset B_{\sqrt{2(T+\varepsilon-t)}}$ for every $t \in [t_i, T)$.
By a compactness argument, the curvature is then uniformly bounded as $t \to T^-$, which is impossible as T is the maximal time of existence of the flow.

Remark 4.3.7. The key point in getting a bound on the curvature by means of this argument is due to the C^1_{loc}–convergence of the rescaled curves to a line (by the L^2 bound on the curvature), which cannot be deduced in higher dimensions.

The Case $\Sigma > 1$.
By what we said above we can find $r_i \nearrow +\infty$ such that the curves γ_{r_i} converge in C^1_{loc} to the unit circle. Moreover, being the unit circle compact, the convergence is actually C^1 with equibounded curvatures in L^2 (not only locally).

Fixing $i \in \mathbb{N}$ and letting $\rho = r - r_i$, we look at the evolution of the following quantity,

$$
\begin{aligned}
\frac{d}{dr}\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+\rho\widetilde{k}_s^2)\,ds =\,&2(T-t)\frac{d}{dt}\int_{\gamma_t}\sqrt{2(T-t)}\,k^2\,ds+\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds\\
&+2(T-t)\rho\frac{d}{dt}\int_{\gamma_t}(\sqrt{2(T-t)})^3\,k_s^2\,ds\\
=&-\sqrt{2(T-t)}\int_{\gamma_t}k^2\,ds+(\sqrt{2(T-t)})^3\int_{\gamma_t}(2kk_{ss}+k^4)\,ds+\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds\\
&-3\big(\sqrt{2(T-t)}\big)^3\rho\int_{\gamma_t}k_s^2\,ds+\big(\sqrt{2(T-t)}\big)^5\rho\int_{\gamma_t}(2k_sk_{sss}+7k^2k_s^2)\,ds\\
=&\int_{\widetilde{\gamma}_r}[-\widetilde{k}^2+2\widetilde{k}\widetilde{k}_{ss}+\widetilde{k}^4+\widetilde{k}_s^2-3\rho\widetilde{k}_s^2+2\rho\widetilde{k}_s\widetilde{k}_{sss}+7\rho\widetilde{k}^2\widetilde{k}_s^2]\,ds\\
\leq&\int_{\widetilde{\gamma}_r}[-\widetilde{k}_s^2+\widetilde{k}^4-2\rho\widetilde{k}_{ss}^2+7\rho\widetilde{k}^2\widetilde{k}_s^2]\,ds\\
=&\int_{\widetilde{\gamma}_r}[-\widetilde{k}_s^2+\widetilde{k}^4+\rho(-2\widetilde{k}_{ss}^2+7\widetilde{k}^3\widetilde{k}_{ss}/3)]\,ds\\
\leq&\int_{\widetilde{\gamma}_r}[-\widetilde{k}_s^2+\widetilde{k}^4+\rho(-2\widetilde{k}_{ss}^2+C\widetilde{k}^6+\widetilde{k}_{ss}^2)]\,ds\\
=&\int_{\widetilde{\gamma}_r}[-\widetilde{k}_s^2+\widetilde{k}^4+C\rho\widetilde{k}^6]\,ds\,.
\end{aligned}
$$

Using the following interpolation inequalities for any closed curve in the plane of length L (see Aubin [**1**, p. 93]),

$$
\|\widetilde{k}\|_{L^4}^4\leq C\|\widetilde{k}_s\|_{L^2}\|\widetilde{k}\|_{L^2}^3+\frac{C}{L}\|\widetilde{k}\|_{L^2}^4\qquad\text{and}\qquad\|\widetilde{k}\|_{L^6}^6\leq C\|\widetilde{k}_s\|_{L^2}^2\|\widetilde{k}\|_{L^2}^4+\frac{C}{L^2}\|\widetilde{k}\|_{L^2}^6
$$

which imply, by means of Young inequality,

$$
\begin{aligned}
\int_{\widetilde{\gamma}_r}\widetilde{k}^4\,ds\,&\leq 1/2\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds+C\left(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\right)^3+\left(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\right)^3+\frac{C}{L^3(\widetilde{\gamma}_r)}\\
C\rho\int_{\widetilde{\gamma}_r}\widetilde{k}^6\,ds\,&\leq\left(\rho\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds\right)^3+C\left(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\right)^3+\frac{C}{L^2(\widetilde{\gamma}_r)}\left(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\right)^3,
\end{aligned}
$$

we can conclude, as we know that $L(\widetilde{\gamma}_r)\geq\int_{\widetilde{\gamma}_r}e^{-\frac{|y|^2}{2}}\,ds\geq\sqrt{2\pi}$,

$$
\frac{d}{dr}\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+\rho\widetilde{k}_s^2)\,ds\leq C\left(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\right)^3+\left(\rho\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds\right)^3+C\leq C\left(\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+\rho\widetilde{k}_s^2)\,ds\right)^3+C\,,
$$

for a constant C independent of $r\geq r_i$ and $i\in\mathbb{N}$.
Integrating this differential inequality for the quantity $Q_i(r)=\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+(r-r_i)\widetilde{k}_s^2)\,ds$ in the interval $[r_i,r_i+\delta]$ it is easy to see that if $\delta>0$ is small enough, we have $Q_i(r)\leq$

$C(\delta, Q_i(r_i)) = C\left(\delta, \int_{\widetilde{\gamma}_{r_i}} \widetilde{k}^2\,ds\right) = C(\delta)$, for every $r \in [r_i, r_i + 2\delta]$, as the curves $\widetilde{\gamma}_{r_i}$ have uniformly bounded curvature in L^2. Hence, if $r \in [r_i + \delta, r_i + 2\delta]$ we have the estimate

$$\int_{\widetilde{\gamma}_r} (\widetilde{k}^2 + \delta\widetilde{k}_s^2/2)\,ds \leq \int_{\widetilde{\gamma}_r} (\widetilde{k}^2 + (r - r_i)\widetilde{k}_s^2)\,ds \leq C(\delta)$$

which implies

$$\int_{\widetilde{\gamma}_r} \widetilde{k}^2\,ds \leq C(\delta) \qquad \text{and} \qquad \int_{\widetilde{\gamma}_r} \widetilde{k}_s^2\,ds \leq \frac{2C(\delta)}{\delta}\,.$$

We can now, as before, find a sequence of values $q_i \in [r_i + \delta/2, r_i + \delta]$ such that

$$\lim_{i\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\gamma}_{q_i}} e^{-\frac{|y|^2}{2}} \left|\widetilde{k} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 ds = 0\,.$$

As this new sequence of rescaled curves $\widetilde{\gamma}_{q_i}$ also satisfies the length estimate (4.3.6) and has $\widetilde{k}$ and $\widetilde{k}_s$ uniformly bounded in L^2, we can extract another subsequence (not relabeled) that, after a possible reparametrization, converges in C^2 to a limit curve which is still the unit circle.
Then, the curves $\widetilde{\gamma}_{q_i}$ definitely have positive curvature, hence, they are convex. This means that the same hold for γ_t for some time t, which is Grayson's result.

