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INDIRIZZO MATEMATICA

CICLO XXVIII

Sharp and Quantitative Isoperimetric Inequalities in Carnot-Carathéodory spaces.

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Abstract

This thesis is dedicated to the study of isoperimetric inequalities in some Carnot-Carathéodory spaces, related to the Heisenberg geometry.

The thesis is organized as follows. Chapter 1 is concerned with some preliminaries: we consider Carnot-Carathéodory spaces, and we define Grushin spaces and H-type groups. Then, we introduce the notion of X-perimeter, showing the validity of a non-sharp isoperimetric inequality. In Chapter 2, we study the sharp isoperimetric inequality in H-type groups and Grushin spaces. Several techniques are needed here, such as representation formulas for the X-perimeter, a concentration-compactness type argument, and non-classical rearrangements. In Chapter 3, we prove quantitative isoperimetric inequalities in the Heisenberg group $\mathbb{H}^n$ and in some Grushin spaces. To this purpose, we use a technique, known in the Calculus of Variations as subcalibration, in a suitable class of sets of finite X-perimeter. Finally, in Chapter 4, we address the problem of studying quantitative isoperimetric inequalities in the Grushin plane in a class of symmetric sets, starting from Euclidean techniques. Crucial differences arise from the lack of invariance under translation of the X-perimeter and lead us to study a variational problem, which has connections with the study of soap bubbles in the Grushin plane.

Sunto

La presente tesi è dedicata allo studio di disuguaglianze isoperimetriche in alcuni spazi di Carnot-Carathéodory, connessi con la geometria dei gruppi di Heisenberg.

La tesi è organizzata come segue. Il Capitolo 1 è introduttivo: consideriamo gli spazi di Carnot-Carathéodory e definiamo gli spazi di Grushin e i gruppi di tipo H. Introduciamo quindi la nozione di X-perimetro, mostrando la validità di una disuguaglianza isoperimetrica non ottimale. Nel Capitolo 2 studiamo la disuguaglianza isoperimetrica ottimale in gruppi di tipo H e spazi di Grushin. Sono necessarie a questo scopo diverse tecniche, tra cui formule di rappresentazione per l'X-perimetro, un argomento di tipo concentrazione-compattezza e riarrangiamenti non standard. Nel Capitolo 3 dimostriamo una disuguaglianza isoperimetrica quantitativa nel gruppo di Heisenberg $\mathbb{H}^n$ e in alcuni spazi di Grushin. Per farlo usiamo una tecnica, nota nel Calcolo delle Variazioni come subcalibrazione, in una opportuna classe di insiemi di X-perimetro finito. Infine, nel Capitolo 4, consideriamo il problema dello studio della disuguaglianza isoperimetrica quantitativa nel piano di Grushin, in una classe di insiemi simmetrici, a partire da tecniche Euclidee. Si presentano alcune differenze sostanziali, dovute alla mancanza di invarianza per traslazioni dell'X-perimetro, e ci conducono allo studio di un problema variazionale, collegato allo studio delle bolle di sapone nel piano di Grushin.

Introduction

Isoperimetric inequalities arise as a natural relation between quantities representing volume and perimeter of regions, and provide both an analytical and a geometrical description of the ambient space. In the Euclidean space $\mathbb{R}^n$, $n \geq 2$, the *isoperimetric inequality* states that if $E \subset \mathbb{R}^n$ is a Lebesgue measurable set with finite measure, then for some dimensional constant $C = C(n) > 0$,

$$P(E)^{\frac{n}{n-1}} \geq C\mathcal{L}^n(E).$$

Here, $P(E)$ is the perimeter of E. For the *sharp constant* $C = C_I(n) = P(B)^{\frac{n}{n-1}}/\mathcal{L}^n(B)$, being B the open unit ball in $\mathbb{R}^n$, equality occurs if and only if E is a Euclidean ball. Such a set is therefore called an *isoperimetric set* in $\mathbb{R}^n$. Despite the ancient origins of the isoperimetric inequality, a complete formulation and proof in the generality of Lebesgue measurable sets in $\mathbb{R}^n$ was given only in the 1958 work [40] by E. De Giorgi. This was done thanks to the powerful notion of perimeter, introduced in the paper [38] by De Giorgi, see also Caccioppoli [26]. As a consequence of the isoperimetric inequality in $\mathbb{R}^n$, the *Gagliardo-Nirenberg inequality*

$$\Big(\int_{\mathbb{R}^n} |Du|\, dx \Big)^{\frac{n}{n-1}} \geq C(n) \int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\, dx \quad u \in W^{1,1}(\mathbb{R}^n)$$

holds true, see Fleming and Rishel [47]. Such inequality is, in fact, equivalent to the isoperimetric inequality. In particular, an embedding theorem for the Sobolev space $W^{1,1}(\mathbb{R}^n)$ into the space of $n/(n-1)$-summable functions $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ is valid (see also Ambrosio, Fusco and Pallara [7, Section 3.4] for the embedding of $BV(\mathbb{R}^n)$ into $L^{\frac{n}{n-1}}(\mathbb{R}^n)$).

A progress on the isoperimetric inequality in $\mathbb{R}^n$ is the study of its stability. After some contributions on the subject (see for instance Fuglede [62], Hall [73], Hall, Hayman and Weitsman [74]), in [63], Fusco, Maggi and Pratelli prove existence of a dimensional constant $C_Q(n) > 0$ such that any Lebesgue measurable set $E \subset \mathbb{R}^n$ satisfies

$$P(E) - P(B(0, r_E)) \geq C_Q(n) \Big(\min_{x \in \mathbb{R}^n} \mathcal{L}^n(E \triangle B(x, r_E)) \Big)^2.$$

The quantity $r_E \geq 0$ is chosen in order to have $\mathcal{L}^n(E) = \mathcal{L}^n(B(0, r_E))$. We refer to such inequality as the sharp *quantitative isoperimetric inequality* in $\mathbb{R}^n$.

In the last two decades an intense investigation on *Analysis and Geometry in Metric Spaces* has been carried out by many authors and led to a generalization of classical theories to these structures: Sobolev spaces (see Hajłasz [71], Hajłasz and Koskela [72]), quasiconformal mappings (see Heinnonen and Koskela [76]), functions of bounded variations and sets of finite perimeter (see Miranda [93], Ambrosio [4], [5], Ambrosio Miranda and Pallara [10], Korte Lahti and Shanmugalingam [82]), currents and rectifiable sets (see Ambrosio and Kircheim [8], [9]), see also Heinonen [75], Ambrosio and Tilli [11]. A very general framework to study isoperimetric inequalities is therefore established. An important class of Metric Spaces is given by *Carnot-Carathéodory spaces*, whose definition is attributed to Gromov: in [67] (see [68] for the english version, see also Pansu [109], Gromov [69]) the author studies various distances defined on manifolds, naming "Carnot-Carathéodory" a length distance used by C. Carathéodory in an ancestral form to axiomatically formalize Thermodynamics (see [30]), and already present in the literature of hypoelliptic differential operators and nonholonomic mechanics. Given a family of vector fields $X = \{X_1, \dots, X_r\}$ defined in a open set $\Omega \subset \mathbb{R}^n$, the Carnot-Carathéodory distance d_{cc} between two points in Ω is defined as the shortest length of *horizontal* curves connecting them, i.e., absolutely continuous curves that are almost everywhere tangent to the distribution of planes generated by $X_1, \dots, X_r$. If no horizontal curves connect the two points, d_{cc} is defined to be ∞. Before a formal definition of Carnot-Carathéodory spaces was given, a sufficient condition to connect any two points by means of horizontal curves was proved independently by Chow, [33] and Rashewski [113], involving the rank of the Lie algebra generated by $X_1, \dots, X_r$. The same condition has a key role for the hypoellipticity of the *subelliptic Laplacian*

$$\Delta_X = \sum_{i=1}^{r} X_i^2,$$

proved by Hörmander, in the 1967 paper [77]. Such a condition is known in the literature as *Hörmander condition*, (also *bracket generating condition*). This result has motivated many authors to study *hypoelliptic operators* defined as sum of squares of vector fields satisfying Hörmander condition, see Bony [21], Kohn [81], Rotshild and Stein [119], Folland [48], [49], Nagel, Stein and Weinger [106], Jerison [79], Varopoulos [123]. In the work by Fefferman and Phong [45], dated 1981, the study of *subelliptic operators* which are not assumed to be written as sum of squares is accomplished associating them with a suitable metric d. This idea gives an impulse to the study of *degenerate elliptic operators*, via associated Carnot-Carathéodory metrics, see Franchi and Lanconelli [54], [55], and Sobolev and Poincaré inequalities are studied in view of a regularity theory for weak solutions and estimates of the fundamental solution, see Franchi [51], Franchi, Gutierrez and Wheeden [53], Franchi, Gallot and Wheeden [52], Capogna, Danielli and Garofalo [28], [29], Lanconelli and Morbidelli [86]. Isoperimetric inequalities with non-sharp constants follow as a result of this research branch, see Varopoulos, Saloff-Coste and Couhlon [124], Garofalo and Nhieu [65], where perimeter in Carnot-Carathéodory spaces is defined following De Giorgi definition, see Capogna, Danielli and Garofalo [28], Franchi, Serapioni and Serra Cassano[56] and corresponds to the definition of Miranda in more general metric spaces [93].

From another point of view, the study of isoperimetric inequalities in Carnot-Carathéodory spaces is induced by geometrical motivations. When a Riemannian metric is defined on a manifold, the volume of a region and of its boundary are well defined according to the metric. In [3] (see also Gromov [68]) Ahlfors proves that if M is a complete n-dimensional Riemannian manifold satisfying for any $D \subset M$

$$\mathrm{vol}(D) \le C\mathrm{vol}(\partial D)^{\alpha},$$

with $\alpha < \frac{n}{n-1}$, then there are no *quasiregular mappings* from $\mathbb{R}^n$ into M. Motivated by this result, in [108], Pansu proves an isoperimetric inequality in the Heisenberg group $\mathbb{H}^1$. Such a group is an example of Carnot-Carathéodory space, being a Lie group on $\mathbb{R}^3$ endowed with a left-invariant metric. Pansu's isoperimetric inequality states that for any Lebesgue measurable set $E \subset \mathbb{H}^1$

$$\mathcal{L}^3(E) \le \left(\frac{12}{\pi}\right)^{\frac{1}{3}} P_H(E)^{\frac{4}{3}},$$

where P_H denotes the perimeter in the Heisenberg group, (see Franchi Serapioni and Serra Cassano [59] for a systematical study). The constant $(12/\pi)^{1/3}$ is not sharp, and, still in [108], Pansu conjectures the form of isoperimetric sets in $\mathbb{H}^1$ as topological balls that are not metric balls. The *sharp isoperimetric inequality* in $\mathbb{H}^1$, which is equivalent to the characterization of minimizers for the Heisenberg perimeter under volume constraint is still an open problem.

The present thesis can be included in this framework of Calculus of Variation in Metric Spaces. Before our work, the only Carnot-Carathéodory space where a sharp isoperimetric inequality was known to hold in the generality of Lebesgue measurable sets, was the Grushin plane, see [99], and no quantitative inequalities were available. The intention of this dissertation is to present sharp and quantitative isoperimetric inequalities in Carnot-Carathéodory spaces connected to the Heisenberg geometry. The original work of the thesis is presented in Chapters 2, 3, 4. The results of Chapters 2 are published in the work [60], and part of the results in Chapter 3 are at the base of the work [61]. The results in Chapter 4, instead, are not yet published.

The main features of Carnot-Carathéodory spaces are described in Chapter 1, together with the generalization of classical tools from Calculus of Variation to such structures, as the notion of functions of bounded variation. In Section 1.1 we describe the notion of length of horizontal curves, given a family of vector fields $X = \{X_1, \dots, X_r\}$, defining Carnot-

Caratheodory and sub-Riemannian spaces, and introducing Carnot Groups as examples. We then focus on H-type groups and Grushin spaces that are subsequent object of study in this thesis. An H-type Lie algebra is a stratified nilpotent Lie algebra $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ of dimension $n \geq 3$ and step 2, such that if $\langle \cdot, \cdot \rangle$ is a scalar product that makes $\mathfrak{h}_1$ and $\mathfrak{h}_2$ orthogonal, we have

$$\langle J_Y(X), J_Y(X') \rangle = |Y|^2 \langle X, X' \rangle.$$

Here $J : \mathfrak{h}_2 \to \mathrm{End}(\mathfrak{h}_1)$ is the Kaplan mapping, introduced in [80] (see 1.1.12) and $|Y| = \langle Y, Y \rangle^{1/2}$. We call *rank* of the Lie algebra the dimension of the first layer $\mathfrak{h}_1$ as a real vector space. An *H-type group* is a Lie group whose Lie algebra is an H-type Lie algebra, and it is a generalization of the n-dimensional Heisenberg group.

Given $\alpha \geq 0$, $h, k \geq 1$ integers and $n = h + k$ we define a *Grushin space* to be $\mathbb{R}^n$ endowed with the Carnot-Carathéodory distance d_α associated to the family of vector fields $X_\alpha = \{X_1, \ldots, X_h, Y_1, \ldots, Y_k\}$ where

$$X_i = \partial_{x_i}, \quad Y_j = |x|^\alpha \partial_{y_j}, \quad i = 1, \ldots, h, \ j = 1, \ldots, k.$$

If $h = k = 1$, $(\mathbb{R}^2, d_\alpha)$ is called the *Grushin plane*. In Section 1.2 we recall how to define the X-gradient of a function u, Xu, and BV_X-functions, with a particular attention to lower semicontinuity of the total variation and compactness of $BV_X(\mathbb{R}^n)$ into $L^1_{loc}(\mathbb{R}^n)$. We then define the X-perimeter of a set $E \subset \mathbb{R}^n$, denoted by $P_X(E)$, showing via an example in the Grushin plane its difference from the length associated to the family X. Section 1.3 is devoted to a proof of the global non-sharp isoperimetric inequality in Grushin spaces and Carnot groups, see Proposition 1.3.4, that follows the classical approach given by [7] in $\mathbb{R}^n$. The starting point is the validity of a global Poincaré inequality for balls, see in particular [86]. The chapter is concluded with a review of the most important results about the isoperimetric problem in Heisenberg groups.

Chapter 2 is devoted to the study of the *sharp isoperimetric inequality in Grushin spaces and H-type groups*, under a suitable symmetry assumption that depends on the dimensions of the layers. For $h, k \geq 1$ integers and $n = h + k$, we endow $\mathbb{R}^n$ with the Carnot-Carathéodory distance associated to the family $X = X_\alpha$, or to the family $X = X_H$ of Lie generators of an H-type Lie algebra of dimension $n \geq 3$ and rank $h \geq 2$. We say that a set $E \subset \mathbb{R}^n$ is *x-spherically symmetric* if there exists a set $F \subset [0, \infty) \times \mathbb{R}^k$ such that $E = \{(x, y) \in \mathbb{R}^n : (|x|, y) \in F\}$. Instead, we say that a set is *x-Schwarz symmetric* if for any $y \in \mathbb{R}^k$ the section $E^y = \{x \in \mathbb{R}^h : (x, y) \in E\}$ is a euclidean ball centered at zero in $\mathbb{R}^h$. The definition of y-Schwarz symmetry is analogous (see Section 2.1). For any given $v > 0$ we consider the isoperimetric problem

$$\inf\{P_X(E) : \mathcal{L}^n(E) = v, \ E \in \mathcal{S}_x \text{ if } h > 1\} \qquad (\mathrm{IP}_X)$$

where $\mathcal{S}_x$ is the class of x-spherically symmetric sets in $\mathbb{R}^n$. In the case $h = 1$ no symmetry assumptions are required. In Theorem 2.1.4 we prove existence of solutions to the isoperimetric problem and we characterize them via a differential equation for the profile function.

Theorem 1. *Let $h, k \geq 1$ and $n = h + k$. There exist minimizers for the isoperimetric problem* (IP$_X$). *Moreover, any isoperimetric set $E \subset \mathbb{R}^n$ is x- and y-Schwarz symmetric, and up to a vertical translation and a null set, it is of the form*

$$E = \{(x, y) \in \mathbb{R}^n : |y| < f(|x|)\},$$

for a decreasing function $f \in C([0, r_0]) \cap C^\infty(0, r_0) \cap C^2([0, r_0))$, for some $0 < r_0 < \infty$. The function f satisfies the following differential equation

$$\frac{f'}{\sqrt{r^{2\alpha} + f'^2}} = r^{1-h} \int_0^r s^{2\alpha+h-1} \frac{k-1}{f\sqrt{s^{2\alpha} + f'^2}}\, ds - \frac{C_{hk\alpha}}{h} r$$

with $C_{hk\alpha} = \frac{QP_\alpha(E)}{(Q-1)\mathcal{L}^n(E)}$, being Q the homogeneous dimension of $(\mathbb{R}^n, d_{cc}^X)$ (see Definitions (1.1.7) *and* (1.1.19)*).*

The proof is done through several Lemmas and Propositions. In Section 2.2, we prove *Representation formulas* for the X-perimeter of sets with regular boundary, in terms of their outer unit normal (see Proposition 2.2.1 and equation 2.2.5). As a consequence, in Proposition 2.2.3 we realize that the H-type perimeter of $E \in \mathcal{S}_x$, denoted by $P_H(E)$, is equal to the Grushin perimeter of E for $\alpha = 1$, denoted by $P_\alpha(E)$. This leads us to study the isoperimetric problem (IP$_X$) only for P_α. In Theorems 2.3.1 and 2.3.2, we prove non-classical *rearrangements* for the α-perimeter, summarized in the next statement:

Theorem 2. *For any set $E \subset \mathbb{R}^n$ with $E \in S_x$, such that $P_\alpha(E) < \infty$ and $0 < \mathcal{L}^n(E) < \infty$ there exists an x- and y-Schwarz symmetric set $E^* \subset \mathbb{R}^n$ such that $P_\alpha(E^*) \leq P_\alpha(E)$ and $\mathcal{L}^n(E^*) = \mathcal{L}^n(E)$. Moreover, if $P_\alpha(E^*) = P_\alpha(E)$ then E is x-Schwarz symmetric and there exist functions $c : [0, \infty) \to \mathbb{R}^k$ and $f : [0, \infty) \to [0, \infty]$ such that, up to a negligible set, we have*

$$E = \{(x, y) \in \mathbb{R}^n : |y - c(|x|)| < f(|x|)\}.$$

Theorem 2 leads to the proof of existence of isoperimetric sets in Section 2.4. This is based on a Concentration-Compactness type argument, adapted to the lack of invariance under translations of the perimeter P_α. Combining Theorem 2 together with the proof of x- and y-Schwarz symmetry of isoperimetric sets (see Proposition 2.5.5), we provide in Section 2.5 a characterization of isoperimetric sets through differential equations for their profile functions. Studying such equations we deduce uniqueness of the isoperimetric set in the case $k = 1$ (see Remark 2.5.2): up to dilations and vertical translations, the unique isoperimetric set in $(\mathbb{R}^{h+1}, d_\alpha)$ is

$$E^\alpha_{\text{isop}} = \Big\{(x,y) \in \mathbb{R}^h \times \mathbb{R} : |y| < \int_{\arcsin|x|}^{\pi/2} \sin^{\alpha+1} t \, dt, \ |x| < 1\Big\}.$$

We also prove asymptotic and convexity properties of isoperimetric sets for general values of k (see Proposition 2.5.3 and Section 2.6): the profile function f of an x- and y-Schwarz symmetric isoperimetric set $E \subset \mathbb{R}^n$ satisfies for some $0 < r_0 < \infty$, $f' \leq 0$, $f(r_0) = 0$, and

$$\lim_{r \to r_0^-} f'(r) = -\infty, \quad \text{and} \quad \lim_{r\to 0^+} \frac{f'(r)}{r^{\alpha+1}} = -\frac{C_{hk\alpha}}{h}.$$

Moreover, it is concave around zero. Uniqueness of isoperimetric sets for general dimensions may require some further work as explained in Remark 2.6.3.

The proof of the sharp isoperimetric inequality for x-spherically symmetric sets in Grushin spaces $\mathbb{R}^{h+k}$, $k = 1$ motivates us to study its stability, starting from the set E^α_{isop}. Chapter 3 is dedicated to several applications of a sub-calibration technique yielding to quantitative isoperimetric inequalities: our proof applies to Heisenberg groups (Section 3.2) and Grushin spaces (Section 3.3), providing the first examples of quantitative isoperimetric inequalities in Carnot-Carathéodory spaces, see Theorems 3.2.1 and 3.3.1. See also Section 3.4 where a quantitative inequality in the Euclidean space $\mathbb{R}^n$ is recovered. We endow $\mathbb{R}^h \times \mathbb{R}$ with the Carnot-Carathéodory metric associated to the Grushin space $(\mathbb{R}^{h+1}, d_\alpha)$ for $\alpha \geq 0$, or associated to the n-dimensional Heisenberg group, $h = 2n$. In the first case, we use the notation $E_{\text{isop}} = E^\alpha_{\text{isop}}$, whereas in the second case we denote by E_{isop} the Pansu set. The general scheme is the following: motivated by Theorem 1 and by Ritoré's proof of the sharp isoperimetric inequality in $\mathbb{H}^n$ in a suitable class of sets, [116], for any $0 \leq \varepsilon < 1$ we introduce the cylinder

$$C_\varepsilon = \{(x,y) \in \mathbb{R}^h \times \mathbb{R} : |x| < 1, \ y > \varepsilon\}.$$

We prove the following quantitative estimates.

Theorem 3. *Let $F \subset \mathbb{R}^{h+1}$, be any measurable set with $\mathcal{L}^{h+1}(F) = \mathcal{L}^{h+1}(E_{\text{isop}})$.*

i) If $F \Delta E_{\text{isop}} \subset\subset C_0$ then there exists a dimensional constant $C(h) > 0$ such that

$$P_X(F) - P_X(E_{\text{isop}}) \geq C(h) \mathcal{L}^{h+1}(F \Delta E_{\text{isop}})^3.$$

ii) If $F \Delta E_{\text{isop}} \subset\subset C_\varepsilon$ for $0 < \varepsilon < 1$, then there exists a constant $C(h, \varepsilon) > 0$ depending on the dimension h and on ε, such that

$$P_X(F) - P_X(E_{\text{isop}}) \geq C(h,\varepsilon) \mathcal{L}^{h+1}(F \Delta E_{\text{isop}})^2.$$

Precise expressions of the constant $C(h)$ and $C(h,\varepsilon)$ are given in Therorems 3.2.1, 3.3.1, and 3.4.1. The proof is based on the existence of a family of hyper-surfaces $\{\Sigma_s\}_{s\in\mathbb{R}}$ foliating the cylinder C_ε which have constant mean curvature. The mean curvature here has to be defined according to the ambient space as in (3.2.5) and (3.3.1). The surfaces Σ_s are constructed as the level sets of a smooth function obtained following the next theorem.

Theorem 4. *Let $0 \le \varepsilon < 1$. There exists a continuous function $u : C_\varepsilon \to \mathbb{R}$ with level sets $\Sigma_s = \big\{(x,y) \in C_\varepsilon : u(x,y) = s\big\}$, $s \in \mathbb{R}$, such that:*

i) $u \in C^1(C_\varepsilon \cap E_{\text{isop}}) \cap C^1(C_\varepsilon \setminus E_{\text{isop}})$ *and $Xu/|Xu|$ is continuously defined on $C_\varepsilon \setminus \{x = 0\}$;*

ii) $\bigcup_{s>1} \Sigma_s = C_\varepsilon \cap E_{\text{isop}}$ *and* $\bigcup_{s\le 1} \Sigma_s = C_\varepsilon \setminus E_{\text{isop}}$*;*

iii) Σ_s is a hypersurface of class C^2 with constant mean curvature $H_{\Sigma_s} = 1/s$ for $s > 1$ and $H_{\Sigma_s} = 1 = H_{\partial E_{\text{isop}}}$ for $s \le 1$.

Theorem 3 is implied by Theorem 4, via further estimates for the mean curvature that are also proved in Theorems 3.2.2, 3.3.4 (see relations (3.2.32), (3.2.33), (3.2.34)).

Chapter 4 contains the results so far obtained as a preliminary research in view of a quantitative isoperimetric inequality for a class of symmetric sets in the Grushin plane. In Theorem 4.1.1 we prove qualitative stability of the isoperimetric inequality.

Theorem 5. *For every $\varepsilon > 0$ there exists $\delta = \delta(\alpha,\varepsilon) > 0$ such that, for any measurable set $E \subset \mathbb{R}^2$ with finite α-perimeter and $\mathcal{L}^2(E) = \omega_\alpha$, if $P_\alpha(E) - P_\alpha(E^\alpha_{\text{isop}}) < \delta$ then*

$$A_\alpha(E) = \min_{y\in\mathbb{R}} \mathcal{L}^2\big(E \,\triangle\, (E^\alpha_{\text{isop}} + (0,y))\big) < \varepsilon.$$

The quantity $A_\alpha(E)$ is called the *α-asymmetry* of E. We say that the sharp *quantitative isoperimetric inequality in the Grushin plane* holds if

$$A_\alpha(E)^2 \le C(\alpha)(P_\alpha(E) - P_\alpha(E^\alpha_{\text{isop}}))$$

for any Lebesgue measurable set $E \subset \mathbb{R}^2$ satisfying $\mathcal{L}^2(E) = \mathcal{L}^2(E^\alpha_{\text{isop}})$. Our purpose is to understand if a quantitative isoperimetric inequality in the Grushin plane holds in the class of x- and y-Schwarz symmetric sets in $\mathbb{R}^2$. The starting plan in this direction was to follow the scheme given by the Euclidean proof presented in Section 4 of [63] for axially symmetric sets in $\mathbb{R}^n$. Crucial differences arise from the very beginning, as explained in Section 4.2. The proof in [63] starts observing that, if $E \subset \mathbb{R}^2$ is x- and y-Schwarz symmetric, satisfying $\mathcal{L}^2(E) = \mathcal{L}^2(B)$, the quantitative inequality

$$\Big(\min_{p\in\mathbb{R}^2} \mathcal{L}^n(E \,\triangle\, B(p,1))\Big)^2 \le C(P(E) - P(B))$$

follows from

$$\mathcal{L}^2((B \setminus E) \cap Z)^2 \le C(P(E) - P(B)), \tag{Z}$$

where $Z = \{(x,y) \in \mathbb{R}^2 : |x| < \sqrt{2}/2\}$. Since the α-perimeter is not invariant under rotation of the axis, relation (Z) fails to be sufficient for the quantitative isoperimetric inequality in the Grushin plane, and it needs to be proved in two *stripes*, namely:

$$\mathcal{L}^2((E^\alpha_{\text{isop}} \vartriangle E) \cap Z)^2 \le C(P_\alpha(E) - P_\alpha(E^\alpha_{\text{isop}})) \tag{Z grushin}$$

and

$$\mathcal{L}^2((E^\alpha_{\text{isop}} \vartriangle E) \cap Z_2)^2 \le C(P_\alpha(E) - P_\alpha(E^\alpha_{\text{isop}})), \tag{Z$_2$ grushin}$$

where $Z_2 = \{(x,y) \in \mathbb{R}^2 : |y| < b\}$ for a suitable $b > 0$. The proof of (Z) adapts to prove (Z$_2$ grushin), while (Z grushin) hides deep differences between the α-perimeter and the euclidean one. The proof of (Z) is divided into three steps:

1. The volume $\mathcal{L}^2((B \setminus E) \cap Z)$ is estimated in terms of the length difference between the intervals $E_{\bar{x}} = \{y \in \mathbb{R} : (\bar{x}, y) \in E\}$ and $B_{x'} = \{y \in \mathbb{R} : (x', y) \in B\}$ for some $\bar{x}, x' > 0$. Namely, there exists $C_1 > 0$ such that

   $$\mathcal{L}^2((E \vartriangle B) \cap Z) \le C_1 |\mathcal{H}^1(E_{\bar{x}}) - \mathcal{H}^1(B_{x'})|$$

   where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. This step can be proved for both stripes Z and Z_2, as shown in Lemma 4.2.4.

2. For any x- and y-Schwarz symmetric set $E \subset \mathbb{R}^2$ satisfying $\mathcal{L}^2(E) = \mathcal{L}^2(B)$ and any suitable choice of $\bar{x}$ and x' as above, there exists an x-spherically symmetric and y-Schwarz symmetric set $E' \subset \mathbb{R}^2$ satisfying $\mathcal{L}^2(E') = \mathcal{L}^2(E)$, such that

   $$P(E') \le P(E) \quad \text{and} \quad |\mathcal{H}^1(E_{\bar{x}}) - \mathcal{H}^1(B_{x'})| \le C_2 |\mathcal{H}^1(E'_0) - \mathcal{H}^1(B_0)|. \tag{Step 2}$$

   The technical key point here is what motivates our further analysis. The set E' is constructed to be equal to a Euclidean ball centered at 0 in the stripe $\{(x,y) \in \mathbb{R}^2 : |x| < x_0\}$ for a suitable $x_0 > 0$ such that

   $$\mathcal{H}^1(E'_{x_0}) = \mathcal{H}^1(E_{\bar{x}}) \quad \text{and} \quad \mathcal{L}^2(E) = \mathcal{L}^2(E \cap \{x < \bar{x}\}). \tag{constr.}$$

   The construction of E' doesn't apply to the Grushin plane in the stripe Z, see Remark 4.2.7. Nonetheless, if it is possible to construct E' to be equal to a dilation of the set E^α_{isop}, then estimate (Step 2) holds true, see Remark 4.2.8. As it is shown in Lemma 4.2.6, this happens in the case of the stripe Z_2.

3. For any x-spherically symmetric and y-Schwarz symmetric set $E \subset \mathbb{R}^2$ satisfying $\mathcal{L}^2(E) = \mathcal{L}^2(B)$, there exists a constant $C_3 > 0$ such that

$$|\mathcal{H}^1(E_0) - \mathcal{H}^1(B_0)|^2 \leq C_3(P(E) - P(B)).$$

The proof adapts to the α-perimeter in both stripes (see Proposition 4.4.1).

The failure of the Euclidean techniques at Step 2, leads us to introduce a minimization problem for a functional involving the Grushin perimeter, in a class of sets that satisfy volume and trace constraints representing (constr.). Given $h_1, h_2, v_1, v_2 > 0$, we define the class $\mathcal{A}_x = \mathcal{A}_x(v_1, v_2, h_1, h_2)$ of all Lebesgue measurable sets $E \subset \mathbb{R}^2$ that are x-symmetric, y-Schwarz symmetric and such that there exists $x_0 = x_0(v_1, v_2, h_1, h_2) \geq 0$ satisfying

$$\begin{aligned} &\mathcal{L}^2(E^x_{x_0-}) = v_1, \quad \mathcal{L}^2(E \setminus E^x_{x_0-}) = v_2, \\ &[-h_1, h_1] \subset \mathrm{tr}^x_{x_0-} E, \quad [-h_2, h_2] \subset \mathrm{tr}^x_{x_0+} E, \end{aligned} \tag{class}$$

where $\mathrm{tr}^x_{x_0\pm} E$ are the traces of E at x_0 in the x-direction introduced in Definition 4.3.2. Introducing the functional $\mathscr{F}_\alpha(E) = P_\alpha(E^x_{x_0-}) + P_\alpha(E^x_{x_0+}) - 4h_1 - 4h_2$, for $E \in \mathcal{A}_x$, we consider

$$\inf\{\mathscr{F}_\alpha(E) : E \in \mathcal{A}_x\}. \tag{Min $\mathscr{F}_\alpha$}$$

The study of such a minimization problem is presented in Section 4.3 and constitutes the heart of the chapter, having possible connections with the study of soap bubbles and minimal clusters in the Grushin plane.

Theorem 6. *Let $h_1, h_2, v_1, v_2 > 0$. There exists a bounded set $E \in \mathcal{A}_x$ realizing the infimum in* (Min $\mathscr{F}_\alpha$) *and such that, for $x_0 \geq 0$ defined by* (class)*, the sets $E \cap \{|x| < x_0\}$, $E \cap \{x > x_0\}$, $E \cap \{x < -x_0\}$ are convex sets.*

If $f : [0, \infty) \to [0, \infty)$ is such that the set $E = \{(x, y) \in \mathbb{R}^2 : |y| < f(|x|)\} \in \mathcal{A}_x$ is such a minimizer, then f is C^2-smooth almost everywhere on $[0, \infty)$ and there exist constants $c \geq 0$, $k \leq 0$, $d \in \mathbb{R}$ such that

$$f'(x) = -\frac{\operatorname{sgn} x \, c|x|^{\alpha+1}}{\sqrt{1 - c^2x^2}} \qquad \text{if} \quad |x| < x_0,$$

$$f'(x) = \frac{(kx + d)\, x^\alpha}{\sqrt{1 - (kx + d)^2}} \quad \text{if} \quad x > x_0, \qquad f'(x) = \frac{(kx - d)\, |x|^\alpha}{\sqrt{1 - (kx - d)^2}} \quad \text{if} \quad x < -x_0.$$

Moreover, if $\lim_{x \to x_0^-} f'(x) > -\infty$, $\mathrm{tr}_{x_0^-} E = [-h_1, h_1]$. Analogously if $\lim_{x \to x_0^+} f'(x) > -\infty$, $\mathrm{tr}_{x_0^+} E = [-h_2, h_2]$.

Existence of minimizers is proved in Theorem 4.3.6, via subsequent adjustments of a minimizing sequence. Characterization of the minimizers via differential equations for the profile function is instead proved in Sections 4.3.2 and 4.3.3.

The question arising from the construction at Step 2 is whether a minimizer for (Min $\mathscr{F}_\alpha$) is equal to a suitable dilation of E^α_{isop} in the stripe $\{(x,y) \in \mathbb{R}^2 : |x| < x_0\}$ or not. In Proposition 4.3.11 we show that this property is not satisfied for the particular choice of the parameter $v_2 = 0$. Hence, it is not clear if the techniques in [63] can be adapted to the Grushin perimeter.

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Basic notation

$\mathbb{R}^n$	n-dimensional Euclidean space
Δ^X	horizontal bundle generated by family of vector fields X
ℓ_X	length of curves in Δ^X, i.e., horizontal curves
d_{cc}^X	Carnot-Carathéodory distance associated to the family of vector fields X
$B_{cc}^X(x,r)$	open ball with respect to d_{cc} of center $x \in \mathbb{R}^n$ and radius $r \geq 0$
d_E	Euclidean distance
$B_E(x,r)$	open ball with respect to d_E of center $x \in \mathbb{R}^n$ and radius $r \geq 0$
$\mathbb{G}$	a Carnot group
τ_x	left translation by x in a Carnot group
$\mathfrak{g}$	Lie algebra of the Carnot group $\mathbb{G}$
Q	homogeneous dimension
$\delta_\lambda^{\mathbb{G}}$	dilation associated to the Carnot group $\mathbb{G}$ by $\lambda \geq 0$
$X_{\mathbb{G}}$	family of canonically generating vector fields of the Carnot group $\mathbb{G}$
$B_{\mathbb{G}}(x,r)$	open ball with respect to $d_{cc}^{\mathbb{G}}$ of center $x \in \mathbb{G}$ and radius $r > 0$
$\mathbb{H}^n$	n-dimensional Heisenberg group
$\delta_\lambda^{\mathbb{H}}$	anisotropic dilation associated with the Heisenberg group $\mathbb{H}^n$
X_α	family of vector fields associated with a Grushin space, for $\alpha \geq 0$
d_α	Carnot-Carathéodory distance in a Grushin space, for $\alpha \geq 0$
ℓ_α	length of horizontal curves in a Grushin space, for $\alpha \geq 0$
δ_λ^α	dilation by $\lambda \geq 0$ in a Grushin space for $\alpha \geq 0$
$B_\alpha(p,r)$	ball with respect to the distance d_α of center $p \in \mathbb{R}^n$ and radius $r \geq 0$
$\mathrm{Box}_\alpha(p,r)$	box of center $p \in \mathbb{R}^n$ and radius $r \geq 0$
$\mathcal{L}^n$	n-dimensional Lebesgue measure
Xu	horizontal gradient of the scalar function u
$\mathrm{div}_X \psi$	horizontal divergence of the vector function ψ
$\mathrm{div}_{\mathbb{G}} \psi$	horizontal divergence in a Carnot group $\mathbb{G}$ of the vector function ψ

$L^p(\Omega)$	p-summable functions in Ω, $1 \le p \le \infty$
$L^p_{loc}(\Omega)$	locally p-sumable functions in Ω
$W^{1,p}_X$	anisotropic Sobolev space associated with X
$\lvert Xu\rvert(\Omega)$	X-variation of the scalar function u in $\Omega \subset \mathbb{R}^n$
$BV_X(\Omega)$	space of functions of bounded X-variation in Ω
$P_X(E;\Omega)$	X-perimeter of E in $\Omega \subset \mathbb{R}^n$
P	Euclidean perimeter
$P_{\mathbb{G}}$	X-perimeter, with $X = X_{\mathbb{G}}$, where $\mathbb{G}$ is a Carnot group
P_H	H-perimeter
P_α	α-perimeter
$P_\alpha(E;\Omega)$	α-perimeter of E in Ω, with $\alpha \ge 0$
C_I	sharp isoperimetric constant
E_{isop}	conjectured isoperimetric set in $\mathbb{H}^n$, called Pansu ball
φ	profile function of E_{isop}
E^α_{isop}	isoperimetric set in Grushin spaces
φ_α	profile function of E^α_{isop}
$\mathcal{S}_x$	class of x-spherically symmetric sets in $\mathbb{R}^n$
N^E	outer unit normal to $E \subset \mathbb{R}^n$
N_α	α-normal to $E \subset \mathbb{R}^n$
N^E_H	H-normal to E
$\mathcal{H}^k$	k-dimensional Hausdorff measure with respect to the Euclidean metric
$D(E)$	isoperimetric deficit of the set $E \subset \mathbb{R}^n$
$A(E)$	Fraenkel asymmetry of $E \subset \mathbb{R}^n$
$D_H(E)$	H-isoperimetric deficit with respect to the Pansu ball of the set $E \subset \mathbb{R}^n$
$A_H(E)$	H-asymmetry with respect to the Pansu ball of $E \subset \mathbb{R}^n$
$D(E)$	αisoperimetric deficit of the set $E \subset \mathbb{R}^2$
$A(E)$	α-asymmetry of $E \subset \mathbb{R}^2$
$\mathrm{tr}^x_{x_0-}E$	left trace of $E \subset \mathbb{R}^2$ at $x_0 > 0$ in the x-direction
$\mathrm{tr}^x_{x_0+}E$	right trace of $E \subset \mathbb{R}^2$ at $x_0 > 0$ in the x-direction
$\mathrm{tr}^x_{x_0}E$	trace of $E \subset \mathbb{R}^2$ at $x_0 > 0$ in the x-direction

Contents

CHAPTER 1

An introduction to Carnot-Carathéodory structures in view of the Calculus of Variations

1.1 Carnot-Carathéodory structures: definitions and examples

In this section we introduce the notion of Carnot-Caratheodory and sub-Riemannian structure on $\mathbb{R}^n$. We also introduce as examples H-type groups and Grushin spaces where we are going to define and study the notion of perimeter in the subsequent Chapters. For the general definition of a *sub-Riemannian manifold* see for instance [69], [2], [85], [94].

1.1.1. Vector fields on $\mathbb{R}^n$

A *vector field on* $\mathbb{R}^n$ is a section of the tangent bundle of $\mathbb{R}^n$, which can be thought of as a function

$$X : \mathbb{R}^n \to T\mathbb{R}^n, \qquad X(x) = \sum_{i=1}^{n} a_i(x)\partial_{x_i} \in T_x\mathbb{R}^n$$

where, given $x \in \mathbb{R}^n$, ∂_{x_i} is the i-th element of the standard basis of the tangent space to $\mathbb{R}^n$ at x, and it can be identified with the partial derivative with respect to x_i evaluated at x. A vector field on $\mathbb{R}^n$ is said to be *(Lipschitz, C^k,...-) continuous* in $\mathbb{R}^n$ if its coefficients $a_i : \mathbb{R}^n \to \mathbb{R}$ are (Lipschitz, C^k,...-) continuous. The idea to define a Carnot-Carathéodory structure on $\mathbb{R}^n$ is to consider some privileged directions and define an associated distance. In sub-Riemannian geometry this is done by fixing a sub-bundle $\Delta \subset T\mathbb{R}^n$ as the span of vector fields on $\mathbb{R}^n$, $X_1, \dots, X_r$ $(1 \le r \le n)$ satisfying the so called Hörmander condition (see (1.1.2) below). To be more precise we need to recall the following notions.

Definition 1.1.1 (Lie Algebra)**.** A real *Lie Algebra* is a real vector space V endowed with an operation

$$\{\ ,\ \} : V \times V \to V, \qquad (v,w) \mapsto \{v,w\}$$

which satisfies the following properties

1. $\{\ ,\ \}$ is $\mathbb{R}$-bilinear;

2. $\{v, w\} = -\{w, v\}$ for any $v, w \in V$ (*skew symmetry*);

3. the following identity, called the *Jacobi identity* holds:

$$\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0 \qquad u, v, w \in V.$$

We say that a vector space $W \subset V$ is a *Lie subalgebra* of V if it is closed under the operation $\{\ ,\ \}$ and properties 1-3 are satisfied.

Given a subset $A \subset V$, we call *Lie subalgebra generated by A (in $(V, \{\ ,\ \})$)* the smallest Lie subalgebra of V containing A and we denote it by $\mathrm{Lie}(A)$.

There is a classical way to associate to a family of vector fields on $\mathbb{R}^n$ a Lie algebra (see for instance [19, Section 1.1]). Given two smooth vector fields on $\mathbb{R}^n$

$$X = \sum_{i=1}^n a_i \partial_{x_i}, \qquad Y = \sum_{j=1}^n b_j \partial_{x_j},$$

we define their composition law as the composition of partial differential operators, which is denoted by $\circ$, namely:

$$X \circ Y = \sum_{i=1}^n \left(a_i (\partial_i b_j) \partial_{x_j} + a_i b_j \partial^2_{x_i x_j} \right)$$

where $\partial^2_{x_i x_j}$ is the second order derivative with respect to x_i and x_j. The *commutator* $[X, Y]$ between X and Y is defined as

$$[X, Y] = X \circ Y - Y \circ X.$$

The set of C^∞ vector fields on $\mathbb{R}^n$, $\mathfrak{X}(\mathbb{R}^n) = \{X = \sum_{i=1}^n a_i \partial_{x_i} : a_i \in C^\infty(\mathbb{R}^n)\}$, endowed with the bracket operation $[\ ,\ ]$ is a Lie-algebra. In particular the commutator of vector fields is again a vector field. Henceforth, given a family of smooth vector fields $X = \{X_1, \dots, X_r\}$, we consider the Lie algebra generated by X in $(\mathfrak{X}, [\ ,\ ])$, and we denote it by $\mathrm{Lie}(X)$. It is easy to see that it coincides with the real span of the iterated brackets of the elements of X, namely:

$$\mathrm{Lie}(X) = \mathrm{span}\{[X_i, [...[X_j, X_k]]] : i, j, k = 1, \dots, r\}. \tag{1.1.1}$$

Definition 1.1.2 (Hörmander vector fields)**.** We say that the smooth vector fields on $\mathbb{R}^n$ $X_1, \dots, X_r \in \mathfrak{X}(\mathbb{R}^n)$ satisfy the *Hörmander condition* if the Lie algebra that they generate has full rank on $\mathbb{R}^n$, namely if

$$\mathrm{rank}(\mathrm{Lie}(X_1, \dots, X_r))(x) = n \qquad x \in \mathbb{R}^n, \tag{1.1.2}$$

where $\mathrm{rank}(W)$ denotes the dimension of W as vector space.

The following result is known as the *Chow-Rashevsky theorem* or *Chow connectivity theorem*, and it was independently proved by Chow in [33] and Rashevsky in [113].

Theorem 1.1.3 (Rashevsy 1938, Chow 1939). *Let $p, q \in \mathbb{R}^n$. If $X = \{X_1, \dots, X_r\}$ is a family of vector fields satisfying the Hörmander condition, there exists an absolutely continuous curve $\gamma : [0,1] \to \mathbb{R}^n$ such that*

$$\gamma(0) = p,\ \gamma(1) = q \qquad \dot\gamma(t) = \sum_{i=1}^{r} a_i(t) X_i(\gamma(t)) \text{ for some coefficients } a_i \text{ for a.e. } t \in [0,1].$$

For a proof we refer to [2, Theorem 3.29].

1.1.2. Sub-Riemannian and Carnot-Carathéodory structures on $\mathbb{R}^n$

Associated to the family $X = \{X_1, \dots, X_r\}$, we define a sub-bundle of the tangent bundle:

$$\Delta^X = \bigcup_{p \in \mathbb{R}^n} \Delta_p^X \qquad \Delta_p^X = \operatorname{span}\{X_1(p), \dots, X_r(p)\}$$

and we call it the *horizontal bundle*. A vector field $Y \in \Delta^X$ is called a *horizontal vector field*. We drop the upper index X if no confusion arises. We say that an absolutely continuous curve $\gamma : [0,1] \to \mathbb{R}^n$ is *horizontal* if $\dot\gamma(t) \in \Delta^X_{\gamma(t)}$ for every $t \in [0,1]$. Given, for every $p \in \mathbb{R}^n$, a scalar product g_p on Δ_p such that $X_1, \dots, X_r$ are orthonormal, we define the *length* of an horizontal curve γ as

$$\ell_X(\gamma) = \int_0^1 g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))\, dt = \int_0^1 \sqrt{\sum_{i=1}^{r} a_i^2(\gamma(t))}\, dt \tag{1.1.3}$$

where $\dot\gamma(t) = \sum_{i=1}^r a_i(\gamma(t)) X_i(\gamma(t)) \in \Delta^X_{\gamma(t)}$.

If the family X satisfies the Hörmander condition, it is possible to associate to this structure a distance through the following steps. By Chow's theorem (Theorem 1.1.3), any two points $p, q \in \mathbb{R}^n$ can be connected by means of horizontal curves. We can therefore define the following distance on $\mathbb{R}^n$, which is called the *Carnot-Carathéodory* (also *sub-Riemannian* or *CC* for short) *distance* associated to X:

$$d_{cc}^X(p,q) = \inf\left\{\ell_X(\gamma) : \gamma \text{ horizontal}, \gamma(0) = p,\ \gamma(1) = q\right\}. \tag{1.1.4}$$

We denote the open ball of center $x \in \mathbb{R}^n$ and radius $r > 0$ with respect to the CC distance by

$$B_{cc}^X(x,r) = \{y \in \mathbb{R}^n : d_{cc}^X(x,y) < r\}.$$

Also in the case of ℓ_X, d_{cc}^X and B_{cc}^X we may drop the index X if no confusion arises. From now on, by *sub-Riemannian structure* we mean $\mathbb{R}^n$ endowed with a family of vector fields on $\mathbb{R}^n$, $X = \{X_1, \dots, X_r\}$ satisfying the Hörmander condition.

If a Carnot-Carathéodory distance d_{cc}^X can be constructed on $\mathbb{R}^n$ starting from a set of vector fields $X = \{X_1, \ldots, X_m\}$ that do not necessarily satisfy Hörmander condition, we call $\mathbb{R}^n$ endowed with the family X a *Carnot-Carathéodory structure on* $\mathbb{R}^n$. Clearly sub-Riemannian structures are Carnot-Carathéodory ones.

Example 1.1.4 (Euclidean distance)**.** The euclidean space $\mathbb{R}^n$ endowed with the family $X = \{\partial_{x_1}, \ldots, \partial_{x_n}\}$ is a sub-Riemannian structure and the CC distance is the euclidean one $d_E(p,q) = \sqrt{\sum_{i=1}^n (p_i - q_i)^2}$. In this case we use the notation $B_E(x,r) = B_{cc}^X(x,r)$ for a euclidean ball of center $x \in \mathbb{R}^n$ and radius $r > 0$.

Example 1.1.5 (Grushin spaces)**.** In Subsection 1.1.4 we introduce Grushin spaces which are Carnot-Carathéodory structures on $\mathbb{R}^n$ endowed with a family of vector fields depending on a parameter $\alpha \geq 0$.

1.1.3. Carnot Groups

An important class of sub-Riemannian structures on $\mathbb{R}^n$ is given by *Carnot groups on* $\mathbb{R}^n$, which are Lie groups. We say that $\mathbb{G}$ is a *Lie group* if it is a smooth manifold endowed with a group operation $*$ such that the composition map $(x,y) \mapsto x * y$ and the inverse map $x \mapsto x^{-1}$ ($x * x^{-1} = x^{-1} * x = e$, unit element) are smooth on $\mathbb{G}$. Fixed $x \in \mathbb{G}$ we call *left translation by* x the map

$$\tau_x : \mathbb{G} \to \mathbb{G},\ \tau_x(y) = x * y$$

and *right translation by* x the map

$$\varrho_x : \mathbb{G} \to \mathbb{G},\ \varrho_x(y) = y * x.$$

The maps τ_x, ϱ_x are clearly C^∞ diffeomorphisms of $\mathbb{G}$ into itself for any $x \in \mathbb{G}$. We say that a vector field X on $\mathbb{G}$ is *left-invariant* if the following holds

$$(Xf) \circ \tau_x = X(f \circ \tau_x) \quad \text{for every } f \in C^\infty(\mathbb{G}),\ x \in \mathbb{G}. \tag{1.1.5}$$

The set of all left invariant vector fields is a Lie algebra, which is called the *Lie algebra of* $\mathbb{G}$ and it is denoted by $\mathrm{Lie}(\mathbb{G})$ or $\mathfrak{g}$. In the theory of Lie groups, the *exponential map* is a map from the Lie algebra of a group to the group itself defined using integral curves of the vector fields in $\mathfrak{g}$:

$$\mathrm{Exp} : \mathfrak{g} \to \mathbb{G}, \quad \mathrm{Exp}(X) = \gamma^X(1) \tag{1.1.6}$$

where $\gamma^X : [0,1] \to \mathbb{G}$ is the unique curve such that

$$\gamma^X(0) = e, \quad \dot{\gamma}^X(t) = X(\gamma(t)) \quad t \in [0,1].$$

Notice that here we use smoothness and completeness of left invariant vector fields (see for instance [19, Remark 1.1.3]).

Definition 1.1.6 (Carnot Group)**.** A *Carnot group* of *step* s is a connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a step s stratification, i.e., there exist linear subspaces $V_1, \dots, V_s$ such that

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s, \; [V_1, V_i] = V_{i+1}, \; V_s \neq \{0\},$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$.

Remark 1.1.7. The Lie algebra of a Carnot group is *nilpotent*, i.e., there exists $m \in \mathbb{N}$ such that $\mathfrak{g}^{(j)} = \{0\}, \; j \geq m$, where $\mathfrak{g}^{(j)}$ is defined recursively as follows:

$$\mathfrak{g}^{(1)} = \mathfrak{g}, \; \mathfrak{g}^{(j+1)} = [\mathfrak{g}, \mathfrak{g}^{(j)}] = \operatorname{span}\{[X, Y] : X \in \mathfrak{g}, \; Y \in \mathfrak{g}^{(j)}\}, \; j \geq 2.$$

We call the *homogeneous dimension* of $\mathbb{G}$ the number

$$Q = \sum_{i=1}^{s} i \dim V_i, \tag{1.1.7}$$

and the *rank* of $\mathbb{G}$, denoted by r, the dimension of V_1, which is the number of Lie-generators of the algebra.

Group operation

If $\mathbb{G}$ is a Carnot group, the exponential map defined in (1.1.6) is a global diffeomorphism. This guarantees that any n-dimensional Carnot group can be identified with $\mathbb{R}^n$. In fact, any point $x \in \mathbb{G}$ can be represented by $\operatorname{Exp}(X)$ for a unique $X \in \mathfrak{g}$. If $X = \sum_{i=1}^{r} x_i X_i$, we call *exponential coordinates* of x the vector $(x_1, \dots, x_n)$. In addition, the *Baker-Campbell-Hausdorff formula* holds for any $X, Y \in \mathfrak{g}$, (see [19, Theorem 2.2.13], [37, Theorem 1.2.1]), namely:

$$\operatorname{Exp}(X) * \operatorname{Exp}(Y) = \operatorname{Exp}(X \diamond Y) \qquad X, Y \in \mathfrak{g}$$

where $\diamond$ is the Baker-Campbell-Hausdorff operation whose first terms are the following

$$X \diamond Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\Big([X, [X, Y]] - [Y, [X, Y]]\Big) + \cdots.$$

The group operation of $\mathbb{G}$ can be written through the Baker-Campbell-Hausdorff formula as

$$x * y = x + y + \mathcal{Q}(x, y) \tag{1.1.8}$$

where $\mathcal{Q} : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ has polinomial, skew-symmetric components $(\mathcal{Q}(-x, -y) = -\mathcal{Q}(x, y))$, such that $\mathcal{Q}_1 = \cdots = \mathcal{Q}_r = 0$ and $\mathcal{Q}_j$, $j > r$ depends only on x_i and y_i for $i < j$ (see [19, Proposition 2.2.22]. This implies in particular that, identifying $\mathbb{G}$ with $\mathbb{R}^n$ via exponential coordinates, the unit element of any Carnot group on $\mathbb{R}^n$ is $0 \in \mathbb{R}^n$ and the inverse map is given by $x \mapsto -x$.

Dilations

The Lie algebra $\mathfrak{g}$ of a Carnot group $\mathbb{G}$ is naturally endowed with a family of *dilations* modeled on its stratification:

$$\delta_\lambda^{\mathfrak{g}}\Big(\sum_{i=1}^{s} Y_i\Big) = \sum_{i=1}^{s} \lambda^i Y_i, \quad Y_i \in V_i, \quad \lambda > 0.$$

The group $\mathbb{G}$ inherits a family of *anisotropic dilations* parametrized by $\lambda > 0$ and defined as

$$\delta_\lambda^{\mathbb{G}}(x) = \delta_\lambda^{\mathbb{G}}\Big(\mathrm{Exp}\Big(\sum_{i=1}^{n} Y_i\Big)\Big) = \mathrm{Exp}\Big(\sum_{i=1}^{s} \lambda^i Y_i\Big).$$

Using exponential coordinates, $\delta_\lambda^{\mathbb{G}}$ turns out to be of the following form (see [19, relations (2.49c), (2.53)])

$$\delta_\lambda^{\mathbb{G}}(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_r, \lambda^{\sigma_{r+1}} x_{r+1}, \dots, \lambda^{\sigma_n} x_n):$$

with $\sigma_j = i$ if $Y_j \in V_i$, $j = r+1, \dots, n$, $i = 2, \dots, r$. The components of the polynomial $\mathcal{Q}$ appearing in the group operation are homogeneous with respect to the intrinsic dilation $\delta_\lambda^{\mathbb{G}}$: $\mathcal{Q}_j(\delta_\lambda^{\mathbb{G}} x, \delta_\lambda^{\mathbb{G}} y) = \lambda^{\alpha_j} \mathcal{Q}_j(x, y)$, $j = 1, \dots, n$. From (1.1.8) we also deduce that $\delta_\lambda^{\mathbb{G}}$ is a family of automorphisms of $\mathbb{G}$, namely

$$\delta_\lambda^{\mathbb{G}} x * \delta_\lambda^{\mathbb{G}} y = \delta_\lambda^{\mathbb{G}}(x * y).$$

Moreover $(\delta_\lambda^{\mathbb{G}})^{-1} = \delta_{1/\lambda}^{\mathbb{G}}$.

Sub-Riemannian structure on $\mathbb{G}$

A sub-Riemannian structure on $\mathbb{G}$ is given considering the first layer V_1 of the stratification of the Lie algebra as the horizontal bundle. Consider a basis for the Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$,

$$X_1, \dots, X_r, X_1^{(2)}, \dots, X_{r_2}^{(2)}, \dots, X_1^{(s)}, \dots, X_{r_s}^{(s)}$$

where $X_1, \dots X_r$ generates V_1, $X_1^{(j)}, \dots, X_{r_j}^{(j)}$ generates V_j for $j = 2, \dots, s$, $r_1 + \cdots + r_s = n$ and such that at the origin it is the canonical orthonormal basis of $\mathbb{R}^n$ in the coordinate system

$$x = (x_1, \dots, x_{r_1}, x_{r_1+1}, \dots, x_{r_1+r_2}, \dots, x_{r_1+\dots+r_{s-1}+1}, \dots, x_n).$$

Namely,

$$X_1^{(j)}(0) = \frac{\partial}{\partial x_{r_1+\dots+r_{j-1}+1}}, \ \dots \ , X_{r_i}^{(j)}(0) = \frac{\partial}{\partial x_{r_1+\dots+r_j}}, \quad j = 1, \dots s.$$

We extend the scalar product that makes $\partial_{x_1}, \dots, \partial_{x_n}$ orthonormal at the origin in a left invariant way and we call it g_p on Δ_p, $p \in \mathbb{G}$. Using left invariance, X_i can be written in the coordinate system $x = (x_1, \dots, x_n)$ as follows (see [50, Proposition 1.26]):

$$X_i(x) = \partial_{x_i} + \sum_{j=r+1}^{n} q_{ji}(x)\partial_{x_j} \qquad i = 1\dots r,\ x \in \mathbb{G}, \tag{1.1.9}$$

where $q_{ji} = \partial_{x_i}\mathcal{Q}_j(x,0)$ is $(\alpha_j - 1)$-homogeneous with respect to $\delta_\lambda^{\mathbb{G}}$ and $q_{ji}(x) = q_{ji}(x_1, \dots, x_{j-1})$. In fact, by left invariance, for any $u \in C^\infty(\mathbb{R}^n)$, $j = 1, \dots, r$, we have

$$X_i u(x) = (X_i u) \circ \tau_x(0) = X_i(u \circ \tau_x)(0) = \partial_{x_i}(u \circ \tau_x)(0) = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(x)\frac{\partial(\tau_x)_j}{\partial x_i}(0).$$

Then, using (1.1.8)

$$\begin{aligned} X_i u(x) &= \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(\delta_{ij} + \partial_{x_i}\mathcal{Q}_j(x,0)) \\ &= \sum_{j=1}^{i-1} \frac{\partial u}{\partial x_j}(\partial_{x_i}\mathcal{Q}_j) + \frac{\partial u}{\partial x_i}(1 + \partial_i \mathcal{Q}_i) + \sum_{j=i+1}^{n} \frac{\partial u}{\partial x_j}(\partial_{x_i}\mathcal{Q}_j) \\ &= \frac{\partial u}{\partial x_i} + \sum_{j=r+1}^{n} q_{ji}\frac{\partial u}{\partial x_i}. \end{aligned}$$

We refer to $X_1, \dots, X_r$ as the family of *canonically generating vector fields* and we use the notation

$$X_{\mathbb{G}} = \{X_1, \dots, X_r\}.$$

We call the *Carnot-Carathéodory distance of the Carnot group* $\mathbb{G}$, and denote it by $d_{cc}^{\mathbb{G}}$, the one defined in (1.1.4) and associated to a family of canonically generating vector fields:

$$d_{cc}^{\mathbb{G}}(p,q) = \inf\Big\{ \int_0^1 \sqrt{\sum_{i=1}^{s} a_i(\gamma(t))^2}\, dt : \gamma(0) = p,\ \gamma(1) = q,\ \dot\gamma = \sum_{i=1}^{s} a_i X_i \Big\}.$$

We use the notation $B_{\mathbb{G}} = B_{cc}^X$ where X is a family of canonical generators for $\mathbb{G}$. The following properties of $d_{cc}^{\mathbb{G}}$ hold:

- The topology induced on $\mathbb{G}$ by $d_{cc}^{\mathbb{G}}$ is the topology of the manifold;
- $d_{cc}^{\mathbb{G}}$ is *left invariant*:

$$d_{cc}^{\mathbb{G}}(\tau_x y, \tau_x z) = d_{cc}^{\mathbb{G}}(y,z);$$

- $d_{cc}^{\mathbb{G}}$ is 1-*homogeneous with respect to intrinsic dilations*

$$d_{cc}^{\mathbb{G}}(\delta_\lambda^{\mathbb{G}} x, \delta_\lambda^{\mathbb{G}} y) = \lambda d_{cc}^{\mathbb{G}}(x,y), \qquad x, y, z \in \mathbb{G}\ \lambda > 0.$$

Metric characterization of Carnot groups

On a Carnot group we can consider a *sub-Finsler distance* instead of a sub-Riemannian one, choosing a left invariant norm $\{\|\cdot\|_p\}_{p\in\mathbb{R}^n}$ on the horizontal bundle V_1, instead of a scalar product $\{g_p\}_{p\in\mathbb{R}^n}$. The length of an horizontal curve $\gamma:[0,1]\to\mathbb{G}$ is then defined as

$$\ell_X(\gamma)=\int_0^1\|\dot\gamma(t)\|_{\gamma(t)}\,dt.$$

Carnot groups endowed with a sub-Finsler distance have a precise characterization as metric spaces, pointed out by E. Le Donne in [84]: they are the only metric spaces S that are:

1. *locally compact*, i.e., every point of the space has a compact neighborhood (involves only topology).
2. *geodesic*, i.e., for all $p,q\in S$, there exists an isometric embedding $i:[0,T]\to S$ with $T\geq 0$ such that $i(0)=p$, $i(T)=q$.
3. *isometrically homogeneous*, i.e., for all $p,q\in S$ there exists a *distance preserving homeomorphism* $f:S\to S$ ($d(f(x),f(y))=d(x,y)$ for ever $x,y\in S$) such that $f(p)=q$.
4. *self similar*, i.e., the space admits a *dilation*, namely there exists $\lambda>1$ and a homeomorphism $f:S\to S$ such that $d(f(p),f(q))=\lambda d(p,q)$ for all $p,q\in S$.

The fact that any Carnot group with a sub-Finsler distance $(\mathbb{G},d)$ is such a metric space is easy.

1. The topology of a Carnot group with the sub-Finsler CC-distance is the euclidean topology;
2. $(\mathbb{G},d)$ is complete and d is defined as a length distance;
3. For any $p,q\in\mathbb{G}$, the left translation $\tau_z:\mathbb{G}\to\mathbb{G}$, $z=\tau_q(p^{-1})$ is distance preserving and satisfies $\tau_z(p)=q$;
4. Carnot groups admits dilations for any $\lambda>0$.

1.1.4. Examples

I) Heisenberg groups

Besides the Euclidean space, the most important example of Carnot group is the Heisenberg group $\mathbb{H}^1$. The *n-dimensional Heisenberg group*, denoted by $\mathbb{H}^n$, is $\mathbb{C}^n\times\mathbb{R}$ endowed with the following group operation:

$$(z,t)*(z',t')=(z+z',t+t'+2\mathrm{Im}(z\bar{z}')),$$

where $\bar{z}'$ denotes the conjugate of z'. Identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ through $z = x + iy \mapsto (x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$, the operation can be also written as

$$(x, y, t) * (x', y', t') = \Big(x + x', y + y', t + t' + 2\sum_{i=1}^{n}\left(x_i' y_i - x_i y_i'\right)\Big). \tag{1.1.10}$$

To find a family of canonically generating vector fields of the Lie algebra $\mathfrak{h}$ of $\mathbb{H}^n$, we look for a family of left invariant vector fields $X = \{X_1, \dots, X_n, Y_1, \dots, Y_n, T\}$ which correspond to the canonical basis of $\mathbb{R}^{2n+1}$ at the origin

$$X_i(0) = \partial_{x_i}, \qquad Y_i(0) = \partial_{y_i}, \qquad T(0) = \partial_t,\ i = 1, \dots, n.$$

This leads to

$$X_i(x, y, t) = \partial_{x_i} + 2y_i\partial_t, \quad Y_i(x, y, t) = \partial_{y_i} - 2x_i\partial_t, \quad T(0) = \partial_t,\ i = 1, \dots, n.$$

Notice that the only nonzero commutator of the family X is $[X_i, Y_i] = -4\partial_t = -4T$. Brackets of order bigger than 2 are zero. Hence, $\mathfrak{h} = \mathrm{Lie(X)}$ with $X = \{X_i, Y_i : i = 1, \dots, n\}$ and the family X satisfies the Hörmander condition (1.1.2): $\mathrm{rank}(\mathrm{Lie}(X)) = 2n + 1$. The horizontal bundle is therefore given by $\Delta = \mathrm{span}\{X_i, Y_i : i = 1, \dots, n\}$. Moreover the Lie algebra $\mathfrak{h}$ admits the stratification

$$\mathfrak{h} = \Delta \oplus [\Delta, \Delta], \quad \Delta = \mathrm{span}\{X_i, Y_i : i = 1, \dots n\},$$

so that $\mathbb{H}^n$ is a Carnot group of step 2 and rank $2n$. The homogeneous dimension of $\mathbb{H}^n$ is

$$Q = 2n + 2$$

and the dilations of the group are

$$\delta_\lambda^{\mathbb{H}}(z, t) = (\lambda z, \lambda^2 t), \quad \lambda > 0,\ (z, t) \in \mathbb{H}^n.$$

Derivations of the Heisenberg group

While talking about sub-Riemannian structures, it is often said that the Heisenberg group is the *"easiest" example*, apart from the euclidean space. In fact, we can view the Heisenberg Lie algebra $\mathfrak{h}$ as the unique three dimensional nilpotent Lie algebra, with a step 2 stratification $\mathfrak{h} = V_1 \oplus V_2$, and rank 2 such that

$$[V_1, V_1] = V_2,\ [V_1, V_2] = \{0\}$$

In particular, if $V_1 = \mathrm{span}\{\mathrm{e}_1, \mathrm{e}_2\}$, $V_2 = \mathrm{span}\{\epsilon\}$ it is sufficient to impose

$$[\mathrm{e}_1, \mathrm{e}_2] = \epsilon. \tag{1.1.11}$$

The group law of the corresponding Carnot group is induced by relation (1.1.11) as follows. Let $\xi, \eta \in \mathfrak{h}$, $\xi = x\mathrm{e}_1 + y\mathrm{e}_2 + t\epsilon$, $\eta = x'\mathrm{e}_1 + y'\mathrm{e}_2 + t'\epsilon$. Then

$$[\xi, \eta] = xy'[e_1, e_2] + x'y[e_2, e_1] = (xy' - x'y)\epsilon.$$

In exponential coordinates, if $p = \mathrm{Exp}(\xi), q = \mathrm{Exp}(\eta) \in \mathbb{H}^1$, we write $p = (x, y, t)$ and $q = (x', y', t')$. Therefore, since the Exponential map is a global diffeomorphism, the group law $\bullet$ of the correspondent Lie group is given by

$$\begin{aligned}(x, y, t) \bullet (x', y', t') &= \mathrm{Exp}(\xi) * \mathrm{Exp}(\eta) = \mathrm{Exp}(\xi \diamond \eta) = \mathrm{Exp}\Big(\xi + \eta + \frac{1}{2}[\xi, \eta]\Big) \\ &= \mathrm{Exp}\Big((x + x')\mathrm{e}_1 + (y + y')\mathrm{e}_2 + (t + t')\epsilon + \frac{1}{2}(xy' - x'y)\epsilon\Big) \\ &= \Big(x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y)\Big)\end{aligned}$$

Condition (1.1.11) leads to the group law $\bullet$: the group law $*$, defined in (1.1.10), is obtained imposing $[\mathrm{e}_1, \mathrm{e}_2] = -4\epsilon$, which still implies $[V_1, V_1] = V_2$.

II) H-type groups

A generalization of the Heisenberg groups is given by H-type groups. Let $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ be a stratified nilpotent real Lie algebra of dimension $n \geq 3$ and step 2. Thus we have $\mathfrak{h}_2 = [\mathfrak{h}_1, \mathfrak{h}_1]$. We fix on $\mathfrak{h}$ a scalar product $\langle \cdot, \cdot \rangle$ that makes $\mathfrak{h}_1$ and $\mathfrak{h}_2$ orthogonal. The *Kaplan mapping*, introduced in [80], is the mapping $J : \mathfrak{h}_2 \to \mathrm{End}(\mathfrak{h}_1)$ defined via the identity

$$\langle J_Y(X), X' \rangle = \langle Y, [X, X'] \rangle, \tag{1.1.12}$$

holding for all $X, X' \in \mathfrak{h}_1$ and $Y \in \mathfrak{h}_2$. The algebra $\mathfrak{h}$ is called an *H-type algebra* if for all $X, X' \in \mathfrak{h}_1$ and $Y \in \mathfrak{h}_2$ there holds

$$\langle J_Y(X), J_Y(X') \rangle = |Y|^2 \langle X, X' \rangle, \tag{1.1.13}$$

where $|Y| = \langle Y, Y \rangle^{1/2}$. An *H-type group* is a Lie group whose Lie algebra is an H-type Lie algebra, clearly an H-type group is a Carnot group. We can identify $\mathfrak{h}$ with $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$, $\mathfrak{h}_1$ with $\mathbb{R}^h \times \{0\}$, and $\mathfrak{h}_2$ with $\{0\} \times \mathbb{R}^k$, where $h \geq 2$ and $k \geq 1$ are integers. In fact, h is always even.

Remark 1.1.8. The subspace $\mathfrak{z} \subset \mathfrak{h}$

$$\mathfrak{z} = \{Z \in \mathfrak{h} : [Z, X] = 0 \ \forall X \in \mathfrak{h}\},$$

is called the *center* of $\mathfrak{h}$. The following general result holds true (see [80, Corollary 1]). Let $h, k \in \mathbb{N} \smallsetminus \{0\}$. Then there exists an H-type Lie algebra of dimension $n = h+k$ whose center has dimension k if and only if $k < \rho(h)$ where ρ is the so called *Hurwitz–Radon* function, which is such that $\rho(n) = 0$ if n is odd. We deduce that there cannot be H-type Lie algebras of dimension $n = h + k$, $h, k \geq 1$ whith center of dimension $k \geq 1$ and h odd, hence h has to be an even integer.

We can assume that $\langle\cdot,\cdot\rangle$ is the standard scalar product of $\mathbb{R}^n$. By the third fundamental theorem of Lie (see [19, Theorem 2.2.14]), since $\mathfrak{h}$ is a finite dimensional Lie algebra, there exists a connected and simply connected Lie group whose Lie algebra is isomorphic to $\mathfrak{h}$. This Lie group is therefore an H-type group and, using exponential coordinates, it can be identified with $\mathbb{R}^n$. Denoting points of $\mathbb{R}^n$ as $(x, y) \in \mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$, the Lie group product $* : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is of the form $(x, y) * (x', y') = (x + x', y + y' + \mathcal{Q}(x, x'))$, where $\mathcal{Q} : \mathbb{R}^h \times \mathbb{R}^h \to \mathbb{R}^k$ is a bilinear skew-symmetric mapping. Let $Q^\ell_{ij} \in \mathbb{R}$ be the numbers

$$Q^\ell_{ij} = \langle \mathcal{Q}(\mathrm{e}_i, \mathrm{e}_j), \mathrm{e}_\ell \rangle, \quad i, j = 1, \dots, h, \ \ell = 1, \dots, k,$$

where $\mathrm{e}_i, \mathrm{e}_j \in \mathbb{R}^h$ and $\mathrm{e}_\ell \in \mathbb{R}^k$ are the standard coordinate versors. An orthonormal basis of the Lie algebra of left-invariant vector fields of the H-type group $(\mathbb{R}^n, \cdot)$ is given by

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i} - \sum_{\ell=1}^{k} \sum_{j=1}^{h} Q^\ell_{ij} x_j \frac{\partial}{\partial y_\ell}, \quad i = 1, \dots, h, \\ Y_j &= \frac{\partial}{\partial y_j}, \quad j = 1, \dots, k. \end{aligned} \tag{1.1.14}$$

Example 1.1.9. Heisenberg groups are H-type groups with $\mathfrak{h} = \mathrm{Lie}(\{X_j, Y_j : j = 1, \dots, N\})$, $\mathfrak{h}_1 = \mathrm{span}\{X_j, Y_j\}$, $\mathfrak{h}_2 = [\mathfrak{h}_1, \mathfrak{h}_1] = \mathrm{span}\{\partial_t\}$. If $T = -4\partial_t \in \mathfrak{h}_2$,

$$J_T\Big(\sum_{j=1}^{N}(a_j X_j + b_j Y_j)\Big) = \sum_{j=1}^{N}(-b_j X_j + a_j Y_j)$$

and (1.1.13) is satisfied.

