REPRESENTATION OF ATOMIC OPERATORS AND EXTENSION PROBLEMS

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Abstract. The notion of an atomic operator between spaces of measurable functions was introduced in [2] in order to provide a reasonable generalisation of local operators useful for applications. It has been shown that, roughly speaking, atomic operators amount to compositions of local operators with shifts. A natural problem is then when a continuous in measure atomic operator can be represented as a composition of a Nemytskiĭ (composition) operator generated by a Carathéodory function, and a shift operator. In this paper we will show that the answer to this question is inherently related to the possibility of extending an atomic operator with continuity from a space of functions measurable with respect to some σ-algebra to a larger space of functions measurable with respect to a larger σ-algebra, as well as to the possibility of extending any σ-homomorphism from a smaller measure algebra to a σ-homomorphism on a larger measure algebra. We characterise precisely the condition on the respective σ-algebras which provides such possibilities and induces the positive answer to the above representation problem.

Keywords: local operators, atomic operators, σ-homomorphism, extension

1. Introduction

In [2] a particular class of operators called atomic operators and generalising the notion of a local operator between ideal function spaces was introduced and many of its analytic properties like acting, boundedness, continuity, compactness and convergence in various topologies were studied. We recall here that this class contains local (in particular, Nemytskiĭ) operators, shift operators and is closed with respect to compositions. This study was to a large extent inspired by stochastic applications. For instance, atomic operators arise naturally in the problem of finding periodic (in distribution) solutions to stochastic differential equations. Moreover, many problems for stochastic dynamical systems can be reduced to the study of atomic operators.

The representation theorem proved in [2] asserts that, roughly speaking, an atomic operator is always a composition of a local operator and a shift. However, even if the atomic operator is continuous, the respective local operator is not necessarily a Nemytskiĭ operator generated by some Carathéodory function, since in general it will be acting between the spaces of functions measurable with respect to different σ-algebras. In fact, the same paper provides examples of continuous atomic operators arising in stochastic applications that cannot be represented as a composition of a Nemytskiĭ operator generated by a Carathéodory function, and

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a shift. Nevertheless, the question of when such a representation exists is of utmost importance for applications, for instance, for the purpose of seeking invariant measures or fixed points (maybe appropriately generalised) of atomic operators.

The above representation problem is inherently related to another two problems. Namely, suppose that (Ω, Σ', μ) is a measure space and $\Sigma \subset \Sigma'$ is some smaller σ-algebra. The first problem is when an atomic operators defined over a space of functions measurable with respect to Σ can be extended with continuity to the bigger space of functions measurable with respect to Σ'. The second problem is when a nullset preserving σ-homomorphism between Σ and any other σ-algebra of sets can be extended to the whole Σ'. A related question is when such a σ-homomorphism can be represented by a point map [8]. In this paper we show that all the above problems are in fact equivalent and admit a positive answer, if and only if Σ satisfies the so-called Ω-condition, which was recently introduced in [7]. Moreover, for standard measure spaces we are able to completely characterise the Ω-condition on σ-subalgebras in terms of the existence of a piecewise injective generating function.

2. Notation and preliminaries

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two measure spaces, and $\Sigma_1^0 \subset \Sigma_1$, $\Sigma_2^0 \subset \Sigma_2$ be the σ-ideals of μ_1- and μ_2-nullsets respectively. We denote by $\tilde{\Sigma}_i := \Sigma_i/\Sigma_i^0$, $i = 1, 2$ the respective *measure algebras* (see § 42 of [8]). The elements of $\tilde{\Sigma}_i$ (i.e. the equivalence classes of sets) will be denoted $\tilde{e}_i$ or $[e_i]$, $i = 1, 2$. Further on we will however frequently abuse the notation and identify the elements of the measure algebras $\tilde{\Sigma}_i$ with the elements of the respective original σ-algebras of sets Σ_i. A map $F\colon \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ is called a *σ-homomorphism*, if $F(\Omega_1) = \Omega_2$, $F(\Omega_1 \backslash e) = \Omega_2 \backslash F(e)$ whenever $e \in \tilde{\Sigma}_1$ and

$$F\left(\bigsqcup_{i=1}^{\infty} e_i\right) = \bigsqcup_{i=1}^{\infty} F(e_i),$$

for any pairwise disjoint collection of $\{e_i\}_{i=1}^{\infty} \subset \tilde{\Sigma}_1$, where $\sqcup$ stands for the disjoint union. Every (Σ_2, Σ_1)-measurable map $g\colon \Omega_2 \to \Omega_1$ satisfying

$$\mu_2(g^{-1}(e_1)) = 0 \text{ when } \mu_1(e_1) = 0 \tag{1}$$

generates a σ-homomorphism according to the formula $F(\tilde{e}_1) := [g^{-1}(e_1)]$. The latter σ-homomorphism is said to be induced by a point map g, and in this case we will write $F = g^{-1}$.

A measure space (Ω, Σ, μ) is called standard, if Ω is a Polish space and Σ is either the Borel σ-algebra or its completion with respect to finite Borel measure μ. In the sequel all the measure spaces we will be dealing with are assumed to be complete and all the measures to be finite.

By $L^0(\Omega, \Sigma, \mu; \mathcal{X})$, where $\mathcal{X}$ is a separable metric space, we denote the space of (classes of) $\mathcal{X}$-valued measurable functions equipped with the usual metric inducing the topology of convergence in measure. Whenever there is no possibility of confusion, the references to $\mathcal{X}$, Ω, Σ and/or μ will be omitted.

3. Extension of atomic operators

For the sake of brevity denote $X_i := L^0(\Omega_i, \Sigma_i; \mathcal{X}_i)$, $i = 1, 2$. Recall the following notion introduced in [2].

