

# NECKPINCH SINGULARITIES IN FRACTIONAL MEAN CURVATURE FLOWS

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ABSTRACT. In this paper we consider the evolution of boundaries of sets by a fractional mean curvature flow. We show that, for any dimension  $n \geq 2$ , there exist embedded hypersurfaces in  $\mathbb{R}^n$  which develop a singularity without shrinking to a point. Such examples are well known for the classical mean curvature flow for  $n \geq 3$ . Interestingly, when  $n = 2$ , our result provides instead a counterexample in the nonlocal framework to the well known Grayson's Theorem [18], which states that any smooth embedded curve in the plane evolving by (classical) MCF shrinks to a point. The essential step in our construction is an estimate which ensures that a suitably small perturbation of a thin strip has positive fractional curvature at every boundary point.

## 1. INTRODUCTION

This paper is concerned with the study of a *nonlocal mean curvature flow*. More precisely, we want to study the evolution  $E_t$ , for time  $t > 0$  of an initial set  $E_0$ , such that the velocity of a point  $x \in \partial E_t$  in the outer normal direction  $\nu$  is given by the quantity  $-H_E^s$ , where  $H_E^s$  denotes the fractional mean curvature of  $E$ , that is, for any  $x \in \partial E_t$  we have

$$\partial_t x \cdot \nu = -H_{E_t}^s. \quad (1)$$

For a real parameter  $s \in (0, 1)$ , we recall that the fractional mean curvature of a set  $E$  at a point  $x \in \partial E$  is defined as follows

$$H_E^s(x) := \int_{\mathbb{R}^n} \frac{\chi_{\mathcal{C}E}(y) - \chi_E(y)}{|x - y|^{n+s}} dy, \quad (2)$$

where  $\chi_A$  denotes the characteristic function of the set  $A$ ,  $\mathcal{C}A$  denotes the complement of  $A$ , and the integral above has to be understood in the principal value sense.

This evolution is the natural analogue in the nonlocal setting of the classical mean curvature flow, which has been widely studied in the last decades, see e.g. [14, 22]. While the classical mean curvature flow is the  $L^2$ -gradient flow of the usual perimeter functional, it can be proved that (1) is the  $L^2$ -gradient flow of the *fractional perimeter*, which was first introduced in [5] on the basis of motivations coming from interfaces in physical models and probabilistic processes. In the same paper, suitable density estimates, a monotonicity formula, and some regularity results for minimizers were established. The question of the regularity for minimizers of the fractional perimeter was also addressed in several recent

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works, see [3, 12, 7, 24]. Further motivations for the study of (1) come from dislocation dynamics and phase-field theory for fractional reaction-diffusion equations, see [21].

The classical mean curvature flow is a quasilinear parabolic problem and a local existence result holds for smooth solutions starting from any compact regular initial surface, see [16, 22]. As time evolves, solutions typically develop singularities due to curvature blowup. Several notions of generalized solutions have been introduced to study the flow after the onset of singularities. Particularly relevant for our purposes are the definitions by Chen, Giga, Goto [11] and by Evans and Spruck [15], based on the level set approach and the notion of viscosity solutions.

For the fractional mean curvature flow, local smooth solutions are also expected to exist, but no proof of this property is available yet. There are existence and uniqueness results for viscosity solutions, first obtained by Imbert [21], and later extended to more general nonlocal flows by Chambolle, Morini, Ponsiglione in [9, 8] and by Chambolle, Novaga, Ruffini [10].

The formation of singularities has been widely studied in the case of the *classical* mean curvature flow. A pioneering result in this framework was obtained by Huisken [19], who showed that a closed convex surface in  $\mathbb{R}^n$ , with  $n > 2$ , remains convex along the evolution and shrinks to a point in finite time. If convexity is dropped, then other kinds of singular behaviour may occur. A standard example is the so called *neckpinch*. The idea is to consider a surface which looks like two large balls connected by a very thin cylindrical neck, so that, in dimension  $n > 2$ , the mean curvature in the neck is much larger than the one in the balls, hence the radius of the neck goes to zero faster than the radius of the balls. The existence of this type of surfaces was first proved by Grayson [17] and later considered with a simplified proof by Ecker [14], see also Angenent [2]. A similar construction for a flow driven by a different curvature function was done in [1].