Example 1.1.10 (Complexified Heisenberg group)**.** Another example of H-type group is given in [19, Example 18.1.3], (see also [114]), and it is $\mathbb{R}^6$ with the following group law

$$x \circ y = \Big(x^{(1)} + y^{(1)}, x^{(2)}_1 + y^{(2)}_1 + \frac{1}{2}\langle P_1 x^{(1)}, y^{(1)}\rangle, x^{(2)}_2 + y^{(2)}_2 + \frac{1}{2}\langle P_2 x^{(1)}, y^{(1)}\rangle\Big)$$

where $x = (x^{(1)}, x^{(2)}_1, x^{(2)}_2) \in \mathbb{R}^6$, $x^{(1)} \in \mathbb{R}^4$ and $x^{(2)}_1, x^{(2)}_2 \in \mathbb{R}$,

$$P_1 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \qquad P_2 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

III) Grushin spaces

We now introduce a Carnot-Carathéodory structure on $\mathbb{R}^n$ which is related to Heisenberg groups without being a Carnot group. Let $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$, where $h, k \geq 1$ are integers and $n = h + k$. Let $\alpha \geq 0$ be a real number. A *Grushin space* is $\mathbb{R}^n$ endowed with the following structure on $\mathbb{R}^n$: $X_\alpha = \{X_1, \dots, X_h, Y_1, \dots, Y_k\}$,

$$\begin{aligned} X_i &= \partial_{x_i}, \quad i = 1, \dots, h, \\ Y_j &= |x|^\alpha \partial_{y_j}, \quad j = 1, \dots, k, \end{aligned} \tag{1.1.15}$$

where $|x|$ is the standard norm of $x \in \mathbb{R}^h$. When $h = k = 1$, $\mathbb{R}^2$ endowed with the family X_α is called the *Grushin plane* and it has been considered by Franchi and Lanconelli in [54] to prove Hölder regularity of the weak solutions of $Lu = 0$,

$$L = \frac{\partial^2}{\partial x^2} + |x_1|^{2\alpha} \frac{\partial^2}{\partial x_2^2},$$

using Moser's technique. The differential operator L is known in the literature as the *Grushin operator* (see also [70]), and it is hypoelliptic for $\alpha \in \mathbb{N}$.

In the paper [22], the Grushin plane is identified as a two dimensional *almost-Riemannian manifold* (see [2], [22], [24] and references therein). Endowing the Grushin plane with the Riemannian volume, that degenerates on the y-axis, the authors consider, in the case $\alpha = 1$, the *Laplace-Beltrami operator*, that is not the Grushin operator, and study solutions to the heat equation in such a structure. In the paper [23], it is more generally considered the case $\alpha \in \mathbb{R}$: the authors characterize the solutions of the heat equation showing that they can flow through the y-axis if and only if $\alpha \in (-3, 1)$.

Remark 1.1.11. Notice that the vector fields Y_j, $j = 1, \dots, k$ are not smooth for every $\alpha \geq 0$ and we can not test the Hörmander condition. Nonetheless, the only non-horizontal curves in a Grushin structure on $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ are the curves $\gamma : [0,1] \to \mathbb{R}^2$ laying on the vertical axis for some time $[t_1, t_2]$, $0 \leq t_1 < t_2 \leq 1$. Hence, any two points $p_1, p_2 \in \mathbb{R}^n$, can be connected by means of horizontal curves and the Carnot-Carathéodory distance can be defined together with all the other tools as the X-perimeter (see Section 1.2 below).

On the other hand, if α is an integer, the family of vector fields $X(x,y) = \partial_x, Y(x,y) = x^\alpha \partial_y$, is a sub-Riemannian structure on $\mathbb{R}^2$. In fact, for every $(x,y) \in \mathbb{R}^2$

$$[X, Y](x,y) = \alpha x^{\alpha-1} \partial_y$$

hence, with α iterated brackets we obtain $[X, \dots, [X, [X, Y]]](x,y) = \partial_y$, which leads to

$$\begin{aligned}\mathrm{Lie}(\{X,Y\}) &= \mathrm{span}\{[[X,\dots,[X,[X,Y]]]]:\ \text{commutators of order} \le \alpha\}\\ &= \mathrm{span}\{\partial_x,\partial_y\} \equiv \mathbb{R}^2.\end{aligned}$$

Condition (1.1.2) is therefore proved.

We show a formula for the length of horizontal cuves in Grushin structures. For $(x,y)\in\mathbb{R}^n=\mathbb{R}^h\times\mathbb{R}^k$, $x\neq 0$, and $\alpha>0$ consider the metric

$$ds_\alpha^2 = dx_1^2+\dots+dx_h^2+\frac{1}{|x|^{2\alpha}}(dy_1^2+\dots+dy_k^2) \tag{1.1.16}$$

where dx_i, dy_j denote the elements of the canonical basis of the cotangent bundle to $\mathbb{R}^n$ in the coordinate system $(x_1,\dots,x_h,y_1,\dots,y_k)$. Then ds_α^2 makes $X_1,\dots,X_h,Y_1,\dots,Y_k$ orthonormal. Following (1.1.3), we define the α-length of an horizontal curve $\gamma:[0,1]\to\mathbb{R}^n=\mathbb{R}^h\times\mathbb{R}^k$ as

$$\ell_\alpha(\gamma)=\int_0^1\sqrt{\sum_{i=1}^h\gamma_i'(t)^2+\frac{1}{|(\gamma_1(t),\dots,\gamma_h(t))|^{2\alpha}}\sum_{j=1}^k\gamma_{1+j}'(t)^2}\,dt. \tag{1.1.17}$$

The Carnot-Carathéodory distance on $\mathbb{R}^n$ associated to the family X is denoted by d_α. The Grushin space $\mathbb{R}^n=\mathbb{R}^h\times\mathbb{R}^k$, with d_α can be endowed with a family of non-isotropic dilations parametrized by $\lambda>0$

$$\delta_\lambda^\alpha(x,y)=(\lambda x,\lambda^{\alpha+1}y),\quad (x,y)\in\mathbb{R}_x^h\times\mathbb{R}_y^k=\mathbb{R}^n \tag{1.1.18}$$

such that $d_\alpha(\delta_\lambda^\alpha p,\delta_\lambda^\alpha q)=\lambda d_\alpha(p,q)$, for $p,q\in\mathbb{R}^n$. We define the *homogeneous dimension of the Grushin space* $\mathbb{R}^n=\mathbb{R}^h\times\mathbb{R}^k$ *with* d_α as

$$Q=h+(\alpha+1)k. \tag{1.1.19}$$

Remark 1.1.12. Grushin spaces are not Carnot groups! In fact, $(\mathbb{R}^n,d_\alpha)$ is a locally compact and geodesic metric space which admits a dilation. These are three up to four properties needed in the metric characterization of Carnot groups given by Le Donne, see [84, Theorem 1.1]. On the other hand, Grushin spaces are not isometrically homogeneous. We show it for $(\mathbb{R}^2,d_\alpha)$: let $P=(0,0)$, $Q=(1,0)$ and consider a distance preserving homeomorphism $f:\mathbb{R}^2\to\mathbb{R}^2$, i.e., $d_\alpha(f(p),f(q))=d_\alpha(p,q)$ for every $p,q\in\mathbb{R}^2$. Since f is a distance preserving homeomorphism, it is a *conformal map*, i.e., there exists a function $u:\mathbb{R}^2\to(0,\infty)$ such that

$$\lim_{p\to q}\frac{d_\alpha(f(p),f(q))}{d_\alpha(p,q)}=u(q)^{-1}.$$

Applying the characterization of conformal maps in the Grushin metric, proved by Morbidelli in [102], we deduce $f(0,0) = (0,y)$ for some $y \in \mathbb{R}$, hence $f(P) \neq f(Q)$. Then the points P and Q cannot be joined through a distance preserving homeomorphism, which implies that $(\mathbb{R}^2, d_\alpha)$ is not isometrically homogeneous.

Connection with the Heisenberg geometry

The Grushin plane $(\mathbb{R}^2, d_\alpha)$, $\alpha = 1$, can be identified with a metric quotient of $\mathbb{H}^1$. Namely, we consider the quotient of $\mathbb{H}^1$ with the one parameter subgroup $\Xi = \{(\xi,0,0) \in \mathbb{H}^1,\ \xi \in \mathbb{R}\}$, i.e.,

$$\mathbb{X} = \{\Xi p : p \in \mathbb{H}^1\},$$

where Ξp is the right coset $\Xi p = \{(\xi,0,0) * (x,y,t) : \xi \in \mathbb{R}\} = \{(\xi + x, y, t - 2\xi y) : \xi \in \mathbb{R}\}$ for every $p = (x,y,t) \in \mathbb{H}^1$. Consider $\varphi : \mathbb{H}^1 \to \mathbb{R}^2$, $\varphi(x,y,t) = (y, t+2xy)$. Since $\varphi((\xi,0,0) * (x,y,t)) = (y, t+2xy)$ for every $\xi \in \mathbb{R}$, φ induces a map

$$\tilde{\varphi} : \mathbb{X} \to \mathbb{R}^2,\ \tilde{\varphi}(\Xi p) = (y, t+2xy).$$

In particular, choosing for any coset Ξp the unique representative $(\xi,0,0) * p = (0,u,v) \in \{(x,y,t) \in \mathbb{H}^1 : x = 0\}$, we identify $\mathbb{X}$ with $\mathbb{R}^2$ via the map $\tilde{\varphi}$ as follows

$$\tilde{\varphi}(\Xi p) = \varphi(0,u,v) = (u,v), \quad (u,v) \in \mathbb{R}^2.$$

On $\mathbb{X}$ we define the quotient metric

$$d_{\mathbb{X}}(\Xi p, \Xi q) = \inf_{(\xi,0,0)\in\Xi} d_{\mathbb{H}^1}((\xi,0,0) * p, q),\ p,q \in \mathbb{H}^1.$$

The map $\tilde{\varphi} : (\mathbb{X}, d_{\mathbb{X}}) \to (\mathbb{R}^2, d_\alpha)$, $\alpha = 1$, is an isometry, i.e.,

$$d_{\mathbb{X}}(\Xi p, \Xi q) = d_\alpha(\tilde{\varphi}(\Xi p), \tilde{\varphi}(\Xi q)), \quad p,q \in \mathbb{H}^1. \tag{1.1.20}$$

To prove it, we first introduce new coordinates in $\mathbb{H}^1$. Denote $[\xi,u,v] = \Psi(\xi,u,v)$ where $\Psi : \mathbb{R}^3 \to \mathbb{H}^1$ is the analytic change of variables

$$\Psi(\xi,u,v) = (\xi,0,0) * (0,u,v) = (\xi, u, v - 2\xi u)$$

so that, if $\pi : \mathbb{H}^1 \to \mathbb{X}$ denotes the projection on the quotient, $\pi([\xi,u,v]) = \Xi(0,u,v)$. Hence in the new coordinates, $\tilde{\varphi} \circ \pi([\xi,u,v]) = (u,v)$. We write $X = \partial_x + 2y\partial_t$, $Y = \partial_y - 2x\partial_t$. For any $f \in C_c^\infty(\mathbb{R}^3)$ and $(x,y,t) = [\xi,u,v] \in \mathbb{H}^1$ we have

$$\begin{aligned}
\partial_\xi f([\xi,u,v]) &= \partial_x f([\xi,u,v]) - 2u\partial_t f([\xi,u,v]) = \partial_x f - 2y\partial_t f \\
\partial_u f([\xi,u,v]) &= \partial_y f([\xi,u,v]) - 2\xi\partial_t f([\xi,u,v]) = \partial_y f - 2x\partial_t f \\
\partial_v f([\xi,u,v]) &= \partial_t f
\end{aligned}$$

hence $X = \partial_\xi + 4u\partial_v$, $Y = \partial_u$, and the push-forward through the map $\tilde{\varphi} \circ \pi$ read $(\tilde{\varphi} \circ \pi)_* X = 4u\partial_v$ and $(\tilde{\varphi} \circ \pi)_* Y = \partial_u$. In fact, for $g : \mathbb{R}^2 \to \mathbb{R}$

$$
\begin{aligned}
&(\tilde{\varphi}_* \pi_* X)(g) = X(g(\tilde{\varphi}(\pi([\xi, u, v])))) = (\partial_\xi + 4u\partial_v)(g(u, v)) = 4u\partial_v g(u, v), \\
&(\tilde{\varphi}_* \pi_* Y)(g) = \partial_u g(u, v).
\end{aligned}
$$

Notice that the Carnot-Carathéodory distance associated to the vector fields $4u\partial_v$ and ∂_u is d_α.

Now, given any horizontal curve $\gamma = (u(t), v(t))$, $t \in (0,1)$ in $(\mathbb{R}^2, d_\alpha)$, the curve $\tilde{\gamma}(t) = [\xi(t), u(t), 4v(t)] \in \mathbb{H}^1$, with $\xi(t) = \xi_0 + \int_0^t \frac{\dot{v}(s)}{u(s)}\, ds$, $\xi_0 \in \mathbb{R}$, is a horizontal lift of γ in $\mathbb{H}^1$, i.e. it is horizontal in $\mathbb{H}^1$ and $\tilde{\varphi} \circ \pi \circ \tilde{\gamma} = \gamma$. In fact,

$$
\dot{\tilde{\gamma}} = \dot{\xi}\partial_\xi + \dot{u}\partial_u + 4\dot{v}\partial_v = \frac{\dot{v}}{u}(X - 4u\partial v) + \dot{u}Y + 4\dot{v}\partial_v = \frac{\dot{v}}{u}X + \dot{u}Y.
$$

Moreover, by definition of length of a horizontal curve, (see (1.1.3) and (1.1.17))

$$
\ell_{\mathbb{H}^1}(\tilde{\gamma}) = \int_0^1 \sqrt{\frac{\dot{v}(s)^2}{u(s)^2} + \dot{u}(s)^2}\, ds = \ell_\alpha(\gamma).
$$

From the lifting, we deduce (1.1.20) as follows

$$
\begin{aligned}
d_\alpha(\tilde{\varphi}(\Xi p), \tilde{\varphi}(\Xi q)) = d_\alpha((u_1, v_1), (u_2, v_2)) &= \inf_{\xi_1, \xi_2 \in \mathbb{R}} d_{\mathbb{H}^1}([\xi_1, u_1, v_1], [\xi_2, u_2, v_2]) \\
&= \inf_{(\xi, 0, 0) \in \Xi} d_{\mathbb{H}^1}((\xi, 0, 0) * (0, u_1, v_1), (0, u_2, v_2)) = d_{\mathbb{X}}(\Xi p, \Xi q).
\end{aligned}
$$

In this explanation we followed [12], where Arcozzi and Baldi review Rotschild and Stein lifting techniques for vector fields satisfying Hörmander condition (see [119, Theorem 4]).

CC-Balls in the Grushin plane

In this Section, we resume estimates for the measure of the Carnot-Carathéodory balls in Grushin spaces, that are used in Section 1.3 to prove a global isoperimetric inequality. We denote by $B_\alpha(x, r)$ the open ball with respect to d_α of center $x \in \mathbb{R}^n$ and radius $r > 0$, i.e.,

$$
B_\alpha(x, r) = \{y \in \mathbb{R}^n : d_\alpha(x, y) < r\}.
$$

When $n = 2$, Franchi and Lanconelli (see [54, Theorem 2.7]) proved estimates for CC-balls in terms of the boxes

$$
\mathrm{Box}_\alpha(p, r) = [x - r, x + r] \times [y - r(|x| + r)^\alpha, y + r(|x| + r)^\alpha], \qquad p = (x, y) \in \mathbb{R}^2,\ r > 0.
$$

Namely, they prove existence of constants $c_1, c_2 > 0$ such that

$$\mathrm{Box}_\alpha(p, c_1 r) \subset B_\alpha(x, r) \subset \mathrm{Box}_\alpha(p, c_2 r), \quad r > 0,\ p \in \mathbb{R}^2.$$

These inclusions are generalized to Grushin spaces $\mathbb{R}^n = \mathbb{R}^h_x \times \mathbb{R}^k_y$ by Franchi, Gutiérrez and Wheeden in [53, Proposition 2.2] as follows. For every $p_0 = (x_0, y_0) \in \mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ and $r > 0$, let

(i) $\Lambda_\alpha(p_0, r) = \sup_{|x-x_0|<r} |x|^\alpha = (|x_0| + r)^\alpha$;

(ii) $F_\alpha(p_0, r) = r\Lambda_\alpha(p_0, r)$;

(iii) $\mathrm{Box}_\alpha(p_0, r) = B_E(x_0, r) \times B_E(y_0, F_\alpha(p_0, r))$.

Then, there exists $b > 1$ such that

$$\mathrm{Box}_\alpha(p, r/b) \subset B_\alpha(p, r) \subset \mathrm{Box}_\alpha(p, br), \quad r > 0,\ p \in \mathbb{R}^n. \tag{1.1.21}$$

In particular, there exist $c_1, c_2 > 0$ such that

$$c_1 r^n \Lambda_\alpha(p, r)^k \le \mathcal{L}^n(B_\alpha(p, r)) \le c_2 r^n \Lambda_\alpha(p, r)^k \quad r > 0,\ p_0 \in \mathbb{R}^n. \tag{1.1.22}$$

Remark 1.1.13. Recalling that $\Lambda_\alpha(p, r)^k = (|x| + r)^{\alpha k} \ge r^{\alpha k}$, the latter estimate implies

$$\mathcal{L}^n(B_\alpha(p, r)) \ge c_1 r^Q \text{ for every } p \in \mathbb{R}^n,\ r > 0.$$

1.1.5. Lebesgue measure in Grushin spaces and Carnot groups: dilations and translations

In this Section, we resume how the Lebesgue measure interacts with the metric structure in Carnot groups and in Grushin spaces.

Proposition 1.1.14 (Q-homogeneity of the Lebesgue measure). *Let $\{\delta_\lambda\}_{\lambda>0}$ be a family of anisotropic dilations on $\mathbb{R}^n$ defined by $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$*

$$\delta_\lambda(p_1, \dots, p_n) = (\lambda^{\sigma_1} p_1, \dots, \lambda^{\sigma_n} p_n), \quad \sigma_i \in \mathbb{N}, \quad 1 = \sigma_1 \le \sigma_2 \le \dots \le \sigma_n. \tag{1.1.23}$$

Let $Q = \sum_{i=1}^n \sigma_i$. Then the n-dimensional Lebesgue measure $\mathcal{L}^n$ is δ_λ-homogeneous of degree Q, i.e., for every measurable set $E \subset \mathbb{R}^n$

$$\mathcal{L}^n(\delta_\lambda(E)) = \lambda^Q \mathcal{L}^n(E). \tag{1.1.24}$$

Proof. Compute the Jacobian determinant of δ_λ at $p \in \mathbb{R}^n$:

$$J_{\delta_\lambda} p = \det \begin{pmatrix} \lambda^{\sigma_1} & 0 & \dots & 0 \\ 0 & \lambda^{\sigma_2} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & \lambda^{\sigma_n} \end{pmatrix} = \lambda^Q. \tag{1.1.25}$$

Hence

$$\mathcal{L}^n(E) = \int_{\delta_\lambda(E)} dq = \int_E \lambda^Q \, dp = \lambda^Q \mathcal{L}^n(E).$$

□

Let X be one of the families of vector fields below:

- $X_{\mathbb{G}} = \{X_1, \dots, X_r\}$ generators of a stratified nilpotent Lie algebra associated to a Carnot group $\mathbb{G}$. In this case, the dilations of the group $\delta_\lambda = \delta_\lambda^{\mathbb{G}}$ are as in (1.1.23), hence the Lebesgue measure is δ_λ-homogeneous of degree Q, which is the number defined in (1.1.7).
- $X_\alpha = \{X_1, \dots, X_h, Y_1, \dots, Y_k\}$ defining a Grushin strcture on $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ as in (1.1.15). The dilations $\delta_\lambda = \delta_\lambda^\alpha = (\lambda x, \lambda^{\alpha+1} y)$ are of type (1.1.23), hence the Lebesgue measure is δ_λ-homogeneous of degree $Q = h + (\alpha + 1)k$.

The Carnot-Carathéodory distance d_{cc} associated to X is 1-homogeneous with respect to δ_λ: $d_{cc}(\delta_\lambda x, \delta_\lambda y) = \lambda d_{cc}(x, y)$ for every $x, y \in \mathbb{R}^n$. In particular, if $B(x, r)$ denotes the CC-ball with center $x \in \mathbb{R}^n$ and radius $r > 0$, we have that

$$\begin{aligned} B(x, \lambda r) &= \{y \in \mathbb{R}^n : d_{cc}(x, y) \le \lambda r\} = \left\{y \in \mathbb{R}^n : d_{cc}\left(\delta_{\frac{1}{\lambda}} x, \delta_{\frac{1}{\lambda}} y\right) \le r\right\} \\ &= \left\{\delta_\lambda y \in \mathbb{R}^n : d_{cc}\left(\delta_{\frac{1}{\lambda}} x, y\right) \le r\right\} = \delta_\lambda B(\delta_{1/\lambda} x, r), \qquad \lambda > 0. \end{aligned} \tag{1.1.26}$$

From (1.1.26) and (1.1.24), we deduce that for every $r > 0$, $x \in \mathbb{R}^n$

$$\mathcal{L}^n(B(x, 2r)) = 2^Q \mathcal{L}^n(B(\delta_{1/2} x, r)). \tag{1.1.27}$$

If $X = X_{\mathbb{G}}$, left and right translations interplay with the Lebesgue measure in the following way.

Proposition 1.1.15 (Haar measure in Carnot groups). *Let $\mathbb{G}$ be a Carnot group identified with $\mathbb{R}^n$. The n-dimensional Lebesgue measure, $\mathcal{L}^n$, is the Haar measure of $\mathbb{G}$, namely it is invariant with respect to left and right translations on $\mathbb{G}$:*

$$\mathcal{L}^n(\tau_x(E)) = \mathcal{L}^n(E) = \mathcal{L}^n(\varrho_x(E)) \quad x \in \mathbb{G}, \ E \subset \mathbb{G} \text{ measurable.} \tag{1.1.28}$$

Proof. The proof is a change of variables - see Proposition 1.3.21 in [19]. □

Proposition 1.1.15 together with (1.1.27) lead to the doubling property of the Lebesgue measure in any Carnot group.

Definition 1.1.16 (Doubling measure)**.** A Borel measure μ on a metric space (S, d) is *doubling* if there exists a constant $C_D > 0$ such that

$$\mu(B(p, 2R)) \leq C_D \mu(B(p, R)) \text{ for every } p \in S,\ R > 0,$$

where $B(p, R) = \{q \in S : d(p, q) \leq R\}$.

For a reference see [72].

Proposition 1.1.17 (Doubling property of the Lebesgue measure - Carnot groups)**.** *Let $\mathbb{G}$ be a Carnot group and Q its homogeneous dimension. Then*

$$\mathcal{L}^n(B_{\mathbb{G}}(x, 2r)) = 2^Q \mathcal{L}^n(B_{\mathbb{G}}(x, r)) \textit{ for every } r > 0,\ x \in \mathbb{G}.$$

Proof. We use (1.1.27) and (1.1.28):

$$\mathcal{L}^n(B_{\mathbb{G}}(x, 2r)) = 2^Q \mathcal{L}^n(B_{\mathbb{G}}(\delta^{\mathbb{G}}_{1/2} x, r)) = 2^Q \mathcal{L}^n(B_{\mathbb{G}}(\tau_x \tau_{-\delta^{\mathbb{G}}_{1/2} x} \delta^{\mathbb{G}}_{1/2} x, r)) = 2^Q \mathcal{L}^n(B_{\mathbb{G}}(x, r)).$$

□

Since Grushin spaces are not isometrically homogeneous, the doubling property of the Lebesgue measure is not straightforward, and it is based on the estimates for CC balls proved in [53] (see (1.1.22)).

Proposition 1.1.18 (Doubling property of the Lebesgue measure - Grushin)**.** *Let $h, k \geq 1$ be integers and $n = h + k$. Then there exists a constant $C > 0$ such that*

$$\mathcal{L}^n(B_\alpha(p, 2r)) \leq C_D \mathcal{L}^n(B_\alpha(p, r)) \textit{ for every } r > 0,\ p \in \mathbb{R}^n.$$

Proof. Let $p = (x, y) \in \mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ and $r > 0$. From (1.1.27) we have $\mathcal{L}^n(B(p, 2r)) = 2^Q \mathcal{L}^n(B(\delta^\alpha_{1/2} p, r))$, with $Q = h + (\alpha + 1)k$. We claim that there exists $c > 0$ such that

$$\mathcal{L}^n(B(\delta^\alpha_{1/2} p, r)) \leq c \mathcal{L}^n(B(p, r)).$$

In fact, from (1.1.22)

$$\mathcal{L}^n(B(\delta^\alpha_{1/2} p, r)) \leq b r^n \Lambda_\alpha(\delta^\alpha_{1/2} p, r)^k \leq b r^n \Lambda_\alpha(p, r)^k = \frac{b^2}{b} r^n \Lambda_\alpha(p, r)^k \leq b^2 \mathcal{L}^n(B_\alpha(p, r))$$

where we used $\Lambda_\alpha(\delta^\alpha_{1/2} p, r)^k \leq \Lambda_\alpha(p, r)^k$ which holds by definition. The statement follows with $C_D = b^2 2^Q$. □

Remark 1.1.19. In 1983, Nagel, Stein and Wainger proved that the Lebesgue measure is (locally) doubling in any sub-Riemannian space, see [106, Theorem 1]. Namely, if $X = \{X_1, \dots, X_r\}$ is a family of vector fields on $\mathbb{R}^n$ satisfying the Hörmander condition, and d_{cc}^X is the associated Carnot-Carathéodory distance, for any compact set $K \subset\subset \mathbb{R}^n$ there exist constants $C = C(K) > 0$, $R_0 = R_0(K) > 0$ such that

$$\mathcal{L}^n(B_{cc}^X(p, 2R)) \le C\mathcal{L}^n(B_{cc}^X(p, R)), \quad p \in K, 0 < R \le R_0.$$

1.2 Functions of bounded X-variation and X-perimeter

In this section we recall the basic notions of the theory of perimeters in the setting of Carnot-Carathéodory structures on $\mathbb{R}^n$, following [28], [65] and [56].

Let $X = \{X_1, \dots, X_r\}$ be a family Lipschitz continuous vector fields on $\mathbb{R}^n$ which are self adjoint, namely $X_j^* = X_j$ where

$$\int_{\mathbb{R}^n} \varphi X_j \psi \, dx = -\int_{\mathbb{R}^n} \psi X_j^* \varphi \, dx \quad \varphi, \psi \in C^\infty, \; j = 1, \dots, r.$$

If $u : \mathbb{R}^n \to \mathbb{R}$ is a $C^1(\mathbb{R}^n)$-function we call the *X-gradient* or *horizontal gardient* of u the following vector field

$$Xu = \sum_{i=1}^{r} (X_i u) X_i. \tag{1.2.1}$$

Another notation for the X-gradient of a function u is $D_X u = Xu$. Moreover, given $\varphi = (\varphi_1, \dots, \varphi_r)$ with $C^1(\mathbb{R}^n)$-components such that $X_i \varphi_i \in L^1_{loc}(\mathbb{R}^n)$ we define the *X-divergence* or *horizontal divergence* as

$$\mathrm{div}_X \varphi = \sum_{i=1}^{r} X_i \varphi_i. \tag{1.2.2}$$

Remark 1.2.1 (Horizontal gradient and divergence in Carnot groups). If $X = X_{\mathbb{G}}$ generates a stratified nilpotent Lie algebra, associated to a Carnot group $\mathbb{G} = (\mathbb{R}^n, *)$, the notion of X-gradient depends on the choice of the family X, while the horizontal divergence only depends on the first layer V_1 of the Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$. Henceforth we use the notation $\mathrm{div}_X = \mathrm{div}_{\mathbb{G}}$. Namely, let $Y = \{Y_1, \dots, Y_r\}$ be a set of canonically generating vector fields, different from X and denote by $\langle \ , \ \rangle_X$ (respectively $\langle \ , \ \rangle_Y$) the left-invariant scalar product on the tangent bundle such that $X_1, \dots, X_r$ (respectively $Y_1, \dots, Y_r$) are orthonormal. We have in general $Xu = \sum_{i=1}^r (X_i u) X_i \neq \sum_{i=1}^r (Y_i u) Y_i = Yu$, and the gradients Xu and Yu are equal if $X_1, \dots, X_r$ are orthonormal with respect to $\langle \ , \ \rangle_Y$ and $Y_1, \dots, Y_r$ are orthonormal with respect to $\langle \ , \ \rangle_X$ (see [120, Remark 3.9]).

On the other hand, let $y_1,\dots,y_n$ denote the the coordinates for the family Y. Then, by (1.1.9), $Y_j(x) = \partial_{y_j} + \sum_{i\geq r+1} q_{ij}\partial_{y_i}$. We write φ as a vector field in the basis Y and compare it to the basis $\partial_{y_1},\dots,\partial_{y_n}$:

$$\varphi = \sum_{i=1}^{r}\varphi_j X_j = \sum_{i=1}^{r}\big(\varphi_j\partial_{y_j} + \sum_{i\geq r+1}\varphi_j q_{ij}\partial_{y_i}\big) = \sum_{j=1}^{r}\varphi_j\partial_{y_j} + \sum_{i\geq r+1}\big(\sum_{j=1}^{r}\varphi_j q_{ij}\big)\partial_{y_i}.$$

Using the independence of q_{ji} from x_i, we obtain that the Y-divergence is equal to the euclidean divergence as follows:

$$\begin{aligned}\mathrm{div}\varphi &= \sum_{j=1}^{r}\partial_{y_j}\varphi_j + \sum_{i=r+1}^{n}\partial_{y_i}\big(\sum_{j=1}^{r}\varphi_j q_{ij}\big) = \sum_{j=1}^{r}\partial_{y_j}\varphi_j + \sum_{i=r+1}^{n}\sum_{j=1}^{r} q_{ij}\partial_{y_i}\varphi_j\\ &= \sum_{j=1}^{r}\big(\partial_{y_j} + \sum_{i=r+1}^{n} q_{ij}\partial_{y_i}\big)\varphi_j = \sum_{j=1}^{r} Y_j\varphi_j = \mathrm{div}_Y\varphi.\end{aligned}$$

Let $\Omega\subset\mathbb{R}^n$ be an open set. For any $m\in\mathbb{N}$, let us define the family of test functions

$$\mathcal{F}_m(\Omega) = \Big\{\varphi\in C^1_c(\Omega;\mathbb{R}^m) : \max_{x\in\Omega}|\varphi(x)| = \max_{x\in\Omega}\sqrt{\sum_{j=1}^{r}\varphi_j^2(x)} \leq 1\Big\}.$$

Definition 1.2.2 (X-variation)**.** For any $u\in L^1_{loc}(\Omega)$ the *X-variation of u in Ω* is defined as

$$|Xu|(\Omega) = \sup\Big\{\int_\Omega u(x)\mathrm{div}_X\varphi(x)\,dx : \varphi\in\mathcal{F}_r(\Omega)\Big\} \tag{1.2.3}$$

A function $u\in L^1_{loc}(\Omega)$ is said to be *of bounded X-variation in Ω* if $|Xu|(\Omega)<\infty$.

Another notation for the X-variation of a function u in Ω is $|D_Xu|(\Omega) = |Xu|(\Omega)$. By Riesz representation theorem, if $|Xu|(\Omega)<\infty$, then the open sets functional $A\mapsto|Xu|(A)$ extends to a finite Radon measure $|Xu|$ in Ω and there exists a $|Xu|$-measurable function $\sigma:\Omega\to\mathbb{R}^r$, with $|\sigma|=1$ $|Xu|$-a.e. such that for any $\varphi\in\mathcal{F}_r(\Omega)$ there holds

$$\int_\Omega u\,\mathrm{div}_X\varphi\,dx = -\int_\Omega\langle\varphi,\sigma\rangle\,d|Xu|,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^r$.

The vector space of functions of bounded X-variation is denoted by $BV_X(\Omega)$. The space $BV_{X,loc}(\Omega)$ is the set of functions belonging to $BV_X(A)$ for every $A\subset\subset\Omega$.

Remark 1.2.3. The following holds

$$|Xu|(\Omega) = \int_\Omega|Xu|(x)\,dx \qquad u\in W^{1,1}_X(\Omega) \tag{1.2.4}$$

where for $1\leq p\leq\infty$, $W^{1,p}_X(\Omega)$ is the *anisotropic Sobolev space* $W^{1,p}_X(\Omega) = \big\{u\in L^p(\Omega) : X_ju\in L^p(\Omega),\ j=1,\dots,r\big\}$ which is a Banach space endowed with the norm

$$\|u\|_{W^{1,p}_X(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{r}\|X_ju\|_{L^p(\Omega)}.$$

For $u \in L^p(\Omega)$, $X_j u$ denotes the *weak derivative in direction* X_j defined through the equality

$$\int_\Omega u \cdot X_j \psi \, dx = -\int_\Omega (X_j u) \cdot \psi \, dx \quad \forall \psi \in C_c^\infty(\Omega).$$

We recall the following approximation result which is proved by Franchi, Serapioni and Serra Cassano in [56, Theorem 2.2.2].

Theorem 1.2.4 (Approximation of BV_X functions)**.** *Let $u \in BV_X(\Omega)$. Then there exists a sequence $(u_h)_{h\in\mathbb{N}} \subset C^\infty(\Omega)$ such that*

$$\lim_{h\to\infty} \|u_h - u\|_{L^1(\Omega)} = 0, \qquad \lim_{h\to\infty} |Xu_h|(\Omega) = |Xu|(\Omega).$$

1.2.1. Lower semicontinuity and compactness of BV_X functions

In the Calculus of Variations, a classical technique to prove existence of minimizers for a minimum problem

$$C_F = \inf\{F(A) : A \in \mathcal{A}\}$$

involving a functional F defined on a nonempty class $\mathcal{A}$ of measurable sets in $\mathbb{R}^n$, is to consider a *minimizing sequence* $A_k \in \mathcal{A}$, i.e.,

$$F(A_k) \le C_F\Big(1 + \frac{1}{k}\Big), \; k \in \mathbb{N}$$

and prove its convergence in the L^1-topology to a minimum, i.e., $A \in \mathcal{A}$ such that $F(A) = C_F$. The necessary tools to use this strategy are a *compactness theorem* to extract a subsequence A_{k_m} converging to $A \in \mathcal{A}$ and *lower semi-continuity* of the functional F to obtain

$$F(A) \le \liminf_{m\to\infty} F(A_{k_m}) \le C_F.$$

We show that the definition of X-variation can be used, in this sense, to address problems from the Calculus of Variations. The following properties hold.

- *Lower semi-continuity of the total X-variation.* Let $X = \{X_1, \dots, X_r\}$ be a family of Lipschitz continuous and self adjoint vector fields on $\mathbb{R}^n$. Suppose $u_m \in BV_X(\Omega)$ for $m \in \mathbb{N}$ and $u_m \to u$ in $L^1(\Omega)$ as $m \to \infty$. Then

$$|Xu|(\Omega) \le \liminf_{m\to\infty} |Xu_m|(\Omega). \tag{1.2.5}$$

To prove (1.2.5), let $\varphi \in \mathcal{F}_r(\Omega)$. Then we have:

$$\begin{aligned}\int_\Omega u \, \mathrm{div}_X \varphi \, dx &= \lim_{m\to\infty} \int_\Omega u_m \, \mathrm{div}_X \varphi \, dx \\ &\le \liminf_{m\to\infty} \sup\Big\{ \int_\Omega u_m \, \mathrm{div}_X \varphi \, dx : \varphi \in \mathcal{F}_r(\Omega)\Big\} = \liminf_{m\to\infty} |Xu_m|(\Omega).\end{aligned}$$

The claim follows considering the supremum over all $\varphi \in \mathcal{F}_r(\Omega)$ in the left-hand side.

- *Compactness of $BV_{X,loc}(\mathbb{R}^n)$ in $L^1_{loc}(\mathbb{R}^n)$.* Assume that X generates a stratified nilpotent Lie algebra associated to a Carnot group $\mathbb{G}$ or X is the family depending on $\alpha \geq 0$ defined in (1.1.15) for Grushin spaces. Then $BV_{X,loc}(\mathbb{R}^n)$ is compactly embedded in $L^q_{loc}(\mathbb{R}^n)$ for $1 \leq q < \frac{Q}{Q-1}$. This result is proved by Garofalo and Nhieu for more general families of vector fields X, see [65, Theorem 1.28].

1.2.2. X-perimeter

Given a Lebesgue measurable set $E \subset \mathbb{R}^n$, we denote by χ_E its characteristic function

$$\chi_E : \mathbb{R}^n \to \{0,1\}, \ \chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

Clearly $\chi_E \in L^1(\mathbb{R}^n)$ if and only if $\mathcal{L}^n(E) < \infty$.

Definition 1.2.5 (X-perimeter)**.** We say that a Lebesgue measurable set $E \subset \mathbb{R}^n$, has *finite X-perimeter in Ω* if $\chi_E \in BV_X(\Omega)$ and we call *X-perimeter of E in Ω* the quantity

$$P_X(E;\Omega) = |X\chi_E|(\Omega).$$

If $\Omega = \mathbb{R}^n$ we say that E has finite X-perimeter and we use the notation $P_X(E) = P_X(E,\mathbb{R}^n)$.

As in the case of the X-variation defined in (1.2.3), by Riesz representation theorem, if $P_X(E,\Omega) < \infty$, then the open sets functional $A \mapsto P_X(E,A)$ extends to a finite Radon measure μ_E in Ω and there exists a μ_E-measurable function $\nu_E : \Omega \to \mathbb{R}^r$, called the *horizontal normal* of E, with $|\nu_E| = 1$ μ_E-a.e. such that the following *Gauss-Green integration by parts formula*

$$\int_E \operatorname{div}_X \varphi \, dx = -\int_\Omega \langle \varphi, \nu_E \rangle \, d\mu_E \tag{1.2.6}$$

holds true for any $\varphi \in C^1_c(\mathbb{R}^n;\mathbb{R}^r)$, where $\langle \cdot,\cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^r$.

Remark 1.2.6. Let ∂E be a C^1 surface, i.e., $\partial E = S = \{p \in \mathbb{R}^n : u(p) = 0\}$, for a smooth function $u \in C^1(\mathbb{R}^n)$ such that $\nabla u \neq 0$. In this case the horizontal normal is defined in terms of the outer unit normal outside a subset of S, which is called the *characteristic set* of S. The latter is the set of points where the horizontal hyperplane and the tangent plane to S coincide:

$$\Sigma(S) = \{p \in S : Xu(p) = 0\}.$$

For every $p \in S \setminus \Sigma(S)$, we have

$$\nu_E(p) = \frac{Xu}{|Xu|}.$$

Unfortunately, the characteristic set might be "big". In the paper [13], in fact, the author shows that in the Heisenberg group $\mathbb{H}^1$ for any $\varepsilon > 0$ there exists a surface $S = \{t = f(z), z \in [-1,1]^2\}$ with $f \in \bigcap_{0<\alpha<1} C^{1,\alpha}([-1,1]^2)$ such that $\mathcal{L}^2(\{z \in [-1,1]^2 : (z, f(z)) \in \Sigma(S))\} > 1 - \varepsilon$.

The notion of X-perimeter is given according to the De Giorgi definition of perimeter in the euclidean space $\mathbb{R}^n$: if $X = \{\partial_{x_1}, \dots \partial_{x_n}\}$, we use the standard notation $P = P_X$.

Notation: If $X = X_{\mathbb{G}}$ generates a stratified nilpotent Lie algebra associated to a Carnot group $\mathbb{G}$, we denote by $P_{\mathbb{G}}$ the X-perimeter and we call it $\mathbb{G}$-*perimeter*. In particular if $\mathbb{G}$ is an H-type group, we denote $P_{\mathbb{G}} = P_H$ and we call it H-*perimeter*. Moreover, if $X = X_\alpha$ is the family depending on $\alpha \geq 0$ defined in (1.1.15) for Grushin spaces, we denote the X-perimeter as P_α and we call it α-*perimeter*. Moreover in this case we call the α-variation the X_α-variation and we denote it with $|D_\alpha u|(\Omega)$.

Remark 1.2.7. In the paper [14] the authors prove a *Steiner-type formula* in the first Heisenberg group. Here, the H-perimeter appears as the first term of the Taylor expansion as $\varepsilon \to 0^+$ of the volume $\mathcal{L}^3(\Omega_\varepsilon)$, where, for a set $\Omega \subset \mathbb{H}^1$, having C^∞-smooth boundary, we let

$$\Omega_\varepsilon = \{p \in \mathbb{H}^1 : d_{cc}^{\mathbb{H}}(p, \Omega) < \varepsilon\}, \quad d_{cc}^{\mathbb{H}}(p, \Omega) = \inf\{d_{cc}^{\mathbb{H}}(p, q) : q \in \Omega\}.$$

We also define the signed distance from the boundary,

$$\delta(p) = \begin{cases} d_{cc}(p, \partial\Omega) & \text{if } p \in \mathbb{H}^1 \setminus \Omega \\ -d_{cc}(p, \partial\Omega) & \text{if } p \in \bar{\Omega} \end{cases}$$

The Steiner-type formula proved in [14] asserts that, in a suitable open set $Q \subset \mathbb{H}^1$, including no characteristic points of $\partial\Omega$, and for $\varepsilon \geq 0$ small enough, the following Taylor expansion holds true:

$$\mathcal{L}^3(\Omega_\varepsilon \cap Q) = \mathcal{L}^3(\Omega \cap Q) + P_H(\Omega, Q) \cdot \varepsilon + \sum_{i=2}^{\infty} a_i \frac{\varepsilon^i}{i!},$$

where

$$a_i = \int_{\partial\Omega\cap Q} \mathrm{div}_{\mathbb{H}}{}^{(i-1)}(X_{\mathbb{H}}\delta)\|\nabla\delta\|^{-1}\, d\mathcal{H}^2,$$
$$\mathrm{div}^{(0)}_{\mathbb{H}}(X_{\mathbb{H}}\delta) = 1, \quad \mathrm{div}^{(i)}_{\mathbb{H}}(X_{\mathbb{H}}\delta) = \mathrm{div}_{\mathbb{H}}\big(\mathrm{div}^{i-1}_{\mathbb{H}}(X_{\mathbb{H}}\delta)\cdot X_{\mathbb{H}}\delta\big).$$

The notation $\|\nabla\delta\|$ indicates the Euclidean norm of the Euclidean gradient of δ and the measure $\|\nabla\delta\|^{-1}\, d\mathcal{H}^2$ is the one appearing in the representation of the perimeter of smooth sets as in (2.2.5) below and [28, Equation (3.2)]. The coefficients a_i involve the intrinsic mean curvatures of the set Ω (see (3.2.5) below).

X-perimeter in Grushin spaces and Carnot groups: dilations and translations

In the next proposition we show the homogeneity properties of α-perimeter and $\mathbb{G}$-perimeter.

Proposition 1.2.8. *Let $X = \{X_1,\dots,X_r\}$ be either one of the families $X = X_{\mathbb{G}}$ associated to a Carnot group $\mathbb{G}$ or $X = X_\alpha$ for $\alpha \geq 0$ and call Q the associated homogeneous dimension (see* (1.1.7), (1.1.19)*). Let $\delta_\lambda = \delta^{\mathbb{G}}_\lambda$ if $X = X_{\mathbb{G}}$ and $\delta_\lambda = \delta^\alpha_\lambda$ if $X = X_\alpha$. Then the X-perimeter is δ_λ-homogeneous of degree $Q-1$, i.e., for every measurable set $E \subset \mathbb{R}^n$*

$$P_X(\delta_\lambda(E)) = \lambda^{Q-1}P_X(E). \tag{1.2.7}$$

Proof. Let $\lambda > 0$. As in (1.1.25), notice that $J_{\delta_\lambda}p = \lambda^Q$. Hence

$$\int_{\delta_\lambda(E)} \mathrm{div}_X\varphi\, dq = \lambda^Q \int_E (\mathrm{div}_X\varphi)(\delta_\lambda(p))\, dp.$$

Moreover, it easy to see that the vector fields X_j are δ_λ-homogeneous of degree 1, namely

$$X_j(\varphi\circ\delta_\lambda) = \lambda(X_j\varphi)(\delta_\lambda). \tag{1.2.8}$$

Then $\mathrm{div}_X(\varphi\circ\delta_\lambda)(p) = \lambda\, \mathrm{div}_X\varphi(\delta_\lambda(p))$, therefore

$$\int_{\delta_\lambda(E)} \mathrm{div}_X\varphi\, dp = \lambda^{Q-1}\int_E \mathrm{div}_X(\varphi\circ\delta_\lambda)(p)\, dp$$

and then, since δ_λ is a C^1-diffeomorphism,

$$\begin{aligned} P_X(\delta_\lambda(E)) &= \lambda^{Q-1}\sup\Big\{\int_E \mathrm{div}_X(\varphi\circ\delta_\lambda)\, dp : \varphi\in\mathcal{F}_r(\mathbb{R}^n)\Big\} \\ &= \lambda^{Q-1}\sup\Big\{\int_E \mathrm{div}_X\varphi\, dp \;:\; \varphi\in\mathcal{F}_r(\mathbb{R}^n)\Big\} = \lambda^{Q-1}P_X(E). \end{aligned}$$

□

If $X = X_{\mathbb{G}}$ for a Carnot group $\mathbb{G}$ identified with $(\mathbb{R}^n, *)$, the following holds.

Proposition 1.2.9. *Let E be a set of finite $\mathbb{G}$-perimeter. Then*

$$P_{\mathbb{G}}(\tau_p(E)) = P_{\mathbb{G}}(E), \qquad p \in \mathbb{R}^n. \tag{1.2.9}$$

Proof. For any $p \in \mathbb{G}$ and $\varphi \in \mathcal{F}_r(\mathbb{R}^n)$ we have by left invariance of X_i:

$$\begin{aligned}\int_{\tau_p(E)} \mathrm{div}_{\mathbb{G}}\varphi \, dq &= \int_{\tau_p(E)} \sum_{i=1}^r X_i\varphi_i \, dq = \int_E \sum_{i=1}^r X_i\varphi_i(\tau_p(w)) \, dw = \int_E \sum_{i=1}^r X_i(\varphi_i \circ \tau_p)(w) \, dw \\ &= \int_E \mathrm{div}_{\mathbb{G}}(\varphi \circ \tau_p) \, dw.\end{aligned}$$

Hence, since τ_p is a C^1-diffeomorphism,

$$\begin{aligned}P_{\mathbb{G}}(\tau_p(E)) &= \sup\Big\{ \int_E \mathrm{div}_{\mathbb{G}}(\varphi \circ \tau_p) \, dp \ : \ \varphi \in \mathcal{F}_r(\mathbb{R}^n)\Big\} \\ &= \sup\Big\{ \int_E \mathrm{div}_{\mathbb{G}}\varphi \, dp \ : \ \varphi \in \mathcal{F}_r(\mathbb{R}^n)\Big\} = P_{\mathbb{G}}(E).\end{aligned}$$

□

1.2.3. Relations between sub-Riemannian distance and perimeter

In the euclidean setting $(\mathbb{R}^2, d_E)$, the perimeter of a smooth set and the length of its boundary as a curve coincide. On one hand, the euclidean metric represented as a 2-covector on the tangent bundle is $ds^2 = dx^2 + dy^2$. Hence the length of a smooth curve $\gamma : [0,1] \to \mathbb{R}^2$ defined in (1.1.3) is $\ell_E(\gamma) = \int_0^1 \sqrt{\gamma_1(t)^2 + \gamma_2(t)^2}\, dt$. On the other hand, if $E \subset \mathbb{R}^2$ is a bounded smooth set and $\gamma : [0,1] \to \mathbb{R}^2$ is a parametrization of its boundary ∂E, by De Giorgi Structure theorem for sets of finite perimeter (see [89, Theorem 15.9]), $P(E) = \mathcal{H}^1(\partial E)$ where $P(E)$ denotes the euclidean perimeter of E and $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. The connection between these two sides is given by the Area Formula (see [44, Theorem 1, pag. 96]) as follows.

$$\begin{aligned} & P(E) = \\ \Big(\text{De Giorgi}\Big) \quad & = \mathcal{H}^1(\partial E) \\ \Big(\text{Area Formula}\Big) \quad & = \int_0^1 |\dot\gamma|(t)\, dt \\ \Big(\text{definition of } \ell_E(\gamma)\Big) \quad & = \ell_E(\gamma) \end{aligned} \tag{1.2.10}$$

In a Carnot-Carathéodory structure there is no connection in general between the length of smooth curves and perimeter. We show it with the next example in the case of the Grushin plane $(\mathbb{R}^2, d_\alpha)$.

Example 1.2.10 ($P_\alpha(E) \neq \ell_\alpha(\partial E)$)**.** We recall that, in the case of the Grushin plane $(\mathbb{R}^2, d_\alpha)$, the horizontal bundle is given by $\Delta = \operatorname{span}\{\partial_x, |x|^\alpha \partial_y\}$ where (x,y) denotes a point in $\mathbb{R}^2$. The metric

$$ds^2 = dx^2 + \frac{1}{|x|^{2\alpha}} dy^2,$$

defined for $x \neq 0$, is such that $ds^2(\partial_x, |x|^\alpha \partial_y) = 0$, $ds^2(\partial_x, \partial_x) = 1$, $ds^2(|x|^\alpha \partial_y, |x|^\alpha \partial_y) = 1$. Hence the length of a curve $\gamma = (\gamma_1, \gamma_2)$ parametrized on $[0,1]$, is defined as

$$\ell_\alpha(\gamma) = \int_0^1 \sqrt{\gamma_1'(t)^2 + \frac{\gamma_2'(t)^2}{\gamma_1(t)^{2\alpha}}}\, dt.$$

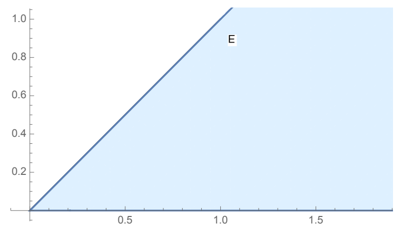


Figure 1.1: The curve γ and a set E having γ as a part of its boundary.

Let $\gamma : [0,1] \to \mathbb{R}^2$, $\gamma(t) = (t,t)$ and suppose $\gamma^* = \gamma([0,1]) \subset \partial E$ where $E \subset \{(x,y) \in \mathbb{R}^2 : x > y\}$ is a smooth set with finite α-perimeter. We have $\gamma'(t) = (1,1)$ and the outer unit normal to E is $N^E = (-1/\sqrt{2}, 1/\sqrt{2})$ at any point in γ^*. We have

$$\ell_\alpha(\gamma) = \int_0^1 \sqrt{1 + \frac{1}{t^{2\alpha}}}\, dt = \int_0^1 \frac{1}{t^\alpha} \sqrt{t^{2\alpha} + 1}\, dt$$

Using the representation formula for the α-perimeter of a smooth set that we will prove in Chapter 2, (see Proposition 2.2.1 below), we have for $\alpha > 0$,

$$\begin{aligned} P_\alpha(E; \{x \le y,\ 0 < x < 1\}) &= \int_{\partial E \cap \{x=y,\ 0<x<1\}} \sqrt{(N_1^E)^2 + |x|^{2\alpha}(N_2^E)^2}\, d\mathcal{H}^1 \\ &= \int_{\gamma^*} \sqrt{\frac{1}{2} + \frac{|x|^{2\alpha}}{2}}\, d\mathcal{H}^1 = \int_0^1 \sqrt{\frac{1}{2} + \frac{t^{2\alpha}}{2}} \sqrt{2}\, dt \\ &= \int_0^1 \sqrt{1 + t^{2\alpha}}\, dt < \int_0^1 \frac{1}{t^\alpha} \sqrt{1 + t^{2\alpha}} = \ell_\alpha(\gamma). \end{aligned}$$

Notice that when $\alpha = 0$ we find $P_\alpha(E) = \ell_\alpha(E)$: in this case, in fact, $\ell_\alpha = \ell_E$ and $P_\alpha = P$.

The step which fails to hold outside the euclidean setting in (1.2.10) is the structure theorem for sets of finite X-perimeter: the relations between X-perimeter and Hausdorff measure in spite of a structure theorem is a current subject of investigation, see [6], [92], [82], [90], [91], [57], [58]. In particular, some recent literature shows that if $\mathbb{G}$ is a Carnot group, the sub-Riemannian perimeter measure can be written in terms of the *spherical Hausdorff measure relative to the CC-distance*. For $0 \le s < \infty$, and $A \subset \mathbb{G}$, this is defined as

$$\mathcal{S}^s_{\mathbb{G}}(A) = \lim_{\delta \to 0} \mathcal{S}^s_{\mathbb{G},\delta}(A),$$

where for $0 < \delta \le \infty$

$$\mathcal{S}^s_{\mathbb{G},\delta}(A) = \inf \Big\{ \sum_{n=1}^{\infty} \operatorname{diam}(B_{\mathbb{G}}(x_n r_n)) \le \delta,\ A \subset \bigcup_{n \in \mathbb{N}} B_{\mathbb{G}}(x_n, r_n) \Big\},$$

and for $E \subset \mathbb{G}$, $\operatorname{diam}(E) = \inf\{d^{\mathbb{G}}_{cc}(x,y),\ x,y \in E\}$ is the *diameter* of E. For any finite $\mathbb{G}$-perimeter set $E \subset \mathbb{G}$ the following representation of the perimeter measure μ_E holds true:

$$\mu_E = \beta(\nu_E)\ \mathcal{S}^{Q-1}_{\mathbb{G}} \llcorner \partial^*_H E,$$

where Q is the homogeneous dimension of the Carnot group $\mathbb{G}$, $\partial^*_H E$ is the so called reduced boundary, and β is a measurable function of the horizontal normal ν_E (see [92, Theorem 1.2]).

The study of the spherical Hausdorff measures relative to the Carnot-Carathéodory distance in comparison with the Hausdorff measures relative to the Euclidean distance in Carnot groups is carried out in [15], [16].

1.3 Non-Sharp Isoperimetric Inequalities

We present here the *isoperimetric inequality* relative to the X-perimeter and the Lebesgue measure for Grushin spaces and Carnot groups on $\mathbb{R}^n$, which relates the X-perimeter and the Lebesgue measure through the homogeneous dimension Q, associated to the dilations. Namely, we show existence of a constant $C > 0$ such that

$$\mathcal{L}^n(E) \le C P_X(E)^{\frac{Q}{Q-1}} \tag{1.3.1}$$

for every set $E \subset \mathbb{R}^n$ of finite Lebesgue measure. In Section 1.1.5, the homogeneous dimension Q, associated to the dilations of Carnot groups and Grushin spaces, is involved to prove the doubling property of the Lebesgue measure

$$\mathcal{L}^n(B^X_{cc}(p,2r)) \le C_D \mathcal{L}^n(B^X_{cc}(p,r)) \quad r > 0$$

which explains the behavior of the Lebesgue measure with respect to the metric structure given by the Carnot-Carathéodory distance. As we will stress in Remark 1.3.5, the main tools to prove an isoperimetric inequality are the *doubling condition* of the Lebesgue measure and a global *Poincaré inequality* on metric balls. Inequality (1.3.1) is obtained for Grushin spaces and Carnot groups in Propositions 1.3.4 and 1.3.6 where we review the classical technique presented in [7, Theorem 3.46].

1.3.1. Poincaré inequalities

Let $X = \{X_1, \dots, X_r\}$ be a Carnot-Carathéodory structure on $\mathbb{R}^n$ and denote by d_{cc} its Carnot-Carthéodory distance. Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that the space $(\mathbb{R}^n, d_{cc}, \mathcal{L}^n)$ *supports a Poincaré inequality in* Ω if every pair u, Xu of a C^1 function and its X-gradient (see (1.2.1)) satisfies

$$\int_B |u(p) - u_B| \, dp \le C_P R \int_B |Xu(p)| \, dp \tag{1.3.2}$$

on each ball $B = B_{cc}(p_0, R) \subset \Omega$ where $C_P = C_P(\Omega) > 0$ is a fixed constant and

$$u_B = \fint_B u(p) \, dp = \frac{1}{\mathcal{L}^n(B)} \int_B u(p) \, dp.$$

The notion of Poincaré inequality has been introduced in the wider class of *metric measure spaces* using *upper gradients*, see for instance [72]. A metric measure space is (S, d, μ) where (S, d) is a metric space and μ is a Borel measure on S. An upper gradient of a Lipschitz function u is a measurable function $g \ge 0$ satisfying

$$|u(x) - u(y)| \le \int_\gamma g \, ds$$

for every pair $x, y \in S$ and all rectifiable curves joining x to y. In [72, Proposition 11.6] it is proved that the notion of *minimal upper gradient* and of X-gradient coincide.

Poincaré inequalities in Carnot-Carathéodory spaces have been first studied motivated by the analysis of second order degenerate elliptic operators $L = \sum_{i,j=1}^n \partial_{x_i}(a_{ij}\partial_{x_j})$ where a_{ij} are measurable coefficients modeled on the Grushin operator and such that a Carnot-Carthéodory distance can be constructed, see [54], [55, Theorem 3.2]. If $X = \{X_1, \dots, X_r\}$, is a family of vector fields on $\mathbb{R}^n$ satisfying the Hörmander condition, the second order operator $L = \sum_{i=1}^r X_j^* X_j$ is known to be hypoelliptic, as established by Hörmander in [77]. In this case, a general result by Jerison (see [79]) guarantees the validity of the Poincaré inequality (1.3.2) in every bounded set $\Omega \subset \mathbb{R}^n$. The constant C_P appearing in (1.3.2) depends on Ω. Poincaré inequalities in bounded sets $\Omega \subset \mathbb{R}^n$ for non smooth vector fields modeled on the Grushin operator are proved in [53].

We concentrate our attention on the families $X = X_\alpha$, defined in (1.1.15), or $X = X_{\mathbb{G}}$, associated to a Carnot group $\mathbb{G}$. In these cases, global Poincaré inequalities are also valid, namely it is possible to replace Ω with $\mathbb{R}^n$ in (1.3.2). For a proof we refer to [83, Section 3] if $X = X_\alpha$, and [72, Proposition 11.17] if $X = X_{\mathbb{G}}$(see also [122]).

Remark 1.3.1. We show that passing from local Poincaré inequalities to global ones is basically due to the presence of dilations.

The constant appearing in the Poincaré inequality in $B_{cc}(0,R)$, $R>0$ is independent of R, namely there exists a constant $C>0$ depending only on the dimension such that for any smooth function u

$$\int_{B_{cc}(0,R)} |u-(u)_{B_{cc}(0,R)}|\,dp \le CR\int_{B_{cc}(0,R)} |Xu|\,dq \quad R>0.$$

Indeed, let $R_1, R_2 > 0$ and $\lambda = R_2/R_1$, so that $R_2 = \lambda R_1$. We call $B_i = B_{cc}(0,R_i)$ for $i=1,2$. Then, by (1.1.26), $B_2 = \delta_\lambda B_1$. Let $u:\mathbb{R}^n \to \mathbb{R}$ be a C^1 function. We have

$$u_{B_2} = \frac{1}{\mathcal{L}^n(B_2)}\int_{B_2} u(q)\,dq = \frac{1}{\lambda^Q\mathcal{L}^n(B_1)}\int_{B_1} u(\delta_\lambda p)\,\lambda^Q\,dp = (u\circ\delta_\lambda)_{B_1},$$

hence

$$\int_{B_2}|u-u_{B_2}|\,dq = \lambda^Q\int_{B_1}|u(\delta_\lambda(p)) - (u\circ\delta_\lambda)_{B_1}|\,dp = \lambda^Q\int_{B_1}|u\circ\delta_\lambda - (u\circ\delta_\lambda)_{B_1}|\,dp.$$

On the other hand, it is easy to see that, by (1.1.15) and (1.1.9), the X-gradient of a C^1-function u is 1-homogeneous with respect to δ_λ, i.e., $X(u\circ\delta_\lambda)(p) = \lambda(Xu)(\delta_\lambda(p))$ for any $p\in\mathbb{R}^n$. Then we get

$$\int_{B_2}|Xu|\,dq = \lambda^Q\int_{B_1}|Xu(\delta_\lambda(p))|\,dp = \lambda^{Q-1}\int_{B_1}|X(u\circ\delta_\lambda)|\,dp.$$

Therefore, using the Poincaré inequality in B_1, since $u\circ\delta_\lambda$ is still C^1, we have

$$\begin{aligned}\int_{B_2}|u-u_{B_2}|\,dq &= \lambda^Q\int_{B_1}|u\circ\delta_\lambda-(u\circ\delta_\lambda)_{B_1}|\,dp \le C_P(B_1)\lambda^Q R_1\int_{B_1}|X(u\circ\delta_\lambda)|\,dp\\ &= C_P(B_1)R_2\int_{B_2}|Xu|(q)\,dq.\end{aligned}$$

Thanks to this Remark, given any ball $B_{cc}(p,r)$, we consider $\Omega = B_{cc}(0,R) \supset\supset B_{cc}(p,r)$. Then, the Poincaré inequality (1.3.2) applies on $B_{cc}(p,r)$ with a constant $C_P = C_P(\Omega) = C_P(n)$, namely

$$\int_{B_{cc}(p,r)}|u-u_{B_{cc}(p,r)}|\,dq \le C_P(n)r\int_{B_{cc}(p,r)}|Xu|(q)\,dq.$$

Remark 1.3.2. By the approximation theorem for BV_X-functions (see Theorem (1.2.4)), and from the Poincaré inequality on $\mathbb{R}^n$, we deduce the following inequality, which we still call a Poincaré inequality

$$\int_B|u-u_B|\,dp \le C_P R|Xu|(B), \qquad u\in BV_X(B), \tag{1.3.3}$$

for every ball $B = B^X_{cc}(p, R)$ where $C_P > 0$ is a fixed constant only depending on the dimension n.

1.3.2. Isoperimetric inequality in Grushin spaces and Carnot groups.

In the following proposition we review and adapt to $(\mathbb{R}^n, d_\alpha)$ the argument given in [7, Theorem 3.46] to prove the global isoperimetric inequality starting from the validity of a Poincaré inequality. We recall that in Grushin spaces the global Poincaré inequality (1.3.3) is valid, implied by [83], and the Lebesgue measure has the following behavior

$$\mathcal{L}^n(B_\alpha(p, R)) \geq c_1 R^Q \quad R > 0 \tag{1.3.4}$$

from Remark 1.1.13. We start from the following covering Lemma.

Lemma 1.3.3. *Let $\alpha \geq 0$, $h, k \in \mathbb{N}$ and $n = h + k$. Let $\bar{R} > 0$. Then there exists a family of balls of radius $\bar{R}$ covering $\mathbb{R}^n$, $\{B_i = B_\alpha(p_i, \bar{R})\}_{i\in\mathbb{N}}$, $p_i = (x_i, y_i) \in \mathbb{R}^n$ and a constant $M = M(\alpha) > 0$ such that*

$$\sum_{i\in\mathbb{N}} P_\alpha(E, B_i) \leq M P_\alpha(E)$$

for any set of finite α-perimeter $E \subset \mathbb{R}^n$.

Proof. We give the proof in the case $h = k = 1$, hence $n = 2$. We recall (1.1.21): $\mathrm{Box}_\alpha(p, R/b) \subset B_\alpha(p, R) \subset \mathrm{Box}_\alpha(p, bR)$ for every $R > 0$, $p = (x, y) \in \mathbb{R}^2$, where for $r > 0$,

$$\mathrm{Box}_\alpha(p, r) = B_E(x, r) \times B_E(y, r(|x| + r)^\alpha).$$

We define the following grid in $\mathbb{R}^2$: for $i, j \in \mathbb{Z}$, let

$$p_{ij} = (x_i, y_{ij}), \quad x_i = 2i\frac{\bar{R}}{b}, \quad y_{ij} = 2jD(i)$$

with

$$D(i) = \left(\frac{\bar{R}}{b}\right)^{\alpha+1} (2|i| + 1)^\alpha.$$

Notice that $\bigcup_{i,j\in\mathbb{Z}} \overline{\mathrm{Box}_\alpha(p_{ij}, \bar{R}/b)} = \mathbb{R}^n$ and

$$\mathrm{Box}_\alpha(p_{ij}, \bar{R}/b) \cap \mathrm{Box}_\alpha(p_{i'j'}, \bar{R}/b) = \varnothing \text{ for } (i, j) \neq (i', j').$$

In fact, given $(i, j) \neq (i', j')$, if $i \neq i'$

$$|x_i - x_{i'}| = 2\frac{\bar{R}}{b}|i - i'| \geq 2\frac{\bar{R}}{b},$$

hence $B_{eucl}(x_i, \frac{\bar{R}}{b}) \cap B_{eucl}(x_{i'}, \frac{\bar{R}}{b}) = \varnothing$. If $i = i'$, $j \neq j'$,

$$|y_{ij} - y_{ij'}| = 2D(i)|j - j'| \geq 2\Big(\frac{\bar{R}}{b}\Big)^{\alpha+1}(2|i|+1)^\alpha = 2\frac{\bar{R}}{b}\Big(|x_i| + \frac{R}{b}\Big)^\alpha,$$

hence $B_{eucl}\Big(y_{ij}, \frac{\bar{R}}{b}\Big(|x_i| + \frac{\bar{R}}{b}\Big)^\alpha\Big) \cap B_{eucl}\Big(y_{ij'}, \frac{\bar{R}}{b}\Big(|x_i| + \frac{\bar{R}}{b}\Big)^\alpha\Big) = \varnothing$.

Now, consider the covering

$$B_{ij} := B_\alpha(p_{ij}, \bar{R}), \quad i, j \in \mathbb{Z}.$$

and define for every $(i,j) \in \mathbb{Z}^2$ the following number

$$\mathcal{I}(i,j) = \#\{(i',j') \in \mathbb{Z}^2 : B_{ij} \cap B_{i'j'} \neq \varnothing\}$$

which represents the number of the balls of the family intersecting B_{ij}. We claim that there exists $M = M(\alpha)$ such that

$$\mathcal{I}(i,j) \leq M(\alpha). \tag{1.3.5}$$

To prove the claim, first notice that

$$\mathcal{I}(i,j) \leq \#\{(i',j') \in \mathbb{Z}^2 : \mathrm{Box}_\alpha(p_{ij}, bR) \cap \mathrm{Box}_\alpha(p_{i'j'}, bR) \neq \varnothing\} =: \mathcal{J}(i,j).$$

Observe that any two boxes $\mathrm{Box}_\alpha(p_{ij}, b\bar{R})$ and $\mathrm{Box}_\alpha(p_{i'j'}, b\bar{R})$ have nonempty intersection if and only if

$$|x_i - x_{i'}| < 2b\bar{R} \quad \text{and} \quad |y_{ij} - y_{i'j'}| < R(i) + R(i') \tag{1.3.6}$$

where

$$R(i) = bR\Big(2|i|\frac{R}{b} + Rb\Big)^\alpha.$$

The first inequality in (1.3.6) is equivalent to $|i - i'| \leq b^2$. If $i = i'$, from (1.3.6) we deduce $|j - j'| < b^{2(\alpha+1)}$ in fact

$$\begin{aligned} 2\Big(\frac{\bar{R}}{b}\Big)^{\alpha+1}(2|i|+1)^\alpha|j-j'| &= |y_{ij} - y_{ij'}| \\ &< 2bR\Big(2|i|\frac{\bar{R}}{b} + \bar{R}b\Big)^\alpha < 2bR\Big(2|i|b\bar{R} + \bar{R}b\Big)^\alpha \\ &= 2(b\bar{R})^{\alpha+1}(2|i|+1)^\alpha. \end{aligned}$$

Notice that the last estimate (for $i = i'$) corresponds to scan the grid vertically, and count the number of boxes $\mathrm{Box}_\alpha(p_{ij'}, b\bar{R})$ intersecting the one centered at p_{ij}. Since in this case $|j - j'|$ has a bound independent on i, to estimate $\mathcal{J}(i,j)$ it is sufficient to multiply such number for the number of stripes such that $|x_i - x_{i'}| < 2b\bar{R}$, namely

$$\mathcal{J}(i,j) < (2b^2)(2b^{2(\alpha+1)}) = 4b^{2(\alpha+2)} = M(\alpha)$$

and (1.3.5) follows. We deduce that there are at most M subfamilies $\{\{B^{\iota}_{ij}\}_{i,j\in Z(\iota)}\}_{\iota=1,\dots,M}$ such that $B^{\iota}_{ij}\cap B^{\iota}_{i'j'}=\varnothing$ for $(i,j)\neq(i',j')$ and

$$\bigcup_{\iota=1}^{M}\bigcup_{i,j\in Z(\iota)} B_{i,j}=\mathbb{R}^n.$$

We are ready for the conclusion. Let $E\subset\mathbb{R}^n$ be a set of finite α-perimeter. Recall that the map $A\mapsto P_\alpha(E,A)$ is a Radon measure measure (see Section 1.2.2), then it is countably additive. Therefore

$$\begin{aligned}\sum_{i,j\in\mathbb{N}} P_\alpha(E,B_{ij}) &= \sum_{\iota=1}^{M}\sum_{i,j\in Z(\iota)} P_\alpha(E,B^{\iota}_{ij})\\ &= \sum_{i,j\in Z(1)} P_\alpha(E,B^{1}_{ij})+\dots+\sum_{i,j\in Z(M)} P_\alpha(E,B^{M}_{ij})\\ &= P_\alpha\Big(E,\bigcup_{i,j\in Z(1)} B^{1}_{ij}\Big)+\dots+P_\alpha\Big(E,\bigcup_{i,j\in Z(M)} B^{M}_{ij}\Big)\le MP_\alpha(E).\end{aligned}$$

□

Proposition 1.3.4. *Let $h,k\ge 1$ be integers and $n=h+k$. Let $\alpha>0$ and $Q=h+(\alpha+1)k$. Then there exists a constant $C>0$ such that*

$$\mathcal{L}^n(E)\le CP_\alpha(E)^{\frac{Q}{Q-1}}$$

for every set $E\subset\mathbb{R}^n$ with finite α-perimeter.

