On the singular part of measures constrained by linear PDEs and applications

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Abstract. The aim of this note is to present some recent results on the structure of the singular part of measures satisfying a PDE constraint, and to describe some applications.

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1. Introduction

In this note we describe some recent advances obtained by the author in collaboration with Filip Rindler concerning the structure of singularities of measures satisfying a (linear) PDE constraint. Besides its own theoretical interest, understanding the structure of singularities of PDE constrained measures turns out to have several (sometimes surprising) applications in the Calculus of Variations and in Geometric Measure Theory. Some of these applications will be described in Sections 2 and 3 below.

Let us consider the following problem: Let $\mathscr{A}$ be a k'th-order linear constant-coefficient PDE operator acting on $\mathbb{R}^m$-valued functions:

$$\mathscr{A}u = \sum_{|\alpha|\le k} A_\alpha \partial^\alpha u \qquad \text{for all } u \in C^\infty(\Omega;\mathbb{R}^m)$$

where $A_\alpha \in \mathbb{R}^{n\times m}$ are matrices and $\partial^\alpha = \partial_1^{\alpha_1}\dots\partial_d^{\alpha_d}$, for every multindex $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d$.

Question 1. *Let $\mu \in \mathcal{M}(\Omega,\mathbb{R}^m)$ be a $\mathbb{R}^m$-valued Radon measure on an open set $\Omega \subset \mathbb{R}^d$ and let us assume that μ is $\mathscr{A}$-free, i.e. that it soves the following system of linear PDE in the sense of distribution:*

$$\mathscr{A}\mu = \sum_{|\alpha|\le k} A_\alpha \partial^\alpha \mu = 0 \qquad \textit{in } \mathcal{D}'(\Omega;\mathbb{R}^n). \tag{1}$$

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What can be said about the singular part [1] of μ?

In answering the above question a prominent role is played by the *wave cone* associated with the differential operator $\mathscr{A}$:

$$\Lambda_{\mathscr{A}} = \bigcup_{|\xi|=1} \operatorname{Ker} \mathbb{A}^k(\xi) \subset \mathbb{R}^m \qquad \text{with} \qquad \mathbb{A}^k(\xi) = (2\pi \mathrm{i})^k \sum_{|\alpha|=k} A_\alpha \xi^\alpha,$$

and we have set $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$.

Roughly speaking, $\Lambda_{\mathscr{A}}$ contains all the amplitudes along which the system (1) is *not elliptic*. Indeed if we assume that $\mathscr{A}$ is homogeneous, $\mathscr{A} = \sum_{|\alpha|=k} A_\alpha \partial^\alpha$, then it is immediate to check that $\lambda \in \mathbb{R}^m$ belongs to $\Lambda_{\mathscr{A}}$ if and only if there exists a non zero $\xi \in \mathbb{R}^d \setminus \{0\}$ such that $\lambda h(x \cdot \xi)$ is $\mathscr{A}$-free for all $h : \mathbb{R} \to \mathbb{R}$. In other words "one dimensional" oscillations and concentrations are possible only if the amplitudes belongs to the wave cone.

For these reasons the wave cone plays a crucial role in the compensated compactness theory for sequences of $\mathscr{A}$-free maps, [25, 35, 36, 47, 48] and in convex integration, see for instance [17–21, 46] and the references therein.

Since the singular part of a measure can be thought as containing "condensed" concentrations, it is quite natural to conjecture that for $|\mu|^s$-almost everywhere the polar vector $d\mu/d|\mu|$ shall belong to $\Lambda_{\mathscr{A}}$. This is indeed the main result of [23]:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set, let $\mathscr{A}$ be a k'th-order linear constant-coefficient differential operator as above, and let $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$ be an $\mathscr{A}$-free Radon measure on Ω with values in $\mathbb{R}^m$. Then,*

$$\frac{d\mu}{d|\mu|}(x) \in \Lambda_{\mathscr{A}} \qquad \text{for } |\mu|^s\text{-a.e. } x \in \Omega.$$