REMARK 4.3.8. Pushing this analysis a little forward, one can also prove along the same lines the asymptotic convergence of the *full* sequence of rescaled curves to the unit circle in C^∞, as done by Gage and Hamilton in [**13, 14**].

REMARK 4.3.9. We remark that the interesting point of this line in proving Grayson's Theorem (or equivalently, in analysing the possible singularities) is the fact that we did not distinguish between *type I* and *type II* singularities (the type I case is characterized by the estimate $\max_{\gamma_t} k^2 \leq C/\sqrt{2(T-t)}$ for some constant C). Indeed, the curvature of the rescaled curves can be unbounded, but the control in L^2_{loc} is enough to imply the C^1_{loc}–convergence which is sufficient to have the smoothness of the limit curve. This is one of the main reasons why this unitary line of analysis is difficult to be pursued in higher dimensions, where the control of the mean curvature in L^2_{loc} is not strong enough to immediately give the C^1_{loc}–convergence.

CHAPTER 5

Evolution of Codimension one Submanifolds with Boundary

Let M be a smooth n-manifold with boundary ∂M and ϕ_0 a smooth embedding of the pair $(M, \partial M)$ into $\mathbb{R}^{n+1}$. We are interested in studying the following evolution equation for which we suppose to have existence and uniqueness of the solution for small times:

$$\begin{aligned}\partial_t \phi(\cdot, t) &= \mathrm{H}\nu + \Lambda = \Delta\phi + \Lambda \\ \phi(\cdot, 0) &= \phi_0 \,.\end{aligned} \tag{5.0.1}$$

where Λ is a smooth tangent vector field on $\phi(M, t)$.

Remark 5.0.10. It is necessary to consider a manifold with non-empty boundary, otherwise any motion with an arbitrary tangential speed component and normal speed component equal to the mean curvature vector is just a (possibly time dependent) reparametrization of the MCF.

5.1. Evolution of geometric Quantities

First of all we compute the evolution of the induced metric on M:

$$\begin{aligned}\partial_t g_{ij} &= \partial_t \langle \partial_i\phi, \partial_j\phi\rangle = \langle \partial_i(\mathrm{H}\nu + \Lambda), \partial_j\phi\rangle + \langle \partial_i\phi, \partial_j(\mathrm{H}\nu + \Lambda)\rangle \\ &= \mathrm{H}\langle \partial_i\nu, \partial_j\phi\rangle + \mathrm{H}\langle \partial_i\phi, \partial_j\nu\rangle + \langle \partial_i\Lambda, \partial_j\phi\rangle + \langle \partial_j\Lambda, \partial_i\phi\rangle \\ (1.2.4) &= -2\mathrm{H}\mathrm{h}_{ij} + \langle \partial_i\Lambda, \partial_j\phi\rangle + \langle \partial_j\Lambda\partial_i\phi\rangle \\ &= -2\mathrm{H}\mathrm{h}_{ij} + \partial_i\langle \Lambda, \partial_j\phi\rangle + \partial_j\langle \Lambda, \partial_i\phi\rangle - 2\langle \Lambda, \partial^2_{ij}\phi\rangle \\ (1.2.4) &= -2\mathrm{H}\mathrm{h}_{ij} + \partial_i\langle \Lambda, \partial_j\phi\rangle + \partial_j\langle \Lambda, \partial_i\phi\rangle - 2\langle \Lambda, \Gamma^k_{ij}\partial_k\phi\rangle \\ &= -2\mathrm{H}\mathrm{h}_{ij} + \nabla_i\omega_j + \nabla_j\omega_i =: S_{ij}\,,\end{aligned} \tag{5.1.1}$$

where, given any smooth vector field X on M, ω is the one form defined by

$$\omega(X) := \langle \Lambda, X\rangle\,. \tag{5.1.2}$$

If we consider $(\partial_1\phi, ..., \partial_n\phi)$ as a local basis for the tangent space at a generic point on $\phi(M, t)$, we can write Λ and ω in components as follows:

$$\Lambda = \lambda^k\partial_k\phi\,, \qquad \omega_k = \lambda^r g_{rk}\,; \tag{5.1.3}$$

while, recalling that the metric is a parallel tensor field, we have

$$\nabla_p \omega_q = \nabla_p \lambda^l g_{lq}\,. \tag{5.1.4}$$

If we set $\partial_t g^{ij} = T^{ij}$, since $g_{ij}g^{jk} = \delta_i^k$ we have

$$\partial_t g_{ij} g^{jk} = S_{ij} g^{jk} + g_{ij} T^{jk} = 0\,,$$

which immediately implies that

$$T^{ij} = -S^{ij}\,.$$

Applying this result to the evolution equation (5.1.1), we obtain

$$\partial_t g^{ij} = 2\mathrm{H}\mathrm{h}^{ij} - (\nabla_p \omega_q + \nabla_q \omega_p) g^{pi} g^{jq}\,. \tag{5.1.5}$$

By means of a direct computation, we have

$$\begin{aligned}\partial_t \sqrt{g} &= \frac{\sqrt{g}}{2} tr(g^{il}\partial_t g_{lj}) = \frac{\sqrt{g}}{2} g^{ij}(-2\mathrm{H}\mathrm{h}_{ij} + \partial_i \omega_j + \partial_j \omega_i)\\ &= \sqrt{g}(\mathrm{div}\,\Lambda - \mathrm{H}^2)\end{aligned} \tag{5.1.6}$$

Using (5.1.1) and normal coordinates, we can compute the evolution for the Christoffel symbols of the Levi-Civita connection:

(5.1.7)

$$\begin{aligned}\partial_t \Gamma_{ij}^k &= \frac{1}{2} g^{kl}(\nabla_i S_{lj} + \nabla_j S_{li} - \nabla_l S_{ij})\\
(\text{using}(1.2.6)) &= -\mathrm{H} g^{kl}\nabla_i \mathrm{h}_{jl} - g^{kl}(\nabla_i \mathrm{H}\mathrm{h}_{jl} + \nabla_j \mathrm{H}\mathrm{h}_{il} - \nabla_l \mathrm{H}\mathrm{h}_{ij})\\
&\quad + \frac{1}{2} g^{kl}(\nabla_i\nabla_l\omega_j - \nabla_l\nabla_i\omega_j + \nabla_j\nabla_l\omega_i - \nabla_l\nabla_j\omega_i + \nabla_i\nabla_j\omega_l + \nabla_j\nabla_i\omega_l)\\
(\text{using}(1.2.5)) &= -\mathrm{H} g^{kl}\nabla_i \mathrm{h}_{jl} - g^{kl}(\nabla_i \mathrm{H}\mathrm{h}_{jl} + \nabla_j \mathrm{H}\mathrm{h}_{il} - \nabla_l \mathrm{H}\mathrm{h}_{ij})\\
&\quad + \frac{1}{2} g^{kl}[(2\mathrm{h}_{ij}\mathrm{h}_l^p - \mathrm{h}_i^p\mathrm{h}_{jl} - \mathrm{h}_j^p\mathrm{h}_{li})\omega_p + \nabla_i\nabla_j\omega_l + \nabla_j\nabla_i\omega_l]\\
(\text{using}(5.1.3)) &= -\mathrm{H} g^{kl}\nabla_l \mathrm{h}_{ij} - g^{kl}(\nabla_i \mathrm{H}\mathrm{h}_{jl} + \nabla_j \mathrm{H}\mathrm{h}_{il} - \nabla_l \mathrm{H}\mathrm{h}_{ij})\\
&\quad + \frac{1}{2} g^{kl}[2\mathrm{h}_{ij}\mathrm{h}_{rl} - \mathrm{h}_{ri}\mathrm{h}_{jl} - \mathrm{h}_{rj}\mathrm{h}_{li}]\lambda^r + \frac{1}{2}[\nabla_i\nabla_j\lambda^k + \nabla_j\nabla_i\lambda^k]\,.\end{aligned}$$

To compute the evolution for the normal vector, it is sufficient to compute the following quantities:

$$\begin{aligned}\langle \partial_t \nu, \partial_i \phi\rangle &= -\langle \nu, \partial_i\partial_t \phi\rangle = -\langle \nu, \partial_i(\mathrm{H}\nu + \Lambda)\rangle\\
&= -\partial_i \mathrm{H} - \langle \nu, \partial_i \Lambda\rangle\\
&= -\partial_i \mathrm{H} + \langle \partial_i \nu, \Lambda\rangle = -\partial_i \mathrm{H} - \langle \mathrm{h}_{il} g^{lp}\partial_p \phi, \Lambda\rangle = -\partial_i \mathrm{H} - \mathrm{h}_{ik}\lambda^k\,,\end{aligned} \tag{5.1.8}$$

which implies

$$\partial_t \nu = -\nabla \mathrm{H} - V\,, \tag{5.1.9}$$

where

$$V := \lambda^p h_{kp} g^{kl} \partial_l \phi = \lambda^p h_p^l \partial_l \phi\,. \tag{5.1.10}$$

It is important to notice that:

$$\begin{aligned} \langle \nu, \partial^2_{ij} \Lambda \rangle &= \langle \nu, \partial^2_{ij}(\lambda^k \partial_k \phi) \rangle \\ &= \langle \nu, \partial_i \lambda^k \partial^2_{kj} \phi + \partial_j \lambda^k \partial^2_{ik} \phi + \lambda^k \partial^3_{ijk} \phi \rangle \\ &= \partial_i \lambda^k \mathrm{h}_{jk} + \partial_j \lambda^k \mathrm{h}_{ik} + \lambda^k \partial_i \mathrm{h}_{jk} + \lambda^k \Gamma^l_{jk} \mathrm{h}_{il}\,. \end{aligned} \tag{5.1.11}$$

We can now compute the evolution equation for the second fundamental form of the embedding:

LEMMA 5.1.1. *The following evolution equation holds:*

$$\partial_t \mathrm{h}_{ij} = \Delta \mathrm{h}_{ij} - 2\mathrm{H}\mathrm{h}_{jl} g^{lk} \mathrm{h}_{ki} + |\mathrm{A}|^2 \mathrm{h}_{ij} + \lambda^k \nabla_k \mathrm{h}_{ij} + \nabla_i \lambda^k \mathrm{h}_{jk} + \nabla_j \lambda^k \mathrm{h}_{ik} \quad \blacksquare \tag{5.1.12}$$

PROOF. Using (5.1.9)-(5.1.11) we have

$$\begin{aligned} \partial_t \mathrm{h}_{ij} &= \partial_t \langle \nu, \partial^2_{ij} \phi \rangle = \langle -\nabla \mathrm{H} - V, \partial^2_{ij} \phi \rangle + \langle \nu, \partial^2_{ij}(\mathrm{H}\nu + \Lambda) \rangle \\ &= \partial^2_{ij} \mathrm{H} + \mathrm{H} \langle \nu, \partial_i (\mathrm{h}_{ij} g^{kl} \partial_l \phi) \rangle + \langle \nu, \partial^2_{ij} \Lambda \rangle - \langle \nabla \mathrm{H} + V, \partial^2_{ij} \phi \rangle \\ &= \Delta \mathrm{h}_{ij} - 2\mathrm{H}\mathrm{h}_{jl} g^{lk} \mathrm{h}_{ki} + |\mathrm{A}|^2 \mathrm{h}_{ij} + \langle \nu, \partial^2_{ij} \Lambda \rangle - \langle V, \partial^2_{ij} \phi \rangle \\ &= \Delta \mathrm{h}_{ij} - 2\mathrm{H}\mathrm{h}_{jl} g^{lk} \mathrm{h}_{ki} + |\mathrm{A}|^2 \mathrm{h}_{ij} + \partial_i \lambda^k \mathrm{h}_{jk} + \partial_j \lambda^k \mathrm{h}_{ik} \\ &\quad + \lambda^k \partial_i \mathrm{h}_{jk} + \lambda^k \Gamma^l_{jk} \mathrm{h}_{il} - \lambda^k \Gamma^r_{ij} \mathrm{h}_{kr} \\ &= \Delta \mathrm{h}_{ij} - 2\mathrm{H}\mathrm{h}_{jl} g^{lk} \mathrm{h}_{ki} + |\mathrm{A}|^2 \mathrm{h}_{ij} + \partial_i \lambda^k \mathrm{h}_{jk} + \partial_j \lambda^k \mathrm{h}_{ik} \\ &\quad + \lambda^k \nabla_k \mathrm{h}_{ij} + \lambda^k (\Gamma^r_{ik} \mathrm{h}_{rj} + \Gamma^r_{jk} \mathrm{h}_{ri}) \\ &= \Delta \mathrm{h}_{ij} - 2\mathrm{H}\mathrm{h}_{jl} g^{lk} \mathrm{h}_{ki} + |\mathrm{A}|^2 \mathrm{h}_{ij} + \lambda^k \nabla_k \mathrm{h}_{ij} + \nabla_i \lambda^k \mathrm{h}_{jk} + \nabla_j \lambda^k \mathrm{h}_{ik} \end{aligned} \tag{5.1.13}$$