Proof. Let $E\subset\mathbb{R}^n$ be as in the statement. By assumption, χ_E is a function of bounded α-variation. Let $p=(x,y)\in\mathbb{R}^n$ and $R>0$. Then, calling $\beta^\alpha_R(p)=(\chi_E)_{B_\alpha(p,R)}$,

$$\beta^\alpha_R(p)=\frac{1}{\mathcal{L}^n(B_\alpha(p,R))}\int_{B_\alpha(p,R)}\chi_E\,dq=\frac{\mathcal{L}^n(E\cap B_\alpha(p,R))}{\mathcal{L}^n(B_\alpha(p,R))},$$

we have

$$\begin{aligned}&\int_{B_\alpha(p,R)}|\chi_E(q)-(\chi_E)_{B_\alpha(p,R)})|\,dq\\ &\quad=\int_{E\cap B_\alpha(p,R)}|1-\beta^\alpha_R(p)|\,dq+\int_{B_\alpha(p,R)\setminus E}\beta^\alpha_R(p)\,dq\\ &\quad=\mathcal{L}^n(E\cap B_\alpha(p,R))(1-\beta^\alpha_R(p))+\beta^\alpha_R(p)[\mathcal{L}^n(B_\alpha(p,R))-\mathcal{L}^n(E\cap B_\alpha(p,R))]\\ &\quad=\mathcal{L}^n(B_\alpha(p,R))[2\beta^\alpha_R(p)(1-\beta^\alpha_R(p))].\end{aligned}$$

Hence, using the Poincaré inequality (1.3.3), since $\min\{a,1-a\}\le 2a(1-a)$ for $a\in[0,1]$ and $\beta^\alpha_R(p)\in[0,1]$, we obtain

$$\min\{\beta^\alpha_R(p),1-\beta^\alpha_R(p)\}\le 2\beta^\alpha_R(p)(1-\beta^\alpha_R(p))\le\frac{C_P R}{\mathcal{L}^n(B_\alpha(p,R))}P_\alpha(E,B_\alpha(p,R)) \quad (1.3.7)$$

for every $p \in \mathbb{R}^n$ and $R > 0$. Let

$$\bar{R} = \Big(3\frac{C_p}{c_1}P_\alpha(E)\Big)^{\frac{1}{Q-1}}$$

where c_1 is the constant appearing in (1.3.4). Using (1.3.4) with $R = \bar{R}$, we deduce from (1.3.7) that

$$\begin{aligned}\min\{\beta^\alpha_{\bar{R}}(p), 1-\beta^\alpha_{\bar{R}}(p)\} &\le \frac{C_P\bar{R}}{\mathcal{L}^n(B_\alpha(p,\bar{R}))}P_\alpha(E, B_\alpha(p,\bar{R}))\\ &\le \frac{C_P}{C_1\bar{R}^{Q-1}}P_\alpha(E, B_\alpha(p,\bar{R})) \\ &= \frac{C_P P_\alpha(E, B_\alpha(p,\bar{R}))}{C_1\big(3\frac{C_p}{c_1}P_\alpha(E)\big)} \le \frac{1}{3}.\end{aligned} \tag{1.3.8}$$

By a continuity argument, either $\beta^\alpha_{\bar{R}}(p) \in [0, 1/2)$ for every $p \in \mathbb{R}^n$ or $\beta^\alpha_{\bar{R}}(p) \in (1/2, 1]$ for every $p \in \mathbb{R}^n$. Assume $\beta^\alpha_{\bar{R}}(p) \in [0, 1/2)$ for every $p \in \mathbb{R}^n$. Then, by (1.3.7) for $R = \bar{R}$

$$\frac{\mathcal{L}^n(E \cap B_\alpha(p,\bar{R}))}{\mathcal{L}^n(B_\alpha(p,\bar{R}))} = \beta^\alpha_{\bar{R}}(p) \le \frac{C_P\bar{R}}{\mathcal{L}^n(B_\alpha(p,\bar{R}))}P_\alpha(E, B_\alpha(p,\bar{R})).$$

Therefore

$$\mathcal{L}^n(E \cap B_\alpha(p,\bar{R})) \le C_P\bar{R}P_\alpha(E, B_\alpha(p,\bar{R})), \text{ for all } p \in \mathbb{R}^n. \tag{1.3.9}$$

Choosing the covering $\{B_i\}_{i\in\mathbb{N}}$ given by Lemma 1.3.3, we have

$$\begin{aligned}\mathcal{L}^n(E) &\le \sum_{i\in\mathbb{N}}\mathcal{L}^n(E \cap B_\alpha(p_i,\bar{R})) \le C_P\bar{R}\sum_{i\in\mathbb{N}}P_\alpha(E, B_i)\\ &\le C_P\Big(\frac{3C_P}{c_1}\Big)^{\frac{1}{Q-1}}P_\alpha(E)^{\frac{1}{Q-1}}MP_\alpha(E) = CP_\alpha(E)^{\frac{Q}{Q-1}}.\end{aligned}$$

□

Remark 1.3.5. To prove the isoperimetric inequality we used:

(i) the validity of a global Poincaré inequality in $\mathbb{R}^n$ to obtain (1.3.7);

(ii) the lower bound for the Lebesgue measure of CC-balls

$$\mathcal{L}^n(B(p,R)) \ge C_1R^Q \tag{1.3.10}$$

to obtain (1.3.8);

(iii) a covering with bounded overlap.

The same computation can be therefore performed in a metric measure space $(\mathbb{R}^n, d, \mu)$ when a Poincaré inequality holds for the elements of a disjoint covering of $\mathbb{R}^n$, and the measure μ satisfies (1.3.10). In particular if μ is a doubling measure on $\mathbb{R}^n$, condition (1.3.10) is obtained as a consequence.

Proposition 1.3.6. *Let $\mathbb{G}$ be a Carnot group identified with $\mathbb{R}^n$ and Q its homogeneous dimension. Then*

$$\mathcal{L}^n(E) \le C P_{\mathbb{G}}(E)^{\frac{Q}{Q-1}}$$

for any set $E \subset \mathbb{G}$ of finite $\mathbb{G}$-perimeter.

Proof. The proof is the same as in Propostion 1.3.4, where $\mathcal{L}^n(B_\alpha(p,\bar{R})) \ge C_1 R^Q$ in (1.3.8) has to be replaced by $\mathcal{L}^n(B_{\mathbb{G}}(p,\bar{R})) = C_1 R^Q$. The validity of a global Poincaré inequality in Carnot groups is established by Varopoulos in [122]. □

Propositions 1.3.4 and 1.3.6 are related to some well known isoperimetric inequalities. In particular we recover the isoperimetric inequality in $\mathbb{H}^1$, due to Pierre Pansu in 1982, see [108], [111].

Corollary 1.3.7 (Pansu's isoperimetric inequality)**.** *There exists a constant $C > 0$ such that*

$$\mathcal{L}^3(E) \le C P_H(E)^{\frac{4}{3}}$$

for any set $E \subset \mathbb{H}^1$ of finite H-perimeter.

Pansu's proof of the isoperimetric inequality is based on a Santaló type formula in $\mathbb{H}^1$. Santaló type formulas with applications to isoperimetric inequalities in sub-Riemannian spaces are proved in [112].

In 1994, Franchi, Gallot and Wheeden proved the isoperimetric inequality for bounded sets in a class of Carnot-Carathéodory spaces, including Grushin spaces, see [52, Theorem 3.1]. Their proof is based on the weighted Sobolev-Poincaré inequality proved by Franchi, Gutierréz and Wheeden in [53]: the latter holds for generalized Grushin spaces where a weight $\lambda = \lambda(x)$ satisfying some regularity and growth assumptions is involved instead of $|x|^\alpha$. Finally, we recall a result due to Garofalo and Nhieu in 1996, who proved an isoperimetric inequality for Carnot-Carathéodory spaces in [65, Theorem 1.18]. In this Theorem, a proof of a *relative isoperimetric inequality* for the X-perimeter and the Lebesgue measure is also given, i.e., there exists $R_1 > 0$ such that for any CC ball $B = B_{cc}(x_0, R)$ with $0 < R < R_1$, one has

$$\min\{\mathcal{L}^n(E \cap B), \mathcal{L}^n(B \setminus E)\} \le C R |B|^{\frac{-1}{Q}} P_X(E, B).$$

1.4 Isoperimetric problem

1.4.1. Sub-Riemannian isoperimetric problem

Given $v > 0$, the minimization problem

$$\min\{P_X(E) : E \subset \mathbb{R}^n, \text{measurable}, \ \mathcal{L}^n(E) = v\} \tag{1.4.1}$$

is called the *isoperimetric problem* relative to the X-perimeter and the Lebesgue measure. A solution to (1.4.1) is a measurable set $E \subset \mathbb{R}^n$ such that $\mathcal{L}^n(E) = v$ and $P_X(E) \le P_X(F)$ for any measurable set $F \subset \mathbb{R}^n$ such that $\mathcal{L}^n(F) = v$. We call such a set E an *isoperimetric set*.

Let $X = X_{\mathbb{G}}$ be the family of canonical generators for a Carnot group $\mathbb{G}$ or $X = X_\alpha$ for $\alpha \ge 0$. In this case the Lebesgue measure and the X-perimeter are homogeneous with respect to a family of dilations δ_λ as we showed in Propositions 1.1.14 and 1.2.8. This allows us to formulate the isoperimetric problem (1.4.1) in a scale invariant form

$$\min\left\{\mathscr{I}_X(E) = \frac{P_X(E)^Q}{\mathcal{L}^n(E)^{Q-1}} : 0 < \mathcal{L}^n(E) < \infty\right\} \tag{1.4.2}$$

where Q is the homogeneous dimension. Notice that the *isoperimetric ratio* $\mathscr{I}_X$ is δ_λ-homogeneous of degree zero:

$$\mathscr{I}_X(\delta_\lambda(E)) = \frac{P_X(\delta_\lambda(E))^Q}{\mathcal{L}^n(\delta_\lambda(E))^{Q-1}} = \frac{\lambda^{Q(Q-1)}P_X(E)^Q}{\lambda^{Q(Q-1)}\mathcal{L}^n(E)^{Q-1}} = \mathscr{I}_X(E).$$

In this case, isoperimetric sets induce a *sharp isoperimetric inequality*. Namely, defining the *sharp isoperimetric constant*

$$C_I = \inf\{P_X(E) : \mathcal{L}^n(E) = 1\},$$

existence of isoperimetric sets in the class of Lebesgue measurable sets implies the following inequality for any Lebesgue measurable $E \subset \mathbb{R}^n$:

$$P_X(E) \ge C_I \mathcal{L}^n(E)^{\frac{Q-1}{Q}}. \tag{1.4.3}$$

Equality holds if and only if E is an isoperimetric set. The characterization of the equality case in the sharp isoperimetric inequality is therefore equivalent to the isoperimetric problem.

Studying isoperimetric problems means to prove existence of isoperimetric sets, identifying their geometric or topological properties and, if possible, to characterize them.

In Carnot groups, existence of isoperimetric sets is proved by Leonardi and Rigot in [87, Theorem 3.2] using lower semicontinuity and compactness of functions of bounded X-variation as explained in Section 1.2.1. In particular, the compactness theorem gives a minimum once shown that a minimizing sequence can be assumed to be essentially bounded. This is proved in [87] via a *concentration-compactness-type argument*, based on the invariance of the perimeter with respect to left-translations.

The Heisenberg Isoperimetric Problem

The only sub-Riemannian spaces where the isoperimetric problem has been solved are some types of Grushin structures. The first result is in the Grushin plane: in [99, Theorem 1.1], the authors prove existence of solutions to

$$\min\{P_\alpha(E) : E \subset \mathbb{R}^2, \mathcal{L}^2(E) = v\}, \text{ for } v > 0 \text{ fixed,}$$

and they characterize them. Namely, minimizers are unique up to vertical translations and they are obtained through a dilation δ_λ^α of the following set

$$E_{isop}^\alpha = \Big\{(x, y) \in \mathbb{R}^2 : |y| < \varphi_\alpha(|x|) = \int_{\arcsin|x|}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)\, dt,\ |x| < 1\Big\}. \tag{1.4.4}$$

In Chapter 2 we generalize this result to Grushin structures on $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ for $k = 1$ (see Theorem 2.1.4 and Remark 2.5.2) and to H-type groups.

There is a famous conjecture about the shape of isoperimetric sets in the Heisenberg groups, which was formulated by Pansu in 1982 in $\mathbb{H}^1$, see [108], [111]. *Pansu's conjecture* is the following: up to a null set, a left translation, and a dilation, the only isoperimetric set in $\mathbb{H}^1$ is

$$E_{\text{isop}} = \{(z, t) \in \mathbb{H}^1 : |t| < \arccos|z| + |z|\sqrt{1 - |z|^2},\ |z| < 1\}. \tag{1.4.5}$$

Pansu didn't write the formula (1.4.5) for the set E_{isop}, but he described how to construct it as the surface obtained by rotating a geodesic with respect to the distance $d_{cc}^{\mathbb{H}^1}$ between the origin and the point $(0, \pi) \in \mathbb{C} \times \mathbb{R}$. Formula (1.4.5) makes sense in $\mathbb{H}^n$ for any $n \in \mathbb{N}$ and the conjecture can be naturally extended Heisenberg groups of any dimensions.

Only partial proofs of the Pansu's conjecture are known in the literature. The first results on the Heisenberg isoperimetric problem date back to 2008. In [96, Theorem 1.2], the conjecture is confirmed in the class

$$\mathcal{R} = \{E \subset \mathbb{H}^n : \text{ if } (z, t) \in E, \text{ then } (\zeta, t) \in E \text{ for } |\zeta| = |z|\} \tag{1.4.6}$$

of *axially symmetric* sets. Namely, it is proved that the infimum

$$\mathrm{Isop}(\mathcal{R}) = \inf\left\{\frac{P_H(E)^{2n+2}}{\mathcal{L}^{2n+1}(E)^{2n+1}} : E \in \mathcal{R}\right\}$$

is attained. Moreover, up to a dilation, a vertical translation and a $\mathcal{L}^{2n+1}$-negligible set, any axially symmetric isoperimetric set (i.e., a set $E \in \mathcal{R}$ such that the infimum in $\mathrm{Isop}(\mathcal{R})$ is attained) coincides with E_{isop}. On the other hand, in [118, Theorem 7.2] it is proved that if $E \subset \mathbb{H}^1$ is an isoperimetric set, whose boundary is a C^2 smooth surface, then up to a dilation and a left translation, $E = E_{\mathrm{isop}}$.

In [100, Theorem 1.1] Pansu's conjecture is proved in $\mathbb{H}^1$ assuming convexity of the isoperimetric set.

In [116, Theorem 3.1], the following geometric situation is considered. For any $r > 0$, let $D_r = \{(z, 0) \in \mathbb{H}^n : |z \leq r|\}$ be the closed Euclidean disk of radius r contained in $\{z = 0\}$, and $C_r = \{(z, t) \in \mathbb{H}^n : |z| \leq r\}$ be the vertical cylinder over D_r. Let $E \subset \mathbb{H}^n$ be a finite H-perimeter set such that $D_r \subset E \subset C_r$ for some $r > 0$. The author uses a calibration argument to prove that $P_H(E) \geq P_H(E_{\mathrm{isop}})$, and equality holds if and only if $E = E_{\mathrm{isop}}$. In Chapter 3 we refine this argument to prove a stability result for the isoperimetric inequality in $\mathbb{H}^n$.

For a detailed review on the Heisenberg isoperimetric problem we refer to the book [31] and to the lecture notes [97].

CHAPTER 2

Isoperimetric problem in Grushin spaces and H-type groups

In this chapter, we study the isoperimetric problem relative to the Lebesgue measure and the sub-Riemannian perimeter associated to H-type groups and Grushin spaces (see Section 1.1.4 for the notation). For $h, k \geq 1$ integers and $n = h + k$, we consider $\mathbb{R}^n$ endowed with the Carnot-Carathéodory distance associated with the family $X = X_\alpha$, for some $\alpha \geq 0$, or with the family $X = X_{\mathbb{H}}$ of canonical generators of an H-type group $\mathbb{H}$. We recall from Section 1.1.4 that X_α is the family

$$X_\alpha = \{X_1, \dots, X_h, Y_1, \dots, Y_k\}, \qquad X_i = \partial_{x_i}, \qquad Y_j = |x|^\alpha \partial_{y_j}$$

while $X_{\mathbb{H}}$ is defined through (1.1.13). For any $v > 0$ we consider the minimization problem

$$\inf\{P_X(E) : E \subset \mathbb{R}^n \text{ measurable}, \mathcal{L}^n(E) = v\}$$

where P_X is the X-perimeter, denoted by P_α if $X = X_\alpha$ and by P_H if $X = X_{\mathbb{H}}$ (see Section 1.2 for more details on perimeters).

As Example 1.1.9 shows, the m-dimensional Heisenberg group $\mathbb{H}^m$ is itself an H-type group with $h = 2m$ and $k = 1$, then the isoperimetric problem associated to its perimeter is included in our study.

A first connection between the isoperimetric problem relative to P_α and P_H is stressed in [99] (see Subsection 1.4.1). The relation between P_α and P_H comes out noticing that, when $\alpha = 1$, the profile function introduced in (1.4.4)

$$r \mapsto \varphi_\alpha(r) = \int_{\arcsin r}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)\, dt, \quad r \in [0, 1]$$

is $\frac{1}{2} \arccos r + r\sqrt{1 - r^2}$ which is the profile function of the Pansu ball (up to a multiplicative constant 1/2). Moreover, the boundary of E^α_{isop} consists of two geodesics in the metric d_α

for $\alpha = 1$: this facts corresponds to the conjectured property of the Heisenberg isoperimetric set to be foliated by geodesics.

Another point of view is proposed in [12]. Inspired by the lifting technique introduced by Rotschild and Stein (see Section 1.1.4.III), the authors prove a connection between geodesics in $\mathbb{H}^1$ and the following geodesic problems in $(\mathbb{R}^2, d_\alpha)$. Let $A > 0$, $(a,0),(b,0) \in \mathbb{R}^2$, $0 < a \le b$. If $\gamma : [0,1] \to \mathbb{R}^2$ connects $(a,0)$ and $(b,0)$, denote by $\Omega \subset \mathbb{R}^2$ the set whose boundary is composed by γ and the segment connecting $(a,0)$ and $(b,0)$. Consider the minimum problems

$$\begin{aligned} &\inf\{\ell_\alpha(\gamma) : \gamma : [0,1] \to \mathbb{R}^2, \text{absolutely continuous} \\ &\qquad \gamma(0) = (a,0), \gamma(1) = (b,0), \int_\Omega \frac{dx\,dy}{x^2} = A\}, \end{aligned} \tag{AB1}$$

$$\begin{aligned} &\inf\{\ell_\alpha(\gamma) : \gamma : [0,1] \to \mathbb{R}^2, \text{absolutely continuous} \\ &\qquad \gamma(0) = (a,0), \gamma(1) = (b,0), -\int_\gamma \frac{dy}{x} = A\}, \end{aligned} \tag{AB2}$$

where ℓ_α is defined in (1.1.17). The result in [12] is the following: if Γ is the geodesic in $\mathbb{H}^1$ connecting $(0,a,0)$ to $(A,b,0)$, then the solution to (AB2) is the projection of Γ on the Grushin plane (identified with $\mathbb{X}$, the metric quotient of $\mathbb{H}$ introduced in Section 1.1.4.III). Moreover, under some further geometrical assumptions on A, Problem (AB1) has the same unique solution of problem (AB2). The problems addressed here, are not of the same type of the isoperimetric problem considered in [99]: even for a smooth set E, $P_\alpha(E)$ is different from the length of ∂E as a curve in $(\mathbb{R}^2, d_\alpha)$ (see Example 1.2.10) and the volumes in (AB1)-(AB2) are not the Lebesgue measure (neither the Riemannian one).

2.1 Symmetries and statement of the main result

Our approach starts from the analysis of the relation between P_α and P_H, introduced in Section 1.2.2, under some symmetry assumptions. To this purpose we introduce the following notation. Let $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ where $h,k \ge 1$ are integers and $n = h+k$. A point in $\mathbb{R}^n$ is denoted by (x,y) with $x \in \mathbb{R}^h$ and $y \in \mathbb{R}^k$: in the following we use the notation $\mathbb{R}^n = \mathbb{R}^h_x \times \mathbb{R}^k_y$.

Definition 2.1.1 (x-spherical symmetry)**.** We say that a set $E \subset \mathbb{R}^h_x \times \mathbb{R}^k_y$ is x-spherically symmetric if there exists a set $F \subset \mathbb{R}^+ \times \mathbb{R}^k$, called generating set of E, such that

$$E = \Big\{(x,y) \in \mathbb{R}^n : (|x|, y) \in F\Big\}.$$

We denote by $\mathcal{S}_x$ the class of $\mathcal{L}^n$-measurable, x-spherically symmetric sets in $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$.

In Proposition 2.2.3, we prove that if $E \in \mathcal{S}_x$, then $P_H(E) = P_\alpha(E)$ for $\alpha = 1$. This equality leads us to focus on two families of isoperimetric problems involving only the α-perimeter. Given $v > 0$, we set up the isoperimetric problem for the α-perimeter in two different classes:

$$\text{if } h = 1,\ k \geq 1 \text{ and } n = h + k \qquad \min\{P_\alpha(E) : \mathcal{L}^n(E) = v\}, \tag{2.1.1a}$$

$$\text{if } h, k \geq 1 \text{ and } n = h + k \qquad \min\{P_\alpha(E) : E \in \mathcal{S}_x,\ \mathcal{L}^n(E) = v\}. \tag{2.1.1b}$$

Notice that Problem (2.1.1b) includes (in the case $\alpha = 1$, $h = 2m$ and $k = 1$) the axially symmetric case of the isoperimetric problem in Heisenberg group (see [96]).

Recalling the notion of homogeneous dimension $Q = h + (\alpha+1)k$ (see (1.1.19)), we notice that, for any measurable set $E \subset \mathbb{R}^n$ and for all $\lambda > 0$ we have $\mathcal{L}^n(\delta_\lambda^\alpha(E)) = \lambda^Q \mathcal{L}^n(E)$ and $P_\alpha(\delta_\lambda^\alpha(E)) = \lambda^{Q-1} P_\alpha(E)$. Then the isoperimetric ratio

$$\mathscr{I}_\alpha(E) = \frac{P_\alpha(E)^Q}{\mathcal{L}^n(E)^{Q-1}}$$

is homogeneous of degree 0 and the isoperimetric problems (2.1.1a) and (2.1.1b) can be formulated in a scale invariant form as follows

$$\text{if } h = 1,\ k \geq 1 \text{ and } n = h + k \qquad \min\{\mathscr{I}_\alpha(E) : 0 < \mathcal{L}^n(E) < \infty\}, \tag{2.1.2a}$$

$$\text{if } h, k \geq 1 \text{ and } n = h + k \qquad \min\{\mathscr{I}_\alpha(E) : E \in \mathcal{S}_x,\ 0 < \mathcal{L}^n(E) < \infty\}. \tag{2.1.2b}$$

In Sections 2.3-2.5, we study existence, symmetry and regularity of the solutions to Problems (2.1.2a) and (2.1.2b). We call the solutions isoperimetric sets. We introduce the following notion.

Definition 2.1.2 (Schwarz symmetry)**.** Let $E \subset \mathbb{R}^h_x \times \mathbb{R}^k_y$. We say that E is x-Schwarz symmetric if for any $y \in \mathbb{R}^k$ the section of E at y, $E^y = \{x \in \mathbb{R}^h : (x, y) \in E\}$, is a ball centered at 0 in $\mathbb{R}^h$. Namely if for any $y \in \mathbb{R}^k$ there is $r(y) \geq 0$ such that

$$E^y = \{x \in \mathbb{R}^h : |x| < r(y)\}.$$

Equivalently, we say that E is y-Schwarz symmetric if for any $x \in \mathbb{R}^h$ there exists $s(x) \geq 0$ such that

$$E^x := \{y \in \mathbb{R}^k : (x, y) \in E\} = \{y \in \mathbb{R}^k : |y| < s(x)\}.$$

Remark 2.1.3. A set $E \subset \mathbb{R}^h \times \mathbb{R}^k$ is x- and y- Schwarz symmetric if and only if there exists a decreasing function $f : [0, \infty) \to [0, \infty)$ such that $E = \{(x, y) \in \mathbb{R}^n : |y| < f(|x|)\}$.

Indeed, let us first assume that $E = \{(x, y) \in \mathbb{R}^n : |y| < f(|x|)\}$. Let $y_0 \in \mathbb{R}^k$, $x_0 \in \mathbb{R}^h$. For every $x \in \mathbb{R}^h$ such that $|x| < |x_0|$ there holds $f(|x|) \geq f(|x_0|)$. Hence

$$x_0 \in E^{y_0} \iff (x_0, y_0) \in E \iff |y_0| < f(|x_0|) \implies |y_0| < f(|x|) \iff x \in E^{y_0},$$

which proves that E is x-Schwarz symmetric. On the other hand, for any $y \in \mathbb{R}^k$ such that $|y| < |y_0|$ we have

$$y_0 \in E^{x_0} \iff |y_0| < (|x_0|) \implies |y| < f(|x_0|) \iff y \in E^{x_0},$$

which proves that E is y-Schwarz symmetric.

If E is x- and y- Schwarz symmetric, then, for every $x \in \mathbb{R}^h$ there exists $s(x)$ such that $E^x = \{|y| \le s(|x|)\}$. We define $f(|x|) = s(x)$. We are left to prove that $r \mapsto f(r)$ is decreasing. Let $0 < r_1 < r_2$. Assume by contradiction that $f(r_1) < f(r_2)$. Let $y \in \mathbb{R}^k$ be such that $|y| = f(r_2)$. Then the closed set $\overline{E}^y$ contains $\{x \in \mathbb{R}^h : |x| = r_2\}$. Since $r_2 > r_1$, it also contains $\{x \in \mathbb{R}^h : |x| = r_1\}$, hence $(x, y) \in E$ for every $|x| = r_1$ and $|y| = f(r_2)$. Namely, $|y| = f(r_2) \le f(r_1) = f(|x|)$, which contradicts $f(r_1) < f(r_2)$.

The main result of this chapter is the following theorem (see [60]).

Theorem 2.1.4. *Let $h, k \ge 1$ and $n = h + k$. There exist x- and y-Schwarz symmetric minimizers for the isoperimetric problems* (2.1.2a) *and* (2.1.2b)*. Moreover, up to a vertical translation and a null set, any isoperimetric set $E \subset \mathbb{R}^n$ is of the form*

$$E = \{(x, y) \in \mathbb{R}^n : |y| < f(|x|)\}.$$

for a decreasing function $f \in C([0, r_0]) \cap C^\infty(0, r_0) \cap C^2([0, r_0))$, for some $0 < r_0 < \infty$. The function f satisfies the following equation

$$\frac{f'}{\sqrt{r^{2\alpha} + f'^2}} = r^{1-h} \int_0^r s^{2\alpha+h-1} \frac{k-1}{f\sqrt{s^{2\alpha} + f'^2}}\, ds - \frac{C_{hk\alpha}}{h} r \tag{2.1.3}$$

with $C_{hk\alpha} = \frac{QP_\alpha(E)}{(Q-1)\mathcal{L}^n(E)}$.

The proof of Theorem 2.1.4 is given in Theorem 2.4.3 and Proposition 2.5.3.

Notice that equation (2.1.3) is scale invariant, as isoperimetric problems (2.1.2a) (2.1.2b) are formulated in a scale invariant form. In fact, equation (2.1.3) will be derived from the following (see Subsection 2.5.2):

$$f'' = \frac{\alpha f'}{r} + (f'^2 + r^{2\alpha}) \left(\frac{k-1}{f} - (h-1)\frac{f'}{r^{2\alpha+1}} \right) - C_{hk\alpha} \frac{(f'^2 + r^{2\alpha})^{\frac{3}{2}}}{r^{2\alpha}}. \tag{2.1.4}$$

It easy to check that, given $C_1, C_2 > 0$ and a solution f of (2.1.4) for $C = C_1$, the function

$$f_\lambda(r) = \lambda^{\alpha+1} f\Big(\frac{r}{\lambda}\Big), \quad \lambda^\alpha = \frac{C_1}{C_2}.$$

is a solution to (2.1.4) for $C = C_2$. Thanks to this property, in the special case $k = 1$, equation (2.1.3) can be integrated and we have an explicit formula for isoperimetric sets. Namely, using scale invariance of the equation, we can choose the normalization $C_{hk\alpha} = h$, that implies $r_0 = 1$, and in this case the profile function solving (2.1.3) gives the isoperimetric set

$$E^\alpha_{\text{isop}} = \left\{(x,y) \in \mathbb{R}^n : |y| < \int_{\arcsin|x|}^{\pi/2} \sin^{\alpha+1}(s)\, ds, |x| < 1\right\}.$$

This formula generalizes to dimensions $h \geq 2$ the result of [99]. When $k = 1$ and $\alpha = 1$, the profile function satisfying the final condition $f(1) = 0$ is $f(r) = \frac{1}{2}(\arccos r + r\sqrt{1-r^2})$, $r \in [0,1]$. This is the profile function of the Pansu's ball in the Heisenberg group.

With a careful study of the geometric equation (2.1.3) we will highlight some important properties of isoperimetric sets (see Figure 2.1 below and Proposition 2.5.3).

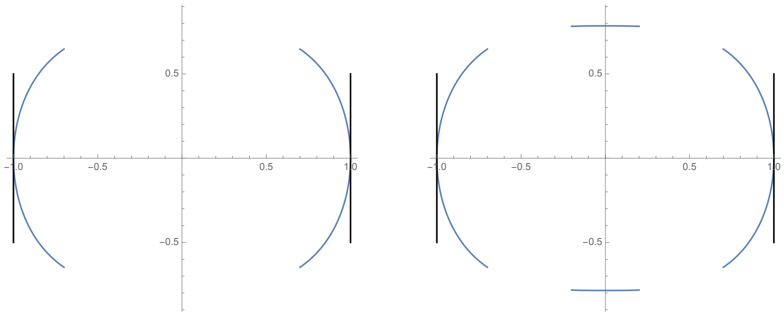


Figure 2.1: The profile of any isoperimetric set closes vertically at r_0, namely $f(r_0) = 0$, $\lim_{r\to r_0^-} f'(r) = -\infty$. Moreover f has the following asymptotic behavior around zero $\lim\limits_{r\to 0^+} \dfrac{f'(r)}{r^{\alpha+1}} = -\dfrac{C_h k\alpha}{h}$. In particular, from this asymptotic behavior we deduce that f is concave around zero (see Section 2.6.1).

In Section 2.6 we comment on the problem of uniqueness of isoperimetric sets.

2.2 Representation and reduction formulas

In this section, we derive some formulas for the representation of α- and H-perimeter of smooth sets and of sets with symmetry. For any open set $A \subset \mathbb{R}^n$ and $m \in \mathbb{N}$, let us define the family of test functions

$$\mathcal{F}_m(A) = \left\{\varphi \in C^1_c(A;\mathbb{R}^m) : \max_{(x,y)\in A} |\varphi(x,y)| \leq 1\right\}.$$

2.2.1. Relation between H-perimeter and α-perimeter

For an open set $E \subset \mathbb{R}^n$ with Lipschitz boundary, the Euclidean outer unit normal $N^E : \partial E \to \mathbb{R}^n$ is defined at $\mathcal{H}^{n-1}$-a.e. point of ∂E, and it can be split in the following way

$$N^E = (N^E_x, N^E_y) \quad \text{with } N^E_x \in \mathbb{R}^h \text{ and } N^E_y \in \mathbb{R}^k.$$

For any $\alpha > 0$, we call the mapping $N_\alpha^E : \partial E \to \mathbb{R}^n$

$$N_\alpha^E = (N_x^E, |x|^\alpha N_y^E) \tag{2.2.1}$$

the α-normal to ∂E.

Proposition 2.2.1. *If $E \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary then the α-perimeter of E in $\mathbb{R}^n$ is*

$$P_\alpha(E) = \int_{\partial E} |N_\alpha^E(x,y)| \, d\mathcal{H}^{n-1}, \tag{2.2.2}$$

where $\mathcal{H}^{n-1}$ is the standard $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

Proof. The inequality

$$P_\alpha(E) \le \int_{\partial E} |N_\alpha^E(x,y)| \, d\mathcal{H}^{n-1}, \tag{2.2.3}$$

follows from Cauchy-Schwarz inequality and the divergence theorem as follows: for any $\varphi \in \mathcal{F}_n(\mathbb{R}^n)$

$$\begin{aligned}\int_E \sum_{i=1}^h \partial_{x_i}\varphi_i + |x|^\alpha \sum_{j=1}^k \partial_{y_j}\varphi_{h+j} \; dxdy &= \int_{\partial E} \sum_{i=1}^h \varphi_i N_{x_i}^E + |x|^\alpha \sum_{j=1}^k N_{y_{h+j}}^E \varphi_{h+j} \; d\mathcal{H}^k(x,y) \\ &= \int_{\partial E} \langle N_\alpha(x,y), \varphi \rangle \; d\mathcal{H}^k(x,y) \le \int_{\partial E} |N_\alpha(x,y)| \; d\mathcal{H}^k(x,y).\end{aligned}$$

By taking the supremum over all $\varphi \in \mathcal{F}_n(\mathbb{R}^n)$ we obtain (2.2.3).

The opposite inequality follows by approximating $N_\alpha^E/|N_\alpha^E|$ with functions in $\mathcal{F}_n(\mathbb{R}^n)$. In fact, by a Lusin-type and Tietze-extension argument, for any $\varepsilon > 0$ there exists $\varphi \in \mathcal{F}_n(\mathbb{R}^n)$ such that

$$\int_{\partial E} \langle N_\alpha^E, \varphi \rangle d\mathcal{H}^{n-1} \ge \int_{\partial E} |N_\alpha^E(x,y)| \, d\mathcal{H}^{n-1} - \varepsilon.$$

The proof of this fact is rather classical and we write it here just for the sake of completeness. By the monotone convergence theorem, it is sufficient to prove that for any $\varepsilon > 0$ and $R > 0$ there exist $\varphi \in \mathcal{F}_n(\mathbb{R}^n)$ such that

$$\int_{\partial E \cap B_R} \langle N_\alpha^E, \varphi \rangle \; d\mathcal{H}^{n-1} \ge \int_{\partial E \cap B_R} |N_\alpha^E| \; d\mathcal{H}^{n-1} - \varepsilon, \tag{2.2.4}$$

where B_R is the open ball centered at 0 with radius R. Given $R > 0$ we define the sets

$$\begin{aligned}\mathcal{N} &= \left\{(x,y) \in \partial E \cap B_R : N^E \text{ is defined}\right\}, \\ \mathcal{Z} &= \{(x,y) \in \mathcal{N} : x = 0 \text{ and } N_x^E(x,y) = 0\}.\end{aligned}$$

Define on $\mathcal{N} \setminus \mathcal{Z}$ the measurable function

$$\tilde{\nu}_\alpha : \mathcal{N} \setminus \mathcal{Z} \to \mathbb{R}^n, \;\; \tilde{\nu_\alpha}(x,y) = \frac{N_\alpha^E(x,y)}{||N_\alpha^E(x,y)||}$$

and extend it to 0 on $((E \cap B_R) \setminus \mathcal{N}) \cup \mathcal{Z}$. This gives a function $\nu_\alpha : \partial E \to \mathbb{R}^n$, which is measurable because it differs from $\tilde{\nu}_\alpha$ on a set of measure zero. Since $\partial E \cap B_R$ has finite $\mathcal{H}^{n-1}$-measure we can apply Lusin's Theorem to get that, given $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \partial E \cap B_R$ such that $\mathcal{H}^{n-1}((\partial E \cap B_R) \setminus K_\varepsilon) < \varepsilon$ and $(\nu_\alpha)_{|K_\varepsilon}$ is continuous.

Now consider a homeomorphism $g : B \to Q$, where B is the unit ball in $\mathbb{R}^n$ and $Q = [-1,1]^n$. By Tietze-Uryshon Theorem applied to each component of ν_α, we can extend the map $g \circ \nu_\alpha : K_\varepsilon \to Q$ to a continuous function from $\mathbb{R}^n$ to Q with compact support in B_R. Composing it with g^{-1} yields a continuous function $\psi \in C_0(B_R, B)$ such that $\psi = \nu_\alpha$ on K_ε. Write

$$\begin{aligned}\int_{\partial E \cap B_R} |N_\alpha^E(x,y)|\, d\mathcal{H}^{n-1}(x,y) &= \int_{K_\varepsilon} \langle \nu_\alpha, N_\alpha^E \rangle d\mathcal{H}^{n-1} + \int_{(\partial E \cap B_R)\setminus K_\varepsilon} |N_\alpha^E|\, d\mathcal{H}^{n-1} \\ &= \int_{(\partial E \cap B_R)} \langle \psi, N_\alpha^E \rangle d\mathcal{H}^{n-1} - \int_{(\partial E \cap B_R)\setminus K_\varepsilon} \left(\langle \psi, N_\alpha^E\rangle - |N_\alpha^E|\right) d\mathcal{H}^{n-1}.\end{aligned}$$

Since $\mathcal{H}^{n-1}((\partial E \cap B_R) \setminus K_\varepsilon) \le \varepsilon$, $\|\psi\|_\infty \le 1$ and $\|N_\alpha^E\|_\infty$ is bounded, there exists $C > 0$ such that

$$\int_{(\partial E \cap B_R)\setminus K_\varepsilon} \left|\langle \psi, N_\alpha^E\rangle - |N_\alpha^E|\right| d\mathcal{H}^{n-1} \le C\varepsilon.$$

Then it follows that

$$\int_{(\partial E \cap B_R)} \langle \psi, N_\alpha^E \rangle d\mathcal{H}^{n-1} \ge \int_{\partial E \cap B_R} |N_\alpha^E(x,y)|\, d\mathcal{H}^{n-1}(x,y) - C\varepsilon.$$

If we approximate uniformly ψ with $\{\psi_s\}_{s\in\mathbb{N}} \in C_0^\infty(B_R, \mathbb{R}^n)$, by using Friederichs mollifiers, we can then fix a test function $\varphi = \psi_s \in \mathcal{F}_n(B_R)$ for s which satisfies (2.2.4). □

If $X_1, \dots, X_h$ are the generators of an H-type Lie algebra, thought of as left-invariant vector fields in $\mathbb{R}^n$ as in (1.1.14), for a set $E \subset \mathbb{R}^n$ with Lipschitz boundary we define the mapping $N_H^E : \partial E \to \mathbb{R}^h$

$$N_H^E = (\langle N^E, X_1 \rangle, \dots, \langle N^E, X_h \rangle).$$

Here, $\langle \cdot, \cdot \rangle$ is the standard scalar product of $\mathbb{R}^n$ and X_i is thought of as an element of $\mathbb{R}^n$ with respect to the standard basis $\partial_1, \dots, \partial_n$. The same argument used to prove (2.2.2) also shows that

$$P_H(E) = \int_{\partial E} |N_H^E(x,y)|\, d\mathcal{H}^{n-1}, \tag{2.2.5}$$

for any set $E \subset \mathbb{R}^n$ with Lipschitz boundary.

Remark 2.2.2. Formulas (2.2.5) and (2.2.2) hold also when ∂E is $\mathcal{H}^{n-1}$-rectifiable.

Proposition 2.2.3. *For any x-spherically symmetric set $E \in \mathcal{S}_x$ there holds $P_H(E) = P_\alpha(E)$ with $\alpha = 1$.*

Proof. By a standard approximation, using the results of [56], it is sufficient to prove the claim for smooth sets, e.g., for a bounded set $E \subset \mathbb{R}^n$ with Lipschitz boundary. By (2.2.5) and (2.2.2), the claim $P_H(E) = P_\alpha(E)$ with $\alpha = 1$ reads

$$P_H(E) = \int_{\partial E} \sqrt{|N_x^E|^2 + |x|^2 |N_y^E|^2} d\mathcal{H}^{n-1}, \tag{2.2.6}$$

where $N^E = (N_x^E, N_y^E) \in \mathbb{R}^h \times \mathbb{R}^k$ is the unit Euclidean normal to ∂E. By the representation formula (2.2.5), we have

$$P_H(E) = \int_{\partial E} \Big(\sum_{i=1}^h \langle X_i, N^E \rangle^2 \Big)^{1/2} d\mathcal{H}^{n-1},$$

where, by (1.1.14), for any $i = 1, \dots, h$

$$\begin{aligned} \langle X_i, N^E \rangle^2 &= \Big(N_{x_i}^E - \sum_{\ell=1}^k \sum_{j=1}^h Q_{ij}^\ell x_j N_{y_\ell}^E \Big)^2 \\ &= (N_{x_i}^E)^2 - 2 N_{x_i}^E \sum_{\ell=1}^k \sum_{j=1}^h Q_{ij}^\ell x_j N_{y_\ell}^E + \Big(\sum_{\ell=1}^k \sum_{j=1}^h Q_{ij}^\ell x_j N_{y_\ell}^E \Big)^2, \end{aligned}$$

and thus

$$\sum_{i=1}^h \langle X_i, N^E \rangle^2 = |N_x^E|^2 - 2 \sum_{\ell=1}^k \sum_{i,j=1}^h Q_{ij}^\ell x_j N_{x_i}^E N_{y_\ell}^E + \sum_{i=1}^h \sum_{\ell,m=1}^k \sum_{j,p=1}^h Q_{ij}^\ell Q_{ip}^m x_j x_p N_{y_\ell}^E N_{y_m}^E. \tag{2.2.7}$$

Since the set E is x-spherically symmetric, the component N_x^E of the normal satisfies the identity

$$N_x^E = \frac{x}{|x|} |N_x^E|. \tag{2.2.8}$$

The bilinear form $Q : \mathbb{R}^h \times \mathbb{R}^h \to \mathbb{R}^k$ is skew-symmetric, i.e., we have $Q(x, x') = -Q(x', x)$ for all $x, x' \in \mathbb{R}^h$ or, equivalently, $Q_{ij}^\ell = -Q_{ji}^\ell$. Using (2.2.8), it follows that for any $\ell = 1, \dots, k$ we have

$$\sum_{i,j=1}^h Q_{ij}^\ell x_j N_{x_i}^E = \frac{|N_x^E|}{|x|} \sum_{i,j=1}^h Q_{ij}^\ell x_i x_j = 0. \tag{2.2.9}$$

Next, we insert into identity (1.1.13), that defines an H-type group, the vector fields

$$X = X' = \sum_{i=1}^h x_i X_i, \quad Y = \sum_{\ell=1}^k N_{y_\ell}^E Y_\ell,$$

where $x \in \mathbb{R}^h$, $N_y^E = (N_{y_1}^E, \dots, N_{y_k}^E)$, and X_i, Y_j are the orthonormal vector fields in (1.1.14). We obtain, together with the definition (1.1.12) of H-type algebra

$$|x|^2 |N_y^E|^2 = \langle X, X' \rangle |Y|^2 = \langle J_Y(X), J_Y(X') \rangle.$$

We write

$$J_Y(X') = J_Y(X) = \sum_{j=1}^h \langle J_Y(X), X_j \rangle X_j = \sum_{j=1}^h \langle Y, [X, X_j] \rangle X_j.$$

where, for any $j = 1, \dots, h$

$$
\begin{aligned}
[X, X_j] &= \sum_{i=1}^{h} \Big[x_i \Big(\partial_{x_i} - \sum_{\ell=1}^{k} \sum_{p=1}^{h} Q^{\ell}_{ip} x_p \partial_{y_\ell} \Big), \partial_{x_j} - \sum_{m=1}^{k} \sum_{s=1}^{h} Q^{m}_{js} x_s \partial_{y_m} \Big] \\
&= \sum_{i=1}^{h} \Big\{ x_i \Big(- \sum_{m=1}^{k} Q^{m}_{ji} \partial_{y_m} \Big) - \delta_{ij} X_i - x_i \Big(- \sum_{\ell=1}^{k} Q^{\ell}_{ij} \partial_{y_\ell} \Big) \Big\} \\
&= 2 \sum_{i=1}^{h} \sum_{m=1}^{k} x_i Q^{m}_{ij} \partial_{y_m} - X_j,
\end{aligned}
$$

hence

$$
\begin{aligned}
\langle Y, [X, X_j] \rangle &= \langle \sum_{\ell=1}^{k} N^E_{y_\ell} \partial_{y_\ell}, 2 \sum_{i=1}^{h} \sum_{m=1}^{k} x_i Q^{m}_{ij} \partial_{y_m} \rangle - \langle \sum_{\ell=1}^{k} N^E_{y_\ell} \partial_{y_\ell}, - \sum_{i=1}^{h} \sum_{m=1}^{k} Q^{m}_{ji} x_i \partial_{y_m} \rangle \\
&= \langle \sum_{\ell=1}^{k} N^E_{y_\ell} \partial_{y_\ell}, \sum_{i=1}^{h} \sum_{m=1}^{k} x_i Q^{m}_{ij} \partial_{y_m} \rangle = \sum_{\ell=1}^{k} \sum_{i=1}^{h} N_{y_\ell} Q^{\ell}_{ij} x_i
\end{aligned}
$$

Therefore

$$
J_Y(X) = \sum_{i,j=1}^{h} \sum_{\ell=1}^{k} N_{y_\ell} Q^{\ell}_{ij} x_i X_j
$$

and we obtain the identity

$$
|x|^2 |N^E_y|^2 = \sum_{\ell,m=1}^{k} \sum_{i,j,p=1}^{h} Q^{\ell}_{ij} Q^{m}_{ip} N^E_{y_\ell} N^E_{y_m} x_j x_p. \tag{2.2.10}
$$

From (2.2.7), (2.2.9), and (2.2.10) we deduce that

$$
\sum_{i=1}^{h} \langle X_i, N^E \rangle^2 = |N^E_x|^2 + |x|^2 |N^E_y|^2,
$$

and formula (2.2.6) follows. □

2.2.2. α-Perimeter for symmetric sets

Thanks to Proposition 2.2.3, from now on we will consider only α-perimeter.

We say that a set $E \subset \mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ is x- and y-spherically symmetric if there exists a set $G \subset \mathbb{R}^+ \times \mathbb{R}^+$ such that

$$
E = \big\{ (x, y) \in \mathbb{R}^n : (|x|, |y|) \in G \big\}.
$$

We call G the generating set of E. In the following we will use the constant

$$
c_{hk} = hk \omega_h \omega_k,
$$

where $\omega_m = \mathcal{L}^m(\{x \in \mathbb{R}^m : |x| < 1\})$, for $m \in \mathbb{N}$.

Proposition 2.2.4. *Let $E \subset \mathbb{R}^n$ be a bounded open set with finite α-perimeter that is x- and y-spherically symmetric with generating set $G \subset \mathbb{R}^+ \times \mathbb{R}^+$. Then we have:*

$$P_\alpha(E) = c_{hk} \sup_{\psi \in \mathcal{F}_2(\mathbb{R}^+ \times \mathbb{R}^+)} \int_G \Big(s^{k-1} \partial_r \big(r^{h-1} \psi_1 \big) + r^{h-1+\alpha} \partial_s \big(s^{k-1} \psi_2 \big) \Big) \, drds. \tag{2.2.11}$$

In particular, if E has Lipschitz boundary then we have:

$$P_\alpha(E) = c_{hk} \int_{\partial G} |(N_r^G, r^\alpha N_s^G)| r^{h-1} s^{k-1} \, d\mathcal{H}^1(r,s), \tag{2.2.12}$$

where $N^G = (N_r^G, N_s^G) \in \mathbb{R}^2$ is the outer unit normal to the boundary $\partial G \subset \mathbb{R}^+ \times \mathbb{R}^+$.

Proof. We prove a preliminary version of (2.2.11). We claim that if E is of finite α-perimeter and x-spherically symmetric with generating set $F \subset \mathbb{R}^+ \times \mathbb{R}^k$, then we have:

$$P_\alpha(E) = h\omega_h \sup_{\psi \in \mathcal{F}_{1+k}(\mathbb{R}^+ \times \mathbb{R}^k)} \int_F \Big(\partial_r \big(r^{h-1} \psi_1 \big) + r^{h-1+\alpha} \sum_{j=1}^k \partial_{y_j} \psi_{1+j} \Big) drdy = Q(F), \tag{2.2.13}$$

where Q is defined via the last identity. For any test function $\psi \in \mathcal{F}_{1+k}(\mathbb{R}^+ \times \mathbb{R}^k)$ we define the test function $\varphi \in \mathcal{F}_n(\mathbb{R}^n)$

$$\varphi(x,y) = \left(\frac{x}{|x|} \psi_1(|x|,y), \psi_2(|x|,y), \dots, \psi_{1+k}(|x|,y) \right) \text{ for } |x| \neq 0, \tag{2.2.14}$$

and $\varphi(0,y) = 0$. For any $i = 1, \dots, h$, $j = 1, \dots, k$, and $x \neq 0$, we have the identities

$$\begin{aligned} \partial_{x_i} \varphi_i(x,y) &= \left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3} \right) \psi_1(|x|,y) + \frac{x_i^2}{|x|^2} \partial_r \psi_1(|x|,y), \\ \partial_{y_j} \varphi_{h+j}(x,y) &= \partial_{y_j} \psi_{1+j}(|x|,y), \end{aligned}$$

and thus, the α-divergence defined by

$$\mathrm{div}_\alpha \varphi(x,y) = \sum_{i=1}^h \frac{\partial \varphi_i(x,y)}{\partial x_i} + |x|^\alpha \sum_{j=1}^k \frac{\partial \varphi_{h+j}(x,y)}{\partial y_j} \tag{2.2.15}$$

satisfies

$$\mathrm{div}_\alpha \varphi(x,y) = \frac{h-1}{|x|} \psi_1(|x|,y) + \partial_r \psi_1(|x|,y) + |x|^\alpha \sum_{j=1}^k \partial_{y_j} \psi_{1+j}(|x|,y). \tag{2.2.16}$$

For any $y \in \mathbb{R}^k$ we define the section $F^y = \{ r > 0 : (r,y) \in F \}$. Using Fubini-Tonelli theorem, spherical coordinates in $\mathbb{R}^h$, the symmetry of E, and (2.2.16) we obtain

$$\begin{aligned} \int_E \mathrm{div}_\alpha \varphi \, dxdy &= \int_{\mathbb{R}^k} \int_{F^y} \int_{|x|=r} \Big(\frac{h-1}{r} \psi_1 + \partial_r \psi_1 + r^\alpha \sum_{j=1}^k \partial_{y_j} \psi_{1+j} \Big) \, d\mathcal{H}^{h-1}(x) drdy \\ &= h\omega_h \int_{\mathbb{R}^k} \int_{F^y} r^{h-1} \Big(\frac{h-1}{r} \psi_1 + \partial_r \psi_1 + r^\alpha \sum_{j=1}^k \partial_{y_j} \psi_{1+j} \Big) \, drdy \\ &= h\omega_h \int_F \partial_r (r^{h-1} \psi_1) + r^{\alpha+h-1} \sum_{j=1}^k \partial_{y_j} \psi_{1+j} \, drdy. \end{aligned} \tag{2.2.17}$$

Because ψ is arbitrary, this proves the inequality $\geq$ in (2.2.13).

We prove the opposite inequality when $E \subset \mathbb{R}^n$ is an x-symmetric bounded open set with smooth boundary. The unit outer normal $N^E = (N_x^E, N_y^E)$ is continuously defined on ∂E. At points $(0,y) \in \partial E$, however, we have $N_x^E(0,y) = 0$ and thus $N_\alpha^E(0,y) = 0$. For any $\varepsilon > 0$ we consider the compact set $K = \{(x,y) \in \partial E : |x| \geq \delta\}$, where $\delta > 0$ is such that $P_\alpha(E; \{|x| = \delta\}) = 0$ and

$$\int_{\partial E \setminus K} |N_\alpha^E(x,y)|\, d\mathcal{H}^{n-1} < \varepsilon. \tag{2.2.18}$$

Let $H \subset \mathbb{R}^+ \times \mathbb{R}^k$ be the generating set of K. By standard extension theorems, there exists $\psi \in \mathcal{F}_{1+k}(\mathbb{R}^+ \times \mathbb{R}^k)$ such that

$$\psi(r,y) = \frac{(N_r^F(r,y), r^\alpha N_y^F(r,y))}{|(N_r^F(r,y), r^\alpha N_y^F(r,y)|} \quad \text{for } (r,y) \in H.$$

The mapping $\varphi \in \mathcal{F}_n(\mathbb{R}^n)$ introduced in (2.2.14) satisfies

$$\varphi(x,y) = \frac{N_\alpha^E(x,y)}{|N_\alpha^E(x,y)|}, \quad \text{for } (x,y) \in K. \tag{2.2.19}$$

Then, by identity (2.2.17), the divergence theorem, (2.2.19), (2.2.18), and (2.2.2) we have

$$\begin{aligned}
Q(F) &\geq \int_F \Big(\partial_r\big(r^{h-1}\psi_1\big) + r^{h-1+\alpha}\sum_{j=1}^k \partial_{y_j}\psi_{1+j}\Big) dr dy \\
&= \int_E \mathrm{div}_\alpha \varphi \, dx dy = \int_{\partial E} \langle \varphi, N_\alpha^E\rangle \, d\mathcal{H}^{n-1} \\
&= \int_K |N_\alpha^E(x,y)|\, d\mathcal{H}^{n-1} + \int_{\partial E \setminus K} \langle \varphi, N_\alpha^E\rangle \, d\mathcal{H}^{n-1} \\
&\geq P_\alpha(E) - 2\varepsilon.
\end{aligned}$$

This proves (2.2.13) when ∂E is smooth. The general case follows by approximation. Let $E \subset \mathbb{R}^n$ be a set of finite α-perimeter and finite Lebesgue measure that is x-symmetric with generating set $F \subset \mathbb{R}^+ \times \mathbb{R}^k$. By [56, Theorem 2.2.2], there exists a sequence $(E_j)_{j\in\mathbb{N}}$ such that each E_j is of class C^∞

$$\lim_{j\to\infty} \mathcal{L}^n(E_j \Delta E) = 0 \quad \text{and} \quad \lim_{j\to\infty} P_\alpha(E_j) = P_\alpha(E).$$

Each E_j can be also assumed to be x-spherically symmetric with generating set $F_j \subset \mathbb{R}^+ \times \mathbb{R}^k$. Then we also have

$$\lim_{j\to\infty} \mathcal{L}^{1+k}(F_j \Delta F) = 0.$$

By lower semicontinuity and (2.2.13) for the smooth case, we have

$$Q(F) \leq \liminf_{j\to\infty} Q(F_j) = \lim_{j\to\infty} P_\alpha(E_j) = P_\alpha(E).$$

This concludes the proof of (2.2.13) for any set E with finite α-perimeter.

The general formula (2.2.11) for sets that are also y-spherically symmetric can be proved in a similar way and we can omit the details.

Formula (2.2.12) for sets E with Lipschitz boundary follows from (2.2.11) with the same argument sketched in the proof of Proposition 2.2.2. □

2.2.3. α-Perimeter in the case $h = 1$

When $h = 1$ there exists a change of coordinates that transforms α-perimeter into the standard perimeter (see [99] for the case of the plane $h = k = 1$). Let $n = 1 + k$ and consider the mappings $\Phi, \Psi : \mathbb{R}^n \to \mathbb{R}^n$

$$\Psi(x,y) = \left(\operatorname{sgn}(x)\frac{|x|^{\alpha+1}}{\alpha+1}, y\right) \quad \text{and} \quad \Phi(\xi,\eta) = \left(\operatorname{sgn}(\xi)|(\alpha+1)\xi|^{\frac{1}{\alpha+1}}, \eta\right).$$

Then we have $\Phi \circ \Psi = \Psi \circ \Phi = \mathrm{Id}_{\mathbb{R}^n}$.

Proposition 2.2.5. *Let $h = 1$ and $n = 1 + k$. For any measurable set $E \subset \mathbb{R}^n$ we have*

$$P_\alpha(E) = \sup\left\{ \int_{\Psi(E)} \operatorname{div}\psi \, d\xi d\eta \, : \, \psi \in \mathcal{F}_n(\mathbb{R}^n)\right\}. \tag{2.2.20}$$

Proof. First notice that the supremum in the right hand side can be equivalently computed over all vector fields $\psi : \mathbb{R}^n \to \mathbb{R}^n$ in the Sobolev space $W_0^{1,1}(\mathbb{R}^n;\mathbb{R}^n)$ such that $\|\psi\|_\infty \leq 1$.

For any $\varphi \in \mathcal{F}_n(\mathbb{R}^n)$, let $\psi = \varphi \circ \Phi$. Then for any $j = 1, \ldots, k = n - 1$, we have

$$\begin{aligned} \partial_\xi \psi_1(\xi,\eta) &= \partial_\xi\big(\varphi_1 \circ \Phi\big)(\xi,\eta) = |(\alpha+1)\xi|^{-\frac{\alpha}{\alpha+1}} \partial_x \varphi_1(\Phi(\xi,\eta)), \\ \partial_{\eta_j} \psi_{1+j}(\xi,\eta) &= \partial_{\eta_j}\big(\varphi_{1+j} \circ \Phi\big)(\xi,\eta) = \partial_{y_j} \varphi_{1+j}(\Phi(\xi,\eta)). \end{aligned} \tag{2.2.21}$$

In particular, we have $\psi \in W_0^{1,1}(\mathbb{R}^n;\mathbb{R}^n)$ and $\|\psi\|_\infty \leq 1$. Then, the standard divergence of ψ satisfies

$$\operatorname{div}\psi(\xi,\eta) = |(\alpha+1)\xi|^{-\frac{\alpha}{\alpha+1}} \operatorname{div}_\alpha \phi(\Phi(\xi,\eta)).$$

The determinant Jacobian of the change of variable $(x,y) = \Phi(\xi,\eta)$ is

$$|\det J\Phi(\xi,\eta)| = |(\alpha+1)\xi|^{-\frac{\alpha}{\alpha+1}}. \tag{2.2.22}$$

and thus we obtain

$$\begin{aligned} \int_E \operatorname{div}_\alpha \varphi(x,y)\, dxdy &= \int_{\Psi(E)} \operatorname{div}_\alpha \varphi(\Phi(\xi,\eta)) |\det J\Phi(\xi,\eta)| \, d\xi d\eta \\ &= \int_{\Psi(E)} \operatorname{div}\psi(\xi,\eta)\, d\xi d\eta. \end{aligned} \tag{2.2.23}$$

The claim follows.

□

2.3 Rearrangements

In this section, we prove various rearrangement inequalities for α-perimeter in $\mathbb{R}^n$. We consider first the case $h = 1$. In this case, there are a Steiner type rearrangement in the x-variable and a Schwarz rearrangement in the y variables that reduce the isoperimetric problem in $\mathbb{R}^n$ to a problem for Lipschitz graphs in the first quadrant $\mathbb{R}^+ \times \mathbb{R}^+$. Then we consider dimensions $h \geq 2$, where we can rearrange sets in $\mathbb{R}^h$ that are already x-spherically symmetric.

2.3.1. Rearrangement in the case $h = 1$

Let $h = 1$ and $n = 1 + k$. We say that a set $E \subset \mathbb{R}^n$ is x-symmetric if $(x, y) \in E$ implies $(-x, y) \in E$; we say that E is x-convex if the section $E^y = \{x \in \mathbb{R} : (x, y) \in E\}$ is an interval for every $y \in \mathbb{R}^k$; finally, we say that E is y-Schwarz symmetric if for every $x \in \mathbb{R}$ the section $E^x = \{y \in \mathbb{R}^k : (x, y) \in E\}$ is an (open) Euclidean ball in $\mathbb{R}^k$ centered at the origin.

Theorem 2.3.1. *Let $h = 1$ and $n = 1 + k$. For any set $E \subset \mathbb{R}^n$ such that $P_\alpha(E) < \infty$ and $0 < \mathcal{L}^n(E) < \infty$ there exists an x-symmetric, x-convex, and y-Schwarz symmetric set $E^* \subset \mathbb{R}^n$ such that $P_\alpha(E^*) \leq P_\alpha(E)$ and $\mathcal{L}^n(E^*) = \mathcal{L}^n(E)$.*

Moreover, if $P_\alpha(E^*) = P_\alpha(E)$ then E is x-symmetric, x-convex and there exist functions $c : [0, \infty) \to \mathbb{R}^k$ and $f : [0, \infty) \to [0, \infty]$ such that for $\mathcal{L}^1$-a.e. $x \in \mathbb{R}$ we have

$$E^x = \{y \in \mathbb{R}^k : |y - c(|x|)| < f(|x|)\}. \tag{2.3.1}$$

Proof. By Proposition 2.2.5, the set $F = \Psi(E) \subset \mathbb{R}^n$ satisfies $P(F) = P_\alpha(E)$, where P stands for the standard perimeter in $\mathbb{R}^n$. We define the measure μ on $\mathbb{R}^n$

$$\mu(F) = \int_F |(\alpha + 1)\xi|^{-\frac{\alpha}{\alpha+1}}\, d\xi d\eta. \tag{2.3.2}$$

Then, by (2.2.22) we also have the identity $\mu(F) = \mathcal{L}^n(E)$.

We rearrange the set F using Steiner symmetrization in direction ξ. Namely, we let

$$F_1 = \{(\xi, \eta) \in \mathbb{R}^n : |\xi| < \mathcal{L}^1(F^\eta)/2\},$$

where $F^\eta = \{\xi \in \mathbb{R} : (\xi, \eta) \in F\}$. The set F_1 is ξ-symmetric and ξ-convex. By classical results on Steiner symmetrization we have $P(F_1) \leq P(F)$ and the equality $P(F_1) = P(F)$ implies that F is ξ-convex: namely, a.e. section F^η is (equivalent to) an interval.

The μ-volume of F_1 is

$$\mu(F_1) = \int_{F_1} |(\alpha + 1)\xi|^{-\frac{\alpha}{\alpha+1}} d\xi d\eta = \int_{\mathbb{R}^k} \Big(\int_{F_1^\eta} |(\alpha + 1)\xi|^{-\frac{\alpha}{\alpha+1}} d\xi \Big) d\eta.$$

For any measurable set $I \subset \mathbb{R}$ with finite measure, the symmetrized set $I^* = (-\mathcal{L}^1(I)/2, \mathcal{L}^1(I)/2)$ satisfies the following inequality (see [99], page 361)

$$\int_I |\xi|^{-\frac{\alpha}{\alpha+1}} d\xi \le \int_{I^*} |\xi|^{-\frac{\alpha}{\alpha+1}} d\xi. \tag{2.3.3}$$

Moreover, if $\mathcal{L}^1(I \Delta I^*) > 0$ then the inequality is strict. This implies that $\mu(F_1) \ge \mu(F)$ and the inequality is strict if F is not equivalent to an ξ-symmetric and ξ-convex set.

We rearrange the set F_1 using Schwarz symmetrization in $\mathbb{R}^k$, namely we let

$$F_2 = \Big\{(\xi, \eta) \in \mathbb{R}^n : |\eta| < \Big(\frac{\mathcal{L}^k(F_1^\xi)}{\omega_k}\Big)^{\frac{1}{k}}\Big\}.$$

By classical results on Schwarz rearrangement, we have $P(F_2) \le P(F_1)$ and the equality $P(F_2) = P(F_1)$ implies that a.e. section F_1^ξ is an Euclidean ball

$$F_1^\xi = \{\eta \in \mathbb{R}^k : |\eta - d(|\xi|)| < \varrho(|\xi|)\} \tag{2.3.4}$$

for some $d(|\xi|) \in \mathbb{R}^k$ and $\varrho(|\xi|) \in [0, \infty]$. By Fubini-Tonelli theorem, the μ-volume is preserved:

$$\mu(F_2) = \int_{\mathbb{R}} |(\alpha+1)\xi|^{-\frac{\alpha}{\alpha+1}} \mathcal{L}^k(F_2^\xi) d\xi = \int_{\mathbb{R}} |(\alpha+1)\xi|^{-\frac{\alpha}{\alpha+1}} \mathcal{L}^k(F_1^\xi) d\xi = \mu(F_1). \tag{2.3.5}$$

Recall that $\delta_\lambda^\alpha(x, y) = (\lambda x, \lambda^{\alpha+1} y)$. The set $E^* = \delta_\lambda^\alpha(\Phi(F_2))$, with $\lambda > 0$ such that $\mathcal{L}^n(E^*) = \mathcal{L}^n(E)$, satisfies the claims in the statement of the theorem. In fact, we have $0 < \lambda \le 1$ because

$$\mathcal{L}^n(\Phi(F_2)) = \mu(F_2) = \mu(F_1) \ge \mu(F) = \mathcal{L}^n(E),$$

and then, by the scaling property of α-perimeter we have

$$P_\alpha(E^*) = \lambda^{Q-1} P_\alpha(\Phi(F_2)) \le P_\alpha(\Phi(F_2)) = P(F_2) \le P(F_1) \le P(F) = P_\alpha(E).$$

This proves the first part of the theorem.

If $P_\alpha(E^*) = P_\alpha(E)$ then we have $P(F_2) = P(F_1)$ and $\lambda = 1$. From the first equality we deduce that the sections F_1^ξ are of the form (2.3.4) and claim (2.3.1) holds with $c(|x|) = d\big(|x|^{\alpha+1}/(\alpha+1)\big)$ and $f(|x|) = \varrho\big(|x|^{\alpha+1}/(\alpha+1)\big)$. From $\lambda = 1$ we deduce that

$$\mu(F) = \mathcal{L}^n(E) = \mathcal{L}^n(E^*) = \mathcal{L}^n(\Phi(F_2)) = \mu(F_2) = \mu(F_1),$$

and thus F is ξ-symmetric and ξ-convex. The same holds then for E.

$\square$

2.3.2. Rearrangement in the case $h \geq 2$

We prove the analogous of Theorem 2.3.1 when $h \geq 2$. We need to start from a set $E \subset \mathbb{R}^n$ that is x-spherically symmetric

$$E = \{(x, y) \in \mathbb{R}^n : (|x|, y) \in F\}$$

for some generating set $F \subset \mathbb{R}^+ \times \mathbb{R}^k$.

By the proof of Proposition 2.2.4, see (2.2.13), we have the identity $P_\alpha(E) = Q(F)$, where

$$Q(F) = h\omega_h \sup_{\psi \in \mathcal{F}_{1+k}(\mathbb{R}^+ \times \mathbb{R}^k)} \int_F \Big(\partial_r \big(r^{h-1} \psi_1 \big) + r^{h-1+\alpha} \sum_{j=1}^k \partial_{y_j} \psi_{1+j} \Big) dr dy. \tag{2.3.6}$$

Our goal is to improve the x-spherical symmetry to the x-Schwarz symmetry. To obtain the Schwarz symmetry, we use the radial rearrangement technique introduced in [98].

Theorem 2.3.2. *Let $h \geq 2$, $k \geq 1$ and $n = h + k$. For any set $E \subset \mathbb{R}^n$ that is x-spherically symmetric and such that $P_\alpha(E) < \infty$ and $0 < \mathcal{L}^n(E) < \infty$ there exists an x- and y-Schwarz symmetric set $E^* \subset \mathbb{R}^n$ such that $P_\alpha(E^*) \leq P_\alpha(E)$ and $\mathcal{L}^n(E^*) = \mathcal{L}^n(E)$.*

Moreover, if $P_\alpha(E^*) = P_\alpha(E)$ then E is x-Schwarz symmetric and there exist functions $c : [0, \infty) \to \mathbb{R}^k$ and $f : [0, \infty) \to [0, \infty]$ such that, up to a negligible set, we have

$$E = \{(x, y) \in \mathbb{R}^n : |y - c(|x|)| < f(|x|)\}. \tag{2.3.7}$$

Proof. Let $F \subset \mathbb{R}^+ \times \mathbb{R}$ be the generating set of E. We define the volume of F via the following formula

$$V(F) = \omega_h \int_F r^{h-1} dr dy = \mathcal{L}^n(E).$$

We rearrange F in the coordinate r using the linear density $r^{h-1+\alpha}$ that appears, in (2.3.6), in the part of divergence depending on the coordinates y. Namely, we define the function $g : \mathbb{R}^k \to [0, \infty]$ via the identity

$$\frac{1}{h+\alpha} g(y)^{h+\alpha} = \int_0^{g(y)} r^{h-1+\alpha} dr = \int_{F_y} r^{h-1+\alpha} dr, \tag{2.3.8}$$

and we let

$$F^\sharp = \big\{ (r, y) \in \mathbb{R}^+ \times \mathbb{R}^k : 0 < r < g(y) \big\}.$$

We claim that $Q(F^\sharp) \leq Q(F)$ and $V(F^\sharp) \geq V(F)$, with equality $V(F^\sharp) = V(F)$ holding if and only if $F^\sharp = F$, up to a negligible set.

For any open set $A \subset \mathbb{R}^+ \times \mathbb{R}^k$, we define

$$\begin{aligned} Q_0(F; A) &= \sup_{\psi \in \mathcal{F}_1(A)} \int_F \partial_r \big(r^{h-1} \psi \big)\, dr dy, \\ Q_j(F; A) &= \sup_{\psi \in \mathcal{F}_1(A)} \int_F r^{h-1+\alpha} \partial_{y_j} \psi \, dr dy, \quad j = 1, \dots, k. \end{aligned} \tag{2.3.9}$$

The open sets mappings $A \mapsto Q_j(F;A)$, $j = 0,1,\dots,k$, extend to Borel measures. For any Borel set $B \subset \mathbb{R}^k$ and $j = 0,1,\dots,k$, we define the measures

$$\mu_j(B) = Q_j(F;\mathbb{R}^+ \times B),$$
$$\mu_j^\sharp(B) = Q_j(F^\sharp;\mathbb{R}^+ \times B).$$

By Step 1 and Step 2 of the proof of Theorem 1.5 in [98], see page 106, we have $\mu_j^\sharp(B) \le \mu_j(B)$ for any Borel set $B \subset \mathbb{R}^k$ and for any $j = 0,1,\dots,k$. It follows that the vector valued Borel measures $\mu = (\mu_0,\dots,\mu_k)$ and $\mu^\sharp = (\mu_0^\sharp,\dots,\mu_k^\sharp)$ satisfy

$$|\mu^\sharp|(\mathbb{R}^k) \le |\mu|(\mathbb{R}^k),$$

where $|\cdot|$ denotes the total variation. This is equivalent to $Q(F^\sharp) \le Q(F)$.

We claim that for any $y \in \mathbb{R}^k$ we have

$$\frac{1}{h} g(y)^h = \int_{F_y^\sharp} r^{h-1}\, dr \ge \int_{F_y} r^{h-1}\, dr, \tag{2.3.10}$$

with strict inequality unless $F_y^\sharp = F_y$ up to a negligible set. From (2.3.10), by Fubini-Tonelli theorem it follows that $V(F^\sharp) \ge V(F)$ with strict inequality unless $F^\sharp = F$ up to a negligible set. By (2.3.8), claim (2.3.10) is equivalent to

$$\Big((h+\alpha)\int_{F_y} r^{h-1+\alpha} dr\Big)^{\frac{1}{h+\alpha}} \ge \Big(h\int_{F_y} r^{h-1} dr\Big)^{\frac{1}{h}}, \tag{2.3.11}$$

and this inequality holds for any measurable set $F_y \subset \mathbb{R}^+$, for any $h \ge 2$, and $\alpha > 0$, by Example 2.5 in [98]. Moreover, we have equality in (2.3.11) if and only if $F_y = (0, g(y))$.

Let $E_1^\sharp \subset \mathbb{R}^n$ be the x-Schwarz symmetric set with generating set $F^\sharp$. Then we have

$$\mathcal{L}^n(E_1^\sharp) = V(F^\sharp) \ge V(F) = \mathcal{L}^n(E),$$

with strict inequality unless $F^\sharp = F$. Then there exists $0 < \lambda \le 1$ such that the set $E^\sharp = \delta_\lambda^\alpha(E_1^\sharp)$ satisfies $\mathcal{L}^n(E^\sharp) = \mathcal{L}^n(E)$. Since $\lambda \le 1$, we also have

$$P_\alpha(E^\sharp) = \lambda^{Q-1} P_\alpha(E_1^\sharp) \le P_\alpha(E_1^\sharp) = Q(F^\sharp) \le Q(F) = P_\alpha(E).$$

If $P_\alpha(E^\sharp) = P_\alpha(E)$ then it must be $\lambda = 1$ and thus $F^\sharp = F$, that in turn implies $E^\sharp = E$, up to a negligible set.

Now the theorem can be concluded applying to $E^\sharp$ a Schwarz rearrangement in the variable $y \in \mathbb{R}^k$. This rearrangement is standard, see the general argument in [98]. The resulting set $E^* \subset \mathbb{R}^n$ satisfies $P_\alpha(E^*) \le P_\alpha(E)$ and also the other claims in the theorem.

□

2.4 Existence of isoperimetric sets

In this section, we prove existence of solutions to the isoperimetric problem for α-perimeter and H-perimeter. When $h \geq 2$, we prove the existence of solutions in the class of x-spherically symmetric sets. The proof is based on a concentration-compactness argument. This is a classical tool to prove existence of solutions to variational problems, used for example by Leonardi and Rigot to prove existence of isoperimetric sets in Carnot groups in [87]. Differently from their proof, in the case of Grushin spaces we have to deal with the lack of left-translations: we use the same "cutting technique" as Fusco Maggi and Pratelli in [63, Lemma 5.1], where invariance under left translations of the perimeter measure is not necessary.