Definition 3.1. *An operator $T\colon X_1 \to X_2$ is called* **atomic** *with respect to the σ-homomorphism $F\colon \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$, if $T(x)\,\big|_{F(e_1)} = T(y)\,\big|_{F(e_1)}$ whenever $x\,|_{e_1} = y\,|_{e_1}$ for $x, y \in X_1$.*

We will further omit the reference to the particular σ-homomorphism, if it is unnecessary.

We remark that in [2] a slightly different definition of an atomic operator was given, namely, the operator $T\colon X_1 \to X_2$ was called atomic with respect to the nullset preserving σ-homomorphism between the original σ-algebras $F\colon \Sigma_1 \to \Sigma_2$, if $T(x)\,\big|_{F(e_1)} = T(y)\,\big|_{F(e_1)}$ whenever $x\,|_{e_1} = y\,|_{e_1}$ for $x, y \in X_1$. The requirement that F be nullset preserving means that $\mu_2(F(e_1)) = 0$ whenever $\mu_1(e_1) = 0$. Clearly, every nullset-preserving σ-homomorphism $F\colon \Sigma_1 \to \Sigma_2$ between the original σ-algebras generates a σ-homomorphism $\tilde{F}$ of the respective measure algebras according to the formula $\tilde{F}(\tilde{e}_1) := [F(e_1)]$. Vice versa, given a σ-homomorphism between the measure algebras $\tilde{F}\colon \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$, provided that the measure space $(\Omega_2, \Sigma_2, \mu_2)$ is complete, we can construct a nullset-preserving σ-homomorphism $F\colon \Sigma_1 \to \Sigma_2$ between the original σ-algebras satisfying $[F(e_1)] := \tilde{F}(\tilde{e}_1)$. The respective construction is provided by the formula $F(e_1) := \pi(\tilde{F}(\tilde{e}_1))$, where $\pi\colon \tilde{\Sigma}_2 \to \Sigma_2$ is the lifting map satisfying $[\pi(\tilde{e}_2)] = \tilde{e}_2$. The existence of the latter is ensured by the von Neumann-Maharam lifting theorem (theorem 4.4 in [4]), for which completeness of $(\Omega_2, \Sigma_2, \mu_2)$ is essential. Therefore, the definition of atomic operator given in [2] is equivalent to the above one provided one works with complete measure spaces.

The notion of an atomic operator seems to be just a reasonable generalisation of the classical notion of a local operator for many applications. In fact, a local operator between spaces of measurable functions is just an atomic one with respect to the identity σ-homomorphism. In particular, a Nemytskiĭ operator $N_f\colon L^0(\Omega; \mathcal{X}_1) \to L^0(\Omega; \mathcal{X}_2)$ defined by the formula $(N_f u)(\omega) := f(\omega, u(\omega))$ for a given function $f\colon \Omega \times \mathcal{X}_1 \to \mathcal{X}_2$ is local, and hence, atomic. But the class of atomic operators is, in fact, much wider and embodies also a lot of operators which though being nonlocal have many properties similar to those of local operators. Here we enlist some of the most important examples of atomic operators.

Example 3.1. The examples below were pointed out in [2].

(1) Let $g\colon \Omega_2 \to \Omega_1$ be a (Σ_2, Σ_1)-measurable function satisfying (1). Then the shift operator $T_g\colon X_1 \to X_2$ defined by the formula

$$(T_g u)(\omega_2) := u(g(\omega_2)) \text{ for } \mu_2\text{-a.e. } \omega_2 \in \Omega_2$$

is atomic with respect to the σ-homomorphism induced by g.

(2) A composition of two atomic operators is still atomic. In particular, a composition of a local (e.g. Nemytskiĭ) operator and a shift is atomic, though, generally speaking, not local.

(3) Every linear continuous in measure *disjointness preserving* (D-) operator (i.e. an operator sending functions with disjoint supports into functions with disjoint supports) is atomic. Note that nonlinear D-operators in general need not to be atomic.

(4) Atomic operators arise naturally in stochastic applications. For instance, a problem of finding a strong periodic solution to a stochastic differential equation can be naturally reduced to a fixed point problem for an operator which is a composition of the so-called Wiener shift and the evolution

(propagation) operator along the trajectories of the stochastic differential equation. Since the latter operator is known to be local, the original problem is reduced to finding fixed points of an atomic operator.

We now consider the following problems which are quite closely related to each other.

Problem 1. *In [2] it has been shown that every atomic operator T can in fact be represented as a composition $T = N \circ T_F$ of a local operator and some generalised shift operator T_F with respect to a σ-homomorphism F. Suppose that the atomic operator $T\colon X_1 \to X_2$ is continuous in measure. We would like to know when the local operator N involved in the respective representation is a Nemytskiĭ operator generated by a Carathéodory function.*

Problem 2. *Let $\Sigma_1 \subset \Sigma_1'$, where Σ_1' is some larger σ-algebra of subsets of Ω. We would like to know under which condition every continuous in measure atomic operator $T\colon X_1 \to X_2$ can be extended to a continuous in measure atomic operator $T'\colon X_1' \to X_2$ over the larger space $X_1' := L^0(\Omega_1, \Sigma_1'; \mathcal{X}_1)$.*

Problem 3. *Let $F\colon \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ be a σ-homomorphism. We would like to know when it can be extended to a σ-homomorphism $F'\colon \tilde{\Sigma}_1' \to \tilde{\Sigma}_2$.*

The answers to the above problems we provide here involves the following notion which first appeared in [7].