Remark 1.2. Let us point out that Theorem 1.1 is also valid in the situation

$$\mathscr{A}\mu = \sigma \quad \text{for some} \quad \sigma \in \mathcal{M}(\Omega; \mathbb{R}^n). \tag{2}$$

This can be reduced to the setting of Theorem 1.1 by defining $\tilde{\mu} = (\mu, \sigma) \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^{m+n})$ and $\tilde{\mathscr{A}}$ (with an additional 0'th-order term) such that (2) is equivalent to $\tilde{\mathscr{A}}\tilde{\mu} = 0$. It is easy to check that, if $k \geq 1$, $\Lambda_{\tilde{\mathscr{A}}} = \Lambda_{\mathscr{A}} \times \mathbb{R}^n$ and that for $|\mu|$-almost every point $d\mu/d|\mu|$ is proportional to $d\mu/d|\tilde{\mu}|$.

In the next two Sections we will describe some applications of Theorem 1.1 to the following two problems:

- The description of the singular part of derivatives of BV and BD functions.
- The study of the sharpness of Rademacher Theorem.

Eventually in Section 4 we will sketch the proof of Theorem 1.1 .

[1] If not specified, the terms singular and absolutely continuous always refer to the Lebesgue measure. We also recall that thanks to the Radon-Nikodym Theorem, a vector valued measure can be written as

$$\mu = \frac{d\mu}{d|\mu|} d|\mu| = g d\mathcal{L}^d + \frac{d\mu}{d|\mu|} d|\mu|^s$$

where $|\mu|$ is the total variation measure, $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\mathcal{L}^d$ is the Lebesgue measure.

2. Linear growth variational problem and the structure of singular derivatives

Let $W : \mathbb{R}^{\ell\times d} \to \mathbb{R}_+$ be a linear growth integrand, $W(A) \gtrsim |A|$ for $|A|$ large, and let us consider the following variational problem:

$$\min\Big\{ \int_\Omega W(Du) \quad u \in C^1(\Omega;\mathbb{R}^\ell)\text{+boundary conditions}\Big\}.$$

It is well known that in order to apply the direct methods of the Calculus of Variations one has to relax the above problem in a setting where it is possible to obtain both compactness of minimising sequences and lower semicontinuity of the functional. Due to the linear growth of the integrand the only easily available estimate on a minimising sequence $\{u_k\}$ is an a-priori bound on the L^1 norm of their derivative:

$$\sup_k \int_\Omega |Du_k| < +\infty.$$

It is then quite natural to relax the functional to the space $BV(\Omega,\mathbb{R}^\ell)$ of functions of bounded variations, i.e. those functions $u \in L^1(\Omega)$ whose distributional gradient is a matrix-valued Radon measure. A fine understanding of the possible behaviour of measures arising as gradients is then fundamental for instance to study the lower semicontinuity of the functional as well as its relaxation to the space BV.

In this respect in [10] Ambrosio and De Giorgi proposed the following conjecture:

Is the singular part of the derivative $D^s u$ of a function $u \in BV(\Omega;\mathbb{R}^\ell)$ of bounded variation always of rank one? Namely is it true that

$$\frac{dD^s u}{d|Du|}(x) = a(x) \otimes b(x) \qquad \textit{for } |D^s u|\textit{-a.e. } x \text{ ?}$$

Their conjecture was motivated by the fact that this structure was known for a part of the singular derivative, the so called jump part, see [11, Chapter 3] for a complete reference concerning functions of bounded variations.

A positive answer to the above question was given by Alberti in [1] with his celebrated *rank-one theorem*. Recently a very short proof Alberti rank one Theorem has been given by Massaccesi and Vittone in [34]. At the the end of this Section we will see that Alberti rank one Theorem is a straightforward consequence of Theorem 1.1.

Besides its theoretical interest, the rank-one theorem has many applications in the theory of BV functions: lower-semicontinuity and relaxation [9, 26, 33], integral representation theorems [15], Young measure theory [32,44], and the study of continuity equations with BV-vector fields [7]. We refer to [11, Chapter 5] for further history.