For any set $E \subset \mathbb{R}^n$ and $t > 0$, we let

$$\begin{aligned} E^x_{t-} &= \{(x,y) \in E : |x| < t\} \quad \text{and} \quad E^x_t = \{(x,y) \in E : |x| = t\}, \\ E^y_{t-} &= \{(x,y) \in E : |y| < t\} \quad \text{and} \quad E^y_t = \{(x,y) \in E : |y| = t\}. \end{aligned} \tag{2.4.1}$$

We also define

$$v^x_E(t) = \mathcal{H}^{n-1}(E^x_t), \tag{2.4.2}$$

and

$$v^y_E(t) = \int_{E^y_t} |x|^\alpha d\mathcal{H}^{n-1}. \tag{2.4.3}$$

In the following, we use the short notation $\{|x| < t\} = \{(x,y) \in \mathbb{R}^n : |x| < t\}$ and $\{|y| < t\} = \{(x,y) \in \mathbb{R}^n : |y| < t\}$.

Proposition 2.4.1. *Let $E \subset \mathbb{R}^n$ be a set with finite measure and finite α-perimeter. Then for a.e. $t > 0$ we have*

$$P_\alpha(E^x_{t-}) = P_\alpha(E; E^x_{t-}) + v^x_E(t) \quad \textit{and} \quad P_\alpha(E^y_{t-}) = P_\alpha(E; E^y_{t-}) + v^y_E(t). \tag{2.4.4}$$

Proof. We prove the claim for E^y_{t-}. Let $\{\phi_\varepsilon\}_{\varepsilon>0}$ be a standard family of mollifiers in $\mathbb{R}^n$ and let

$$f_\varepsilon(z) = \int_E \phi_\varepsilon(|z-w|)dw, \quad z \in \mathbb{R}^n.$$

Then $f_\varepsilon \in C^\infty(\mathbb{R}^n)$ and $f_\varepsilon \to \chi_E$ in $L^1(\mathbb{R}^n)$ for $\varepsilon \to 0$. Therefore, by the coarea formula we also have, for a.e. $t > 0$ and possibly for a suitable infinitesimal sequence of ε's,

$$\lim_{\varepsilon \to 0} \int_{\{|y|=t\}} |f_\varepsilon - \chi_E| d\mathcal{H}^{n-1} = 0. \tag{2.4.5}$$

Since E has finite α-perimeter, the set $\{t > 0 : P_\alpha(E; \{|y| = t\}) > 0\}$ is at most countable, and thus

$$P_\alpha(E; \{|y| = t\}) = 0 \quad \text{for a.e. } t > 0. \tag{2.4.6}$$

We use the notation $\nabla_\alpha f_\varepsilon = (X_1 f_\varepsilon, \dots, X_h f_\varepsilon, Y_1 f_\varepsilon, \dots Y_k f_\varepsilon)$, where X_i, Y_j are the vector fields (1.1.15). By the divergence Theorem, for any $\varphi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ we have

$$\begin{aligned}\int_{\{|y|<t\}} f_\varepsilon(z)\mathrm{div}_\alpha\varphi(z)\,dz &= \int_{\{|y|<t\}} \big(\mathrm{div}_\alpha(f_\varepsilon\varphi) - \langle\nabla_\alpha f_\varepsilon, \varphi\rangle\big)dz \\ &= -\int_{\{|y|=t\}} f_\varepsilon(z)|x|^\alpha\langle N, \varphi(z)\rangle d\mathcal{H}^{n-1} - \int_{\{|y|<t\}} \langle\nabla_\alpha f_\varepsilon, \varphi\rangle dz,\end{aligned} \tag{2.4.7}$$

where $N = (0, -y/|y|)$ is the inner unit normal of $\{|y| < t\}$. For any $t > 0$, we have

$$\lim_{\varepsilon\to 0}\int_{\{|y|<t\}} f_\varepsilon(z)\mathrm{div}_\alpha\varphi(z)\,dz = \int_{E^y_{t-}} \mathrm{div}_\alpha\varphi(z)\,dz, \tag{2.4.8}$$

and, for any $t > 0$ satisfying (2.4.5),

$$\lim_{\varepsilon\to 0}\int_{\{|y|=t\}} f_\varepsilon(z)|x|^\alpha\langle N, \varphi(z)\rangle d\mathcal{H}^{n-1} = \int_{E^y_t} |x|^\alpha\langle N, \varphi(z)\rangle d\mathcal{H}^{n-1}. \tag{2.4.9}$$

On the other hand, we claim that

$$\lim_{\varepsilon\to 0}\int_{\{|y|<t\}} \langle\nabla_\alpha f_\varepsilon, \varphi\rangle dz = \int_{\{|y|<t\}} \Big\{\sum_{i=1}^h \varphi_i d\mu_E^{x_i} + \sum_{\ell=1}^k \varphi_{h+\ell}|x|^\alpha d\mu_E^{y_\ell}\Big\}, \tag{2.4.10}$$

where $\mu_E^{x_i}$ and $\mu_E^{y_\ell}$ are the distributional partial derivatives of χ_E, that are Borel measures on $\mathbb{R}^n$, because E has finite α-perimeter. For the coordinate y_ℓ, we have

$$\begin{aligned}\int_{\{|y|<t\}} \varphi_{h+\ell}(z)|x|^\alpha\partial_{y_\ell} f_\varepsilon(z)dz &= \int_{\{|y|<t\}} \varphi_{h+\ell}(z)|x|^\alpha \int_E \partial_{y_\ell}\phi_\varepsilon(|z-w|)dw\,dz \\ &= -\int_{\{|y|<t\}} \varphi_{h+\ell}(z)|x|^\alpha \int_E \partial_{\eta_\ell}\phi_\varepsilon(|z-w|)dw\,dz \\ &= \int_{\{|y|<t\}} \varphi_{h+\ell}(z)|x|^\alpha \int_{\mathbb{R}^n} \phi_\varepsilon(|z-w|)d\mu_E^{y_\ell}(w)\,dz \\ &= \int_{\mathbb{R}^n}\int_{\{|y|<t\}} \varphi_{h+\ell}(z)|x|^\alpha\phi_\varepsilon(|z-w|)dz\,d\mu_E^{y_\ell}(w),\end{aligned}$$

where we let $w = (\xi, \eta) \in \mathbb{R}^h \times \mathbb{R}^k$. By (2.4.6), the measure $\mu_E^{y_\ell}$ is concentrated on $\{|y| \neq t\}$. It follows that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\int_{\{|y|<t\}} \varphi_{h+\ell}(z)|x|^\alpha\phi_\varepsilon(|z-w|)dz\,d\mu_E^{y_\ell}(w) = \int_{\{|\eta|<t\}} \varphi_{h+\ell}(w)|\xi|^\alpha d\mu_E^{y_\ell}(w).$$

This proves (2.4.10).

Now, from (2.4.7)–(2.4.10) we deduce that

$$\begin{aligned}\int_{E\cap\{|y|<t\}} \mathrm{div}_\alpha\varphi(z)\,dz = &-\int_{E\cap\{|y|=t\}} |x|^\alpha\langle N, \varphi(z)\rangle d\mathcal{H}^{n-1} \\ &- \int_{\{|y|<t\}} \Big\{\sum_{i=1}^h \varphi_i d\mu_E^{x_i} + |x|^\alpha\sum_{\ell=1}^k \varphi_{h+\ell} d\mu_E^{y_\ell}\Big\},\end{aligned} \tag{2.4.11}$$

and the claim follows by optimizing the right hand side over $\varphi \in \mathcal{F}_n(\mathbb{R}^n)$.

□

Proposition 2.4.2. *Let $E \subset \mathbb{R}^n$ be a set with finite measure and finite α-perimeter. For a.e. $t > 0$ we have $P_\alpha(E^x_{t-}) \le P_\alpha(E)$ and $P_\alpha(E^y_{t-}) \le P_\alpha(E)$.*

Proof. The proof is a calibration argument. Notice that

$$\begin{aligned} P_\alpha(E^y_{t-}) &= P_\alpha(E^y_{t-}; \{|y| < t\}) + P_\alpha(E^y_{t-}; \{|y| \ge t\}) \\ &= P_\alpha(E; \{|y| < t\}) + P_\alpha(E^y_{t-}; \{|y| = t\}). \end{aligned}$$

Let $t > 0$ be such that $P_\alpha(E; \{|y| = t\}) = 0$; a.e. $t > 0$ has this property, see (2.4.6). It is sufficient to show that

$$P_\alpha(E^y_{t-}; \{|y| = t\}) \le P_\alpha(E; \{|y| \ge t\}) = P_\alpha(E; \{|y| > t\}).$$

The function $\varphi(x, y) = (0, -y/|y|) \in \mathbb{R}^n$, $|y| \ne 0$, has negative divergence:

$$\mathrm{div}_\alpha \varphi(x, y) = -|x|^\alpha \sum_{\ell=1}^{k} \Big(\frac{1}{|y|} - \frac{y_\ell^2}{|y|^3} \Big) = -\frac{(k-1)|x|^\alpha}{|y|} \le 0.$$

As in the proof of (2.4.11), we have

$$\begin{aligned} 0 \ge \int_{E \cap \{|y| > t\}} \mathrm{div}_\alpha \varphi \, dz &= \int_{E^y_t} |x|^\alpha d\mathcal{H}^{n-1} - \int_{\{|y| > t\}} |x|^\alpha \sum_{\ell=1}^{k} \varphi_{h+\ell} d\mu_E^{y_\ell} \\ &\ge \int_{E \cap \{|y| = t\}} |x|^\alpha d\mathcal{H}^{n-1} - P_\alpha(E; \{|y| > t\}). \end{aligned}$$

By the representation formula (2.2.2), we obtain

$$P_\alpha(E^y_{t-}; \{|y| = t\}) = \int_{E^y_t} |x|^\alpha d\mathcal{H}^{n-1} \le P_\alpha(E; \{|y| > t\}).$$

This ends the proof. □

We prove the existence of isoperimetric sets using the validity of the following isoperimetric inequality, holding for any $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ with finite measure

$$P_\alpha(E) \ge C \mathcal{L}^n(E)^{\frac{Q-1}{Q}} \tag{2.4.12}$$

for some geometric constant $C > 0$, see Proposition 1.3.4.

By the homogeneity properties of Lebesgue measure and α-perimeter, we can define the constant

$$C_I = \inf\{P_\alpha(E) : \mathcal{L}^n(E) = 1 \text{ and } E \in \mathcal{S}_x, \text{ if } h \ge 2\}. \tag{2.4.13}$$

Only when $h \ge 2$ we are adding the constraint $E \in \mathcal{S}_x$. We have $C_I > 0$ by the validity of (2.4.12) for some $C > 0$. Our goal is to prove that the infimum in (2.4.13) is attained.

Theorem 2.4.3. *Let $h, k \ge 1$ and $n = h + k$. There exists an x- and y-Schwarz symmetric set $E \subset \mathbb{R}^n$ realizing the infimum in* (2.4.13).

Proof. Let $(E_m)_{m\in\mathbb{N}}$ be a minimizing sequence for the infimum in (2.4.13), with the additional assumption that the sets involved in the minimization are x-spherically symmetric when $h \geq 2$. Namely,

$$\mathcal{L}^n(E_m) = 1 \text{ and } P_\alpha(E_m) \leq C_I\left(1 + \frac{1}{m}\right), \quad m \in \mathbb{N}. \tag{2.4.14}$$

By Theorems 2.3.1 and 2.3.2, we can assume that every set E_m is x- and y-Schwarz symmetric. We claim that the minimizing sequence can be also assumed to be in a bounded region of $\mathbb{R}^n$.

Fix $m \in \mathbb{N}$ and let $E = E_m$. For any $t > 0$ such that (2.4.4) holds we consider the set $E^x_{t-} = E \cap \{|x| < t\} \in \mathcal{S}_x$.

We apply the isoperimetric inequality (2.4.12) with the constant $C_I > 0$ in (2.4.13) to the sets E^x_{t-} and $E \setminus E^x_{t-}$, and we use Proposition 2.4.1:

$$\begin{aligned} C_I \mathcal{L}^n(E^x_{t-})^{\frac{Q-1}{Q}} &\leq P_\alpha(E^x_{t-}) = P_\alpha(E; \{|x| < t\}) + v^x_E(t) \\ C_I(1 - \mathcal{L}^n(E^x_{t-}))^{\frac{Q-1}{Q}} &\leq P_\alpha(E \setminus E^x_{t-}) = P_\alpha(E; \{|x| > t\}) + v^x_E(t). \end{aligned} \tag{2.4.15}$$

As in (2.4.2), we let $v^x_E(t) = \mathcal{H}^{n-1}(E^x_t)$. Adding up the two inequalities we get

$$C_I(\mathcal{L}^n(E^x_{t-})^{\frac{Q-1}{Q}} + (1 - \mathcal{L}^n(E^x_{t-}))^{\frac{Q-1}{Q}}) \leq P_\alpha(E) + 2v^x_E(t). \tag{2.4.16}$$

The function $g : [0, \infty) \to \mathbb{R}$, $g(t) = \mathcal{L}^n(E^x_{t-})$ is continuous, $(0,1) \subset g([0,\infty)) \subset [0,1]$, and it is increasing. In particular, g is differentiable almost everywhere. For any $t > 0$ such that $P_\alpha(E; \{|x| = t\}) = 0$, also the standard perimeter vanishes, namely $P(E; \{|x| = t\}) = 0$. With the vector field $\varphi = (x/|x|, 0)$, and for $t < s$ satisfying $P_\alpha(E; \{|x| = t\}) = P_\alpha(E; \{|x| = s\}) = 0$, we have

$$\begin{aligned} \int_{E^x_{s-} \setminus E^x_{t-}} \frac{h-1}{|x|}\, dz &= \int_{E^x_{s-} \setminus E^x_{t-}} \operatorname{div}\varphi\, dz \\ &= \mathcal{H}^{n-1}(E^x_s) - \mathcal{H}^{n-1}(E^x_t) + \int_{\partial^* E \cap \{s<|x|<t\}} \langle \varphi, \nu_E \rangle d\mathcal{H}^{n-1} \end{aligned}$$

where $\partial^* E$ is the reduced boundary of E, see [89, Chapter 15] for a definition. This implies that

$$\lim_{s\to t} \mathcal{H}^{n-1}(E^x_s) = \mathcal{H}^{n-1}(E^x_t),$$

with limit restricted to s satisfying the above condition, and thus

$$g'(t) = \lim_{s\to t} \frac{1}{s-t} \int_t^s \mathcal{H}^{n-1}(E^x_\tau)\, d\tau = \mathcal{H}^{n-1}(E^x_t). \tag{2.4.17}$$

At this point, by (2.4.14), inequality (2.4.16) gives

$$C_I\Big(g(t)^{\frac{Q-1}{Q}} + (1 - g(t))^{\frac{Q-1}{Q}} - 1 - \frac{1}{m}\Big) \leq 2g'(t). \tag{2.4.18}$$

The function $\psi : [0,1] \to \mathbb{R}$, $\psi(s) = s^{\frac{Q-1}{Q}} + (1-s)^{\frac{Q-1}{Q}} - 1$ is concave, it attains its maximum at $s = 1/2$ with $\psi(1/2) = 2^{\frac{1}{Q}} - 1$, and it satisfies $\psi(s) = \psi(1-s)$, $\psi(0) = \psi(1) = 0$. By (2.4.18) we have

$$g'(t) \geq \frac{C_I}{2}\Big(\psi(g(t)) - \frac{1}{m}\Big) \geq \frac{C_I}{4}\psi(g(t)) + \frac{C_I}{4}\Big(\psi(g(t)) - \frac{2}{m}\Big), \tag{2.4.19}$$

for almost every $t \in \mathbb{R}$ and every $m \in \mathbb{N}$. Provided that $m \in \mathbb{N}$ is such that $2/m \leq \max\psi = 2^{1/Q} - 1$, we show that there exist constants $0 < a_m < b_m < \infty$ such that inequality (2.4.19) implies the following:

$$g'(t) \geq \frac{C_I}{4}\psi(g(t)) \text{ for a.e. } t \in [a_m, b_m]. \tag{2.4.20}$$

In fact, by continuity of g and ψ, and by symmetry of ψ with respect to the line $\{s = 1/2\}$, for m large enough, there exist $0 < a_m < b_m < \infty$ such that

$$0 < g(a_m) = 1 - g(b_m) < \frac{1}{2} \quad \text{and} \quad \psi(g(a_m)) = \psi(g(b_m)) = \frac{2}{m}.$$

By concavity of ψ and monotonicity of g, it follows that $\psi(g(t)) \geq \frac{2}{m}$ for every $t \in [a_m, b_m]$, and (2.4.20) follows. As $m \to \infty$ we have $g(b_m) \to 1$, that implies

$$\lim_{m\to\infty} b_m = \sup\{b > 0 : g(b) < 1\} > 0.$$

Moreover, as $m \to \infty$ we also have $g(a_m) \to 0$. Since the set E is x-Schwarz symmetric, there holds $g(a) > 0$ for all $a > 0$. Therefore, we deduce that $a_m \to 0$.

We infer that, for m large enough, we have $a_m < b_m/2$. Integrating inequality (2.4.20) on the interval $[b_m/2, b_m]$, we find

$$\frac{b_m}{2} \leq \frac{4}{C_I}\int_{b_m/2}^{b_m} \frac{g'(t)}{\psi(g(t))}dt \leq \frac{4}{C_I}\int_{g(b_m/2)}^{g(b_m)} \frac{1}{\psi(s)}ds \leq \frac{4}{C_I}\int_0^1 \frac{1}{\psi(s)}ds = \ell_1. \tag{2.4.21}$$

We consider the set $\widehat{E}_m = E^x_{b_m-}$. By (2.4.21), $\widehat{E}_m$ is contained in the cylinder $\{|x| < 2\ell_1\}$ and, by Proposition 2.4.2, it satisfies $P_\alpha(\widehat{E}_m) \leq P_\alpha(E_m)$. Define the set $E^\dagger_m = \delta_{\lambda_m}(\widehat{E}_m)$, where $\lambda_m \geq 1$ is chosen in such a way that $\mathcal{L}^n(\widehat{E}^\dagger_m) = 1$; namely, λ_m is the number

$$\lambda_m = \Big(\frac{1}{\mathcal{L}^n(\widehat{E}_m)}\Big)^{\frac{1}{Q}},$$

where

$$\mathcal{L}^n(\widehat{E}_m) = \mathcal{L}^n(E_m \cap \{|x| < b_m\}) = g(b_m) = 1 - g(a_m). \tag{2.4.22}$$

By concavity of ψ, for $0 < s < 1/2$ the graph of ψ lays above the straight line through the origin passing through the maximum $(1/2, \psi(1/2))$, i.e., $\psi(s) > 2(2^{1/Q} - 1)s$. Therefore, since $g(a_m) < 1/2$ and $\psi(g(a_m)) = 2/m$, then

$$g(a_m) \leq \frac{1}{m(2^{1/Q} - 1)},$$

and thus

$$\lambda_m \le \Big(\frac{1}{1-\frac{1}{m(2^{1/Q}-1)}}\Big)^{1/Q} = \Big(\frac{m}{m-\frac{1}{2^{1/Q}-1}}\Big)^{1/Q}.$$

By homogeneity of α-perimeter,

$$\begin{aligned}P_\alpha(E_m^\dagger) &= \lambda_m^{Q-1}P_\alpha(\hat{E}_m) \le \lambda_m^{Q-1}P_\alpha(E_m) \le \lambda_m^{Q-1}C_I\Big(1+\frac{1}{m}\Big)\\ &\le C_I\Big(1+\frac{1}{m}\Big)\Big(\frac{m}{m-\frac{1}{2^{1/Q}-1}}\Big)^{\frac{Q-1}{Q}}.\end{aligned}$$

In conclusion, $(E_m^\dagger)_{m\in\mathbb{N}}$ is a minimizing sequence for C_I and, for m large enough, it is contained in the cylinder $\{|x|<\ell\}$, where $\ell = 2^{1/Q+1}\ell_1$.

Now we consider the case of the y-variable. We start again from (2.4.15) for the sets E_{t-}^y for $t>0$. Now the set E can be assumed to be contained in the cylinder $\{|x|<\ell\}$. In this case, we have

$$v_E^y(t) = \int_{E_t^y}|x|^\alpha\, d\mathcal{H}^{n-1} \le \ell^\alpha\mathcal{H}^{n-1}(E_t^y) = \ell^\alpha g'(t).$$

So inequality (2.4.16) reads

$$C_I\Big(g(t)^{\frac{Q-1}{Q}} + (1-g(t))^{\frac{Q-1}{Q}} - 1 - \frac{1}{m}\Big) \le 2\ell^\alpha g'(t). \tag{2.4.23}$$

Now the argument continues exactly as in the first case. The conclusion is that there exists a minimizing sequence $(E_m)_{m\in\mathbb{N}}$ for (2.4.13) and there exists $\ell>0$ such that we have:

i) $\mathcal{L}^n(E_m) = 1$ for all $m\in\mathbb{N}$;

ii) $P_\alpha(E_m) \le C_I(1+1/m)$ for all $m\in\mathbb{N}$;

iii) $E_m \subset \{(x,y)\in\mathbb{R}^n : |x|<\ell \text{ and } |y|<\ell\}$ for all $m\in\mathbb{N}$;

iv) Each E_m is x- and y-Schwarz symmetric.

By the compactness theorem for sets of finite α-perimeter (see [65] for a general statement that covers our case), there exists a set $E\subset\mathbb{R}^n$ of finite α-perimeter which is the L^1-limit of (a subsequence of) the sequence $(E_m)_{m\in\mathbb{N}}$. Then we have

$$\mathcal{L}^n(E) = \lim_{m\to\infty}\mathcal{L}^n(E_m) = 1.$$

Moreover, by lower semicontinuity of α-perimeter

$$P_\alpha(E) \le \liminf_{m\to\infty} P_\alpha(E_m) = C_I.$$

The set E is x- and y-Schwarz symmetric, because these symmetries are preserved by the L^1-convergence. This concludes the proof. $\square$

2.5 Profile of isoperimetric sets

In Theorem 2.4.3, we proved existence of isoperimetric sets, in fact in the class of x-spherically symmetric sets when $h \geq 2$. By the characterization of the equality case in Theorems 2.3.1 and 2.3.2, any isoperimetric set E is x-Schwarz symmetric and there are functions $c : [0,\infty) \to \mathbb{R}^k$ and $f : [0,\infty) \to [0,\infty)$ such that

$$E = \{(x,y) \in \mathbb{R}^n : |y - c(|x|)| < f(|x|)\}. \tag{2.5.1}$$

The function f is decreasing. We will prove in Proposition 2.5.5 that, for isoperimetric sets, the function c is constant.

We start with the characterization of an isoperimetric set E with constant function $c = 0$. Let $F \subset \mathbb{R}^+ \times \mathbb{R}^+$ be the generating set of E

$$E = \{(x,y) \in \mathbb{R}^n : (|x|,|y|) \in F\}.$$

The set F is of the form

$$F = \{(r,s) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < s < f(r),\ r \in (0,r_0)\}, \tag{2.5.2}$$

where $f : (0,r_0) \to (0,\infty)$ is a decreasing function, for some $0 < r_0 \leq \infty$.

By the regularity theory of Λ-minimizers of perimeter, the boundary ∂E is a C^∞ hypersurface where $x \neq 0$, see Theorems 26.3, 27.4 [89, Part III]. We do not need the general regularity theory, and we prove this fact in our case by an elementary method that gives also the C^∞-smoothness of the function f in (2.5.2).

2.5.1. Smoothness of f

We prove that the boundary $\partial F \subset \mathbb{R}^+ \times \mathbb{R}^+$ is the graph of a smooth function $s = f(r)$.

We rotate clockwise by 45 degrees the coordinate system $(r,s) \in \mathbb{R}^2$ and we call the new coordinates (ϱ,σ); namely, we let

$$r = \frac{\sigma + \varrho}{\sqrt{2}}, \quad s = \frac{\sigma - \varrho}{\sqrt{2}}.$$

There exist $-\infty \leq a < 0 < b \leq \infty$ and a function $g : (a,b) \to \mathbb{R}$ such that the boundary $\partial F \subset \mathbb{R}^+ \times \mathbb{R}^+$ is a graph $\sigma = g(\varrho)$; namely, we have

$$\partial F = \Big\{\big(r(\varrho), s(\varrho)\big) = \Big(\frac{g(\varrho)+\varrho}{\sqrt{2}}, \frac{g(\varrho)-\varrho}{\sqrt{2}}\Big) : \varrho \in (a,b)\Big\}. \tag{2.5.3}$$

Since the function f is decreasing, the function g is 1-Lipschitz continuous.

By formula (2.2.12) and by the standard length formula for Lipschitz graphs, the α-perimeter of E is

$$P_\alpha(E) = c_{hk} \int_a^b \sqrt{s'^2 + r^{2\alpha} r'^2}\, r^{h-1} s^{k-1}\, d\varrho,$$

where $c_{hk} = hk\omega_h\omega_k$. On the other hand, the volume of E is

$$\mathcal{L}^n(E) = c_{hk} \int_a^b \left(\int_{|\varrho|}^{g(\varrho)} \left(\frac{\sigma+\varrho}{\sqrt{2}}\right)^{h-1} \left(\frac{\sigma-\varrho}{\sqrt{2}}\right)^{k-1} d\sigma \right) d\varrho.$$

For $\varepsilon \in \mathbb{R}$ and $\psi \in C_c^\infty(a,b)$, let $g_\varepsilon = g + \varepsilon\psi$, $s_\varepsilon = s + \varepsilon\frac{\psi}{\sqrt{2}}$, $r_\varepsilon = r + \varepsilon\frac{\psi}{\sqrt{2}}$ and let $F_\varepsilon \subset \mathbb{R}^+ \times \mathbb{R}^+$ be the subgraph in $\sigma > |\varrho|$ of the function g_ε. The set $E_\varepsilon \subset \mathbb{R}^n$ with generating set F_ε has α-perimeter

$$\begin{aligned} p(\varepsilon) &= P_\alpha(E_\varepsilon) \\ &= c_{hk} \int_a^b \sqrt{\left(s' + \varepsilon\frac{\psi'}{\sqrt{2}}\right)^2 + \left(r + \varepsilon\frac{\psi}{\sqrt{2}}\right)^{2\alpha} \left(r' + \varepsilon\frac{\psi'}{\sqrt{2}}\right)^2} \left(r + \varepsilon\frac{\psi}{\sqrt{2}}\right)^{h-1} \left(s + \varepsilon\frac{\psi}{\sqrt{2}}\right)^{k-1} d\varrho, \end{aligned}$$

and volume

$$v(\varepsilon) = \mathcal{L}^n(E_\varepsilon) = c_{hk} \int_a^b \left(\int_{|\varrho|}^{g(\varrho)+\varepsilon\psi(\varrho)} \left(\frac{\sigma+\varrho}{\sqrt{2}}\right)^{h-1} \left(\frac{\sigma-\varrho}{\sqrt{2}}\right)^{k-1} d\sigma \right) d\varrho.$$

Since E is an isoperimetric set, we have

$$0 = \frac{d}{d\varepsilon} \frac{p(\varepsilon)^Q}{v(\varepsilon)^{Q-1}}\bigg|_{\varepsilon=0} = \frac{Qp^{Q-1}p'v^{Q-1} - p^Q(Q-1)v^{Q-2}v'}{v^{2Q-2}}\bigg|_{\varepsilon=0},$$

that gives

$$p'(0) - C_{hk\alpha} v'(0) = 0, \quad \text{where} \quad C_{hk\alpha} = \frac{Q-1}{Q} \frac{P_\alpha(E)}{\mathcal{L}^n(E)}. \tag{2.5.4}$$

After some computations, we find

$$\begin{aligned} p'(0) = \frac{c_{hk}}{\sqrt{2}} \int_a^b \Big\{ & \frac{(r^{2\alpha} r' + s')\psi' + \alpha r^{2\alpha-1} r'^2 \psi}{\sqrt{s'^2 + r^{2\alpha} r'^2}} + \\ & + \sqrt{s'^2 + r^{2\alpha} r'^2} \Big[\frac{h-1}{r} + \frac{k-1}{s} \Big] \psi \Big\} r^{h-1} s^{k-1}\, d\varrho, \end{aligned} \tag{2.5.5}$$

and

$$v'(0) = c_{hk} \int_a^b r^{h-1} s^{k-1} \psi\, d\varrho. \tag{2.5.6}$$

From (2.5.4), (2.5.5), and (2.5.6) we deduce that g is a 1-Lipschitz function that, via the auxiliary functions r and s, solves in a weak sense the ordinary differential equation

$$\begin{aligned} \frac{d}{d\varrho}\Big(r^{h-1} s^{k-1} \frac{r^{2\alpha} r' + s'}{\sqrt{s'^2 + r^{2\alpha} r'^2}} \Big) = r^{h-1} s^{k-1} \Big\{ & \frac{\alpha r^{2\alpha-1} r'^2}{\sqrt{s'^2 + r^{2\alpha} r'^2}} + \\ & + \sqrt{s'^2 + r^{2\alpha} r'^2} \Big[\frac{h-1}{r} + \frac{k-1}{s} \Big] - \sqrt{2} C_{hk\alpha} \Big\}. \end{aligned} \tag{2.5.7}$$

By an elementary argument, if follows that $g \in C^\infty(a,b)$. We show this statement in the next proposition.

Proposition 2.5.1. *Let $r, s : (a,b) \to \mathbb{R}$ are Lipschitz functions solving (2.5.7). Then the Lipschitz function g defined via r, s through* (2.5.3) *is of class $C^\infty(a,b)$.*

Proof. Fix a test function $\chi \in C_c^\infty(a,b)$ which has zero mean: $\int_a^b \chi(\varrho)d\varrho = 0$. Define a test function $\psi \in \mathcal{F}_1((a,b))$ as the integral of χ:

$$\psi(\varrho) = \int_a^\varrho \chi(t)dt,$$

hence $\psi'(\varrho) = \chi(\varrho)$ and $\psi \in C_c^\infty((a,b))$. Equation (2.5.7) reads

$$0 = \int_a^b \eta(\varrho)\psi'(\varrho) + \xi(\varrho)\psi(\varrho)\, d\varrho \tag{2.5.8}$$

where

$$\begin{aligned} \eta(\varrho) &= \frac{r^{2\alpha}r' + s'}{\sqrt{s'^2 + r^{2\alpha}r'^2}} r^{h-1}s^{k-1}, \\ \xi(\varrho) &= \Big\{\frac{\alpha r^{2\alpha-1}r'^2}{\sqrt{s'^2 + r^{2\alpha}r'^2}} + \sqrt{s'^2 + r^{2\alpha}r'^2}\Big[\frac{h-1}{r} + \frac{k-1}{s}\Big] - \sqrt{2}C_{hk\alpha}\Big\} r^{h-1}s^{k-1} \end{aligned}$$

Applying (2.5.8) to ψ gives the following, where $\mathcal{I}_{[p,q]}$ denotes the characteristic function of $[p,q] \subset \mathbb{R}$

$$\begin{aligned} 0 &= \int_a^b \Big(\eta(\varrho)\chi(\varrho) + \xi(\varrho)\int_a^\varrho \chi(t)dt\Big)\, d\varrho \\ &= \int_a^b \eta(\varrho)\chi(\varrho)d\varrho + \int_a^b \Big(\int_a^b \mathcal{I}_{[a,\varrho]}\xi(\varrho)\chi(t)dt\Big)\, d\varrho \\ &= \int_a^b \eta(t)\chi(t)dt + \int_a^b \Big(\int_a^b \mathcal{I}_{[t,b]}\xi(\varrho)\chi(t)d\varrho\Big)\, dt \\ &= \int_a^b \Big(\eta(t) + \int_t^b \xi(\varrho)d\varrho\Big)\chi(t)dt. \end{aligned}$$

By arbitrariness of χ in the set of zero mean value test functions we deduce that there exists a constant $c \in \mathbb{R}$ such that $\eta(t) + \int_t^b \xi(\varrho)d\varrho = c$. Notice that the function

$$\phi(t) = \int_t^b \Big\{\frac{\alpha r^{2\alpha-1}r'^2}{\sqrt{s'^2 + r^{2\alpha}r'^2}} + \sqrt{s'^2 + r^{2\alpha}r'^2}\Big[\frac{h-1}{r} + \frac{k-1}{s}\Big] - \sqrt{2}C_{hk\alpha}\Big\} r^{h-1}s^{k-1} d\varrho$$

is Lipschitz because ϕ' is bounded, being r, s Lipschitz functions. Moreover, using the definition of r and s in (2.5.3), we write

$$\eta(t) = c - \phi \iff \frac{r^{2\alpha}(g'+1) + (g'-1)}{\sqrt{(g'-1)^2 + r^{2\alpha}(g'+1)^2}} r^{h-1}s^{k-1} = c - \phi.$$

Squaring both sides we obtain a second order equation in g'. Solving this equation we write g' as a product of Lipschitz functions. We conclude that g' is Lipschitz, hence $g \in C^{1,1}$. □

We claim that for all $\varrho \in (a,b)$ there holds $g'(\varrho) \neq -1$. By contradiction, assume that there exists $\bar\varrho \in (a,b)$ such that $g'(\bar\varrho) = -1$, i.e., $r'(\bar\varrho) = 0$ and $s'(\bar\varrho) = -\sqrt{2}$. Inserting these values into the differential equation (2.5.7) we can compute $g''(\bar\varrho)$ as a function of $g(\bar\varrho)$; namely, we obtain

$$g''(\bar\varrho) = 2^{\alpha+1}\frac{2(h-1) - \sqrt{2}C_{hk\alpha}r(\bar\varrho)}{r(\bar\varrho)^{2\alpha+1}}. \tag{2.5.9}$$

Now there are three possibilities:

(1) $g''(\bar\varrho) < 0$. In this case, g is strictly concave at $\bar\varrho$ and this contradicts the fact that E is y-Schwarz symmetric.

(2) $g''(\bar\varrho) > 0$. In this case, g' is strictly increasing at $\bar\varrho$ and since $g'(\bar\varrho) = -1$ this contradicts the fact the g is 1-Lipschitz, equivalently, the fact that E is x-Schwarz symmetric.

(3) $g''(\bar\varrho) = 0$. In this case, the value of g at $\bar\varrho$ is, by (2.5.9),

$$g(\bar\varrho) = -\bar\varrho + \frac{\sqrt{2}(h-1)}{C_{hk\alpha}}. \tag{2.5.10}$$

The function $\widehat{g}(\varrho) = -\varrho + \dfrac{\sqrt{2}(h-1)}{C_{hk\alpha}}$, $\varrho \in \mathbb{R}$, is the unique solution to the ordinary differential equation (2.5.7) with initial conditions $g(\bar\varrho)$ given by (2.5.10) and $g'(\bar\varrho) = -1$. It follows that $g = \widehat{g}$ and this contradicts the boundedness of the isoperimetric set; namely, the fact that isoperimetric sets have finite volume.

This proves that $g'(\varrho) \neq -1$ for all $\varrho \in (a,b)$.

2.5.2. Differential equations for the profile function

By the discussion in the previous section, the function f appearing in the definition of the set F in (2.5.2) is in $C^\infty(0,r_0)$. The function f is decreasing, $f' \le 0$. By formula (2.2.12), the perimeter of the set E with generating set F is

$$P_\alpha(E) = c_{hk}\int_0^{r_0}\sqrt{f'(r)^2 + r^{2\alpha}}\, r^{h-1}f(r)^{k-1}dr, \tag{2.5.11}$$

and the volume of E is

$$\mathcal{L}^n(E) = \frac{c_{hk}}{k}\int_0^{r_0} r^{h-1}f(r)^k dr. \tag{2.5.12}$$

As in the previous section, for $\psi \in C_c^\infty(0,r_0)$ and $\varepsilon \in \mathbb{R}$, we consider the perturbation $f + \varepsilon\psi$ and we define the set

$$E_\varepsilon = \big\{(x,y) \in \mathbb{R}^n : |y| < f(|x|) + \varepsilon\psi(|x|)\big\}.$$

Then we have

$$
\begin{aligned}
p(\varepsilon) &= P_\alpha(E_\varepsilon) = c_{hk}\int_0^{r_0}\sqrt{(f'+\varepsilon\psi')^2+r^{2\alpha}}\,(f+\varepsilon\psi)^{k-1}r^{h-1}dr,\\
v(\varepsilon) &= \mathcal{L}^n(E_\varepsilon) = \frac{c_{hk}}{k}\int_0^{r_0}(f+\varepsilon\psi)^k r^{h-1}dr,
\end{aligned}
$$

and from these formulas we compute the first derivatives at $\varepsilon = 0$:

$$
\begin{aligned}
p'(0) &= c_{hk}\int_0^{r_0}\Big[\frac{f^{k-1}f'}{\sqrt{f'^2+r^{2\alpha}}}\psi' + (k-1)f^{k-2}\sqrt{f'^2+r^{2\alpha}}\psi\Big]\,r^{h-1}\,dr,\\
v'(0) &= c_{hk}\int_0^{r_0} f^{k-1}\psi\, r^{h-1}\,dr.
\end{aligned}
$$

The minimality equation (2.5.4) reads

$$
\int_0^{r_0}\Big(\frac{f'f^{k-1}}{\sqrt{f'^2+r^{2\alpha}}}\psi' + \Big[(k-1)f^{k-2}\sqrt{f'^2+r^{2\alpha}} - C_{hk\alpha}f^{k-1}\Big]\psi\Big)\,r^{h-1}\,dr = 0. \tag{2.5.13}
$$

Integrating by parts the term with ψ' and using the fact that ψ is arbitrary, we deduce that f solves the following second order ordinary differential equation:

$$
-\frac{d}{dr}\left(r^{h-1}\frac{f'f^{k-1}}{\sqrt{f'^2+r^{2\alpha}}}\right) + r^{h-1}\Big[(k-1)\sqrt{f'^2+r^{2\alpha}}\;f^{k-2} - C_{hk\alpha}f^{k-1}\Big] = 0. \tag{2.5.14}
$$

The normal form of this differential equation is

$$
f'' = \frac{\alpha f'}{r} + (f'^2+r^{2\alpha})\left(\frac{k-1}{f} - (h-1)\frac{f'}{r^{2\alpha+1}}\right) - C_{hk\alpha}\frac{(f'^2+r^{2\alpha})^{\frac{3}{2}}}{r^{2\alpha}}, \tag{2.5.15}
$$

and it can be rearranged in the following ways:

$$
\begin{aligned}
\frac{\partial}{\partial r}\Big(\frac{f'}{r^\alpha}\Big) &= (f'^2+r^{2\alpha})\Big(\frac{k-1}{fr^\alpha} - (h-1)\frac{f'}{r^{3\alpha+1}}\Big) - C_{hk\alpha}\frac{(f'^2+r^{2\alpha})^{\frac{3}{2}}}{r^{3\alpha}}\\
&= r^\alpha\Big(\Big(\frac{f'}{r^\alpha}\Big)^2+1\Big)\Big(\frac{k-1}{f} - \frac{(h-1)}{r^{\alpha+1}}\frac{f'}{r^\alpha}\Big) - C_{hk\alpha}\Big(\Big(\frac{f'}{r^\alpha}\Big)^2+1\Big)^{\frac{3}{2}}.
\end{aligned} \tag{2.5.16}
$$

Call $\theta = \arctan\frac{f'}{r^\alpha}$. Hence

$$
\tan\vartheta = \frac{f'}{r^\alpha} \implies \sin\vartheta = \frac{f'}{\sqrt{f'^2+r^{2\alpha}}},\ \cos\vartheta = \frac{r^\alpha}{\sqrt{f'^2+r^{2\alpha}}}.
$$

With the substitution

$$
z = \sin\arctan\Big(\frac{f'}{r^\alpha}\Big) = \frac{f'}{\sqrt{r^{2\alpha}+f'^2}}, \tag{2.5.17}
$$

equation (2.5.16) transforms as follows

$$
\begin{aligned}
(r^{h-1}z)' &= (h-1)r^{h-2}z + r^{h-1}\Big(\frac{r^\alpha}{\sqrt{f'^2+r^{2\alpha}}}\frac{r^{2\alpha}}{f'^2+r^{2\alpha}}\partial_r\Big(\frac{f'^2}{r^\alpha}\Big)\Big)\\
&= (h-1)r^{h-2}\frac{f'}{\sqrt{f'^2+r^{2\alpha}}} + r^{h-1}\Big(\frac{r^{3\alpha}}{(f'^2+3^{2\alpha})^{\frac{3}{2}}}\cdot\\
&\quad\cdot\Big\{\frac{f'+r^{2\alpha}}{r^\alpha}\Big(\frac{k-1}{f}-\frac{h-1}{r^{2\alpha+1}}f'\Big) - C_{hk\alpha}\frac{(f'^2+r^{2\alpha})^{\frac{3}{2}}}{r^{3\alpha}}\Big\}\Big)\\
&= r^{h-1}\Big\{\frac{h-1}{r}\frac{f'}{\sqrt{f'^2+r^{2\alpha}}} - \frac{(h-1)r^{3\alpha}}{(f'^2+r^{2\alpha})^{\frac{3}{2}}}\frac{(f'^2+r^{2\alpha})}{r^\alpha}\frac{f'}{r^{2\alpha+1}}\\
&\quad - C_{hk\alpha}\frac{r^{3\alpha}}{(f'^2+r^{2\alpha})^{\frac{3}{2}}}\frac{(f'^2+r^{2\alpha})^{\frac{3}{2}}}{r^{3\alpha}} + \frac{r^{3\alpha}}{(f'^2+r^{2\alpha})^{\frac{3}{2}}}\frac{f'^2+r^{2\alpha}}{r^\alpha}\frac{k-1}{f}\Big\}\\
&= r^{h-1}\Big\{-C_{hk\alpha} + r^\alpha\frac{(k-1)}{f}\frac{r^\alpha}{\sqrt{f'^2+r^{2\alpha}}}\Big\}
\end{aligned}
$$

Hence we get an equivalent equation for f:

$$
(r^{h-1}z)' = r^{\alpha+h-1}\frac{k-1}{f}\sqrt{1-z^2} - C_{hk\alpha}r^{h-1}. \tag{2.5.18}
$$

We integrate equation (2.5.18) on the interval $(0,r)$. When $h>1$ we use the fact that $r^{h-1}z=0$ at $r=0$. When $h=1$ we use the fact that z has a finite limit as $r\to 0^+$. In both cases, we deduce that there exists a constant $D\in\mathbb{R}$ such that

$$
z(r) = r^{1-h}\int_0^r s^{\alpha+h-1}\frac{k-1}{f}\sqrt{1-z^2}\,ds - \frac{C_{hk\alpha}}{h}r + Dr^{1-h}. \tag{2.5.19}
$$

Inserting (2.5.17) into (2.5.19), we get

$$
\frac{f'}{\sqrt{r^{2\alpha}+f'^2}} = r^{1-h}\int_0^r s^{2\alpha+h-1}\frac{k-1}{f\sqrt{s^{2\alpha}+f'^2}}\,ds - \frac{C_{hk\alpha}}{h}r + Dr^{1-h}. \tag{2.5.20}
$$

If $h\ge 2$, from (2.5.20) we deduce that $D=0$. In fact, the left-hand side of (2.5.20) is bounded as $r\to 0^+$, while the right-hand side diverges to $\pm\infty$ according to the sign of $D\neq 0$. In the next section, we prove that $D=0$ also when $h=1$, provided that f is the profile of an isoperimetric set.

Remark 2.5.2 (Computation of the solution when $k=1$). When $k=1$ and $D=0$, equation (2.5.20) reads

$$
\frac{f'}{\sqrt{r^{2\alpha}+f'^2}} = -\frac{C_{hk\alpha}}{h}r.
$$

and this is equivalent to

$$
f'(r) = -\frac{C_{hk\alpha}r^{\alpha+1}}{\sqrt{h^2-C_{hk\alpha}^2r^2}}, \quad r\in[0,r_0). \tag{2.5.21}
$$

Without loss of generality we can assume that $r_0 = 1$ and this holds if and only if $C_{hk\alpha} = h$. Integrating (2.5.21) with $f(1) = 0$ we obtain the solution

$$f(r) = \int_r^1 \frac{s^{\alpha+1}}{\sqrt{1-s^2}}\,ds = \int_{\arcsin r}^{\pi/2} \sin^{\alpha+1}(s)\,ds.$$

This is the profile function for the isoperimetric set when $k = 1$.

2.5.3. Proof that $D = 0$.

We prove that $D = 0$ in (2.5.20) in the case $h = 1$. We assume by contradiction that $D \neq 0$. For a small parameter $s > 0$, let $f_s : [0, r_0) \to \mathbb{R}^+$ be the function

$$f_s(r) = \begin{cases} f(s) & \text{for } 0 < r \le s \\ f(r) & \text{for } r > s, \end{cases}$$

and define the set

$$E_s = \big\{(x,y) \in \mathbb{R}^n : |y| < f_s(|x|)\big\}.$$

Recall that the isoperimetric ratio is $\mathcal{I}_\alpha(E) = P_\alpha(E)^Q/\mathcal{L}^n(E)^{Q-1}$. We claim that for $s > 0$ small, the difference of isoperimetric ratios

$$\begin{aligned} \mathcal{I}_\alpha(E_s) - \mathcal{I}_\alpha(E) &= \frac{P_\alpha(E_s)^Q}{\mathcal{L}^n(E_s)^{Q-1}} - \frac{P_\alpha(E)^Q}{\mathcal{L}^n(E)^{Q-1}} \\ &= \frac{P_\alpha(E_s)^Q\mathcal{L}^n(E)^{Q-1} - P_\alpha(E)^Q\mathcal{L}^n(E_s)^{Q-1}}{\mathcal{L}^n(E_s)^{Q-1}\mathcal{L}^n(E)^{Q-1}} \end{aligned} \tag{2.5.22}$$

is strictly negative.

The α-perimeter of E_s is

$$\begin{aligned} P_\alpha(E_s) &= c_{hk}\int_0^\infty \sqrt{f_s'^2 + r^{2\alpha}}\, f_s^{k-1}\, r^{h-1}\,dr \\ &= c_{hk}\Big[f(s)^{k-1}\int_0^s r^{\alpha+h-1}\,dr + \int_s^\infty \sqrt{f'^2 + r^{2\alpha}}\, f^{k-1}\, r^{h-1}\,dr\Big] \\ &= P_\alpha(E) + c_{hk}\int_0^s \Big[r^\alpha f(s)^{k-1} - \sqrt{f'^2 + r^{2\alpha}}\, f^{k-1}\Big] r^{h-1}\,dr, \end{aligned}$$

and its volume is

$$\begin{aligned} \mathcal{L}^n(E_s) &= \frac{c_{hk}}{k}\int_0^\infty f_s^k r^{h-1}\,dr = \frac{c_{hk}}{k}\Big(\int_0^s f(s)^k r^{h-1}\,dr + \int_s^\infty f(r)^k\, r^{h-1}\,dr\Big) \\ &= \mathcal{L}^n(E) + \frac{c_{hk}}{k}\int_0^s \big(f(s)^k - f(r)^k\big)\, r^{h-1}\,dr, \end{aligned}$$

so, by elementary Taylor approximations, we find

$$\begin{aligned} &\mathcal{L}^n(E)^{Q-1}P_\alpha(E_s)^Q = \\ &\quad= \mathcal{L}^n(E)^{Q-1}\Big\{P_\alpha(E) + c_{hk}\int_0^s \Big[r^\alpha f(s)^{k-1} - \sqrt{f'^2 + r^{2\alpha}}\, f^{k-1}\Big] r^{h-1}\,dr\Big\}^Q \\ &\quad= \mathcal{L}^n(E)^{Q-1}\Bigg\{P_\alpha(E)^Q + dc_{hk}P_\alpha(E)^{Q-1}\Bigg(\int_0^s \Big[r^\alpha f(s)^{k-1} - \sqrt{f'^2 + r^{2\alpha}}\, f^{k-1}\Big] r^{h-1}\,dr\Bigg) \\ &\qquad\qquad\qquad + R_1(s)\Bigg\}, \end{aligned}$$

where $R_1(s)$ is a higher order infinitesimal as $s \to 0$, and

$$\begin{aligned} P_\alpha(E)^d \mathcal{L}^n(E_s)^{Q-1} &= P_\alpha(E)^Q \Big\{ \mathcal{L}^n(E) + \frac{c_{hk}}{k} \int_0^s \big(f(s)^k - f(r)^k\big) r^{h-1}\,dr \Big\}^{Q-1} \\ &= P_\alpha(E)^Q \Big\{ \mathcal{L}^n(E)^{Q-1} + \frac{c_{hk}(Q-1)}{k} \mathcal{L}^n(E)^{d-2} \int_0^s \big(f(s)^k - f(r)^k\big) r^{h-1}\,dr + R_2(s) \Big\}, \end{aligned}$$

where $R_2(s)$ is a higher order infinitesimal as $s \to 0$. The difference is thus

$$\begin{aligned} \Delta(s) &= P(E_s)^Q \mathcal{L}^n(E)^{Q-1} - P_\alpha(E)^Q \mathcal{L}^n(E_s)^{Q-1} \\ &= c_{hk} P_\alpha(E)^Q \mathcal{L}^n(E)^{Q-1} \Big\{ d \frac{A(s)}{P_\alpha(E)} - (Q-1) \frac{B(s)}{k\mathcal{L}^n(E)} \Big\}, \end{aligned}$$

where we let

$$\begin{aligned} A(s) &= \int_0^s \big[r^\alpha f(s)^{k-1} - \sqrt{f'^2 + r^{2\alpha}} f^{k-1} \big] r^{h-1}\,dr + R_1(s) \\ B(s) &= \int_0^s \big(f(s)^k - f(r)^k\big) r^{h-1}\,dr + R_2(s). \end{aligned}$$

Now we let $h = 1$ and we observe that the differential equation (2.5.19) or its equivalent version (2.5.20) imply that

$$\lim_{r\to 0^+} \frac{f'(r)}{r^\alpha} = D.$$

So for $D \neq 0$ and, in fact, for $D < 0$ (because f is decreasing) we have

$$\lim_{s\to 0^+} \frac{A(s)}{s^{\alpha+h}} = f(0)^{k-1} \frac{1 - \sqrt{D^2+1}}{\alpha + h} < 0,$$

and

$$\lim_{s\to 0^+} \frac{B(s)}{s^{\alpha+h}} = 0.$$

It follows that for $s > 0$ small there holds

$$\frac{\Delta(s)}{s^{h+\alpha}} = f(0)^{k-1} \frac{1 - \sqrt{D^2+1}}{\alpha + h} d c_{hk} P_\alpha(E)^{Q-1} \mathcal{L}^n(E)^{Q-1} + o(1) < 0.$$

Then E is not an isoperimetric set. This proves that $D = 0$.

2.5.4. Initial and final conditions for the profile function

In this section, we study the behavior of f at 0 and r_0.

Proposition 2.5.3. *The profile function f of an x- and y-Schwarz symmetric isoperimetric set $E \subset \mathbb{R}^n$ satisfies $f \in C^\infty(0, r_0) \cap C([0, r_0]) \cap C^2([0, r_0))$ for some $0 < r_0 < \infty$, $f' \leq 0$, $f(r_0) = 0$, it solves the differential equation* (2.5.20) *with $D = 0$, and*

$$\lim_{r\to r_0^-} f'(r) = -\infty, \quad \text{and} \quad \lim_{r\to 0^+} \frac{f'(r)}{r^{\alpha+1}} = -\frac{C_{hk\alpha}}{h}.$$

Proof. By Remark 2.5.2, it is sufficient to prove that $r_0 < \infty$ when $k > 1$. Assume by contradiction that $r_0 = \infty$. In this case, it must be

$$\lim_{r\to\infty} f(r) = 0, \tag{2.5.23}$$

otherwise the set E with profile f would have infinite volume.

For $\varepsilon > 0$ and $M > 0$, let us consider the set

$$K_M = \{r \ge M : f'(r) \ge -\varepsilon\}.$$

Recall that in our case we have $f' \le 0$. The set K_M is closed and nonempty for any M. If $K_M = \varnothing$ for some M, then this would contradict (2.5.23).

Let $\bar r \in K_M$. From (2.5.15) we have

$$\begin{aligned} f''(\bar r) &\ge -\frac{\alpha\varepsilon}{\bar r} + \bar r^{2\alpha}\frac{k-1}{f(\bar r)} - C_{hk\alpha}\frac{(\varepsilon^2 + \bar r^{2\alpha})^{3/2}}{\bar r^{2\alpha}} \\ &\ge \frac{1}{2}M^{2\alpha}\frac{k-1}{f(M)} > 0, \end{aligned} \tag{2.5.24}$$

provided that M is large enough. We deduce that there exists $\delta > 0$ such that $f'(r) \ge -\varepsilon$ for all $r \in [\bar r, \bar r + \delta)$. This proves that K_M is open to the right. It follows that it must be $K_M = [M, \infty)$. This proves that

$$\lim_{r\to\infty} f'(r) = 0,$$

and this in turn contradicts (2.5.24).

Now we have $r_0 < \infty$ and we also have

$$L = \lim_{r\to r_0^-} f(r) = 0.$$

If it were $L > 0$, then the isoperimetric set would have a "vertical part". We would get a contradiction by the argument at point (3) at the end of Section 2.5.1.

We claim that

$$\lim_{r\to r_0^-} f'(r) = -\infty.$$

For $M > 0$ and $0 < s < r_0$, consider the set

$$K_s = \{s \le r < r_0 : f'(r) \ge -M\}.$$

By contradiction assume that there exists $M > 0$ such that $K_s \ne \varnothing$ for all $0 < s < r_0$. If $\bar r \in K_s$, we have as above $f''(\bar r) \ge \frac{1}{2}(k-1)s^{2\alpha}/f(s) > 0$. We deduce that there exists $s < r_0$ such that $0 \ge f'(r) \ge -M$ for all $r \in [s, r_0)$. From (2.5.15), we deduce that there exists a constant $C > 0$ such that

$$f''(r) \ge \frac{C}{f(r)}.$$

Multiplying by $f' \le 0$ and integrating the resulting inequality we find

$$f'(r)^2 \le 2C \log |f(r)| + C_0,$$

for some constant $C_0 \in \mathbb{R}$. This is a contradiction because $\lim\limits_{r\to r_0^-} \log |f(r)| = -\infty$.

To prove that f is C^1 at 0, it is sufficient to use de l'Hopital Theorem. Since f is smooth away from 0 and continuous at 0,

$$f'(0) = \lim_{r\to 0^+} \frac{f(r) - f(0)}{r} = \lim_{r\to 0^+} f'(r),$$

Moreover, by Section 2.5.2, we have $D = 0$ in (2.5.20). In this case, by (2.5.19) we can compute the limit

$$\lim_{r\to 0^+} \frac{f'(r)}{r^{\alpha+1}} = \lim_{r\to 0^+} -\frac{C_{hk\alpha}}{h} + r^{-h} \int_0^r s^{\alpha+h-1} \frac{k-1}{f} \sqrt{1-z^2}\, ds = -\frac{C_{hk\alpha}}{h}.$$

In particular,

$$f'(0) = \lim_{r\to 0^+} f'(r) = \lim_{r\to 0^+} \frac{f'(r)}{r^{\alpha+1}} r^{\alpha+1} = 0.$$

In the same way

$$f''(0) = \lim_{r\to 0^+} \frac{f'(r) - f'(0)}{r} = \lim_{r\to 0^+} f''(r),$$

hence $f \in C^2([0, r_0))$. This concludes the proof. □

Remark 2.5.4. If $\alpha = 0$ we are in the euclidean case and the unique minimizers are balls. If $\alpha > 0$ we can compute $f''(0)$ as follows

$$f''(0) = \lim_{r\to 0^+} \frac{f'(r) - f'(0)}{r} = \lim_{r\to 0^+} \frac{f'(r)}{r} = \lim_{r\to 0^+} \frac{f'(r)}{r^{\alpha+1}} r^{\alpha} = 0.$$

We deduce that any isoperimetric set $E \subset \mathbb{R}^n$ has a C^2 boundary. In fact, since $f'(0) = f''(0) = 0$, the even function

$$[-1,1] \ni r \mapsto \begin{matrix} f(r) & \text{if } r \ge 0 \\ f(-r) & \text{if } r < 0 \end{matrix}$$

is C^2 smooth.

On the other hand, this regularity can not be improved, in general. For instance, in the case $\alpha = 1$, there holds $f'''(0) = -2$, hence the boundary of E is not a C^3 surface at the point $(0, f(0))$. This fact will be clarified later (see Proposition 3.3.3).

Isoperimetric sets are y-Schwarz symmetric

To conclude the proof of Theorem 2.1.4 we are left to show that for an isoperimetric set E of the type (2.5.1), the function c of the centers is constant.

Proposition 2.5.5. *Let $h, k \geq 1$ and $n = h + k$. Let $E \subset \mathbb{R}^n$ be a set of the form*

$$E = \{(x, y) \in \mathbb{R}^n : |y - c(|x|)| < f(|x|)\}$$

for measurable functions $c : [0, \infty) \to \mathbb{R}^k$ and $f : [0, \infty) \to [0, \infty]$. If E is an isoperimetric set for the problem (2.4.13) *then the function c is constant.*

Proof. If E is isoperimetric, then also its y-Schwarz rearrangement $E^* = \{(x, y) \in \mathbb{R}^n : |y| < f(|x|)\}$ is an isoperimetric set, see Theorems 2.3.1 and 2.3.2. Then, by Proposition 2.5.3, we have $f \in C^\infty(0, r_0) \cap C([0, r_0])$ with $f(r_0) = 0$ and $f' \leq 0$. In particular, $f \in \mathrm{Lip}_{\mathrm{loc}}(0, r_0)$. We claim that $c \in \mathrm{Lip}_{\mathrm{loc}}(0, r_0)$.

Since E is x-Schwarz symmetric, for any $0 < r_1 < r_2 < r_0$ we have the inclusion

$$\{y \in \mathbb{R}^k : |y - c(r_2)| \leq f(r_2)\} \subset \{y \in \mathbb{R}^k : |y - c(r_1)| \leq f(r_1)\}.$$

Assume $c(r_2) \neq c(r_1)$ and let $\vartheta = c(r_2) - c(r_1)/|c(r_2) - c(r_1)|$. Then we have

$$c(r_2) + \vartheta f(r_2) \in \{y \in \mathbb{R}^k : |y - c(r_1)| \leq f(r_1)\},$$

and therefore

$$|c(r_2) - c(r_1)| + f(r_2) = |c(r_2) + \vartheta f(r_2) - c(r_1)| \leq f(r_1).$$

This implies that c is locally Lipschitz on $(0, r_0)$.

Let $F \subset \mathbb{R}^+ \times \mathbb{R}^k$ be the generating set of E:

$$E = \{(x, y) \in \mathbb{R}^n : (|x|, y) \in F\}.$$

By the discussion above, the set E and thus also the set F have locally Lipschitz boundary away from a negligible set. By the representation formula (2.2.13), we have

$$P_\alpha(E) = Q(F) = h\omega_h \int_{\partial F} \sqrt{N_r^2 + r^{2\alpha}|N_y|^2}\, r^{h-1}\, d\mathcal{H}^k,$$

where $(N_r, N_y) \in \mathbb{R}^{1+k}$ is the unit normal to ∂F in $\mathbb{R}^+ \times \mathbb{R}^k$, that is defined $\mathcal{H}^k$ almost everywhere on the boundary. By the coarea formula (see [25]) we also have

$$Q(F) = h\omega_h \int_0^\infty r^{h-1} \int_{\partial F_r} \frac{\sqrt{N_r^2 + r^{2\alpha}|N_y|^2}}{\sqrt{1 - N_r^2}}\, d\mathcal{H}^{k-1}\, dr,$$

where $\partial F_r = \partial\{y \in \mathbb{R}^k : (r, y) \in F\} = \{y \in \mathbb{R}^k : |y - c(r)| = f(r)\}$.

A defining equation for ∂F is $|y - c(r)|^2 - f(r)^2 = 0$. From this equation, we find

$$N_r = -\frac{\langle y - c, c'\rangle + ff'}{\sqrt{(\langle y - c, c'\rangle + ff')^2 + |y - c|^2}},$$

$$N_y = \frac{y - c}{\sqrt{(\langle y - c, c'\rangle + ff')^2 + |y - c|^2}},$$

and thus, by translation and scaling in the inner integral,

$$
\begin{aligned}
Q(F) &= h\omega_h \int_0^\infty r^{h-1} \int_{|y-c(r)|=f(r)} \sqrt{\left\{\frac{\langle y-c(r), c'(r)\rangle}{f(r)} + f'(r)\right\}^2 + r^{2\alpha}}\, d\mathcal{H}^{k-1}(y)\, dr \\
&= h\omega_h \int_0^\infty r^{h-1} f(r)^{k-1} \int_{|y|=1} \sqrt{\left\{\langle y, c'(r)\rangle + f'(r)\right\}^2 + r^{2\alpha}}\, d\mathcal{H}^{k-1}(y)\, dr.
\end{aligned}
$$

For any $r > 0$, the function $\Phi : \mathbb{R}^h \to \mathbb{R}^+$

$$
\Phi(z) = \int_{|y|=1} \sqrt{\left(\langle y, z\rangle + f'(r)\right)^2 + r^{2\alpha}}\, d\mathcal{H}^{k-1}(y)
$$

is strictly convex. This follows from the strict convexity of $t \mapsto \sqrt{r^{2\alpha} + t^2}$. The function Φ is also radially symmetric because the integral is invariant under orthogonal transformations. It follows that Φ attains the minimum at the point $z = 0$ and that this minimum point is unique.

Denoting by F^* the generating set of E^*, we deduce that if c' is not 0 a.e., then we have the strict inequality $P_\alpha(E^*) = Q(F^*) < Q(F) = P_\alpha(E)$, and E is not isoperimetric. Hence, c is constant and this concludes the proof.

□

2.6 Remarks about uniqueness and convexity

Let $\alpha \geq 0$, $h, k \geq 1$ be integers and $n = h + k$. When $\alpha = 0$ or $\alpha \geq 0$ and $k = 1$ the isoperimetric set is unique and it is convex (see Remark 2.5.2). In this section we prove that any isoperimetric set is convex in a neighborhood of the origin using its profile function, which satisfies equation (2.5.15):

$$
f'' = \frac{\alpha f'}{r} + (f'^2 + r^{2\alpha})\left(\frac{k-1}{f} - (h-1)\frac{f'}{r^{2\alpha+1}}\right) - C_{hk\alpha}\frac{(f'^2 + r^{2\alpha})^{\frac{3}{2}}}{r^{2\alpha}}.
$$

2.6.1. Convexity

By Proposition 2.5.3 we know that, if f is the profile function of an isoperimetric set $E \subset \mathbb{R}^n$,

$$
f'(r) = -\frac{C_{hk\alpha}}{h} r^{\alpha+1} + o(r^{\alpha+1}) \text{ as } r \to 0.
$$

Inserting the latter asymptotics in (2.5.15), we obtain, as $r \to 0^+$

$$\begin{aligned}
f''(r) &= \frac{\alpha}{r}\Big(-\frac{C_{hk\alpha}}{h}r^{\alpha+1} + o(r^{\alpha+1})\Big) + \Big(\frac{C_{hk\alpha}^2}{h^2}r^{2\alpha+2} + r^{2\alpha} + o(r^{2\alpha+2})\Big) \\
&\quad \cdot \Big(\frac{k-1}{f} - \frac{h-1}{r^\alpha} \cdot \frac{-\frac{C_{hk\alpha}}{h}r^{\alpha+1} + o(r^{\alpha+1})}{r^{\alpha+1}}\Big) - \frac{C_{hk\alpha}}{r^{2\alpha}}\Big(\frac{C_{hk\alpha}^2}{h^2}r^{2\alpha+2} + r^{2\alpha} + o(r^{2\alpha+2})\Big)^{\frac{3}{2}} \\
&= -\frac{C_{hk\alpha}\alpha}{h}r^\alpha + \cdots + r^{2\alpha}\Big(1 + \frac{C_{hk\alpha}^2}{h^2}r^2 + \cdots\Big) \cdot \frac{1}{r^\alpha}\Big(\frac{(k-1)r^\alpha}{f} + (h-1)\frac{C_{hk\alpha}}{h} + \cdots\Big) \\
&\quad - \frac{C_{hk\alpha}r^{3\alpha}}{r^{2\alpha}}\Big(1 + \frac{C_{hk\alpha}^2}{h^2}r^2 + \cdots\Big)^{\frac{3}{2}} \\
&= -\frac{C_{hk\alpha}\alpha}{h}r^\alpha + \cdots + r^\alpha\frac{(h-1)C_{hk\alpha}}{h} + \cdots - C_{hk\alpha}r^\alpha + \cdots \\
&= \Big(\frac{-C_{hk\alpha}\alpha + C_{hk\alpha}h - C_{hk\alpha} - C_{hk\alpha}h}{h}\Big)r^\alpha + o(r^\alpha) = -\frac{C_{hk\alpha}(\alpha+1)}{h}r^\alpha + o(r^\alpha).
\end{aligned}$$

Therefore we deduce that in a neighborhood of 0, the profile function f is concave, since the second order derivative has negative sign.

Remark 2.6.1. According to the behavior of the solution in the case $k = 1$, we expect that the profile function of an isoperimetric set is globally concave.

2.6.2. Uniqueness

The Cauchy Problem for the differential equation (2.5.15), with the initial conditions $f(0) = 1$ and $f'(0) = 0$ has a unique decreasing solution on some interval $[0,\delta]$, with $\delta > 0$, in the class of functions $f \in C^1([0,\delta]) \cap C^\infty((0,\delta])$ such that

$$\lim_{r\to 0^+} \frac{f'(r)}{r^{\alpha+1}} = -\frac{C_{hk\alpha}}{h}.$$

This can be proved using the Banach fixed point Theorem with the norm

$$\|f\| = \max_{r\in[0,\delta]} |f(r)| + \max_{r\in[0,\delta]} \frac{|f'(r)|}{r^{\alpha+1}},$$

as shown in the following proposition.

Proposition 2.6.2. *Let $C > 0$. There exist $\delta > 0$ and $f : [0,\delta] \to \mathbb{R}$ that solve the Cauchy problem*

$$\begin{cases} \text{ode (2.5.15)} & \textit{in } (0,\delta) \\ f(0) = 1 \\ f'(0) = 0. \end{cases} \tag{CP}$$

Moreover, the solution is unique.

To prove Proposition 2.6.2 we need to use another equivalent formulation of equation (2.5.15):

$$
\begin{aligned}
&-f''+\frac{\alpha f'}{\rho}+(f'^2+\rho^{2\alpha})\left(\frac{k-1}{f}-(h-1)\frac{f'}{\rho^{2\alpha+1}}\right)-C_{hk\alpha}\frac{(f'^2+\rho^{2\alpha})^{\frac{3}{2}}}{\rho^{2\alpha}}=0\\
\iff\ &-f''+\frac{\alpha f'}{\rho}-(h-1)\frac{f'}{\rho}+\rho^{2\alpha}\frac{k-1}{f}\\
&\qquad\qquad+f'^2\left(\frac{k-1}{f}-(h-1)\frac{f'}{\rho^{2\alpha+1}}\right)-C_{hk\alpha}\frac{(f'^2+\rho^{2\alpha})^{\frac{3}{2}}}{\rho^{2\alpha}}=0\iff\\
&\left(\rho^{h-1-\alpha}f'\right)'=\rho^{h-1-\alpha}\left(\rho^{2\alpha}\frac{k-1}{f}+f'^2\left(\frac{k-1}{f}-(h-1)\frac{f'}{\rho^{2\alpha+1}}\right)-C_{hk\alpha}\frac{(f'^2+\rho^{2\alpha})^{\frac{3}{2}}}{\rho^{2\alpha}}\right)
\end{aligned}
$$

Integrating the differential equation on the interval with end points 0 and $\rho\in(0,\rho_0]$ we get

$$
f'(\rho)=\frac{1}{\rho^{h-1-\alpha}}\int_0^\rho t^{h-1+\alpha}\frac{k-1}{f}+t^{h-1-\alpha}f'^2\left(\frac{k-1}{f}-(h-1)\frac{f'}{t^{2\alpha+1}}\right)-C_{hk\alpha}t^{h-1-3\alpha}(f'^2+t^{2\alpha})^{\frac{3}{2}}dt. \tag{2.6.1}
$$

On the other hand

$$
f(\rho)=f(0)+\int_0^\rho f'\,dt=1-\int_0^\rho |f'|\,dt. \tag{2.6.2}
$$

Proof. To prove Proposition 2.6.2 we use Banach fixed point Theorem. Let $\delta>0$. The space of functions $C([0,\delta])\times C([0,\delta])$ endowed with the sup-norm

$$
\|(f,g)\|=\|f\|_\infty+\left\|\frac{g}{\rho^{\alpha+1}}\right\|_\infty=:\|f\|_\infty+\|g\|^*
$$

is a complete metric space. The subset X of $C([0,\delta])^2$

$$
X:=\left\{(f,g)\in(C([0,\delta]))^2:\begin{array}{l}(H1)\ f(0)=1\\ (H2)\ \exists\lim\limits_{\rho\to0^+}\dfrac{g(\rho)}{\rho^{\alpha+1}}=-\dfrac{C_{hk\alpha}}{h}\\ (H3)\ f(\rho)\geq\dfrac{1}{2}\text{ for }\rho\in[0,\delta]\end{array}\right\} \tag{2.6.3}
$$

is closed under uniform convergence: clearly, conditions (H1) and (H3) are preserved. To prove that (H2) is also preserved under uniform convergence, let $(g_j)_{j\in\mathbb{N}}$ be a sequence of continuous functions defined on $[0,\delta]$ which converges uniformly with respect to the norm $\|\cdot\|^*$ to a continuous function g. Therefore we can exchange the two limits as follows

$$
\lim_{\rho\to0^+}\lim_{j\to\infty}\frac{g_j(\rho)}{\rho^{\alpha+1}}=\lim_{j\to\infty}\lim_{\rho\to0^+}\frac{g_j(\rho)}{\rho^{\alpha+1}}=\lim_{j\to\infty}\left(-\frac{C_{hk\alpha}}{h}\right)=-\frac{C_{hk\alpha}}{h}
$$

and prove that condition (H2) is stable under uniform convergence. Therefore the metric space (X,d) is complete with respect to the metric $d((f,g),(\hat f,\hat g))=\|(f-\hat f,g-\hat g)\|$. Notice that condition (H2) implies $g(0)=0$.

We define the mapping $T : X \to X$, $T(f,g)(\rho) =: (f^T(\rho), g^T(\rho))$

$$\begin{aligned} f^T(\rho) &= 1 - \int_0^\rho |g|\, dt, \\ g^T(\rho) &= \rho^{\alpha+1-h} \int_0^\rho t^{h-1+\alpha}\frac{k-1}{f} + t^{h-1-\alpha} g^2\Big(\frac{k-1}{f} - (h-1)\frac{g}{t^{2\alpha+1}}\Big) \\ &\quad - C_{hk\alpha} t^{h-1-3\alpha}(g^2 + t^{2\alpha})^{\frac{3}{2}} dt \end{aligned} \tag{2.6.4}$$

for $\rho \in (0, \rho_0]$, the maximal interval of definition of (f,g). If f is a solution to the Cauchy Problem (CP), there exists $\delta > 0$ such that $(f, f') \in X$. In fact, (H1) is obviously satisfied. Moreover arguing as in Propostition 2.5.1 we deduce $f \in C^1([0,\delta])$. Hence, (H3) is satisfied for a suitable $\delta > 0$. In conclusion, assumption (H2) follows as in Proposition 2.5.3. By definition of the mapping T and equations (2.6.1) and (2.6.2), the couple (f, f') is therefore a fixed point of the mapping T. On the other hand, if a couple of continuous function (f,g) solves the fixed point equation $T(f,g) = (f,g)$ then $g = f'$, f is of class C^2 and solves the Cauchy Problem (CP). To show uniqueness of the solution we apply the Banach fixed point Theorem. We shall see that, for a suitable choice of δ, $T(f,g) \in X$ for all $(f,g) \in X$.