Definition 3.2. *Let $\Sigma_1 \subset \Sigma_1'$ be σ-algebras of subsets of Ω_1. Then Σ_1 is said to satisfy the Ω-condition with respect to Σ_1' (written $\Sigma_1 \in \Omega(\Sigma_1')$), if there is an at most countable cover of Ω_1 by pairwise disjoint sets $\Omega_1 = \sqcup_j \Omega_1^j$, $\Omega_1^j \in \Sigma_1'$, such that for each $j \in \mathbb{N}$ one has $\Sigma_1 \cap \Omega_1^j = \Sigma_1' \cap \Omega_1^j$.*

It is worth remarking that the same notion of Ω-condition can be extended in an obvious way to measure algebras, namely, by saying that $\tilde{\Sigma}_1 \in \Omega(\tilde{\Sigma}_1')$, if $\Sigma_1 \cap \Omega_1^j/\Sigma_1^0 = \Sigma_1' \cap \Omega_1^j/\Sigma_1^0$ for each $j \in \mathbb{N}$ and for some partition of Ω_1 by an at most countable number of disjoint sets $\{\Omega_1^j\}_{j\in\mathbb{N}}$, where Σ_1^0 stands for the σ-ideal of μ_1-nullsets.

The following result gives the answer to problem 2. Note that its proof uses also the answer to problem 3 given by proposition 5.1.

Theorem 3.1. *Let $\tilde{\Sigma}_1 \in \Omega(\tilde{\Sigma}_1')$. Then every atomic operator $T\colon X_1 \to X_2$ admits a (not necessarily unique) extension to an atomic operator $T'\colon X_1' \to X_2$. Moreover, T' is continuous in measure whenever T is continuous in measure.*

PROOF: Let $T\colon X_1 \to X_2$ be atomic with respect to the σ-homomorphism $F\colon \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$. According to proposition 5.1 proven in the sequel, the latter admits an extension to $F'\colon \tilde{\Sigma}_1' \to \tilde{\Sigma}_2$. We use now lemma 3.1 below to find the desired extension T' of the operator T, which is atomic with respect to F'. □

Lemma 3.1. *Let $T\colon X_1 \to X_2$ be atomic with respect to the σ-homomorphism $F\colon \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$, while $\tilde{\Sigma}_1 \in \Omega(\tilde{\Sigma}_1')$ and F admits an extension to a σ-homomorphism $F'\colon \tilde{\Sigma}_1' \to \tilde{\Sigma}_2$. Then the operator T admits the unique extension to the operator $T'\colon X_1' \to X_2$, which is atomic with respect to F'. Moreover, the extended operator T' is continuous in measure whenever T is continuous in measure.*

PROOF:

Existence. For an arbitrary $v \in X_1'$ and for every $j \in \mathbb{N}$ there is a $v^j \in X_1$ such that

$$v \Big|_{\Omega_1^j} = v^j \Big|_{\Omega_1^j} \tag{2}$$

Set now

$$(T'v)(x) := (Tv^j)(x) \text{ for } \mu_2 - \text{a.e. } x \in F'(\Omega_1^j).$$

Note that the above definition is correct. In fact, although there might exist another $\tilde{v}^j \in X_1$ such that

$$v \Big|_{\Omega_1^j} = \tilde{v}^j \Big|_{\Omega_1^j},$$

but clearly then

$$\tilde{v}^j \Big|_{\Phi(\Omega_1^j)} = v^j \Big|_{\Phi(\Omega_1^j)}$$

where $\Phi(e_1)$ stands for the minimum (up to a set of μ_1-measure zero) set from Σ_1 which contains $e_1 \in \Sigma_1'$. Therefore, in view of atomicity of T one has

$$(T\tilde{v}^j)(x) = (Tv^j)(x) \text{ for } \mu_2 - \text{a.e. } x \in F(\Phi(\Omega_1^j)),$$

and hence

$$(T\tilde{v}^j)(x) = (Tv^j)(x) \text{ for } \mu_2 - \text{a.e. } x \in F'(\Omega_1^j),$$

since $F'(\Omega_1^j) \subset F'(\Phi(\Omega_1^j)) = F(\Phi(\Omega_1^j))$.

It is easy to observe that T' is the desired extension (namely, it is atomic with respect to F'). In fact, if $u, v \in X_1'$ are such that

$$u\,|_{e_1} = v\,|_{e_1},$$

then for each $j \in \mathbb{N}$ there exist $u^j, v^j \in X_1$ such that

$$u \Big|_{\Omega_1^j} = u^j \Big|_{\Omega_1^j}, \qquad v \Big|_{\Omega_1^j} = v^j \Big|_{\Omega_1^j}.$$

Therefore,

$$u^j \Big|_{\Phi(e_1 \cap \Omega_1^j)} = v^j \Big|_{\Phi(e_1 \cap \Omega_1^j)},$$

which implies

$$Tu^j \Big|_{F(\Phi(e_1 \cap \Omega_1^j))} = Tv^j \Big|_{F(\Phi(e_1 \cap \Omega_1^j))}$$

in view of atomicity of T. Then

$$Tu \Big|_{F'(e_1 \cap \Omega_1^j)} = Tu^j \Big|_{F'(e_1 \cap \Omega_1^j)} = Tv^j \Big|_{F'(e_1 \cap \Omega_1^j)} = Tv \Big|_{F'(e_1 \cap \Omega_1^j)},$$

since $F'(e_1 \cap \Omega_1^j) \subset F'(\Phi(e_1 \cap \Omega_1^j)) = F(\Phi(e_1 \cap \Omega_1^j))$.