□

Contracting with the metric tensor, we obtain the evolution equation for the mean curvature:

$$\begin{aligned} \partial_t \mathrm{H} &= \partial_t (g^{ij} \mathrm{h}_{ij}) \\ &= [2\mathrm{H}\mathrm{h}^{ij} - (\nabla_p \lambda^s g_{sq} + \nabla_q \lambda^s g_{sp}) g^{pi} g^{jq}] \mathrm{h}_{ij} \\ &\quad + [\Delta \mathrm{h}_{ij} - 2\mathrm{h}_{ip} g^{pq} \mathrm{h}_{qj} + |\mathrm{A}|^2 \mathrm{h}_{ij} + \lambda^k \nabla_k \mathrm{h}_{ij} + \nabla_i \lambda^k \mathrm{h}_{kj} + \nabla_j \lambda^k \mathrm{h}_{ki}] g^{ij} \\ &= \Delta \mathrm{H} + |\mathrm{A}|^2 \mathrm{H} + \lambda^k \nabla_k \mathrm{H}\,. \end{aligned} \tag{5.1.14}$$

Using (5.1.5) and (5.1.12) we compute the evolution of the squared norm of the second fundamental form:

(5.1.15)
$$\begin{aligned}\partial_t|\mathrm{A}|^2 &= \partial_t(g^{ia}g^{jb}\mathrm{h}_{ij}\mathrm{h}_{ab})\\ &= 2[2\mathrm{H}\mathrm{h}^{ia} - (\nabla_p\lambda^a g^{pi} + \nabla_q\lambda^i g^{qa})]\mathrm{h}_{ij}\mathrm{h}_{ab} + 2[\Delta\mathrm{h}_{ij} - 2\mathrm{h}_{ip}g^{pq}\mathrm{h}_{qj} + |\mathrm{A}|^2\mathrm{h}_{ij}]\mathrm{h}_{ab}g^{ia}g^{jb}\\ &\quad + 2[\lambda^k\nabla_k\mathrm{h}_{ij} + \nabla_i\lambda^k\mathrm{h}_{kj} + \nabla_j\lambda^k\mathrm{h}_{ki}]\mathrm{h}_{ab}g^{ia}g^{jb}\\ &= \Delta|\mathrm{A}|^2 - 2|\nabla\mathrm{A}|^2 + 2|\mathrm{A}|^4 + \lambda^k\nabla_k|\mathrm{A}|^2\,.\end{aligned}$$

We now want to find out how the evolution of the quantity $|\nabla\mathrm{A}|^2$ does depend on λ and its spatial derivatives.

LEMMA 5.1.2. *The covariant derivative of the second fundamental form evolves according to*

$$\begin{aligned}\partial_t\nabla_k\mathrm{h}_{ij} = \nabla_k\Delta\mathrm{h}_{ij} &- 2\mathrm{H}\nabla_k[\mathrm{h}_{ip}\mathrm{h}_{qj}]g^{pq} + \nabla_k[|\mathrm{A}|^2\mathrm{h}_{ij}]\\ &- g^{rl}[\nabla_i\mathrm{H}\mathrm{h}_{kl} + \nabla_l\mathrm{H}\mathrm{h}_{ki}]\mathrm{h}_{rj} - g^{rl}[\nabla_j\mathrm{H}\mathrm{h}_{kl} + \nabla_l\mathrm{H}\mathrm{h}_{kj}]\mathrm{h}_{ri}\\ &+ \lambda^s\nabla_s\nabla_k\mathrm{h}_{ij} + \nabla_k\lambda^s\nabla_s\mathrm{h}_{ij} + \nabla_i\lambda^s\nabla_k\mathrm{h}_{sj} + \nabla_j\lambda^s\nabla_k\mathrm{h}_{si} \quad\blacksquare\end{aligned}\tag{5.1.16}$$

NOTATION . From now on, the symbol $\hat{=}$ will mean equality modulo terms which do not contain λ and its spatial derivatives of any order.

PROOF. Is is easy to check that

$$\partial_t\nabla_k\mathrm{h}_{ij} = \nabla_k\partial_t\mathrm{h}_{ij} - \partial_t\Gamma^r_{ki}\mathrm{h}_{rj} - \partial_t\Gamma^r_{kj}\mathrm{h}_{ri}\,.\tag{5.1.17}$$

Let us compute the three terms on the rhs of (5.1.17). From (5.1.12) we have:

$$\begin{aligned}\nabla_k\partial_t\mathrm{h}_{ij} &= \nabla_k[\Delta\mathrm{h}_{ij} - 2\mathrm{H}\mathrm{h}_{ip}g^{pq}\mathrm{h}_{qj} + |\mathrm{A}|^2\mathrm{h}_{ij}]\\ &\quad + \nabla_k[\lambda^s\nabla_s\mathrm{h}_{ij} + \nabla_i\lambda^s\mathrm{h}_{sj} + \nabla_j\lambda^s\mathrm{h}_{si}]\,,\end{aligned}\tag{5.1.18}$$

while (5.1.7) gives us:

(5.1.19)
$$\begin{aligned}-\partial_t\Gamma^r_{ki}\mathrm{h}_{rj} &= [\mathrm{H}g^{rl}\nabla_k\mathrm{h}_{il} + g^{rl}(\nabla_k\mathrm{H}\mathrm{h}_{il} - \nabla_i\mathrm{H}\mathrm{h}_{kl} - \nabla_l\mathrm{H}\mathrm{h}_{ki})]\mathrm{h}_{rj}\\ &\quad - \frac{1}{2}g^{rl}[2\mathrm{h}_{ki}\mathrm{h}_{sl} - \mathrm{h}_{sk}\mathrm{h}_{li} - \mathrm{h}_{si}\mathrm{h}_{lk}]\mathrm{h}_{rj}\lambda^s - \frac{1}{2}[\nabla_i\nabla_k\lambda^r + \nabla_k\nabla_i\lambda^r]\mathrm{h}_{rj}\\ \text{(using(1.2.5))} &= [\mathrm{H}g^{rl}\nabla_k\mathrm{h}_{il} + g^{rl}(\nabla_k\mathrm{H}\mathrm{h}_{il} - \nabla_i\mathrm{H}\mathrm{h}_{kl} - \nabla_l\mathrm{H}\mathrm{h}_{ki})]\mathrm{h}_{rj}\\ &\quad + \frac{1}{2}[R_{iks}{}^r - 2R_{ksi}{}^r]\mathrm{h}_{rj}\lambda^s - \nabla_k\nabla_i\lambda^r\mathrm{h}_{rj} - \frac{1}{2}R_{iks}{}^r\mathrm{h}_{rj}\lambda^s\\ &= [\mathrm{H}g^{rl}\nabla_k\mathrm{h}_{il} + g^{rl}(\nabla_k\mathrm{H}\mathrm{h}_{il} - \nabla_i\mathrm{H}\mathrm{h}_{kl} - \nabla_l\mathrm{H}\mathrm{h}_{ki})]\mathrm{h}_{rj} - \nabla_k\nabla_i\lambda^r\mathrm{h}_{rj} - R_{ksi}{}^r\mathrm{h}_{rj}\lambda^s\end{aligned}$$

