TOUGHENING BY CRACK DEFLECTION IN THE HOMOGENIZATION OF BRITTLE COMPOSITES WITH SOFT INCLUSIONS

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Abstract. We present a simple example of toughening mechanism in the homogenization of composites with soft inclusions, produced by crack deflection at microscopic level. We show that the mechanism is connected to the irreversibility of the crack process. Because of that it cannot be detected through the standard homogenization tool of the Γ-convergence.

1. INTRODUCTION

In this paper we focus on the toughening mechanism in composites, and more precisely on that produced by crack-path deflection at microscopic level. It occurs whenever interactions between the crack and an inclusion cause the crack site to evolve in the matrix out of the path expected in an homogeneous material. In this way, the energy required to open and enlarge a crack in the material increases.

Our aim is to replicate this mechanism within the framework of the weak formulation of Griffith's theory of brittle fracture (see [1]) with ad hoc model. According to the weak formulation, we assumed that the displacement u belongs to the class of *Special functions with Bounded Variation*. Within this functional framework, the crack site is identified with the set S_u of the discontinuities of u, the orientation of the crack is described by the normal ν_u to S_u, and the opening of the crack is identified with the jump $[u]$. At microscopic level, the equilibrium configurations of the system are reached by minimizing the sum $F_\varepsilon(u)$ of the elastic energy stored in the uncracked part of the body, and the surface energy dissipated to open the crack. The small parameter ε takes into account the size of the heterogeneity of the composite. It is important to determine, at macroscopic level, the effective material properties, i.e., to replace the composite with an ideal homogeneous material. Among the various properties, we are interested in the toughness. Since the analysis rests on the study of equilibrium states, or minimizers, of the energy F_ε, it is natural to use the Γ-convergence as homogenization tool in order to describe such an effective property. However, the toughening mechanisms cannot be captured by the Γ-limit F of the family (F_ε), as the size ε of the heterogeneities goes to zero. On the other hand, our analysis enlightens what is the missing information, i.e., that the mechanism is originated by the *irreversibility of the crack process.* When we add this constraint, then the asymptotic behavior of the family (F_ε) reproduces the toughening.

Our model is very simple: it is a composite constituted by a brittle matrix with soft inclusions arranged at microscopic level in a sort of chessboard structure (see Figure 1). The matrix and the soft inclusions have different elastic moduli, but the same toughness, here normalized to one.

At the microscale the surface energy part of F_ε depends only on the length of the crack $\mathcal{H}^1(S_u)$, while the Γ-limit F is of cohesive type, in the meaning that the surface energy depends also on the opening of the crack:

$$\int_{S_u} g([u], \nu_u)\, \mathrm{d}\mathcal{H}^1$$

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for a certain surface energy density g. From the physical point of view, the fracture energy is not completely dissipated at crack initiation but, due to the interaction between the crack's faces, also during the opening of the crack.

Because of that the surface energy associate to different displacements with the same crack site could be different. Assume for instance to have a piecewise constant displacement u having a horizontal crack, i.e., ν_u is constantly equal to the vertical direction e, and with opening $[u] = t$ between the crack's faces. How is $F(u)$ determined? From the operative point of view, we have to build a family of displacements (u_ε) converging to u on one side, and minimizing the energy F_ε on the other one. Then $F(u)$ will be the limit of $(F_\varepsilon(u_\varepsilon))$ as ε goes to zero. Now, when t is small, at the microscopic level the soft inclusions can be stretched paying a few amount of bulk energy also in case of high gradients. In a certain sense, from the energetic point of view, the material behaves as if there are perforations in place of soft inclusion. Because of that, the best way to approximate u is with a zig-zag configuration (see Figures 2-4), i.e., a displacement u_ε having crack site going from a soft inclusion to another one in diagonal (and not in horizontal) and that stretches the soft regions without breaking them (a sort of bridges between the two opposite faces of the macroscopic crack). In particular this shows that the limit model F has a positive activation threshold $g(0^+, e) = 1/\sqrt{2}$ strictly smaller than that of F_ε, which is equal to one (as expected, because the presence of the soft inclusions). On the other hand, when one has a displacement v has before, but with an opening $[v] = s$ large, it is no longer energetically convenient to stretch the soft inclusions instead of breaking them. Because of that the best way to approximate v is with v itself. Indeed $g(s, e) = 1$ for s larger than a certain threshold. So, the Γ-limit does no detect any increment of the toughness! However, the point is that if v is an evolution of u, then in building the approximation (v_ε) for v we have to take into account the irreversibility of the crack process, i.e., at every fixed ε the crack site of v_ε has to contain the crack site of u_ε:

$$S_{v_\varepsilon} \supset S_{u_\varepsilon}.$$

If we add this constraint, then we cannot set $v_\varepsilon = v$ but we can only modify the previous sequence u_ε by keeping the zig-zag configuration in the matrix and extending the crack inside the soft inclusions. In this way we obtain an effective surface energy density g_{eff} such that $g_{\text{eff}}(s, e) = 1/2 + 1/\sqrt{2} > 1 = g(s, e)$ for s large: there is an increment in resistance to large crack-opening and to further growth.

2. Setting of the problem and presentation of the results

Let Ω be an open bounded subset of $\mathbb{R}^2$. The space of special functions of bounded variation on Ω will be denoted by $SBV(\Omega)$. For the general theory we refer to [2]. For every $u \in SBV(\Omega)$, ∇u denotes the *approximate gradient* of u, S_u the *approximate discontinuity set* of u (the crack site), and ν_u the *generalized normal* to S_u, which is defined up to the sign. If u^+ and u^- are the traces of u on the sides of S_u determined by ν_u, the difference $u^+ - u^-$ is called the *jump* of u (the opening of the crack) and is denoted by $[u]$.

Our ambient space is the subspace of $SBV(\Omega)$ given by

$$SBV^2(\Omega) := \{u \in SBV(\Omega) \colon \nabla u \in L^2(\Omega; \mathbb{R}^2) \text{ and } \mathcal{H}^1(S_u) < +\infty\}.$$

We consider also the larger space of *generalized special functions of bounded variation* on Ω, $GSBV(\Omega)$, which is made of all the integrable functions $u : \Omega \to \mathbb{R}$ whose truncations $u^m := (u \wedge m) \vee (-m)$ belong to $SBV(\Omega)$ for every $m \in \mathbb{N}$. By analogy with the case of SBV functions, we say that $u \in GSBV^2(\Omega)$ if $u \in GSBV(\Omega)$, $\nabla u \in L^2(\Omega; \mathbb{R}^2)$ and $\mathcal{H}^1(S_u) < +\infty$.

For $r > 0$ we denote by Q_r the square with side-length r, centered at the origin, i.e., $Q_r := (-r/2, r/2)^2$; while we simply write Q in place of Q_1. Finally, we indicate by u_t the function on Q defined by

$$u_t := t\chi_{(-1/2,1/2)\times(0,1/2)}.$$

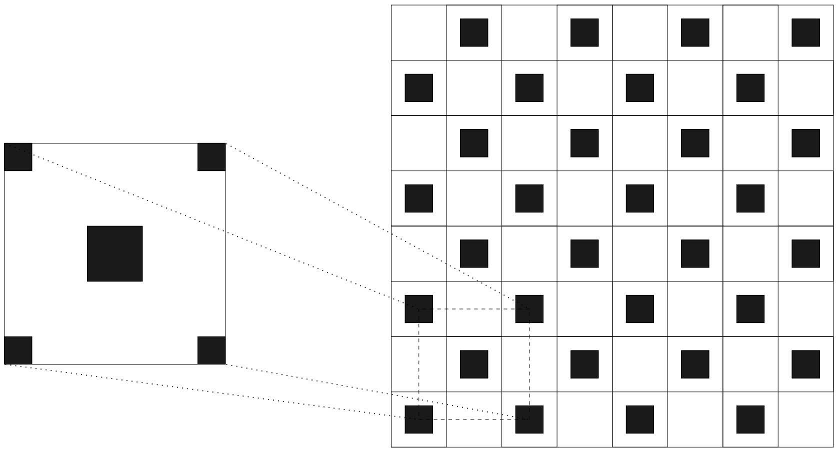


FIGURE 1. In black, the sets D (on the left) and $\mathbb{R}^2 \setminus P$ (on the right).