In problems in elasticity and plasticity one often needs to consider a larger space of functions with respect to the space of bounded variations. Indeed in this setting energies usually only depend on the *symmetric part* of the gradient and one has to consider the following type of variational problem:

$$\min\Big\{\int_\Omega W(Eu) \quad u\in C^1(\Omega;\mathbb{R}^d)\text{+boundary conditions}\Big\}.$$

where $Eu=(Du+Du^T)/2$ is the symmetric gradient and $W:\mathbb{R}^{d\times d}_{\rm sym}\to\mathbb{R}_+$ is a linear growth integrand, $W(A)\gtrsim |A|$. In this case for a minimising sequence $\{u_k\}$, one can only obtain that

$$\sup_k\int_\Omega |Eu_k|<+\infty$$

and, due to the failure of Korn's inequality in L^1, [38], this is not enough to ensure that $\sup_k\int|Du_k|<+\infty$. One then introduces the space $BD(\Omega)$ of functions of *bounded deformation*, i.e. those functions $u\in L^1(\Omega;\mathbb{R}^d)$ such that Eu is a Radon measure, see [8, 49, 50]. Clearly $BV(\Omega,\mathbb{R}^d)\subset BD(\Omega)$ and the inclusion is strict. Note that as a consequence of Alberti rank one Theorem one has :

$$\frac{dE^s u}{d|Eu|}(x)=a(x)\odot b(x)\qquad \text{for all } u\in BV(\Omega;\mathbb{R}^d)$$

where $a\odot b=(a\otimes b+b\otimes a)/2$ is the symmetrised tensor product. One is then naturally led to the following conjecture

Question 2. *Is it true that for a function of bounded deformation $u\in BD(\Omega)$*

$$\frac{dE^s u}{d|Eu|}(x)=a(x)\odot b(x)\qquad \text{for } |E^s u|\text{-a.e. } x.$$

Again besides its theoretical interest a positive answer of the above question would have several applications to the study of lower semicontinuity and relaxation of functionals defined on BD, [13, 24] as well as in establishing the absence of a singular part for minimisers, see for instance [27, Remark 4.8]. Let us however mention that sometimes in the study of lower semicontinuity it is sometimes possible to avoid the use of this fine result, see [42, 43].

Let us conclude this Section by showing how both a positive answer to Question 2 and a new proof of Alberti rank-one Theorem can be easily obtained by applying Theorem 1.1 to suitable differential operators.

Theorem 2.1. *Let $\Omega\subset\mathbb{R}^d$ be an open set, then:*

(i) *For every $u\in BV(\Omega;\mathbb{R}^\ell)$*

$$\frac{dD^s u}{d|Du|}(x)=a(x)\otimes b(x)\qquad \text{for } |D^s u|\text{-a.e. } x.$$

(ii) *For every $u\in BD(\Omega)$*

$$\frac{dE^s u}{d|Eu|}(x)=a(x)\odot b(x)\qquad \text{for } |E^s u|\text{-a.e. } x.$$

Proof. Since $\mu = Du$ is curl-free,

$$0 = \operatorname{curl}(\mu) = \Big(\partial_i \mu_j^k - \partial_j \mu_i^k\Big)_{i,j=1,\dots,d;\ k=1,\dots,\ell}.$$

point (i) above follows from

$$\Lambda_{\mathrm{curl}} = \big\{a \otimes \xi : a \in \mathbb{R}^\ell,\ \xi \in \mathbb{R}^d \setminus \{0\}\big\}.$$

In the same way if $\mu = Eu$, then it satisfies the Saint-Venant compatibility conditions:

$$0 = \operatorname{curl}\operatorname{curl}(\mu) := \left(\sum_{i=1}^d \partial_{ik}\mu_i^j + \partial_{ij}\mu_i^k - \partial_{jk}\mu_i^i - \partial_{ii}\mu_j^k\right)_{j,k=1,\dots,d}.$$