When  $n = 2$ , a result analogous to [19] for convex curves was proved by Gage and Hamilton [16]. However, a stronger result holds in this dimension in the classical case. In fact, Grayson [18] showed that *any smooth closed embedded curve in the plane becomes convex in finite time under the flow*, and therefore, by [16], *shrinks smoothly to a point*. Thus, all other kinds of singularities are ruled out for embedded curves.

In this paper we construct examples of neckpinch singularities for the fractional mean curvature flow. More precisely, we obtain the following result:

**Theorem 1.** *Let  $n \geq 2$ . There exists an embedded hypersurface  $\mathcal{M}_0$  in  $\mathbb{R}^n$  such that the viscosity solution of the fractional mean curvature flow (1) starting from  $\mathcal{M}_0$  does not shrink to a point.*

We point out that, in contrast to the classical case, our construction can be made in any dimension, in particular for  $n = 2$ , showing that Grayson's theorem fails in the nonlocal case. It also shows that the distance comparison property for curves for the classical flow proved by Huisken [20] no longer holds in the nonlocal case. The heuristic reason for this is that, thanks to its nonlocality, the fractional mean curvature of a very thin neck, is strictly positive also in dimension 2: Indeed if we "sit" on the boundary of the neck we see much more complement of  $E$  than  $E$  itself. Our result has some analogies with the one of [4], where Delaunay-type periodic curves in the plane with constant fractional mean curvature are constructed, while in the classical case such objects exist only in dimension  $n \geq 3$ .

Our results deal with the viscosity solution of (1) because we lack a local existence result for smooth solutions. However, Theorem 1 also implies that, if the hypersurface we construct has a local smooth evolution, then it *develops singularities* before shrinking to a point.

A crucial step in the proof of Theorem 1 is provided by the following result, of independent interest: if a set  $E$  is contained in a strip and its boundary  $\partial E$  has sufficiently small classical curvatures, then the fractional mean curvature of  $E$  is positive everywhere. In particular, a thin set can have all negative classical principal curvatures at some point, but positive fractional mean curvature.

We conclude by recalling other recent contributions in the study of the fractional (and more general nonlocal) mean curvature flows. In [6], the convergence of a class of threshold dynamics approximations to moving fronts was established. In particular, threshold dynamics associated to fractional powers of the Laplacian of order  $s \in (0, 2)$  were considered: interestingly, when  $s \in [1, 2)$  the resulting interface moves by a (weighted) mean curvature flow, while when  $s \in (0, 1)$  it moves by a fractional mean curvature flow. In [10], the results contained in [6] have been extended to the anisotropic case, and it was proved that convexity is preserved as in the classical case. Finally, in [23] smooth solutions to the fractional mean curvature flow were studied and the evolution equations for several geometric local and nonlocal quantities were computed. The particular cases of entire graphs and star-shaped surfaces were considered, obtaining striking analogies with the properties of the classical case.

The paper is organized as follows:

- In Section 2 we describe the level set approach in the study of nonlocal mean curvature flows, we recall the notion of viscosity solutions and the statement of the comparison principle;
- In Section 3 we establish the estimates on the nonlocal mean curvature of thin sets, which will be used in the proof of our main result;
- In Section 4, we prove our main result Theorem 1.

## 2. VISCOSITY SOLUTIONS AND COMPARISON PRINCIPLES VIA THE LEVEL SET APPROACH

In this Section we recall the notion of viscosity solutions for the fractional mean curvature flow, which is based on the *level sets approach*. The idea is the following: given an initial surface  $\mathcal{M}_0 = \partial E_0$ , we choose any continuous function  $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

$$\mathcal{M}_0 = \{x \in \mathbb{R}^n : u_0(x) = 0\}. \quad (3)$$

The geometric equation satisfied by the evolution  $\mathcal{M}_t$  of  $\mathcal{M}_0$  can then be translated into an equation satisfied by a function  $u(x, t)$ , where  $u(x, 0) = u_0(x)$  and at each time

$$\mathcal{M}_t = \{x \in \mathbb{R}^n : u(\cdot, t) = 0\}. \quad (4)$$

More precisely, the *level set equation* satisfied by  $u$  is

$$\partial_t u + H^s[x, u(\cdot, t)]|Du(x, t)| = 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty), \quad (5)$$

where  $u$  satisfies the initial condition

$$u(x, t) = u_0(x) \quad \text{in } \mathbb{R}^n.$$

Here and in the following we denote by  $H_s[x, u(\cdot, t)]$  the fractional mean curvature of the superlevel set of  $u(\cdot, t)$  at the point  $x$ , i.e.