In what follows the Γ-convergence of functionals is always understood with respect to the strong L^1-topology. For the general theory about Γ-convergence we refer to the short presentation in [5] and the references therein.

Let us introduce our model. We set (see Figure 1)

$$\begin{aligned} D &:= \overline{Q}_{\frac{1}{4}} \cup \Big(\overline{Q}_{\frac{1}{8}} \pm \big(\tfrac{7}{16}, \tfrac{7}{16}\big)\Big) \cup \Big(\overline{Q}_{\frac{1}{8}} \pm \big(\tfrac{7}{16}, -\tfrac{7}{16}\big)\Big), \\ P &:= \mathbb{R}^2 \setminus \bigcup_{i\in\mathbb{Z}^2} (D+i). \end{aligned}$$

We are interested in the asymptotic behavior of the functionals $F_\varepsilon : L^1(\Omega) \to [0,+\infty]$ defined as

$$F_\varepsilon(u,\Omega) := \begin{cases} \displaystyle\int_{\Omega\cap\varepsilon P} |\nabla u|^2 \,\mathrm{d}x + \varepsilon \int_{\Omega\setminus\varepsilon P} |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^1(S_u) & \text{if } u \in SBV^2(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{2.1}$$

In the setting of linearized elasticity and antiplane shear, Ω represents the cross section of a cylindrical body in its reference configuration, while $F_\varepsilon(u,\Omega)$ represents the energy corresponding to a displacement $u\colon \Omega \to \mathbb{R}$. The body is a periodic brittle composite made of two constituents having different elastic properties. The constituent located in $\Omega \setminus \varepsilon P$ has elastic modulus represented by the vanishing sequence ε. For this reason, in what follows, $\Omega \setminus \varepsilon P$ is referred as the *soft inclusions*.

We also consider the functionals $\hat{F}_\varepsilon : L^1(\Omega) \to [0,+\infty]$ given by

$$\hat{F}_\varepsilon(u,\Omega) := \begin{cases} \displaystyle\int_{\Omega\cap\varepsilon P} |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^1(S_u \cap \Omega \cap \varepsilon P) & \text{if } u|_{\Omega\cap\varepsilon P} \in SBV^2(\Omega \cap \varepsilon P), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

From the physical point of view $\Omega \setminus \varepsilon P$ represents a *perforation*. The asymptotic behavior of functionals like $\hat{F}_\varepsilon$ has been extensively studied in [4, 6, 7]. Specifically, it has been shown that $(\hat{F}_\varepsilon)$ Γ-converges to

$$\hat{F}(u,\Omega) := \begin{cases} \displaystyle\int_{\Omega} f(\nabla u) \,\mathrm{d}x + \int_{S_u} \hat{g}(\nu_u) \,\mathrm{d}\mathcal{H}^1 & \text{if } u \in GSBV^2(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

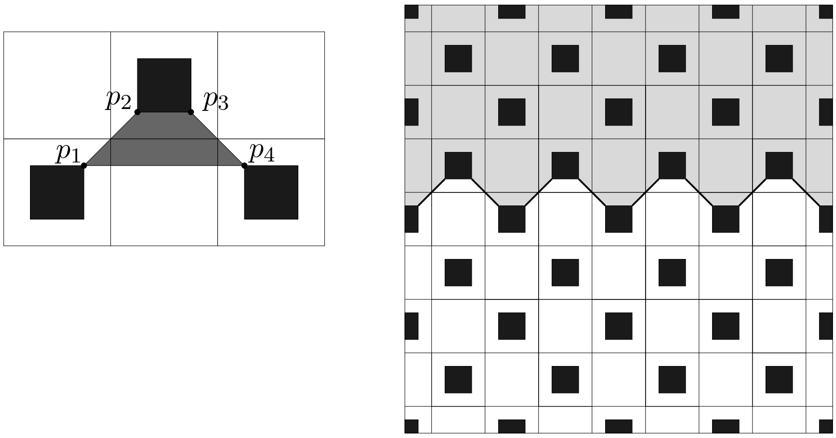


FIGURE 2. The trapezoid T (on the left, in dark grey) and the set $Z_\varepsilon \cap \varepsilon P$ with the "zig-zag" configuration (on the right, in grey).

where $f : \mathbb{R}^2 \to [0, +\infty)$ and $\hat{g} : \mathbb{S}^1 \to [0, +\infty)$ satisfy

$$\begin{aligned} c_1|\xi|^2 \le f(\xi) \le |\xi|^2 \qquad & \text{for every } \xi \in \mathbb{R}^2, \\ c_2 \le \hat{g}(\nu) \le 1 \qquad & \text{for every } \nu \in \mathbb{S}^1, \end{aligned} \tag{2.2}$$

for some constants $c_1, c_2{>}0$ only depending on P. The function f is a quadratic form given by the following homogenization formula (see [4, Theorem 4]):

$$f(\xi) = \inf\left\{ \int_{Q\cap P} |\xi + \nabla w|^2 \, \mathrm{d}x \colon w \in H^1_{\mathrm{per}}(Q) \right\}. \tag{2.3}$$

Also the function $\hat{g}$ is given through an homogenization formula (see again [4, Theorem 4]). In particular, denoted by e the (unitary) vertical vector, one has $\hat{g}(e) = 1/\sqrt{2}$. Indeed,

$$\hat{g}(e) = \lim_{\varepsilon \to 0^+} \inf\bigl\{ \mathcal{H}^1(S_w) \colon w \in SBV_{0,1}(Q \cap \varepsilon P) \bigr\}, \tag{2.4}$$

where $SBV_{0,1}(Q \cap \varepsilon P)$ is the family of functions $w \in SBV(Q \cap \varepsilon P)$ such that $\nabla w = 0$ a.e. in $Q \cap \varepsilon P$, with $w(x) = 1$ [respect. 0] on a neighborhood of $\partial Q \cap \{x_2 \ge 0\}$ [respect. $\partial Q \cap \{x_2 < 0\}$]. The functions $w \in SBV_{0,1}(Q \cap \varepsilon P)$ having the shortest discontinuity set are those such that S_w connects in diagonals two close perforations. Among all the possible configurations, let us consider the simplest one. We define the trapezoid T of vertices $p_1 := (1/8, 1/8), p_2 := (3/8, 3/8), p_3 := (3/8, 5/8), p_4 := (7/8, 1/8)$ (see Figure 2), and the sets

$$\begin{aligned} Z &:= [0,1) \times [\tfrac{1}{8}, +\infty) \setminus T, \\ Z_\varepsilon &:= Q \cap \varepsilon \bigcup_{i \in \mathbb{Z}} \bigl(Z + (i, 0) \bigr). \end{aligned} \tag{2.5}$$

Then, if ε^{-1} is an integer, the function $w = \chi_{Z_\varepsilon \cap \varepsilon P}$ belongs to $SBV_{0,1}(Q \cap \varepsilon P)$. Its discontinuity set S_w is a "zig-zag" configuration (see Figure 2) and $\mathcal{H}^1(S_w) = 1/\sqrt{2}$. If ε^{-1} is not an integer, it is enough to slightly modify w in a neighborhood of the points $(-1/2, 0)$ and $(1/2, 0)$, possibly increasing S_w of a quantity vanishing as ε.