and analogously,

$$
\begin{aligned}
-\partial_t\Gamma^r_{kj}\mathrm{h}_{ri} = {} & [\mathrm{H}g^{rl}\nabla_k\mathrm{h}_{jl} + g^{rl}(\nabla_k\mathrm{H}\mathrm{h}_{jl} - \nabla_j\mathrm{H}\mathrm{h}_{kl} - \nabla_l\mathrm{H}\mathrm{h}_{kj})]\mathrm{h}_{ri} \\
& - \nabla_k\nabla_j\lambda^r\mathrm{h}_{ri} - R_{ksj}{}^r\mathrm{h}_{ri}\lambda^s
\end{aligned} \tag{5.1.20}
$$

Adding (5.1.18), (5.1.19) and (5.1.20), (5.1.17) we have the thesis.

□

We are now able to state and prove

LEMMA 5.1.3. *The squared norm of the second fundamental form evolves as*

$$\partial_t|\nabla\mathrm{A}|^2 \hat{=} \lambda^s\nabla_s|\nabla\mathrm{A}|^2 \ \blacksquare \tag{5.1.21}$$

PROOF. Remembering that

$$|\nabla\mathrm{A}|^2 := \nabla_k\mathrm{h}_{ij}\nabla_c\mathrm{h}_{ab}g^{kc}g^{ia}g^{jb}$$

and using (5.1.16) together with (5.1.5), we obtain

$$
\begin{aligned}
\partial_t|\nabla\mathrm{A}|^2 &= 2\,\partial_t\nabla_k\mathrm{h}_{ij}\nabla_c\mathrm{h}_{ab}g^{kc}g^{ia}g^{jb} + \nabla_k\mathrm{h}_{ij}\nabla_c\mathrm{h}_{ab}\,\partial_t(g^{kc}g^{ia}g^{jb}) \\
&\hat{=} 2[\lambda^s\nabla_s\nabla_k\mathrm{h}_{ij} + \nabla_k\lambda^s\nabla_s\mathrm{h}_{ij} + \nabla_i\lambda^s\nabla_k\mathrm{h}_{sj}\nabla_j\lambda^s + \nabla_k\mathrm{h}_{si}]\nabla_c\mathrm{h}_{ab} \\
&\quad - \nabla_k\mathrm{h}_{ij}\nabla_c\mathrm{h}_{ab}\,((\nabla_p\lambda^k g^{pc} + \nabla_p l^c g^{pk})g^{ia}g^{jb}) \\
&\quad - \nabla_k\mathrm{h}_{ij}\nabla_c\mathrm{h}_{ab}\,(g^{kc}(\nabla_p\lambda^i g^{pa} + \nabla_p\lambda^a g^{pi})g^{jb}) \\
&\quad - \nabla_k\mathrm{h}_{ij}\nabla_c\mathrm{h}_{ab}\,(g^{kc}g^{ia}(\nabla_p\lambda^j g^{pb} + \nabla_p\lambda^b g^{pj})) \\
&= 2\lambda^s\nabla_s\nabla_k\mathrm{h}_{ij}\nabla_c\mathrm{h}_{ab}g^{kc}g^{ia}g^{jb} \\
&= \lambda^s\nabla_s|\nabla\mathrm{A}|^2\,.
\end{aligned}
$$

□

NOTATION . Given an integer $\alpha \geq 1$, we set

$$(\nabla^{(\alpha)}A)_{k_\alpha\cdots k_{-1}} := \nabla_{k_\alpha}\cdots\nabla_{k_1}\mathrm{h}_{ij}\,, \tag{5.1.22}$$

where we have set $k_0 = i$ and $k_{-1} = j$.

It is possible to check the following commutation rule for the time derivative and the covariant differentiation:

$$\partial_t\nabla^{(\alpha)}_{k_\alpha\cdots k_1}\mathrm{h}_{k_0k_{-1}} = \nabla_{k_\alpha}\partial_t\nabla^{(\alpha-1)}_{k_{\alpha-1}\cdots k_1}\mathrm{h}_{k_0k_{-1}} - \sum_{\beta=-1}^{\alpha-1}\partial_t\Gamma^r_{k_\alpha k_\beta}(\nabla^{(\alpha-1)}\mathrm{A})_{k_{\alpha-1}\cdots k_{\beta+1}r\cdots k_{-1}}\,. \tag{5.1.23}$$

Looking at (5.1.12) and (5.1.16), we now prove the following result.

LEMMA 5.1.4. *For any $\alpha \geq 0$ we have*

$$\partial_t(\nabla^{(\alpha)}\mathrm{A}) \hat{=} \lambda^s \nabla_s \nabla^{(\alpha)}\mathrm{A} + \sum_{\gamma=-1}^{\alpha} \nabla_{k_\gamma}\lambda^s (\nabla^{(\alpha)}\mathrm{A})_{k_\alpha \cdots k_{\gamma+1} s k_{\gamma-1} \cdots k_{-1}} \quad \blacksquare \tag{5.1.24}$$

NOTATION . Throughout all the computations we will use the following shortcuts:

$$\Gamma^a_{k_\beta c} * (\nabla^{(\alpha)}\mathrm{A})_{\binom{a}{\beta}} := \sum_{\beta=-1}^{\alpha} \Gamma^a_{k_\beta c} \nabla_{k_\alpha} \cdots \nabla_{k_{\beta+1}} \nabla_a \nabla_{k_{\beta-1}} \cdots \nabla_{k_1} \mathrm{h}_{k_0 k_{-1}}$$

$$\nabla_{k_\gamma}\lambda^s * (\nabla^{(\alpha)}\mathrm{A})_{\binom{s}{\gamma}} := \sum_{\gamma=-1}^{\alpha} \nabla_{k_\gamma}\lambda^s \nabla_{k_\alpha} \cdots \nabla_{k_{\beta+1}} \nabla_{k_a} \nabla_{k_{\beta-1}} \cdots \nabla_{k_1} \mathrm{h}_{k_0 k_{-1}}$$