$$H^s[x, u(\cdot, t)] = H_{\{y \in \mathbb{R}^n : u(y, t) > u(x, t)\}}^s(x).$$

Of course the definition of  $\mathcal{M}_t$  is well posed if one shows that equation (5) has a unique solution, and definition (3) does not depend on the initial choice of the function  $u_0$ . These basic properties were established in [15] for the classical mean curvature flow and in [21] for the fractional one.

Since the nonlocal case is not so standard and for the sake of completeness, we recall here below all the rigorous definitions and the basic results in [21] (see also [9]). Let  $\mathcal{M} = \{x \in \mathbb{R}^n : u(x) = 0\} = \partial\{x \in \mathbb{R}^n : u(x) > 0\}$ . If  $u \in C^{1,1}$  and  $Du \neq 0$ , we can define the following quantities

$$\begin{aligned} k^*[x, \mathcal{M}] &= k^*[x, u] = \int_{\mathbb{R}^n} \frac{\chi_{\{u(x+z) \geq u(x)\}}(z) - \chi_{\{u(x+z) < u(x)\}}(z)}{|z|^{n+s}} dz, \\ k_*[x, \mathcal{M}] &= k_*[x, u] = \int_{\mathbb{R}^n} \frac{\chi_{\{u(x+z) > u(x)\}}(z) - \chi_{\{u(x+z) \leq u(x)\}}(z)}{|z|^{n+s}} dz. \end{aligned} \tag{6}$$

It is easy to see that if  $u \in C^{1,1}$  and its gradient  $Du$  does not vanish on  $\{z \in \mathbb{R}^n : u(z) = u(x)\}$ , then  $k^*$  are finite and

$$k^*[x, u] = k_*[x, u] = -H_s[x, u].$$

We can now give the definition of *viscosity solution* for (5) (see [21], Sec. 3).

**Definition 2.** i) *An upper semicontinuous function  $u : [0, T] \times \mathbb{R}^n$  is a viscosity subsolution of (5) if for every smooth test function  $\phi$  such that  $u - \phi$  admits a global zero maximum at  $(t, x)$ , we have*

$$\partial_t \phi \leq k^*[x, \phi(\cdot, t)] |D\phi|(x, t) \tag{7}$$

*if  $D\phi(x, t) \neq 0$ , and  $\partial_t \phi(x, t) \leq 0$  if not.*

ii) *A lower semicontinuous function  $u : [0, T] \times \mathbb{R}^n$  is a viscosity supersolution of (5) if for every smooth test function  $\phi$  such that  $u - \phi$  admits a global zero minimum at  $(t, x)$ , we have*

$$\partial_t \phi \geq k_*[x, \phi(\cdot, t)] |D\phi|(x, t) \tag{8}$$

*if  $D\phi(x, t) \neq 0$ , and  $\partial_t \phi(x, t) \geq 0$  if not.*

iii) *A locally bounded function  $u$  is a viscosity solution of (5) if its upper semicontinuous envelope is a subsolution and its lower semicontinuous envelope is a supersolution of (5).*

**Remark 3.** It is easy to verify that any classical subsolution (respectively supersolution) is in particular a viscosity subsolution (respectively supersolution).

We can now state the comparison principles, that we will use later on in Section 4.

**Proposition 4** (Theorem 2 in [21]). *Suppose that the initial datum  $u_0$  is a bounded and Lipschitz continuous function. Let  $u$  (respectively  $v$ ) be a bounded viscosity subsolution (respectively supersolution) of (5).*

*If  $u(x, 0) \leq u_0(x) \leq v(x, 0)$ , then  $u \leq v$  on  $\mathbb{R}^n \times (0, +\infty)$ .*

In Theorem 3 of [21] existence and uniqueness of viscosity solutions of (5) were proven. The prove of existence uses Perron's method, while uniqueness relies on the comparison principle stated in Proposition 4. Finally in [21], Theorem 6, the consistency of Definition (4) is established, showing that if  $u$  and  $v$  are two viscosity solutions of (5) with two different initial data  $u_0$  and  $v_0$  which have the same zero-level set, then for every time  $t > 0$  also  $u(\cdot, t)$  and  $v(\cdot, t)$  have the same zero level set.