It is now a direct computation to check that

$$\Lambda_{\mathrm{curl\,curl}} = \big\{a \odot \xi : a \in \mathbb{R}^d,\ \xi \in \mathbb{R}^d \setminus \{0\}\big\}.$$

□

3. The converse of Rademacher's Theorem

Rademacher's Theorem asserts that a Lipschitz function $f \in \mathrm{Lip}(\mathbb{R}^d, \mathbb{R}^\ell)$ is differentiable $\mathcal{L}^d$-almost everywhere. A natural question, which has attracted the attention of several researchers, is to understand how sharp is this result. Namely:

Question 3 (Strong converse of Rademacher Theorem)**.** *Given a Lebesgue null set $E \subset \mathbb{R}^d$ is it possible to find some $\ell \geq 1$ and a Lipschitz function $f \in \mathrm{Lip}(\mathbb{R}^d, \mathbb{R}^\ell)$ such that f is not differentiable in any point of E?*

Question 4 (Weak converse of Rademacher Theorem)**.** *Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ be a positive Radon measure such that every Lipschitz function is differentiable ν-almost everywhere. Is ν necessarily absolutely continuous with respect to the Lebesgue measure?*

Clearly a positive answer to Question 3 implies a positive answer to Question 4. Let us also stress that in answering Question 3 an important role is played by the dimension ℓ of the target set , see point (2) below, while this does not have any influence for what concern Question 4, see [5, Remark 7.2].

We refer to [2, 3, 5] for a detailed account on the history of these problems and here we simply record the following facts:

(1) For $d = 1$ a positive answer to Question 3 is due to Zahorski [51].

(2) For $d \geq 2$ there exists a null set E such that every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is differentiable in at least one point of E. This is was proved by Preiss in [40] for $d = 2$ and later extended by Preiss and Speight in [41] to every dimension.

(3) For $d = 2$ a positive answer to Question 3 has been given by Alberti, Csörnyei and Preiss as a consequence of their deep result concerning the structure of null sets in the plane, [2–4]. Namely they show that for every null set $E \subset \mathbb{R}^2$ there exists a Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that f is not differentiable at every point of E.

(4) For $d \geq 2$ an extension of the result described in point (3) above (i.e. that for every null set $E \subset \mathbb{R}^2$ there exists a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^d$ such that f is not differentiable at every point of E) has been announced in 2011 by Csörnyei and Jones, [30].

Let us now show how Question 4 is related to Question 1. In [5] Alberti and Marchese have shown the following result, see Theorem 1.1 there.

Theorem 3.1 (Alberti-Marchese)**.** *Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ be a positive Radon measure, then there exists a vector space valued ν-measurable map $V(\nu, x)$ (the* decomposability bundle of *ν) such that:*

(i) *Every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable in the directions of $V(\nu, x)$ at ν-almost every x.*

(ii) *There exists a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ such that for ν-almost every x and every $v \notin V(\nu, x)$ the derivative of f in x in the direction of v does not exist.*

Thanks to the above theorem Question 4 is then equivalent to the following:

Question 5. *Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ be a positive Radon measure such that $V(\nu, x) = \mathbb{R}^d$ for ν-almost every x, is ν absolutely continuous with respect to the Lebesgue measure?*

The link between the above question and Theorem 1.1 is due to the following result, again due to Alberti and Marchese, see [5, Corollary 6.5] and [23, Lemma 3.1] [2].