With this notation, equation (5.1.24) becomes

$$\partial_t(\nabla^{(\alpha)}\mathrm{A}) \hat{=} \lambda^s \nabla_s \nabla^{(\alpha)}\mathrm{A} + \nabla_{k_\gamma}\lambda^s * (\nabla^{(\alpha)}\mathrm{A})_{\binom{s}{\gamma}} \, . \tag{5.1.25}$$

PROOF. By (5.1.12) and (5.1.16), our claim is true for $\alpha = 0$ and $\alpha = 1$ respectively. We now suppose that (5.1.24) is true for a generic α and we prove that it holds for $\alpha + 1$ as well.
Using the commutation rule (5.1.23) and the inductive claim (5.1.24), collecting the terms as in (5.1.18), (5.1.19) and (5.1.20) we obtain

$$\begin{aligned}
\partial_t(\nabla^{(\alpha+1)}\mathrm{A}) \hat{=} & \nabla_{k_{\alpha+1}}[\lambda^s \nabla_s \nabla^{(\alpha)}\mathrm{A} + \nabla_{k_\gamma}\lambda^s * (\nabla^{\alpha}\mathrm{A})_{\binom{s}{\gamma}}] - \partial_t \Gamma^r_{k_{\alpha+1}k_\beta} * (\nabla^{(\alpha)})_{\binom{r}{\beta}} \\
\hat{=} & \nabla_{k_{\alpha+1}}\lambda^s \nabla_s \nabla^{(\alpha)}\mathrm{A} + \lambda^s \nabla_s \nabla^{(\alpha+1)}\mathrm{A} + \nabla_{k_\gamma}\lambda^s * \nabla_{k_{\alpha+1}}(\nabla^{(\alpha)}\mathrm{A})_{\binom{s}{\gamma}} \\
& + \lambda^s R_{k_{\alpha+1} s k_\beta}{}^r * (\nabla^{(\alpha)}\mathrm{A})_{\binom{r}{\beta}} + \nabla_{k_{\alpha+1}}\nabla_{k_\gamma}\lambda^s * (\nabla^{(\alpha)}\mathrm{A})_{\binom{s}{\gamma}} \\
& - \frac{1}{2} g^{rl}[2\mathrm{h}_{k_{\alpha+1}k_\beta}\mathrm{h}_{ls} - \mathrm{h}_{k_{\alpha+1}l}\mathrm{h}_{k_\beta s} - \mathrm{h}_{k_{\alpha+1}s}\mathrm{h}_{k_\beta l}]\lambda^s * (\nabla^{(\alpha)}\mathrm{A})_{\binom{r}{\beta}} \\
& - [\nabla_{k_{\alpha+1}}\nabla_{k_\beta}\lambda^r - \frac{1}{2} R_{k_{\alpha+1}k_\beta s}{}^r \lambda^s] * (\nabla^{(\alpha)}\mathrm{A})_{\binom{r}{\beta}} \\
= & \nabla_{k_{\alpha+1}}\lambda^s \nabla_s \nabla^{(\alpha)}\mathrm{A} + \lambda^s \nabla_s \nabla^{(\alpha+1)}\mathrm{A} + \nabla_{k_\gamma}\lambda^s * \nabla_{k_{\alpha+1}}(\nabla^{(\alpha)}\mathrm{A})_{\binom{s}{\gamma}} \, ,
\end{aligned} \tag{5.1.26}$$

which is (5.1.24) for $\alpha + 1$.

□

Using Lemma 5.1.4 we have

THEOREM 5.1.5. *For any integer $\alpha \geq 0$ we have:*

$$\partial_t |\nabla^{(\alpha)}\mathrm{A}|^2 \hat{=} \lambda^s \nabla_s |\nabla^{(\alpha)}\mathrm{A}|^2 \quad \blacksquare \tag{5.1.27}$$

Proof. The proof consists of a computation which makes use of (5.1.24), (5.1.7) and (5.1.25). During the proof we will use the following conventions:

$$\nabla^{(\alpha)}_{\underline{k}}\mathrm{A} := \nabla_{k_\alpha}\cdots\nabla_{k_1}\mathrm{h}_{k_0k_{-1}}\,,$$

$$g^{\langle\underline{k},\underline{c}\rangle} := g^{k_\alpha c_\alpha}\cdots g^{k_1c_1}g^{k_0c_0}g^{k_{-1}c_{-1}}\,.$$

We can now compute

$$\begin{aligned}
\partial_t|\nabla^{(\alpha)}\mathrm{A}|^2 &= 2\partial_t\,\nabla^{(\alpha)}_{\underline{k}}\mathrm{A}\ \nabla^{(\alpha)}_{\underline{c}}\mathrm{A}\ g^{\langle\underline{k},\underline{c}\rangle} + \nabla^{(\alpha)}_{\underline{k}}\mathrm{A}\ \nabla^{(\alpha)}_{\underline{c}}\mathrm{A}\ \partial_t g^{\langle\underline{k},\underline{c}\rangle}\\
&\hat{=}2[\lambda^s\nabla_s\nabla^{(\alpha)}_{\underline{k}}\mathrm{A} + \nabla_{k_\gamma}\lambda^s * (\nabla^{(\alpha)}_{\underline{k}}\mathrm{A})_{\binom{s}{\gamma}}]\nabla^{(\alpha)}_{\underline{c}}\mathrm{A}\ g^{\langle\underline{k},\underline{c}\rangle}\\
&\quad - \nabla^{(\alpha)}_{\underline{k}}\mathrm{A}\ \nabla^{(\alpha)}_{\underline{c}}\mathrm{A}\ \sum_{\beta=-1}^{\alpha}(g^{k_\alpha c_\alpha}\cdots g^{k_{\beta+1}c_{\beta+1}}(\nabla_p\lambda^{k_\beta}g^{pc_\beta} + \nabla_p\lambda^{c_\beta}g^{pk_\beta})\cdots g^{k_{-1}c_{-1}})\\
&= 2\lambda^s\nabla_s\nabla^{(\alpha)}_{\underline{k}}\mathrm{A}\ \nabla^{(\alpha)}_{\underline{c}}\mathrm{A}\ g^{\langle\underline{k},\underline{c}\rangle}\\
&= \lambda^s\nabla_s|\nabla^{(\alpha)}\mathrm{A}|^2\,.
\end{aligned}$$

□

5.2. Partitions of the three dimensional Euclidean Space

In this section we give a possible setting for the evolution of partitions of the Euclidean three dimensional space with immersions of three copies of the bidimensional disk with their boundaries suitably identified. All our definitions are given in the spirit of generalizing the work done in [**28**] to higher dimensional analogues.
Let D^2 be the open unit disk in the Euclidean plane and $\Phi : D^2 \to \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3$, a triple of smooth immersions of D^2 into $\mathbb{R}^3$, which written in components becomes

$$\Phi = (\phi^1,\phi^2,\phi^3),\qquad \phi^\alpha : D^2\to\mathbb{R}^3 \text{ for } \alpha\in\{1,2,3\}\,.$$