$(H1):$ $f^T(0) = 1$.

$(H3):$ Since $\dfrac{g(\rho)}{\rho^{\alpha+1}} \to -\dfrac{C_{hk\alpha}}{h}$ for $\rho \to 0^+$, there exists $\delta > 0:$

$\left|\dfrac{g(t)}{t^{\alpha+1}}\right| \le \dfrac{C_{hk\alpha}}{h} + 1$ for every $t \in [0,\delta]$. Therefore

$$\begin{aligned} f^T(\rho) &= 1 - \int_0^\rho |g(t)|\, dt \ge 1 - \Big(\frac{C_{hk\alpha}}{h} + 1\Big)\int_0^\rho t^{\alpha+1}\, dt = 1 - \Big(\frac{C_{hk\alpha}}{h} + 1\Big)\Big[\frac{t^{\alpha+2}}{\alpha+2}\Big]_0^\rho \\ &= 1 - \Big(\frac{C_{hk\alpha}}{h} + 1\Big)\frac{\rho^{\alpha+2}}{\alpha+2} \ge 1 - \Big(\frac{C_{hk\alpha}}{h} + 1\Big)\frac{\delta^{\alpha+2}}{\alpha+2} \end{aligned}$$

which is greater than $\dfrac{1}{2}$ if and only if $\delta^{\alpha+2} \le \dfrac{1}{2}(\alpha+2)\dfrac{1}{\frac{C_{hk\alpha}}{h} + 1}$.

We are left to see that (H2) holds for g^T. Let $\rho \in [0,\delta]$. We are going to use (H2) for g, (H3) for f and the de l'Hôpital rule. We have:

$$\begin{aligned} &\lim_{\rho\to 0^+} \frac{g^T(\rho)}{\rho^{\alpha+1}} \\ &= \lim_{\rho\to 0^+} \frac{1}{\rho^h}\int_0^\rho t^{h-1+\alpha}\frac{k-1}{f} + t^{h-1-\alpha}g^2\Big(\frac{k-1}{f} - (h-1)\frac{g}{t^{2\alpha+1}}\Big) - C_{hk\alpha}t^{h-1-3\alpha}(g^2+t^{2\alpha})^{\frac{3}{2}} dt \\ &\overset{H}{=} \lim_{\rho\to 0^+} \frac{1}{h\rho^{h-1}}\Big(\rho^{h-1+\alpha}\frac{k-1}{f} + \rho^{h-1-\alpha}g^2\Big(\frac{k-1}{f} - (h-1)\frac{g}{\rho^{2\alpha+1}}\Big) - C_{hk\alpha}\rho^{h-1-3\alpha}(g^2+\rho^{2\alpha})^{\frac{3}{2}}\Big) \\ &= \lim_{\rho\to 0^+} \frac{1}{h}\left[\frac{k-1}{f}\Big(\rho^\alpha + \frac{g^2}{\rho^\alpha}\Big) - (h-1)\frac{g^3}{\rho^{3\alpha+1}} - C_{hk\alpha}\Big(\Big(\frac{g}{\rho^\alpha}\Big)^2 + 1\Big)^{\frac{3}{2}}\right] \\ &= \lim_{\rho\to 0^+} \frac{1}{h}\left[\rho^\alpha\frac{k-1}{f}\Big(\Big(\frac{g}{\rho^\alpha}\Big)^2 + 1\Big) - (h-1)\frac{g^3}{\rho^{3\alpha+1}} - C_{hk\alpha}\Big(\Big(\frac{g}{\rho^\alpha}\Big)^2 + 1\Big)^{\frac{3}{2}}\right] = -\frac{C_{hk\alpha}}{h} \end{aligned}$$

where we have used

$$\lim_{\rho\to 0^+}\frac{g}{\rho^\alpha}=\lim_{\rho\to 0^+}\frac{g}{\rho^{\alpha+1}}\rho=0$$
$$\lim_{\rho\to 0^+}\frac{g^3}{\rho^{3\alpha+1}}=\lim_{\rho\to 0^+}\frac{g}{\rho^{\alpha+1}}\left(\frac{g}{\rho^\alpha}\right)^2=0.$$

In conclusion, we may choose δ small enough to define $T: X\to X$.

To apply Banach fixed point Theorem we are left to show that T is a contraction, namely that there exists $\lambda<1$ such that, given (f,g) and $(\hat f,\hat g)$ in X,

$$\|f^T-\hat f^T\|_\infty+\|g^T-\hat g^T\|^*\le\lambda(\|f-\hat f\|_\infty+\|g-\hat g\|^*)$$

We have, given $\rho\in[0,\delta]$,

$$\begin{aligned}|f^T(\rho)-\hat f^T(\rho)|&=\left|\int_0^\rho|\hat g(t)|-|g(t)|\,dt\right|\le\int_0^\rho\big|\,|\hat g(t)|-|g(t)|\,\big|\,dt\\&\le\int_0^\rho|\hat g(t)-g(t)|\,dt=\int_0^\rho\left|\frac{\hat g(t)-g(t)}{t^{\alpha+1}}\right|t^{\alpha+1}\,dt\le\|g-\hat g\|^*\left[\frac{t^{\alpha+2}}{\alpha+2}\right]_0^\rho\\&\le\frac{\delta^{\alpha+2}}{\alpha+2}\|g-\hat g\|^*\le\frac{\delta^{\alpha+2}}{\alpha+2}(\|f-\hat f\|_\infty+\|g-\hat g\|^*).\end{aligned}$$

By arbitrariness of $\rho\in[0,\delta]$, we get $\|f^T-\hat f^T\|\le\dfrac{\delta^{\alpha+2}}{\alpha+2}\|(f-\hat f,g-\hat g)\|$. Let's now see that the same holds for $\|g-\hat g\|^*$ for a proper $\lambda<1$. We have

$$\begin{aligned}\left|\frac{g^T-\hat g^T}{\rho^{\alpha+1}}\right|=&\frac{1}{\rho^h}\left|\int_0^\rho(k-1)\left[\frac{t^{h-1+\alpha}}{f}-\frac{t^{h-1+\alpha}}{\hat f}+\frac{g^2t^{h-1-\alpha}}{f}-\frac{\hat g^2t^{h-1-\alpha}}{\hat f}\right]dt\right.\\&\left.+\int_0^\rho(h-1)\left[\frac{\hat g^3-g^3}{t^{3\alpha+2-h}}\right]dt+C_{hk\alpha}\int_0^\rho\frac{(\hat g^2+t^{2\alpha})^{\frac32}-(g^2+t^{2\alpha})^{\frac32}}{t^{3\alpha+1-h}}\,dt\right|\\&\le\frac{(k-1)}{\rho^h}\int_0^\rho\frac{\left|t^{h-1+\alpha}(\hat f-f)\right|+\left|t^{h-1-\alpha}(g^2\hat f-\hat g^2f)\right|}{f\hat f}\,dt\\&+\frac{(h-1)}{\rho^h}\int_0^\rho\left|\frac{\hat g^3-g^3}{t^{3\alpha+2-h}}\right|dt+\frac{C_{hk\alpha}}{\rho^h}\int_0^\rho\frac{\left|(\hat g^2+t^{2\alpha})^{\frac32}-(g^2+t^{2\alpha})^{\frac32}\right|}{t^{3\alpha+1-h}}\,dt\\&=:A+B+C,\end{aligned}$$

where A and B can be estimated as follows:

$$\begin{aligned}A=&\frac{(k-1)}{\rho^h}\int_0^\rho\frac{\left|t^{h-1+\alpha}(\hat f-f)\right|+\left|t^{h-1-\alpha}(g^2\hat f-\hat g^2f)\right|}{f\hat f}\,dt\\&\le\frac{4(k-1)}{\rho^h}\int_0^\rho t^{h-1+\alpha}\left|\hat f-f\right|+t^{h-1-\alpha}\left|g^2\hat f-\hat f\hat g^2+\hat f\hat g^2-\hat g^2f\right|dt\\&\le\frac{4(k-1)}{\rho^h}\left\{\left[\frac{t^{h+\alpha}}{h+\alpha}\right]_0^\rho\|f-\hat f\|_\infty+\int_0^\rho t^{h-1-\alpha}\left[|\hat f|\,|\hat g^2-g^2|+|\hat g^2|\,|\hat f-f|\right]dt\right\}\\&\le\frac{4(k-1)}{h+\alpha}\rho^\alpha\|f-\hat f\|_\infty+\frac{4(k-1)}{\rho^h}\|\hat f\|_\infty\left\|\frac{g+\hat g}{\rho^{\alpha+1}}\right\|_\infty\left\|\frac{g-\hat g}{\rho^{\alpha+1}}\right\|_\infty\int_0^\rho t^{h-1-\alpha+2\alpha+2}dt\end{aligned}$$

$$
\begin{aligned}
&+ \frac{4(k-1)}{\rho^h}\left\|\left(\frac{\hat{g}^2}{\rho^{\alpha+1}}\right)^2\right\|_\infty \|f-\hat{f}\|_\infty \int_0^\rho t^{h+\alpha+1}dt \\
\leq & 4(k-1)\left\{\frac{\delta^\alpha\|f-\hat{f}\|_\infty}{h+\alpha} + \frac{[\|\hat{f}\|_\infty \|g+\hat{g}\|^* \|g-\hat{g}\|^* \rho^{\alpha+2} + \|\hat{g}\|^{*2}\|f-\hat{f}\|_\infty \rho^{\alpha+2}}{h+\alpha+2}\right\} \\
\leq & \frac{4(k-1)\delta^\alpha}{h+\alpha}\|f-\hat{f}\|_\infty + \frac{[\|\hat{f}\|_\infty \|g+\hat{g}\|^* \delta^{\alpha+2}}{h+\alpha+2}\|g-\hat{g}\|^* + \frac{\|\hat{g}\|^{*2}\delta^{\alpha+2}}{h+\alpha+2}\|f-\hat{f}\|_\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
B = & \frac{h-1}{\rho^h}\int_0^\rho \frac{|\hat{g}-g|\,|\hat{g}^2+\hat{g}g+g^2|}{t^{3\alpha+2-h}}dt = \frac{h-1}{\rho^h}\int_0^\rho t^{h+1}\frac{|\hat{g}-g|}{t^{\alpha+1}}\frac{|\hat{g}^2+\hat{g}g+g^2|}{t^{2\alpha+2}}dt \\
\leq & \frac{h-1}{\rho^h}\|\hat{g}^2+\hat{g}g+g^2\|^*\|\hat{g}-g\|^*\left[\frac{t^{h+2}}{h+2}\right]_0^\rho \leq \frac{h-1}{h+2}\|\hat{g}^2+\hat{g}g+g^2\|^*\delta^{h+2}\|\hat{g}-g\|^*.
\end{aligned}
$$

Therefore we can choose δ sufficiently small to have $|g^T(x)-\hat{g}^T(x)| \leq \|(f-\hat{f}, g-\hat{g})\|$ which leads to the conclusion that T is a contraction.

We deduce that there exists a unique fixed point for the equation $T(f,g) = (f,g)$, which corresponds to a unique solution of the Cauchy Problem (CP). Moreover, we deduce that (CP) has a unique maximal solution f satisfying the initial conditions $f(0) = 1$ and $f'(0) = 0$. □

Remark 2.6.3. From Theorem 2.4.3 and Proposition 2.5.3, there exists a value of the constant $C_{hk\alpha} > 0$ such that the maximal decreasing solution of the Cauchy Problem (CP) has a maximal interval $[0, r_0]$ such that $f(r_0) = 0$. We expect that there exists only one C, such that the unique soultion to (CP) is the profile function of an isoperimetric set, namely we expect that for only one $C > 0$ the solution closes at r_0. If a characterization of constant mean curvature surfaces in this context were available, we could probably use it to deduce this uniqueness property, as it is done in the case of the Heisenberg group (see page 119 in [96], using [117]).

CHAPTER 3

Quantitative Isoperimetric Inequalities via Subcalibrations

A classical tool to prove minimality of the perimeter measure of minimal surfaces in $\mathbb{R}^n$ is the technique of calibrations. For instance, the original proof of the minimality of the Simons cone $S \subset \mathbb{R}^8$ by Bombieri, De Giorgi and Giusti [18] is based on finding a suitable *calibration*, i.e., a divergence free vector field with norm less then or equal to 1, which extends the unit normal to the surface S to the whole $\mathbb{R}^n$. The "model computation" that proves minimality through calibrations is the following. If g is a calibration for the boundary of a set $E \subset \mathbb{R}^n$ we formally apply the divergence theorem and use Cauchy-Schwartz inequality: for any competitor $F \subset \mathbb{R}^n$ such that $E \,\triangle\, F \subset\subset B_R$

$$\begin{aligned} 0 &= \int_{E\smallsetminus F} \mathrm{div} g \, dx = \int_{(\partial E\smallsetminus F)\cap B_R} \langle g, N^E\rangle \, d\mathcal{H}^{n-1} - \int_{(\partial F\cap E)\cap B_R} \langle g, N^F\rangle \, d\mathcal{H}^{n-1} \\ &= \mathcal{H}^{n-1}(\partial E \smallsetminus F) - \int_{(\partial F\cap E)\cap B_R} \langle g, N^F\rangle \, d\mathcal{H}^{n-1} \\ &\geq \mathcal{H}^{n-1}(\partial E \smallsetminus F) - \mathcal{H}^{n-1}(\partial F \cap E), \end{aligned} \tag{Cal}$$

where N^E (resp. N^F) denotes the outer unit normal to E (resp. F). Equivalently, integrating on $F \smallsetminus E$, $0 \leq \mathcal{H}^{n-1}(\partial F \smallsetminus E) - \mathcal{H}^{n-1}(\partial E \cap F)$, hence

$$\mathcal{H}^{n-1}(\partial E) \leq \mathcal{H}^{n-1}(\partial F).$$

In [42] and [43], the authors notice that to perform such a computation it is enough to consider a vector field g whose divergence has a sign inside E and outside E, according to the inequalities in the model computation above (see (Cal)). Namely, it is enough to consider $g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

- $g(x) = N^E(x)$ is the outer unit normal to ∂E for every $x \in \partial E$;
- $\mathrm{div} g(x) \leq 0$ for $x \in E$, $\mathrm{div} g(x) \geq 0$ for $x \in \mathbb{R}^n \smallsetminus E$;

- $|g| \le 1$ in $\mathbb{R}^n$.

Such a vector field is called a *sub-calibration* (also "*quantitative calibration*") for E. As De Philippis and Maggi show in [42], sub-calibrations also provide estimates for the difference $\mathcal{H}^{n-1}(\partial F) - \mathcal{H}^n(\partial E)$ in terms of the Lebesgue measure of the symmetric difference $E \triangle F$, also known as quantitative estimates.

More generally, given a bounded set E whose boundary has constant mean curvature H, we can look for a a vector field $g \in C^1$ satisfying $\mathrm{div} g = H$, also called calibration for E. If such a calibration exists for $E \subset \mathbb{R}^n$, the computation (Cal) can be reformulated to prove that the set E is isoperimetric.

Ritoré in [116] uses this approach to prove that the Pansu ball

$$E_{\text{isop}} = \{(z,t) \in \mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} : |t| < \arccos|z| + |z|\sqrt{1-|z|^2},\ |z| < 1\}$$

is isoperimetric in the class of sets $E \subset \mathbb{H}^n$ of finite H-perimeter, such that

$$\{(z,0) \in \mathbb{H}^n : |z| < 1\} \subset E \subset \{(z,t) \in \mathbb{H}^n : |z| < 1\}.$$

In this chapter we show how to refine the argument of Ritoré via a sub-calibration to obtain *quantitative isoperimetric inequalities in* $\mathbb{H}^n$ (see Theorem 3.2.1). Moreover we show how to use this technique to prove quantitative inequalities in $\mathbb{R}^n$ and in Grushin spaces (see Sections 3.4, 3.3).

3.1 Isoperimetric deficit and asymmetry

Given a Lebesgue measurable set $E \subset \mathbb{R}^n$, we denote by $P(E)$ its euclidean perimeter. The standard *isoperimetric deficit of* E is defined as the quantity

$$D(E) = \frac{P(E) - P(B_E(0,r(E)))}{P(B_E(0,r(E)))} \tag{3.1.1}$$

where $r(E) = (\mathcal{L}^n(E)/\omega_n)^{1/n}$ satisfies $\mathcal{L}^n(B_E(0,r(E))) = \omega_n r(E)^n = \mathcal{L}^n(E)$. The quantity $D(E)$ describes the gap between the perimeter of E and the perimeter of an isoperimetric set with the same volume. The *Fraenkel asymmetry* measures how far a set is from being isoperimetric through its volume: given $E \subset \mathbb{R}^n$, the latter is defined as

$$A(E) = \min_{x\in\mathbb{R}^n} \frac{\mathcal{L}^n(E \triangle B_E(x,r(E)))}{\mathcal{L}^n(E)}. \tag{3.1.2}$$

A *quantitative isoperimetric inequality* is an estimate of the Fraenkel asymmetry in terms of the isoperimetric deficit, see for instance [62], [73], [74], [63], [35], [36], and the review in [64].

In [73, Theorem 2], Hall proves a quantitative isoperimetric inequality for axially symmetric sets: for any set $E \subset \mathbb{R}^n$ which has an axis of symmetry and such that all the non-empty cross sections of E perpendicular to this axis are $(n-1)$-dimensional balls,

$$D(E) \geq C(n)A(E)^2 \tag{3.1.3}$$

where $C = C(n) > 0$ is a dimensional constant. The exponent 2 is sharp (it cannot be improved). In [63], the sharp quantitative isoperimetric inequality for all sets of finite perimeter is proved, i.e., (3.1.3) holds true for any Borel set $E \subset \mathbb{R}^n$ with finite Lebesgue measure. We refer to Section 4.2 for a review of the proof given in [63] of Hall's inequality.

Remark 3.1.1. For any measurable set $E \subset \mathbb{R}^n$, we have

$$A(E) \leq \frac{\mathcal{L}^n(E \triangle B(0, r(E)))}{\mathcal{L}^n(E)} \leq \frac{\mathcal{L}^n(E) + \mathcal{L}^n(B_E(0, r(E)))}{\mathcal{L}^n(E)} = 2.$$

Hence, to prove (3.1.3) it is possible to assume without loss of generality that $D(E) \leq \delta(n)$ for some dimensional constant $\delta = \delta(n) > 0$. In fact, if $D(E) \geq \delta$, we have

$$A(E) \leq 2 = \frac{2}{\sqrt{\delta}}\sqrt{\delta} \leq C(n)\sqrt{D(E)}$$

and (3.1.3) holds true for $C \geq 2/\sqrt{\delta}$.

3.1.1. Isoperimetric deficit and asymmetry in $\mathbb{H}^n$

As the sharp isoperimetric inequality is not established in full generality (see Subsection 1.4.1), our techniques will give results in a class of sets satisfying certain assumptions.

We introduce the H-isoperimetric deficit, and the H-asymmetry of a Lebesgue measurable set $E \subset \mathbb{H}^n$ with respect to the Pansu ball $E_{\text{isop}} = \{(z,t) \in \mathbb{H}^n : |t| < \varphi(|z|),\ |z| < 1\}$, where

$$\varphi(r) = \arccos r + r\sqrt{1 - r^2}. \tag{3.1.4}$$

Given $E \subset \mathbb{H}^n$, we set

$$B(E) = \delta_{\lambda(E)}(E_{\text{isop}}), \quad \lambda(E) = \Big(\frac{\mathcal{L}^{2n+1}(E)}{\mathcal{L}^{2n+1}(E_{\text{isop}})}\Big)^{\frac{1}{Q}}$$

the dilation of the Pansu ball such that $\mathcal{L}^{2n+1}(B(E)) = \mathcal{L}^{2n+1}(E)$.

Definition 3.1.2 (H-isoperimetric deficit and asymmetry)**.** We call *H-isoperimetric deficit* of E with respect to the Pansu ball, the quantity

$$D_H(E) = \frac{P_H(E) - P_H(B(E))}{P_H(B(E))} = \frac{P_H(E)}{P_H(E_{\text{isop}})}\Big(\frac{\mathcal{L}^{2n+1}(E_{\text{isop}})}{\mathcal{L}^{2n+1}(E)}\Big)^{\frac{Q-1}{Q}} - 1. \tag{3.1.5}$$

We also define the *H-asymmetry* with respect to the Pansu ball, as

$$A_H(E) = \min_{p \in \mathbb{H}^n} \frac{\mathcal{L}^{2n+1}(E \triangle \tau_p(B(E)))}{\mathcal{L}^{2n+1}(E)}. \tag{3.1.6}$$

By the homogeneity properties of P_H and $\mathcal{L}^{2n+1}$ with respect to the dilations $\delta_\lambda(z,t) = (\lambda z, \lambda^2 t)$ (see Proposition 1.1.14 and Proposition 1.2.9), both the asymmetry and the isoperimetric deficit are scale invariant, namely

$$\begin{aligned} D_H(\delta_\lambda(E)) &= \frac{P_H(\delta_\lambda(E)) - P_H(B(\delta_\lambda(E))}{P_H(B(\delta_\lambda(E))} = \frac{P_H(\delta_\lambda(E)) - P_H(\delta_\lambda(B(E))}{P_H(\delta_\lambda(B(E))} = D_H(E) \\ A_H(\delta_\lambda(E)) &= \min_{p\in\mathbb{H}^n} \frac{\mathcal{L}^{2n+1}(\delta_\lambda(E) \vartriangle B(\delta_\lambda(E)))}{\mathcal{L}^{2n+1}(\delta_\lambda(E))} = \min_{p\in\mathbb{H}^n} \frac{\mathcal{L}^{2n+1}(\delta_\lambda(E \vartriangle B(E)))}{\mathcal{L}^{2n+1}(\delta_\lambda(E))} = A_H(E). \end{aligned} \tag{3.1.7}$$

This allows us to consider only sets $E \subset \mathbb{H}^n$ with $\mathcal{L}^{2n+1}(E) = \mathcal{L}^{2n+1}(E_{\text{isop}})$, so that $B(E) = E_{\text{isop}}$.

The result that we would like to prove is the existence of a constant $C = C(n)$ such that

$$A_H(E)^2 \le C(n) D_H(E) \text{ for all sets } E \subset \mathbb{H}^n \tag{3.1.8}$$

satisfying appropriate conditions. In what follows we show some properties of deficit and asymmetry that are listed in view of showing optimality of the exponent 2 in (3.1.8), in Example 3.1.8.

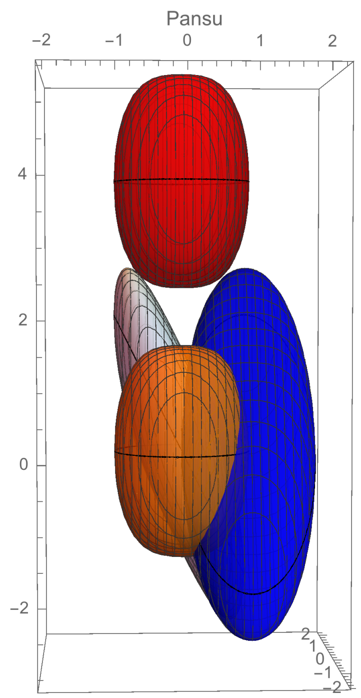


Figure 3.1: Left translations of the Pansu ball, $\tau_p(E_{\text{isop}})$. In orange $p = 0$, in red $p = (0,0,4)$, in blue $p = (1,0,0)$, in light blue $p = (0,1,0)$.

Remark 3.1.3. In this remark we collect some facts that help us in understanding the asymmetry in the sub-Riemannian setting.

First of all, notice that for every $p \in \mathbb{H}^n$, $\tau_p(E_{\text{isop}})$ has the same perimeter of E_{isop} since the H-perimeter is invariant under left-translation (see Proposition 1.2.9). Moreover, since the Lebesgue measure is the Haar measure of every Carnot group, also the volume is preserved under left-translation.

On the other hand, a left translation of E_{isop}, $\tau_p(E_{\text{isop}})$ is a euclidean translation of it if and only if $p = (0, 0, t)$ for some $t \in \mathbb{R}$: roughly speaking the shape of isoperimetric sets changes while left-translating them (see Figure 3.1). Namely, if $p = (x, y, t) = (z, t) \in \mathbb{H}^n$

$$\tau_p(E_{\text{isop}}) = \{(z', t') \in \mathbb{H}^n : |t' - t - 2(xy' - x'y)| < \varphi(|z' - z|), \},$$

hence, if $p = (0, 0, t)$ $\tau_p(E_{\text{isop}}) = E_{\text{isop}} + p$, otherwise $\tau_p(E_{\text{isop}})$ is not obtained through euclidean translations or rotations of E_{isop}. Nonetheless, the "shape" of isoperimetric sets is preserved under a certain type of translations, as we show in the next proposition.

We say that a map $T : \mathbb{H}^n \to \mathbb{H}^n$ is *measure preserving* if $\mathcal{L}^{2n+1}(T(E)) = \mathcal{L}^{2n+1}(E)$ for any measurable set $E \subset \mathbb{H}^n$.

Proposition 3.1.4. *Let $p = (z, t) \in \mathbb{C}^n \times \mathbb{R} = \mathbb{H}^n$, $q = (\zeta, \tau) \in \mathbb{H}^n$ be such that $|z| = |\zeta|$ and $\tau = -t$. Then there exists a measure preserving automorphism $T : \mathbb{H}^n \to \mathbb{H}^n$ such that $\tau_q(E_{\text{isop}}) = T(\tau_p(E_{\text{isop}}))$.*

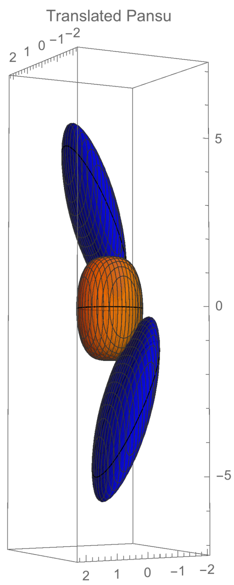


Figure 3.2: $\tau_p(E_{\text{isop}})$ for $p = (1, 0, 3)$ and $(-1, 0, -3)$.

Proof. Since $|z| = |\zeta|$, there exists a unitary transformation $U \in \mathscr{U}(n)$ such that $\overline{U(z)} = \zeta$.

We recall that the *unitary group in* $\mathbb{C}^n$, $\mathscr{U}(n)$, is defined as the set of complex $n \times n$ matrices

$$U = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix} \quad u_{ij} \in \mathbb{C}$$

such that $UU^\dagger = U^\dagger U = I_n$, I_n is the identity matrix and $U^\dagger = (\overline{U})^T$ is the conjugate transposed of U. Notice that the maps

$$\begin{aligned} &\mathcal{I} : \mathbb{H}^n \to \mathbb{H}^n, \quad \mathcal{I}(\xi, \eta) = (\overline{\xi}, -\eta) \\ &W : \mathbb{H}^n \to \mathbb{H}^n, \quad W(\xi, \eta) = (U(\xi), \eta) \end{aligned} \qquad (\xi, \eta) \in \mathbb{H}^n$$

are automorphisms of $\mathbb{H}^n$, i.e., $\mathcal{I}((\xi,\eta) * (\xi',\eta')) = \mathcal{I}(\xi,\eta) * \mathcal{I}(\xi',\eta')$ and $W((\xi,\eta)*(\xi',\eta')) = W(\xi,\eta) * W(\xi',\eta')$ for every $(\xi,\eta),(\xi',\eta') \in \mathbb{H}^n$. Then the map $T = \mathcal{I} \circ W$ is an automorphism of $\mathbb{H}^n$ itself. Moreover,

$$T : \mathbb{H}^n \to \mathbb{H}^n, \; T(\xi, \eta) = (\overline{U(\xi)}, -\eta), \qquad (\xi, \eta) \in \mathbb{H}^n,$$

hence it is measure preserving, $(\zeta, \tau) = T(z,t)$ and for every $(\xi, \eta) \in \mathbb{H}^n$

$$T(\tau_p(\xi,\eta)) = T((z,t) * (\xi,\eta)) = T(z,t) * T(\xi,\eta) = (\zeta,\tau) * T(\xi,\eta) = \tau_q(T(\xi,\eta)). \quad (3.1.9)$$

Notice also that, by symmetry properties of E_{isop}, $(z_0,t_0) \in E_{\text{isop}}$ if and only if $T(z_0,t_0) \in E_{\text{isop}}$, hence $T(E_{\text{isop}}) = E_{\text{isop}}$. We conclude from (3.1.9) that $T(\tau_p(E_{\text{isop}})) = \tau_q(E_{\text{isop}})$. □

Corollary 3.1.5. *Let $p = (z,t), q = (\zeta,\tau) \in \mathbb{H}^n$ be such that $|z| = |\zeta|$, $|t| = |\tau|$. Then there exists a measure preserving automorphism $T : \mathbb{H}^n \to \mathbb{H}^n$ such that*

$$\tau_q(E_{\text{isop}}) = T(\tau_p(E_{\text{isop}})) \quad (3.1.10)$$

Proof. If $t = -\tau$, T is the one given by Proposition 3.1.4. If $t = \tau$, recalling that there exists $\mathcal{U} \in U(n)$ such that $\zeta = U(z)$, we compose the transformation in Proposition 3.1.4 with $W : \mathbb{H}^n \to \mathbb{H}^n$, $W(\xi,\eta) = (U(\xi),\eta)$. □

We recall Definition 2.1.1 in the case of $\mathbb{H}^n$. We say that a set $E \subset \mathbb{H}^n = \mathbb{C}^n_z \times \mathbb{R}_t$ is z-spherically symmetric if there exists a set $F \subset \mathbb{R}^+ \times \mathbb{R}$ such that $E = \{(z,t) \in \mathbb{H}^n : (|z|,t) \in F\}$. Moreover, we say that E is t-symmetric if $(z,t) \in E \iff (z,-t) \in E$.

In the case of a z-Schwartz and t-symmetric set E, the transformation involved in Corollary 3.1.5 is such that $T(E) = E$. Hence the following holds.

Lemma 3.1.6. *If $p_0 = (z_0, t_0) \in \mathbb{H}^n$ is such that*

$$A_H(E) = \frac{\mathcal{L}^{2n+1}(E \triangle \tau_{p_0}(B(E))}{\mathcal{L}^{2n+1}(E)},$$

then we also have

$$A_H(E) = \frac{\mathcal{L}^{2n+1}(E \triangle \tau_p(B(E))}{\mathcal{L}^{2n+1}(E)} \tag{3.1.11}$$

for every $p = (z,t) \in \mathbb{H}^n$ such that $|z| = |z_0|$ and $|t| = |t_0|$.

Proof. Since the asymmetry is scale invariant, it is enough to prove it for sets such that $\mathcal{L}^{2n+1}(E) = \mathcal{L}^{2n+1}(E_{\text{isop}})$. Given $p \in \mathbb{H}^n$ as above,

$$\mathcal{L}^{2n+1}(E \triangle \tau_p(E_{\text{isop}})) = 2\mathcal{L}^{2n+1}(E \setminus \tau_p(E_{\text{isop}})) = 2(\mathcal{L}^{2n+1}(E) - \mathcal{L}^{2n+1}(\tau_p(E_{\text{isop}}) \cap E)).$$

Let $T : \mathbb{H}^n \to \mathbb{H}^n$ be as in (3.1.10), such that $T(\tau_p(E_{\text{isop}})) = \tau_{p_0}(E)$. Recall that T is volume preserving and notice that, by symmetry of E, $T(E) = E$. Then

$$\mathcal{L}^{2n+1}(\tau_p(E_{\text{isop}}) \cap E) = \mathcal{L}^{2n+1}(T(\tau_p(E_{\text{isop}}) \cap E)) = \mathcal{L}^{2n+1}(\tau_{p_0}(E_{\text{isop}}) \cap E),$$

hence $\mathcal{L}^{2n+1}(E \setminus \tau_p(E_{\text{isop}})) = \mathcal{L}^{2n+1}(E \setminus \tau_{p_0}(E_{\text{isop}}))$. We conclude

$$\frac{\mathcal{L}^{2n+1}(E \triangle \tau_p(E_{\text{isop}}))}{\mathcal{L}^{2n+1}(E)} = \frac{\mathcal{L}^{2n+1}(E \triangle \tau_{p_0}(E_{\text{isop}}))}{\mathcal{L}^{2n+1}(E)} = A_H(E).$$

□

In the next proposition we prove a property of symmetric sets using Lemma 3.1.6 and following Lemma 2.2 in [63]: the asymmetry with respect to the Pansu ball is equivalent to the symmetric difference with the Pansu ball itself.

Proposition 3.1.7. *Let $E \subset \mathbb{H}^n = \mathbb{C}^n_z \times \mathbb{R}_t$ be a z-spherically symmetric and t-symmetric, set. Then*

$$\frac{\mathcal{L}^{2n+1}(E \triangle B(E))}{\mathcal{L}^{2n+1}(E)} \geq A_H(E) \geq \frac{1}{2^{2n+1}} \frac{\mathcal{L}^{2n+1}(E \triangle B(E))}{\mathcal{L}^{2n+1}(E)}.$$

Proof. The first inequality comes from the definition of asymmetry.

To prove the other one, we let $Q^+ = \{(z,t) \in \mathbb{H}^n : x_i \geq 0, y_i \geq 0, t \geq 0, \text{ for } i = 1, \dots, n\}$ and $Q^- = \{(z,t) \in \mathbb{H}^n : x_i \leq 0, y_i \leq 0, t \leq 0, \text{ for } i = 1, \dots, n\}$. By definition and by Lemma 3.1.6, there exist $p_0 = (x_0, 0, \dots, 0, t_0) \in Q^-$ such that

$$A_H(E) = \frac{\mathcal{L}^{2n+1}(E \triangle \tau_{p_0}(B(E)))}{\mathcal{L}^{2n+1}(E)}.$$

Moreover $(E \smallsetminus B(E)) \cap Q^+ \subset (E \smallsetminus \tau_{p_0}(B(E))) \cap Q^+$. Hence, we have

$$\begin{aligned}\frac{\mathcal{L}^{2n+1}(E \triangle B(E))}{\mathcal{L}^{2n+1}(E)} &= \frac{2\mathcal{L}^{2n+1}(E \smallsetminus B(E))}{\mathcal{L}^{2n+1}(E)} = \frac{2 \cdot 2^{2n+1}\mathcal{L}^{2n+1}((E \smallsetminus B(E)) \cap Q^+)}{\mathcal{L}^{2n+1}(E)} \\ &\leq 2^{2n+2}\frac{\mathcal{L}^{2n+1}((E \smallsetminus \tau_{p_0}(B(E))) \cap Q^+)}{\mathcal{L}^{2n+1}(E)} \leq 2^{2n+2}\frac{\mathcal{L}^{2n+1}(E \smallsetminus \tau_{p_0}(B(E)))}{\mathcal{L}^{2n+1}(E)} \\ &= 2^{2n+1}\frac{\mathcal{L}^{2n+1}(E \triangle \tau_{p_0}(B(E)))}{\mathcal{L}^{2n+1}(E)} = 2^{2n+1}A_H(E).\end{aligned}$$

□

The next example is concerned with the optimality of the exponent 2 in (3.1.8). The following construction is based on the one introduced by Maggi in [88, page 382] for the euclidean quantitative isoperimetric inequality.

Example 3.1.8 (Sharpness of the exponent 2)**.** Let $0 < \gamma \leq 1/3$. For any $\varepsilon \geq 0$ we consider the set $E_\varepsilon = \{(z,t) \in \mathbb{H}^n : |t| < f_\varepsilon(|z|),\ |z| < 1\}$, with

$$f_\varepsilon(r) = \begin{cases} \varepsilon\Big(1 - \big(\frac{r}{\varepsilon^\gamma}\big)^2\Big) + \varphi(\varepsilon^\gamma) & \text{if } 0 \leq r < \varepsilon^\gamma \\ \varphi(r) & \text{if } \varepsilon^\gamma \leq r < 1. \end{cases}$$

The function φ is the profile function of the Pansu set, defined in (3.1.4). We show that, given any $0 < \sigma < 2$, it is possible to choose the parameter $\gamma \in (0, 1/3]$ in such a way that

$$\lim_{\varepsilon \to 0} \frac{A_H(E_\varepsilon)^{2-\sigma}}{D_H(E_\varepsilon)} = \infty. \tag{3.1.12}$$

This implies that the exponent 2 in (3.1.8) is optimal.

Notice that for every $\varepsilon > 0$, E_ε is a z- and t-symmetric set, hence Propositions 2.2.3 and 2.2.4 apply to calculate the H-perimeter of E_ε. The euclidean unit normal to the generating set of E_ε is $N(r,t) = (-f_\varepsilon'(r), 1)$, where

$$f_\varepsilon'(r) = \begin{cases} -2\varepsilon^{1-2\gamma} r & \text{if } 0 \leq r < \varepsilon^\gamma \\ \varphi'(r) & \text{if } \varepsilon^\gamma \leq r < 1. \end{cases}$$

Hence, using a Taylor expansion as $\varepsilon \to 0$, we obtain

$$\begin{aligned} P_H(E_\varepsilon) &= 4n\omega_{2n}\int_0^1 \sqrt{f_\varepsilon'(r)^2 + r^2}\, r^{2n-1}\, dr \\ &= 4n\omega_{2n}\Big[\int_0^{\varepsilon^\gamma} \sqrt{4\varepsilon^{2-4\gamma}r^2 + r^2}\, r^{2n-1}\, dr + \int_{\varepsilon^\gamma}^1 \sqrt{\varphi'(r)^2 + r^2}\, r^{2n-1}\, dr\Big] \\ &= P_H(E_{\text{isop}}) + 4n\omega_{2n}\Big[\sqrt{4\varepsilon^{2-4\gamma} + 1}\int_0^{\varepsilon^\gamma} r^{2n}\, dr - \int_0^{\varepsilon^\gamma} \sqrt{\varphi'(r)^2 + r^2}\, r^{2n-1}\, dr\Big], \\ &\leq P_H(E_{\text{isop}}) + 4n\omega_{2n}\Big[\Big(1 + 2\varepsilon^{2-4\gamma} + o(\varepsilon^{2-4\gamma})\Big)\frac{\varepsilon^{\gamma(2n+1)}}{2n+1} - \frac{\varepsilon^{\gamma(2n+1)}}{2n+1}\Big] \\ &= P_H(E_{\text{isop}}) + \frac{8n\omega_{2n}}{2n+1}\varepsilon^{2-4\gamma+\gamma(2n+1)} + o(\varepsilon^{2-4\gamma+\gamma(2n+1)}) \end{aligned}$$

and as $\varepsilon \to 0^+$, we therefore have

$$D_H(E_\varepsilon) \le c_1 \varepsilon^{2-3\gamma+2n\gamma} + o(\varepsilon^{2-3\gamma+2n\gamma}), \tag{3.1.13}$$

with $c_1 = c_1(n) > 0$. In the following $c_i = c_i(n) > 0$.

Now, the function φ has the following Taylor expansion

$$\varphi(\varrho) = \frac{\pi}{2} - 4\varrho^3 + o(\varrho^3), \text{ as } \varrho \to 0^+.$$

We deduce the behavior of the Lebesgue measure of E_ε as $\varepsilon \to 0^+$ as follows:

$$\begin{aligned}
\mathcal{L}^{2n+1}(E_\varepsilon) &= 4n\omega_{2n} \int_0^1 f_\varepsilon(r) r^{2n-1}\, dr \\
&= 4n\omega_{2n} \Big[\int_0^{\varepsilon^\gamma} \Big(\varepsilon\big(1 - \frac{r^2}{\varepsilon^{2\gamma}}\big) + \varphi(\varepsilon^\gamma) \Big) r^{2n-1}\, dr + \int_{\varepsilon^\gamma}^1 \varphi(r) r^{2n-1}\, dr \Big] \\
&= \mathcal{L}^{2n+1}(E_{\text{isop}}) + 4n\omega_{2n} \int_0^{\varepsilon^\gamma} \Big(\varepsilon + \varphi(\varepsilon^\gamma) - r^2 \varepsilon^{1-2\gamma} - \varphi(r) \Big) r^{2n-1}\, dr \\
&= \mathcal{L}^{2n+1}(E_{\text{isop}}) + 4n\omega_{2n} \Big[\Big(\varepsilon + \frac{\pi}{2} - 4\varepsilon^{3\gamma} + o(\varepsilon^{3\gamma}) \Big) \frac{\varepsilon^{2n\gamma}}{2n} - \varepsilon^{1-2\gamma} \frac{\varepsilon^{\gamma(2n+2)}}{2n+2} \\
&\qquad\qquad - \frac{\pi}{2} \frac{\varepsilon^{2n\gamma}}{2n} + 4 \frac{\varepsilon^{\gamma(2n+3)}}{2n+3} + o(\varepsilon^{\gamma(2n+3)}) \Big] \\
&= \mathcal{L}^{2n+1}(E_{\text{isop}}) + 4n\omega_{2n} \Big[\varepsilon^{2n\gamma+1} \Big(\frac{1}{2n} - \frac{1}{2n+2} \Big) + \varepsilon^{(2n+3)\gamma} \Big(\frac{4}{2n+3} - \frac{2}{n} \Big) + o(\varepsilon^{\gamma(2n+3)}) \Big].
\end{aligned}$$

Since $\gamma \le 1/3$, then $\varepsilon^{2n\gamma+1} = o(\varepsilon^{(2n+3)\gamma})$, as $\varepsilon \to 0$, hence

$$\mathcal{L}^{2n+1}(E_\varepsilon) = \mathcal{L}^{2n+1}(E_{\text{isop}}) + 16n\omega_{2n} \Big(\frac{1}{2n+3} - \frac{1}{2n} \Big) \varepsilon^{\gamma(2n+3)} + o(\varepsilon^{\gamma(2n+3)}), \tag{3.1.14}$$

which implies, by Proposition 3.1.7,

$$A_H(E_\varepsilon) \ge \frac{1}{2^{2n+1}} \frac{\mathcal{L}^{2n+1}(E_\varepsilon \triangle B(E_\varepsilon))}{\mathcal{L}^{2n+1}(E_\varepsilon)} \ge c_2 \varepsilon^{\gamma(2n+3)} + o(\varepsilon^{\gamma(2n+3)}), \text{ as } \varepsilon \to 0. \tag{3.1.15}$$

Then, by (3.1.13) and (3.1.15), we get for $\varepsilon > 0$ small enough

$$\frac{A_H(E_\varepsilon)^{2-\sigma}}{D_H(E_\varepsilon)} \ge c_3(n) \frac{\varepsilon^{(2-\sigma)\gamma(2n+3)}}{\varepsilon^{2+(2n-3)\gamma}}.$$

In conclusion, the limit in (3.1.12) follows by choosing

$$\gamma < \frac{2}{(2-\sigma)(2n+3) - (2n-3)}.$$

3.2 Subcalibration in $\mathbb{H}^n$

In this section we refine the calibration argument used by Ritoré in [116] via a subcalibration to prove a quantitative isoperimetric inequality for competitors of E_{isop} in half-cylinders.

For any $0 \le \varepsilon < 1$ we define the half-cylinder

$$C_\varepsilon = \Big\{(z,t) \in \mathbb{H}^n : |z| < 1 \text{ and } t > t_\varepsilon\Big\}, \tag{3.2.1}$$

where $t_\varepsilon = \varphi(1-\varepsilon)$ with $\varphi(r) = \arccos(r) + r\sqrt{1-r^2}$. The proof provides an inequality with a variable structure, according to whether $\varepsilon = 0$ or $\varepsilon > 0$. The main result of this chapter is the following

Theorem 3.2.1. *Let $F \subset \mathbb{H}^n$, $n \ge 1$, be any measurable set with $\mathcal{L}^{2n+1}(F) = \mathcal{L}^{2n+1}(E_{\text{isop}})$.*

i) If $F\Delta E_{\text{isop}} \subset\subset C_0$ then

$$P_H(F) - P_H(E_{\text{isop}}) \ge \frac{n}{240\,\omega_{2n}^2}\mathcal{L}^{2n+1}(F\Delta E_{\text{isop}})^3. \tag{3.2.2}$$

ii) If $F\Delta E_{\text{isop}} \subset\subset C_\varepsilon$ for $0 < \varepsilon < 1$, then

$$P_H(F) - P_H(E_{\text{isop}}) \ge \frac{n\sqrt{\varepsilon}}{16\,\omega_{2n}}\mathcal{L}^{2n+1}(F\Delta E_{\text{isop}})^2. \tag{3.2.3}$$

Above, ω_{2n} denotes the Lebesgue measure of the Euclidean unit ball in $\mathbb{R}^{2n}$.

Clearly, since by definition of H-asymmetry $\mathcal{L}^{2n+1}(F\Delta E_{\text{isop}}) \ge A_H(F)$, Theorem 3.2.1 implies the quantitative isoperimetric inequalities

$$\begin{aligned} D_H(F) &\ge \frac{nP_H(E_{\text{isop}})}{240\,\omega_{2n}^2}A_H(F)^3 \qquad \text{if} \qquad F \subset\subset C_0 \\ D_H(F) &\ge \frac{nP_H(E_{\text{isop}})\sqrt{\varepsilon}}{16\,\omega_{2n}}A_H(F)^2 \qquad \text{if} \qquad F \subset\subset C_\varepsilon. \end{aligned}$$

In (3.2.2), the asymmetry index $\mathcal{L}^{2n+1}(F\Delta E_{\text{isop}})$ appears with the power 3. In (3.2.3), the power is 2 but there is a constant that vanishes with ε.

The sub-calibration is constructed in the following way. The set $E_{\text{isop}} \cap C_\varepsilon$ can be foliated by a family of hypersurfaces with constant H-mean curvature that decreases from 1, the H-curvature of ∂E_{isop}, to 0, the curvature of the surface $\{t = t_\varepsilon\}$. The velocity of the decrease depends on the parameter ε. The horizontal unit normal to the leaves gives the sub-calibration.

The H-mean curvature is defined in the following way. Let $\Sigma \subset \mathbb{H}^n$ be a hypersurface that is locally given by the zero set of a function $u \in C^1$ such that $|\nabla_H u| \ne 0$ on Σ, where

$$\nabla_H u = \big(X_1 u, \ldots, X_n u, Y_1 u, \ldots, Y_n u\big) \tag{3.2.4}$$

is the horizontal gradient of u. Then we define the H-mean curvature of Σ at the point $(z,t) \in \Sigma$ as

$$H_\Sigma(z,t) = \frac{1}{2n}\operatorname{div}_H\Big(\frac{\nabla_H u(z,t)}{|\nabla_H u(z,t)|}\Big) \tag{3.2.5}$$

where for any given vector field $W = \sum_{i=1}^n w_i X_i + \omega_i Y_i$ the H-divergence of W is (see Remark 1.2.1)

$$\mathrm{div}_{\mathrm{H}} W = \sum_{i=1}^n X_i w_i + Y_i \omega_i.$$

The definition (3.2.5) depends on a choice of sign. We shall work with orientable embedded hypersurfaces and so we can choose the positive sign, $H(z,t) \geq 0$. Then, the boundary of E_{isop} has constant H-mean curvature 1. For a set $E = \{(z,t) \in \mathbb{H}^n : u(z,t) > 0\}$ the horizontal normal ν_E is given on ∂E by the vector

$$\nu_E = \frac{\nabla_H u}{|\nabla_H u|}.$$

The proof of Theorem 3.2.1 relies on the construction described in the following result.

Theorem 3.2.2. *Let $0 \leq \varepsilon < 1$. There exists a continuous function $u : C_\varepsilon \to \mathbb{R}$ with level sets $\Sigma_s = \big\{(z,t) \in C_\varepsilon : u(z,t) = s\big\}$, $s \in \mathbb{R}$, such that:*

i) $u \in C^1(C_\varepsilon \cap E_{\mathrm{isop}}) \cap C^1(C_\varepsilon \smallsetminus E_{\mathrm{isop}})$ *and* $\nabla_H u / |\nabla_H u|$ *is continuously defined on* $C_\varepsilon \smallsetminus \{z = 0\}$*;*

ii) $\bigcup_{s>1} \Sigma_s = C_\varepsilon \cap E_{\mathrm{isop}}$ *and* $\bigcup_{s\leq 1} \Sigma_s = C_\varepsilon \smallsetminus E_{\mathrm{isop}}$*;*

iii) Σ_s *is a hypersurface of class* C^2 *with constant* H*-mean curvature* $H_{\Sigma_s} = 1/s$ *for* $s > 1$ *and* $H_{\Sigma_s} = 1$ *for* $s \leq 1$*;*

iv) For any point $(z, \varphi(|z|) - t) \in \Sigma_s$ *with* $s > 1$ *we have*

$$1 - H_{\Sigma_s}(z, \varphi(|z|) - t) \geq \frac{1}{20} t^2 \quad \text{when } \varepsilon = 0. \tag{3.2.6}$$

and

$$1 - H_{\Sigma_s}(z, \varphi(|z|) - t) \geq \frac{\sqrt{\varepsilon}}{4} t \quad \text{when } 0 < \varepsilon < 1, \tag{3.2.7}$$

The estimates (3.2.6) and (3.2.7) are the basis of the two inequalities (3.2.2) and (3.2.3), respectively.

3.2.1. Proof of Theorem 3.2.2

In $C_\varepsilon \smallsetminus E_{\mathrm{isop}}$, the leaves Σ_s are vertical translations of the top part of the boundary $\partial E_{\mathrm{isop}}$. In $C_\varepsilon \cap E_{\mathrm{isop}}$, the leaves Σ_s are constructed in the following way: the surface $\partial E_{\mathrm{isop}}$ is first dilated by a factor larger than 1, and then it is translated downwards in such a way that, after the two operations, the sphere $S_\varepsilon = \{(z,t) \in \partial E_{\mathrm{isop}} : t = t_\varepsilon\}$ with $t_\varepsilon = \varphi(1-\varepsilon)$ remains fixed.

The profile function of the set E_{isop} is the function $\varphi : [0,1] \to \mathbb{R}$

$$\varphi(r) = \arccos(r) + r\sqrt{1-r^2} \qquad 0 \le r \le 1. \tag{3.2.8}$$

Its first and second order derivatives are

$$\varphi'(r) = \frac{-2r^2}{\sqrt{1-r^2}} \quad \text{and} \quad \varphi''(r) = \frac{2r(r^2-2)}{(1-r^2)^{3/2}}, \quad 0 \le r < 1. \tag{3.2.9}$$

Notice that $\varphi'''(0) = -4$. We also need the function $\psi : [0,1) \to \mathbb{R}$

$$\psi(r) = 2\varphi(r) - r\varphi'(r) = 2\left(\frac{r}{\sqrt{1-r^2}} + \arccos(r)\right). \tag{3.2.10}$$

Its derivative is

$$\psi'(r) = \varphi'(r) - r\varphi''(r) = \frac{2r^2}{(1-r^2)^{3/2}}. \tag{3.2.11}$$

We start the construction of the function u. On the set $C_\varepsilon \smallsetminus E_{\text{isop}}$ we let

$$u(z,t) = \varphi(|z|) - t + 1, \quad (z,t) \in C_\varepsilon \smallsetminus E_{\text{isop}}. \tag{3.2.12}$$

Notice that $u(z, \varphi(|z|)) = 1$ for all $|z| < 1$ (see Figure 3.2.1 on the left).

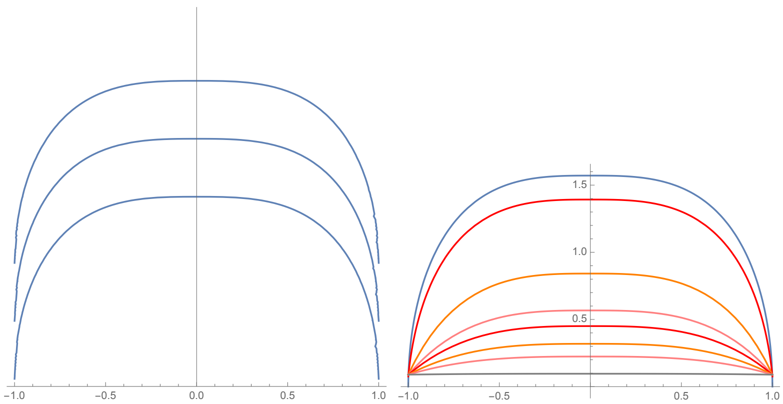


Figure 3.3: The foliation above and below the Pansu ball.

We define the function u in the set

$$D_\varepsilon = C_\varepsilon \cap E_{\text{isop}} = \{(z,t) \in E_{\text{isop}} : |z| < 1 - \varepsilon,\ t_\varepsilon < t < \varphi(|z|)\}.$$

as the restriction of a function defined on its closure (still denoted by u) satisfying

$$S_\varepsilon \subset \{(z,t) \in \overline{D}_\varepsilon : u(z,t) = s\} \text{ for any } s > 1. \tag{3.2.13}$$

We use the short notation $r = |z|$ and $r_\varepsilon = 1 - \varepsilon$.

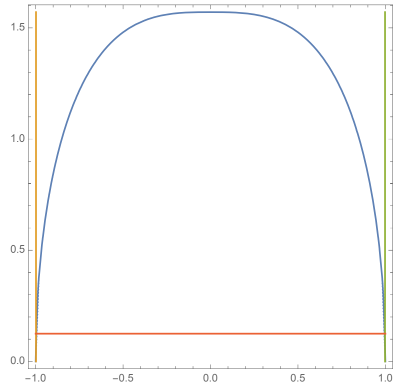


Figure 3.4: The foliation below the Pansu ball is constructed fixing the circle $\{(z,t_\varepsilon) \in \mathbb{H}^n : |z| = r_\varepsilon\}$ on every leaf.

We define the auxiliary function $F_\varepsilon : \overline{D}_\varepsilon \times (1,\infty) \to \mathbb{R}$ in such a way that $F_\varepsilon(z,t,s) = 0$ if and only if

$$t = s^2\varphi(r/s) + t_\varepsilon - s^2\varphi(r_\varepsilon/s),$$

where, for any $s > 1$, the function of $r \in [0, r_\varepsilon)$ on the right hand side describes the boundary of a dilated Pansu ball after a left translation that overlaps the points $\{(z, s^2\varphi(r_\varepsilon/s)), |z| = r_\varepsilon\}$ and S_ε. For instance,

$$F_\varepsilon(z,t,s) = s^2\big(\varphi(r/s) - \varphi(r_\varepsilon/s)\big) + t_\varepsilon - t.$$

We claim that for any point $(z,t) \in D_\varepsilon$ there exists a unique $s > 1$ such that $F_\varepsilon(z,t,s) = 0$. In this case, we can define the function $u(z,t) : D_\varepsilon \to \mathbb{R}$ letting

$$s = u(z,t) \quad \text{if and only if} \quad F_\varepsilon(z,t,s) = 0. \tag{3.2.14}$$

If this holds, of course (3.2.13) is satisfied, in fact

$$u(z,t_\varepsilon) = s \iff F_\varepsilon(z,t_\varepsilon,s) = 0 \iff s^2(\varphi(r/s) - \varphi(r_\varepsilon/s)) + t_\varepsilon - t = 0.$$

We prove the claim. For any $(z,t) \in D_\varepsilon$ we have

$$\lim_{s\to 1^+} F_\varepsilon(z,t,s) = \varphi(r) - t > 0. \tag{3.2.15}$$

Moreover, with a second order Taylor expansion of φ based on (3.2.9) we see that

$$\lim_{s\to\infty} F_\varepsilon(z,t,s) = \lim_{s\to\infty} \Big(s^2\Big[\varphi(0) + o\Big(\frac{r^2}{s^2}\Big) - \varphi(0) + o\Big(\frac{r_\varepsilon^2}{s^2}\Big)\Big]\Big) + t_\varepsilon - t = t_\varepsilon - t < 0.$$

Since $s \mapsto F_\varepsilon(z,t,s)$ is continuous, this proves the existence of a solution of $F_\varepsilon(z,t,s) = 0$. By (3.2.10), the derivative in s of F_ε is

$$\partial_s F_\varepsilon(z,t,s) = s\big(\psi(r/s) - \psi(r_\varepsilon/s)\big), \tag{3.2.16}$$

and thus by (3.2.11) we deduce that $\partial_s F_\varepsilon(z,t,s) < 0$. This proves the uniqueness.

We prove claim iii). Namely, we prove that for any point $(z,t) \in \Sigma_s$ with $s > 1$ and $z \neq 0$, the H-mean curvature of Σ_s at (z,t) is

$$H_{\Sigma_s}(z,t) = -\frac{1}{2n}\mathrm{div}_H\Big(\frac{\nabla_H u}{|\nabla_H u|}\Big) = \frac{1}{s}. \tag{3.2.17}$$

We are using definition (3.2.4). The claim when $s \leq 1$ is analogous because Σ_s is a vertical translation of the top part of $\partial E_{\mathrm{isop}}$.

By the implicit function theorem, the derivatives of u can be computed from the partial derivatives of F_ε. Using $\partial_{x_i} r = x_i/r$ and $\partial_{y_i} r = y_i/r$, with $i = 1,\dots,n$ and $z = (x_1+iy_1,\dots,x_n+iy_n)$, we find

$$\partial_{x_i} F_\varepsilon(z,t,s) = \frac{sx_i}{r}\varphi'(r/s) \quad \text{and} \quad \partial_{y_i} F_\varepsilon(z,t,s) = \frac{sy_i}{r}\varphi'(r/s). \tag{3.2.18}$$

Letting $s = u(z,t)$, thanks to (3.2.14), (3.2.16), (3.2.18), and (3.2.9) we obtain

$$\partial_{x_i} u(z,t) = -\frac{\partial_{x_i} F_\varepsilon(z,t,s)}{\partial_s F_\varepsilon(z,t,s)} = \frac{2rx_i}{s\sqrt{s^2-r^2}\big(\psi(r/s) - \psi(r_\varepsilon/s)\big)}, \tag{3.2.19}$$

$$\partial_{y_i} u(z,t) = -\frac{\partial_{x_i} F_\varepsilon(z,t,s)}{\partial_s F_\varepsilon(z,t,s)} = \frac{2ry_i}{s\sqrt{s^2-r^2}\big(\psi(r/s) - \psi(r_\varepsilon/s)\big)}, \tag{3.2.20}$$

$$\partial_t u(z,t) = -\frac{\partial_t F_\varepsilon(z,t,s)}{\partial_s F_\varepsilon(z,t,s)} = \frac{1}{s\big(\psi(r/s) - \psi(r_\varepsilon/s)\big)}, \tag{3.2.21}$$

and thus

$$\partial_{x_i} u = 2x_i\frac{r}{\sqrt{s^2-r^2}}\partial_t u \quad \text{and} \quad \partial_{y_i} u = 2y_i\frac{r}{\sqrt{s^2-r^2}}\partial_t u. \tag{3.2.22}$$

It is then immediate to compute

$$X_i u = \partial_{x_i} u + 2y_i\partial_t u = \frac{2rx_i + 2y_i\sqrt{s^2-r^2}}{s\sqrt{s^2-r^2}\big(\psi(r/s) - \psi(r_\varepsilon/s)\big)},$$

$$Y_i u = \partial_{y_i} u - 2x_i\partial_t u = \frac{2ry_i - 2x_i\sqrt{s^2-r^2}}{s\sqrt{s^2-r^2}\big(\psi(r/s) - \psi(r_\varepsilon/s)\big)},$$

and the squared length of the horizontal gradient of u in D_ε is

$$\begin{aligned}|\nabla_H u|^2 &= \sum_{i=1}^n (X_i u)^2 + (Y_i u)^2 \\ &= \sum_{i=1}^n \frac{4r^2(x_i^2+y_i^2) + 4(x_i^2+y_i^2)(s^2-r^2)}{s^2(s^2-r^2)\big(\psi(r/s) - \psi(r_\varepsilon/s)\big)^2} \\ &= \frac{4r^2}{(s^2-r^2)\big(\psi(r/s) - \psi(r_\varepsilon/s)\big)^2}.\end{aligned}$$

Note that $|\nabla_H u(z,t)| = 0$ if and only if $z = 0$. So for any $(z,t) \in D_\varepsilon$ with $z \neq 0$ we have

$$a_i(z,t) = -\frac{X_i u}{|\nabla_H u|} = \frac{rx_i + y_i\sqrt{s^2-r^2}}{rs} = \frac{x_i}{s} + y_i\frac{\sqrt{s^2-r^2}}{rs} \tag{3.2.23}$$

and

$$b_i(z,t) = -\frac{Y_i u}{|\nabla_H u|} = \frac{r y_i - x_i\sqrt{s^2-r^2}}{rs} = \frac{y_i}{s} - x_i\frac{\sqrt{s^2-r^2}}{rs}. \tag{3.2.24}$$

If $(z,t) \in E_{\mathrm{isop}}$ tends to $(\bar z,\bar t) \in \partial E_{\mathrm{isop}}$ with $\bar t > 0$ and $\bar z \neq 0$, then $s = u(z,t)$ converges to 1, and from (3.2.23) and (3.2.24) we see that

$$\lim_{(z,t)\to(\bar z,\bar t)} \frac{\nabla_H u(z,t)}{|\nabla_H u(z,t)|} = -\Big(\bar x + \bar y\frac{\sqrt{1-|\bar z|^2}}{|\bar z|}, \bar y - \bar x\frac{\sqrt{1-|\bar z|^2}}{|\bar z|}\Big) = \frac{\nabla_H u(\bar z,\bar t)}{|\nabla_H u(\bar z,\bar t)|},$$

where the right hand side is computed using the definition (3.2.12) of u. This ends the proof of claim i).

Claim ii) is clear. We prove claim iii). The auxiliary function $w(r,s) = \sqrt{s^2-r^2}/rs$ satisfies

$$\partial_{x_i} w = \frac{x_i}{r}\partial_r w + \partial_{x_i} u\,\partial_s w, \quad \partial_{y_i} w = \frac{y_i}{r}\partial_r w + \partial_{y_i} u\,\partial_s w, \quad \partial_s w = \frac{r}{s^2\sqrt{s^2-r^2}}. \tag{3.2.25}$$

By (3.2.23), (3.2.24), (3.2.22), and (3.2.25) we obtain

$$\begin{aligned}
X_i a_i + Y_i b_i &= \partial_{x_i} a_i + 2y_i\partial_t a_i + \partial_{y_i} b_i - 2x_i\partial_t b_i \\
&= \frac{1}{s} - \frac{x_i}{s^2}\partial_{x_i} u + y_i\Big(\frac{x_i}{r}\partial_r w + \partial_{x_i} u\,\partial_s w\Big) + 2y_i\Big(-\frac{x_i}{s^2}\partial_t u + y_i\partial_s w\,\partial_t u\Big) \\
&\quad + \frac{1}{s} - \frac{y_i}{s^2}\partial_{y_i} u - x_i\Big(\frac{y_i}{r}\partial_r w + \partial_{y_i} u\,\partial_s w\Big) - 2x_i\Big(-\frac{y_i}{s^2}\partial_t u - x_i\partial_s w\,\partial_t u\Big) \\
&= \frac{2}{s} - \frac{x_i\partial_{x_i} u + y_i\partial_{y_i} u}{s^2} + 2(x_i^2+y_i^2)\partial_s w\,\partial_t u \\
&= \frac{2}{s} - \frac{x_i\partial_{x_i} u + y_i\partial_{y_i} u}{s^2} + \frac{2r(x_i^2+y_i^2)\partial_t u}{s^2\sqrt{s^2-r^2}} = \frac{2}{s}.
\end{aligned}$$

Summing over $i = 1,\dots,n$ and dividing by $2n$, we obtain (3.2.17).

We prove claim iv). We fix a point z with $|z| < 1-\varepsilon$ and for $0 \le t < \varphi(|z|) - t_\varepsilon$ we define the function

$$f_z(t) = u(z,\varphi(|z|)-t) = s = \frac{1}{H_{\Sigma_s}}, \tag{3.2.26}$$

where $s \ge 1$ is uniquely determined by $(z,\varphi(|z|)-t) \in \Sigma_s$. The function $t \mapsto f_z(t)$ is increasing and $f_z(0) = 1$

By (3.2.21), the function f_z solves the differential equation

$$f_z'(t) = -\partial_t u(z,\varphi(|z|)-t) = \frac{1}{f_z(t)\big(\psi(r_\varepsilon/f_z(t)) - \psi(r/f_z(t))\big)}$$

for all $0 < t < \varphi(|z|) - t_\varepsilon$, and since, by (3.2.11), ψ is strictly increasing, f_z solves the differential inequality

$$f_z'(t) \ge \frac{1}{f_z(t)\big(\psi(r_\varepsilon/f_z(t)) - \pi\big)}.$$

On the other hand, for any $s > 1$ we have

$$\begin{aligned} s\big(\psi(r_\varepsilon/s) - \pi\big) &= s\int_0^{r_\varepsilon/s} \psi'(r)\,dr \\ &= s\int_0^{r_\varepsilon/s} \frac{2r^2}{(1-r^2)^{3/2}}\,dr \\ &\le r_\varepsilon \int_0^{r_\varepsilon/s} \frac{2r}{(1-r^2)^{3/2}}\,dr \\ &= 2r_\varepsilon\Big((1-(r_\varepsilon/s)^2)^{-1/2} - 1\Big) \\ &\le \frac{2}{\sqrt{s-r_\varepsilon}}. \end{aligned} \tag{3.2.27}$$

In the case $\varepsilon = 0$ we have $r_\varepsilon = 1$ and inequality (3.2.27) reads

$$s\big(\psi(1/s) - \pi\big) \le \frac{2}{\sqrt{s-1}}.$$

Hence, the function f_z satisfies the differential inequality

$$f_z'(t) \ge \frac{1}{2}\sqrt{f_z(t) - 1}, \qquad t > 0.$$

An integration with $f_z(0) = 1$ gives

$$\int_0^t \frac{f_z'(\tau)}{\sqrt{f_z(\tau)-1}}\,d\tau \ge \int_0^t \frac{1}{2}\,d\tau \iff \Big[2\sqrt{f_z(\tau)-1}\Big]_0^t \ge \frac{t}{2} \implies f_z(t) \ge 1 + t^2/16,$$

and thus by the relation (3.2.26) and by the bound $t < \pi/2$ we find

$$1 - H_{\Sigma_s}(z, \varphi(|z|) - t) = 1 - \frac{1}{f_z(t)} \ge \frac{t^2}{16+t^2} \ge \frac{1}{20}t^2.$$

This is claim (3.2.6).

When $0 < \varepsilon < 1$, inequality (3.2.27) implies

$$s\big(\psi(r_\varepsilon/s) - \pi\big) \le \frac{2}{\sqrt{\varepsilon}},$$

and thus $f_z'(t) \ge \sqrt{\varepsilon}/2$, that gives $f_z(t) \ge 1 + t\sqrt{\varepsilon}/2$. In this case, we find

$$1 - H_{\Sigma_s}(z, \varphi(|z|) - t) = 1 - \frac{1}{f_z(t)} \ge \frac{2\sqrt{\varepsilon}t}{4+\pi} \ge \frac{\sqrt{\varepsilon}}{4}t.$$

This is claim (3.2.7). This finishes the proof of Theorem 3.2.2.

3.2.2. Proof of Theorem 3.2.1

In this section, we prove the quantitative isoperimetric estimates (3.2.2) and (3.2.3).

Let $u : C_\varepsilon \to \mathbb{R}$, $0 \le \varepsilon < 1$, be the function given by Theorem 3.2.2 and let $\Sigma_s = \{(z,t) \in C_\varepsilon : u(z,t) = s\}$ be the leaves of the foliation, $s \in \mathbb{R}$. On $C_\varepsilon \smallsetminus \{|z| = 0\}$ we define the vector field $X : C_\varepsilon \smallsetminus \{|z| = 0\} \to \mathbb{R}^{2n}$ by

$$X = -\frac{\nabla_H u}{|\nabla_H u|}.$$

Both u and X depend on ε. In particular, X satisfies the following properties:

i) $|X| = 1$;

ii) for $(z,t) \in \partial E_{\mathrm{isop}} \cap C_\varepsilon$ we have $X(z,t) = -\nu_{E_{\mathrm{isop}}}(z,t)$, the horizontal unit normal to $\partial E_{\mathrm{isop}}$ (see Section 1.2).

iii) For any point $(z,t) \in \Sigma_s$, $s \in \mathbb{R}$, we have,

$$\frac{1}{2n}\mathrm{div}_H X(z,t) = H_{\Sigma_s}(z,t) \le H_{\Sigma_0} = 1. \tag{3.2.28}$$

We start the proof. Let $F \subset \mathbb{H}^n$ be a set with finite H-perimeter such that $\mathcal{L}^{2n+1}(F) = \mathcal{L}^{2n+1}(E_{\mathrm{isop}})$ and $F \Delta E_{\mathrm{isop}} \subset\subset C_\varepsilon$. By Theorem 2.5 in [56], we can without loss of generality assume that ∂F is of class C^∞. For $\delta > 0$, let $E^\delta_{\mathrm{isop}} = \{(z,t) \in E_{\mathrm{isop}} : |z| > \delta\}$. By (3.2.28) and by the Gauss-Green formula (1.2.6) for the perimeter measure and the horizontal normal, we have

$$\begin{aligned}
\mathcal{L}^{2n+1}(E^\delta_{\mathrm{isop}} \smallsetminus F) &= \int_{E^\delta_{\mathrm{isop}} \smallsetminus F} 1\, dzdt \ge \int_{E^\delta_{\mathrm{isop}} \smallsetminus F} \frac{\mathrm{div}_H X}{2n}\, dzdt \\
&= \frac{1}{2n}\left\{\int_{\partial F \cap E^\delta_{\mathrm{isop}}} \langle X, \nu_F\rangle d\mu_F - \int_{(\partial E^\delta_{\mathrm{isop}}) \smallsetminus F} \langle X, \nu_{E^\delta_{\mathrm{isop}}}\rangle d\mu_{E^\delta_{\mathrm{isop}}}\right\}.
\end{aligned}$$

Observe that $\mu_{E^\delta_{\mathrm{isop}}} = \mu_{E_{\mathrm{isop}}} \llcorner \{|z| > \delta\} + \mu_{\{|z|>\delta\}} \llcorner E_{\mathrm{isop}}$ and $\mu_{\{|z|>\delta\}}(E_{\mathrm{isop}}) \le C\delta^{2n-1}$. Letting $\delta \to 0^+$ and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\mathcal{L}^{2n+1}(E_{\mathrm{isop}} \smallsetminus F) &\ge \frac{1}{2n}\left\{\int_{\partial F \cap E_{\mathrm{isop}}} \langle X, \nu_F\rangle d\mu_F - \int_{(\partial E_{\mathrm{isop}}) \smallsetminus F} \langle X, \nu_{E_{\mathrm{isop}}}\rangle d\mu_{E_{\mathrm{isop}}}\right\} \\
&\ge \frac{1}{2n}\left\{\mu_{E_{\mathrm{isop}}}(C_\varepsilon \smallsetminus F) - \mu_F(E_{\mathrm{isop}})\right\} \\
&= \frac{1}{2n}\{P_H(E_{\mathrm{isop}}, C_\varepsilon \smallsetminus F) - P_H(F, E_{\mathrm{isop}})\}.
\end{aligned} \tag{3.2.29}$$

By a similar computation we also have

$$\mathcal{L}^{2n+1}(F \smallsetminus E_{\mathrm{isop}}) = \int_{F \smallsetminus E_{\mathrm{isop}}} 1\, dzdt = \int_{F \smallsetminus E_{\mathrm{isop}}} \frac{\mathrm{div}_H X}{2n}\, dzdt \tag{3.2.30}$$

$$\begin{aligned}
&= \frac{1}{2n}\left\{\int_{\partial E_{\mathrm{isop}} \cap F} \langle X, \nu_{E_{\mathrm{isop}}}\rangle d\mu_{E_{\mathrm{isop}}} - \int_{(\partial F) \smallsetminus E_{\mathrm{isop}}} \langle X, \nu_F\rangle d\mu_F\right\} \\
&\le \frac{1}{2n}\left\{\mu_F(C_\varepsilon \smallsetminus E_{\mathrm{isop}}) - \mu_{E_{\mathrm{isop}}}(F)\right\} \\
&= \frac{1}{2n}\{P_H(F, C_\varepsilon \smallsetminus E_{\mathrm{isop}}) - P_H(E_{\mathrm{isop}}, F)\}.
\end{aligned} \tag{3.2.31}$$

On the other hand,

$$\begin{aligned}\int_{E_{\text{isop}}\setminus F} \frac{\operatorname{div}_H X}{2n}\, dzdt &= \int_{E_{\text{isop}}\setminus F} \Big(1 + \Big(\frac{\operatorname{div}_H X}{2n} - 1\Big)\Big)\, dzdt \\ &= \mathcal{L}^{2n+1}(E_{\text{isop}} \setminus F) - \int_{E_{\text{isop}}\setminus F} \Big(1 - \frac{\operatorname{div}_H X}{2n}\Big)\, dzdt \\ &= \mathcal{L}^{2n+1}(E_{\text{isop}} \setminus F) - \mathcal{G}(E_{\text{isop}} \setminus F),\end{aligned}$$

where

$$\mathcal{G}(E_{\text{isop}} \setminus F) = \int_{E_{\text{isop}}\setminus F} \Big(1 - \frac{\operatorname{div}_H X}{2n}\Big)\, dzdt.$$

From (3.2.29) and (3.2.30), we obtain

$$\begin{aligned}\frac{1}{2n}\{P_H(E_{\text{isop}}, C_\varepsilon \setminus F) - P_H(F, E_{\text{isop}})\} &\le \int_{E_{\text{isop}}\setminus F} \frac{\operatorname{div}_H X}{2n}\, dzdt \\ &= \mathcal{L}^{2n+1}(E_{\text{isop}} \setminus F) - \mathcal{G}(E_{\text{isop}} \setminus F) \\ &= \mathcal{L}^{2n+1}(F \setminus E_{\text{isop}}) - \mathcal{G}(E_{\text{isop}} \setminus F) \\ &\le \frac{1}{2n}\{P_H(F, C_\varepsilon \setminus E_{\text{isop}}) - P_H(E_{\text{isop}}, F)\} - \mathcal{G}(E_{\text{isop}} \setminus F),\end{aligned}$$

that is equivalent to

$$P_H(F) - P_H(E_{\text{isop}}) \ge 2n\mathcal{G}(E_{\text{isop}} \setminus F). \tag{3.2.32}$$

For any z with $|z| < 1 - \varepsilon$, we define the vertical sections $E^z_{\text{isop}} = \{t \in \mathbb{R} : (z,t) \in E_{\text{isop}}\}$ and $F^z = \{t \in \mathbb{R} : (z,t) \in F\}$. By Fubini-Tonelli theorem, we have

$$\begin{aligned}\mathcal{G}(E_{\text{isop}} \setminus F) &= \int_{E_{\text{isop}}\setminus F} \Big(1 - \frac{\operatorname{div}_H X}{2n}\Big)\, dzdt \\ &= \int_{\{|z|<1\}} \int_{E^z_{\text{isop}}\setminus F^z} \left(1 - \frac{\operatorname{div}_H X(z,t)}{2n}\right) dt\, dz.\end{aligned}$$

The function $t \mapsto \operatorname{div}_H X(z,t)$ is increasing, and thus letting $m(z) = \mathcal{L}^1(E^z_{\text{isop}} \setminus F^z)$, by monotonicity we obtain

$$\begin{aligned}\mathcal{G}(E_{\text{isop}} \setminus F) &\ge \int_{\{|z|<1\}} \int_{\varphi(|z|)-m(z)}^{\varphi(|z|)} \left(1 - \frac{\operatorname{div}_H X(z,t)}{2n}\right) dt\, dz \\ &= \int_{\{|z|<1\}} \int_0^{m(z)} \left(1 - \frac{1}{f_z(t)}\right) dt\, dz,\end{aligned}$$

where $f_z(t) = u(z, \varphi(|z|) - t)$ is the function introduced in (3.2.26).