If instead we take $w = \chi_{(-1/2,1/2)\times(0,1/2)\cap \varepsilon P}$, then S_w is constituted by horizontal segments and $\mathcal{H}^1(S_w) = 3/4$. The fact that a horizontal discontinuity set is longer than a diagonal one will be *crucial* in our analysis.

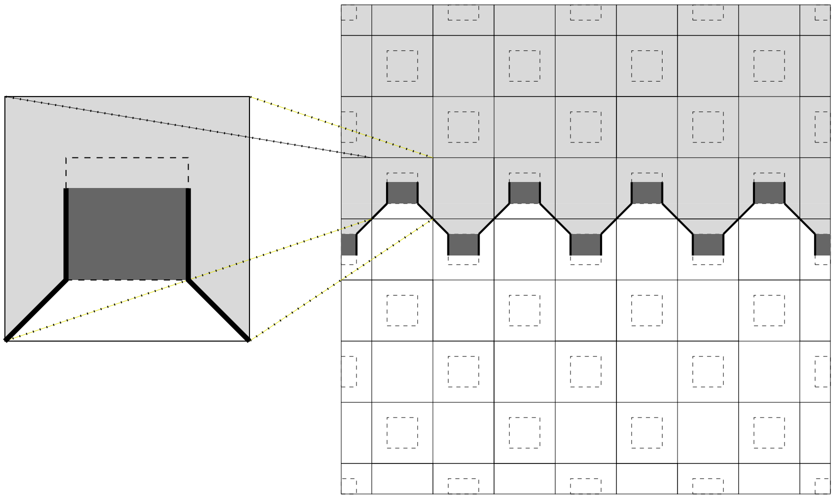


FIGURE 3. In grey the set $Z_k \setminus (R_k \cup \tilde{R}_k)$ where u_k takes value t, in dark grey the set $R_k \cup \tilde{R}_k$ where u_k is affine, and in black the discontinuity set S_{u_k}.

The asymptotic behavior of functionals like F_ε has been instead studied only more recently in [3]. Specifically, it can be shown that for (F_ε) the following result holds. Since the microgeometry considered in [3] is slightly different from the one considered here, we give a short proof highlighting the steps that differ from the original one.

Theorem 1. *For every decreasing sequence of positive numbers converging to zero, there exists a subsequence* (ε_k) *such that* (F_{ε_k}) Γ*-converges to a functional* $F\colon L^1(\Omega) \to [0,+\infty]$ *of the form*

$$F(u,\Omega) := \begin{cases} \displaystyle\int_\Omega f(\nabla u)\,\mathrm{d}x + \int_{S_u} g([u],\nu_u)\,\mathrm{d}\mathcal{H}^1 & \text{if } u \in GSBV^2(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

where f *is as in* (2.2) *and* $g : \mathbb{R} \times \mathbb{S}^1 \to [0,+\infty)$ *is a Borel function satisfying the following properties:*

(i) *for every* $t \neq 0$ *and* $\nu \in \mathbb{S}^1$

$$\hat{g}(\nu) \leq g(t,\nu) \leq 1;$$

(ii) *for any fixed* $\nu \in \mathbb{S}^1$, $g(\cdot,\nu)$ *is nondecreasing and left-continuous in* $(0,+\infty)$ *and satisfies the symmetry condition* $g(-t,-\nu) = g(t,\nu)$*;*

(iii) *for every* $t > 0$, $g(\cdot,e)$ *satisfies the estimate from above*

$$g(t,e) \leq \tfrac{1}{\sqrt{2}} + 2\sqrt{2}\,t. \tag{2.6}$$

In particular, $g(0^+,e) = \hat{g}(e) = 1/\sqrt{2}$*. Moreover, there exists a threshold* $t_0 > 0$ *such that*

$$g(t,e) = 1 \quad \text{for } t \geq t_0. \tag{2.7}$$

Proof. The integral representation of the Γ-limit and points (i) and (ii) follow as a particular case of [3, Theorem 1]. We divide the proof of (iii) into two steps: one for (2.6) and one for (2.7).

Estimate (2.6)**.** Since $g(\cdot,e) \leq 1$, it is enough to show that $g(t,e) \leq 1/\sqrt{2} + 2\sqrt{2}\,t$ whenever $t \leq (\sqrt{2}-1)/4$. To this aim, consider the sets

$$R := (-\tfrac{1}{8},\tfrac{1}{8}) \times (\tfrac{1}{8} - \tfrac{t}{\sqrt{2}},\tfrac{1}{8})$$
$$\tilde{R} := (\tfrac{3}{8},\tfrac{5}{8}) \times (\tfrac{3}{8},\tfrac{3}{8} + \tfrac{t}{\sqrt{2}})$$

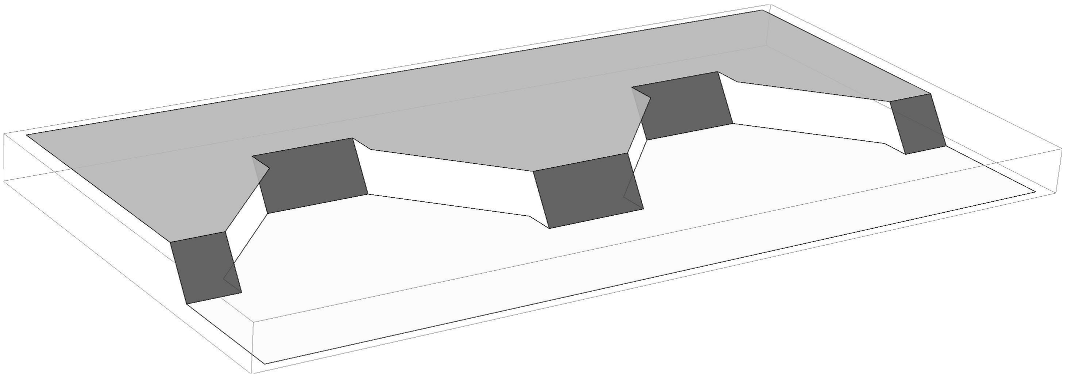


FIGURE 4. The profile of the function u_k.

and

$$R_\varepsilon := Q \cap \varepsilon \bigcup_{i\in\mathbb{Z}} \big(R + (i,0)\big)$$

$$\tilde{R}_\varepsilon := Q \cap \varepsilon \bigcup_{i\in\mathbb{Z}} \big(\tilde{R} + (i,0)\big).$$

Then, with Z_ε as in (2.5), let $(u_k) \subset SBV^2(Q)$ be the sequence of "bridging" functions defined as

$$u_k(x) := \begin{cases} t & \text{if } x \in Z_{\varepsilon_k} \setminus (R_{\varepsilon_k} \cup \tilde{R}_{\varepsilon_k}),\\ t - \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{\varepsilon_k} x_2 & \text{if } x \in R_{\varepsilon_k},\\ -\frac{3\sqrt{2}}{8} + \frac{\sqrt{2}}{\varepsilon_k} x_2 & \text{if } x \in \tilde{R}_{\varepsilon_k},\\ 0 & \text{if } x \in Q \setminus (Z_{\varepsilon_k} \cup R_{\varepsilon_k} \cup \tilde{R}_{\varepsilon_k}), \end{cases}$$

(see Figures 3 and 4). Note that with our choice of t we have $R \subset Q_{1/4}$ and $\tilde{R} \subset Q_{1/4}+(1/2,1/2)$, and therefore $R_\varepsilon \cup \tilde{R}_\varepsilon \subset Q \setminus \varepsilon P$. We clearly have $u_k \to u_t$ in $L^1(Q)$; moreover

$$\int_{R_{\varepsilon_k}} |\nabla u_k|^2 \,\mathrm{d}x \le (\lfloor \tfrac{1}{\varepsilon_k} \rfloor + 1)\sqrt{2}\, t \quad \text{and} \quad \mathcal{H}^1(S_{u_k}) \le \varepsilon_k(\lfloor \tfrac{1}{\varepsilon_k} \rfloor + 1)\big(\tfrac{1}{\sqrt{2}} + \sqrt{2}\, t\big).$$

Thus we readily deduce

$$g(t,e) = F(u_t, Q) \le \limsup_{k\to+\infty} F_{\varepsilon_k}(u_k, Q) \le \tfrac{1}{\sqrt{2}} + 2\sqrt{2}\, t,$$

and hence the estimate from above.