Lemma 3.2. . *Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ be a positive Radon measure, then the following are equivalent*

(i) *The decomposability bundle of ν is full dimensional, i.e. $V(\nu, x) = \mathbb{R}^d$ for ν-almost every x.*

(ii) *There exists d $\mathbb{R}^d$-valued measures $\mu_1, \dots, \mu_d \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$ with measure valued divergence* $\operatorname{div} \mu_i \in \mathcal{M}(\mathbb{R}^d; \mathbb{R})$*, such that $\nu \ll |\mu_i|$ for $1 = 1, \dots, d$ and* [3]

$$\operatorname{span}\left\{\frac{d\mu_1}{d|\mu_1|}(x), \dots, \frac{d\mu_d}{d|\mu_d|}(x)\right\} = \mathbb{R}^d \qquad \textit{for } \nu \textit{ a.e. } x. \tag{3}$$

[2] In the cited references the results are stated in terms of normal currents. By the trivial identifications of the space of normal currents with the space of measure valued vector-fields whose divergence is a measure it is immediate to see that they are equivalent to Lemma 3.2

[3] Note that since $\nu \ll |\mu_i|$ for all $i = 1, \dots, d$, in item (ii) above all the Radon-Nikodym derivatives $d\mu_i/d|\mu_i|$ $i = 1, \dots, d$ exist for ν-a.e. x.

With the above Lemma at hand a positive answer to Question 5 (and thus to Question 4) follows straightforwardly from Theorem 1.1. Indeed let ν be a measure such that $V(\nu, x) = \mathbb{R}^d$ for ν-almost every x an let μ_i the measures provided by Lemma 3.2. Let us consider the matrix-valued measure

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_d \end{pmatrix} \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^{d\times d}).$$

and note that $\operatorname{div} \boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$, where div is the row-wise divergence operator. Since, by direct computation,

$$\Lambda_{\mathrm{div}} = \big\{ M \in \mathbb{R}^{d\times d} \text{ such that } \operatorname{rank} M \le d-1 \big\},$$

Theorem 1.1 and Remark 1.2 imply that $\operatorname{rank}(d\boldsymbol{\mu}/d|\boldsymbol{\mu}|) \le d-1$ for $|\boldsymbol{\mu}|^s$-almost every point. Hence, by (3), $\nu \perp |\boldsymbol{\mu}|^s$. On the other hand, since $\nu \ll |\mu_i|$ for all $i = 1, \dots, d$, $\nu^s \ll |\boldsymbol{\mu}|^s$. This two fact then implies that $\nu^s = 0$ as desired.

Let us conclude by mentioning that the weak converse of Rademacher Theorem, i.e. a positive answer to Question 4, has important implications concerning the structure of Ambrosio–Kirchheim metric currents, [12], and the structure of the so called Lipschitz differentability spaces, [16, 31]. In particular it allows to prove the top-dimensional case of the flat chain conjecture proposed by Ambrosio and Kirchheim in [12], see [23, Theorem 1.15] and [45], and to provide a positive answer to a conjecture raised by Cheeger [16, Conjecture 4.63], see [14, 28, 31] and [22]. We refer the reader to the above mentioned references for more details.

4. Sketch of the proof of Theorem 1.1

In this Section we shall give some details concerning the proof of Theorem 1.1. For simplicity we will only consider the case in which $\mathscr{A}$ is a *first order homogeneous* operator, namely we will assume that μ satisfies

$$\mathscr{A}\mu = \sum_{j=1}^{d} A_j \partial_j \mu = 0 \qquad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^n).$$

Note that in this case we have

$$\Lambda_{\mathscr{A}} = \bigcup_{|\xi|=1} \operatorname{Ker} \mathbb{A}(\xi), \qquad \mathbb{A}(\xi) = \mathbb{A}^1(\xi) = 2\pi \mathrm{i} \sum_{j=1}^{d} A_j \xi_j.$$

Let

$$E = \Big\{ x \in \Omega : \frac{d\mu}{d|\mu|}(x) \notin \Lambda_{\mathscr{A}} \Big\},$$

and let us assume by contradiction that $|\mu|^s(E) > 0$.