By (3.2.6), when $\varepsilon = 0$ the function f_z satisfies the estimate $1 - 1/f_z(t) \ge t^2/20$, and by

Hölder inequality we find

$$
\begin{aligned}
\mathcal{G}(E_{\text{isop}} \smallsetminus F) &\geq \frac{1}{20} \int_{\{|z|<1\}} \int_0^{m(z)} t^2 dt\, dz \\
&= \frac{1}{60} \int_{\{|z|<1\}} m(z)^3\, dz \\
&\geq \frac{1}{60\omega_{2n}^2} \Big(\int_{\{|z|<1\}} m(z)\, dz \Big)^3 \\
&= \frac{1}{480\omega_{2n}^2} \mathcal{L}^{2n+1}(E_{\text{isop}} \Delta F)^3.
\end{aligned}
\tag{3.2.33}
$$

From (3.2.33) and (3.2.32) we obtain (3.2.2).

By (3.2.7), when $0 < \varepsilon < 1$ the function f_z satisfies the estimate $1 - 1/f_z(t) \geq \sqrt{\varepsilon} t/4$ and we find

$$
\begin{aligned}
\mathcal{G}(E_{\text{isop}} \smallsetminus F) &\geq \frac{\sqrt{\varepsilon}}{4} \int_{\{|z|<1\}} \int_0^{m(z)} t\, dt\, dz \\
&= \frac{\sqrt{\varepsilon}}{8} \int_{\{|z|<1\}} m(z)^2\, dz \\
&\geq \frac{\sqrt{\varepsilon}}{8\omega_{2n}} \Big(\int_{\{|z|<1\}} m(z)\, dz \Big)^2 \\
&= \frac{\sqrt{\varepsilon}}{32\omega_{2n}} \mathcal{L}^{2n+1}(E_{\text{isop}} \Delta F)^2.
\end{aligned}
\tag{3.2.34}
$$

From (3.2.34) and (3.2.32) we obtain claim (3.2.3).

3.3 Subcalibration in Grushin spaces and H-type groups.

In Chapter 2, we studied the isoperimetric problem in Grushin spaces and H-type groups. Given $h, k \geq 1$ integers, and $n = h + k$, we consider for $\alpha \geq 0$ the family of vector fields on $\mathbb{R}^n$

$$
\begin{aligned}
&X_\alpha = \{X_1, \dots, X_h, Y_1, \dots, Y_k\}, \\
&X_i = \partial x_i,\ Y_j(x,y) = |x|^\alpha \partial_{y_j}, \quad i = 1 \dots h,\ j = 1, \dots k.
\end{aligned}
$$

In Remark 2.5.2 we proved that, when $k = 1$, the isoperimetric problem for the X-perimeter and the Lebesgue measure has a unique solution up to dilations and vertical translations,

$$
E^\alpha_{\text{isop}} = \{(x,y) \in \mathbb{R}^n : |y| \leq \varphi_\alpha(|x|),\ |x| < 1\}, \qquad \varphi_\alpha(|x|) = \int_{\arcsin |x|}^{\pi/2} \sin^{\alpha+1} t\, dt
$$

in the class of x-spherically symmetric sets

$$
E \subset \mathbb{R}^h_x \times \mathbb{R}_y, \quad E = \{(x,y) \in \mathbb{R}^n : (|x|, y) \in F\} \text{ for a generating set } F \subset \mathbb{R}^+ \times \mathbb{R}^k.
$$

Starting from this profile function, we show that the sub-calibration technique provides a quantitative isoperimetric inequality also in this setting. More details about isoperimetric deficit and asymmetry in Grushin spaces will be given in Chapter 4 in the case of the Grushin plane.

Given a hypersurface $\Sigma \subset \mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}$ which is locally the zero set of a function $u \in C^1$ such that $|\nabla_\alpha u| \neq 0$ on Σ, where $\nabla_\alpha u = (\partial_{x_1} u, \dots, \partial_{x_h} u, |x|^\alpha \partial_y u)$, we define its α-mean curvature as

$$H_\Sigma^\alpha = \frac{1}{h} \operatorname{div}_\alpha\Big(\frac{\nabla_\alpha u}{|\nabla_\alpha u|}\Big). \tag{3.3.1}$$

For any vector field $W = \sum_{i=1}^n w_i \partial_{x_i}$ in $\mathbb{R}^n$, $\operatorname{div}_\alpha$ denotes the α-divergence of W, (see (1.2.2))

$$\operatorname{div}_\alpha W = \sum_{i=1}^h \partial_{x_i} w_i + |x|^\alpha \partial_y w_n.$$

For any $\varepsilon \geq 0$ we define $r_\varepsilon = 1 - \varepsilon$, $y_\varepsilon = \varphi_\alpha(r_\varepsilon)$. Analogously to (3.2.1), we define the cylinder

$$C_\varepsilon = \{(x,y) \in \mathbb{R}^n : x \in \mathbb{R}^h,\ y \in \mathbb{R},\ |x| < r_\varepsilon,\ y > y_\varepsilon\}$$

and

$$D_\varepsilon = C_\varepsilon \cap E_{\mathrm{isop}}^\alpha.$$

We also call $\pi_\alpha := \varphi_\alpha(0)$.

Theorem 3.3.1. *Let $\alpha \geq 0$, $h \geq 1$, $n = h+1$ and $F \subset \mathbb{R}^n$ be any measurable set with $\mathcal{L}^n(F) = \mathcal{L}^n(E_{\mathrm{isop}}^\alpha)$.*

i) If $F \Delta E_{\mathrm{isop}}^\alpha \subset\subset C_0$ then

$$P_\alpha(F) - P_\alpha(E_{\mathrm{isop}}^\alpha) \geq \frac{h}{8(24 + 3\pi_\alpha^2)\omega_h^2} \mathcal{L}^n(F \Delta E_{\mathrm{isop}}^\alpha)^3. \tag{3.3.2}$$

ii) If $F \Delta E_{\mathrm{isop}}^\alpha \subset\subset C_\varepsilon$ for $0 < \varepsilon < 1$, then

$$P_\alpha(F) - P_\alpha(E_{\mathrm{isop}}^\alpha) \geq \frac{h\sqrt{\varepsilon}}{8(1+\pi_\alpha)\omega_h} \mathcal{L}^n(F \Delta E_{\mathrm{isop}}^\alpha)^2. \tag{3.3.3}$$

Remark 3.3.2. In Remark 2.5.2 we proved the sharp isoperimetric inequality for x-spherically symmetric sets in $(\mathbb{R}^n, d_\alpha)$ with $n = h+1$. Theorem 3.3.1 gives a new class of sets for which the sharp isoperimetric inequality in the Grushin space $(\mathbb{R}^n, d_\alpha)$ is valid, for $n = h+1$. Namely, for any set of finite α-perimeter $F \subset \mathbb{R}^n$ such that $\mathcal{L}^n(F) = \mathcal{L}^n(E_{\mathrm{isop}}^\alpha)$ and $F \vartriangle E_{\mathrm{isop}}^\alpha \subset\subset C_0$, we have

$$P_\alpha(E_{\mathrm{isop}}^\alpha) \leq P_\alpha(F).$$

Equality holds if and only if $F = E_{\mathrm{isop}}^\alpha$ (see Theorem 2.1.4 and [116]).

Basically, the argument used to prove Theorem 3.2.1 applies directly to prove Theorem 3.3.1 and we are not going to write it again. The analogous of Theorem 3.2.2 in this context is based on the fact that for α integer, the profile function φ_α satisfies for a constant $C<0$

$$\varphi_\alpha(r)=\varphi_\alpha(0)+Cr^{\alpha+2}+o(r^{\alpha+2}),\ r\to 0^+. \tag{3.3.4}$$

The sign of C is due to the fact that φ_α is strictly decreasing. In the following, for any $m\in\mathbb{N}$ we denote the m-th derivative of φ_α at 0, by $\varphi_\alpha^{(m)}(0)$. The expansion (3.3.4) is justified by the following Proposition, which holds true forcing $\alpha>0$ to be integer.

Proposition 3.3.3. *Let $\alpha\in\mathbb{N}$. Then φ_α is differentiable $\alpha+2$ times at 0 and there holds*

$$\varphi_\alpha^{(m)}(0)=0,\ \textit{if } m=1,2,\dots,\alpha+1 \qquad \varphi_\alpha^{(\alpha+2)}(0)\neq 0. \tag{3.3.5}$$

Proof. By Proposition 2.5.3, $\varphi_\alpha'(r)=O(r^{\alpha+1})$ as $r\to 0^+$. This implies that, for any $m=1,\dots,\alpha+2$, $\varphi_\alpha^{(m)}(r)=O(r^{\alpha+2-m})$. Moreover $\varphi_\alpha'(0)=0$. We conclude the proof recursively for $m=1,\dots,\alpha+2$. At the first stage we have

$$\varphi_\alpha''(0)=\lim_{r\to 0^+}\frac{\varphi_\alpha'(r)-\varphi_\alpha'(0)}{r}=\lim_{r\to 0^+}\frac{\varphi_\alpha'(r)}{r^{\alpha+1}}r^\alpha\begin{cases}=0 & \text{if } \alpha=0\\ \neq 0 & \text{if } \alpha>0.\end{cases}$$

To consider derivatives of order higher than 2 in (3.3.5), we assume $\alpha>0$. Let $m\le\alpha+2$, hence $\varphi_\alpha^{(m-1)}(0)=0$. We have

$$\begin{aligned}\varphi_\alpha^{(m)}(0)&=\lim_{r\to 0^+}\frac{\varphi_\alpha^{(m-1)}(r)-\varphi_\alpha^{(m-1)}(0)}{r}=\lim_{r\to 0^+}\frac{\varphi_\alpha^{(m-1)}(r)}{r}\\ &=\lim_{r\to 0^+}\frac{O(r^{\alpha+2-m+1})}{r}=\lim_{r\to 0^+}O(r^{\alpha+2-m})\begin{cases}=0 & \text{if } m\le\alpha+1\\ \neq 0 & \text{if } m=\alpha+2.\end{cases}\end{aligned}$$

□

Theorem 3.3.4. *Let $0\le\varepsilon<1$. There exists a continuous function $u:C_\varepsilon\to\mathbb{R}$ with level sets $\Sigma_s=\{(x,y)\in C_\varepsilon : u(x,y)=s\}$, $s\in\mathbb{R}$, such that:*

i) $u\in C^1(C_\varepsilon\cap E^\alpha_{\mathrm{isop}})\cap C^1(C_\varepsilon\setminus E^\alpha_{\mathrm{isop}})$ and $\nabla_\alpha u/|\nabla_\alpha u|$ is continuously defined on $C_\varepsilon\setminus\{x=0\}$;

ii) $\bigcup_{s>1}\Sigma_s=C_\varepsilon\cap E_{\mathrm{isop}}$ and $\bigcup_{s\le 1}\Sigma_s=C_\varepsilon\setminus E_{\mathrm{isop}}$;

iii) Σ_s is a hypersurface of class C^2 with constant H-mean curvature $H_{\Sigma_s}=1/s$ for $s>1$ and $H^\alpha_{\Sigma_s}=1$ for $s\le 1$;

iv) For any point $(x,\varphi_\alpha(|x|)-y)\in\Sigma_s$ with $s>1$ we have

$$1-H^\alpha_{\Sigma_s}(x,\varphi_\alpha(|x|)-y)\ge\frac{1}{8+\pi_\alpha}s^2 \quad \textit{when } \varepsilon=0. \tag{3.3.6}$$

and

$$1 - H^{\alpha}_{\Sigma_s}(x, \varphi_\alpha(|x|) - y) \geq \frac{\sqrt{\varepsilon}}{1+\pi_\alpha} s \quad \text{when } 0 < \varepsilon < 1, \tag{3.3.7}$$

Proof. We only recall the main steps of the proof highlighting the main differences with respect to Theorem 3.2.2. Using the dilations $\delta_\lambda(x,y) = (\lambda x, \lambda^{\alpha+1} y)$, we define the function u at x as the zero locus of $F_\varepsilon : D_\varepsilon \to \mathbb{R}$

$$F_\varepsilon(x,y,s) = s^{\alpha+1}\Big\{\varphi_\alpha\big(\frac{r}{s}\big) - \varphi_\alpha\big(\frac{r_\varepsilon}{s}\big)\Big\} + y_\varepsilon - y,$$

where we use the notation $r = |x|$. In this case the limit (3.2.15) becomes

$$\begin{aligned}\lim_{s\to+\infty} F_\varepsilon(x,y,s) &= \lim_{s\to\infty} s^{\alpha+1}\{\varphi_\alpha(0) + o\big(\frac{r^{\alpha+1}}{s^{\alpha+1}}\big) - \varphi_\alpha(0) + o\big(\frac{r_\varepsilon^{\alpha+1}}{s^{\alpha+1}}\big)\} + y_\varepsilon - y \\ &= \lim_{s\to\infty} s^{\alpha+1} o\big(\frac{1}{s^{\alpha+1}}\big) + y_\varepsilon - y = y_\varepsilon - y < 0\end{aligned}$$

while $F_\varepsilon \to \varphi_\alpha(r) - y > 0$ as $s \to 1^+$. Moreover

$$\begin{aligned}\partial_s F_\varepsilon(x,y,s) &= (\alpha+1)s^\alpha\big[\varphi_\alpha\big(\frac{r}{s}\big) - \varphi_\alpha\big(\frac{r_\varepsilon}{s}\big)\big] - \frac{s^{\alpha+1} r_\varepsilon}{s^2}\varphi'_\alpha\big(\frac{r_\varepsilon}{s}\big) \\ &= s^\alpha\big[\psi_\alpha\big(\frac{r}{s}\big) - \psi_\alpha\big(\frac{r_\varepsilon}{s}\big)\big] < 0\end{aligned} \tag{3.3.8}$$

where

$$\psi_\alpha(r) = (\alpha+1)\varphi_\alpha(r) - r\varphi'_\alpha(r) \tag{3.3.9}$$

is a strictly increasing function:

$$\psi'_\alpha(r) = \alpha\varphi'_\alpha(r) - r\varphi''_\alpha(r) = -\alpha\frac{r^{\alpha+1}}{\sqrt{1-r^2}} + \frac{r^{\alpha+1}(\alpha+1-\alpha r^2)}{(1-r^2)^{3/2}} = \frac{r^{\alpha+1}}{(1-r^2)^{3/2}}.$$

Hence

$$u(x,y) = s \text{ if and only if } F(x,y,s) = 0$$

is well defined. Claims i) and ii) derive from this definition of u as in Theorem 3.2.2.

We show claim iii). This follows applying the implicit function theorem to compute

$$\partial_{x_i} u = -\frac{\partial_{x_i} F}{\partial_s F} = \frac{x_i r^\alpha}{s^\alpha\sqrt{s^2-r^2}} \cdot \frac{1}{\Big[\psi_\alpha\big(\frac{r_\varepsilon}{s}\big) - \psi_\alpha\big(\frac{r}{s}\big)\Big]}, \qquad \partial_y u = -\frac{\partial_y F}{\partial_s F} = -\frac{1}{\Big[\psi_\alpha\big(\frac{r_\varepsilon}{s}\big) - \psi_\alpha\big(\frac{r}{s}\big)\Big]s^\alpha}.$$

Hence

$$\partial_{x_i} u = \frac{x_i r^\alpha}{\sqrt{s^2-r^2}}\partial_y u.$$

Therefore

$$|\nabla_\alpha u|^2 = \Big(\sum_{i=1}^h \big(\frac{x_i r^\alpha}{\sqrt{s^2-r^2}}\big)^2 + r^{2\alpha}\Big)(\partial_y u)^2 = \frac{s^2 r^{2\alpha}}{s^2-r^2} \cdot \frac{1}{s^{2\alpha}\Big[\psi_\alpha\big(\frac{r_\varepsilon}{s}\big) - \psi_\alpha\big(\frac{r}{s}\big)\Big]^2}$$

and

$$\frac{\partial_{x_i} u}{|\nabla_\alpha u|} = \frac{x_i r^\alpha}{\sqrt{s^2 - r^2}} \partial_y u \frac{1}{\frac{r^\alpha s}{\sqrt{s^2 - r^2}} \partial_y u} = \frac{x_i}{s}, \qquad \frac{\partial_y u}{|\nabla_\alpha u|} = \frac{\partial_y u}{\frac{r^\alpha s}{\sqrt{s^2 - r^2}} \partial_y u} = \frac{\sqrt{s^2 - r^2}}{s r^\alpha}.$$

Therefore $H^\alpha_{\Sigma_s} = 1/s$:

$$\operatorname{div}_\alpha\Big(\frac{\nabla_\alpha u}{|\nabla_\alpha u|}\Big) = \sum_{i=1}^h \partial_{x_i} \frac{x_i}{s} = \frac{h}{s}.$$

To show the estimates at claim iv), analogously to (3.2.26), we define for $0 < |x| < 1 - \varepsilon$ and $0 < y < \varphi_\alpha(r) - y_\varepsilon$ the function

$$f_x(y) = u(x, \varphi_\alpha(r) - y) = s = \frac{1}{H^\alpha_{\Sigma_s}} \quad \text{if and only if} \quad (x, \varphi_\alpha(r) - y) \in \Sigma_s.$$

Given $|x| < r_\varepsilon$, we have for any $0 < y < \varphi_\alpha(r) - y_\varepsilon$

$$f_x'(y) = -\partial_y u(x, \varphi_\alpha(r) - y) = \frac{1}{f_x(y)^\alpha\Big[\psi_\alpha(r_\varepsilon/s) - \psi_\alpha(r/s)\Big]} \geq \frac{1}{f_x(y)^\alpha\Big[\psi_\alpha(r_\varepsilon/s) - \alpha\pi_\alpha\Big]},$$

where $\alpha\pi_\alpha = \psi_\alpha(0)$. Moreover, (see (3.2.27)) we have

$$\begin{aligned} s^\alpha\big(\psi_\alpha(r_\varepsilon/s) - \alpha\pi_\alpha\big) &= s^\alpha \int_0^{r_\varepsilon/s} \psi_\alpha'(r)\, dr = s^\alpha \int_0^{r_\varepsilon/s} \frac{r^{\alpha+1}}{(1 - r^2)^{3/2}}\, dr \\ &\leq r_\varepsilon^\alpha \int_0^{r_\varepsilon/s} \frac{r}{(1 - r^2)^{3/2}}\, dr = r_\varepsilon^\alpha\Big((1 - (r_\varepsilon/s)^2)^{-1/2} - 1\Big) \\ &\leq \frac{1}{\sqrt{s - r_\varepsilon}}. \end{aligned}$$

Estimates (3.3.7) and (3.3.6) follow from

$$f_x'(y) \geq \sqrt{f_x(y) - r_\varepsilon}$$

exactly as in Theorem 3.2.2. In particular, when $\varepsilon = 0$, an integration with $f_x(0) = 1$ gives $f_y(x) \geq 1 + t^2/8$, hence

$$1 - H_{\Sigma_s}(x, \varphi_\alpha(|x|) - y) = 1 - \frac{1}{f_x(y)} \geq \frac{y^2}{8 + y^2} \geq \frac{y^2}{8 + \pi_\alpha^2}$$

When $\varepsilon > 0$, $f_x'(y) \geq \sqrt{f_x(y) - r_\varepsilon} \geq \sqrt{\varepsilon}$, hence $f_x(y) \geq 1 + y\sqrt{\varepsilon}$ and

$$1 - H_{\Sigma_s}(x, \varphi_\alpha(|x|) - y) = 1 - \frac{1}{f_x(y)} \geq y\frac{\sqrt{\varepsilon}}{1 + y\sqrt{\varepsilon}} \geq \frac{\sqrt{\varepsilon}}{1 + \pi_\alpha}.$$

□

3.4 Subcalibration in the Euclidean space $\mathbb{R}^n$

The proof of Theorems 3.2.1 and 3.3.1 is based on the following model in $\mathbb{R}^n$, $n \geq 2$ with the euclidean perimeter. We consider the unit ball in $\mathbb{R}^n$, $B = \{x \in \mathbb{R}^n : |x| < 1\}$ in $\mathbb{R}^n$ whose profile function is given by

$$\phi : [0,1] \to \mathbb{R}, \ \phi(r) = \sqrt{1-r^2}$$

and it satisfies

$$\phi'(r) = -\frac{r}{\sqrt{1-r^2}}, \ \phi''(r) = -\frac{1}{\sqrt{1-r^2}} - \frac{r^2}{(1-r^2)^{3/2}} \text{ for } r \in [0,1).$$

Given $\varepsilon \geq 0$ we set $r_\varepsilon = 1-\varepsilon$ and $x_n^\varepsilon = \phi(1-\varepsilon) = \sqrt{1-(1-\varepsilon)^2} = \sqrt{1-r_\varepsilon^2}$. We call $C_\varepsilon = \{x = (\hat{x}, x_n) : |\hat{x}| < r_\varepsilon, \ x_n > x_n^\varepsilon\}$. In $C_\varepsilon \setminus B$, the leaves Σ_s are vertical translations of the top part of the boundary B. We define on

$$D_\varepsilon = B \cap \{x = (\hat{x}, x_n) \in \mathbb{R}^n : x_n^\varepsilon < x_n \leq \phi(|\hat{x}|), \ |\hat{x}| < r_\varepsilon\}$$

the functional

$$F_\varepsilon(x,s) = s\big[\phi\big(\frac{|\hat{x}|}{s}\big) - \phi\big(\frac{r_\varepsilon}{s}\big)\big] + x_n^\varepsilon - x_n.$$

As before, for any $x \in \mathbb{R}^n$ there exists a unique $s > 1$ such that $F_\varepsilon(x,s) = 0$. Hence we define $u(x) = s \iff F_\varepsilon(x,s) = 0$ for $x \in D_\varepsilon$. The vector field $\nabla u/|\nabla u|$ turns out to satisfy properties i), ii) and iii) of Theorem 3.2.2 and the estimates at point iv) are the following: for any point $(\hat{x}, \phi(|\hat{x}|) - x_n) \in \Sigma_s$ with $s > 1$ we have

$$1 - H_{\Sigma_s}(|\hat{x}|, \phi(|\hat{x}|) - x_n) \geq \frac{1}{9}x_n^2 \quad \text{when } \varepsilon = 0. \tag{3.4.1}$$

and

$$1 - H_{\Sigma_s}(\hat{x}, \phi(|\hat{x}|) - x_n) \geq \frac{\sqrt{\varepsilon}}{2}x_n \quad \text{when } 0 < \varepsilon < 1. \tag{3.4.2}$$

This construction leads to the quantitative estimates. In the following $P(E)$ denotes the Euclidean perimeter of $E \subset \mathbb{R}^n$.

Theorem 3.4.1. *Let $F \subset \mathbb{R}^n$, $n \geq 2$, be any measurable set with $\mathcal{L}^n(F) = \mathcal{L}^n(B)$.*

i) If $F \Delta B \subset\subset C_0$ then

$$P(F) - P(B) \geq \frac{n-1}{216\,\omega_{n-1}^2}\mathcal{L}^n(F\Delta B)^3. \tag{3.4.3}$$

ii) If $F\Delta B \subset\subset C_\varepsilon$ for $0 < \varepsilon < 1$, then

$$P(F) - P(B) \geq \frac{(n-1)\sqrt{\varepsilon}}{16\,\omega_{n-1}}\mathcal{L}^{2n+1}(F\Delta B)^2. \tag{3.4.4}$$

In the Euclidean setting, the same foliation as in Theorems 3.2.2 and 3.3.4 can be explicitly constructed interpreting $\varepsilon \geq 0$ in the vertical direction. Namely, given $\varepsilon \geq 0$, we call $r_\varepsilon = \phi^{-1}(\varepsilon) = \sqrt{1-\varepsilon^2}$ and

$$C^\varepsilon = \{x = (\hat{x}, x_n) \in \mathbb{R}^n : |\hat{x}| < r_\varepsilon,\ x_n > \varepsilon\}, \quad D^\varepsilon = C^\varepsilon \cap B.$$

In this case, we prove that for any measurable set $F \subset \mathbb{R}^n$ such that $F \vartriangle B \subset\subset C^\varepsilon$,

$$\begin{aligned}&\text{there exists } c_1 = c_1(\varepsilon) > 0,\ c_1(\varepsilon) \to 0 \text{ as } \varepsilon \to 0 \text{ such that}\\ &P(F) - P(B) \geq \frac{c_1(\varepsilon)(n-1)}{n\omega_n}\mathcal{L}^n(F\Delta B)^2 \text{ if } \varepsilon > 0\end{aligned} \tag{3.4.5}$$

and

$$\text{there exists } c_0 > 0 \text{ such that } P(F) - P(B) \geq \frac{c_0(n-1)}{n^2\omega_n^2}\mathcal{L}^n(F\Delta B)^3 \text{ if } \varepsilon = 0. \tag{3.4.6}$$

The foliation above the unit ball is obtained translating the upper part of its boundary: the leaves $\{u = t\}$ are described for $t < 0$ by $(\partial B \cap C^\varepsilon) + (0, -t)$. For any $t > 0$ we consider the sphere centered at $(0, -t) \in \mathbb{R}^n$ passing through the $n-1$-dimensional sphere $S^\varepsilon = \{(\hat{x}, x_n) \in \mathbb{R}^n : |\hat{x}| = r_\varepsilon, x_n = \varepsilon\}$, namely Σ_t is the graph $\{(\hat{x}, h_t^\varepsilon(|\hat{x}|)),\ |\hat{x}| < r_\varepsilon\}$, with $h_t^\varepsilon(r) = -t + \sqrt{r(t)^2 - r^2}$, for $0 < r < r(t)$ and $r(t) > 0$ satisfying $h_t^\varepsilon(r_\varepsilon) = \varepsilon$, i.e.,

$$\sqrt{r(t)^2 - 1 + \varepsilon^2} - t = \varepsilon \iff r(t) = \sqrt{t^2 + 2t\varepsilon + 1}.$$

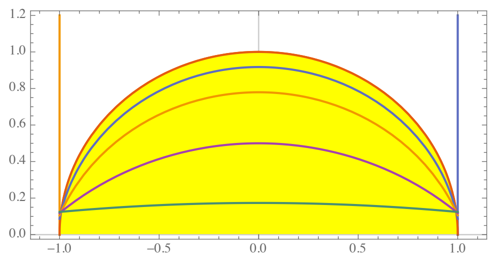


Figure 3.5: The foliation below the unit sphere.

Given $x = (\hat{x}, x_n) \in \mathbb{R}^n$, we define the function $u : D^\varepsilon \to \mathbb{R}$ through the equivalence

$$u(x) = t \iff x_n = h_t^\varepsilon(|\hat{x}|) \text{ i.e., } (x_n + t)^2 = r(t)^2 - |\hat{x}|^2.$$

Hence $u(x) = \frac{1-|x|^2}{2(x_n-\varepsilon)}$. It holds

$$\begin{aligned}
\frac{\partial u}{\partial x_i} &= -\frac{x_i}{x_n-\varepsilon} \text{ for } i = 1,\dots,n-1\\
\frac{\partial u}{\partial x_n} &= -\frac{x_n}{x_n-\varepsilon} - \frac{1-|x|^2}{2(x_n-\varepsilon)^2} = \frac{2\varepsilon x_n - x_n^2 - 1 + |\hat{x}|^2}{2(x_n-\varepsilon)^2} \le -\frac{x_n}{x_n-\varepsilon} \le -1 \text{ in } C_\varepsilon \cap B\\
|\nabla u|^2 &= \frac{\sum_{i=1}^{n-1} x_i^2}{(x_n-\varepsilon)^2} + \frac{(2\varepsilon x_n - x_n^2 - 1 + |\hat{x}|^2)^2}{4(x_n-\varepsilon)^4}\\
&= \frac{1}{4(x_n-\varepsilon)^4}\{4|\hat{x}|^2(x_n-\varepsilon)^2 + (1-2\varepsilon x_n + x_n^2 - |\hat{x}|^2)^2\}
\end{aligned}$$

Therefore

$$\nabla u = 0 \iff |\bar{x}|(x_n-\varepsilon) = 0 \text{ and } 1 - 2\varepsilon x_n + x_n^2 - |\bar{x}|^2 = 0$$

which is satisfied only for $x = (\hat{x}, x_n) \in \mathbb{R}^n$, $x_n = \varepsilon$ and $|\hat{x}|^2 = 1-\varepsilon^2$. Moreover

$$|\nabla u(x)| \ge \left|\frac{\partial u}{\partial x_n}\right| \ge 1 \quad x \in C^\varepsilon \cap B. \tag{3.4.7}$$

We define the vector field

$$X = -\frac{\nabla u}{|\nabla u|}$$

which is C^1 on C^ε. Its divergence at the point $x \in \Sigma_t$ equals the tangential divergence to the vector field on the surface Σ_t, hence

$$\frac{\operatorname{div} X}{n-1} = H_{\Sigma_t} = \frac{1}{r(t)} \text{ for } x \in \Sigma_t,\ t > 0$$

where H_{Σ_t} is the mean curvature of the surface Σ_t.

We start the proof of (3.4.5) and (3.4.6) that makes of use the following elementary Lemma. The strategy that we are going to describe represents an alternative argument to the conclusion of the proof of Theorems 3.2.1 and 3.3.1.

Lemma 3.4.2. *For any $M, L, k > 0$ and for any measurable function $\vartheta : [0, M] \to [0, L]$ there holds*

$$\Big(\int_0^M \vartheta(s)\,ds\Big)^{k+1} \le (k+1)L^k \int_0^M s^k \vartheta(s)\,ds. \tag{3.4.8}$$

Proof. Integrating by parts, we obtain

$$\int_0^M s^k \vartheta(s)\,ds = M^k \int_0^M \vartheta(s)\,ds - k\int_0^M s^{k-1}\int_0^s \vartheta(t)dt\,ds.$$

The function

$$g(s) = \int_0^s \vartheta(t)dt$$

is nonnegative, increasing, L-Lipschitz and with $g(0) = 0$. The claim (3.4.8) is equivalent to the validity of the inequality

$$g(M)^{k+1} \le (k+1)L^k\Big(M^k g(M) - k\int_0^M s^{k-1} g(s)\,ds\Big)$$

which is equivalent to

$$\int_0^M s^{k-1} g(s)\,ds \le \frac{1}{k} M^k g(M) - \frac{1}{k(k+1)L^k} g(M)^{k+1}.$$

For any $0 \le N \le LM$, let $\mathcal{A}_N$ be the set of the functions $g : [0, M] \to [0, L]$ that are nonnegative, increasing, L-Lipschitz, with $g(0) = 0$ and $g(M) = N$. The maximizer of the maximum problem

$$\max\Big\{ \int_0^M s^{k-1} g(s)\,ds \,:\, g \in \mathcal{A}_N \Big\}$$

is the function

$$\bar g(s) = \begin{cases} Ls & 0 \le s \le \frac{N}{L}, \\ N & \frac{N}{L} \le s \le M. \end{cases}$$

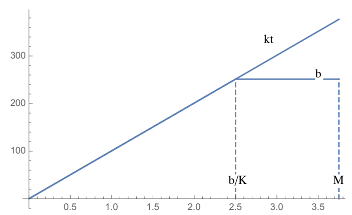


Figure 3.6: The graph of $\bar g$.

The integral of $\bar g$ is

$$\begin{aligned} \int_0^M s^{k-1} \bar g(s)\,ds &= \frac{L}{k+1}\Big(\frac{N}{L}\Big)^{k+1} + \frac{N}{k}\Big(M^k - \Big(\frac{N}{L}\Big)^k\Big) \\ &= \frac{1}{k} M^k g(M) - \frac{1}{k(k+1)L^k} g(M)^{k+1}, \end{aligned}$$

which proves the thesis. □

Let $F \subset\subset C^\varepsilon$. Applying the argument used to prove (3.2.32), we get

$$P(F) - P(B) \ge (n-1)\mathcal{G}(B \setminus F), \quad \mathcal{G}(B \setminus F) = \int_{B\setminus F} \Big(1 - \frac{\mathrm{div} X}{n-1}\Big)\, dzdt.$$

W claim that there exist constants $C_1(\varepsilon)$, $C_0 > 0$ such that

$$\mathcal{G}(B \setminus F) \ge \frac{C_0}{n^2\omega_n^2} \mathcal{L}^n(B \triangle F)^3 \text{ if } \varepsilon = 0, \quad \mathcal{G}(B \setminus F) \ge \frac{C_1}{n\omega_n} \mathcal{L}^n(B \triangle F)^2 \text{ if } \varepsilon > 0.$$

In fact, by the coarea formula, the measure of $B \setminus F$ is

$$\mathcal{L}^n(B \setminus F) = \int_{B\setminus F} dx = \int_0^\infty \Big(\int_{\Sigma_t} \chi_{B\setminus F} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1}\Big) dt$$

The function $\vartheta : [0, \infty) \to \mathbb{R}$,

$$\vartheta(t) = \int_{\Sigma_t} \chi_{B\setminus F} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1}$$

is nonnegative. By (3.4.7) we have

$$0 \le \vartheta(t) \le \int_{\Sigma_t} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1} \le \mathcal{H}^{n-1}(\Sigma_t) \le \mathcal{H}^{n-1}(\partial\Sigma_0).$$

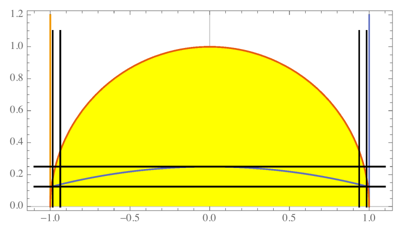
We let $L = \mathcal{H}^{n-1}(\Sigma_0) = \mathcal{H}^{n-1}(\partial B)/2 = n\omega_n/2$. On the other hand

$$\begin{aligned}\mathcal{G}(B \setminus F) &= \int_{B\setminus F} \left(1 - \frac{\mathrm{div} X}{n-1}\right) dz dt \\ &= \int_0^{M(\varepsilon)} \left(\int_{\Sigma_t} \left(1 - \frac{\mathrm{div} X}{n-1}\right) \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1} \right) dt \\ &= \int_0^{M(\varepsilon)} (1 - H_{\Sigma_t})\vartheta(t)\, dt,\end{aligned}$$

We consider the case $\varepsilon > 0$. Without loss of generality we can assume $B \vartriangle F \subset\subset C_{2\varepsilon}$. In this case $\vartheta(t) = 0$ for $t > M(\varepsilon)$,

$$M(\varepsilon) = \frac{1 - 4\varepsilon^2}{2\varepsilon}, \tag{3.4.9}$$

where M is the solution t of the equation $u(0, 2\varepsilon) = t$. If $t > M(\varepsilon)$, the surface Σ_t is contained in $\{t < 2\varepsilon\}$.



We claim that there exists $C = C(\varepsilon) > 0$ such that

$$1 - H_{\Sigma_t} = 1 - \frac{1}{\sqrt{t^2 + 2\varepsilon t + 1}} \ge C(\varepsilon)t \text{ for all } t \in [0, M(\varepsilon)]. \tag{3.4.10}$$

In fact,

$$1 - H_{\Sigma_t} = 1 - \left(1 - \frac{1}{2}(t^2 + 2\varepsilon t) + o(t^2 + 2\varepsilon t)\right) \text{ as } t \to 0^+.$$

Therefore there exists $t_\varepsilon > 0$ such that

$$1 - H_{\Sigma_t} \ge \frac{1}{4}(t^2 + 2\varepsilon t) \ge \frac{\varepsilon}{4} t \text{ for } t < t_\varepsilon.$$

To conclude we use uniform convergence on $[t_\varepsilon, M(\varepsilon)]$. We call $I_\varepsilon := [t_\varepsilon, M(\varepsilon)]$. Let $C_j > 0$ be a sequence of real numbers such that $C_j \to 0$ as $j \to \infty$. Since for any $j \in \mathbb{N}$

$$\max_{t \in I_\varepsilon} C_j t = C_j M(\varepsilon) \to 0 \text{ as } j \to \infty,$$

there exists $\bar{j} \in \mathbb{N}$ such that $C_j t \le 1 - H_{\Sigma_{t_\varepsilon}} > 0$ for any $j \ge \bar{j}$ and $t \in I_\varepsilon$. Then

$$C_{\bar{j}} t \ge 1 - \frac{1}{\sqrt{t_\varepsilon^2 + 2\varepsilon t_\varepsilon + 1}} \ge 1 - \frac{1}{\sqrt{t^2 + 2\varepsilon t + 1}}, \ t \in I_\varepsilon.$$

We prove the claim choosing

$$C(\varepsilon) = \min\{C_{\bar{j}}, \varepsilon/4\} \to 0,\ \varepsilon \to 0^+.$$

In conclusion, applying Lemma 3.4.2 for $k = 1$, when $\varepsilon > 0$ and $F \triangle B \subset\subset C_{2\varepsilon}$

$$\mathcal{G}(B \smallsetminus F) = \int_0^{M(\varepsilon)} (1 - H_{\Sigma_t})\vartheta(t)\, dt \geq C(\varepsilon) \int_0^{M(\varepsilon)} t\vartheta(t)\, dt \geq \frac{C(\varepsilon)}{\mathcal{H}^{n-1}(\partial B)} \Big(\int_0^{M(\varepsilon)} \vartheta(t)\, dt \Big)^2,$$

which proves (3.4.5).

Finally we consider the case when $\varepsilon = 0$. The following estimate holds: there exists $\tilde{C} > 0$ such that

$$1 - H_{\Sigma_t} = 1 - \frac{1}{\sqrt{1-t^2}} \geq \tilde{C} \min\{t^2, 1\},\ \text{for any } t > 0.$$

In fact, since $(1+t^2)^{-1/2} = 1 - t^2/2 + o(t^2)$ as $t \to 0^+$, there exists $t_0 \leq 1$ such that

$$1 - H_{\Sigma_t} \geq \frac{1}{4} t^2 \text{ for } t < t_0.$$

We call $I_0 := [t_0, 1]$. For any $t \in t_0$, we have uniform convergence

$$\sup_{t \in I_0} C_j t^2 = C_j \to 0,\ j \to \infty$$

where $C_j \to 0$ as $j \to \infty$. We conclude as previously that for j big enough $C_j t^2 \leq 1 - H_{\Sigma_{t_0}} \leq 1 - H_{\Sigma_t}$ for any $t \in I_0$. Hence for $t \in [0,1]$, $1 - H_{\Sigma_t} \geq Ct^2$ for some $C > 0$.
On the other hand, $1 - H_{\Sigma_t} \geq 1 - H_{\Sigma_1} = \bar{C}$ for any $t \geq 1$. The claim follows with $\tilde{C} \leq \min\{C, \bar{C}\}$.

By Lemma 3.4.2,

$$\begin{aligned}
\mathcal{G}(B \smallsetminus F) &\geq \tilde{C} \int_0^\infty \vartheta(t) \min\{t^2, 1\}\, dt = \tilde{C} \int_0^1 t^2 \vartheta(t)\, dt + \tilde{C} \int_1^\infty \vartheta(t)\, dt \\
&\geq \frac{\tilde{C}}{3(\mathcal{H}^{n-1}(\partial B)/2)^2} \Big(\int_0^1 \vartheta(t)\, dt \Big)^3 + \tilde{C} \int_1^\infty \vartheta(t)\, ds \\
&\geq \frac{C}{n^2 \omega_n^2} \Big\{ \Big(\int_0^1 \vartheta(t)\, dt \Big)^3 + \Big(\int_1^\infty \vartheta(t)\, dt \Big)^3 \Big\} \\
&\geq \frac{C}{n^2 \omega_n^2} \Big\{ \Big(\int_0^1 \vartheta(t)\, dt + \int_1^\infty \vartheta(t)\, dt \Big)^3 \Big\} \\
&= \frac{C}{n^2 \omega_n^2} \mathcal{L}^n (B \Delta F)^3,
\end{aligned}$$

for $C > 0$, which proves (3.4.6).

CHAPTER 4

A partitioning problem for the isoperimetric stability of the Grushin plane

The purpose of this chapter is to study the stability of the isoperimetric inequality in the particular case of the Grushin plane $(\mathbb{R}^2, d_\alpha)$, where the sharp isoperimetric inequality is known to hold in the wider generality of measurable sets with finite Lebesgue measure (see Chapter 2). So far, we are able to provide only partial results, nonetheless, some new questions arise from our analysis and we describe their solutions.

For any Lebesgue measurable set $E \subset \mathbb{R}^2$, we introduce the notation

$$B_\alpha(E) = \delta_\lambda^\alpha(E_{\mathrm{isop}}^\alpha), \quad \lambda = \Big(\frac{\mathcal{L}^2(E)}{\mathcal{L}^2(E_{\mathrm{isop}}^\alpha)}\Big)^{\frac{1}{Q}},$$

where $Q = \alpha + 2$ is the homogeneous dimension of the Grushin plane and

$$E_{\mathrm{isop}}^\alpha = \{(x,y) \in \mathbb{R}^2 : |y| < \varphi_\alpha(|x|),\ |x| < 1\}, \quad \varphi_\alpha(r) = \int_{\arcsin r}^{\pi/2} \sin^{\alpha+1} t \, dt$$

is the isoperimetric set in the Grushin plane. We define the *α-isoperimetric deficit of E* as

$$D_\alpha(E) = \frac{P_\alpha(E) - P_\alpha(B_\alpha(E))}{P_\alpha(E)},$$

and the *α-asymmetry of E* as

$$A_\alpha(E) = \min_{y \in \mathbb{R}} \frac{\mathcal{L}^2(E \vartriangle (B_\alpha(E) + (0,y)))}{\mathcal{L}^2(E)}.$$

These definitions are given in analogy with the euclidean and the Heisenberg ones (see Section 3.1). Since the isoperimetric set in $(\mathbb{R}^2, d_\alpha)$ is unique up to vertical translations, in the definition of α-asymmetry, comparing E with all the vertical translations of $B_\alpha(E)$ corresponds to compare it with all the isoperimetric sets of the same volume (see Theorem 2.1.4). This situation is slightly different from the Euclidean case (see (3.1.1)) and from the

Heisenberg groups (see Definitions 3.1.2), where invariance under left-translations of the perimeter leads to a wider class of competing sets for E.

A first stability result for the isoperimetric inequality in the Grushin plane is proved in Theorem 4.1.1 where we show that if the α-isoperimetric deficit is small enough, then also the α-asymmetry is arbitrarily close to zero. This connection between isoperimetric deficit and asymmetry is known as the *qualitative stability of the isoperimetric inequality*, and is also valid in more general Grushin spaces $(\mathbb{R}^n, d_\alpha)$ with $n = h + k$ for $h \geq 1$ integer, $k = 1$ where a sharp isoperimetric inequality is established (see Remark 4.1.3).

In the rest of the chapter, we present preliminary results in view of a *Hall-type quantitative isoperimetric inequality* in the Grushin plane, see (3.1.3). Our approach is to consider the techniques of [63] in the class of x- and y-Schwarz symmetric sets in the plane (see Definition 2.1.2), replacing standard perimeter with α-perimeter. Some crucial differences from the classical case arise from the very beginning (see Section 4.2.1 and Remark 4.2.7), and lead us to the study of a partitioning problem for the α-perimeter, that we describe in Section 4.3.

4.1 Qualitative Stability

In this Section, we prove the qualitative stability of the isoperimetric inequality in the Grushin plane. We introduce the notation

$$\omega_\alpha := \mathcal{L}^2(E^\alpha_{\text{isop}}).$$

Theorem 4.1.1 (Qualitative stability). *For every $\varepsilon > 0$ there exists $\delta = \delta(\alpha, \varepsilon) > 0$ such that, for any measurable set $E \subset \mathbb{R}^2$ with finite α-perimeter and $\mathcal{L}^2(E) = \omega_\alpha$, if $D_\alpha(E) \leq \delta$ then $A_\alpha(E) < \varepsilon$.*

Notice that Theorem 4.1.1 implies a qualitative estimate for any choice of $\mathcal{L}^2(E)$, thanks to the invariance under δ^α_λ-dilations of the α-isoperimetric deficit and the α-asymmetry. To prove Theorem 4.1.1 we use a compactness argument relying on the following lemma.

Lemma 4.1.2. *There exist constants $\ell = \ell(\alpha)$, $C = C(\alpha) > 0$ such that for any measurable set $E \subset \mathbb{R}^2$ with finite α-perimeter and such that $\mathcal{L}^2(E) = \omega_\alpha$, there exists a set $E' \subset Q_\ell = [-\ell, \ell]^2$ with $\mathcal{L}^2(E') = \omega_\alpha$ satisfying*

$$D_\alpha(E') \leq C D_\alpha(E), \qquad A_\alpha(E) \leq A_\alpha(E') + C D_\alpha(E). \tag{4.1.1}$$

Lemma 4.1.2 is the same as Lemma 5.1 in [63] reformulated for the α-perimeter. Our proof represents an alternative to the one in [63] and it is based on the choice of *symmetric* *cuts*: to define the set E', we consider a dilation of $\hat{E} = E \cap \{(x,y) \in \mathbb{R}^2 : |x| < \bar{x}\}$ for some $\bar{x} \in \mathbb{R}$ bounded by a constant $\ell_1 = \ell_1(\alpha) > 0$, instead of considering a dilation of $\tilde{E} = E \cap \{(x,y) \in \mathbb{R}^2 : x_1 < x < x_2\}$ for some $x_1, x_2 \in \mathbb{R}$, with $|x_2 - x_1| \le \ell_1$ as in [63]. The last strategy would need a translation of E' to be centered at zero, while our choice does not need any translation and better adapts to the anisotropy of the α-perimeter. Moreover, in the final part we use Proposition 2.4.2 to deduce estimate (4.1.1) for the isoperimetric deficit. We recall the notation introduced in Section 2.4. For any set $E \subset \mathbb{R}^2$, and $t > 0$ we let

$$E^x_{t-} = \{(x,y) \in E : |x| < t\} \quad \text{and} \quad E^x_t = \{(x,y) \in E : |x| = t\}$$
$$E^y_{t-} = \{(x,y) \in E : |y| < t\} \quad \text{and} \quad E^y_t = \{(x,y) \in E : |y| = t\}$$

and

$$v^x_E(t) = \mathcal{H}^1(E^x_t), \quad v^y_E(t) = \int_{E^y_t} |x|^\alpha \, d\mathcal{H}^1.$$

Proof. (of Lemma 4.1.2) Let $E \subset \mathbb{R}^2$ be as in the statement. Following the proof of Theorem 2.4.3, define the function $g : [0,\infty) \to [0,1]$,

$$g(t) = \frac{\mathcal{L}^2(E^x_{t-})}{\omega_\alpha},$$

which is continuous, $(0,1) \subset g(\mathbb{R}) \subset [0,1]$ and it is increasing, hence differentiable almost everywhere. In particular, we deduce as in (2.4.17), that

$$g'(t) = \frac{\mathcal{H}^1(E^x_t)}{\omega_\alpha} = \frac{v^x_E(t)}{\omega_\alpha}.$$

Consider the set $\delta_r(E^x_{t-})$ with $r = g(t)^{-\frac{1}{Q}}$. We have $\mathcal{L}^2(\delta_r(E^x_{t-})) = \omega_\alpha$, hence, by the sharp isoperimetric inequality (see [99] and Remark (2.5.2)),

$$g(t)^{-\frac{Q-1}{Q}} P_\alpha(E^x_{t-}) = P_\alpha(\delta_r(E^x_{t-})) \ge P_\alpha(E^\alpha_{\mathrm{isop}}) \tag{4.1.2}$$

Similarly,

$$P_\alpha(E \setminus E^x_{t-}) \ge (1 - g(t))^{\frac{Q-1}{Q}} P_\alpha(E^\alpha_{\mathrm{isop}}) \tag{4.1.3}$$

Moreover, by Proposition 2.4.1, $P_\alpha(E^x_{t-}) = P_\alpha(E; E^x_{t-}) + v^x_E(t)$ and $P_\alpha(E \setminus E^x_{t-}) = P_\alpha(E; E \setminus E^x_{t-}) + v^x_E(t)$. Then, summing up (4.1.2) and (4.1.3) we obtain

$$P_\alpha(E) + 2v^x_E(t) \ge \left\{ \left(\frac{\mathcal{L}^2(E^x_{t-})}{\omega_\alpha} \right)^{\frac{Q-1}{Q}} + \left(1 - \frac{\mathcal{L}^2(E^x_{t-})}{\omega_\alpha} \right)^{\frac{Q-1}{Q}} \right\} P_\alpha(E^\alpha_{\mathrm{isop}}). \tag{4.1.4}$$

By definition of α-isoperimetric deficit, $P_\alpha(E) = P_\alpha(E^\alpha_{\mathrm{isop}})(1 + D_\alpha(E))$, hence we deduce

$$\left\{ g(t)^{\frac{Q-1}{Q}} + (1 - g(t))^{\frac{Q-1}{Q}} - 1 - D_\alpha(E) \right\} P_\alpha(E^\alpha_{\mathrm{isop}}) \le 2v^x_E(t). \tag{4.1.5}$$

Consider the auxiliary function $\psi : [0,1] \to \mathbb{R}$, $\psi(s) = s^{\frac{Q-1}{Q}} + (1-s)^{\frac{Q-1}{Q}} - 1$ which is concave, with maximum $\psi(1/2) = 2^{\frac{1}{Q}} - 1$ and satisfying $\psi(s) = \psi(1-s)$, $\psi(0) = \psi(1) = 0$. From (4.1.5) we get

$$v_E^x(t) \geq \frac{P_\alpha(E_{\mathrm{isop}}^\alpha)}{4}\psi(g(t)) + \frac{P_\alpha(E_{\mathrm{isop}}^\alpha)}{4}\big\{\psi(g(t)) - 2D_\alpha(E)\big\}.$$

Now, if $2D_\alpha(E) < \max\psi = 2^{\frac{1}{Q}} - 1$, there exist $t_1, t_2 \in \mathbb{R}$ such that $g(t_1) = 1 - g(t_2)$ and $\psi(g(t_1)) = \psi(g(t_2)) = 2D_\alpha(E)$. By concavity of ψ and continuity of g, this leads to $\psi(g(t)) \geq 2D_\alpha(E)$ for any $t_1 \leq t \leq t_2$. Then

$$\omega_\alpha g'(t) = v_E^x(t) \geq \frac{P_\alpha(E_{\mathrm{isop}}^\alpha)}{4}\psi(g(t)), \quad t_1 \leq t \leq t_2. \tag{4.1.6}$$

On the other hand, for $D_\alpha(E)$ small enough, from $\psi(g(t_2)) = 2D_\alpha(E)$ and $1 - g(t_1) = g(t_2)$ we deduce $t_1 \leq t_2/2$. Therefore

$$\frac{t_2}{2} \leq t_2 - t_1 = \int_{t_1}^{t_2} 1 \leq \frac{4}{\omega_\alpha}P_\alpha(E_{\mathrm{isop}}^\alpha)\int_{t_1^{t_2}} \frac{g'(t)}{\psi(g(t))}\,dt \leq \frac{4}{\omega_\alpha}P_\alpha(E_{\mathrm{isop}}^\alpha)\int_0^1 \frac{1}{\psi(s)}\,ds =: \tilde{\ell}.$$

Let $\ell_1 = 2\tilde{\ell}$. We consider the sets $\hat{E} = E_{t_2-}^x$ and $E^\dagger = \delta_\lambda \hat{E}$ with $\lambda \geq 1$ such that $\mathcal{L}^2(E^\dagger) = \omega_\alpha$. We have $\mathcal{L}^2(E^\dagger) = \lambda^Q \lambda(\hat{E})$ and $\lambda(\hat{E}) = \omega_\alpha g(t_2) = \omega_\alpha(1 - g(t_1))$. Notice that, by concavity of ψ, for any $s \in (0, \frac{1}{2})$, $\psi(s) > 2(2^{\frac{1}{Q}} - 1)s$, hence, applied to $g(t_1) < \frac{1}{2}$:

$$\lambda^Q = \frac{1}{1 - g(t_1)} \leq \frac{1}{1 - \frac{\psi(g(t_1))}{2(2^{\frac{1}{Q}}-1)}} = \frac{1}{1 - \frac{D_\alpha(E)}{2^{\frac{1}{Q}}-1}}.$$

In particular, since $2D_\alpha(E) \leq \max\psi = 2^{\frac{1}{Q}-1}$, $\lambda^Q \leq 2$. Hence, by Proposition 2.4.2,

$$P_\alpha(E^\dagger) = \lambda^{Q-1}P_\alpha(\hat{E}) \leq \lambda^{Q-1}P_\alpha(E) \leq 2^{\frac{Q-1}{Q}}P_\alpha(E) \tag{4.1.7}$$

and the first inequality in (4.1.1) is proved.

To prove the second inequality in (4.1.1), let $(0,y) \in \mathbb{R}^2$ be such that $\omega_\alpha A_\alpha(E^\dagger) = \mathcal{L}^2(E^\dagger \vartriangle E_y^\alpha)$ with $E_y^\alpha = E_{\mathrm{isop}}^\alpha + (0,y)$. Then

$$\begin{aligned}
\omega_\alpha A_\alpha(E) &\leq \mathcal{L}^2(E \vartriangle E_y^\alpha) \\
&\leq \mathcal{L}^2(E \vartriangle \hat{E}) + \mathcal{L}^2\big(\hat{E} \vartriangle \delta_{1/\lambda}(E_y^\alpha)\big) + \mathcal{L}^2\big(\delta_{1/\lambda}(E_y^\alpha) \vartriangle E_y^\alpha\big) \\
&= \mathcal{L}^2(E \setminus \hat{E}) + \mathcal{L}^2\big(\delta_{1/\lambda}(E^\dagger) \vartriangle \delta_{1/\lambda}(E_y^\alpha)\big) + \mathcal{L}^2\big(E_y^\alpha \setminus \delta_{1/\lambda}(E_y^\alpha)\big) \\
&\leq \frac{1}{\lambda^Q}\mathcal{L}^2(E^\dagger \vartriangle E_y^\alpha) + \mathcal{L}^2(E \setminus \hat{E}) + \mathcal{L}^2\big(E_y^\alpha \setminus \delta_{1/\lambda}(E_y^\alpha)\big) \\
&\leq \omega_\alpha A_\alpha(E^\dagger) + C(\alpha)D_\alpha(E)
\end{aligned}$$

To conclude we start again from (4.1.5) replacing E with $E^\dagger$ and considering sets $(E^\dagger)_{t-}^y$, $t > 0$, which can be assumed to be in the stripe $|x| < \ell_1$. In this case, for $t > 0$

$$v_E^y(t) = \int_{E_t^y} |x|^\alpha \, d\mathcal{H}^1 \leq \ell_1^\alpha \omega_\alpha g'(t)$$

and inequality (4.1.6) reads in some interval $t_3 \leq t \leq t_4$:

$$\ell_1^\alpha \omega_\alpha g'(t) \geq v_E^y(t) \geq \frac{P_\alpha(E_{\text{isop}}^\alpha)}{4}\psi(g(t)).$$

We deduce (4.1.1) for $\ell = \max\{\ell_1, \ell_2\}$ with $\ell_2 = 2\tilde{\ell}/\ell_1^\alpha$. □

Proof. (of Theorem 4.1.1) It is enough to prove the statement for $E \subset Q_\ell$ where $\ell > 0$ is the constant in Lemma 4.1.2. In fact, if $E \subset \mathbb{R}^2$ is such that $\mathcal{L}^2(E) = \mathcal{L}^2(B)$, from Lemma 4.1.2 there exists $E' \subset Q_\ell$ satisfying (4.1.1) for $C > 0$. If the qualitative estimate in the statement holds for sets contained in Q_ℓ, for $\varepsilon > 0$ fixed, there exists $\delta_1 > 0$ such that if $D_\alpha(E') < \delta_1$, then $\lambda_\alpha(E') < \varepsilon/2$. Defining $\delta = \min\{\frac{\varepsilon}{2C}, \frac{\delta_1}{C}\}$, we get for $D_\alpha(E) \leq \delta$,

$$\lambda_\alpha(E) \leq \lambda_\alpha(E') + C D_\alpha(E) < \frac{\varepsilon}{2} + C\frac{\varepsilon}{2C} = \varepsilon.$$

We assume by contradiction that there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exists $E_n \subset Q_\ell$ with $\lambda_\alpha(E_n) \geq \varepsilon$ and $D_\alpha(E_n) \leq \frac{1}{n}$. Hence

$$P_\alpha(E_n) \leq \Big(1 + \frac{1}{n}\Big) P_\alpha(B_\alpha) \leq 2P_\alpha(B_\alpha) \text{ for every } n \in \mathbb{N} \tag{4.1.8}$$

By the compactness theorem for BV_α-functions, see Section (1.2.1), there exists a subsequence converging in $L^1_{loc}(\mathbb{R}^2)$ to a measurable set E_∞ with finite α-perimeter. Since the sets E_n are all contained in the same compact Q, convergence $\chi_{E_n} \to \chi_{E_\infty}$ is in $L^1(\mathbb{R}^2)$. This implies

$$\mathcal{L}^2(E_\infty) = \int_{\mathbb{R}^2} \chi_{E_\infty} = \lim_{n\to\infty} \int_{\mathbb{R}^2} \chi_{E_n} = \lim_{n\to\infty} \mathcal{L}^2(E_n) = \mathcal{L}^2(B_\alpha). \tag{4.1.9}$$

Moreover, by the lower semicontinuity of the α-perimeter,

$$P_\alpha(E_\infty) \leq \liminf_{n\to\infty} P_\alpha(E_n) = P_\alpha(B_\alpha) \text{ since } D_\alpha(E_n) \to 0. \tag{4.1.10}$$

We have constructed a set $E_\infty \subset \mathbb{R}^2$ such that $\mathcal{L}^2(E_\infty) = \mathcal{L}^2(B_\alpha)$, and $P_\alpha(E_\infty) \leq P_\alpha(B_\alpha)$. Therefore by Theorem 3.2 in [99], E_∞ coincides up to a vertical translation to the set B_α, i.e., $E_\infty = B_\alpha(0, y)$ for some $y \in \mathbb{R}$. Hence, by definition of λ_α,

$$\lambda_\alpha(E_n) \leq \frac{\mathcal{L}^2(E_n \,\triangle\, E_\infty)}{\mathcal{L}^2(B_\alpha)} \to 0 \text{ for } n \to \infty$$

which contradicts the assumption $\lambda_\alpha(E_n) > \varepsilon$. □

Remark 4.1.3. The proofs of Theorem 4.1.1 and Lemma 4.1.2 apply also to the case of $(\mathbb{R}^n, d_\alpha)$ $n = h + k$, with h, k integers such that $h \geq 1$, $k = 1$. In fact, by Remark 2.5.2, the solution of the isoperimetric problem in this case is unique up to dilations and vertical translations, hence a sharp isoperimetric inequality holds and (4.1.2), (4.1.3) can be reproduced. Moreover estimates (4.1.5) and (4.1.7) hold for any choice of $h, k \geq 1$ integers and the same holds for the compactness theorem.

4.2 Euclidean techniques to prove Hall's theorem

In the seminal paper [63], the authors prove the quantitative isoperimetric inequality in $\mathbb{R}^n$. In a central step, contained in Section 4 of [63], they prove the quantitative inequality for axially symmetric sets. In this Section, we describe their techniques in the case of x- and y-Schwarz symmetric sets in $\mathbb{R}^2$. Namely, we show how to prove existence of a constant $C > 0$ such that

$$A(E)^2 \leq CD(E), \tag{4.2.1}$$

for all x- and y-Schwarz symmetric sets $E \subset \mathbb{R}^2$.

In the euclidean setting, the proof of (4.2.1) can be reduced to an estimate of the asymmetry inside the stripe $Z = \left\{(x,y) : |x| \leq \frac{\sqrt{2}}{2}\right\}$ with respect to the isoperimetric deficit, i.e.,

$$\mathcal{L}^2((B \smallsetminus E) \cap Z) \leq C\sqrt{D(E)}, \tag{4.2.2}$$

for any x- and y-Schwarz symmetric set $E \subset \mathbb{R}^2$ with $\mathcal{L}^2(E) = \mathcal{L}^2(B)$, where $B = B_E(0,1)$. In fact, if Z_2 denotes the stripe where $|y| \leq \frac{\sqrt{2}}{2}$, we have $B \subset Z \cup Z_2$. Then $B \smallsetminus E \subset ((B \smallsetminus E) \cap Z) \cup ((B \smallsetminus E) \cap Z_2)$. Now, if $\mathcal{L}^2((B \smallsetminus E) \cap Z) < \mathcal{L}^2((B \smallsetminus E) \cap Z_2)$, the transformation $\theta : \mathbb{R}^2 \to \mathbb{R}^2$, $\theta(x_1, x_2) = (x_2, x_1)$ is such that

i. $\theta(E)$ is x- and y-Schwarz symmetric

ii. $P(E) = P(\theta(E))$, $A(\theta(E)) = A(E)$

iii. $\mathcal{L}^2((B \smallsetminus \theta(E)) \cap Z) > \mathcal{L}^2((B \smallsetminus \theta(E)) \cap Z_2)$.

Hence, up to a rotation of the axes we can assume $\mathcal{L}^2((B \smallsetminus E) \cap Z) \geq \mathcal{L}^2((B \smallsetminus E) \cap Z_2)$ which implies

$$\mathcal{L}^2(B \smallsetminus E) \leq 2\mathcal{L}^2((B \smallsetminus E) \cap Z). \tag{4.2.3}$$

By definition of asymmetry, we hence get

$$A(E) \cdot \omega_n \leq \mathcal{L}^2(E \vartriangle B) = 2\mathcal{L}^2(B \smallsetminus E) \leq 4\mathcal{L}^2((B \smallsetminus E) \cap Z)$$

and (4.2.1) follows if (4.2.2) holds. The proof of (4.2.1) is divided into three steps.

Step 1: From the asymmetry to section estimates

Let $\ell > 0$. There exist $C_1 > 0$ and $\delta_1 = \delta_1(\ell) > 0$ such that, if $E \subset Q_\ell = [-\ell, \ell]^2$, $\mathcal{L}^2(E) = \mathcal{L}^2(B)$, is x-symmetric and y-Schwarz symmetric with $D(E) \leq \delta_1$, there exists $\bar{x} = \bar{x}(E) \in [0, \sqrt{2}/2]$ such that, if $x' = x'(E) > 0$ is defined through the equality

$$\int_0^{\bar{x}} v_E(t)\, dt = \int_0^{x'} v_B(t)\, dt, \tag{4.2.4}$$

then

$$\mathcal{L}^2((B \setminus E) \cap Z) \le C_1 |v_E(\bar{x}(E)) - v_B(x'(E))|. \tag{4.2.5}$$

We will prove this estimate in Lemma 4.2.4 for the α-perimeter and for every $\alpha \ge 0$.

Step 2: From unaligned sections to sections aligned at $\{x_1 = 0\}$.

Let $\ell > 0$. There exists $C_2 > 0$ such that for any x-and y-Schwarz symmetric set $E \subset Q_\ell$, with $\mathcal{L}^2(E) = \mathcal{L}^2(B)$ such that, if $\delta_1 > 0$, $\bar{x}(E)$ and $x'(E)$ are as in Step 1 and $D(E) < \delta_1$, then there exists a set $\tilde{E} \subset \mathbb{R}^2$ so that

$$P(\tilde{E}) \le P(E), \qquad |v_E(\bar{x}(E)) - v_B(x'(E))| \le C_2\, |v_{\tilde{E}}(0) - v_B(0)|. \tag{4.2.6}$$

The proof of this fact is based on the two following remarks.

Remark 4.2.1. Given a x- and y-Schwarz symmetric set $E \subset \mathbb{R}^2$ and $h > 0$ such that $v_E(\bar{x}) = 2h$ for some $\bar{x} > 0$, we construct a x-symmetric and y-Schwarz symmetric set $\tilde{E} \subset \mathbb{R}^2$ that satisfies:

1. $P(\tilde{E}) \le P(E)$;
2. there exists $t > 0$ such that the ball $\tilde{B} = B(0, R)$ centered at $0 \in \mathbb{R}^2$ of some radius $R > 0$ is the central part of $\tilde{E}$, namely $\tilde{E}^x_{t-} = \tilde{B}^x_{t-}$;
3. $\mathcal{L}^2(\tilde{E}^x_{t-}) = \mathcal{L}^2(E^x_{\bar{x}-})$ and $\mathcal{L}^2(\tilde{E} \setminus \tilde{E}^x_{t-}) = \mathcal{L}^2(E \setminus E^x_{\bar{x}-})$;
4. $v_{\tilde{E}}(t) = v_E(\bar{x}) = 2h$.

The set $\tilde{E}$ is therefore obtained as a minimum of the perimeter under the *prescribed partition of volume* in claim 3 and the *one-dimensional constraint* in claim 2.

We begin the proof observing that there exists a unique euclidean ball $\tilde{B} = B_E(0, R) \subset \mathbb{R}^2$ and a unique $t > 0$ such that

$$\mathcal{L}^2(\tilde{B}^x_{t-}) = \mathcal{L}^2(E^x_{\bar{x}-}) \quad \text{and} \quad v_E(\bar{x}) = v_{\tilde{B}}(t)$$

We let

$$E^l = E \cap \{(x, y) \in\in \mathbb{R}^2 : x < \bar{x}\}, \qquad E^r = E \cap \{(x, y) \in \mathbb{R}^2 : x > \bar{x}\},$$
$$\tilde{B}^l = \tilde{B} \cap \{(x, y) \in \mathbb{R}^2 : x < -t\}, \qquad \tilde{B}^r = \tilde{B} \cap \{(x, y) \in \mathbb{R}^2 : x > t\}$$

The set

$$\tilde{E} = \Big(E^l + (\bar{x} - t)\Big) \cup \tilde{B}^x_{t-} \cup \Big(E^r + (t - \bar{x})\Big)$$

satisfies claims 2, 3 and 4. We show that $P(\tilde{E}) \le P(E)$. In fact

$$\mathcal{L}^2\Big(\big(\tilde{B}^l + ((x_0 - \bar{x}), 0)\big) \cup E^x_{\bar{x}-} \cup \big(\tilde{B}^r + ((\bar{x} - x_0), 0)\big)\Big) = \mathcal{L}^2(\tilde{B}),$$

therefore, by the equality case in the isoperimetric inequality in $\mathbb{R}^2$, we obtain

$$\begin{aligned} &P(\tilde{B};\{x<-t\})+P(\tilde{B};\{|x|<t\})+P(\tilde{B};\{x>t\})=P(\tilde{B})\\ &\leq P\Big(\big(\tilde{B}^l+(t-\bar{x},0)\big)\cup E^x_{\bar{x}-}\cup\big(\tilde{B}^r+(\bar{x}-t,0)\big)\Big)\\ &=P(\tilde{B}^l+(t-\bar{x},0);\{x<-\bar{x}\})+P(E,\{|x|<\bar{x}\})+P(\tilde{B}^r+(\bar{x}-t,0);\{x>\bar{x}\})\\ &=P(\tilde{B};\{x<-t\})+P(E,\{|x|<\bar{x}\})+P(\tilde{B};\{x>t\}) \end{aligned} \tag{4.2.7}$$

and claim 1 follows, by writing

$$\begin{aligned} P(\tilde{E})&=P(E^l+(\bar{x}-t,0),\{x<-t\})+P(\tilde{B};\{|x|<t\})+P(E^r+(t-\bar{x},0);\{x>t\})\\ &=P(E^l,\{x<-\bar{x}\})+P(\tilde{B};\{|x|<t\})+P(E^r;\{x>\bar{x}\})\\ &\leq P(E^l,\{x<-\bar{x}\})+P(E;\{|x|<\bar{x}\})+P(E^r;\{x>\bar{x}\})=P(E). \end{aligned} \tag{4.2.8}$$

Remark 4.2.2. Let $B=B_E(0,1)$, $\tilde{B}=B_E(0,R)$ with $R>0$ and $x'\in(0,\sqrt{3}/2)$. Define the point $x_0>0$ through the following equality

$$\int_0^{x_0} v_{\tilde{B}}(x)\,dx=\int_0^{x'} v_B(x)\,dx \tag{4.2.9}$$

Then, there exists a constant $C_2>0$ such that

$$|v_B(x')-v_{\tilde{B}}(x_0)|\leq C_2|v_B(0)-v_{\tilde{B}}(0)|. \tag{4.2.10}$$

We stress that the balls $\tilde{B}$ and B have the same center: this property is central in the argument (see Proposition 4.3.11).