Equality (2.7)**.** Let

$$\tilde{P} := \mathbb{R}^2 \setminus \tfrac{1}{2}\Big(\bigcup_{i\in\mathbb{Z}} \overline{Q}_{\frac{1}{2}} + i\Big)$$

and define $\tilde{F}_\varepsilon$ as F_ε with P replaced by $\tilde{P}$, cf. (2.1). Moreover, let $\tilde{F}$ be the Γ-limit of $(\tilde{F}_{\varepsilon_k})$ and $\tilde{g}$ its surface energy density. Then we can apply (a straightforward modification of) [3, Theorem 2] to $\tilde{F}_\varepsilon$, obtaining that $\tilde{g} = 1$ for t larger than a threshold t_0. On the other hand, since $\tilde{P} \subset P$, we have $\tilde{F}_\varepsilon \le F_\varepsilon$, which implies $\tilde{F} \le F$ and by locality $\tilde{g} \le g$. □

Note that, while the function f is given by the homogenization formula (2.3) (and therefore is independent of the subsequence), at the moment we are not able to provide an explicit representation for the function g (and it could be dependent on the subsequence). In what follows we fix a decreasing sequence (ε_k) of positive numbers such that (F_{ε_k}) Γ-converges to a functional F. Just in order to simplify the construction involved in our results, we assume that (ε_k^{-1}) is is a sequence of odd integers. In this way for any fixed k the unitary cell Q is divided precisely in periodicity cells of side ε_k, one centered in the origin.

We are mainly interested in the local minima of the Γ-limit F. Fixed $t > 0$ and given $\delta > 0$, denote by w_t a solution to the problem

$$\begin{cases} \min F(w,Q)\colon w \in SBV^2(Q), \\ w = 0 \ \text{ in } (-1/2,1/2)\times(-1/2,-\delta/2), \\ w = t \ \text{ in } (-1/2,1/2)\times(\delta/2,1/2). \end{cases} \tag{2.8}$$

Lemma 1. *For any given $t > 0$ and given $\delta > 0$, there exists a solution w_t to the minimum problem* (2.8) *constant in the horizontal direction, i.e.,*

$$w_t(x_1,x_2) = \hat{w}_t(x_2) \tag{2.9}$$

for a certain $\hat{w}_t \in SBV^2((-1/2,1/2))$.

Proof. Since the energy decreases by truncation, in searching for solution to (2.8) we can always assume the additional L^∞-bound $w(x) \in [0,t]$. Then, the compactness in SBV and the direct method of calculus of variations provide the existence of a solution w to (2.8). For any $j \in \mathbb{N}$ and $i \in \{1,\dots,j\}$ let S_j^i be the open strip $(-1/2+(i-1)/j, -1/2+i/j)\times(-1/2,1/2)$. Moreover, let $i_j \in \{1,\dots,j\}$ be a solution to the problem

$$\min\bigl\{F(w,S_j^i)\colon i = 1,\dots,j\bigr\}.$$

We restrict w to $S_j^{i_j}$, and then we extend it to Q by reflection with respect to the axes $x_1 = -1/2 + i/j$, $i = 1,\dots,j-1$; we denote by v_j such an extension. Because the symmetry of the set P, $g(s,(\nu_1,\nu_2)) = g(s,(-\nu_1,\nu_2))$ for any $s \in \mathbb{R}$ and $\nu \in \mathbb{S}^1$. Therefore, we have $F(w,Q) = F(v_j,Q)$ and v_j is still a solution to (2.8). Again by compactness in SBV, up to a subsequence v_j converges to a certain v. By the lower semicontinuity of the functional F, v is still a solution to (2.8). Moreover, since any v_j is $2/j$-periodic in the variable x_1, the function v depends only on x_2 and it is the desired solution. □

In what follows we will work with solutions to (2.8) satisfying condition (2.9), because they are easier to handle. Being f a quadratic form, if $\hat{w}_t \in H^1((-1/2,1/2))$, then $\hat{w}_t$ has to be affine in $(-\delta/2,\delta/2)$. Noted that the function $u_t := t\chi_{(-1/2,1/2)\times(0,1/2)}$ has energy $F(u_t,Q) = g(t,e) \le 1$, we deduce that for δ small enough $\hat{w}_t$ presents a discontinuity, since the energy of an affine function blows-up as δ goes to zero. Since g varies between $1/\sqrt{2}$ and 1, $\hat{w}_t$ cannot have more than one discontinuity point, otherwise

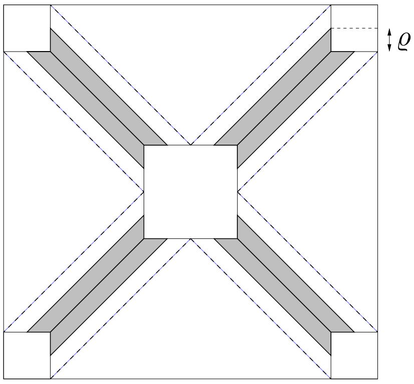
$$F(w_t,Q) \ge \sqrt{2} > F(u_t,Q).$$

Finally, again by minimality, if $\hat{x}_2$ is the discontinuity point of $\hat{w}_t$, we have that $\hat{w}_t$ in affine in $(-\delta/2,\hat{x}_2)$ and $(\hat{x}_2,\delta/2)$. The slopes of the function in this two intervals depend on f, g, t, and δ. Note that, being $g = g(t,e)$ definitively equal to 1 for t large, beyond a certain threshold t_{flat} it is not energetically favorable not to be flat in $(-\delta/2,\hat{x}_2)$ and $(\hat{x}_2,\delta/2)$, since the increment of bulk energy is not compensated by the reduction of the surface energy. Therefore, $\hat{w}_t = t\chi_{(\hat{x}_2,1/2)}$ for t larger than t_{flat}.

Let us now fix a small quantity $\eta > 0$ that we will use later. Since $g(t,e) \le 1/\sqrt{2} + 2\sqrt{2}\,t$, there exists $t_{\text{init}} = t_{\text{init}}(\eta)$ such that

$$F(u_t,Q) = g(t,e) < \tfrac{1}{\sqrt{2}} + \eta \quad \text{for } t \in (0,t_{\text{init}}]. \tag{2.10}$$

For what we said before, we can also choose $\delta = \delta(t)$ so small that any solution w_t to the problem (2.8)-(2.9) has a horizontal discontinuity set $(-1/2,1/2)\times\{\hat{x}_2\}$. Note that the solution is not unique, since the point $\hat{x}_2$ can vary. Now that t and δ are fixed as functions of η, let us also fix a



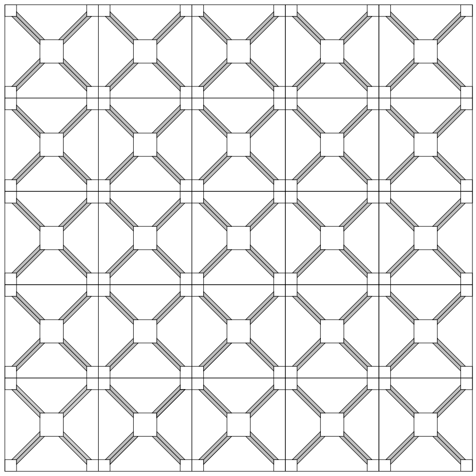


FIGURE 5. In grey the set T (on the left) and the set T_{ε_k} (on the right).

solution $w = w_t$ of the problem (2.8), and consider a recovery sequence (w_k) for w with respect to F_{ε_k}. By definition, $F_{\varepsilon_k}(w_k, Q) \to F(w, Q)$ and $w_k \to w$ strongly in L^1 (and therefore weakly in SBV) as k goes to infinity.