The uniqueness and the consistency results allow to define the fractional mean curvature flow for  $\mathcal{M}_t$  by using the solution  $u(\cdot, t)$  of (5).

### 3. THE NONLOCAL CURVATURE OF THIN SETS

The goal of this section is to give estimates on the nonlocal curvature of sets with small classical curvatures which are contained in a strip. The property is rather general, but we focus on a particular case for the sake of concreteness:

**Proposition 5.** *Let  $\kappa > 0$ . Let  $E_-, E_+ \subset \mathbb{R}^n$  be connected sets. Assume that  $E_+ \cap E_- = \emptyset$ , that*

$$E_- \supseteq \{x_n \leq -1\} \quad \text{and} \quad E_+ \supseteq \{x_n \geq 1\}.$$

*Suppose also that the boundaries of  $E_-$  and  $E_+$  are of class  $C^2$ , with classical directional curvatures bounded in absolute value by  $\kappa$ .*

*Let  $E := \mathbb{R}^n \setminus (E_- \cup E_+)$ . Then, there exist  $c_0$  and  $\kappa_0 > 0$ , depending on  $n, s$  and the  $C^2$  bounds on  $\partial E_-$  and  $\partial E_+$ , such that for any  $x \in \partial E$*

$$H_E^s(x) \geq c_0,$$

*provided that  $\kappa \in [0, \kappa_0]$ .*

*Proof.* As a first step, we show that, if  $\kappa_0$  is small enough, then the normal vector at each boundary point of  $E_+$  and  $E_-$  is close to being vertical and therefore the boundary can be represented as a global graph on  $\mathbb{R}^{n-1}$ .

To see this, let us denote by  $e_n$  the unit vector in the  $x_n$  direction. For any  $x \in \partial E_-$ , we denote by  $\nu(x)$  the outer normal to  $E_-$  at  $x$  and by  $e_n^T(x) = e_n - \langle \nu, e_n \rangle e_n$  be the tangential component of  $e_n$  at  $x$ . Let us fix an arbitrary  $\bar{x} \in \partial E_-$  where the normal is far enough from being vertical, say

$$|\langle \nu(\bar{x}), e_n \rangle| \leq \frac{1}{2}. \tag{9}$$

We then consider the curve  $\gamma(s)$  on  $\partial E_-$  which solves the o.d.e.

$$\gamma' = \frac{e_n^T(\gamma)}{|e_n^T(\gamma)|}, \quad \gamma(0) = \bar{x}.$$

The curve  $\gamma$  has unit speed and is defined as long as  $e_n^T(\gamma) \neq 0$ . We can now estimate the maximal rate of change of the vertical component of  $\nu$  along  $\gamma$ . Let us denote by  $\alpha_x(u, v)$  the second fundamental form at a point  $x \in \partial E_-$  evaluated at two tangent vectors  $u, v$ . Our assumption on the curvature implies that  $|\alpha_x(u, v)| \leq \kappa|u||v|$ . Since

$$\frac{d}{ds} \langle \nu(\gamma(s)), e_n \rangle = \alpha_{\gamma(s)}(\gamma'(s), e_n^T),$$

we obtain

$$\left| \frac{d}{ds} \langle \nu(\gamma(s)), e_n \rangle \right| \leq \kappa |\gamma'(s)| |e_n^T| = \kappa \sqrt{1 - \langle \nu(\gamma(s)), e_n \rangle^2},$$

that is

$$-\kappa \leq \frac{d}{ds} \arcsin \langle \nu(\gamma(s)), e_n \rangle \leq \kappa.$$