We give a proof in the case $R\geq 1$. In this case we have $v_{\tilde{B}}\geq v_B$, that implies $x_0\leq x'$, by (4.2.9). Notice that

$$|v_{\tilde{B}}(0)-v_B(0)|=2(R-1)$$

Therefore it is sufficient to prove $v_{\tilde{B}}(x_0)-v_B(x')\leq C_2(R-1)$. To this purpose, notice that for any $0\leq x\leq x_0$, we have

$$\begin{aligned} v_{\tilde{B}}(x)-v_B(x)&=2\big(\sqrt{R^2-x^2}-\sqrt{1-x^2}\big)=2\frac{R^2-1}{\sqrt{R^2-(\frac{\sqrt{3}}{2})^2}+\sqrt{1-(\frac{\sqrt{3}}{2})^2}}\\ &\leq 2\frac{R+1}{\sqrt{R^2-(\frac{\sqrt{3}}{2})^2}+\frac{1}{2}}(R-1)\leq C(R-1) \end{aligned} \tag{4.2.11}$$

hence

$$C\frac{\sqrt{3}}{2}(R-1)\geq\int_0^{x_0} v_{\tilde{B}}(x)-v_B(x)\,dx=\int_{x_0}^{x'} v_B(x)\,dx\geq v_B(\sqrt{3}/2)|x'-x_0|\geq|x'-x_0|.$$

In conclusion,

$$\begin{aligned} v_{\tilde{B}}(x_0)-v_B(x')&=(v_{\tilde{B}}(x_0)-v_B(x_0))+(v_B(x_0)-v_B(x'))\\ &\leq C(R-1)+v_B(x')+v_B'(x')(x_0-x')-v_B(x')\\ &\leq C_2(R-1). \end{aligned}$$

Proof of (4.2.6). Starting from a x- and y-Schwarz symmetric set $E \subset Q_\ell$, such that $\mathcal{L}^2(E) = \mathcal{L}^2(B)$, we consider $\bar{x}(E) \in (0, \sqrt{2}/2)$ and $x'(E) > 0$ as in (4.2.5). From the proof of (4.2.5), given in Lemma 4.2.4 below, it will be clear that, assuming $D(E)$ small enough ($D(E) < \delta_1$), we have $x'(E) \in (0, \sqrt{3}/2)$. Applying Remark 4.2.1, we obtain a set $\tilde{E} \subset \mathbb{R}^2$ and a point $x_0 > 0$ such that $\mathcal{L}^2(\tilde{E}) = \mathcal{L}^2(B)$, $P(\tilde{E}) \le P(E)$ and $v_{\tilde{E}}(x_0) = v_E(\bar{x}(E))$. In particular, $\tilde{E}$ is a ball centered at zero in its central part, hence applying Remark 4.2.2,

$$|v_E(\bar{x}(E)) - v_B(x'(E))| = |v_{\tilde{E}}(x_0) - v_B(x'(E))| \le C_2 |v_{\tilde{E}}(0) - v_B(0)|$$

and (4.2.6) follows. □

Step 3: Estimate of the section-gap in terms of the isoperimetric deficit.

There exist $C_3, \delta_3 > 0$ such that for any x-symmetric and y-Schwarz symmetric set $E \subset \mathbb{R}^2$, $\mathcal{L}^2(E) = \mathcal{L}^2(B)$ such that $D(E) < \delta_3$, we have

$$|v_E(0) - v_B(0)| \le C_3 \sqrt{D(E)}. \tag{4.2.12}$$

The proof starts from replacing the set E with E', such that $P(E') \le P(E)$, in the same spirit of Step 2. In this case, we consider the unique ball B' centered on the positive x-axis such that

- $\mathcal{L}^2(B' \cap \{x > 0\}) = \mathcal{L}^2(E)/2$;
- $v_{B'}(0) = v_E(0)$.

The set E' is obtained as the union of $B' \cap \{x > 0\}$ with its reflection with respect to the y-axis, and it is a minimizer for the perimeter in the class of x-symmetric and y-Schwarz symmetric sets in $\mathbb{R}^2$ with prescribed volume and section $\beta = v_E(0)$. The final estimate (4.2.12) follows from a Taylor development of $D(E')$ as a function of β, to obtain

$$|v_{E'}(0) - v_B(0)| \le \sqrt{D(E')}.$$

Proof of (4.2.1). Let $E \subset \mathbb{R}^2$ be a x- and y-Schwarz symmetric set. As observed in Remark 3.1.1, we can assume without loss of generality that $D(E) \le \min\{\delta_1, \delta_3, 1\}$. By invariance of $A(E)$ and $D(E)$ with respect to dilations we also assume $\mathcal{L}^2(E) = \mathcal{L}^2(B_E(0,1))$. By Lemma 4.1.2, it is enough to prove (4.2.1) for $E' \subset Q_\ell$, with $\ell > 0$ as in the lemma, for $\alpha = 0$. In fact, by (4.1.1), if (4.2.1) holds for every set contained in Q_ℓ, we have, increasing the constant C, if necessary:

$$A(E) \le A(E') + CD(E) \le C(\sqrt{D(E')} + D(E)) \le C(\sqrt{D(E)} + D(E)) \le C\sqrt{D(E)}.$$

Assuming $E \subset Q_\ell$, we hence apply the three previous steps in the following way

$$\mathcal{L}^2((E \triangle E^\alpha_{\text{isop}}) \cap Z) \le |v_E(\bar{x}) - v_B(x')| \le |v_{\tilde{E}}(0) - v_B(0)| \le \sqrt{D(\tilde{E})} \le \sqrt{D(E)},$$

where $\tilde{E} \subset \mathbb{R}^2$ is the x-symmetric and y-Schwarz symmetric set found at Step 2. □

4.2.1. Comments on possible adaptations to the Grushin plane

In the Grushin plane the isoperimetric set

$$E^\alpha_{\text{isop}} = \Big\{(x,y) \in \mathbb{R}^2 : |y| \le \varphi_\alpha(|x|), |x| < 1\Big\},\ \varphi_\alpha(r) = \int_{\arcsin r}^{\frac{\pi}{2}} \sin^{\alpha+1} t\, dt$$

is contained in $Z \cup Z_2$ where

$$Z = \Big\{(x,y) \in \mathbb{R}^2 : |x| \le \frac{\sqrt{2}}{2}\Big\}, \quad Z_2 = \Big\{(x,y) \in \mathbb{R}^2 : |y| \le \varphi_\alpha\Big(\frac{\sqrt{2}}{2}\Big)\Big\}$$

Hence, for any set $E \subset \mathbb{R}^2$

$$\begin{aligned}\mathcal{L}^2(E \triangle E^\alpha_{\text{isop}}) = 2\mathcal{L}^2(E^\alpha_{\text{isop}} \setminus E) &\le 2\{\mathcal{L}^2((E^\alpha_{\text{isop}} \setminus E) \cap Z) + \mathcal{L}^2((E^\alpha_{\text{isop}} \setminus E) \cap Z_2)\}\\ &\le 4 \max\{\mathcal{L}^2((E^\alpha_{\text{isop}} \setminus E) \cap Z),\ \mathcal{L}^2((E^\alpha_{\text{isop}} \setminus E) \cap Z_2)\}\end{aligned}$$

Differently from the euclidean case, in general, a rotation of the axes changes the α-perimeter of a set (see Example 4.2.3 below). To use Fusco Maggi and Pratelli approach to Hall's inequality we must therefore prove estimate (4.2.2) in both stripes Z, Z_2. In Lemma 4.2.4 we show that the estimate (4.2.5) at Step 1, holds for the α-perimeter for every $\alpha \ge 0$, in both stripes Z, Z_2. In Lemma 4.2.6, we show that the proof of estimate (4.2.6) at Step 2 can be easily extended to the α-perimeter in the case of the stripe Z_2, while it cannot be reproduced in the stripe Z since the α-perimeter is not invariant under translations by vectors $(x, 0)$, $x \in \mathbb{R}$. This fact leads us to study a partitioning problem to replace the argument at Step 2 (see Section 4.3).

Example 4.2.3. The map $(x,y) \mapsto \theta(x,y) = (y,x)$ fails to preserve α-perimeter, namely the following property does not hold in general:

$$P_\alpha(E) = P_\alpha(\theta(E)). \tag{4.2.13}$$

We show an example. Consider the set $E \subset \mathbb{R}^2$, $E = [-1,1] \times [-2,2]$ and define $E^\theta = \theta(E) = [-2,2] \times [-1,1]$. By the representation formula for the α-perimeter, we have

$$\begin{aligned}P_\alpha(E) &= \int_{\partial E} |N^E_\alpha(x,y)|\, d\mathcal{H}^1\\ &= \int_{\{1\}\times[-2,2]} d\mathcal{H}^1 + \int_{\{-1\}\times[-2,2]} d\mathcal{H}^1 + \int_{[-1,1]\times\{2\}} |x|^\alpha\, d\mathcal{H}^1 + \int_{[-1,1]\times\{-2\}} |x|^\alpha\, d\mathcal{H}^1\\ &= 8 + 2\int_{-1}^{1} |x|^\alpha\, dx = 8 + 4\int_0^1 x^\alpha\, dx = 8 + \frac{4}{\alpha+1},\end{aligned}$$

while

$$P_\alpha(E^\theta) = 2\mathcal{H}^1([-1,1]\times\{2\}) + 2\int_{[-2,2]\times\{1\}} |x|^\alpha \, d\mathcal{H}^1 = 4 + 4\int_0^2 x^\alpha \, dx = 4 + 4\frac{2^{\alpha+1}}{\alpha+1}.$$

Hence $P_\alpha(E) = P_\alpha(\theta(E))$ if and only if

$$\alpha + 2 = 2^{\alpha+1}$$

which has the only solution $\alpha = 0$.

From the asymmetry to section estimates in vertical and horizontal stripes

For any set $E \subset \mathbb{R}^2$ and any $t > 0$ we let

$$w_E^y(t) = \mathcal{H}^1(E_t^y) \tag{4.2.14}$$

and we introduce from (2.4.2) the notation

$$v_\alpha^x(t) = v^x_{E^\alpha_{\text{isop}}}(t), \quad w_\alpha^y(t) = w^x_{E^\alpha_{\text{isop}}}(t).$$

Lemma 4.2.4. *Let $\alpha \geq 0$, $\ell > 0$. Then, there exist $\delta_1 = \delta_1(\alpha,\ell) > 0$, $C_1 = C_1(\alpha,\ell) > 0$ with the property that for any x- and y-Schwarz symmetric set $E \subset Q_\ell$, satisfying $D_\alpha(E) < \delta_1$, there exists $\bar{x} = \bar{x}(E) \in [0, \frac{\sqrt{2}}{2}]$ such that if $x' = x'(E)$ is defined through the equality*

$$\int_0^{\bar{x}} v_E^x(t)\, dt = \int_0^{x'} v_\alpha^x(t)\, dt \tag{4.2.15}$$

then

$$\mathcal{L}^2((E \vartriangle B_\alpha) \cap Z_1) \leq C_1 |v_E(\bar{x}) - v_\alpha^x(x')|. \tag{4.2.16}$$

Moreover there exists $\bar{y} = \bar{y}(E) \in [0, \varphi_\alpha(\frac{\sqrt{2}}{2})]$ such that if $y' = y'(E)$ is defined through the equality

$$\int_0^{\bar{y}} w_E^y(t)\, dt = \int_0^{y'} w_\alpha^y(t)\, dt \tag{4.2.17}$$

then

$$\mathcal{L}^2((E \vartriangle B_\alpha) \cap Z_2) \leq C_1 |w_E^y(\bar{y}) - w_\alpha^y(y')|. \tag{4.2.18}$$

Proof. We prove (4.2.16). We begin using x-symmetry and y-Schwarz symmetry of E to write:

$$\mathcal{L}^2((E \vartriangle E^\alpha_{\text{isop}}) \cap Z) = \int_0^{\sqrt{2}/2} \mathcal{H}^1(E_t^x \vartriangle (E^\alpha_{\text{isop}})^x_t)\, dt = \int_0^{\sqrt{2}/2} |v_E^x(t) - v_\alpha^x(t)|\, dt.$$

By the Mean Value Theorem, there exists $x_0 \in [0, \frac{\sqrt{2}}{2}]$ such that

$$|v_E^x(x_0) - v_\alpha^x(x_0)| \geq \sqrt{2}\mathcal{L}^2((E \vartriangle E^\alpha_{\text{isop}}) \cap Z). \tag{4.2.19}$$

We define

$$v_\Delta(x_0) = v_E^x(x_0) - v_\alpha^x(x_0), \quad I_\Delta(x_0) = \int_0^{x_0} v_E^x(t) - v_\alpha^x(t)\,dt$$

If $v_\Delta(x_0) \le 0$ and $I_\Delta(x_0) \le 0$, we set $\bar{x} = x_0$. Then,

$$\int_0^{x'} v_\alpha^x(t)\,dt = \int_0^{\bar{x}} v_E^x(t)\,dt \le \int_0^{\bar{x}} v_\alpha^x(t)\,dt$$

hence $0 \le x' \le \bar{x} \le \frac{\sqrt{2}}{2}$. By convexity of E_{isop}^α, we deduce from (4.2.19) that

$$v_\alpha^x(x') \ge v_\alpha^x(\bar{x}) \ge v_E^x(\bar{x}) + \sqrt{2}\mathcal{L}^2((E \vartriangle E_{\text{isop}}^\alpha) \cap Z)$$

and (4.2.16) is proved. The same argument applies if $v_\Delta(x_0) \ge 0$ and $I_\Delta(x_0) \ge 0$.

If $v_\Delta(x_0) \le 0$ and $I_\Delta(x_0) \ge 0$, we consider two cases.

Case 1: The quantity $I_\Delta(x_0) \ge 0$ satisfies the following bound

$$I_\Delta(x_0) \le m(\alpha)\mathcal{L}^2((E \vartriangle E_{\text{isop}}^\alpha) \cap Z), \quad \text{where} \quad m(\alpha) = \frac{1}{8\sqrt{2}}\Big(\frac{2}{\sqrt{3}}\Big)^{\alpha+1} v_\alpha^x\Big(\frac{\sqrt{3}}{2}\Big). \tag{4.2.20}$$

We let $\bar{x}(E) = x_0$. Since $I_\Delta(x_0) \ge 0$, we deduce by (4.2.15) that $x'(E) \ge \bar{x}(E)$. Nevertheless, by Theorem 4.1.1, for any $\varepsilon > 0$, we can choose $\delta > 0$ such that $|\bar{x}(E) - x'(E)| \le \varepsilon$ if $D_\alpha(E) < \delta$. In particular, we can choose $\delta_1 > 0$ such that

$$x'(E) \le \sqrt{3}/2 \quad \text{if} \quad D_\alpha(E) < \delta_1 \tag{4.2.21}$$

In fact, assume by contradiction that for some $\varepsilon > 0$, there exist a sequence of sets $E_n \subset Q_\ell$, $n \in \mathbb{N}$ satisfying

$$D_\alpha(E) \le \frac{1}{n} \quad \text{and} \quad |\bar{x}(E_n) - x'(E_n)| \ge \varepsilon. \tag{4.2.22}$$

Applying the same compactness argument used in the proof of Theorem 4.1.1, (see relations (4.1.8), (4.1.9), (4.1.10)), we deduce that the L^1-limit of the sequence E_n is the isoperimetric set E_{isop}^α. Hence, by (4.2.15), $\lim_{n\to\infty} \bar{x}(E_n) = \lim_{n\to\infty} x'(E_n)$ which contradicts (4.2.22). For $D_\alpha(E) < \delta_1$, by (4.2.21), we therefore have the following estimates

$$\begin{aligned}
&v_\alpha^x(t) \ge v_\alpha^x\Big(\frac{\sqrt{3}}{2}\Big) = 4\int_{\pi/3}^{\pi/2} \sin^{\alpha+1} x\,dx \quad \text{for} \quad 0 \le t \le x' \\
&|\bar{x}(E) - x'(E)| = \int_{\bar{x}}^{x'} dx \le \frac{1}{v_\alpha^x(\sqrt{3}/2)}\int_{\bar{x}}^{x'} v_\alpha^x(t)\,dt = \frac{I_\Delta(x_0)}{v_\alpha^x(\frac{\sqrt{3}}{2})} \\
&\frac{dv_\alpha^x(t)}{dt} = \frac{d}{dt}\Big(4\int_{\arcsin t}^{\pi/2} \sin^{\alpha+1} x\,dx\Big) = -4\frac{t^{\alpha+1}}{\sqrt{1-t^2}} \ge -8\Big(\frac{\sqrt{3}}{2}\Big)^{\alpha+1} \quad \text{for} \quad 0 < t < x'
\end{aligned} \tag{4.2.23}$$

We deduce, using (4.2.20) and (4.2.19)

$$\begin{aligned}
v_\alpha^x(x'(E)) &= v_\alpha^x(\bar{x}(E)) + \int_{\bar{x}(E)}^{x'(E)} \frac{dv_\alpha^x(t)}{dt}\,dt \\
&\ge v_\alpha^x(\bar{x}(E)) - \frac{8}{v_\alpha(\frac{\sqrt{3}}{2})}\Big(\frac{\sqrt{3}}{2}\Big)^{\alpha+1} m(\alpha)\mathcal{L}^2((E \vartriangle E_{\text{isop}}^\alpha) \cap Z) \\
&\ge v_E(\bar{x}(E)) + \Big(\sqrt{2} - \frac{1}{\sqrt{2}}\Big)\mathcal{L}^2((E \vartriangle E_{\text{isop}}^\alpha) \cap Z)
\end{aligned}$$

and (4.2.16) holds with $C_1(\alpha) = \sqrt{2}/2$.

Case 2: We have $I_\triangle(x_0) \geq m(\alpha)\mathcal{L}^2((E \triangle E^\alpha_{\text{isop}}) \cap Z)$. In this case we have

$$\frac{1}{x_0}\int_0^{x_0} v_E^x(t) - v_\alpha^x(t)\,dt \geq \sqrt{2}m(\alpha)\mathcal{L}^2((E \triangle E^\alpha_{\text{isop}}) \cap Z).$$

and we can define

$$\bar{x} = \max\left\{t \in [0, x_0] : v_E^x(t) - v_\alpha^x(t) \geq \sqrt{2}m(\alpha)\mathcal{L}^2((E \triangle E^\alpha_{\text{isop}}) \cap Z)\right\}. \tag{4.2.24}$$

Then

$$\int_{\bar{x}}^{x_0} v_E^x(t) - v_\alpha^x(t)\,dt \leq \sqrt{2}m(\alpha)\mathcal{L}^2((E \triangle E^\alpha_{\text{isop}}) \cap Z)|\bar{x} - x_0| \leq m(\alpha)\mathcal{L}^2((E \triangle E^\alpha_{\text{isop}}) \cap Z).$$

We deduce

$$\int_0^{\bar{x}} v_E^x(t) - v_\alpha^x(t)\,dt \geq 0,$$

that reads, through (4.2.15), $x' \geq \bar{x}$. By convexity of E^α_{isop} and definition of $\bar{x}$, we deduce (4.2.16) for $C_1 = 1/\sqrt{2}m(\alpha)$:

$$v_\alpha^x(x') \leq v_\alpha^x(\bar{x}) \leq v_E^x(\bar{x}) - \sqrt{2}m(\alpha)\mathcal{L}^2((E \triangle E^\alpha_{\text{isop}}) \cap Z).$$

If $v_\triangle(x_0) \geq 0$ and $I_\triangle(x_0) \leq 0$, the same argument works considering two cases depending on the validity of the inequality

$$I_\triangle(x_0) \geq -m_2(\alpha)\mathcal{L}^2((E \triangle E^\alpha_{\text{isop}}) \cap Z), \quad \text{with} \quad m_2(\alpha) = \frac{v_\alpha^x(\frac{1}{\sqrt{2}})}{8}\left(\sqrt{2}\right)^{\alpha+1}. \tag{4.2.25}$$

The same reasoning applies to prove (4.2.18), where the constants m and m_2 have to be replaced observing that

$$w_\alpha^y(t) = 4\psi_\alpha(t), \quad \text{for} \quad 0 < t < \varphi_\alpha(0)$$

where ψ_α is the inverse function of φ_α, and it is still decreasing and concave. □

Remark 4.2.5. By the proof of Lemma 4.2.4, it is clear that it is possible to choose $\delta_1 > 0$ small enough to have $x'(E) \leq \frac{\sqrt{3}}{2}$ (see (4.2.21)) and $y'(E) \leq \varphi_\alpha(\frac{\sqrt{3}}{2})$.

From unaligned sections to sections aligned at {y=0} in Z_2

Lemma 4.2.6. *Let $\alpha \geq 0$ and $\ell > 0$. There exists a constant $C_2 = C_2(\alpha, \ell) > 0$ such that for any x- and y-Schwarz symmetric set $E \subset Q_\ell$ with $\mathcal{L}^2(E) = \mathcal{L}^2(E^\alpha_{\text{isop}})$, if $\delta_1, \bar{y}(E)$ and $y'(E)$ are as in Lemma 4.2.4, there exists a set $\tilde{E} \subset \mathbb{R}^2$ so that*

$$P_\alpha(\tilde{E}) \leq P_\alpha(E), \qquad |w_E^y(\bar{y}(E)) - w_\alpha^y(y'(E))| \leq C_2|w_{\tilde{E}}^y(0) - w_\alpha^y(0)|. \tag{4.2.26}$$

Proof. The proof follows the scheme given by Remarks 4.2.1 and 4.2.2 in the euclidean case. We consider the unique pair of numbers $\lambda, y_0 > 0$ such that

$$\mathcal{L}^2\Big((\delta_\lambda^\alpha E_{\text{isop}}^\alpha)_{y_0-}^y\Big) = \mathcal{L}^2(E_{\bar{y}-}^y) \quad \text{and} \quad \mathcal{H}^1(\{(x,y) \in \delta_\lambda^\alpha E_{\text{isop}}^\alpha : |y| = y_0\}) = w_E^y(\bar{y})$$

and we call $\tilde{E}_\lambda = \delta_\lambda^\alpha E_{\text{isop}}^\alpha$. We let

$$E^d = \{(x,y) \in E : y < -\bar{y}\}, \quad E^u = \{(x,y) \in E : y > \bar{y}\},$$
$$\tilde{E}^d = \{(x,y) \in \tilde{E}_\lambda : y < -y_0\}, \quad \tilde{E}^u = \{(x,y) \in \tilde{E}_\lambda : y > y_0\}.$$

With the same reasoning as in (4.2.7) and (4.2.8), by invariance of the α-perimeter under vertical translations, we deduce that the set $\tilde{E} = \big(E^d + (0, \bar{y} - y_0)\big) \cup (\tilde{E}_\lambda)_{y_0-}^y \cup \big(E^u + (0, y_0 - \bar{y})\big)$ satisfies $P_\alpha(\tilde{E}) \le P_\alpha(E)$.

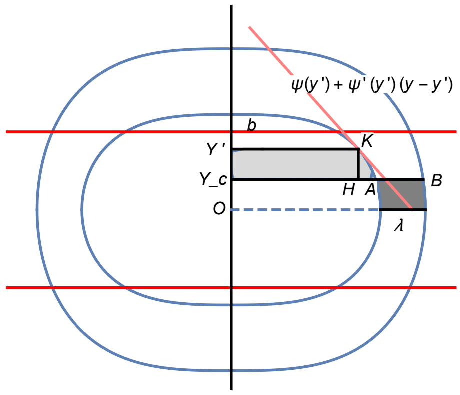


Figure 4.1: Situation in the proof of Lemma 4.2.6 in the stripe Z_2.

To conclude we are left to show existence of a constant C_2 such that $|w_\alpha^y(y') - w_{\tilde{E}}^y(y_0)| \le C_2|w_\alpha^y(0) - w_{\tilde{E}}^y(0)|$. As in Remark 4.2.2, assuming $\lambda > 1$, we have $|w_\alpha^y(0) - w_{\tilde{E}}^y(0)| = 2\pi_\alpha(\lambda - 1)$, hence we only have to prove $|w_\alpha^y(y') - w_{\tilde{E}}^y(y_0)| \le C(\lambda - 1)$. We can assume without loss of generality that $y'(E) \le \varphi_\alpha(\sqrt{3}/2)$, as observed in Remark 4.2.5 (see Figure 4.1). This allows to prove the following estimate: let $0 < t < y_0 \le \varphi_\alpha(\sqrt{3}/2)$ and define $J_t(\lambda) = \lambda\psi_\alpha(t/\lambda^{\alpha+1}) - \psi_\alpha(t)$ for $\lambda \ge 1$. We have $w_{\tilde{E}_\lambda}^y(t) - w_\alpha^y(t) = 4J_t(\lambda)$, then there exists $\lambda^* \ge 1$ such that

$$w_{\tilde{E}_\lambda}^y(t) - w_\alpha^y(t) = 4J_t'(\lambda^*)(\lambda - 1) \le C(\lambda - 1)$$

since for any $\lambda \ge 1$, $0 \le t \le \varphi_\alpha(\sqrt{3}/2)$ we have

$$J_t'(\lambda) = \psi_\alpha\Big(\frac{t}{\lambda^{\alpha+1}}\Big) - \frac{t}{\lambda^{\alpha+1}}\psi_\alpha'\Big(\frac{t}{\lambda^{\alpha+1}}\Big) \le 1 + \psi_\alpha'\Big(\frac{\sqrt{3}}{2}\Big)\varphi_\alpha\Big(\frac{\sqrt{3}}{2}\Big).$$

Then, we deduce

$$\begin{aligned} C\varphi_\alpha(\sqrt{3}/2)(\lambda - 1) &\ge \int_0^{y_0} w_{\tilde{E}_\lambda}^y(t) - w_\alpha^y(t)\, dt \\ &= \int_{y_0}^{y'} w_\alpha^y(t)\, dt \ge w_\alpha^y(\varphi_\alpha(\sqrt{3}/2))|y' - y_0| = 2\sqrt{3}|y' - y_0|. \end{aligned}$$

The rest of the proof runs as in Remark 4.2.2. □

Remark 4.2.7. Since the α-perimeter is not invariant under horizontal translations, namely there exist $E \subset \mathbb{R}^2$ and $x \in \mathbb{R}$ such that $P_\alpha(E + (x,0)) \neq P_\alpha(E)$, it is not possible to argue as in the previous lemma (or as in Remark 4.2.1) to prove an estimate analogous to 4.2.26 in the stripe Z.

In Section 4.3, we show our attempts to solve this problem, studying a minimal partition problem that replaces the construction in the proof of Lemma 4.2.6. In the next Remark we show that for any dilation of the set E^α_{isop}, an estimate as (4.2.26) can be proved.

Remark 4.2.8. For any $\lambda > 0$, we let $\tilde{E} = \delta^\alpha_\lambda E^\alpha_{\text{isop}}$. For any $\bar{x}' \in (0, \sqrt{3}/2)$ we let $x_0 > 0$ defined by

$$\int_0^{x_0} v^x_{\tilde{E}}(t)\,dt = \int_0^{x'} v^x_\alpha(t)\,dt$$

Then, the estimate

$$|v^x_\alpha(x') - v_{\tilde{E}}(x_0)| \leq C_2 |v^x_\alpha(0) - v_{\tilde{E}}(0)|$$

can be proved for some constant $C_2 > 0$ as in the proof of Lemma 4.2.6.

4.3 A minimal partition problem

In this Section, we describe a minimum problem for the α-perimeter of sets in the plane having a prescribed partition of volumes. Motivated by the proof of Step 2 and 3 in Section 4.2, we want to add a *one dimensional constraint* to the minimization problem, that can be described using the notion of trace of a Schwarz symmetric set. To give the definition of trace we use the result of the following Lemma.

Lemma 4.3.1. *Let $E \subset \mathbb{R}^2$ be a y-Schwarz symmetric set and let $x_0 \in \mathbb{R}$. Then there exist $y^+, y^- \geq 0$ such that if $T^\pm$ is the real interval $[-y^\pm, y^\pm] \subset \mathbb{R}$, the following holds*

$$\lim_{x \to x_0^\pm} \int_{\mathbb{R}} |\chi_E(x,y) - \chi_{T^\pm}(y)|\,dy = 0.$$

Proof. We prove the statement for the limit as $x \to x_0^-$. Let $u \in C^1(\mathbb{R}^2)$ and $x_1, x_2 \in (-\infty, x_0)$. The α-gradient of u is $D_\alpha u = (\partial_x u, |x|^\alpha \partial_y u)$. We have

$$\begin{aligned}\int_{\mathbb{R}} \big(u(x_2,y) - u(x_1,y)\big)dy &= \int_{\mathbb{R}} \int_{x_1}^{x_2} \partial_x u(\xi, y)\,d\xi dy \\ &\leq \int_{(x_1,x_2)\times\mathbb{R}} |\partial_x u|(\xi,y)\,d\xi dy \leq |D_\alpha u|((x_1,x_2)\times\mathbb{R}).\end{aligned}$$

By the approximation theorem for BV_α-functions (see Theorem 1.2.4 and [56]), the last inequality can be extended to $u \in BV(\mathbb{R}^2)$. For $u = \chi_E$ we obtain

$$\int_{\mathbb{R}} \big(\chi_E(x_2,y) - \chi_E(x_1,y)\big)dy \leq P_\alpha(E; (x_1,x_2)\times\mathbb{R}),$$

which implies that for any $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\|\chi_E(x_2,\cdot) - \chi_E(x_1,\cdot)\|_{L^1(\mathbb{R})} \le \varepsilon \text{ for } x_0 - \delta < x_1 < x_2 < x_0,$$

which is a Cauchy condition in the complete space $L^1(\mathbb{R})$. We deduce that there exists a function $u \in L^1(\mathbb{R})$ which is the limit of $\chi_E(x,\cdot)$ as $x \to x_0^-$. Moreover, since the sections of E in the vertical direction are real intervals centered at zero, $u = \chi_T$, for some symmetric interval $T = [-y^-, y^-]$. □

Definition 4.3.2 (Traces of Schwarz symmetric sets)**.** Let $E \subset \mathbb{R}^2$ be a y-Schwarz symmetric set and let $x_0 \in \mathbb{R}$. The interval T^- (resp. T^+) defined in Lemma 4.3.1 is called the *left (resp. right) trace of E at x_0 in the x-direction* and it is denoted by $\mathrm{tr}^x_{x_0-}$ (reap $\mathrm{tr}^x_{x_0+}$). If $\mathrm{tr}^x_{x_0-}E$ and $\mathrm{tr}^x_{x_0+}E$ are the same interval, we call it the *trace of E at x_0 in the x-direction* and we denote it by $\mathrm{tr}^x_{x_0}E$. In this case we say that the set E *has trace at x_0 in the x-direction.*

Remark 4.3.3. In the same way we can define *left and right traces at $y_0 > 0$ in the y-direction* for x-Schwarz symmetric sets through the formula $\|\chi_E(\cdot,y_2) - \chi_E(\cdot,y_1)\|_{L^1(\mathbb{R})} \le \varepsilon$ for $\hat{y} - \delta < y_1 < y_2 < \hat{y}$.

Remark 4.3.4. If $E = \{(x,y) \in \mathbb{R}^2 : |y| < f(|x|)\}$ is a x-symmetric and y-Schwarz symmetric set, then for any $x_0 > 0$, $\mathrm{tr}^x_{x_0\pm}E = [-y_0^\pm, y_0^\pm]$ with

$$\lim_{x \to x_0^\pm} f(x) = y_0^\pm.$$

In fact, by definition of left and right traces

$$\begin{aligned} 0 &= \lim_{x \to x_0^\pm} \int_{\mathbb{R}} |\chi_E(x,y) - \chi_{[-y_0^\pm, y_0^\pm]}(y)|\, dy \\ &= \lim_{x \to x_0^\pm} \mathcal{L}^1((E)_x \bigtriangleup [-y_0^\pm, y_0^\pm]) = 2 \lim_{x \to x_0^\pm} |f(x) - y_0^\pm|. \end{aligned}$$

For any given $v_1, v_2, h_1, h_2 \ge 0$, we define the class $\mathcal{A}_x = \mathcal{A}_x(v_1,v_2,h_1,h_2)$ of all Lebesgue measurable sets $E \subset \mathbb{R}^2$ that are x-symmetric, y-Schwarz symmetric and such that there exists $x_0 = x_0(v_1,v_2,h_1,h_2) \ge 0$ satisfying

$$\begin{aligned} &\mathcal{L}^2(E^x_{x_0-}) = v_1, \quad \mathcal{L}^2(E \setminus E^x_{x_0-}) = v_2, \\ &[-h_1,h_1] \subset \mathrm{tr}^x_{x_0-}E, \quad [-h_2,h_2] \subset \mathrm{tr}^x_{x_0+}E. \end{aligned} \tag{4.3.1}$$

We define the following functional on the class $\mathcal{A}_x$:

$$\mathscr{F}_\alpha(E) = P_\alpha(E^x_{x_0-}) + P_\alpha(E^x_{x_0+}) - 4h_1 - 4h_2, \quad E \subset \mathcal{A}_x. \tag{4.3.2}$$

Remark 4.3.5. The functional $\mathscr{F}_\alpha$ is non-negative on the sets of the class $\mathcal{A}_x$. In fact, using the Representation formula for the α-perimeter of smooth sets, combined with the approximation result in Theorem 1.2.4, we get

$$\begin{aligned}\mathscr{F}_\alpha(E) = P_\alpha(E; E^x_{x_0-}) + P_\alpha(E^x_{x_0-}; \{|x| = x_0\}) - 4h_1 \\ + P_\alpha(E; E \setminus E^x_{x_0-}) + P_\alpha(E \setminus E^x_{x_0-}; \{|x| = x_0\}) - 4h_2\end{aligned}$$

which is non-negative thanks to the trace assumption.

Moreover, if $h_1 = h_2 = h$, for any set $E \in \mathcal{A}_x$ such that $\mathrm{tr}^x_{x_0-}E = \mathrm{tr}^x_{x_0+}E = [-h, h]$, we have

$$\mathscr{F}_\alpha(E) = P_\alpha(E; E^x_{x_0-}) + 4h + P_\alpha(E; E \setminus E^x_{x_0-}) + 4h - 8h = P_\alpha(E).$$

We consider the minimization problem for $\mathscr{F}_\alpha$ in the class $\mathcal{A}_x$, defining the constant

$$C_{IP} = \inf\{\mathscr{F}_\alpha(E) : E \in \mathcal{A}_x\}. \tag{4.3.3}$$

The constant C_{IP} is strictly positive thanks to the isoperimetric inequality in the Grushin plane (see Proposition 1.3.4).

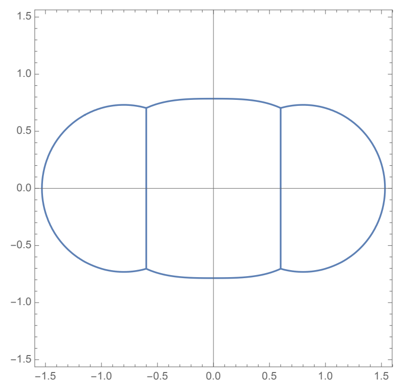


Figure 4.2: Example of a set in $\mathcal{A}_x(v_1, v_2, h_1, h_2)$, for $h_1 = h_2$ i.e., a x-symmetric and y-Schwarz symmetric set whose volume is partitioned into three parts that touch each other in sections of prescribed length.

Section 4.3.1 is dedicated to the proof of existence of bounded minimizers for Problem (4.3.3), satisfying suitable convexity properties. In Section 4.3.2 we deduce differential equations for the profile function of such minimizers and we use them to prove some elementary properties of minimizers, see Remark 4.3.8.

In Section 4.3.3 we are concerned with trace properties of the minimizers for (4.3.3). What we expect is that given $v_1, v_2, h_1, h_2 \geq 0$, a minimizer $E \subset \mathcal{A}_x(v_1, v_2, h_1, h_2)$ for problem (4.3.3) satisfies $\mathrm{tr}^x_{x_0-}E = [-h_1, h_1]$ and $\mathrm{tr}^x_{x_0+}E = [-h_2, h_2]$ where $x_0 > 0$ is defined by (4.3.1). So far we are able to prove only part of the claim. In Proposition 4.3.9 we prove that if the profile function of E has finite derivative at x_0, then $\mathrm{tr}^x_{x_0^-} E = [-h_1, h_1]$. The technique that we use here does not work in the case of infinite derivative, see Remark 4.3.10.

In Section 4.3.4 we focus on the following question, motivated by Remark 4.2.8: are minimizers obtained as dilations of E^α_{isop} in their central part? In Proposition 4.3.11 we prove that this property fails to hold when $v_2 = 0$ and $h_1 > 0$.

4.3.1. Existence of solutions to the partitioning problem

Theorem 4.3.6. *Let $v_1, v_2, h_1, h_2 \geq 0$. Then there exists a bounded set $E \in \mathcal{A}_x$ realizing the infimum in* (4.3.3) *and such that, for $x_0 \geq 0$ as in* (4.3.1)*, $E^x_{x_0-}$, $E \cap \{x > x_0\}$, $E \cap \{x < -x_0\}$ are convex sets.*

Proof. Let $(E_m)_{m\in\mathbb{N}}$ be a minimizing sequence for the infimum in (4.3.3), namely

$$E_m \in \mathcal{A}_x \quad \mathscr{F}_\alpha(E_m) \leq C_{IP}\Big(1 + \frac{1}{m}\Big) \quad m \in \mathbb{N}.$$

Since $E_m \in \mathcal{A}_x$, there exists $x_m > 0$ such that (4.3.1) is satisfied for $E = E_m$, $x_0 = x_m$. Moreover, by the symmetry properties of the sets in $\mathcal{A}_x$, there exists a measurable function f_m such that $E_m = \{(x, y) \in \mathbb{R}^2 : |y| < f_m(|x|)\}$. We define $y_m^-, y_m^+ \geq 0$ such that $\mathrm{tr}^x_{x_m\pm} E_m = [-y_m^\pm, y_m^\pm]$. By Remark 4.3.4,

$$\lim_{x\to x_m^+} f_m(x) = y_m^+, \quad \lim_{x\to x_m^-} f_m(x) = y_m^-, \quad \text{with} \quad y_m^+ \geq h_2,\ y_m^- \geq h_1. \tag{4.3.4}$$

Fix $m \in \mathbb{N}$ and let $E = E_m$, $x_0 = x_m$, $f = f_m$, $y_0^- = y_m^-$, $y_0^+ = y_m^+$. The proof is divided into several steps.

Step 1. (Approximation by smooth sets). We claim that there exists a sequence of x-symmetric and y-Schwarz symmetric sets $(\mathcal{E}_j)_{j\in\mathbb{N}}$ such that

1. $\partial\mathcal{E}_j$ is a locally C^∞ curve, i.e., for any $(x, y) \in \partial\mathcal{E}_j$ there exists $r > 0$ such that $\partial\mathcal{E}_j \cap B((x, y), r)$ is a C^∞ curve;
2. $\lim\limits_{j\to\infty} P_\alpha((\mathcal{E}_j)^x_{x_0-}) = P_\alpha(E^x_{x_0-})$ and $\lim_{j\to\infty} P_\alpha(\mathcal{E}_j \smallsetminus (\mathcal{E}_j)^x_{x_0-}) = P_\alpha(E \smallsetminus E^x_{x_0-})$;
3. $\lim\limits_{j\to\infty} \mathcal{L}^2((\mathcal{E}_j)^x_{x_0-}) = \mathcal{L}^2(E^x_{x_0-})$ and $\lim\limits_{j\to\infty} \mathcal{L}^2(\mathcal{E}_j \smallsetminus (\mathcal{E}_j)^x_{x_0-}) = \mathcal{L}^2(E \smallsetminus E^x_{x_0-})$;
4. if $\mathrm{tr}^x_{x_0\pm}\mathcal{E}_j = [-q_j^\pm, q_j^\pm]$, for some $q_j^\pm \geq 0$, we have $q_j^\pm \to y_0^\pm$ as $j \to \infty$.

To construct the sequence $(\mathcal{E}_j)_{j\in\mathbb{N}}$ we introduce a positive symmetric mollifier $J \in C^\infty(\mathbb{R}^2)$, i.e., $J \in C_c^\infty(B_E(0, 1))$, $J \geq 0$, $\int_{\mathbb{R}^2} J(p)\, dp = 1$, and $J(p) = J(q)$, for $p, q \in \mathbb{R}^2$, $|p| = |q|$. For any $\varepsilon > 0$, let $J_\varepsilon(p) = \frac{1}{\varepsilon^2} J(|p|/\varepsilon)$, $p \in \mathbb{R}^2$ and define the mollified function $h_\varepsilon = J_\varepsilon * \chi_E$. For any $t \in (0, 1)$, let $E_{\varepsilon t} = \{(x, y) \in \mathbb{R}^2 : h_\varepsilon(x, y) > t\}$. Consider a sequence $\varepsilon_j \to 0$ as $j \to \infty$.

Then, following [66, Theorem 1.24], we can choose $t \in (0,1)$ such that the set $\mathcal{E}_j = E_{\varepsilon_j t}$ satisfies 1, 3, and, in addition

$$\lim_{j\to\infty} P_\alpha(\mathcal{E}_j; (\mathcal{E}_j)^x_{x_0-}) = P_\alpha(E; E^x_{x_0-}), \quad \lim_{j\to\infty} P_\alpha(\mathcal{E}_j; \mathcal{E}_j \smallsetminus (\mathcal{E}_j)^x_{x_0-}\}) = P_\alpha(E; E \smallsetminus E^x_{x_0-}). \tag{4.3.5}$$

Observe that the sets $\mathcal{E}_j$, $j \in \mathbb{N}$ are y-Schwarz symmetric, i.e., for any $(\bar{x},\bar{y}) \in \mathcal{E}_j$, $(\bar{x},y) \in \mathcal{E}_j$ if $|y| < \bar{y}$. In fact, since E is y-Schwarz symmetric, $\chi_E(\bar{x}-x',\bar{y}-y') \le \chi_E(\bar{x}-x',y-y')$ for every $(x',y') \in \mathbb{R}^2$ and $|y| < |\bar{y}|$. Hence

$$\begin{aligned} t < & h_{\varepsilon_j}(\bar{x},\bar{y}) = \int_{B_{\varepsilon_j}(0)} J_\varepsilon(x',y')\chi_E(\bar{x}-x',\bar{y}-y')\,dx'dy' \\ & \le \int_{B_{\varepsilon_j}(0)} J_\varepsilon(x',y')\chi_E(\bar{x}-x',y-y')\,dx'dy' = h_{\varepsilon_j}(\bar{x},y), \end{aligned}$$

which implies $(\bar{x},y) \in \mathcal{E}_j$. Moreover by symmetry of the mollifier J, for every $j \in \mathbb{N}$, $\mathcal{E}_j$ is also x-symmetric. Hence the left and right traces of $\mathcal{E}_j$ are well defined. Let ϕ_j denote the profile function of $\mathcal{E}_j$, i.e., $\mathcal{E}_j = \{(x,y) \in \mathbb{R}^2 : |y| < \phi_j(|x|)\}$, and define

$$q_j^- = \lim_{x\to x_0^-} \phi_j(x_0), \quad q_j^+ = \lim_{x\to x_0^+} \phi_j(x_0).$$

By Remark 4.3.4, $\mathrm{tr}^x_{x_0\pm}\mathcal{E}_j = [-q_j^\pm, q_j^\pm]$. We prove that $q_j^+ \to y_0^+$ as $j \to \infty$. The same reasoning applies to prove $q_j^- \to y_0^-$, $j \to \infty$, and claim 4 follows. Let $0 < \sigma < y_0^+$, by (4.3.4) there exists $\delta = \delta(\sigma) > 0$ such that

$$|f(x) - y_0^+| < \sigma \quad \text{for} \quad x_0 < x < x_0 + \delta. \tag{4.3.6}$$

Choose $\bar{j}(\sigma) \in \mathbb{N}$ to have $\varepsilon_j < \min\{\sigma,\ \delta(\sigma)/4\}$ for $j \ge \bar{j}(\sigma)$. We first claim that for any $j \ge \bar{j}(\sigma)$, if $x \in (x_0 + \varepsilon_j, x_0 + \frac{\delta}{2})$ and $y \in (0, y_0^+ - 2\sigma)$, then

$$(x-\xi, y-\eta) \in E, \quad \text{for} \quad (\xi,\eta) \in B(0,\varepsilon_j). \tag{4.3.7}$$

In fact, the following estimates holds true for $j \ge \bar{j}(\sigma)$, $x \in (x_0 + \varepsilon_j, x_0 + \frac{\delta}{2})$, $-\varepsilon_j < \xi < \varepsilon_j$:

$$x_0 < x - \xi < x_0 + \frac{\delta}{2} - \xi < x_0 + \frac{\delta}{2} + \varepsilon_j < x_0 + \delta$$

hence, by (4.3.6), for $y \in (0, y_0^+ - 2\sigma)$ and $-\varepsilon_j < \eta < \varepsilon_j$,

$$y - \eta < y + \varepsilon_j < y_0^+ - 2\sigma + \sigma < f(x-\xi).$$

We now deduce from (4.3.7) that

$$A_\sigma = \big(x_0 + \varepsilon_j, x_0 + \frac{\delta}{2}\big) \times (0, y_0 - 2\sigma) \subset \mathcal{E}_j, \quad \text{for } j \ge j(\sigma). \tag{4.3.8}$$

This follows applying the definition of the set $\mathcal{E}_j$, since, for any $j \ge \bar{j}(\sigma)$, if $(x,y) \in A_\sigma$ we have

$$h_{\varepsilon_j}(x,y) = \int_{B(0,\varepsilon_j)} J_{\varepsilon_j}(\xi,\eta)\chi_E(x-\xi,y-\eta)\,d\xi d\eta = \int_{B(0,\varepsilon_j)} J_{\varepsilon_j}(\xi,\eta)\,d\xi d\eta = 1 > t.$$

In particular, (4.3.8) implies

$$(-y_0^+ + 2\sigma, y_0^+ - 2\sigma) \subset \mathrm{tr}^x_{(x_0+\varepsilon_j)+}\mathcal{E}_j \text{ for every } j > \bar{j}(\sigma). \tag{4.3.9}$$

Similarly, we can choose $\bar{\bar{j}}(\sigma) \in \mathbb{N}$ such that

$$\mathrm{tr}_{(x_0+\varepsilon_j)+}\mathcal{E}_j \subset (-y_0^+ - 2\sigma, y_0 + 2\sigma) \text{ for } j \geq \bar{\bar{j}}(\sigma). \tag{4.3.10}$$

We deduce claim 4 from (4.3.9) and (4.3.10). Claim 2 follows from claim 4 and (4.3.5).

Step 2. (Gluing around the y-axis). We claim that there exist x-symmetric and y-Schwarz symmetric sets $\hat{\mathcal{E}}_j$, $j \in \mathbb{N}$, such that for every $j \in \mathbb{N}$, there exist $0 < \hat{x}_j \leq x_0$ satisfying:

1. the euclidean outer unit normal to $\hat{\mathcal{E}}_j$ exists outside a set of $\mathcal{H}^1$-measure zero;
2. if $\hat{\phi}_j : [0,\infty) \to [0,\infty)$ denotes the profile function of $\hat{\mathcal{E}}_j$ and $D_j = \inf\{\bar{x} \geq 0 : \phi_j(x) = 0 \text{ for } x \geq \bar{x}\}$, then $\mathcal{H}^1(\{x \in [0, D_j] : \hat{\phi}_j(x) = 0\}) = 0$;
3. $P_\alpha((\hat{\mathcal{E}}_j)^x_{\hat{x}_j-}) \leq P_\alpha((\mathcal{E}_j)^x_{x_0-})$ and $P_\alpha(\hat{\mathcal{E}}_j \setminus (\hat{\mathcal{E}}_j)^x_{\hat{x}_j-}) \leq P_\alpha(\mathcal{E}_j \setminus (\mathcal{E}_j)^x_{x_0-})$;
4. $\mathcal{L}^2((\hat{\mathcal{E}}_j)^x_{\hat{x}_j-}) = \mathcal{L}^2((\mathcal{E}_j)^x_{x_0-})$ and $\mathcal{L}^2(\hat{\mathcal{E}}_j \setminus (\hat{\mathcal{E}}_j)^x_{\hat{x}_j-}) = \mathcal{L}^2(\mathcal{E}_j \setminus (\mathcal{E}_j)^x_{x_0-})$;
5. $\mathrm{tr}^x_{x_0-}\hat{\mathcal{E}}_j = \mathrm{tr}^x_{x_0-}\mathcal{E}_j$ and $\mathrm{tr}^x_{x_0+}\hat{\mathcal{E}}_j = \mathrm{tr}^x_{x_0+}\mathcal{E}_j$.

For any $j \in \mathbb{N}$, define the set $Z_j := \{x \in \mathbb{R} : \phi_j(x) = 0\}$ and write $Z_j = Z_j^1 \cup Z_j^2$ with

$$Z_j^1 = \{x \in [0, D_j] : \phi_j(x) = 0 \text{ and } \phi_j(\xi) \neq 0 \text{ for } \xi \in (x-\delta, x+\delta) \setminus \{x\} \text{ for some } \delta > 0\},$$
$$Z_j^2 = \{x \in [0, D_j] : \exists \delta > 0 : \phi_j(\xi) = 0 \text{ for } \xi \in (x-\delta, x] \text{ or } \xi \in [x, x+\delta)\}.$$

By symmetry and smoothness of $\mathcal{E}_j$, we have $Z_j^1 = \varnothing$. In fact, suppose by contradiction that there exists $x \in Z_j^1$, and let $p = (x, 0) \in \partial\mathcal{E}_j$. Since $\partial\mathcal{E}_j$ is smooth, there exists the outer unit normal ν at p. By y-symmetry of E, $\nu = (\pm 1, 0)$. Moreover, there exists a smooth function $\theta : B_r(p) \to \mathbb{R}$, defined on a euclidean ball $B_r(p) = \{q \in \mathbb{R}^2 : |q - p| < r\}$, for some radius $r > 0$ such that

$$\begin{aligned}
\theta(q) = 0 &\iff q \in \partial\mathcal{E}_j \cap B_r(p),\\
\theta(q) > 0 &\iff q \in \mathcal{E}_j \cap B_r(p),\\
\theta(q) < 0 &\iff q \in (\mathbb{R}^2 \setminus \overline{\mathcal{E}_j}) \cap B_r(p).
\end{aligned}$$

We deduce that $p + \tau\nu \notin \mathcal{E}_j$ for $0 < \tau < \min\{r, \delta\}$, which contradicts $\phi_j(x \pm \tau) \neq 0$.

On the other hand, the set Z_j^2 is the complement in $\mathbb{R}$ of the set $\{x \in \mathbb{R} : (x, 0) \in \overline{\mathcal{E}_j}\}$, therefore it is open in the $\mathbb{R}$-topology. Hence, Z_j^2 is the union of at most countably many open intervals. We diversify the notation for the intervals in $Z_j^2 \cap \{x \in \mathbb{R} : |x| < x_0\}$ and in

$Z_j^2 \cap \{x \in \mathbb{R} : |x| > x_0\}$: there exists a sequence of points $0 \le a_j^1 < b_j^1 < a_j^2 < b_j^2 < \cdots \le x_0 < c_j^1 < d_j^1 < c_j^2 < d_j^2 < \cdots \le D_j$, such that

$$Z_j^2 = \bigcup_{k\in\mathfrak{I}}(a_j^k, b_j^k) \cup \bigcup_{k\in\mathfrak{J}}(c_j^k, d_j^k) \cup \bigcup_{k\in\mathfrak{I}}(-b_j^k, -a_j^k) \cup \bigcup_{k\in\mathfrak{J}}(-d_j^k, -c_j^k),$$

where $\mathfrak{I}, \mathfrak{J} \subset \mathbb{N}$. We rearrange $\mathcal{E}_j$ in at most countably many steps, each one corresponding to an interval (a_j^k, b_j^k) for $k \in \mathfrak{I}$.

Base step. We define the set

$$\begin{aligned}\mathcal{E}_j^1 = (\mathcal{E}_j)^x_{a_j^1-} &\cup \{(x + a_j^1 - b_j^1, y) : (x,y) \in \mathcal{E}_j,\ x > b_j^1\} \\ &\cup \{(x + b_j^1 - a_j^1, y) : (x,y) \in \mathcal{E}_j,\ x < -b_j^1\}\end{aligned}$$

which is x-symmetric and y-Schwarz symmetric. Let $x_j^1 = x_0 + a_j^1 - b_j^1 < x_0$. Since $\mathcal{E}_j \cap \big((a_j^1, b_j^1) \times \mathbb{R}\big) = \varnothing$, we have

$$\mathcal{L}^2((\mathcal{E}_j^1)^x_{x_j^1-}) = \mathcal{L}^2((\mathcal{E}_j)^x_{x_0-}) \quad \text{and} \quad \mathcal{L}^2(\mathcal{E}_j^1 \smallsetminus (\mathcal{E}_j^1)^x_{x_j^1-}) = \mathcal{L}^2(\mathcal{E}_j \smallsetminus (\mathcal{E}_j)^x_{x_0-}).$$

Moreover $\mathrm{tr}^x_{x_j^1\pm}\mathcal{E}_j^1 = \mathrm{tr}^x_{x_0\pm}\mathcal{E}_j$. We prove that $P_\alpha((\mathcal{E}_j^1)^x_{x_j^1-}) \le P_\alpha((\mathcal{E}_j)^x_{x_0-})$. Since $\partial\mathcal{E}_j^1$ is locally smooth outside the set $\{(x,y) \in \mathbb{R}^2 :\ |x| = a_j^1\}$, let $N_j^1(p) = (N_{jx}^1(p), N_{jy}^1(p))$ be euclidean outer unit normal to $\mathcal{E}_j^1$ at $p = (x,y) \in \mathbb{R}^2$, for $|x| \ne a_j^1$. If N_j is the euclidean outer unit normal to $\partial\mathcal{E}_j$, for $(x,y) \in \partial\mathcal{E}_j$, we have

$$N_j^1(x - b_j^1 + a_j^1, y) = N_j(x,y) \quad \text{if} \quad x > b_j^1; \quad N_j^1(x + b_j^1 - a_j^1, y) = N_j(x,y) \quad \text{if} \quad x < -b_j^1.$$

Hence, by the representation formula in Proposition 2.2.1 we get

$$\begin{aligned}&P_\alpha((\mathcal{E}_j^1)^x_{x_j^1-}) \le P_\alpha((\mathcal{E}_j^1)^x_{a_j^1-}) + P_\alpha((\mathcal{E}_j^1)^x_{x_j^1-} \smallsetminus (\mathcal{E}_j^1)^x_{a_j^1-}) \\ &\le \int_{\partial(\mathcal{E}_j^1)^x_{a_j^1-}} \sqrt{N_{jx}^1(x,y)^2 + |x|^{2\alpha} N_{jy}^1(x,y)^2}\, d\mathcal{H}^1(x,y) \\ &+ \int_{\partial\left((\mathcal{E}_j^1)^x_{x_j^1-} \smallsetminus (\mathcal{E}_j^1)^x_{a_j^1-}\right)} \sqrt{(N_{jx}^1(x - b_j^1 + a_j^1, y))^2 + |x - b_j^1 + a_j^1|^{2\alpha}(N_{jy}^1(x - b_j^1 + a_j^1, y))^2}\, d\mathcal{H}^1 \\ &\le \int_{\partial(\mathcal{E}_j)^x_{a_j-}} ({N_{jx}^1}^2 + |x|^{2\alpha}{N_{jy}^1}^2)^{\frac12}\, d\mathcal{H}^1 + \int_{\partial\left((\mathcal{E}_j)^x_{x_0-} \smallsetminus (\mathcal{E}_j)^x_{b_j-}\right)} (N_{jx}^2 + |x|^{2\alpha} N_{jy}^2)^{\frac12}\, d\mathcal{H}^1(x,y) \\ &= P_\alpha(\mathcal{E}_j).\end{aligned}$$

In the same way follows

$$P_\alpha(\mathcal{E}_j^1 \smallsetminus (\mathcal{E}_j^1)^x_{x_j^1-}) \le P_\alpha(\mathcal{E}_j \smallsetminus (\mathcal{E}_j)^x_{x_0-}).$$

Second step. Let $\mathcal{E}_j^2$ be the x-symmetric and y-Schwarz symmetric set

$$\begin{aligned}\mathcal{E}_j^2 = &\{(x,y) \in \mathcal{E}_j^1 : |x| \le a_j^2 - (b_j^1 - a_j^1)\} \\ &\cup \Big\{\Big(x - \sum_{i=1}^2 (b_j^i - a_j^i), y\Big) : (x,y) \in \mathcal{E}_j^1,\ x > b_j^2 - (b_j^1 - a_j^1)\Big\} \\ &\cup \Big\{\Big(x + \sum_{i=1}^2 (b_j^i - a_j^i), y\Big) : (x,y) \in \mathcal{E}_j^1,\ x < -b_j^2 + (b_j^1 - a_j^1)\Big\},\end{aligned}$$

and $x_j^2 = x_0 - \sum_{i=1}^2 (b_j^i - a_j^1)$. Then,

$$\mathcal{L}^2((\mathcal{E}_j^2)^x_{x_j^2-}) = \mathcal{L}^2((\mathcal{E}_j)^x_{x_0-}) \quad \text{and} \quad \mathcal{L}^2(\mathcal{E}_j^1 \setminus (\mathcal{E}_j^2)^x_{x_j^2-}) = \mathcal{L}^2(\mathcal{E}_j \setminus (\mathcal{E}_j)^x_{x_0-}),$$

and $\mathrm{tr}^x_{x_j^2\pm}\mathcal{E}_j^2 = \mathrm{tr}^x_{x_j^1\pm}\mathcal{E}_j^1 = \mathrm{tr}^x_{x_0\pm}\mathcal{E}_j$. Moreover, as in the previous step, $\partial\mathcal{E}_j^2$ is locally smooth outside the set $\{(x,y) \in \mathbb{R}^2 : \ |x| = a_j^1, \ |x| = a_j^2 - (b_j^1 - a_j^1)\}$, hence, $P_\alpha((\mathcal{E}_j^2)^x_{x_j^2-}) \le P_\alpha((\mathcal{E}_j)^x_{x_0-})$ and $P_\alpha(\mathcal{E}_j^2 \setminus (\mathcal{E}_j^2)^x_{x_j^2-}) \le P_\alpha(\mathcal{E}_j \setminus (\mathcal{E}_j)^x_{x_0-})$.

Inductive step. Let $\mathcal{E}_j^k$ be the x-symmetric and y-Schwarz symmetric set

$$\begin{aligned}\mathcal{E}_j^k = &\Big\{(x,y) \in \mathcal{E}_j^{k-1} : |x| \le a_j^k - \sum_{i=1}^{k-1}(b_j^i - a_j^i)\Big\} \\ &\cup \Big\{\Big(x - \sum_{i=1}^{k}(b_j^i - a_j^i), y\Big) : (x,y) \in \mathcal{E}_j^{k-1}, \ x > b_j^k - \sum_{i=1}^{k-1}(b_j^i - a_j^i)\Big\} \\ &\cup \Big\{\Big(x + \sum_{i=1}^{k}(b_j^i - a_j^i), y\Big) : (x,y) \in \mathcal{E}_j^{k-1}, \ x < -b_j^k + \sum_{i=1}^{k-1}(b_j^i - a_j^i)\Big\},\end{aligned}$$

and define

$$x_j^k = x_0 - \sum_{i=1}^{k}(b_j^i - a_j^i) < x_0.$$

Then

$$\mathcal{L}^2((\mathcal{E}_j^k)^x_{x_j^k-}) = \mathcal{L}^2((\mathcal{E}_j)^x_{x_0-}) \quad \text{and} \quad \mathcal{L}^2(\mathcal{E}_j^1 \setminus (\mathcal{E}_j^k)^x_{x_j^k-}) = \mathcal{L}^2(\mathcal{E}_j \setminus (\mathcal{E}_j)^x_{x_0-}),$$

$P_\alpha((\mathcal{E}_j^k)^x_{x_j^k-}) \le P_\alpha((\mathcal{E}_j)^x_{x_0-})$, $P_\alpha(\mathcal{E}_j^k \setminus (\mathcal{E}_j^k)^x_{x_j^k-}) \le P_\alpha(\mathcal{E}_j \setminus (\mathcal{E}_j)^x_{x_0-})$, and $\partial\mathcal{E}_j^k$ is locally smooth outside $\{(x,y) \in \mathbb{R}^2 : |x| = a_j^1 - \sum_{i=2}^{\ell}(b_j^i - a_j^i) \text{ for } \ell = 2, \dots, k\}$.

Iterating this procedure at most countably many times, we obtain a x-symmetric and y-Schwarz symmetric set $\hat{\mathcal{E}}_j$ satisfying claims 3, 4 an 5 for

$$\hat{x}_j = x_0 - \sum_{i \in \mathcal{I}}(b_j^i - a_j^i). \tag{4.3.11}$$

Repeating this argument for the intervals (c_j^k, d_j^k), $k \in \mathfrak{J}$, we obtain a set, which we still call $\hat{\mathcal{E}}_j$, that satisfies also claims 1 and 2. In fact, let

$$\hat{Z}_j = \Big\{a_j^k - \sum_{i=1}^{k-1}(b_j^i - a_j^i) : k \in \mathfrak{I}\Big\} \cup \{c_j^k - \sum_{i=1}^{k-1}(d_j^i - c_j^i) : k \in \mathfrak{J}\}$$

which is at most countable, and denote by $\hat{\phi}_j$ be the profile function of $\hat{\mathcal{E}}_j$. Then the outer unit normal to $\hat{\mathcal{E}}_j$ exists outside the set $\{(x,y) \in \mathbb{R}^2 : \ |x| \in \hat{Z}_j\}$, and $\{x \in \mathbb{R} : \hat{\phi}_j(x) = 0\} \subset \hat{Z}_j$.

Step 3. (Reflection in the vertical direction) For any $j \in \mathbb{N}$, we rearrange the set $\hat{\mathcal{E}}_j$ into a x-symmetric and y-Schwarz symmetric set $\hat{\hat{\mathcal{E}}}_j$ with profile function $\hat{\hat{\phi}}_j : [0,\infty) \to [0,\infty)$ such that

1. The euclidean outer unit normal to $\hat{\hat{\mathcal{E}}}_j$ exists outside a set of $\mathcal{H}^1$-measure zero;

2. $\hat{\hat{\phi}}_j(|x|) \geq q_j$ for $x \in \mathbb{R}$, $|x| < \hat{x}_j$;

3. $P_\alpha((\hat{\hat{\mathcal{E}}}_j)^x_{\hat{x}_j -}) \leq P_\alpha((\hat{\mathcal{E}}_j)^x_{\hat{x}_j -})$ and $P_\alpha(\hat{\hat{\mathcal{E}}}_j \smallsetminus (\hat{\hat{\mathcal{E}}}_j)^x_{\hat{x}_j -}) = P_\alpha(\hat{\mathcal{E}}_j \smallsetminus (\hat{\mathcal{E}}_j)^x_{\hat{x}_j -})$;

4. $\mathcal{L}^2((\hat{\hat{\mathcal{E}}}_j)^x_{\hat{x}_j -}) \geq \mathcal{L}^2((\hat{\mathcal{E}}_j)^x_{\hat{x}_j -})$ and $\mathcal{L}^2(\hat{\hat{\mathcal{E}}}_j \smallsetminus (\hat{\hat{\mathcal{E}}}_j)^x_{\hat{x}_j -}) = \mathcal{L}^2(\hat{\mathcal{E}}_j \smallsetminus (\hat{\mathcal{E}}_j)^x_{\hat{x}_j -})$;

5. $\mathrm{tr}^x_{\hat{x}_j -}\hat{\hat{\mathcal{E}}}_j = \mathrm{tr}^x_{\hat{x}_j -}\hat{\mathcal{E}}_j$ and $\mathrm{tr}^x_{\hat{x}_j +}\hat{\hat{\mathcal{E}}}_j = \mathrm{tr}^x_{\hat{x}_j +}\hat{\mathcal{E}}_j$.

We define the rearranged function $\hat{\hat{\phi}}_j : [0,\infty) \to [0,\infty)$,

$$\hat{\hat{\phi}}_j(x) = \begin{cases} |\hat{\phi}_j(x) - q_j^-| + q_j^- = \begin{cases} \hat{\phi}_j(x) & \text{if } \hat{\phi}_j(x) \geq q_j^- \\ 2q_j^- - \hat{\phi}_j(x) & \text{if } \hat{\phi}_j(x) < q_j^- \end{cases} & \text{if } |x| < \hat{x}_j, \\ \hat{\phi}_j(x) & \text{if } |x| > \hat{x}_j. \end{cases}$$

Let $\hat{\hat{\mathcal{E}}}_j$ be the x- and y-symmetric set generated by $\hat{\hat{\phi}}_j$ (see Figure 4.3). Clearly $\hat{\hat{\mathcal{E}}}_j \smallsetminus (\hat{\hat{\mathcal{E}}}_j)^x_{\hat{x}_j -} = \hat{\mathcal{E}}_j \smallsetminus (\hat{\mathcal{E}}_j)^x_{\hat{x}_j -}$ and claims 2 and 5 are satisfied. Claim 4 follows, observing that $\hat{\hat{\phi}} \in L^1(\mathbb{R})$ and $\hat{\hat{\phi}}_j \geq \hat{\phi}_j$, thus $\mathcal{L}^2(\hat{\hat{\mathcal{E}}}_j) \geq \mathcal{L}^2(\hat{\mathcal{E}}_j)$.

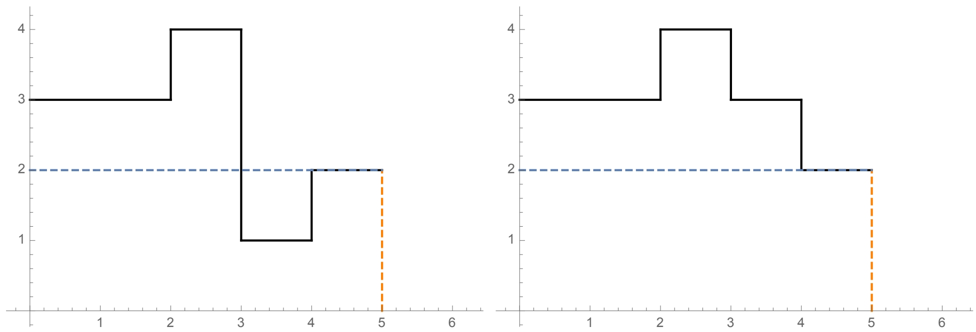


Figure 4.3: The set $\hat{\mathcal{E}}_j$ and the rearranged $\hat{\hat{\mathcal{E}}}_j$.

Denote by $\mathcal{K}_j$, the set of points $(x,y) \in \partial\hat{\mathcal{E}}_j$ with $\hat{\phi}_j(x) = q_j^-$ and for which there exists $\delta > 0$ such that $\hat{\phi}(\xi) \neq q_j^-$ for $\xi \in (x, x+\delta)$ or $\xi \in (x-\delta, x)$. Then, $\mathcal{H}^1(\mathcal{K}_j) = 0$ for any $j \in \mathbb{N}$ since $\mathcal{K}_j$ is countable.

By construction, the outer unit normal to $\hat{\hat{\mathcal{E}}}_j$ exists outside $\mathcal{K}_j \cup \hat{Z}_j$, which has $\mathcal{H}^1$-measure zero, and we denote it by $\hat{\hat{N}}_j = (\hat{\hat{N}}_{jx}, \hat{\hat{N}}_{jy})$. Moreover, if $\hat{N}_j = (\hat{N}_{jx}, \hat{N}_{jy})$ is the outer unit normal to $\partial\hat{\mathcal{E}}_j$, we have for any $(x,y) \in \partial\hat{\mathcal{E}}_j \smallsetminus \big(\mathcal{K}_j \cup \hat{Z}_j\big)$, $|x| < \hat{x}_j$,

$$\hat{\hat{N}}_j(x, |y - q_j| + q_j) = \big(\hat{N}_{jx}(x,y), \mathrm{sgn}(y - q_j)\hat{N}_{jy}(x,y)\big),$$

hence

$$\begin{aligned}
P_\alpha\big((\hat{\hat{\mathcal{E}}}_j)^x_{\hat{x}_j-}\big) &= \int_{\partial_j(\hat{\hat{\mathcal{E}}}_j)^x_{\hat{x}_j-}} \sqrt{\hat{\hat{N}}^2_{jx} + |x|^{2\alpha}\hat{\hat{N}}^2_{jy}}\, d\mathcal{H}^1 \\
&= \int_{\{p\in\partial(\hat{\mathcal{E}}_j)^x_{\tilde{x}_j-}:\ \hat{N}_{jy}(p)\neq 0\}} \sqrt{\hat{N}^2_{jx} + |x|^{2\alpha}\hat{N}^2_{jy}}\, d\mathcal{H}^1 + \mathcal{H}^1\big(\{p\in\partial(\hat{\hat{\mathcal{E}}}_j)^x_{\hat{x}_j-}:\ \hat{\hat{N}}_{jy}(p)=0\}\big) \\
&\le \int_{\{p\in\partial(\hat{\mathcal{E}}_j)^x_{\tilde{x}_j-}:\ \hat{N}_{jy}(p)\neq 0\}} \sqrt{\hat{N}^2_{jx} + |x|^{2\alpha}\hat{N}^2_{jy}}\, d\mathcal{H}^1 + \mathcal{H}^1\big(\{p\in\partial(\hat{\mathcal{E}}_j)^x_{\hat{x}_j-}:\ \hat{N}_{jy}(p)=0\}\big) \\
&= P_\alpha(\hat{\mathcal{E}}_j).
\end{aligned}$$

Step 4. (Convexification) For any $j\in\mathbb{N}$, we rearrange the set $\hat{\hat{\mathcal{E}}}_j$ into a x-symmetric and y-Schwarz symmetric set $\tilde{\mathcal{E}}_j$, such that there exists $0<\tilde{x}_j<x_0$ satisfying:

1. the sets $(\tilde{\mathcal{E}}_j)^x_{\tilde{x}_j-}$, $\tilde{\mathcal{E}}_j\cap\{(x,y)\in\mathbb{R}^2 : x<-\tilde{x}_j\}$ and $\tilde{\mathcal{E}}_j\cap\{(x,y)\in\mathbb{R}^2 : x>\tilde{x}_j\}$ are convex;
2. $P_\alpha((\tilde{\mathcal{E}}_j)^x_{\tilde{x}_j-}) \le P_\alpha((\mathcal{E}_j)^x_{x_0-})$ and $P_\alpha(\tilde{\mathcal{E}}_j\setminus(\tilde{\mathcal{E}}_j)^x_{\tilde{x}_j-}) \le P_\alpha(\mathcal{E}_j\setminus(\mathcal{E}_j)^x_{x_0-})$;
3. $\mathcal{L}^2((\tilde{\mathcal{E}}_j)^x_{\tilde{x}_j-}) = \mathcal{L}^2((\mathcal{E}_j)^x_{x_0-})$ and $\mathcal{L}^2(\tilde{\mathcal{E}}_j\setminus(\tilde{\mathcal{E}}_j)^x_{\tilde{x}_j-}) = \mathcal{L}^2(\mathcal{E}_j\setminus(\mathcal{E}_j)^x_{x_0-})$;
4. $\mathrm{tr}^x_{\tilde{x}_j-}\tilde{\mathcal{E}}_j = \mathrm{tr}^x_{\tilde{x}_j+}\tilde{\mathcal{E}}_j \supset [-q_j,q_j]$.