Our main result is to show where the discontinuity set of w_k concentrates for k going to infinity. Let us introduce some other sets in order to better explain the geometrical setting.

We fix another small quantity $\varrho \in (0, 1/8)$. We denote by T^1 the set given by the union of two isosceles trapezoids sharing the same short base constituted by the segment R^1 of endpoints $(1/8, 1/8)$ and $(3/8, 3/8)$. They have long base constituted respectively by the segment of endpoints $(1/8, 1/8 - \varrho)$ and $(3/8 + \varrho, 3/8)$, and the segment of endpoints $(1/8 - \varrho, 1/8)$ and $(3/8, 3/8 + \varrho)$. We set $T^3 := -T^1$, while we denote by T^2 [respect. T^4] the reflection of T^1 with respect to the axis $\{x_1 = 0\}$ [respect. $\{x_2 = 0\}$]. Finally, we set $T := \bigcup_{h=1}^4 T^h$ (see Figure 5) and

$$T_\varepsilon := \varepsilon \bigcup_{i \in \mathbb{Z}^2} (T + i). \tag{2.11}$$

Theorem 2. *Given $\eta > 0$ and $t \in (0, t_{\text{init}}]$, choose $\delta > 0$ small enough so that the solutions to the problem (2.8)-(2.9) are not affine and discontinuous. Let w be one of this solutions, with discontinuity set $(-1/2, 1/2) \times \{\hat{x}_2\}$ for a certain $\hat{x}_2 \in [-\delta/2, \delta/2]$, and (w_k) one of its recovery sequences. Then, given $\varrho \in (0, 1/7)$ and defined T_ε as in (2.11),*

$$\liminf_{k \to +\infty} \mathcal{H}^1(S_{w_k} \cap T_{\varepsilon_k}) \geq \frac{1}{\sqrt{2}} - \frac{\eta}{4\varrho}. \tag{2.12}$$

Proof. The basic idea is that for t small the behaviors of F_{ε_k} and $\hat{F}_{\varepsilon_k}$ are similar. For the Γ-limit $\hat{F}$, the cell formula (2.4) suggests that the functions of a recovery sequence for w should have the discontinuity set concentrate in T_ε. Indeed, this is the best way to cut the hard region εP (since it is thinner in the diagonals between two close soft inclusions). In order to simplify the description of the proof, we assume $\hat{x}_2 = 0$.

First of all, let us define for each $m \in M := (-1/2, -1/8) \cup (1/8, 1/2)$ the fiber l^m passing through m. If $m \in (1/8, 1/2)$, consider the points (see Figure 6)

$$\begin{aligned} &p_1 := (m, 0),\ p_2 := (m, m - 1/8),\ p_3 := (1/24 + 2m/3, 1/24 + 2m/3), \\ &p_4 := (m - 1/8, m),\ p_5 := (m - 1/8, 1/2). \end{aligned}$$

The point p_3 belongs to the segment R^1 of endpoints $(1/8, 1/8)$ and $(3/8, 3/8)$, while p_2 belongs to the segment S^1 of endpoints $(0, 1/8)$ and $(1/2, 3/8)$. The middle point of R^1 is $p_7 := (1/4, 1/4)$,

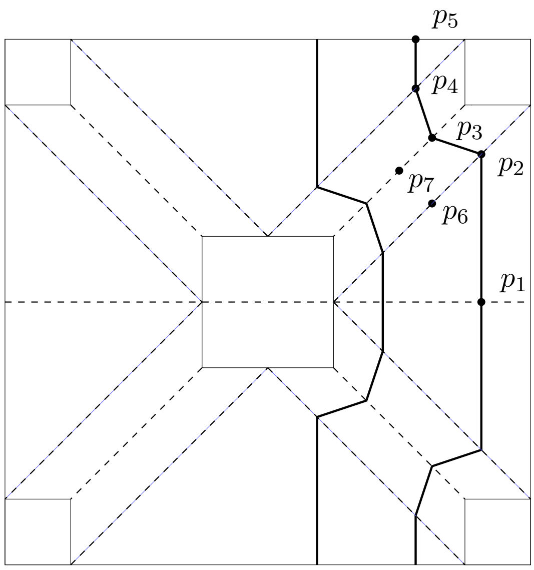


FIGURE 6. A couple of fibers l^m.

while the middle point of S^1 is $p_6 := (5/16, 3/16)$. The ratio between the distance of p_3 from p_7 and the distance of p_2 from p_6 is $2/3$, i.e., the same ration between the lengths of R^1 and S^1.

We define $\tilde{l}^m$ as the union of the segments of endpoints (p_1, p_2), (p_2, p_3),..., (p_4, p_5), and then

$$l^m := \tilde{l}^m \cup \{(x_1, x_2)\colon (x_1, -x_2) \in \tilde{l}^m\}.$$

If $m \in (-1/2, -1/8)$, we define l^m via reflection:

$$l^m := \{(x_1, x_2)\colon (-x_1, x_2) \in l^{-m}\}.$$

Note that $Q \setminus D \overset{a.e.}{=} \bigcup\{l^m\colon m \in M\}$ and that the bundle of fibers undergo a sort of compression of ratio $2/3$ in passing from $S^1 \cup S^2$ to $R^1 \cup R^2$, where R^2 [respect. S^2] is the reflection of R^1 [respect. S^1] with respect to the axis $\{x_1 = 0\}$. We also set $R^3 := -R^1$, while we denote by R^4 the reflection of R^3 with respect to the axis $\{x_2 = 0\}$, and by $R := \bigcup_{h=1}^4 R^h$ the union.

Finally, let us define the periodic and rescaled versions of the sets above:

$$M_\varepsilon := \varepsilon \bigcup_{i \in \mathbb{Z}} (M + i), \quad R_\varepsilon := \varepsilon \bigcup_{i \in \mathbb{Z}^2} (R + i), \tag{2.13}$$

and for $m \in M_\varepsilon$

$$l^m_\varepsilon := \varepsilon \bigcup_{i \in \mathbb{Z}} (l^{\frac{m}{\varepsilon} - [\frac{m}{\varepsilon}]} + i).$$

Note that $\varepsilon P \overset{a.e.}{=} \bigcup\{l^m_\varepsilon\colon m \in M_\varepsilon\}$. In the next two steps we will show that S_{w_k} intersects asymptotically any fiber $l^m_{\varepsilon_k}$, $m \in M_{\varepsilon_k} \cap (-1/2, 1/2)$, and that such an intersection takes place mainly close to $Q \cap R_{\varepsilon_k}$, in the region $Q \cap T_{\varepsilon_k}$.