Uniqueness. Let T_1' and T_2' be both extensions of T atomic with respect to F'. Again, for every $v \in X_1'$ and for every $j \in \mathbb{N}$ there is a $v^j \in X_1$ such that (2) holds. In view of atomicity of T_1' and T_2' the restrictions of $T_1'v$ and $T_2'v$ to each $F'(\Omega_1^j)$ coincide with those of $T_1'v^j$ and $T_2'v^j$. But since $T_1'v^j = T_2'v^j = Tv^j$, then in view of the arbitrariness of j one has $T_1'v = T_2'v$.

Continuity. The verification of this property is straightforward. Suppose in fact that T is continuous and let $v_\nu \to v$ in X_1'. For each $j \in \mathbb{N}$, let v_ν^j, $v^j \in X_1$ be such that

$$v_\nu \Big|_{\Omega_1^j} = v_\nu^j \Big|_{\Omega_1^j}, \qquad v \Big|_{\Omega_1^j} = v^j \Big|_{\Omega_1^j}.$$

Then one has $v^j_\nu \to v^j$ in $L^0(\Phi(\Omega_1^j), \Sigma_1 \cap \Phi(\Omega_1^j), \mu_1)$. Hence $T'v_\nu = Tv^j_\nu \to Tv_j = T'v$ in $L^0(F'(\Omega_1^j), \Sigma_2 \cap F(\Omega_1^j), \mu_2)$ and, since the latter holds for all $j \in \mathbb{N}$, one has $T'v_\nu \to Tv$ in X_2. $\square$

Note that one has uniqueness of the extended operator only when the extension of the respective σ-homomorphism is fixed. In fact, even a local operator can have several extensions to atomic operators (among which, clearly, there is exactly one local extension), as the following example shows.

Example 3.2. Let $\Omega_1 = \Omega_2 = S^1$ be a unit circumference, let $\mathcal{X}_1 = \mathcal{X}_2 = \mathbb{R}$, and let $\Sigma_1' = \Sigma_2$ be a completion of the Borel σ-algebra with respect to the one-dimensional Haussdorf measure. Let g: $S^1 \to S^1$ be the rotation of the circumference by the angle $2\pi\alpha$. Consider any measurable map h: $S^1 \to S^1$ such that $(h \circ g)(x) = h(x)$ for a.e. $x \in S^1$. Let $\Sigma_1 := h^{-1}(\Sigma_1')$. The operator T: $X_1 \to X_2$ defined by

$$(Tu)(x) := f(x, u(g(x))) \tag{3}$$

for some Carathéodory function f: $S^1 \times \mathbb{R} \to S^1$ is clearly local. Fore every $u \in X_1$ one has $u \circ g = u$, and hence $(Tu)(x) = f(x, u(x))$. Therefore, a continuous local extension T': $X_1' \to X_2$ of the operator T is clearly given by the formula

$$(T'u)(x) = f(x, u(x)).$$

On the other hand, another continuous atomic extension of this operator to the whole space X_1' is given by the same formula (3). This operator is atomic with respect to F' defined by $F'(\tilde{e}_1') := [g^{-1}(e_1')]$, but it is not local.

We remark that lemma 3.1 though being extremely simple, has interesting consequences on its own. For instance, one has to mention that its immediate corollary is the following assertion on the extension of local operators proved in [7].

Corollary 3.1. *Let N: $X_1 \to X_2$ be a local operator, and $\tilde{\Sigma}_1 \in \Omega(\tilde{\Sigma}_1')$. Then N admits the unique extension to the local operator N': $X_1' \to X_2$. Moreover, the extended operator N' preserves the continuity of N in measure.*

Now we can claim the following representation result for continuous in measure atomic operators which gives an answer to problem 1.

Theorem 3.2. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be standard measure spaces and F: $\tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ be a σ-homomorphism. Then any continuous operator T: $X_1 \to X_2$ atomic with respect to F can be represented as*

$$(Tu)(x) = f(x, u(g(x))) \text{ for } \mu_2 - a.e.\ x \in \Omega_2$$

for some Carathéodory function f: $\Omega_2 \times \mathcal{X}_1 \to \mathcal{X}_2$, a measurable function g: $\Omega_2 \to \Omega_1$ satisfying (1) and every $u \in X_1$, if and only if $F(\tilde{\Sigma}_1) \in \Omega(\tilde{\Sigma}_2)$.

PROOF: The theorem 32.3 of [8] implies the existence of a measurable function g: $\Omega_2 \to \Omega_1$ satisfying (1) and $[g^{-1}(e_1)] = F(\tilde{e}_1)$ for every $e_1 \in \Sigma_1'$. According to the representation theorem 3.1 from [2], one has $T = N \circ T_g$, where N is a local operator defined over $L^0(\Omega_2, F(\tilde{\Sigma}_1), \mu_2; \mathcal{X}_1)$. If $F(\tilde{\Sigma}_1) \in \Omega(\tilde{\Sigma}_2)$, then according to the corollary 3.1 the operator N admits a unique continuous local extension to the whole space $L^0(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_1)$, and hence is representable as

$$(Nv)(x) = f(x, v(x)) \text{ for } \mu_2 - \text{a.e. } x \in \Omega_2,$$

where f: $\Omega_2 \times \mathcal{X}_1 \to \mathcal{X}_2$ is a Carathéodory function.

Otherwise, if $F(\tilde{\Sigma}_1) \notin \Omega(\tilde{\Sigma}_2)$, then theorem 7 from [7] asserts the existence of a continuous local operator $N\colon L^0(\Omega_2, F(\tilde{\Sigma}_1), \mu_2; \mathcal{X}_1) \to L^0(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_2)$ which cannot be represented as a Nemytskiĭ operator generated by a Carathéodory function. Therefore, the operator $T := N \circ T_g$ is atomic with respect to F but cannot be represented as indicated in the statement of the theorem. □

4. Ω-condition and piecewise injective functions

Before studying problem 3, we recall a notion of ω-condition for a function acting between two standard measure spaces. This notion was introduced by M.E. Drakhlin in [3] to study a class of shifts in Lebesgue spaces which have "nice" representation of adjoints.