A natural strategy in Geometric Measure Theory consits in "zooming" around a generic point of E in order to see what happens. In particular one can show that for $|\mu|^s$-almost every point $x_0 \in E$ there exists a sequence of radii $r_k \downarrow 0$ such that

$$w^* - \lim_{k\to\infty} \frac{(T^{x_0,r_k})_\sharp \mu}{|\mu|(B_{r_k}(x_0))} = w^* - \lim_{k\to\infty} \frac{(T^{x_0,r_j})_\sharp \mu^s}{|\mu|^s(B_{r_k}(x_0))} = P_0 \nu,$$

where $T^{x,r} : \mathbb{R}^d \to \mathbb{R}^d$ is the dilation map $T^{x,r}(y) = (y-x)/r$, $T^{x,r}_\sharp$ denotes the push-forward operator [4], $\nu \in \mathrm{Tan}(x_0, |\mu|) = \mathrm{Tan}(x_0, |\mu|^s)$ is a non-zero tangent measure in the sense of Preiss [39],

$$P_0 = \frac{d\mu}{d|\mu|}(x_0) \notin \Lambda_{\mathscr{A}}$$

and the limit is intended in the weak-* topology of Radon measures (i.e. in duality with compactly supported continuous functions). Moreover, one easily checks that

$$\sum_{j=1}^d A_j \frac{d\mu}{d|\mu|}(x_0)\, \partial_j \nu = 0 \qquad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^n).$$

By taking the Fourier transform of the above equation, we get

$$\mathbb{A}(\xi) P_0\, \hat{\nu}(\xi) = 0, \qquad \xi \in \mathbb{R}^d.$$

where $\hat{\nu}(\xi)$ is the Fourier transform of ν in the sense of distribution (actually ν needs not to be a tempered distribution, hence some care is needed, see below for more details). Having assumed that $P_0 \notin \Lambda_{\mathscr{A}}$, i.e. that

$$\mathbb{A}(\xi) P_0 \neq 0 \text{ for all } \xi \neq 0,$$

this implies $\operatorname{supp} \hat{\nu} = \{0\}$ and thus $\nu \ll \mathcal{L}^d$. The latter fact, however, is not by itself a contradiction to $\nu \in \mathrm{Tan}(x_0, |\mu|^s)$. Indeed, Preiss [39] provided an example of a purely singular measure that has only multiples of Lebesgue measure as tangents (we also refer to [37] for a measure that has *every* measure as a tangent at almost every point).

The above reasoning provides a sort of *rigidity* property for $\mathscr{A}$-measures: if, for a constant polar vector $P_0 \notin \Lambda_{\mathscr{A}}$ and a measure $\nu \in \mathcal{M}_+(\mathbb{R}^d)$, the measure $P_0\nu$ is $\mathscr{A}$-free, then necessarily $\nu \ll \mathcal{L}^d$. However, since as we commented above this is not enough to conclude, in order to prove the Theorem we need to enforce this rigidity property in a *stability* property.

In this respect note that since $P_0 \notin \Lambda_{\mathscr{A}}$ implies that $\mathbb{A}(\xi) P_0 \neq 0$, one can hope for some sort of "elliptic regularisation" that forces not only $\nu \ll \mathcal{L}^d$ but also

$$\mu_k = \frac{(T^{x_0,r_k})_\sharp \mu}{|\mu|^s(B_{r_k}(x_0))} \ll \mathcal{L}^d,$$

[4] That is, for any measure σ and Borel set B, $[(T^{x,r})_\sharp \sigma](B) := \sigma(x + rB))$

at least in a neighbourhood of x_0. This is actually the case: Inspired by Allard's Strong Constancy Lemma in [6], we can show that the ellipticity of the system at the limit (i.e. that $\mathbb{A}(\xi)P_0 \neq 0$) improves the weak-* convergence of μ_k to $P_0\nu$ to the convergence in the total variation norm:

$$|\mu_k - P_0\nu|(B_{1/2}) \to 0. \tag{4}$$

Since the singular part of μ_k is asymptotically predominant around x_0, see (5) below, this latter fact implies that

$$|\mu_k^s - P_0\nu|(B_{1/2}) \to 0.$$

easily gives a contradiction to $\nu \ll \mathcal{L}^d$ and concludes the proof.