We denote the three correspondent induced Riemannian metrics on $\phi^\alpha(D^2)$ with ${}^\alpha g_{ij}$.
Let $\Psi : \partial D^2 \to \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3$ to be three given smooth embeddings of ∂D^2 in $\mathbb{R}^3$ and set $G_{ij} = ({}^1g_{ij},{}^2g_{ij},{}^3g_{ij})$ with inverse $G^{ij} = ({}^1g^{ij},{}^2g^{ij},{}^3g^{ij})$.
If $\Phi = (\phi_1,\phi_2,\phi_3) : D^2\to\mathbb{R}^3$, we will use the following notation for considering the triple of the associated mean curvature vectors:

$$\mathbf{H}(s) = \Delta_g\phi(s) = (\Delta_g\phi_1(s),\Delta_g\phi_2(s),\Delta_g\phi_3(s)) = H(s)\nu(s)\,,$$

where in components we have

$$\Delta_{{}^\alpha g}\phi^\alpha := {}^\alpha g^{ij}\left(\frac{\partial^2\phi^\alpha}{\partial s^i\partial s^j} - \Gamma^h_{ij}\frac{\phi^\alpha}{\partial s^h}\right) \tag{5.2.1}$$

and ν is the outward unit normal vector.

Let now $\overline{\Phi} : D^2 \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ be a given initial datum

$$
\begin{cases}
\Phi_t & = G^{ij}\Phi_{ij} & \text{in } D^2 \times (0,T)\,, \\
\Phi & = \Psi & \text{on } \partial D^2 \times (0,T)\,, \\
\sum_{k=1}^{3} \nu^\alpha & = (0,0,0) & \text{on } \partial D^2 \times (0,T)\,, \\
\Phi(0) & = \overline{\Phi} & \text{in } D^2\,.
\end{cases}
\tag{5.2.2}
$$

The first equation shorthands three systems of equations, which expressed in coordinates read:

$$\phi^\alpha_t = {}^\alpha g^{ij}\phi^\alpha_{ij} =: H\nu^\alpha + \Lambda^\alpha, \qquad \alpha \in \{1,2,3\}$$

where, according to (5.2.1),

$$\Lambda^\alpha = {}^\alpha g^{ij}\, {}^\alpha\Gamma^h_{ij}\frac{\partial {\varphi^\alpha}_h}{\partial s^h} =: \lambda^k \partial_k \phi \in \nu^{\alpha\perp}, \qquad \alpha \in \{1,2,3\}$$

and

$$\lambda^k = g^{ij}\Gamma^k_{ij}\,. \tag{5.2.3}$$

In Section 5.1 we deduced all the evolution equations for the geometric quantities associated to a generic immersion ϕ of a codimension-one submanifold into an Euclidean space. It is important at this point to notice that in principle there would be an obstacle at applying those computations to our case: from (5.2.3) it is actually clear that Λ is not a vector field (since the Christoffel symbols are not the component of any tensor). In the case of immersions of D^2, since we have a single well defined global chart, the non tensoriality of Λ does not affect the results. Moreover, even if Λ is not a tensor, we can formally define its covariant derivative with respect to the Levi-Civita connection by mean of its expression in components and check that all the algebraic properties which are used in the computations hold true.

We can now ready to compute the evolution for the components of Λ:

LEMMA 5.2.1. *The components of Λ evolve according to*

$$
\begin{aligned}
\partial_t\lambda^k = {} & \Delta\lambda^k - 2\Gamma^k_{pq}\nabla_r\lambda^q g^{pr} + [h_{pq}h_{sl} - h_{sp}h_{ql}]g^{pq}g^{kl}\lambda^s \\
& + 2g^{pq}g^{kl}\nabla_p H h_{ql} + 2Hh^{pq}\Gamma^k_{pq}
\end{aligned}
\tag{5.2.4}
$$

PROOF. According to (5.2.3) we have

$$\partial_t\lambda^k = \partial_t g^{ij}\Gamma^k_{ij} + g^{ij}\partial_t\Gamma^k_{ij}.$$

Using (5.1.5) and (5.1.7) we obtain

$$\partial_t g^{ij}\Gamma^k_{ij} = [2Hh^{ij} - (\nabla_p\lambda^j g^{pi} + \nabla_q\lambda^i g^{qj})]\Gamma^k_{ij} \tag{5.2.5}$$

and

$$g^{ij}\partial_t\Gamma^k_{ij} = -g^{ij}g^{kl}(\nabla_i H h_{jl} + \nabla_j H h_{il}) + g^{kl}Hh^p_l\omega_p - h^p_i h^i_l g^{kl}\omega_p + g^{kl}\Delta\omega_l\,. \tag{5.2.6}$$

The thesis follows summing the last two equations.

□

We can now compute the evolution of the squared norm of the tangential speed Λ.

LEMMA 5.2.2. *The following equation holds:*

$$\begin{aligned}\partial_t|\Lambda|^2 = {}&\Delta|\Lambda|^2 - 2|\nabla\Lambda|^2 + 2\langle\lambda^l\nabla_l\Lambda,\Lambda\rangle - 4\Gamma^k_{ij}\nabla_p\lambda^j\lambda^l g^{ip}g_{kl} - 2\lambda^r\lambda^s h_{si}h_{jr}g^{ij}\\&+ 4\nabla_i H h_{jr}g^{ij}g^{kr} + 4Hh^{ij}\Gamma^k_{ij}\lambda^l g_{kl}\,.\end{aligned} \tag{5.2.7}$$

PROOF. We begin by observing that

$$\Delta|\Lambda|^2 = g^{ij}\nabla_i\nabla_j(\lambda^k\lambda^l g_{kl}) = 2\Delta\lambda^k\lambda^l g_{kl} + 2|\nabla\Lambda|^2\,. \tag{5.2.8}$$

Now, using (5.2.4) and (5.2.8), we can compute

(5.2.9)

$$\begin{aligned}\partial_t|\Lambda|^2 &= \partial_t(\lambda^k\lambda^l g_{kl}) = 2\partial_t\lambda^l\lambda^k g_{kl} + \lambda^k\lambda^l\partial_t g_{kl}\\&= 2[\Delta\lambda^k - 2\Gamma^k_{ij}\nabla_p\lambda^j g^{ip} + (h_{ij}h_{sr} - h_{si}h_{jr})g^{ij}g^{kr}\lambda^s + 2g^{ij}g^{kr}\nabla_i H h_{jr} + 2Hh^{ij}\Gamma^k_{ij}]\lambda^l g_{kl}\\&\quad - [2Hh_{kl} - (\nabla_k\lambda^r g_{rl} + \nabla_l\lambda^r g_{rk})]\lambda^k\lambda^l\\&= \Delta|\Lambda|^2 - 2|\nabla\Lambda|^2 + 2\langle\lambda^l\nabla_l\Lambda,\Lambda\rangle - 4\Gamma^k_{ij}\nabla_p\lambda^j\lambda^l g^{ip}g_{kl} - 2\lambda^r\lambda^s h_{si}h_{jr}g^{ij}\\&\quad + 4\nabla_i H h_{jr}g^{ij}g^{kr} + 4Hh^{ij}\Gamma^k_{ij}\lambda^l g_{kl}\,,\end{aligned}$$

which is the thesis.