Definition 4.1. *Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$ be standard measure spaces. We say that a measurable function $g\colon \Omega_2 \to \Omega_1$*

(i) *is piecewise injective, if there exists a disjoint at most countable covering of Ω_2 by measurable sets $\Omega_2 = \sqcup_j \Omega_2^j$ such that over each Ω_2^j the function is injective;*
(ii) *satisfies the ω-condition, if it is piecewise injective, satisfies (1), and the respective inverses $\gamma_j\colon g(\Omega_2^j) \to \Omega_2^j$, $\gamma_j \circ g|_{\Omega_2^j} = id$, satisfy (1).*

Note that a piecewise injective function $g\colon \Omega_2 \to \Omega_1$ satisfying (1) can always be changed on a set of zero μ_2-measure in order to satisfy the ω-condition. In fact, consider the measures μ_1^j on Ω_1 defined by

$$\mu_1^j(e_1) := \mu_2(g^{-1}(e_1) \cap \Omega_2^j)$$

for every $e_1 \in \Sigma_1'$. Since μ_1^j are absolutely continuous with respect to μ_1, we may consider the respective Radon-Nikodym derivatives $\frac{d\mu_1^j}{d\mu_1}$ and set

$$E_2^j := g^{-1}\left(\left\{y \,:\, \frac{d\mu_1^j}{d\mu_1}(y) = 0\right\}\right) \bigcap \Omega_2^j.$$

Setting now

$$\tilde{g}(x) := \begin{cases} g(x), & x \in \Omega_2 \setminus \bigcup_j E_2^j, \\ x_0, & x \in \bigcup_j E_2^j, \end{cases}$$

where $x_0 \in \Omega_1$ is arbitrary, and observing that $\mu_2(E_2^j) = 0$, one arrives at a conclusion that $g(x) = \tilde{g}(x)$ for μ_2-a.e. $x \in \Omega_2$, while $\tilde{g}$ satisfies the ω-condition. Therefore, we will just identify the piecewise injective functions satisfying (1) with the functions satisfying ω-condition.

We find it convenient to provide also the following simple and rather folkloric lemma.

Lemma 4.1. *Let $g\colon \Omega_2 \to \Omega_1$ satisfy ω-condition. Then there is a Borel set $\Omega_2' \subset \Omega_2$ with $\mu_2(\Omega_2 \setminus \Omega_2') = 0$, such that $\Omega_2' = \sqcup_j \Omega_2^{'j}$, where all $\Omega_2^{'j}$ are Borel sets, while the restrictions $g|_{\Omega_2^{'j}}$ are Borel measurable, injective, and possess Borel measurable inverses γ_j which satisfy (1).*

Proof: Let $\Omega_2 = \sqcup_j \Omega_2^j$, where Ω_2^j are injectivity sets for $g\colon \Omega_2 \to \Omega_2$, as indicated in definition 4.1. It is enough to show that for each j there is a Borel set $\Omega_2^{'j} \subset \Omega_2^j$ with $\mu_2(\Omega_2^j \setminus \Omega_2^{'j}) = 0$, while the restriction of g to $\Omega_2^{'j}$ is Borel measurable. Borel measurability of inverses γ_j will follow then from the Kuratowski-Suslin theorem (theorem 8.3.7 of [1]), and putting $\Omega_2' := \sqcup_j \Omega_2^{'j}$ we will conclude the proof of the lemma.

To prove this claim, pick up a countable set of generators $\mathcal{G} = \{G_\nu\}$ of $\mathcal{B}_1$, i.e. $\sigma(\mathcal{G}) = \mathcal{B}_1$, where $\sigma(\mathcal{G})$ stands for the minimum σ-algebra generated by $\mathcal{G}$, and $\mathcal{B}_i$ stands for the Borel σ-algebra of Ω_i, $i = 1, 2$. For each $\nu \in \mathbb{N}$ one has

$$g^{-1}(G_\nu) \cap \Omega_2^j = B'_{j,\nu} \cup C'_{j,\nu},$$

where $B'_{j,\nu} \in \mathcal{B}_2$ and $C'_{j,\nu} \subset C_{j,\nu} \in \mathcal{B}_2$, $\mu_2(C_{j,\nu}) = 0$. Let $D_j \in \mathcal{B}_2$ be such that $\mu_2(\Omega_2^j \setminus D_j) = 0$ and set

$$\Omega_2^{'j} := D_j \setminus \bigcup_\nu C_{j,\nu}.$$

Clearly, $\Omega_2^{'j}$ is a Borel set, while $\mu_2(\Omega_2^j \setminus \Omega_2^{'j}) = 0$. One has now

$$g^{-1}(G_\nu) \cap \Omega_2^{'j} = B'_{j,\nu} \cap \Omega_2^{'j} \in \mathcal{B}_2,$$

for every $\nu \in \mathbb{N}$, which proves that the restriction of g to $\Omega_2^{'j}$ is Borel measurable, and hence concludes the proof. □

The above lemma has a very easy but rather significant consequence formulated in the proposition below.