Step 1. Let us define

$$\tilde{M}_{\varepsilon_k} := \big\{m \in M_{\varepsilon_k} \cap (-1/2, 1/2)\colon l^m_{\varepsilon_k} \cap S_{w_k} = \emptyset\big\},$$

i.e., the set of the points m whose fibers $l^m_{\varepsilon_k}$ do not intersect S_{w_k}. We will show in this step that $\tilde{M}_{\varepsilon_k}$ tends to vanish:

$$\lim_{k\to\infty} \mathcal{H}^1(\tilde{M}_{\varepsilon_k}) = 0. \tag{2.14}$$

The key in this step is that along these fibers w_k is regular, and therefore it cannot converge to w, since it is not regular. We prove (2.14) by contradiction assuming that there exists $\lambda > 0$ and a subsequence not relabeled such that

$$\mathcal{H}^1(\tilde{M}_{\varepsilon_k}) \geq \lambda \text{ for any } k \in \mathbb{N}. \tag{2.15}$$

First of all, we straighten the fibers. Fixed $m \in M_\varepsilon$, we define $\psi_{\varepsilon,m}$ as the unique isometry that transform l_ε^m in the segment $\{m\} \times (-|l_\varepsilon^m|/2, |l_\varepsilon^m|/2)$ keeping the direction. Then, we define $\varphi_{\varepsilon,m} : (-1/2, 1/2) \to l_\varepsilon^m$ by setting $\varphi_{\varepsilon,m}(s) := \psi_{\varepsilon,m}^{-1}((m, |l_\varepsilon^m|s))$. We also assume that w_k coincides with its precise representative defined as in [2, Remark 3.79 and Corollary 3.80]. Then, by [2, Theorems 3.28, 3.107 and 3.108], for a.e. $m \in \tilde{M}_{\varepsilon_k}$ the composition $\omega_{k,m} := w_k \circ \varphi_{\varepsilon_k,m}$ is a Sobolev map and its derivative is given by $\omega'_{k,m} = \partial_1 w_k (\varphi_{\varepsilon_k,m})'_1 + \partial_2 w_k (\varphi_{\varepsilon_k,m})'_2$. In particular, since $|\varphi'_{\varepsilon_k,m}| \leq |l_\varepsilon^m| \leq 2$, we have

$$\int_{\tilde{M}_{\varepsilon_k}} \int_{-1/2}^{1/2} |\omega'_{k,m}|^2 \, \mathrm{d}s \, \mathrm{d}m \leq 4 \int_Q |\nabla w_k|^2 \, \mathrm{d}x.$$

Up to a subsequence, there exists a set $W \subset Q$ of null measure such that $w_k \to w$ pointwise in $Q \setminus W$. Fixed a $\rho > 0$, we select for each $k \in \mathbb{N}$ a $m_k \in \tilde{M}_{\varepsilon_k}$ so that ω_{k,m_k} is a Sobolev map, $\mathcal{H}^1(l_{\varepsilon_k}^{m_k} \cap W) = 0$ and

$$\int_{-1/2}^{1/2} |\omega'_{k,m_k}|^2 \, \mathrm{d}s - \rho \leq \inf_{m \in \tilde{M}_{\varepsilon_k}} \int_{-1/2}^{1/2} |\omega'_{k,m}|^2 \, \mathrm{d}s. \tag{2.16}$$

By (2.15)-(2.16) the sequence (ω'_{k,m_k}) is bounded in $L^2((-1/2, 1/2), \mathbb{R}^2)$. On the other hand, since $w_k \to w$ pointwise in $Q \setminus W$, (ω_{k,m_k}) is also converging pointwise a.e. to a function with discontinuity set $\{0\}$ and this is a contradiction. Therefore (2.14) has to hold true.

Step 2. As we already said, in order to cut the bundle of fibers, the best choice is to make the cut in T_ε, and more precisely along the set R_ε as defined in (2.13). Indeed, here the hard region εP is thin just $1/\sqrt{2}$. On the other hand, outside T_ε the best choice is to make the cut along the diagonal part of the boundary of T_ε itself. Indeed, here the hard region εP is thin $(1 + 4\varrho)/\sqrt{2}$ (that it is smaller than $3/4$, since $\varrho < 1/7$). *The key in this step is the fact that the ratio of the costs between the optimal cuts outside and inside T_ε is $1 + 4\varrho$.*

Let us define

$$M_{\varepsilon_k}^1 := \{m \in M_{\varepsilon_k} \cap (-1/2, 1/2) \colon l_{\varepsilon_k}^m \cap S_{w_k} \cap T_{\varepsilon_k} \neq \emptyset\},$$

i.e., the set of the points m whose fibers $l_{\varepsilon_k}^m$ intersect S_{w_k} in T_{ε_k}, and $M_{\varepsilon_k}^2 := (M_{\varepsilon_k} \cap (-1/2, 1/2)) \setminus (M_{\varepsilon_k}^1 \cup \tilde{M}_{\varepsilon_k})$. Note that by the previous step

$$\lim_{k \to \infty} \mathcal{H}^1(M_{\varepsilon_k}^1) + \mathcal{H}^1(M_{\varepsilon_k}^2) = \frac{3}{4}. \tag{2.17}$$

The bundle of the fibers $\{l_\varepsilon^m \colon m \in M_\varepsilon \cap (-1/2, 1/2)\}$ has cross section $3/4$ in $M_\varepsilon \times \{0\}$, $1/\sqrt{2}$ in $\varepsilon \bigcup_{i \in \mathbb{Z}} ((R^1 \cup R^2) + i)$, and $(1 + 4\varrho)/\sqrt{2}$ in $\varepsilon \bigcup_{i \in \mathbb{Z}} ((\tilde{R}^1 \cup \tilde{R}^2) + i)$, where $\tilde{R}^1$ is the segment of endpoints $(1/8, 1/8 - \varrho)$ and $(3/8 + \varrho, 3/8)$, and $\tilde{R}^2$ is the reflection of $\tilde{R}^1$ with respect to the axis $\{x_1 = 0\}$. Therefore, if we first project $S_{w_k} \cap T_{\varepsilon_k}$ along the fibers on $\varepsilon_k \bigcup_{i \in \mathbb{Z}} ((R^1 \cup R^2) + i)$, and then back to $M_{\varepsilon_k} \times \{0\}$, we get the estimate

$$\mathcal{H}^1(S_{w_k} \cap T_{\varepsilon_k}) \geq \frac{2\sqrt{2}}{3} \mathcal{H}^1(M_{\varepsilon_k}^1), \tag{2.18}$$

while if we first project $S_{w_k} \setminus T_{\varepsilon_k}$ along the fibers on $\varepsilon_k \bigcup_{i \in \mathbb{Z}} ((\tilde{R}^1 \cup \tilde{R}^2) + i)$, and then back to $M_{\varepsilon_k} \times \{0\}$, we get the estimate

$$\mathcal{H}^1(S_{w_k} \setminus T_{\varepsilon_k}) \geq (1 + 4\varrho) \frac{2\sqrt{2}}{3} \mathcal{H}^1(M_{\varepsilon_k}^2). \tag{2.19}$$

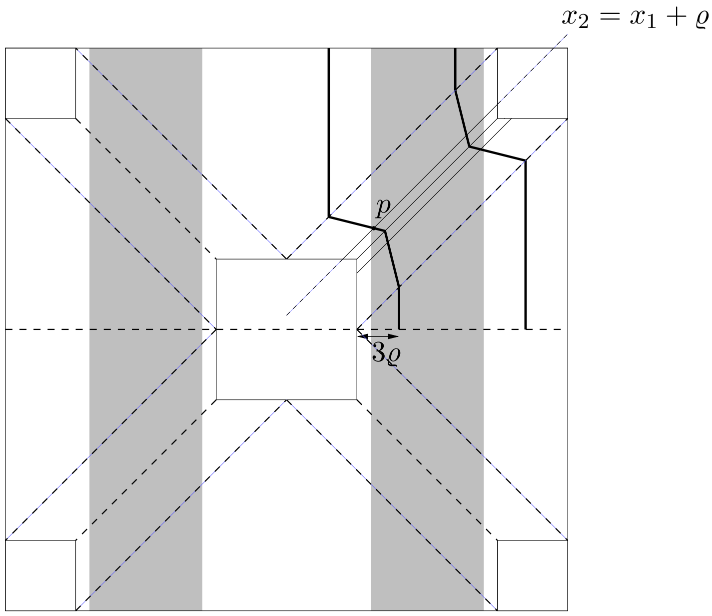


FIGURE 7. In grey the set U.