Let us briefly sketch how (4) is obtained. For $\chi \in \mathcal{D}(B_1)$, $0 \le \chi \le 1$ consider the measures

$$P_0\chi\nu_k \qquad \text{where} \qquad \nu_k = \frac{(T^{x_0,r_k})_\sharp|\mu|^s}{|\mu|^s(B_{r_k}(x_0))}$$

and note that, since we can assume that for the chosen x_0,

$$\frac{|\mu|^a(B_{r_k}(x_0))}{|\mu|^s(B_{r_k}(x_0))} \to 0 \quad \text{and} \fint_{B_{r_k}(x_0)} \left|\frac{d\mu}{d|\mu|}(x) - \frac{d\mu}{d|\mu|}(x_0)\right| d|\mu|^s(x) \to 0; \tag{5}$$

we have that

$$|P_0\chi\nu_k - \chi\mu_k|(\mathbb{R}^d) \le |P_0\nu_k - \mu_k|(B_1) \to 0. \tag{6}$$

Using the $\mathscr{A}$-freeness of μ_k (which trivially follows from the on of μ) we can write down an equation for $\chi\nu_k$:

$$\sum_{j=1}^{d} A_j P_0 \partial_j(\chi\nu_k) = \sum_{j=1}^{d} A_j \partial_j(P_0\chi\nu_k - \chi\mu_k) + \sum_{j=1}^{d} A_j \mu_k \partial_j \chi. \tag{7}$$

Since we essentially deal with a-priori estimates, in the following we treat measures as they were smooth L^1 functions, this can be achieved by a sufficiently fast regularisation, see [23] for more details.

Taking the Fourier transform of equation (7) (note that we are working with compactly supported functions) we obtain:

$$\mathbb{A}(\xi)P_0\widehat{\chi\nu_k}(\xi) = \mathbb{A}(\xi)\widehat{V_k}(\xi) + \widehat{R_k}(\xi) \tag{8}$$

where

$$V_k = P_0\chi\nu_k - \chi\mu_k \qquad \text{satisfies} \qquad |V_k|(\mathbb{R}^d) \to 0 \tag{9}$$

and

$$R_k = \sum_{j=1}^{d} A_j \mu_k \partial_j \chi \qquad \text{satisfies} \qquad \sup_k |R_k|(\mathbb{R}^d) \le C. \tag{10}$$

By scalar multiplying (8) by $\overline{\mathbb{A}(\xi)P_0}$, adding to both members $\widehat{\chi\nu_k}$ and rearranging the terms we arrive to

$$\begin{aligned}\widehat{\chi\nu_k}(\xi) &= \frac{\overline{\mathbb{A}(\xi)P_0}\mathbb{A}(\xi)\widehat{V_k}(\xi)}{1+|\mathbb{A}(\xi)P_0|^2} + \frac{\overline{\mathbb{A}(\xi)P_0}\cdot\widehat{R_k}(\xi)}{1+|\mathbb{A}(\xi)P_0|^2} + \frac{\widehat{\chi\nu_k}(\xi)}{1+|\mathbb{A}(\xi)P_0|^2} \\ &= T_0(V_k) + T_1(R_k) + T_2(\chi\nu_k)\end{aligned} \tag{11}$$

where

$$\begin{aligned}T_0[V] &= \mathcal{F}^{-1}\Big[m_0(\xi)\hat{V}(\xi)\Big], \\ T_1[R] &= \mathcal{F}^{-1}\Big[m_1(\xi)(1+4\pi^2|\xi|^2)^{-1/2}\hat{R}(\xi)\Big], \\ T_2[u] &= \mathcal{F}^{-1}\Big[m_2(\xi)(1+4\pi^2|\xi|^2)^{-1}\hat{u}(\xi)\Big],\end{aligned}$$

and we have set

$$\begin{aligned}m_0(\xi) &= (1+|\mathbb{A}(\xi)P_0|^2)^{-1}\overline{\mathbb{A}(\xi)P_0}\mathbb{A}(\xi) \\ m_1(\xi) &= (1+|\mathbb{A}(\xi)P_0|^2)^{-1}(1+4\pi^2|\xi|^2)^{1/2}\overline{\mathbb{A}(\xi)P_0}, \\ m_2(\xi) &= (1+|\mathbb{A}(\xi)P_0|^2)^{-1}(1+4\pi^2|\xi|^2).\end{aligned}$$