Proposition 4.1. *Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$ be standard measure spaces, while g: $\Omega_2 \to \Omega_1$ satisfy ω-condition. Then $g^{-1}(\tilde{\Sigma}_1) \in \Omega(\tilde{\Sigma}_2)$.*

PROOF: Let $\Omega_2' = \sqcup_j \Omega_2^{'j}$ be as in the statement of lemma 4.1. We will show that $g^{-1}(\Sigma_1) \cap \Omega_2^{'j}/\Sigma_2^0 = \Sigma_2 \cap \Omega_2^{'j}/\Sigma_2^0$ for all j, where Σ_2^0 stands for the σ-ideal of μ_2-nullsets. The only nontrivial inclusion to be proven is

$$\Sigma_2 \cap \Omega_2^{'j}/\Sigma_2^0 \subset g^{-1}(\Sigma_1) \cap \Omega_2^{'j}/\Sigma_2^0.$$

To show the latter, consider an arbitrary $\tilde{e}_2 \in \Sigma_2 \cap \Omega_2^{'j}/\Sigma_2^0$ and let $e_2 \subset \tilde{e}_2$ be such a Borel set that $[e_2] = \tilde{e}_2$. Clearly, $e_2 \cap \Omega_2^{'j}$ is a Borel set and since g is injective over $\Omega_2^{'j}$ and has a Borel measurable inverse, then

$$e_2 \cap \Omega_2^{'j} = g^{-1}(g(e_2 \cap \Omega_2^{'j})) \cap \Omega_2^{'j}.$$

Minding that $[e_2 \cap \Omega_2^{'j}] = \tilde{e}_2$ and that $g(e_2 \cap \Omega_2^{'j}) = \gamma_j^{-1}(e_2) \in \Sigma_1$ is a Borel set, suffices to conclude the proof. □

The above proposition asserts, roughly speaking, that a function between standard measure spaces satisfying ω-condition, generates a σ-algebra satisfying Ω-condition. In the sequel we will prove that in a sense a reverse statement is valid, namely, that every σ-subalgebra of a standard measure space which satisfies Ω-condition, is generated by some function satisfying ω-condition.

5. Extensions of σ-homomorphisms

In this section we will study problem 3 and we will see that it is in fact quite closely related to problems 1 and 2. The answer to this problem is given by the following proposition.

Proposition 5.1. *Let $\tilde{\Sigma}_1 \in \Omega(\tilde{\Sigma}_1')$. Then every σ-homomorphism F: $\tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ admits an extension to a σ-homomorphism F': $\tilde{\Sigma}_1' \to \tilde{\Sigma}_2$.*

PROOF: According to the assumptions there is an at most countable covering $\Omega_1 = \sqcup_j \Omega_1^j$, $\Omega_1^j \in \Sigma_1'$, such that for each $j \in \mathbb{N}$ one has $\Sigma_1 \cap \Omega_1^j/\Sigma_1^0 = \Sigma_1' \cap \Omega_1^j/Sigma_1^0$. Observe that for each $e_1' \in \tilde{\Sigma}_1'$ the minimum Σ_1-measurable set containing e_1' is defined uniquely up to a set of measure zero, and thus denote the equivalence class of the latter by $\Phi(e_1')$. Note that Φ: $\Sigma_1' \cap \Omega_1^j/\Sigma_1^0 \to \tilde{\Sigma}_1$ is a σ-homomorphism. In fact, the only nontrivial property to verify is $\Phi(e_1') \cap \Phi(e_1'') = \emptyset$ whenever $e_1' \cap e_1'' = \emptyset$. Suppose this is not true, i.e. there exist such disjoint e_1', $e_1'' \in \Sigma_1' \cap \Omega_1^j/\Sigma_1^0$ that the set $T := \Phi(e_1') \cap \Phi(e_1'')$ has positive measure. In view of the Ω-condition, one has $\Phi(e_1') \cap \Omega_1^j = e_1'$ and $\Phi(e_1'') \cap \Omega_1^j = e_1''$, and hence, since e_1' and e_1'' are disjoint, $T \cap (e_1' \cup e_1'') = \emptyset$. It is easy to observe now that the set $\Phi(e_1') \setminus T \in \Sigma_1$ contains e_1' but has measure strictly less than that of $\Phi(e_1')$, which contradicts the definition of Φ.

If $e_1^{'j} \in \Sigma_1' \cap \Omega_1^j$, define

$$F^{'j}(e_1^{'j}) := F(e_1^j) \cap \Omega_2^j, \text{ where } e_1^j := \Phi(e_1^{'j}) \text{ and } \Omega_2^j := F(\Phi(\Omega_1^j)).$$

Then clearly $F^{'j}$: $\Sigma_1' \cap \Omega_1^j/\Sigma_1^0 \to \Sigma_2 \cap \Omega_2^j/\Sigma_2^0$ is a σ-homomorphism. The desired extension is now given by

$$F'(e_1') := \bigsqcup_j F^{'j}(e_1' \cap \Omega_1^j)$$

for every $e_1' \in \tilde{\Sigma}_1'$. □

We can claim more, namely, that the above proposition in a certain sense completely characterises the Ω-condition. For this purpose consider first the following example.

Example 5.1. Let $\Omega_1 = (0,1) \times (0,1)$, Σ_1' being the usual Lebesgue σ-algebra of Ω_1 and μ_1 being the two-dimensional Lebesgue measure, while $\Omega_2 = (0,1)$, Σ_2 is the Lebesgue σ-algebra of Ω_2 and μ_2 is the one-dimensional Lebesgue measure. Let $\Sigma_1 \subset \Sigma_1'$ stand for the σ algebra of the sets $e_2 \times (0,1)$, where $e_2 \in \Sigma_2$, and define the σ-homomorphism F: $\tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ by setting

$$F([e_2 \times (0,1)]) := \tilde{e}_2.$$

Clearly, $\tilde{\Sigma}_1 \notin \Omega(\tilde{\Sigma}_1')$. Moreover, in this case F cannot be extended to the whole $\tilde{\Sigma}_1'$. In fact, if it were possible, then by the theorem 32.3 of [8] there would exist a measurable function g: $\Omega_2 \to \Omega_1$ satisfying (1) and $[g^{-1}(e_1)] = F(\tilde{e}_1)$ for every $e_1 \in \Sigma_1'$, that is,