□

We now can compute the evolution for the gradient of the tangential speed.

LEMMA 5.2.3. *The following evolution equation holds true:*

$$\begin{aligned}\partial_t\nabla_k\lambda^i = {}&\Delta\nabla_k\lambda^i + \lambda^r\nabla_r\nabla_k\lambda^i + \lambda^l\nabla_s(h_{kp}h_{ql})g^{pq}g^{si}\\&- \lambda^l\nabla_l(h_{kp}h_{qs})g^{pq}g^{si} - \lambda^s\nabla_k(h_{sp}h_{ql})g^{pq}g^{il} + H(h^i_s\nabla_k\lambda^s - h^l_k\nabla_l\lambda^i)\\&+ 2[h_{ql}h_{kr} - h_{kl}h_{qr}]\nabla_p\lambda^l g^{pq}g^{ri} + h_{kp}h_{qs}\nabla_l\lambda^i g^{pq}g^{ls} - h_{sp}h_{ql}\nabla_k\lambda^s g^{pq}g^{il}\\&- 2\nabla_k[\Gamma^i_{pq}\nabla_r\lambda^q g^{rp}] + 2\nabla_k[Hh^{pq}\Gamma^i_{pq}]\,.\end{aligned} \tag{5.2.10}$$

PROOF. We begin by noticing that we have the following commutation rule:

(5.2.11)
$$
\begin{aligned}
\nabla_i\Delta\lambda^k &= g^{pq}\nabla_p\nabla_i\nabla_q\lambda^k + g^{pq}R_{ipq}{}^l\nabla_l\lambda^k - g^{pq}R_{ipl}{}^k\nabla_q\lambda^l\\
&= \Delta\nabla_i\lambda^k - g^{pq}\nabla_p[R_{iql}{}^k\lambda^l] - R_i^l\nabla_l\lambda^k - g^{pq}R_{ipl}{}^k\nabla_q\lambda^l\\
&= \Delta\nabla_i\lambda^k + [\nabla_l R_{is} - \nabla_s R_{il}]\lambda^k g^{ks} + g^{pq}R_{qil}{}^k\nabla_p\lambda^l - R_i^l\nabla_l\lambda^k - g^{pq}R_{ipl}{}^k\nabla_q\lambda^l\\
&= \Delta\nabla_i\lambda^k + [\nabla_l H h_{is} - \nabla_s H h_{il} + \nabla_s(h_{ip}h_{ql})g^{pq} - \nabla_l(h_{ip}h_{qs})g^{pq}]\lambda^l g^{sk}\\
&\quad + 2g^{pq}[h_{ql}h_i{}^k - h_{il}h_q{}^k]\nabla_p\lambda^l - Hh_i{}^l\nabla_l\lambda^k + h_{ip}h_{qs}g^{ls}g^{pq}\nabla_l\lambda^k\,.
\end{aligned}
$$

Using (5.2.4) and (5.2.11) we can compute as follows:

(5.2.12)
$$
\begin{aligned}
\nabla_k\partial_t\lambda^i &= \nabla_k\Delta\lambda^i - 2\nabla_k[\Gamma^i_{pq}\nabla_r\lambda^q g^{rp}] + \nabla_k[(h_{pq}h_{sl} - h_{sp}h_{ql})\lambda^s]g^{pq}g^{il} + 2\nabla_k[Hh^{pq}\Gamma^I_{pq}]\\
&= \Delta\nabla_k\lambda^i + [\nabla_l H h_{ks} - \nabla_s H h_{kl} + \nabla_s(h_{kp}h_{ql})g^{pq} - \nabla_l(h_{kp}h_{qs})g^{pq}]\lambda^l g^{si}\\
&\quad + 2g^{pq}g^{is}[h_{ql}h_{ks} - h_{kl}h_{qs}]\nabla_p\lambda^l - Hh^l_k\nabla_l\lambda^i + h_{kp}h_{qs}g^{pq}g^{ls}\nabla_l\lambda^i - 2\nabla_k[\Gamma^i_{pq}\nabla_r\lambda^q g^{pr}]\\
&\quad + \nabla_k[(h_{pq}h_{sl} - h_{sp}h_{ql})\lambda^s]g^{pq}g^{il} + 2\nabla_k[Hh^{pq}\Gamma^i_{pq}]\,.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\partial_t\Gamma^i_{kr}\lambda^r &= -Hg^{il}\nabla_k h_{lr}\lambda^r - g^{il}(\nabla_k H h_{rl} + \nabla_r H h_{kl} - \nabla_l H h_{kr})\lambda^r\\
&\quad + \frac{1}{2}g^{il}[2h_{kr}h_{sl} - h_{sk}h_{lr} - h_{sr}h_{lk}]\lambda^s\lambda^r + \frac{1}{2}[2\nabla_r\nabla_k\lambda^i - R_{krs}{}^i\lambda^s]\lambda^r\,.
\end{aligned}
\tag{5.2.13}
$$

Adding the two last equations and simplifying we obtain the thesis.

□

We would like to conclude making some observations on the methods that we will use to go on with the analysis.

Remark 5.2.4. From equation (5.1.24) we can see that it is sufficient to control uniformly the norm of the second fundamental form and of the vector field Λ to have a control on the derivatives of all orders on A itself. Anyway, from (5.2.7), we can see that if we want to control $|\Lambda|$, we have to control its higher covariant derivatives too. If we now suppose to have existence and uniqueness for small times in (5.2.2), we can let the partition evolve until $|A|$ does not blow up. In case that during this evolution $|\Lambda|$ blows up, since |A| is still uniformly bounded , we can always reparametrize and go on with the evolution.

Remark 5.2.5. If we want to obtain stronger results, since we are studying the evolution of manifolds with boundary, we can not use maximum principles in a straightforward way, as it is done in the case of curvature motion of closed manifolds. We will be forced to use integral estimates for the relevant quantities as it has been done in [**28**].

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