We now note that since $P_0 \notin \Lambda_{\mathscr{A}}$, by homogeneity there exists $c > 0$ such that $|\mathbb{A}(\xi)P_0| \geq c|\xi|$ (this is the ellipticity condition we were mentioning at the beginning!). Hence the symbols m_i, $i = 1, 2, 3$ satisfies the assumption of Hörmander–Mihlin multiplier Theorem, [29, Theorem 5.2.7]:

$$|\partial^\beta m_i(\xi)| \leq K_{\beta,d}|\xi|^{-|\beta|} \qquad \beta \in \mathbb{N}^d.$$

This implies that T_0 is a bounded operator from L^1 to $L^{1,\infty}$ and thus, thanks to (9),

$$\|T_0(V_k)\|_{L^{1,\infty}} \leq C|V_k|(\mathbb{R}^d) \to 0. \tag{12}$$

Moreover,

$$\langle T_0(V_k), \varphi\rangle = \langle V_k, T_0^*(\varphi)\rangle \to 0 \qquad \text{for every } \varphi \in \mathcal{D}(\mathbb{R}^d). \tag{13}$$

where T_0^* is the adjoint operator of T_0. We now note that

$$T_1 = Q_{m_1} \circ (\mathrm{Id} - \Delta)^{-1/2} \qquad \text{and} \qquad T_2 = Q_{m_2} \circ (\mathrm{Id} - \Delta)^{-1}$$

where Q_{m_1} and Q_{m_2} are the Fourier multipliers operators associated with the symbols m_1 and m_2 respectively. In particular, again by the Hörmander–Mihlin multiplier Theorem, they are bounded from L^p to L^p for every $p \in (1, \infty)$. Moreover, $(\mathrm{Id} - \Delta)^{-s/2}$ is a compact operator from[5] $L^1_c(B_1)$ to L^q for some $q = q(d, s) > 1$, see for instance [23, Lemma 2.1]. In conclusion, by (10) and $\sup_k |\chi\nu_k|(\mathbb{R}^d) \leq C$ we infer that

$$\big\{T_1(R_k) + T_2(\chi\nu_k)\big\}_{k\in\mathbb{N}} \qquad \text{is pre-compact in } L^1(B_1). \tag{14}$$

[5] Here we denote by $L^1_c(B_1)$ the space of L^1 functions vanishing outside B_1.

Hence, equation (11) together with (12), (13) and (14) imply that

$$\chi\nu_k = u_k + w_k$$

where $u_k \to 0$ in $L^{1,\infty}$, $u_k \overset{*}{\rightharpoonup} 0$ in the sense of distributions and $\{w_k\}$ pre-compact in $L^1(B_1)$. By $\chi\nu_k \geq 0$,

$$(u_k)^- = \max\{-u_k, 0\} \leq |w_k|,$$

so that $\{(u_k)^-\}$ is pre-compact in $L^1(B_1)$. Since $u_k \to 0$ in $L^{1,\infty}$, Vitali convergence Theorem implies that $(u_k)^- \to 0$ in $L^1(B_1)$ which, combined with $u_k \overset{*}{\rightharpoonup} 0$, easily implies that $u_k \to 0$ in $L^1(B_1)$, see [23, Lemma 2.2]. In conclusion $\{\chi\nu_k\}$ is pre-compact in $L^1(B_1)$ that, together with (4) and the weak-* convergence of μ_k to $P_0\nu$, implies

$$|\mu_k - P_0\nu|(B_{1/2}) \to 0,$$

and concludes the proof.

5. References

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