Observe that  $\gamma$  is well defined as long as  $e_n^T \neq 0$ , that is,  $|\arcsin \langle \nu, e_n \rangle| < \pi/2$ . Taking into account (9), we obtain

$$|\arcsin \langle \nu(\gamma(s)), e_n \rangle| \leq \frac{\pi}{6} + \kappa s$$

for all  $s$  where  $\gamma$  is defined. Therefore  $\gamma(s)$  is defined at least for  $s \in [0, \pi/(6\kappa)]$  and satisfies

$$|\langle \nu(\gamma(s)), e_n \rangle| \leq \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad s \in \left[0, \frac{\pi}{6\kappa}\right].$$

Let us denote by  $\gamma_n$  the  $n$ -th component of  $\gamma$ . We can estimate

$$\gamma'_n(s) = \langle \gamma'(s), e_n \rangle = |e_n^T(\gamma(s))| = \sqrt{1 - \langle \nu(\gamma(s)), e_n \rangle^2} \geq \frac{1}{2}, \quad s \in \left[0, \frac{\pi}{6\kappa}\right],$$

which implies

$$\gamma_n \left( \frac{\pi}{6\kappa} \right) - \gamma_n(0) \geq \frac{\pi}{12\kappa}.$$

Since  $\gamma(s) \in \partial E_-$  is confined in the strip with  $|x_n| \leq 1$ , the right hand side cannot be larger than 2. This gives a contradiction if  $\kappa < \pi/24$ , showing that (9) cannot be satisfied. By the arbitrariness of  $\bar{x}$ , we obtain that  $|\langle \nu(\bar{x}), e_n \rangle| > 1/2$  at any boundary point of  $\partial E_-$ . In particular,  $\partial E_-$  cannot have point where the tangent plane is vertical. The assumptions on  $E_-$  then easily imply that  $\partial E_-$  can be written as the graph of a  $C^2$  function on  $f_- : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ . The property  $|\langle \nu, e_n \rangle| > 1/2$  gives a bound on the Lipschitz constant of  $f_-$ . Writing the curvatures of  $\partial E_-$  in terms of the derivatives of  $f_-$ , it is easy to find that  $\|D^2 f_-\|_\infty \leq C\kappa$  for some constant  $C = C(n)$ . In a similar way, we obtain that  $\partial E_+$  is the graph of a function  $f_+ : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  with analogous properties, such that  $-1 \leq f_-(x') < f_+(x') \leq 1$  for all  $x' \in \mathbb{R}^n$ .

After establishing these properties, we can prove our statement by contradiction. If the desired result were false, there would exist a sequence of pairs of functions  $f_\pm^{(j)} : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ , such that

$$-1 \leq f_-^{(j)}(x') < f_+^{(j)}(x') \leq 1 \text{ for all } x' \in \mathbb{R}^n, \quad (10)$$

$$\|Df_\pm^{(j)}\|_\infty \leq C, \quad \|D^2 f_\pm^{(j)}\|_\infty \rightarrow 0 \text{ as } j \rightarrow \infty \quad (11)$$

and such that, if  $E^{(j)} = \{x = (x', x_n) \in \mathbb{R}^n : f_-^{(j)}(x') \leq x_n \leq f_+^{(j)}(x')\}$  then there exists  $x^{(j)} \in \partial E^{(j)}$  with

$$\lim_{j \rightarrow +\infty} H_{E^{(j)}}^s(x^{(j)}) \leq 0. \quad (12)$$

Since these properties are invariant with respect to translation in the first  $(n-1)$  components, we can further assume that the point  $x^{(j)}$  lies over the origin in  $\mathbb{R}^{n-1}$ , that is, it has the form  $x^{(j)} := (0, \dots, 0, x_n^{(j)})$ . Since  $|x_n^{(j)}| \leq 1$ , up to a subsequence, we suppose that  $x^{(j)} \rightarrow \bar{x} = (0, \dots, 0, \bar{x}_n)$  as  $j \rightarrow +\infty$  for a suitable  $\bar{x}_n \in [-1, 1]$ .