$$[g^{-1}(e_2 \times (0,1))] = \tilde{e}_2.$$

Since $g = (\phi, \psi)$, where ϕ, ψ: $\Omega_2 \to \Omega_2$ are the components of the function g, then the above relationship implies

$$[\phi^{-1}(e_2)] = \tilde{e}_2.$$

for all $\tilde{e}_2 \in \tilde{\Sigma}_2$, and therefore $\phi(z) = z$ a.e. on Ω_2. In other words, $g(z) = (z, \psi(z))$ for some measurable function ψ: $(0,1) \to (0,1)$. The set $G := \operatorname{graph} \phi$ has then Lebesgue measure zero in Ω_1, but $g^{-1}(G) = \Omega_2$, hence contradicting (1).

We now extend the above example to the more general situation.

Proposition 5.2. *Let $(\Omega_1, \Sigma_1', \mu_1)$ be a standard measure space and $\tilde{\Sigma}_1 \notin \Omega(\tilde{\Sigma}_1')$. Then there exists a σ-homomorphism F: $\tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ which admits no extension to a σ-homomorphism F': $\tilde{\Sigma}_1' \to \tilde{\Sigma}_2$.*

PROOF: We take $(\Omega_2, \Sigma_2, \mu_2) = (\Omega_1, \Sigma_1', \mu_1)$. There are two possibilities: either $\tilde{\Sigma}_1 = g^{-1}(\tilde{\Sigma}_1')$ for some measurable function g: $\Omega_2 \to \Omega_1$ satisfying (1), or else $\tilde{\Sigma}_1$ is not generated by any function. In the latter case it is enough to take the identity σ-homomorphism for F. In fact, if it were extendable to the whole $\tilde{\Sigma}_1'$, then by the theorem 32.3 in [8] one would have $F(\tilde{e}_1) = [g^{-1}(e_1)]$ for some measurable g: $\Omega_2 \to \Omega_1$ satisfying (1), and hence, $\tilde{\Sigma}_1 = g^{-1}(\tilde{\Sigma}_1')$ contradicting the assumption. We concentrate therefore on the former case, namely, when $\tilde{\Sigma}_1 = g^{-1}(\tilde{\Sigma}_1')$. Let us remark that the function g cannot be piecewise injective, since otherwise Σ_1 would clearly satisfy the Ω-condition according to proposition 4.1. Restricting, if necessary, to a set of full measure in Ω_2, we may suppose that g is Borel measurable. Let $E_2' \subset \Omega_2$, $\mu_2(E_2') > 0$ be such that the restriction of g to E_2' is antiinjective, that is, for every $e_2 \subset E_2'$, $e_2 \in \Sigma_2$ one has $\mu_2(e_2) = 0$ whenever $g|_{e_2}$ is injective. Suppose also without loss of generality that $\mu_2(E_2') = 1$ (otherwise just renorm the measure). By proposition 2.2 from [6], which is a consequence of Maharam theorem on homogeneous measure algebras, there exist then a Borel set $E_2 \subset E_2'$, $\mu_2(E_2) = \mu_2(E_2') = 1$, a compact metric space M, a nonatomic Borel probability measure $\mathbb{P}$ on M, and a Borel measurable map τ: $E_2 \to M$ such that the following conditions hold

(a) the map $g \times \tau$: $E_2 \to \Omega_1 \times M$ defined by

$$(g \times \tau)(x) := (g(x), \tau(x))$$

is invertible;

(b) the image measure μ' over $\Omega_1 \times M$ defined by

$$\mu'(B) := \mu_2((g \times \tau)^{-1}(B))$$

for all Borel sets $B \subset \Omega \times M$ satisfies

$$\mu' = \mu_g \otimes \mathbb{P}, \text{ where } \mu_g(e_1) := \mu_2(g^{-1}(e_1)).$$

For the moment we identify g with its restriction to E_2, so that g: $E_2 \to \Omega_1$. It is easy to see that

$$g(x) = p_{\Omega_1}((g \times \tau)(x)), \tag{4}$$

where p_{Ω_1}: $\Omega_1 \times M \to \Omega_1$ is the projection map. Set now $F([g^{-1}(e_1)]) := \tilde{e}_1$ for $e_1 \in \Sigma_1'$. Defined in this way, F: $\tilde{\Sigma}_1 \cap E_2 \to \tilde{\Sigma}_2$ is a σ-homomorphism in view of (4). Denote by $\hat{\Sigma}_1'$ the completion with respect to μ' of the Borel σ-algebra of $\Omega_1 \times M$, and by $\hat{\Sigma}_1$ the σ-algebra of subsets of $\Omega_1 \times M$ of the form $e_1 \times M$, $e_1 \in \Sigma_1$. We note that F induces a σ-homomorphism $\hat{F}$: $\tilde{\hat{\Sigma}}_1 \to \tilde{\Sigma}_2$ according to the formula

$$\hat{F}(\hat{e}_1) := F((g \times \tau)^{-1}(\hat{e}_1)).$$

In fact, $\hat{F}$: $p_{\Omega_1}^{-1}(e_1) \mapsto e_1$. Suppose that F can be extended to F' defined over the whole $\tilde{\Sigma}_1' \cap E_2$. Then $\hat{F}$ would be extendible to an $\hat{F}'$: $\tilde{\hat{\Sigma}}_1' \to \tilde{\Sigma}_2$ according to the formula

$$\hat{F}(\hat{e}_1') := F((g \times \tau)^{-1}(\hat{e}_1')),$$

and the reasoning of the example 5.1 shows that this is impossible. □

6. Characterisation of the Ω-condition

Proposition 5.1 has an interesting consequence which extends the famous theorem of J. von Neumann on the representation of σ-homomorphisms (see [10, 5] or theorem 32.3 in [8]).