We now prove that

$$\liminf_{k\to+\infty} \mathcal{H}^1(M^1_{\varepsilon_k}) \geq \frac{3}{4} - \frac{3\eta}{8\sqrt{2}\varrho}. \tag{2.20}$$

This, together with (2.18) will provide (2.12). We proceed by contradiction assuming that (2.20) is false. Thanks to (2.17), this is equivalent to assume

$$\liminf_{k\to+\infty} \mathcal{H}^1(M^2_{\varepsilon_k}) \geq \frac{3\eta}{8\sqrt{2}\varrho}.$$

Then, by using (2.18)-(2.19) and again (2.17),

$$\begin{aligned} F(w,Q) &\geq \liminf_{k\to+\infty} \mathcal{H}^1(S_{w_k}) \\ &= \liminf_{k\to+\infty} \mathcal{H}^1(S_{w_k} \cap T_{\varepsilon_k}) + \mathcal{H}^1(S_{w_k} \setminus T_{\varepsilon_k}) \\ &\geq \frac{2\sqrt{2}}{3} \liminf_{k\to+\infty} \mathcal{H}^1(M^1_{\varepsilon_k}) + (1+4\varrho)\frac{2\sqrt{2}}{3} \liminf_{k\to+\infty} \mathcal{H}^1(M^2_{\varepsilon_k}) \\ &= \frac{1}{\sqrt{2}} + 4\varrho\frac{2\sqrt{2}}{3} \liminf_{k\to+\infty} \mathcal{H}^1(M^2_{\varepsilon_k}) \geq \frac{1}{\sqrt{2}} + \eta. \end{aligned}$$

On the other hand, by (2.10) we have, since w is a solution to (2.8), $F(w,Q) < 1/\sqrt{2} + \eta$, thus a contradiction. □

Remark 1. *Note that while* (2.12) *just says that the discontinuity set is mainly localized in* T_{ε_k}*,* (2.20) *is stronger and it says that the discontinuity set is spread so to cut the fibers.*

In particular, consider the set $U := ((-1/2+\varrho, -1/8-\varrho) \cup (1/8+\varrho, 1/2-\varrho)) \times \mathbb{R}$ *and*

$$U_\varepsilon := \varepsilon \bigcup_{i\in\mathbb{Z}} (U + (i,0)).$$

The fiber l^m *intersects the straight line* $\{x_2 = x_1 + \varrho\}$ *(see Figure 7) at the point*

$$p := (1/24 + 2m/3 + 8m\varrho/3 - 4\varrho/3, 1/24 + 2m/3 + 8m\varrho/3 - \varrho/3).$$

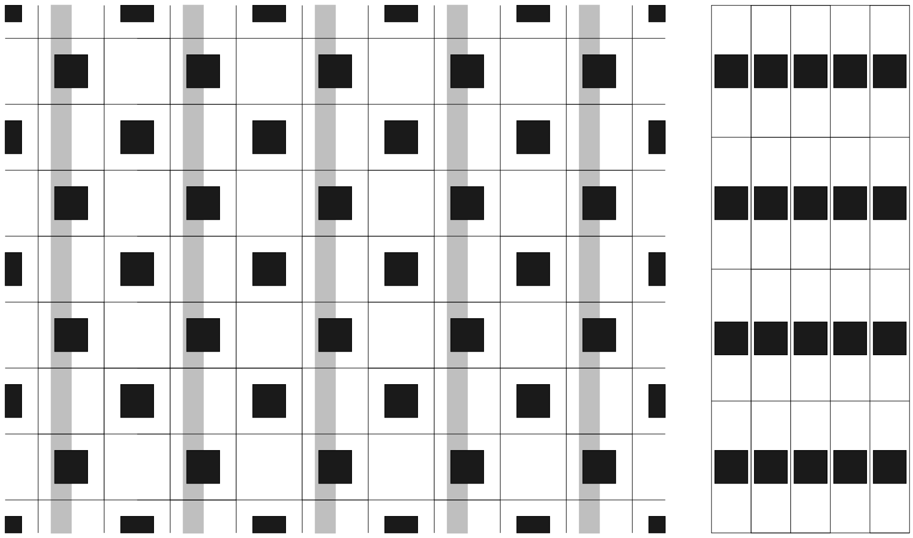


FIGURE 8. In grey the strips S_k^i. On the right, in black, the set $\mathbb{R}^2 \setminus \tilde{P}$.

Therefore, if $1/8 + 3\varrho \leq |m| \leq 1/2 - 3\varrho$, *then* $l^m \cap T \subset U$. *Let us define*

$$\tilde{M}^1_{\varepsilon_k} := \left\{ m \in M^1_{\varepsilon_k} : 1/8 + 3\varrho \leq \tfrac{m}{\varepsilon_k} - [\tfrac{m}{\varepsilon_k}] \leq 1/2 - 3\varrho \right\}.$$

The set $\tilde{M}^1_{\varepsilon_k}$ *is constituted by points* m *whose fiber* $l^m_{\varepsilon_k}$ *intersect* S_{w_k} *in* $T_{\varepsilon_k} \cap U_{\varepsilon_k}$, *i.e.,* $l^m_{\varepsilon_k} \cap S_{w_k} \cap T_{\varepsilon_k} \cap U_{\varepsilon_k} \neq \emptyset$. *Arguing as for* (2.18) *we get*

$$\mathcal{H}^1(S_{w_k} \cap T_{\varepsilon_k} \cap U_{\varepsilon_k}) \geq \frac{2\sqrt{2}}{3} \mathcal{H}^1(\tilde{M}^1_{\varepsilon_k}).$$

By (2.20) *we have also*

$$\liminf_{k \to +\infty} \mathcal{H}^1(\tilde{M}^1_{\varepsilon_k}) \geq \frac{3}{4} - \frac{3\eta}{8\sqrt{2}\varrho} - 6\varrho$$

and then

$$\liminf_{k \to +\infty} \mathcal{H}^1(S_{w_k} \cap T_{\varepsilon_k} \cap U_{\varepsilon_k}) \geq \frac{1}{\sqrt{2}} - \frac{\eta}{4\varrho} - 4\sqrt{2}\varrho. \tag{2.21}$$

As we observed before, when t is larger than a threshold t_{flat} (depending on f, g and δ), the solutions to the problem (2.8)-(2.9) have the form $w = t\chi_{(-1/2,1/2)\times(\hat{x}_2,1/2)}$ for some $\hat{x}_2 \in (1/2, 1/2)$. Let us give for these solutions a complementary estimate to (2.21), in the sets $Q \setminus U_{\varepsilon_k}$ and when t is large. The proof is based on [3, Theorem 2]. The point is that for t large at the microscopic level it is energetically convenient to break also the soft inclusions, instead to stretch them.