The estimates in (10), (11) show that both sequences  $f_-^{(j)}, f_+^{(j)}$  are bounded in  $C^2$  and their second derivatives tend uniformly to zero. Then, up to a subsequence, they converge

in  $C_{loc}^2(\mathbb{R}^{n-1})$  to some limit functions  $\bar{f}_- \leq \bar{f}_+$ . Let us set  $\bar{E} = \{x = (x', x_n) \in \mathbb{R}^n : \bar{f}_-(x') \leq x_n \leq \bar{f}_+(x')\}$ . Then we have  $\bar{x} \in \partial\bar{E}$  and, by known results about the continuity of the fractional curvature with respect to smooth convergence (see e.g. Theorem 1.1 in [13]), we have that

$$\lim_{j \rightarrow +\infty} H_{E^{(j)}}^s(x^{(j)}) = H_{\bar{E}}^s(\bar{x}).$$

Comparing with (12), we thus conclude that

$$H_{\bar{E}}^s(\bar{x}) \leq 0. \quad (13)$$

On the other hand, by (10) and (11), we see that the limit functions  $f_{\pm}$  are bounded and linear, since their second derivatives vanish identically. Hence, they are constant and we obtain that  $\bar{E}$  is a slab of the form  $\{x_n \in (a, b)\}$  with  $-1 \leq a \leq b \leq 1$ . However, it is well known, and it follows easily from the definition of nonlocal curvature, that such a slab satisfies  $H_{\bar{E}}^s > 0$  at all boundary points, in contradiction with (13).  $\square$

**Corollary 6.** *Let  $\epsilon, \delta > 0$  and*

$$E := \left\{ x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } |x_n| < \epsilon + \frac{1}{\pi} \arctan(\delta |x'|^2) \right\}.$$

*Then, if  $\epsilon$  and  $\delta$  are sufficiently small, we have that*

$$\inf_{x \in \partial E} H_E^s(x) \geq c_0 > 0,$$

*for some  $c_0$  depending only on  $s$  and  $n$ .*

*Proof.* We define

$$E_- := \left\{ x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } x_n \leq -\epsilon - \frac{1}{\pi} \arctan(\delta |x'|^2) \right\}$$

and

$$E_+ := \left\{ x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } x_n \geq \epsilon + \frac{1}{\pi} \arctan(\delta |x'|^2) \right\}.$$

Notice that the boundaries of  $E_{\pm}$  are contained in the strip  $|x_n| \leq 1$  for  $\epsilon \leq 1/2$ , and the curvatures are of size  $O(\delta^2)$ . Thus, we are in the position of exploiting Proposition 5, from which we obtain the desired result.  $\square$

#### 4. NECKPINCH

In this section we prove Theorem 1. More precisely, we provide an example of surface evolving by fractional mean curvature flow, which develops a singularity before it can shrink to a point. To this purpose, we recall a property proved in [23].

**Lemma 7** (Lemma 2 and Corollary 3 in [23]). *Given  $s \in (0, 1)$  and an integer  $n \geq 2$ , there exists  $\bar{\omega} = \bar{\omega}(s, n) > 0$  such that the fractional mean curvature of a ball of radius  $R$  in  $\mathbb{R}^n$  is given by*

$$H_{B_R}^s(x) = \bar{\omega} R^{-s}$$

*for any  $x \in \partial B_R(0)$ . Moreover, if we set  $R(t) := (R_0^{s+1} - (\bar{\omega}(1+s))t)^{\frac{1}{s+1}}$ , then  $B_{R(t)}$  is a solution to the fractional mean curvature flow starting from  $B_{R_0}$  and it collapses to a*

point in the finite time

$$T_{B_{R_0}} = \frac{R_0^{s+1}}{\bar{\omega}(s+1)}. \quad (14)$$

Observe that, while for the classical mean curvature flow, the extinction time of a sphere of radius  $R_0$  is proportional to  $R_0^2$ , in the fractional case it is proportional to  $R_0^{s+1}$ .

We can now give the proof of our main result.

*Proof of Theorem 1.* We consider now the set  $E_\epsilon$  defined in Corollary 6:

$$E_\epsilon := \left\{ x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } |x_n| < \epsilon + \frac{2}{\pi} \arctan(\delta |x'|^2) \right\}.$$

We know that there exists  $\bar{\epsilon}$  and  $\bar{\delta}$  positive such that, for any  $0 < \epsilon \leq \bar{\epsilon}$  and  $0 < \delta \leq \bar{\delta}$

$$\inf_{x \in \partial E_\epsilon} H_{E_\epsilon}^s(x) \geq c_0 > 0, \quad (15)$$

for some  $c_0$  depending only on  $n$  and  $s$ .