We introduce the function

$$\Psi:\mathbb{R}^2\to\mathbb{R}^2,\ \ \Psi(x,y)=\Big(\mathrm{sgn}(x)\frac{|x|^{\alpha+1}}{\alpha+1},y\Big),$$

which is a homeomorphism with inverse

$$\Phi:\mathbb{R}^2\to\mathbb{R}^2,\ \ \Phi(\xi,\eta)=\Big(\mathrm{sgn}(\xi)|(\alpha+1)\xi|^{\frac{1}{\alpha+1}},\eta\Big).$$

As shown in [99, Proposition 2.3], for any measurable set $F\subset\mathbb{R}^2$, we have

$$P_\alpha(F)=P(\Psi(F)) \quad\text{and}\quad \mathcal{L}^2(F)=\mu(\Psi(F)),$$

where P denotes the Euclidean perimeter and μ is a Borel measure on $\mathbb{R}^2$ defined on Borel sets as follows:

$$\mu(A)=\int_A |(\alpha+1)\xi|^{-\frac{\alpha}{\alpha+1}}\, d\xi d\eta, \quad A\subset\mathbb{R}^2 \text{ Borel}.$$

For any $j\in\mathbb{N}$, let $F^c_j=\Psi((\hat{\hat{\mathcal{E}}}_j)^x_{\hat{x}_j-})\subset\mathbb{R}^2$ and consider its convex envelope in $\mathbb{R}^2$, $\mathrm{co}(F^c_j)$. We show that the transformed set $\mathcal{F}^c_j=\Phi(\mathrm{co}(F^c_j))$ is convex itself (see [99, page 362]). First of all, notice that the maps Φ, Ψ preserve the symmetries, namely, since $\mathrm{co}(F^c_j)$ is x- and y-Schwarz symmetric, also $\mathcal{F}^c_j$ has such symmetries. We show that it is also a convex set. In fact, let $t\in(0,1)$, for any $q_1,q_2\in\Phi(\mathrm{co}(F^c_j))$ there exist $p_1=(\xi_1,\eta_1),p_2=(\xi_2,\eta_2)\in\mathrm{co}(F^c_j)$. Since $\mathrm{co}(F^c_j)$ is convex, we have

$$\begin{aligned}
&\Big(\mathrm{sgn}\big(t\xi_1+(1-t)\xi_2\big)(\alpha+1)|t\xi_1+(1-t)\xi_2|^{\frac{1}{\alpha+1}}, t\eta_1+(1-t)\eta_2\Big) \\
&\quad=\Phi(tp_1+(1-t)p_2)\in\Phi(\mathrm{co}(F^c_j))
\end{aligned} \tag{4.3.12}$$

On the other hand, by the concavity inequality

$$|t\xi_1 + (1-t)\xi_2|^{\frac{1}{\alpha+1}} \geq t|\xi_1|^{\frac{1}{\alpha+1}} + (1-t)|\xi_2|^{\frac{1}{\alpha+1}}, \quad t \in (0,1), \quad \xi_1, \xi_2 \geq 0$$

and by x- and y-Schwarz symmetry of $\Phi(\mathrm{co}(F_j^{\mathrm{c}}))$, we get from (4.3.12)

$$tq_1+(1-t)q_2 = \Big((\alpha+1)\big\{\mathrm{sgn}(t\xi_1)|t\xi_1|^{\frac{1}{\alpha+1}}+\mathrm{sgn}((1-t)\xi_2)|(1-t)\xi_2|^{\frac{1}{\alpha+1}}\big\}, t\eta_1+(1-t)\eta_2\Big) \in \Phi(\mathrm{co}(F_j^{\mathrm{c}})),$$

which proves that $\mathcal{F}_j^{\mathrm{c}}$ is convex. The set $\mathcal{F}_j^{\mathrm{c}}$ satisfies

$$\mathcal{L}^2(\mathcal{F}_j^{\mathrm{c}}) = \mu(\mathrm{co}(F_j^{\mathrm{c}})) \geq \mu(F_j^{\mathrm{c}}) = \mathcal{L}^2((\hat{\hat{\mathcal{E}}}_j)^x_{\hat{x}_j-}) \geq \mathcal{L}^2((\hat{\mathcal{E}}_j)^x_{\hat{x}_j-}) \geq \mathcal{L}^2((\mathcal{E}_j)^x_{x_0-}), \tag{4.3.13a}$$

$$P_\alpha(\mathcal{F}_j^{\mathrm{c}}) = P(\mathrm{co}(F_j^{\mathrm{c}})) \leq P(F_j^{\mathrm{c}}) = P_\alpha((\hat{\hat{\mathcal{E}}}_j)^x_{\hat{x}_j-}), \tag{4.3.13b}$$

$$\mathrm{tr}^x_{\hat{x}_j-}\mathcal{F}_j^{\mathrm{c}} = [-q_j^-, q_j^-]. \tag{4.3.13c}$$

By (4.3.13a), we define $\tilde{x}_j \in [0, \hat{x}_j]$ such that $\mathcal{L}^2\big((\mathcal{F}_j^{\mathrm{c}})^x_{\tilde{x}_j-}\big) = \mathcal{L}^2\big((\mathcal{E}_j)^x_{x_0-}\big)$. By x- and y-Schwarz symmetry of $\mathcal{F}_j^c$, its profile function is decreasing, hence

$$[-q_j^-, q_j^-] \subset \mathrm{tr}^x_{\tilde{x}_j-}\mathcal{F}_j^{\mathrm{c}}.$$

Define the set

$$\begin{aligned}\mathcal{F}_j = (\mathcal{F}_j^{\mathrm{c}})^x_{\tilde{x}_j-} &\cup \{(x-\hat{x}_j+\tilde{x}_j, y) \in \mathbb{R}^2 : (x,y) \in \hat{\hat{\mathcal{E}}}_j,\ x > \hat{x}_j\} \\ &\cup \{(x+\hat{x}_j-\tilde{x}_j, y) \in \mathbb{R}^2 : (x,y) \in \hat{\hat{\mathcal{E}}}_j,\ x < -\hat{x}_j\}.\end{aligned}$$

As in Step 1, we have $P_\alpha(\mathcal{F}_j \setminus (\mathcal{F}_j)^x_{\tilde{x}_j-}) \leq P_\alpha(\hat{\hat{\mathcal{E}}}_j \setminus (\hat{\hat{\mathcal{E}}}_j)^x_{\hat{x}_j-})$, $\mathcal{L}^2(\mathcal{F}_j \setminus (\mathcal{F}_j)^x_{\tilde{x}_j-}) = \mathcal{L}^2(\hat{\hat{\mathcal{E}}}_j \setminus (\hat{\hat{\mathcal{E}}}_j)^x_{\hat{x}_j-})$ and $\mathrm{tr}^x_{\tilde{x}_j+}\mathcal{F}_j = \mathrm{tr}^x_{\hat{x}_j+}\hat{\hat{\mathcal{E}}}_j$.

Let $\psi_j : [0,\infty) \to [0,\infty)$ be the profile function of $\mathcal{F}_j$. The same argument used to prove (4.3.13a)-(4.3.13c), shows that the sets

$$\mathcal{F}_j^r = \Phi(\mathrm{co}(\Psi(\mathcal{F}_j \cap \{x > \tilde{x}_j\}))) \quad \text{and} \quad \mathcal{F}_j^l = \Phi(\mathrm{co}(\Psi(\mathcal{F}_j \cap \{x < -\tilde{x}_j\})))$$

are convex sets satisfying

$$\begin{aligned}&\mathcal{F}_j^l = \{(-x,y) : (x,y) \in \mathcal{F}_j^r\}, \quad \mathcal{L}^2(\mathcal{F}_j^r \cup \mathcal{F}_j^l) \geq \mathcal{L}^2(\mathcal{F}_j \setminus (\mathcal{F}_j)^x_{\tilde{x}_j-}) \\ &\mathrm{tr}_{\tilde{x}_j+}\mathcal{F}_j^r = \mathrm{tr}_{\tilde{x}_j+}\mathcal{F}_j, \quad P_\alpha(\mathcal{F}_j^l \cup \mathcal{F}_j^r) \leq P_\alpha(\mathcal{F}_j \setminus (\mathcal{F}_j)^x_{\tilde{x}_j-}).\end{aligned}$$

For any $j \in \mathbb{N}$, let $r_j \geq \tilde{x}_j$ be such that $\mathcal{L}^2((\mathcal{F}_j^r \cup \mathcal{F}_j^l)^x_{r_j-}) = \mathcal{L}^2(\mathcal{E}_j \setminus (\mathcal{E}_j)^x_{x_0-})$. The sequence

$$\tilde{\mathcal{E}}_j = (\mathcal{F}_j^{\mathrm{c}})^x_{\tilde{x}_j-} \cup (\mathcal{F}_j^r \cup \mathcal{F}_j^l)^x_{r_j-}, \quad j \in \mathbb{N}, \tag{4.3.14}$$

satisfies all the claims of this Step.

Step 5. (Boundedness) We go back to the sequence E_m, $m \in \mathbb{N}$ and use the following notation:

- $\mathcal{E}_j^m \subset \mathbb{R}^2$ is the sequence at Step 1, with $\mathrm{tr}^x_{x_m\pm}\mathcal{E}_j^m = [-q_j^{m\pm}, q_j^{m\pm}]$;
- $\tilde{\mathcal{E}}_j^m \subset \mathbb{R}^2$ is the sequence at Step 4 and $\tilde{x}_j^m > 0$ is such that $\mathrm{tr}^x_{\tilde{x}_j^m\pm} = [-\tilde{q}_j^{m\pm}, \tilde{q}_j^{m\pm}]$.

By Step 1, for any $m \in \mathbb{N}$ there exists $J(m) \in \mathbb{N}$ such that for $j \geq J(m)$, we have

$$|q_j^\pm - y_m^\pm| \leq \frac{1}{m} \tag{4.3.15}$$

and

$$\begin{aligned}
&\Big|P_\alpha\big((\tilde{\mathcal{E}}_j^m)^x_{\tilde{x}_j^m-}\big) - P_\alpha((E_m)^x_{x_m-})\Big| \leq \frac{1}{m}, \quad \Big|P_\alpha\big(\tilde{\mathcal{E}}_j^m \setminus (\tilde{\mathcal{E}}_j^m)^x_{\tilde{x}_j^m-}\big) - P_\alpha\big(E_m \setminus (E_m)^x_{x_m-}\big)\Big| \leq \frac{1}{m},\\
&\Big|\mathcal{L}^2\big((\tilde{\mathcal{E}}_j^m)^x_{\tilde{x}_j^m-}\big) - \mathcal{L}^2((E_m)^x_{x_m-})\Big| \leq \frac{1}{m}, \quad \Big|\mathcal{L}^2\big(\tilde{\mathcal{E}}_j^m \setminus (\tilde{\mathcal{E}}_j^m)^x_{\tilde{x}_j^m-}\big) - \mathcal{L}^2(E_m \setminus (E_m)^x_{x_m-})\Big| \leq \frac{1}{m}.
\end{aligned} \tag{4.3.16}$$

Let $(j_m)_{m\in\mathbb{N}}$ be an increasing sequence of integer numbers such that $j_m \geq J(m)$ for any $m \in \mathbb{N}$. We choose the diagonal sequence $\tilde{E}_m = \tilde{\mathcal{E}}_{j_m}^m$, $m \in \mathbb{N}$. We prove that there exists $\ell > 0$ such that

$$\tilde{E}_m \subset [-\ell, \ell] \times [-\ell, \ell], \quad \text{for any} \quad m \in \mathbb{N}. \tag{4.3.17}$$

First of all, letting $\tilde{x}_m = \tilde{x}_{j_m}^m$, we have

$$\sup\{P_\alpha((\tilde{E}_m)^x_{\tilde{x}_m-}) : m \in \mathbb{N}\} < \infty, \quad \text{and} \quad \sup\{P_\alpha(\tilde{E}_m \setminus (\tilde{E}_m)^x_{\tilde{x}_m-}) : m \in \mathbb{N}\} < \infty \tag{4.3.18}$$

by definition of $\mathscr{F}_\alpha$ in (4.3.2), and minimality of $(E_m)_{m\in\mathbb{N}}$. In fact:

$$\begin{aligned}
\max\{P_\alpha\big((\tilde{E}_m)^x_{\tilde{x}_m-}\big), P_\alpha\big(\tilde{E}_m \setminus (\tilde{E}_m)^x_{\tilde{x}_m-}\big)\} &\leq P_\alpha((\tilde{E}_m)^x_{\tilde{x}_m-}) + P_\alpha(\tilde{E}_m \setminus (\tilde{E}_m)^x_{\tilde{x}_m-})\\
&\leq P_\alpha((E_m)^x_{x_m-}) + P_\alpha(E \setminus (E_m)^x_{x_m-}) + \frac{2}{m}\\
&= \mathscr{F}_\alpha(E_m) + 4h_1 + 4h_2 + \frac{2}{m} \leq 2C_{IP} + 4h_1 + 4h_2 + 2.
\end{aligned}$$

We prove that the sequence $\tilde{x}_m$ is bounded. Let $\tilde{\phi}_m$ be the profile function of $\tilde{E}_m$ and assume by contradiction that $x_m \to \infty$ as $m \to \infty$. In this case, by the Representation formula in Proposition 2.2.1 we have:

$$P_\alpha((\tilde{E}_m)^x_{\tilde{x}_m-}) = \int_0^{\tilde{x}_m} \sqrt{\tilde{\phi}_m(x)^2 + |x|^{2\alpha}}\, dx \geq \int_0^{\tilde{x}_m} |x|^\alpha = \frac{\tilde{x}_m^{\alpha+1}}{\alpha+1} \to \infty, \quad m \to \infty$$

which is in contradiction with (4.3.18). In the same way we can see that, if r_m is defined by (4.3.14) so that $\tilde{E}_m \subset (\tilde{E}_m)^x_{r_m-}$, the sequence $(r_m)_{m\in\mathbb{N}}$ is bounded.

Now, we show boundedness in the vertical direction, namely we show that there exists $L \geq 0$ such that $\tilde{E}_m \subset (\tilde{E}_m)^y_{L-}$. Suppose by contradiction that for any $L \geq 0$, there exists $m = m(L) \in \mathbb{N}$ such that $(\tilde{E}_m)^x_{\tilde{x}_m-} \setminus (\tilde{E}_m)^y_{L-} \neq \varnothing$, then by convexity of $(\tilde{E}_m)^x_{\tilde{x}_m-}$, it is equivalent to assume that for any $L \geq 0$ there exists $j(L) \geq 0$ such that

$$\tilde{\phi}_m(0) > L \text{ for } m \geq m(L), \tag{4.3.19}$$

We write for $x \in (0, \tilde{x}_m)$

$$\tilde{\phi}_m(x) = -\int_x^{\tilde{x}_m} \tilde{\phi}'_m(\xi)\, d\xi = \int_x^{\tilde{x}_m} |\tilde{\phi}'_m(\xi)|\, d\xi,$$

then

$$\tilde{\phi}_m(0) = \lim_{x\to 0} \int_x^{\tilde{x}_m} |\tilde{\phi}'_m(\xi)|\, d\xi = \int_0^{\tilde{x}_m} |\tilde{\phi}'_m(\xi)|\, d\xi,$$

which implies, by (4.3.19)

$$\lim_{m\to\infty} \int_0^{\tilde{x}_m} |\tilde{\phi}'_m(\xi)|\, d\xi = \lim_{m\to\infty} \tilde{\phi}_m(0) = \infty.$$

Therefore

$$P_\alpha((\tilde{E}_m)^x_{\tilde{x}_m-}) = 4\int_0^{\tilde{x}_m} \sqrt{(\tilde{\phi}'_m(x))^2 + x^{2\alpha}}\, dx \geq \int_0^{\tilde{x}_m} |\tilde{\phi}'_m(x)|\, dx \to \infty \text{ as } m \to \infty,$$

which is in contradiction with (4.3.18). Similarly, we exclude the case that for any $L > 0$ $\big(\tilde{E}_m \smallsetminus (\tilde{E}_m)^x_{\tilde{x}_m-}\big) \smallsetminus (\tilde{E}_m)^y_{L-} \neq \varnothing$.

Step 6. (Existence of a minimum) From (4.3.18), by the compactness theorem for BV_α functions (see [65, Theorem 1.28]), there exists a set E_∞ which is the L^1_{loc}-limit of $\tilde{E}_m$ as $m \to \infty$. By (4.3.17), convergence $\chi_{\tilde{E}_m} \to \chi_{E_\infty}$ is in $L^1(\mathbb{R}^2)$. Moreover, since the sequence $(\tilde{x}_m)_{m\in\mathbb{N}}$ is bounded, we let $x_\infty \geq 0$ be the limit up to subsequences of $\tilde{x}_m$ as $m \to \infty$. We have, by (4.3.16),

$$\mathcal{L}^2((E_\infty)_{\tilde{x}_\infty}) = \lim_{m\to\infty} \mathcal{L}^2((\tilde{E}_m)^x_{\tilde{x}_m-}) = \lim_{m\to\infty} \mathcal{L}^2((E_m)^x_{x_m-}) = v_1$$

and

$$\mathcal{L}^2(E_\infty \smallsetminus (E_\infty)_{\tilde{x}_\infty}) = \lim_{m\to\infty} \mathcal{L}^2(\tilde{E}_m \smallsetminus (\tilde{E}_m)^x_{\tilde{x}_m-}) = \lim_{m\to\infty} \mathcal{L}^2(E_m \smallsetminus (E_m)^x_{x_m-}) = v_2.$$

Now, since $(\tilde{E}_m)^x_{\tilde{x}_m-}$, $\tilde{E}_m \cap \{x > \tilde{x}_m\}$, $\tilde{E}_m \cap \{x < -\tilde{x}_m\}$ are convex, we can choose a representative for E_∞ such that $(E_\infty)^x_{x_\infty-}$, $E_\infty \cap \{x > \tilde{x}_\infty\}$, $E_\infty \cap \{x < -\tilde{x}_\infty\}$ are convex. By boundedness of the sequence $\tilde{E}_m$, let $y^\pm_\infty \geq 0$ be such that $\mathrm{tr}^x_{x_\infty\pm} E_\infty = [-y^\pm_\infty, y^\pm_\infty]$. Then, by (4.3.15) and claim 4 at Step 4, we have

$$y^-_\infty \geq \lim_{m\to\infty} \tilde{q}^-_m \geq \lim_{m\to\infty} q^-_m \geq \lim_{m\to\infty} y^-_m - \frac{1}{m} \geq h_1,$$

equivalently $y^+_\infty \geq h_2$. Hence $E_\infty \in \mathcal{A}_x$.

By the lower semi-continuity of the α-perimeter together with (4.3.16), we have

$$\begin{aligned}
\mathscr{F}_\alpha(E_\infty) &= P_\alpha((E_\infty)^x_{x_\infty-}) + P_\alpha(E_\infty \smallsetminus (E_\infty)^x_{x_\infty-}) - 4h_1 - 4h_2 \\
&\leq \liminf_{m\to\infty} P_\alpha((\tilde{E}_m)^x_{\tilde{x}_m-}) + \liminf_{m\to\infty} P_\alpha(\tilde{E}_m \smallsetminus (\tilde{E}_m)^x_{\tilde{x}_m-}) - 4h_1 - 4h_2 \\
&\leq \liminf_{m\to\infty} P_\alpha((E_m)^x_{x_m-}) + \liminf_{m\to\infty} P_\alpha(E_m \smallsetminus (E_m)^x_{x_m-}) + \frac{2}{m} - 4h_1 - 4h_2 \\
&= \liminf_{m\to\infty} \mathscr{F}_\alpha(E_m) + \frac{2}{m} \leq C_{IP}.
\end{aligned} \tag{4.3.20}$$

In conclusion the set $E_\infty \subset \mathbb{R}^2$ is such that

$$P_\alpha(E_\infty) = \inf\{P_\alpha(E), E \in \mathcal{A}_x\}$$

with $E_\infty \in \mathcal{A}_x$. It is therefore a bounded minimizer for (4.3.3) such that $E^x_{x_\infty-}$, $E_\infty \cap \{|x| < x_\infty\}$, and $E_\infty \cap \{|x| > x_\infty\}$ are convex. □

4.3.2. Differential equations for the profile function

In the following proposition we deduce differential equations for a minimizer for Problem (4.3.3) as in Theorem 4.3.6.

Proposition 4.3.7. *Let $v_1, v_2, h_1, h_2 \geq 0$ and $E \in \mathcal{A}_x$ be a bounded minimizer for (4.3.3) such that for $x_0 \geq 0$ as in (4.3.1), $E^x_{x_0-}$, $E \cap \{x > x_0\}$, $E \cap \{x < -x_0\}$ are convex sets. Then its profile function $f : [0,\infty) \to [0,\infty)$ is C^2 smooth almost everywhere on $[0,\infty)$ and there exist constants $c \geq 0$, $k \leq 0$, $d \in \mathbb{R}$ such that*

$$f'(x) = -\frac{\operatorname{sgn} x\; c|x|^{\alpha+1}}{\sqrt{1-c^2x^2}} \qquad \text{if} \quad |x| < x_0, \tag{4.3.21a}$$

$$f'(x) = \frac{(kx+d)\, x^\alpha}{\sqrt{1-(kx+d)^2}} \qquad \text{if} \quad x > x_0. \tag{4.3.21b}$$

$$f'(x) = \frac{(kx-d)\, |x|^\alpha}{\sqrt{1-(kx-d)^2}} \qquad \text{if} \quad x < -x_0. \tag{4.3.21c}$$

Proof. Let $E \subset \mathbb{R}^2$ be a bounded minimizer as in the statement and let $r_0 = \inf\{r > 0 : E \subset E^x_{r-}\}$, then $r_0 < \infty$. By the convexity properties of E, the profile function of E, $f : [0, r_0) \to [0,\infty)$ is C^2-smooth a.e.

We first prove (4.3.21a). For $\psi_1 \in C^\infty_c(0, x_0)$ with $\int \psi_1 = 0$, and $\varepsilon \in \mathbb{R}$, consider the perturbation $x \mapsto f(|x|) + \varepsilon\psi_1(|x|)$ and define the set

$$E_\varepsilon = \{(x,y) \in \mathbb{R}^2 : |y| < f(|x|) + \varepsilon\psi_1(|x|)\}$$

Notice that the variation involves only $E^x_{x_0-}$. The set E_ε is still in the class $\mathcal{A}_x$, hence $\mathscr{F}_\alpha(E_\varepsilon) \leq \mathscr{F}_\alpha(E)$. Therefore

$$\frac{d}{d\varepsilon}\mathscr{F}_\alpha(E_\varepsilon)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}P_\alpha((E_\varepsilon)^x_{x_0-})\Big|_{\varepsilon=0} = 0$$

where, by the Representation formula for the α-perimeter of x- and y-symmetric sets (2.2.12) (see also (2.5.11)), we have

$$p_1(\varepsilon) = P_\alpha((E_\varepsilon)^x_{x_0-}) = 4\Big\{\int_0^{x_0} \sqrt{(f' + \varepsilon\psi')^2 + x^{2\alpha}}\, dx + \lim_{x\to x_0^-} f(x)\Big\},$$

that leads to

$$\begin{aligned} 0 = p_1'(\varepsilon)\Big|_{\varepsilon=0} &= 4\int_0^{x_0} \frac{d}{d\varepsilon}\Big(\sqrt{(f'+\varepsilon\psi_1')^2+x^{2\alpha}}\Big)\Big|_{\varepsilon=0} dx \\ &= 4\int_0^{x_0} \frac{f'(x)\psi_1'(x)}{\sqrt{f'^2(x)+x^{2\alpha}}}\, dx = -4\int_0^{x_0} \frac{d}{dx}\left(\frac{f'}{\sqrt{f'^2+x^{2\alpha}}}\right)\psi_1(x)\, dx. \end{aligned}$$

Using the fact that ψ_1 is arbitrary we deduce the following second order ordinary differential equation satisfied for some $C \in \mathbb{R}$

$$\frac{d}{dx}\left(\frac{f'(x)}{\sqrt{f'(x)^2+x^{2\alpha}}}\right) = C \quad \text{for a.e. } 0 < x < x_0. \tag{4.3.22}$$

The normal form of (4.3.22) is

$$f''(x) = \frac{c}{x^{2\alpha}}(f'(x)^2 + x^{2\alpha})^{\frac{3}{2}}, \tag{4.3.23}$$

which implies, by convexity of $E^x_{x_0-}$ that $C \le 0$. We let $c = -C \ge 0$. Now, since E is x-symmetric, the function f is even, hence f' is odd and f'' is even. This allows us to extend (4.3.22) to $|x| < x_0$. Moreover, integrating (4.3.22), we obtain existence of a constant $d \in \mathbb{R}$ such that for some $\delta > 0$,

$$\frac{f'(x)}{\sqrt{f'(x)^2+x^{2\alpha}}} = Cx + d \qquad \text{for } |x| < \delta.$$

Since f' is odd we deduce that $d = 0$, in fact for $|x| < \delta$

$$Cx + d = \frac{f'(x)}{\sqrt{f'^2(x) - x^{2\alpha}}} = -\frac{f'(-x)}{\sqrt{f'^2(-x)+(-x)^2\alpha}} = -(C(-x)+d) = Cx - d.$$

We therefore get to an ode for f:

$$\frac{f'(x)}{\sqrt{f'^2(x)+x^{2\alpha}}} = Cx \text{ for } |x| < \delta,$$

which is equivalent to:

$$f'(x) = -\operatorname{sgn}(x)\frac{c|x|^{\alpha+1}}{\sqrt{1-c^2x^2}} \quad \text{for } |x| < \delta.$$

A solution to the latter equation can be extended up to $(-1/c, 1/c)$. This implies $0 < x_0 \le 1/c$ and (4.3.21a) is proved.

To prove (4.3.21b) and (4.3.21c), we proceed in the same way, considering a function $\psi_2 \in C_c^\infty(x_0, r_0)$, with $\int \psi_2 = 0$ and the associated perturbation $f + \eta\psi_2$ for $\eta \in \mathbb{R}$. The set $E_\eta = \{(x,y) \in \mathbb{R}^2 : |y| < f(|x|) + \varepsilon\psi_2(|x|)\}$ is inside the class $\mathcal{A}_x$, hence, as previously, minimality of E leads to

$$\frac{d}{d\eta}\mathscr{F}_\alpha(E_\eta)\Big|_{\eta=0} = \frac{d}{d\eta}P_\alpha(E_\eta \setminus (E_\eta)^x_{x_0-})\Big|_{\eta=0} = 0$$

and we obtain existence of a constant $k \in \mathbb{R}$ such that

$$\frac{d}{dx}\left(\frac{f'(x)}{\sqrt{f'(x)^2 + x^{2\alpha}}}\right) = k \quad \text{for} \quad x_0 < |x| < r_0. \tag{4.3.24}$$

In particular, $k \le 0$. Let $x_0 < x < r_0$. An integration between x_0 and x shows that, letting

$$d = \lim_{x \to x_0^+} \frac{f'(x)}{\sqrt{f'(x)^2 + x^{2\alpha}}} - kx_0,$$

we have

$$\frac{f'(x)}{\sqrt{f'(x)^2 + x^{2\alpha}}} = kx + d \quad \text{for} \quad x_0 < x < r_0$$

which is equivalent to (4.3.21b). In particular, $|kx + d| < 1$ for $x_0 < x < r_0$.

Analogously, for any $x \in (-r_0, -x_0)$, an integration between x and $-x_0$ shows that

$$\frac{f'(x)}{\sqrt{f'(x)^2 + x^{2\alpha}}} = kx - d \quad \text{for} \quad -r_0 < x < -x_0,$$

which leads to (4.3.21c) and $|kx - d| < 1$ for $-r_0 < x < -x_0$. □

Remark 4.3.8. If E is a minimizer as in Proposition 4.3.7, using the ordinary differential equations (4.3.21a)-(4.3.21c) we can show, as in Proposition 2.5.1, that the function f is indeed $C^2([0, x_0)) \cap C^2(x_0, r_0) \cap C^2(-r_0, -x_0)$, with $r_0 = \inf\{r > 0 : E \subset E^x_{r-}\} < \infty$.

By (4.3.21a), we deduce that a minimizer $E \in A_x$ for Problem (4.3.3) as in Theorem 4.3.6, is obtained in its central part through the composition of a dilation and a vertical translation of E^α_{isop}, namely there exists $y \in \mathbb{R}$ such that

$$E^x_{x_0-} = \left(\delta^\alpha_\lambda(E^\alpha_{\text{isop}}) + (0, y)\right)^x_{x_0-}. \tag{4.3.25}$$

In particular

$$\lim_{x \to 0^+} \frac{f'(x)}{|x|^{\alpha+1}} = 0.$$

Moreover, by (4.3.21b),

$$f(r_0) = 0, \quad \lim_{x \to r_0^+} f'(x) = -\infty$$

In particular, r_0 is characterized by the following equality:

$$-1 = \lim_{x \to r_0^-} \frac{f'(x)}{\sqrt{f'(x)^2 + x^{2\alpha}}} = kr_0 + d,$$

namely,

$$r_0 = -\frac{1 + d}{k}. \tag{4.3.26}$$

4.3.3. Traces of minimizers

In this Section, we study traces of minimizers. What we expect is that if $E \in \mathcal{A}_x$ is a minimizer for (4.3.3) as in Theorem 4.3.6, then

$$\mathrm{tr}^x_{x_0-}E = [-h_1, h_1] \quad \text{and} \quad \mathrm{tr}^x_{x_0+}E = [-h_2, h_2],$$

where x_0 is defined by (4.3.1). In Proposition 4.3.9 we prove the claim for the left-trace assuming that the profile function of E does not have infinite derivative at x_0. The argument used here does not apply to the case of infinite derivative as shown in Remark 4.3.10. The case of the right trace is analogous.

Proposition 4.3.9. *Let $v_1, v_2, h_1, h_2 \geq 0$ and $E \in \mathcal{A}_x$ be a bounded minimizer for* (4.3.3) *such that for $x_0 \geq 0$ as in* (4.3.1)*, $E^x_{x_0-}$, $E \cap \{x > x_0\}$, $E \cap \{x < -x_0\}$ are convex sets. Let $f : [0, r_0) \to [0, \infty)$ be its profile function for $r_0 > 0$ as in Remark 4.3.8. If*

$$\lim_{x \to x_0^-} f'(x) > -\infty,$$

then

$$\mathrm{tr}^x_{x_0-}E = [-h_1, h_1].$$

Proof. Assume by contradiction that $f(x_0^-) > h_1$, where

$$f(x_0^-) = \lim_{x \to x_0^-} f(x).$$

We show that in this case, there exists a set $F \in \mathcal{A}_x$ such that $P_\alpha(F^x_{x_0-}) < P_\alpha(E^x_{x_0-})$, $P_\alpha(F \setminus F^x_{x_0-}) = P_\alpha(E \setminus E^x_{x_0-})$, hence $\mathscr{F}_\alpha(F) < \mathscr{F}_\alpha(E)$, which is in contradiction with the minimality of E.

For a small parameter $\varepsilon > 0$, let $f_\varepsilon : [0, x_0) \to [0, \infty)$ be the function defined by

$$f_\varepsilon(x) = \begin{cases} f(x), & \text{if } 0 < x < x_0 - \varepsilon \\ r_\varepsilon(x), & \text{if } x_0 - \varepsilon < x < x_0 \end{cases}$$

where r_ε is the segment connecting the points $\big(x_0 - \varepsilon, f(x_0 - \varepsilon)\big)$ and $(x_0, f(x_0^-) - \varepsilon)$, i.e.,

$$r_\varepsilon(x) = m(\varepsilon)(x - x_0) + f(x_0^-) - \varepsilon, \quad m(\varepsilon) = \frac{1}{\varepsilon}(f(x_0^-) - \varepsilon - f(x_0 - \varepsilon)) < 0.$$

By convexity of $E^x_{x_0-}$, $f(x) \geq r_\varepsilon(x)$ for $x_0 - \varepsilon < x < x_0$. We define the set $E_\varepsilon = \{(x, y) \in \mathbb{R}^2 : |y| < f_\varepsilon(|x|)\}$.

We compute the difference $P_\alpha(E) - P_\alpha(E_\varepsilon)$, using the Representation formula for symmetric sets (2.2.12). Since $\partial E^x_{x_0} = \partial E \cap \{|x| = x_0\}$ is a vertical segment, the outer unit

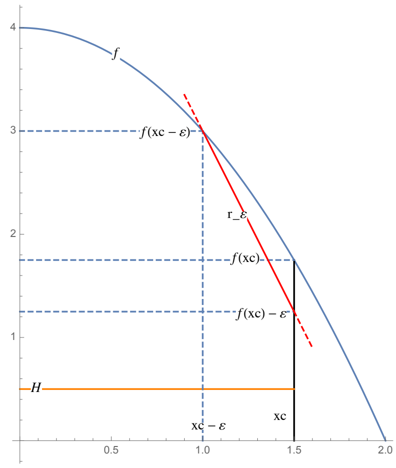


Figure 4.4: Construction of the set E_ε.

normal to E is constant on $\partial E^x_{x_0}$, $N^E = (1,0)$. In the same way the outer unit normal to E_ε is constant on $\partial(E_\varepsilon)^x_{x_0}$, $N^{E_\varepsilon} = (1,0)$. Then, we have

$$\begin{aligned} P_\alpha(E^x_{x_0-}) - P_\alpha(E_\varepsilon) &= 4\int_{x_0-\varepsilon}^{x_0} \sqrt{f'(x)^2 + x^{2\alpha}} - \sqrt{m(\varepsilon)^2 + x^{2\alpha}}\, dx + \int_{\partial E^x_{x_0}} d\mathcal{H}^1 - \int_{\partial(E_\varepsilon)^x_{x_0}} d\mathcal{H}^1 \\ &= 4\int_{x_0-\varepsilon}^{x_0} \sqrt{f'(x)^2 + x^{2\alpha}} - \sqrt{m(\varepsilon)^2 + x^{2\alpha}}\, dx + 4\big(f(x_0^-) - (f(x_0^-) - \varepsilon)\big) \\ &= 4\Big\{\int_{x_0-\varepsilon}^{x_0} \sqrt{f'(x)^2 + x^{2\alpha}} - \sqrt{m(\varepsilon)^2 + x^{2\alpha}}\, dx + \varepsilon\Big\}. \end{aligned}$$

Let $A(\varepsilon) = (P_\alpha(E^x_{x_0-}) - P_\alpha(E_\varepsilon))/4$. On the other hand,

$$\mathcal{L}^2(E^x_{x_0-}) - \mathcal{L}^2(E_\varepsilon) = 4\int_{x_0-\varepsilon}^{x_0} f(x) - r_\varepsilon(x)\, dx,$$

and we define $B(\varepsilon) = (\mathcal{L}^2(E^x_{x_0-}) - \mathcal{L}^2(E_\varepsilon))/4$. For any $\varepsilon > 0$, let $y_\varepsilon = B(\varepsilon)/x_0$. We claim that for $\varepsilon > 0$ small enough the set

$$F_\varepsilon = (E_\varepsilon + (0, y_\varepsilon)) \cup \big([-x_0, x_0] \times [-y_\varepsilon, y_\varepsilon]\big),$$

obtained by translating E_ε in the vertical direction of the quantity y_ε, satisfies

$$P_\alpha(F_\varepsilon) < P_\alpha(E). \tag{4.3.27}$$

It follows that the set $F = F_\varepsilon \cup (E \setminus E^x_{x_0-})$ satisfies $\mathscr{F}_\alpha(F) < \mathscr{F}_\alpha(E)$. Moreover $F \in \mathcal{A}_x$, since $\mathcal{L}^2(F^x_{x_0-}) = \mathcal{L}^2(E_\varepsilon) + 4x_0 y_\varepsilon = \mathcal{L}^2(E^x_{x_0-})$, and the other properties in (4.3.1) are clear by construction. By invariance under vertical translations of the α-perimeter,

$$P_\alpha(F_\varepsilon) = P_\alpha(E_\varepsilon) + 4y_\varepsilon = P_\alpha(E^x_{x_0-}) - 4\Big(A(\varepsilon) - \frac{B(\varepsilon)}{x_0}\Big)$$

To prove (4.3.27), it is therefore sufficient to show that for $\varepsilon > 0$ small enough

$$x_0 A(\varepsilon) > B(\varepsilon). \tag{4.3.28}$$

First of all, notice that, by Lebesgue dominated convergence Theorem,

$$\lim_{\varepsilon\to 0^+} A(\varepsilon) = \lim_{\varepsilon\to 0^+} B(\varepsilon) = 0. \tag{4.3.29}$$

Let $f'(x_0^-) = \lim_{x\to x_0^-} f'(x)$. By convexity of $E^x_{x_0-}$, we have $f'(x_0^-) \le 0$ $f''(x_0^-) = \lim_{x\to x_0^-} f''(x) \le 0$. Moreover, the following limit exists

$$\lim_{\varepsilon\to 0^+} m(\varepsilon) = \lim_{\varepsilon\to 0^+} \frac{f(x_0^-) - f(x_0-\varepsilon)}{\varepsilon} - 1 = f'(x_0^-) - 1.$$

Now, since $f'(x_0^-) > -\infty$, we also have $f''(x_0^-) > -\infty$, hence the following limit exists

$$\begin{aligned} m'(0) &= \lim_{\varepsilon\to 0^+} m'(\varepsilon) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\Big\{f'(x_0-\varepsilon) - \frac{1}{\varepsilon}\big[f(x_0^-) - f(x_0-\varepsilon)\big]\Big\} \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\Big\{f'(x_0^-) - f''(x_0^-)\varepsilon - \frac{1}{\varepsilon}\Big[f(x_0^-) - \Big(f(x_0^-) - f'(x_0^-)\varepsilon + \frac{f''(x_0^-)}{2}\varepsilon^2\Big)\Big] + o(\varepsilon)\Big\} \\ &= -\frac{f''(x_0^-)}{2}. \end{aligned}$$

On the other hand, by the chain rule

$$\begin{aligned} A'(\varepsilon) &= 1 + \sqrt{f'^2(x_0-\varepsilon) + (x_0-\varepsilon)^{2\alpha}} - \sqrt{m(\varepsilon)^2 + (x_0-\varepsilon)^{2\alpha}} - \int_{x_0-\varepsilon}^{x_0} \frac{m(\varepsilon)m'(\varepsilon)}{\sqrt{m(\varepsilon^2) + x^{2\alpha}}}\, dx \\ &\ge 1 + \sqrt{f'^2(x_0-\varepsilon) + (x_0-\varepsilon)^{2\alpha}} - \sqrt{m(\varepsilon)^2 + (x_0-\varepsilon)^{2\alpha}} \end{aligned}$$

that gives

$$A'(0) = \lim_{\varepsilon\to 0} A'(\varepsilon) \ge 1 + \sqrt{f'^2(x_0^-)^2 + x_0^{2\alpha}} - \sqrt{(f'(x_0^-) - 1)^2 + x_0^{2\alpha}} > 0, \tag{4.3.30}$$

where the last inequality is justified by the following: for any $c < a < 0$, and $b \in \mathbb{R}$,

$$\sqrt{a^2+b^2} - \sqrt{c^2+b^2} > c - a.$$

We conclude observing that $B'(0) = \lim_{\varepsilon\to 0^+} B'(\varepsilon) = 0$, in fact:

$$\begin{aligned} B'(\varepsilon) &= f(x_0-\varepsilon) - r_\varepsilon(x_0-\varepsilon) + \int_{x_0-\varepsilon}^{x_0} \frac{d}{d\varepsilon}\Big\{f(x) - m(\varepsilon)(x-x_0) - f(x_0^-) + \varepsilon\Big\}\, dx \\ &= f(x_0-\varepsilon) + \varepsilon m(\varepsilon) - f(x_0^-) + \varepsilon - \int_{x_0-\varepsilon}^{x_0} 1 - m'(\varepsilon)(x-x_0)\, dx \xrightarrow[\varepsilon\to 0]{} 0 \end{aligned}$$

Then, (4.3.28) follows by (4.3.30) and (4.3.29). □

Remark 4.3.10. Assume that the profile function of a minimizer as in Theorem 4.3.6 satisfies

$$f'(x_0^-) = \lim_{x\to x_0^-} f'(x) = -\infty.$$

Using the notation of Proposition 4.3.9, we claim that, for $\varepsilon > 0$, small enough we have

$$x_0 A(\varepsilon) - B(\varepsilon) = -\frac{x_0^{\alpha+2}}{6\sqrt{2}}\Big(\frac{\varepsilon}{x_0}\Big)^{\frac{3}{2}} + o(\varepsilon^{3/2}) < 0 \qquad \text{for } \varepsilon < \varepsilon_0. \tag{4.3.31}$$

Hence, the construction proposed in the latter proposition does not apply to this case. To prove (4.3.31), we observe that if $f'(x_0^-) = -\infty$, we are in the case when $c = 1/x_0$ in equation (4.3.21a). Hence, for any $|x| < x_0$, we have

$$f(x) = x_0^{\alpha+1}\varphi_\alpha\Big(\frac{x}{x_0}\Big) + f(x_0^-) \tag{4.3.32}$$

where φ_α is the profile function of the isoperimetric set E^α_{isop},

$$\varphi_\alpha(r) = \int_{\arcsin r}^{\pi/2} \sin t^{\alpha+1}\, dt, \quad 0 \le r \le 1.$$

Notice that the map $x \mapsto \Phi_\alpha(x) = x_0^{\alpha+1}\varphi_\alpha(\frac{x}{x_0})$ is the profile function of a dilation of E^α_{isop} such that it closes at x_0, i.e., $\Phi_\alpha(x_0) = 0$. For any $\varepsilon > 0$, we have, by (4.3.32) and definition of $m(\varepsilon)$

$$m(\varepsilon) = \frac{1}{\varepsilon}\Big(f(x_0^-) - \varepsilon - f(x_0 - \varepsilon)\Big) = -\Big(1 + \frac{x_0^{\alpha+1}\varphi_\alpha(1-\frac{\varepsilon}{x_0})}{\varepsilon}\Big)$$

We write $A(\varepsilon) = \varepsilon + A_1(\varepsilon) - A_2(\varepsilon)$, where, using the differential equation (4.3.21a), we let

$$A_1(\varepsilon) = \int_{x_0-\varepsilon}^{x_0} \sqrt{f'^2 + x^{2\alpha}}\, dx = x_0 \int_{x_0-\varepsilon}^{x_0} \frac{x^\alpha}{\sqrt{x_0^2 - x^2}}\, dx, \quad A_2(\varepsilon) = \int_{x_0-\varepsilon}^{x_0} \sqrt{m(\varepsilon)^2 + x^{2\alpha}}\, dx.$$

Moreover using the definition of r_ε, and (4.3.32)

$$\begin{aligned} B(\varepsilon) &= \int_{x_0-\varepsilon}^{x_0} f(x) - r_\varepsilon(x)\, dx = \int_{x_0-\varepsilon}^{x_0} x_0^{\alpha+1}\varphi_\alpha\Big(\frac{x}{x_0}\Big) + f(x_0^-) - m(\varepsilon)(x - x_0) - f(x_0^-) + \varepsilon\, dx \\ &= x_0^{\alpha+1} \int_{x_0-\varepsilon}^{x_0} \varphi_\alpha\Big(\frac{x}{x_0}\Big)\, dx + \Big(\frac{m(\varepsilon)}{2} + 1\Big)\varepsilon^2. \end{aligned}$$

Using De L'Hopital Theorem, we prove the following asymptotic behavior of $A_1(\varepsilon)$ as $\varepsilon \to 0^+$.

$$A_1(\varepsilon) = \sqrt{2}x_0^{\alpha+1}\Big(\frac{\varepsilon}{x_0}\Big)^{\frac{1}{2}} + \frac{1-4\alpha}{6\sqrt{2}}x_0^{\alpha+1}\Big(\frac{\varepsilon}{x_0}\Big)^{\frac{3}{2}} + o(\varepsilon^{\frac{3}{2}})$$

To find developments for A_2 and B we first notice that

$$\varphi_\alpha\Big(1 - \frac{\varepsilon}{x_0}\Big) = \sqrt{2}\Big(\frac{\varepsilon}{x_0}\Big)^{\frac{1}{2}} - \frac{4\alpha+3}{6\sqrt{2}}\Big(\frac{\varepsilon}{x_0}\Big)^{\frac{3}{2}} + o(\varepsilon^{\frac{3}{2}}), \quad \text{as } \varepsilon \to 0^+,$$

then

$$m(\varepsilon) = -1 - \sqrt{2}x_0^\alpha\Big(\frac{\varepsilon}{x_0}\Big)^{-1/2} + \frac{4\alpha+3}{6\sqrt{2}}x_0^\alpha\Big(\frac{\varepsilon}{x_0}\Big)^{\frac{1}{2}} + o(\sqrt{\varepsilon}), \quad \text{as } \varepsilon \to 0^+,$$

and

$$\int_{x_0-\varepsilon}^{x_0} \varphi_\alpha\Big(\frac{x}{x_0}\Big)\, dx = \frac{2x_0\sqrt{2}}{3}\Big(\frac{\varepsilon}{x_0}\Big)^{\frac{3}{2}} + o(\varepsilon^{\frac{3}{2}}), \quad \text{as } \varepsilon \to 0^+.$$

After some computations that are omitted we obtain

$$A_2(\varepsilon) = \sqrt{2}x_0^{\alpha+1}\sqrt{\frac{\varepsilon}{x_0}} + \varepsilon - \frac{\sqrt{2}\alpha x_0^{\alpha+1}}{3}\Big(\frac{\varepsilon}{x_0}\Big)^{\frac{3}{2}} + o(\varepsilon^{\frac{3}{2}}), \quad B(\varepsilon) = \frac{x_0^{\alpha+2}}{3\sqrt{2}}\Big(\frac{\varepsilon}{x_0}\Big)^{\frac{3}{2}} + o(\varepsilon^{\frac{3}{2}}).$$

We then deduce (4.3.31) since

$$A(\varepsilon) = \varepsilon + A_1(\varepsilon) - A_2(\varepsilon) = \frac{x_0^{\alpha+1}}{6\sqrt{2}}\Big(\frac{\varepsilon}{x_0}\Big)^{\frac{3}{2}} + o(\varepsilon^{\frac{3}{2}}).$$

4.3.4. Center of the solution to the partitioning problem with $v_2 = 0$

In this section, we present a preliminary analysis of the minimizers obtained in Theorem 4.3.6. The question that arises in view of Remark 4.2.8 is to understand if we can assume a minimizer $E \in \mathcal{A}_x$ to satisfy for some $\lambda > 0$:

$$E^x_{x_0-} = (\delta^\alpha_\lambda E^\alpha_{\text{isop}})^x_{x_0-}. \tag{4.3.33}$$

Let f be the profile function of E. As observed in Remark 4.3.8, from the study of the differential equations in Proposition 4.3.7, it follows

$$f(x) = \lambda^{\alpha+1}\varphi_\alpha\Big(\frac{x}{\lambda}\Big) + y, \quad |x| < x_0 \tag{4.3.34}$$

for some $\lambda > 0$, and $y \in \mathbb{R}$, where φ_α is the profile function of the isoperimetric set E^α_{isop}.

We address the problem of characterizing λ and $y \in \mathbb{R}$ in the easier case of minimizers for Problem (4.3.3) with $h_2 = v_2 = 0$. Namely, given $h, v \geq 0$ we consider a convex set $E \in \mathcal{A}_x(h,v) = \mathcal{A}_x(v,0,h,0)$ (see (4.3.1) for definitions), that is x- and y-Schwarz symmetric, $E \subset E^x_{x_0-}$ and satisfies $P_\alpha(E) \leq P_\alpha(F)$ for any $F \in \mathcal{A}_x(h,v)$. In the next proposition we show that, for $\alpha = 1$, the vertical translation y is in fact strictly negative, i.e., $y < 0$, and property (4.3.33) fails with $v_2 = 0$.

Proposition 4.3.11. *Let $\alpha \in \{0,1\}$, $h \geq 0$, $v > 0$. Let E be a convex minimizer for* (4.3.3) *with $h_1 = h$, $v_1 = v$, $h_2 = v_2 = 0$ and let $x_0 > 0$ be such that $\operatorname{tr}^x_{x_0^-} E = [-h,h]$. Let $c > 0$ be as in* (4.3.21a) *and λ, y be as in* (4.3.34)*. Then, letting $d = cx_0$, λ and y satisfy*

$$\lambda = \frac{1}{c} = \frac{x_0}{d}, \quad y = h\Big\{1 - \frac{\varphi_\alpha(d)}{d^\alpha\sqrt{1-d^2}}\Big\} \tag{4.3.35}$$

Moreover

$$\begin{cases} x_0^{\alpha+1} = \frac{dh}{\sqrt{1-d^2}} \\ x_0^{\alpha+2} G_\alpha(d) + hx_0 - v = 0, \end{cases} \tag{4.3.36}$$

where

$$G_\alpha(d) = \frac{1}{d^{\alpha+2}} \int_0^d \int_{\arcsin t}^{\arcsin d} \sin^{\alpha+1} t \, d\vartheta \, dt.$$

Remark 4.3.12. Relation (4.3.35) can be extended to any $\alpha \geq 0$ such that (4.3.43) is true (see the proof below). If this happens for any $\alpha \geq 0$, we notice that $y = 0$ if and only if $\alpha = 0$. In fact the function $d \mapsto \varphi_\alpha(d) - d^\alpha\sqrt{1-d^2}$ is 0 at $d = 1$ and it is strictly monotone decreasing since

$$(\varphi_\alpha(d) - d^\alpha\sqrt{1-d^2})' = -\frac{d^{\alpha+1}}{\sqrt{1-d^2}} - \alpha d^{\alpha-1}\sqrt{1-d^2} + \frac{d^{\alpha+1}}{\sqrt{1-d^2}} < 0.$$

Hence if $\alpha > 0$, $\varphi_\alpha(d) - d^\alpha\sqrt{1-d^2} > 0$ for $0 < d < 1$.

Proof. Let $E \subset \mathbb{R}^2$ be a convex set such that for some $x_0 = x_0(E) > 0$,

$$E \subset E^x_{x_0-}, \quad \mathrm{tr}^x_{x_0-}E = [-h, h], \quad \mathcal{L}^2(E) = v_1$$

and

$$P_\alpha(E) = \min\{P_\alpha(F) : F \in \mathcal{A}_x\}.$$

By Proposition 4.3.7, the profile function of E, $f : [0, x_0] \to [0, \infty)$, satisfies (4.3.21a), for some $c \geq 0$, we then have $\lambda = 1/c$. By (4.3.34), we deduce $f(t) = \lambda^{\alpha+1}\varphi_\alpha\big(\frac{x}{\lambda}\big) + y$. Let $\beta = f(0) > 0$. From Propositon 2.2.1, we define the functional

$$p(\beta, c, x_0) = P_\alpha(E) = \int_0^{x_0} \sqrt{f'^2(t) + t^{2\alpha}}\, dt = \int_0^{x_0} \frac{t^\alpha}{\sqrt{1-(ct)^2}}\, dt = \frac{1}{c^{\alpha+1}} \int_0^{\arcsin cx_0} \sin^\alpha \vartheta\, d\vartheta.$$

Notice that p is independent of β, since P_α is independent of vertical translations. Let $d = cx_0$. We have

$$p(d, x_0) = x_0^{\alpha+1} g_\alpha(d), \text{ with } g_\alpha(d) = \frac{1}{d^{\alpha+1}} \int_0^{\arcsin d} \sin^\alpha \vartheta\, d\vartheta. \tag{4.3.37}$$

We write the volume and trace constraint in terms of the parameters d and x_0. For any $t \in (0, x_0)$ we write

$$f(t) = \beta + \int_0^t f'(s)\, ds = \beta - \int_0^t \frac{cs^{\alpha+1}}{\sqrt{1-(cs)^2}}\, ds = \beta - \frac{1}{c^{\alpha+1}} \int_0^{\arcsin ct} \sin^{\alpha+1} \vartheta\, d\vartheta$$

Hence the trace constraint $f(x_0) = h$ is equivalent to

$$\begin{aligned} &\beta = \beta(d, x_0) = h + x_0^{\alpha+1}\sigma_\alpha(d), \text{ with} \\ &\sigma_\alpha(d) = \frac{1}{d^{\alpha+1}} \int_0^{\arcsin d} \sin^{\alpha+1} \vartheta\, d\vartheta > 0 \text{ for } d \in (0,1), \end{aligned} \tag{4.3.38}$$

and, plugging $\beta = \beta(d, x_0)$ in the expression for f,

$$f(t) = h + x_0^{\alpha+1}\sigma_\alpha(d) - x_0^{\alpha+1} \frac{1}{d^{\alpha+1}} \int_0^{\arcsin(\frac{d}{x_0}t)} \sin^{\alpha+1} \vartheta\, d\vartheta, \tag{4.3.39}$$

that implies

$$\begin{aligned} y = f(\lambda) &= h + x_0^{\alpha+1} b_\alpha(d), \text{ with} \\ b_\alpha(d) &= \frac{1}{d^{\alpha+1}} \Big\{ \int_0^{\arcsin d} \sin^{\alpha+1} \vartheta\, d\vartheta - \int_0^{\pi/2} \sin^{\alpha+1} \vartheta\, d\vartheta \Big\} \\ &= -\frac{1}{d^{\alpha+1}} \int_{\arcsin d}^{\pi/2} \sin^{\alpha+1} \vartheta; d\vartheta = -\frac{\varphi_\alpha(d)}{d^{\alpha+1}} < 0 \text{ for } d \in (0,1). \end{aligned} \tag{4.3.40}$$

The volume constraint $\int f = v$ reads, using the trace constraint $\beta = \beta(d, x_0)$ in (4.3.38)

$$\begin{aligned} v &= \int_0^{x_0} \Big(\beta - \frac{1}{c^{\alpha+1}} \int_0^{\arcsin ct} \sin^{\alpha+1} \vartheta\, d\vartheta\Big)\, dt = \beta x_0 - \frac{1}{c^{\alpha+2}} \int_0^{cx_0} \int_0^{\arcsin r} \sin^{\alpha+1} \vartheta\, d\vartheta\, dr \\ &= \Big(h + x_0^{\alpha+1}\sigma_\alpha(d)\Big)x_0 - x_0^{\alpha+2} \frac{1}{d^{\alpha+2}} \int_0^d \int_0^{\arcsin t} \sin^{\alpha+1} \vartheta\, d\vartheta\, dt, \end{aligned}$$

hence

$$\begin{aligned} v &= hx_0 + x_0^{\alpha+2} G_\alpha(d), \text{ with} \\ G_\alpha(d) &= \frac{1}{d^{\alpha+2}} \Big(d \int_0^{\arcsin d} \sin^{\alpha+1} \vartheta \, d\vartheta - \int_0^d \int_0^{\arcsin t} \sin^{\alpha+1} \vartheta \, d\vartheta \, dt \Big) \\ &= \frac{1}{d^{\alpha+2}} \int_0^d \int_{\arcsin t}^{\arcsin d} \sin^{\alpha+1} \vartheta \, d\vartheta \, dt > 0 \text{ for } d \in (0,1) \end{aligned} \tag{4.3.41}$$

The function

$$F(d, x_0) = x_0^{\alpha+2} G_\alpha(d) + hx_0 - v.$$

defines implicitly the constraints of the problem. Existence of a minimizer together with the Lagrange Multipliers theorem imply that there exists $\mu \in \mathbb{R}$ such that $\nabla p(d, x_0) = \mu \nabla F(d, x_0)$, namely

$$\begin{cases} \partial_d p = \mu \partial_d F \\ \partial_{x_0} p = \mu \partial_{x_0} F \\ F(d, x_0) = 0 \end{cases} \iff \begin{cases} g_\alpha'(d) x_0^{\alpha+1} = \mu x_0^{\alpha+2} G_\alpha'(d) \\ (\alpha+1) x_0^\alpha g_\alpha(d) = \mu\big((\alpha+2) x_0^{\alpha+1} G_\alpha(d) + h\big) \\ x_0^{\alpha+2} G_\alpha(d) + hx_0 - v = 0 \end{cases} \tag{4.3.42}$$

Recalling the definitions of g_α and G_α in (4.3.37) and (4.3.41), we write the expressions for the derivatives

$$\begin{aligned} g_\alpha'(d) &= -\frac{\alpha+1}{d^{\alpha+2}} \int_0^{\arcsin d} \sin^\alpha \vartheta \, d\vartheta + \frac{1}{d\sqrt{1-d^2}} \\ G_\alpha'(d) &= -\frac{\alpha+2}{d^{\alpha+3}} \int_0^d \int_{\arcsin t}^{\arcsin d} \sin^{\alpha+1} \vartheta \, d\vartheta + \frac{1}{\sqrt{1-d^2}} \end{aligned}$$

We claim that when $\alpha = 1$ (or $\alpha = 0$) we have

$$g_\alpha'(d) = d G_\alpha'(d) \tag{4.3.43}$$

In fact, we have when $\alpha = 1$, $g_\alpha'(d) = -\frac{2}{d^3}[\sqrt{1-d^2} - 1] + \frac{1}{d\sqrt{1-d^2}} = \frac{2-d^2-2\sqrt{1-d^2}}{d^3\sqrt{1-d^2}}$, hence

$$\begin{aligned} G_\alpha'(d) &= -\frac{3}{d^4} \int_0^d \frac{1}{2} \Big[\vartheta - \frac{\sin(2\vartheta)}{2}\Big]_{\arcsin t}^{\arcsin d} dt + \frac{1}{\sqrt{1-d^2}} \\ &= \frac{3}{2d^4} \int_0^d \big(\arcsin t - t\sqrt{1-t^2} - \arcsin d + d\sqrt{1-d^2} \big) + \frac{1}{\sqrt{1-d^2}} \\ &= \frac{3}{2d^4} \Big\{ \Big[t \arcsin t + \sqrt{1-t^2} + \frac{1}{3}(1-t^2)^{3/2} \Big]_0^d - d \arcsin d + d^2\sqrt{1-d^2} \Big\} \\ &= \frac{3}{2d^4} \Big\{ d \arcsin d + \sqrt{1-d^2} + \frac{1}{3}(1-d^2)^{3/2} - 1 - \frac{1}{3} \\ &\quad - d \arcsin d + d^2 \sqrt{1-d^2} \Big\} \\ &= \frac{1}{d^4} \{ (2+d^2)\sqrt{1-d^2} - 2 \} + \sqrt{1}\sqrt{1-d^2} = \frac{2 - d^2 - 2\sqrt{1-d^2}}{d^4\sqrt{1-d^2}}. \end{aligned}$$

Going back to any $\alpha \geq 0$, we notice that (4.3.43) is equivalent to

$$\begin{aligned} 0 &= \frac{1}{d^{\alpha+2}} \Big\{ (\alpha+2) \int_0^d \int_{\arcsin t}^{\arcsin d} \sin^{\alpha+1} \vartheta \, d\vartheta \, dt - (\alpha+1) \int_0^{\arcsin d} \sin^\alpha \vartheta \, d\vartheta \Big\} + \frac{1-d^2}{d\sqrt{1-d^2}} \\ &= (\alpha+2) G_\alpha(d) - \frac{1}{d}(\alpha+1) g_\alpha(d) + \frac{1}{d}\sqrt{1-d^2} \end{aligned} \tag{4.3.44}$$

From (4.3.43), the first equation in system (4.3.42) gives

$$\mu = \frac{g'_\alpha(d)}{x_0 G'_\alpha(d)} = \frac{d}{x_0} = \frac{1}{\lambda}.$$

Plugging μ into the second equation of (4.3.42) we obtain

$$(\alpha+1)x_0^{\alpha+1} g_\alpha(d) = d\{(\alpha+2)x_0^{\alpha+1}G_\alpha(d) + h\},$$

hence, using (4.3.44),

$$x_0^{\alpha+1} = \frac{dh}{(\alpha+1)g_\alpha(d) - d(\alpha+2)G_\alpha(d)} = \frac{dh}{\sqrt{1-d^2}} \tag{4.3.45}$$

We are left to calculate $y = f(\lambda) = f(x_0/d)$ with $x_0 = x_0(h,d)$ given by (4.3.45) and d satisfying the last equation in (4.3.42), which reads

$$\Big(\frac{dh}{\sqrt{1-d^2}}\Big)^{\frac{\alpha+2}{\alpha+1}} G_\alpha(d) + h\Big(\frac{dh}{\sqrt{1-d^2}}\Big)^{\frac{1}{d^{\alpha+1}}} - v = 0. \tag{4.3.46}$$

Expression (4.3.40) for y, combined with the last equation for x_0 in (4.3.45)

$$\begin{aligned} y = f\Big(\frac{x_0}{d}\Big) &= h + x_0^{\alpha+1} b_\alpha(d) = h - \frac{dh}{\sqrt{1-d^2}} \frac{\varphi_\alpha(d)}{d^{\alpha+1}} \\ &= -\frac{h}{d^\alpha\sqrt{1-d^2}}\Big\{\varphi_\alpha(d) - d^\alpha\sqrt{1-d^2}\Big\} \end{aligned}$$

which concludes the proof. □

4.4 Estimates of the section-gap in terms of the α-isoperimetric deficit

In this Section, we show that the conclusive estimate of the techniques in [63] holds true for the α-perimeter in both stripes Z and Z_2, in the case when $\alpha = 1$. In this case we have

$$\mathcal{L}^2(E^\alpha_{\text{isop}}) = \frac{8}{3} \quad \text{and} \quad P_\alpha(E^\alpha_{\text{isop}}) = 4.$$

Proposition 4.4.1. *Let $\alpha = 1$. There exists $\beta_0 > 0$ such that for any $0 \le \beta \le \beta_0$, any minimizer $E \in \mathcal{A}_x(0, \omega_\alpha, 0, \beta)$ for Problem (4.3.3) as in Theorem 4.3.6, satisfies*

$$|v^x_E(0) - v^x_\alpha(0)| \le C\sqrt{D_\alpha(E)}$$

for some constant $C > 0$.

Proof. If E is a minimizer as in Theorem 4.3.6, then $E \cap \{x > 0\}$ and $E \cap \{x < 0\}$ are convex and, by Proposition 4.3.7, the profile function of E, $f : [0, r_0) \to [0, \infty)$ satisfies for some $k \le 0$ and $d \in \mathbb{R}$ the following ordinary differential equation:

$$f'(x) = \frac{(kx + d)x}{\sqrt{1 - (kx + d)^2}}, \quad 0 \le x < r_0.$$

We deduce that $|d| < 1$, by $|kx + d| < 1$ for $0 < x < r_0$. Moreover, as observed in Remark 4.3.25, $r_0 = -(1 + d)/k$. Notice that if $x = -d/k$, $f'(x) = 0$, hence $-d/k$ is a critical point for f and we can deduce for geometrical reasons that it is in fact a maximum point.

By the Representation formula for the α-perimeter, we have

$$\begin{aligned} P_\alpha(E)/4 &= \int_0^{r_0} \sqrt{f'^2 + x^2}\, dx = \int_0^{r_0} \frac{x}{\sqrt{1 - (kx + d)^2}}\, dx = \frac{1}{k^2} \int_{-1}^{d} \frac{d - z}{\sqrt{1 - z^2}}\, dz \\ &= \frac{1}{k^2}\Big[d \arcsin z + \sqrt{1 - z^2} \int \Big]_{-1}^{d} = \frac{1}{k^2}\Big[\sqrt{1 - d^2} + d\big(\arcsin d + \frac{\pi}{2}\big)\Big] \end{aligned} \tag{4.4.1}$$

Let $x \in [-d/k, r_0)$. Then, by Remark 4.3.8

$$\begin{aligned} f(x) = f(x) - f(r_0) &= -\int_x^{r_0} \frac{(kt + d)t}{\sqrt{1 - (kt + d)^2}}\, dt = \frac{1}{k^2} \int_{-1}^{kx+d} \frac{z^2 - d \cdot z}{\sqrt{1 - z^2}}\, dz \\ &= \frac{1}{k^2}\Big[\frac{1}{2}\Big(\arcsin z - z\sqrt{1 - z^2}\Big) + d\sqrt{1 - z^2}\Big]_{-1}^{kx+d} \\ &= \frac{1}{2k^2}\Big\{\frac{\pi}{2} + \arcsin(kx + d) + (d - kx)\sqrt{1 - (kx + d)^2}\Big\} \end{aligned}$$

Hence, we have

$$f(-d/k) = \frac{1}{2k^2}\Big(\frac{\pi}{2} + 2d\Big)$$

and, with a similar argument we deduce the same expression for f at $x \in [0, -d/k]$. Hence

$$f(x) = \frac{1}{2k^2}\Big\{\frac{\pi}{2} + \arcsin(kx + d) + (d - kx)\sqrt{1 - (kx + d)^2}\Big\}, \quad x \in [0, r_0). \tag{4.4.2}$$

In particular, there holds

$$\beta = f(0) = \frac{1}{2k^2}\Big\{\frac{\pi}{2} + \arcsin d + d\sqrt{1 - d^2}\Big\} \tag{4.4.3}$$

On the other hand, by (4.4.2)

$$\begin{aligned} \frac{2}{3} = \frac{\omega_\alpha}{4} &= \int_0^{r_0} f(x)\, dx = -\frac{1}{2k^3} \int_{-1}^{d} \Big(\frac{\pi}{2} + \arcsin z + 2d\sqrt{1 - z^2} - z\sqrt{1 - z^2}\Big)\, dz \\ &= -\frac{1}{2k^3}\Big\{\frac{\pi}{2}(d + 1) + \Big[z \arcsin z + \sqrt{1 - z^2} + d\big(\arcsin z + z\sqrt{1 - z^2}\big) + \frac{1}{3}(1 - z^2)^{3/2}\Big]\Big\} \\ &= -\frac{1}{k^3}\Big[d\Big(\arcsin d + \frac{\pi}{2}\Big) + \frac{1}{3}(2 + d^2)\sqrt{1 - d^2}\Big]. \end{aligned}$$

Hence, deducing k in terms of β and d in (4.4.3) and inserting it into the last equation, we deduce

$$F(\beta, d) = 0, \tag{4.4.4}$$

where $F : [0,\infty) \times [-1,1] \to \mathbb{R}$ is defined as

$$F(\beta,d) = \frac{2}{3}\Big(\frac{1}{2\beta}\Big\{\frac{\pi}{2} + \arcsin d + d\sqrt{1-d^2}\Big\}\Big)^{3/2} - d\Big(\frac{\pi}{2} + \arcsin d\Big) - \frac{2+d^2}{3}\sqrt{1-d^2}.$$

We let

$$\ell : [-1,1] \to \mathbb{R}, \quad \ell(d) = \frac{1}{2}\Big\{\frac{\pi}{2} + \arcsin d + d\sqrt{1-d^2}\Big\}$$

hence, we have

$$\ell'(d) = \sqrt{1-d^2} \quad \text{and} \quad \ell''(d) = -\frac{d}{\sqrt{1-d^2}}.$$

Notice that $F(\pi/4, 0) = 0$. We have

$$\frac{\partial}{\partial d}F(\beta,d) = \ell(d)^{\frac{1}{2}}\beta^{-3/2}\Big\{\ell'(d) - 2\beta^{3/2}\ell(d)^{1/2}\Big\}$$

and

$$\frac{\partial}{\partial \beta}F = -\ell(d)^{\frac{3}{2}}\beta^{-\frac{5}{2}}$$

In particular,

$$\frac{\partial}{\partial d}F(\pi/4,0) = \frac{4}{\pi}\frac{8-\pi^2}{8}, \quad \frac{\partial}{\partial \beta}F(\pi/4,0) = -\frac{4}{\pi}.$$

Therefore, by the implicit function theorem there exists a neighborhood $U_{\pi/4} \subset \mathbb{R}$ of $\pi/4$ such that d can be written as a function of β around $\pi/4$ with $d(\pi/4) = 0$ and

$$\dot{d}(\beta) = \frac{d}{d\beta}d(\beta) = -\frac{\partial_\beta F(\beta, d(\beta))}{\partial_d F(\beta, d(\beta))} = \frac{\ell(d)}{\beta}\Big\{\ell'(d) - 2\beta^{3/2}\ell(d)^{1/2}\Big\}^{-1}. \tag{4.4.5}$$

Moreover,

$$\begin{aligned}\ddot{d}(\beta) = &\frac{\ell'(d)\dot{d}(\beta)}{\beta\big\{\ell'(d) - 2\beta^{3/2}\ell(d)^{1/2}\big\}} - \frac{\ell(d)}{\beta^2\big\{\ell'(d) - 2\beta^{3/2}\ell(d)^{1/2}\big\}}\\ &- \frac{\ell(d)\big[\ell''(d)\dot{d} - 2\big(\frac{3}{2}\beta^{1/2}\ell(d)^{1/2} + \frac{1}{2}\beta^{3/2}\ell'(d)\dot{d}\ell(d)^{-1/2}\big)\big]}{\beta\{\ell'(d) - 2\beta^{3/2}\ell(d)^{1/2}\}^2}\end{aligned}$$

Now, we consider the functional Δ that associates to any $\beta \in U_{\pi/4}$ the α-isoperimetric deficit of a minimizer for (4.3.3) with trace $[-\beta,\beta]$ at $x_0 = 0$: from (4.4.1), since $P_\alpha(E^\alpha_{\text{isop}}) = 4$, we write

$$\Delta(\beta) = D_\alpha(E) = \frac{\beta\big(\sqrt{1-d(\beta)^2} + d\big(\arcsin d + \frac{\pi}{2}\big)\big)}{\frac{1}{2}\big\{\frac{\pi}{2} + \arcsin d + d\sqrt{1-d^2}\big\}} - 1.$$

The proof is concluded once we show that

$$\Delta(\pi/4) = \dot{\Delta}(\pi/4) = 0 \quad \text{and} \quad \ddot{\Delta}(\pi/4) > 0 \tag{4.4.6}$$

In fact, in this case

$$D_\alpha(E) = \Delta(\beta) = \ddot{\Delta}(\pi/4)(\beta - \pi/4)^2 + o\big((\beta-\pi/4)^2\big) \geq C\Big(v^x_E(0) - v^x_\alpha(0)\Big)^2, \quad \beta \in U_{\pi/4}.$$

To prove (4.4.6) we simply use the expression of Δ and the derivatives of d. We have

$$\Delta\Big(\frac{\pi}{4}\Big) = \frac{\frac{\pi}{4}}{\frac{1}{2}\cdot\frac{\pi}{2}} - 1 = 0$$

Moreover, a computation yields

$$\begin{aligned}\dot{\Delta}(\beta) &= \frac{1}{\ell(d)}\Big(\sqrt{1-d^2} + d(\arcsin d + \pi/2)\Big) + \frac{\beta\dot{d}}{\ell(d)}\Big(\arcsin d + \pi/2\Big)\\ &\quad - \frac{\beta\Big(\sqrt{1-d^2} + d(\arcsin d + \pi/2)\Big)}{\ell(d)^2}\ell'(d)\dot{d}\\ &= \frac{\sqrt{1-d^2} + (\arcsin d + \pi/2)(d + \dot{d}\beta)}{\ell(d)} - \frac{\dot{d}\ell'(d)}{\ell(d)^2}\beta\Big(\sqrt{1-d^2} + d(\arcsin d + \pi/2)\Big).\end{aligned}$$

Hence, since $\dot{d}(\pi/4) = 8/(8-\pi^2)$ (see (4.4.5)), we have

$$\dot{\Delta}\Big(\frac{\pi}{4}\Big) = \Big(1 + \frac{\pi}{2}\frac{8}{8-\pi^2}\frac{\pi}{4}\Big)\frac{4}{\pi} - \frac{16}{\pi^2}\frac{8}{8-\pi}\frac{\pi}{4} = \frac{4(8-\pi^2+\pi^2) - 32}{(8-\pi^2)\pi} = 0.$$

A computation done with the help of the software *Wolfram Mathematica* gives the values

$$\ddot{d}\Big(\frac{\pi}{4}\Big) = \frac{6\pi(16-\pi^2)}{(\pi^2-8)^2}, \quad \ddot{\Delta}\Big(\frac{\pi}{4}\Big) = \frac{-64 + 40\pi^2 - 3\pi^4}{(\pi^2-8)^2} \sim 11 > 0.$$

This completes the proof. □

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