Corollary 6.1. *Let $\tilde{\Sigma}_1 \in \Omega(\tilde{\Sigma}_1')$, while $(\Omega_1, \Sigma_1', \mu_1)$ is a standard measure space. Then every σ-homomorphism $F\colon \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ is generated by a measurable map $g\colon \Omega_2 \to \Omega_1$ satisfying (1), namely,*

$$[g^{-1}(e_1)] = F(\tilde{e}_1)$$

for all $e_1 \in \Sigma_1$. In particular, there is a measurable map $h\colon \Omega_1 \to \Omega_1$ satisfying (1) such that $\tilde{\Sigma}_1 = h^{-1}(\tilde{\Sigma}_1')$.

PROOF: Every σ-homomorphism $F\colon \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ can be extended by proposition 5.1 to a σ-homomorphism $F'\colon \tilde{\Sigma}_1' \to \tilde{\Sigma}_2$. It is enough to apply the representation theorem 32.3 of [8] to the latter to conclude the proof of the first assertion. To prove the existence of the map h which generates the measure algebra $\tilde{\Sigma}_1'$, one applies the representation result to the identity σ-homomorphism. □

Now we can completely characterise the Ω-condition on σ-subalgebras in standard measure spaces. In fact, the theorem below gives the desired result on characterisation of Ω-condition on σ-algebras in terms of the existence of a generating function satisfying the ω-condition.

Theorem 6.1. *Let (Ω, Σ', μ) be a standard measure space with nonatomic measure and $\Sigma \subset \Sigma'$. Then $\tilde{\Sigma} \in \Omega(\tilde{\Sigma}')$, if and only if there is a function $h\colon \Omega \to \Omega$ satisfying the ω-condition, such that $\tilde{\Sigma} = h^{-1}(\tilde{\Sigma}')$. Moreover, every measurable function $h\colon \Omega \to \Omega$ satisfying (1) such that $\tilde{\Sigma} = h^{-1}(\tilde{\Sigma}')$ satisfies also the ω-condition.*

PROOF: If $h\colon \Omega \to \Omega$ satisfies the ω-condition, then according top proposition 4.1 one has $h^{-1}(\tilde{\Sigma}') \in \Omega(\tilde{\Sigma}')$. We prove therefore the reverse statement. Namely, suppose that $\tilde{\Sigma} \in \Omega(\tilde{\Sigma}')$. Then according to the corollary 6.1 there is a measurable $h\colon \Omega \to \Omega$ satisfying (1) such that $\tilde{\Sigma} = h^{-1}(\tilde{\Sigma}')$. On the other hand, the Ω-condition implies the existence of an at most countable cover $\Omega = \sqcup_j \Omega^j$ consisting of sets $\Omega_j \in \Sigma'$ of positive measure such that

$$h^{-1}(\Sigma') \cap \Omega^j / \Sigma^0 = \Sigma' \cap \Omega^j / \Sigma^0 \text{ for all } j \in \mathbb{N},$$

where $\Sigma^0 \subset \Sigma'$ stands for the σ-ideal of μ-nullsets. In other words for all $e \in \Sigma'$, $e \subset \Omega^j$ (with $j \in \mathbb{N}$ fixed) there is an $e' \in \Sigma'$ such that $h^{-1}(e') \cap \Omega^j = e$ modulo a μ-nullset. In order to prove that h satisfies the ω-condition, it suffices to show that so does its restriction to each Ω^j. Suppose that the latter is not true for some $j \in \mathbb{N}$. In view of proposition 3.1 from [9], there is then a set $E \subset \Omega^j$ with $\mu(E) > 0$ and a nonatomic σ-algebra $\hat{\Sigma}$ of subsets of E such that $h^{-1}(\Sigma') \cap E$ and $\hat{\Sigma}$ are independent, that is, for all $e_1 \in h^{-1}(\Sigma') \cap E$ and $e_2 \in \hat{\Sigma}$ one has

$$\mu_E(e_1 \cap e_2) = \mu_E(e_1)\mu_E(e_2), \text{ where } \mu_E(e) := \mu(e)/\mu(E).$$

Taking an arbitrary $e \in \hat{\Sigma}$, one has for the respective e' satisfying $h^{-1}(e') \cap E = e$ the relationship

$$\mu_E(e) = \mu_E(e \cap (h^{-1}(e') \cap E)) = \mu_E(e)\mu_E(h^{-1}(e') \cap E) = \mu_E(e)^2,$$

which implies that either $\mu_E(e) = 0$ or $\mu_E(e) = 1$, contradicting the nonatomicity of $\hat{\Sigma}$. □

Clearly the requirement of nonatomicity of the measure μ is essential in the above theorem. In fact, consider the following example.

Example 6.1. Let $\Omega = (0,1)$, Σ' be the Lebesgue σ-algebra of Ω and μ be the point measure defined by

$$\mu(e) := \begin{cases} 1, & x_0 \in e, \\ 0, & x_0 \notin e \end{cases}$$

for a given $x_0 \in \Omega$. Then clearly $\tilde{\Sigma}' = \{\tilde{\emptyset}, \tilde{\Omega}\}$. We consider $\Sigma := \Sigma'$. Then the function g: $\Omega \to \Omega$ defined by $g(x) := x_0$ for all $x \in \Omega$ generates $\tilde{\Sigma} = \tilde{\Sigma}'$ but is not piecewise injective.

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