Theorem 3. *let* $w := t\chi_{(-1/2,1/2)\times(\hat{x}_2,1/2)}$ *and* (w_k) *be a sequence in* $SBV^2(Q)$ *converging to* v *in* $L^1(Q)$. *Given* $\varrho > 0$, *there exists a threshold* $t_{\text{end}} = t_{\text{end}}(\varrho)$ *such that, if* $t \geq t_{\text{end}}$, *then*

$$\liminf_{k \to +\infty} F_{\varepsilon_k}(w_k, Q \setminus U_{\varepsilon_k}) \geq \frac{1}{2} + 4\varrho. \tag{2.22}$$

Proof. Let S_k^i be the open strip $[(-\varepsilon_k(1/8+\varrho), 0) \times (-1/2, 1/2)] + (\varepsilon_k i, 0)$, and S_k the union of the strips S_k^i included in Q (see Figure 8). Moreover, let $\lambda > 0$ be a small quantity and let (λ_k) be a sequence such that

$$\lambda_k \to +\infty \quad \text{and} \quad \sup_k \left(\lambda_k \int_Q |w_k - w| \, dx \right) \leq \lambda.$$

Let $S_k^{i_k} \in \mathbb{Z}$ be a solution to

$$\min\Big\{F_{\varepsilon_k}(w_k, S_k^i) + \lambda_k \int_{S_k^i} |w_k - w| \,\mathrm{d}x \colon S_k^i \subset Q\Big\}.$$

We first restrict w_k to $S_k^{i_k}$, and then we extend it to the strip $[(-\varepsilon_k(1/8+\varrho), \varepsilon_k(1/8+\varrho)) \times (-1/2, 1/2)] + (\varepsilon_k i, 0)$ by reflection with respect to the axis $x_1 = i_k\varepsilon_k$; we denote by $\tilde{w}_k$ such an extension. Then we extend further $\tilde{w}_k$ by periodicity in the x_1-variable to the whole $\mathbb{R} \times (-1/2, 1/2)$, with period $\varepsilon_k(1/4 + 2\varrho)$. The penalization term ensures that $\tilde{w}_k \to w$ in $L^1(Q)$.

Consider now the sets $\Omega := (-1/8 - \varrho, 1/8 + \varrho) \times (-1/2, 1/2)$,

$$\begin{aligned} \tilde{D} &:= [-\tfrac{1}{8}, \tfrac{1}{8}] \times [-\tfrac{1}{8}, \tfrac{1}{8}], \\ \tilde{P} &:= \mathbb{R}^2 \setminus \bigcup_{(i,j)\in\mathbb{Z}^2} \big(\tilde{D} + \big((\tfrac{1}{4} + 2\varrho)i, j\big)\big) \end{aligned}$$

(see Figure 8), and the functional $\tilde{F}_\varepsilon : L^1(\Omega) \to [0, +\infty]$ defined as

$$\tilde{F}_\varepsilon(u, \Omega) := \begin{cases} \displaystyle\int_{\Omega\cap\varepsilon\tilde{P}} |\nabla u|^2 \,\mathrm{d}x + \varepsilon \int_{\Omega\setminus\varepsilon\tilde{P}} |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^1(S_u) & \text{if } u \in SBV^2(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Note that $\tilde{P}$ is connected. By construction of the sequence $(\tilde{w}_k)$ we have

$$F_{\varepsilon_k}(w_k, S_k) + \lambda_k \int_{S_k} |w_k - w| \,\mathrm{d}x \geq \frac{1}{2}\tilde{F}_{\varepsilon_k}(\tilde{w}_k, \Omega), \tag{2.23}$$

while by [3, Theorem 2], with some slight modifications due to the different cell of periodicity and the different size of the soft inclusions, for t larger than a threshold $t_{\mathrm{end}} = t_{\mathrm{end}}(\varrho)$

$$\liminf_{k\to+\infty} \tilde{F}_{\varepsilon_k}(\tilde{w}_k, \Omega) \geq \frac{1}{4} + 2\varrho. \tag{2.24}$$

By (2.23) and (2.24), and being λ arbitrary, we get

$$\liminf_{k\to+\infty} F_{\varepsilon_k}(w_k, S_k) \geq \frac{1}{8} + \varrho.$$

By repeating a similar estimate on the remaining part of $Q\setminus U_{\varepsilon_k}$, we have the full estimate (2.22). □

3. Conclusions

Let us summarize estimates (2.21) and (2.22) in a comprehensive result.

Main Theorem. *Given $\varrho \in (0, 1/7)$, let $\eta = 4\varrho^2$. Fixed $t \in (0, t_{\mathrm{init}}]$, choose $\delta > 0$ small enough so that the solutions to the problem (2.8)-(2.9) are not affine and discontinuous. Let w be one of this solutions, with discontinuity set $(-1/2, 1/2) \times \{\hat{x}_2\}$ for a certain $\hat{x}_2 \in [-\delta/2, \delta/2]$, and (w_k) one of its recovery sequences. Moreover, let $v := s\chi_{(-1/2,1/2)\times(\hat{x}_2,1/2)}$ and (v_k) be a sequence in $SBV^2(Q)$ converging to v in $L^1(Q)$. There exists a threshold $t_{\mathrm{end}} = t_{\mathrm{end}}(\varrho)$ such that, if $s \geq t_{\mathrm{end}}$ and*

$$S_{v_k} \supset S_{w_k}, \tag{3.1}$$

then

$$\liminf_{k\to+\infty} F_{\varepsilon_k}(v_k, Q) \geq \frac{1}{2} + \frac{1}{\sqrt{2}} + (3 - 4\sqrt{2})\varrho. \tag{3.2}$$

We can assume $t_{\mathrm{end}} \geq t_{\mathrm{flat}}$, so that v is a solutions to the problem (2.8)-(2.9). We take $t = t_{\mathrm{init}}$ and $s = t_{\mathrm{end}}$ and we rename w as $w_{t_{\mathrm{init}}}$ and v as $w_{t_{\mathrm{end}}}$. We see the displacement $w_{t_{\mathrm{end}}}$ as an evolution of $w_{t_{\mathrm{init}}}$ at macroscopic scale when we change the boundary condition from t_{init} to t_{end}. Note that the discontinuity set $(-1/2, 1/2) \times \{\hat{x}_2\}$ remains the same. At the microscopic level, i.e., for k fixed, we should impose that also v_k is an evolution of w_k, so that in particular condition (3.1) holds. Under this constraint the energy in the final configuration $F_{\varepsilon_k}(v_k, Q)$ is asymptotically bounded from below as in (3.2). Since ϱ can be taken arbitrarily small, inequality (3.2) shows that the effective toughness of the material increases from one to $1/2 + 1/\sqrt{2}$. From the physical point of view, the explanation is that the bridging of the soft inclusions, being energetically favorable when the opening of the macroscopic crack is small, originates a deflection of the crack path with respect to the straight one. Because of the irreversibility of the crack process, this deflection persists also when the opening of the crack is large and a straight path should be energetically favorable with respect to the deflected one. This behavior cannot be captured by the Γ-limit F, since it is obtained by a minimization process at microscopic level for any fixed opening of the crack, without taking into account condition (3.1). Indeed we have in the final configuration

$$F(w_{t_{\mathrm{end}}}, Q) = 1.$$

Note also that our result shows that for (F_ε) homogenization and evolution of cracks do not commute: as we said, at the microscopic level the energy of the final configuration is asymptotically close to $1/2 + 1/\sqrt{2}$, while the homogenized energy of the final configuration is one. This is in contrast with what happens in the case of a family of functionals that do not depend on the opening of the crack, but have standard growth conditions: not only the Γ-limit still does not depend on the opening, but homogenization and evolution commute (see [8]).

To conclude, in our specific example an effective model that takes into account the irreversibility of the crack process at microscopic level, i.e., condition (3.1), should provide accordingly to (3.2) an effective surface energy density g_{eff} such that

$$g_{\mathrm{eff}}(s, e) = \frac{1}{2} + \frac{1}{\sqrt{2}} > 1 = g(s, e)$$

for s large enough. Therefore, some generalizations must be envisaged in order to combine Γ-convergence of energies and irreversibility of the crack process at microscopic level. However, this seems to be a challenging problem at the moment.

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