Let now  $\kappa$  and  $\epsilon_0$  be two positive parameters satisfying

$$\kappa < c_0 \quad \text{and} \quad \epsilon_0 < \min \left\{ \bar{\epsilon}, \frac{1}{4} \kappa T_{B_1} \right\}, \quad (16)$$

where  $T_{B_1}$  is the extinction time of the ball of radius 1 given in (14).

The idea is to consider the set  $E_{\epsilon_0}$  and to make it evolve with constant velocity  $\kappa$  in the inner vertical direction. More precisely, we set

$$\epsilon(t) := \epsilon_0 - \kappa t,$$

and, for any  $t$ , we consider the set

$$E_{\epsilon(t)} := \left\{ x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } |x_n| < \epsilon(t) + \frac{2}{\pi} \arctan(\delta |x'|^2) \right\}. \quad (17)$$

Hence, we have that any point  $x \in \partial E_{\epsilon(t)}$  satisfies

$$\partial_t x \cdot \nu = V \cdot \nu,$$

where

$$V = \begin{cases} -\kappa e_n & \text{if } x_n > 0 \\ \kappa e_n & \text{if } x_n < 0. \end{cases}$$

Thus,

$$\partial_t x \cdot \nu \geq -\kappa > -c_0 \geq -H_{E_{\epsilon(t)}}^s,$$

where in the last inequality we have used (15) and the fact that  $E_{\epsilon(t)} \subset E_{\epsilon_0}$  for any  $t > 0$ . Therefore, the set  $E_{\epsilon(t)}$  is a smooth supersolution (hence also a viscosity supersolution) to (1).

By the definition of the set  $E_{\epsilon_0}$  we have that the infimum distance between the two disconnected components of its boundary  $\{(x', x_n) \in \mathbb{R}^n \text{ s.t. } x_n = \epsilon_0 + \arctan(\delta |x'|^2)\}$  and  $\{(x', x_n) \in \mathbb{R}^n \text{ s.t. } x_n = -\epsilon_0 - \arctan(\delta |x'|^2)\}$  is attained at the points  $(0, \dots, 0, \epsilon_0)$



and  $(0, \dots, 0, -\epsilon_0)$ . Since  $E_{\epsilon(t)}$  evolves with constant negative velocity  $\kappa$  along the inner vertical direction, we deduce that the singular time for  $E_{\epsilon(t)}$  is given by

$$T_{E_{\epsilon(t)}} = \frac{2\epsilon_0}{\kappa}. \quad (18)$$

Let now consider any closed set  $A_0$  with the following properties:

- (1)  $A_0$  is rotationally symmetric around the  $x_1$  axis;
- (2)  $A_0$  is symmetric with respect to the  $x_1 = 0$  hyperplane;
- (3)  $A_0$  is contained in  $E_{\epsilon_0}$ ;
- (4)  $A_0$  contains two balls  $B_1^-$  and  $B_1^+$  of radius 1 centered at  $(-L, 0, \dots, 0)$  and  $(L, 0, \dots, 0)$  respectively, where  $L$  is chosen large enough so that (1) and (2) are both satisfied.

We consider now the fractional mean curvature flow  $A_t$  starting from  $A_0$ . By uniqueness,  $A_t$  retains the symmetries of  $A_0$ . By the comparison principle (Proposition 4),  $A_t$  must be contained in  $E_{\epsilon(t)}$ . Moreover it must contain the evolutions  $B_{1,t}^-$  and  $B_{1,t}^+$  of the two balls  $B_1^-$  and  $B_1^+$ .

On the one hand, since  $A_t$  is contained in  $E_{\epsilon(t)}$ , using (18) and the choice of  $\epsilon_0$  (16), we deduce that at any time  $t > T_A$ , where

$$T_A = \frac{2\epsilon_0}{\kappa} \leq \frac{1}{2}T_{B_1},$$

the  $x_1 = 0$  cross section of  $A_t$  is empty.

On the other hand, by assumption (2), at the same time,  $A_t$  contains two balls with positive radius in the  $x_1 > 0$  and  $x_1 < 0$  half-spaces respectively. This shows that, at some time smaller than  $T_A$ , the set  $A_t$  splits into two symmetric disconnected components, hence it cannot shrink to a point. This concludes the proof of Theorem 1.  $\square$

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